Combinatorics of the classic groups with Questions

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Remind. The type BC,

$$BC_n = \{\sigma \in \mathfrak{S}_{\{\pm 1,\ldots,\pm n\}} : \sigma(i) = -\sigma(i)\} = \mathfrak{S}_n \ltimes \{\pm 1\}^n.$$

where

$$s_0 : 1 \leftrightarrow -1, \qquad \text{other fixed}$$

$$s_i : \pm i \leftrightarrow \pm(i+1), \qquad \text{other fixed}$$

We have two notations, mirror notation and $\bullet$-notation. It acts on polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$ by

$$\sigma x_i = x_{\sigma(i)}, \qquad \text{where } x_{-i} = -x_i.$$

The type D

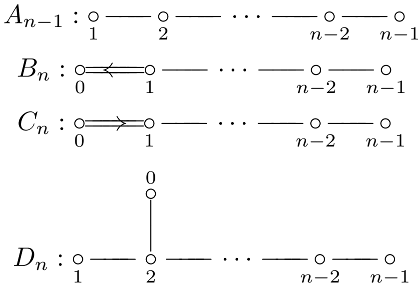
$$D_n = \{\sigma \in \mathfrak{S}_{\{\pm 1,\ldots,\pm n\}} : \sigma(i) = -\sigma(i), \text{and even}\} = \mathfrak{S}_n \ltimes \{\pm 1\}^{n-1}.$$

where

$$s_0 : \pm 1 \leftrightarrow \mp 2, \qquad \text{other fixed}$$

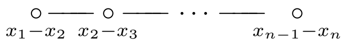
$$s_i : \pm i \overset{\text{exchange}}{\leftrightarrow} \pm(i+1), \qquad \text{other fixed}$$

It is convenient to use $\bullet$-notations. The action is the same as BC_n.



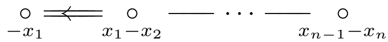
Roots. For A_{n-1} type,

roots	$\{x_i - x_j : i, j = 1, \ldots, n\}$
positive roots	$\{x_i = x_j : i < j\}$
simple roots	$\{x_i - x_{i+1} : i = 1, \ldots, n-1\}$

A_{n-1} :


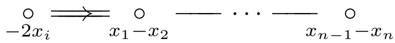
For B_n type,

roots	$\{\pm x_i : i = 1, \ldots, n\} \cup \{\pm x_i \pm x_j : i, j = 1, \ldots, n\}$
positive roots	$\{-x_i : i = 1, \ldots, n\} \cup \{x_i \pm x_j : i < j\}$
simple roots	$\{-x_1\} \cup \{x_i - x_{i+1} : i = 1, \ldots, n-1\}$

B_n :


For C_n type,

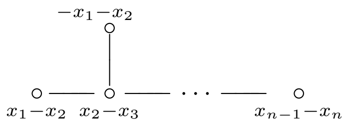
roots	$\{\pm 2x_i : i = 1, \ldots, n\} \cup \{x_i - x_j : i, j = 1, \ldots, n\}$
positive roots	$\{-2x_i : i = 1, \ldots, n\} \cup \{x_i \pm x_j : i < j\}$
simple roots	$\{-2x_1\} \cup \{x_i - x_{i+1} : i = 1, \ldots, n-1\}$

C_n :


For D_n type,

roots	$\{\pm 2x_i : i = 1, \ldots, n\} \cup \{x_i - x_j : i, j = 1, \ldots, n\}$
positive roots	$\{x_i \pm x_j : i < j\}$
simple roots	$\{-x_1 - x_2\} \cup \{x_i - x_{i+1} : i = 1, \ldots, n-1\}$

D_n :



There is other notation by $y_i = -x_{n+1-i}$, this will be positive, but it would not be stable.

The alternating sum. Let $X_i = e^{x_i}$, so the action

$$\sigma X_i = X_{\sigma(i)}, \qquad \text{where } X_{-i} = -X_i^{-1}.$$

Let $W = BC_n$ or D_n. Let us compute

$$\sum_{w \in W} (-1)^w x^\lambda \qquad \sum_{w \in W} (-1)^w X^\lambda$$

where $x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$, and $X^\lambda = X_1^{\lambda_1} \cdots X_n^{\lambda_n}$.

For BC type

$$\begin{aligned}
&\textstyle\sum_{w\in W} (-1)^w \cdot x^\lambda \\
&= \textstyle\sum_{\sigma \in \mathfrak{S}_n} \sum_{|w|=\sigma} (-1)^w x^\lambda && |\cdot| = \text{forgetting } \bullet\text{'s} \\
&= \textstyle\sum_{\sigma \in \mathfrak{S}_n} (-1)^\sigma \sum_{\text{independent } \pm\text{'s}} (-1)^{\#\{\pm=-\}} \prod_{i=1}^n (\pm x_{\sigma(i)})^{\lambda_i} \\
& && \because \#\{\bullet\} + |w| \equiv \ell(w) \mod 2 \\
&= \textstyle\sum_{\sigma \in \mathfrak{S}_n} (-1)^\sigma \sum_{\text{independent } \pm\text{'s}} \prod_{i=1}^n \pm (\pm x_{\sigma(i)})^{\lambda_i} \\
&= \textstyle\sum_{\sigma \in \mathfrak{S}_n} (-1)^\sigma \left(x_{\sigma(1)}^{\lambda_1} - (-x_{\sigma(1)})^{\lambda_1}\right) \cdots \left(x_{\sigma(n)}^{\lambda_n} - (-x_{\sigma(n)})^{\lambda_n}\right) \\
&= \textstyle\sum_{\sigma \in \mathfrak{S}_n} (-1)^\sigma \prod_i \left(x_{\sigma(i)}^{\lambda_i} - x_{\sigma(i)}^{-\lambda_i}\right) \\
&= \det(x_i^{\lambda_j} - (-x_i)^{\lambda_j}).
\end{aligned}$$

Remark 1. If some $\lambda_i \in 2\mathbb{Z}$, then it is zero. This is reasonable, since it is already symmetric under $x_i \leftrightarrow -x_i$.

Remark 2. If all $\lambda_i \in 2\mathbb{Z}+1$, then

$$\sum_{w \in W} (-1)^w \cdot x^\lambda = 2^n \det(x_i^{\lambda_j}).$$

$$
\begin{aligned}
&\textstyle\sum_{w\in W}(-1)^w X^\lambda\\
&= \textstyle\sum_{\sigma\in\mathfrak{S}_k}(-1)^\sigma \sum_{\text{independent } \pm\text{'s}}(-1)^{\#\{\pm=-\}} \prod X_{\sigma(i)}^{\pm\lambda_i}\\
&= \textstyle\sum_{\sigma\in\mathfrak{S}_k}(-1)^\sigma \sum_{\text{independent } \pm\text{'s}} \prod \big(\pm X_{\sigma(i)}^{\pm\lambda_i}\big)\\
&= \textstyle\sum_{\sigma\in\mathfrak{S}_k}(-1)^\sigma \prod_i \big(X_{\sigma(i)}^{\lambda_i} - X_{\sigma(i)}^{-\lambda_i}\big)\\
&= \det(X_i^{\lambda_j} - X_i^{-\lambda_j}).
\end{aligned}
$$

Or write

$$
\sum_{w\in W}(-1)^w X^\lambda = 2^n \det(\sinh \lambda_j x_i).
$$

For D type, note that

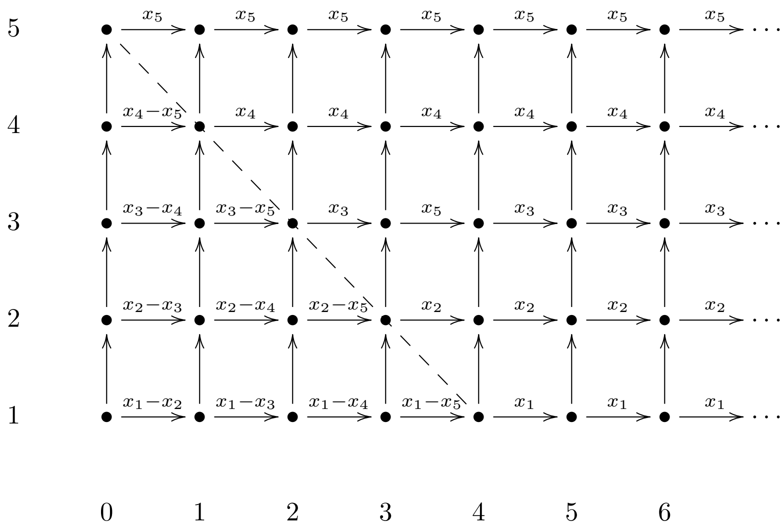
$$
\begin{aligned}
&\textstyle\sum_{w\in W}(-1)^w x^\lambda\\
&= \textstyle\sum_{\sigma\in\mathfrak{S}_k}(-1)^\sigma \sum_{\text{even } \pm=-}(\quad(\pm x_{\sigma(1)})^{\lambda_1})\cdots(\quad(\pm x_{\sigma(n)})^{\pm\lambda_n})\\
&\hspace{6cm} \because \ell(w)\equiv \ell(|w|) \mod 2\\
&= \textstyle\sum_{\sigma\in\mathfrak{S}_k}(-1)^\sigma \sum_{\text{even } \pm=-}(\pm(\pm x_{\sigma(1)})^{\lambda_1})\cdots(\pm(\pm x_{\sigma(n)})^{\pm\lambda_n})\\
&= \tfrac{1}{2}\textstyle\sum_{\sigma\in\mathfrak{S}_k}(-1)^\sigma \sum_{\pm}(\quad(\pm x_{\sigma(1)})^{\lambda_1})\cdots(\quad(\pm x_{\sigma(n)})^{\pm\lambda_n})\\
&\quad+\tfrac{1}{2}\textstyle\sum_{\sigma\in\mathfrak{S}_k}(-1)^\sigma \sum_{\pm}(\pm(\pm x_{\sigma(1)})^{\lambda_1})\cdots(\pm(\pm x_{\sigma(n)})^{\pm\lambda_n})\\
&= \tfrac{1}{2}\big(\det(x_i^{\lambda_j} + (-x_i)^{\lambda_j}) + \det(x_i^{\lambda_j} - (-x_i)^{\lambda_j})\big).
\end{aligned}
$$

$$
\begin{aligned}
&\textstyle\sum_{w\in W}(-1)^w X^\lambda = \sum_{w\in W}(-1)^w e^{\lambda\cdot x}\\
&= \textstyle\sum_{\sigma\in\mathfrak{S}_k}\sum_{|w|=\sigma} e^{\lambda\cdot wx}\\
&= \textstyle\sum_{\sigma\in\mathfrak{S}_k}(-1)^\sigma \sum_{\text{even } \pm=-}(\quad X_{\sigma(1)}^{\pm\lambda_1})\cdots(\quad X_{\sigma(n)}^{\pm\lambda_n})\\
&= \textstyle\sum_{\sigma\in\mathfrak{S}_k}(-1)^\sigma \sum_{\text{even } \pm=-}(\pm X_{\sigma(1)}^{\pm\lambda_1})\cdots(\pm X_{\sigma(n)}^{\pm\lambda_n})\\
&= \tfrac{1}{2}\textstyle\sum_{\sigma\in\mathfrak{S}_k}(-1)^\sigma \sum_{\pm}(\quad X_{\sigma(1)}^{\pm\lambda_1})\cdots(\quad X_{\sigma(n)}^{\pm\lambda_n})\\
&\quad+\tfrac{1}{2}\textstyle\sum_{\sigma\in\mathfrak{S}_k}(-1)^\sigma \sum_{\pm}(\pm X_{\sigma(1)}^{\pm\lambda_1})\cdots(\pm X_{\sigma(n)}^{\pm\lambda_n})\\
&= \tfrac{1}{2}\big(\det(X_i^{\lambda_j} + X_i^{-\lambda_j}) + \det(X_i^{\lambda_j} - X_i^{-\lambda_j})\big).
\end{aligned}
$$

1 Characters

Nonintersecting Paths (A type). The main technique is the Lindström–Gessel–Viennot Lemma See this survey.

For A type, this takes from my previous work.



$$\sum \text{weights of all paths } (0,k)\to(N,n) = x_k^N.$$

$\boxed{\text{Question}}$ I mentioned the relation of Newton interpolation with this grid. Can we say more?

For example

$$\begin{tikzcd}
1 \arrow[r, "x_5"] & x_1 \arrow[r, "x_5"] & x_1^2 \arrow[r, "x_5"] & x_1^3 \arrow[r, "x_5"] & {} \\
1 \arrow[u] \arrow[r, "x_4-x_5"] & x_1-x_5 \arrow[u] \arrow[r, "x_4"] & x_1(x_1-x_5) \arrow[u] \arrow[r, "x_4"] & x_1^2(x_1-x_5) \arrow[u] \arrow[r, "x_4"] & {} \\
1 \arrow[u] \arrow[r, "x_3-x_4"] & x_1-x_4 \arrow[u] \arrow[r, "x_3-x_5"] & (x_1-x_5)(x_1-x_4) \arrow[u] \arrow[r, "x_3"] & x_1(x_1-x_5)(x_1-x_4) \arrow[u] \arrow[r, "x_5"] & {} \\
1 \arrow[u] \arrow[r, "x_2-x_3"] & x_1-x_3 \arrow[u] \arrow[r, "x_2-x_4"] & (x_1-x_4)(x_1-x_3) \arrow[u] \arrow[r, "x_2-x_5"] & (x_1-x_5)(x_1-x_4)(x_1-x_3) \arrow[u] \arrow[r, "x_2"] & {} \\
1 \arrow[u] \arrow[r, "x_1-x_2"] & x_1-x_2 \arrow[u] \arrow[r, "x_1-x_3"] & (x_1-x_3)(x_1-x_2) \arrow[u] \arrow[r, "x_1-x_4"] & (x_1-x_4)(x_1-x_3)(x_1-x_2) \arrow[u] \arrow[r, "x_1-x_5"] & {}
\end{tikzcd}$$

The key induction process is

$$\begin{tikzcd}
\cdots \arrow[r] & (z)_{n+1}^{N-1} \arrow[r, "x_n-x_{n+N-1}"] & (z)_{n+1}^{N} \\
\cdots \arrow[r] & (z)_{n}^{N-1} \arrow[u] \arrow[r, "x_{n-1}-x_{n+N}"] & (z)_{n}^{N} \arrow[u] \\
& \vdots \arrow[u] & \vdots \arrow[u]
\end{tikzcd}$$

Define

$$(z)_n^N = (z-x_n)\cdots(z-x_{n+N-1})$$

We take $0 = x_{n+1} = x_{n+2} = \cdots$. Note that $(z)_n^N = (z-x_n)(z)_{n+1}^{N-1}$, so

$$(z)_n^N + (x_n - x_{n+N-1})(z)_{n+1}^{N-1} = (z)_{n+1}^N$$

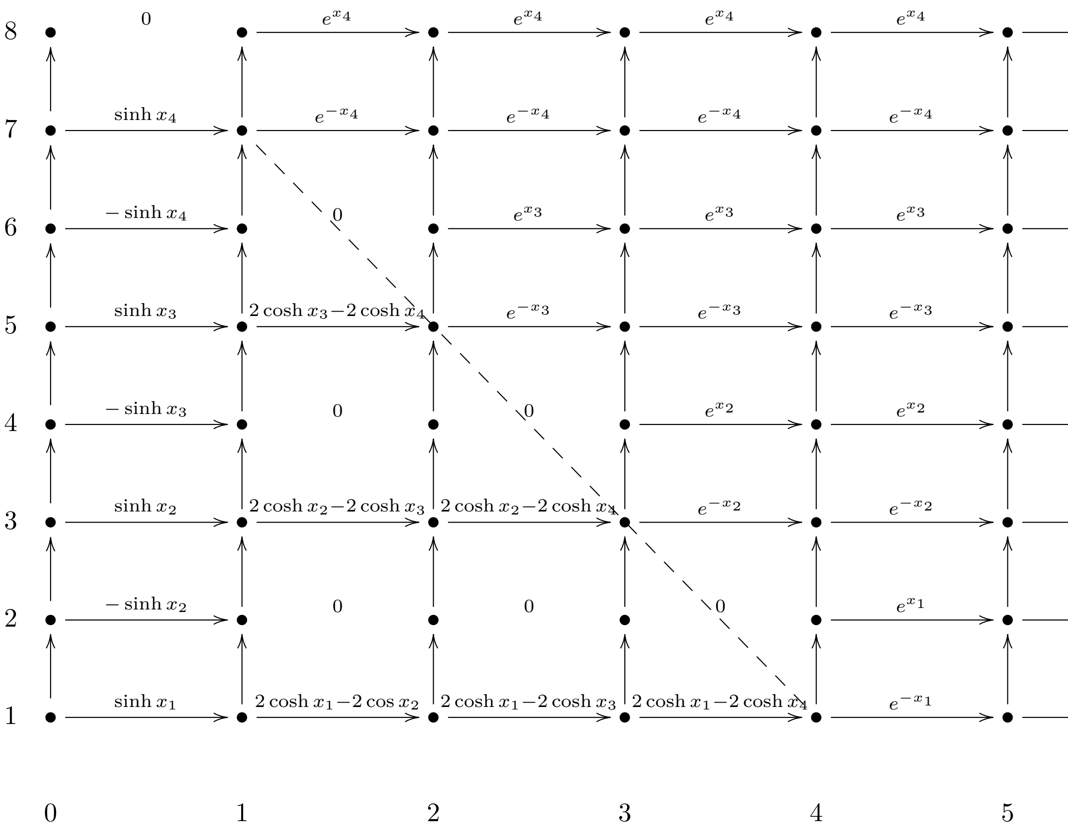
The right hand side.

This can also be used to get the factorial/equivariant Schur polynomials.

Nonintersecting Paths (BC type). Denote $\sinh x = \frac{e^x - e^{-x}}{2}$, and $\cosh x = \frac{e^x + e^{-x}}{2}$. I created a grid such that

$$\sinh(Nx_k) = \sum \text{weights of all paths } (0, 2k-1) \to (N, 2n).$$

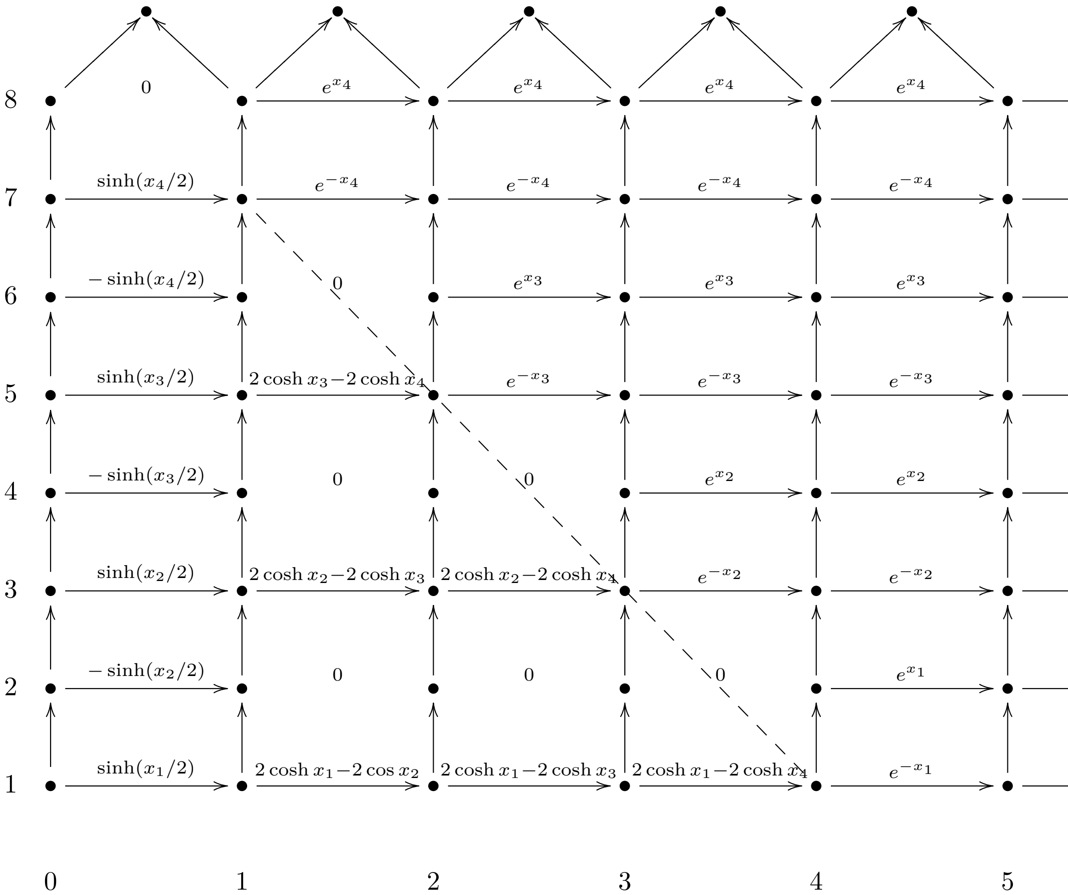
$$0 = \sum \text{weights of all paths } (0, 2k) \to (N, 2n).$$



I do not know how to prove it, but it is true (I spent one day to create it).

Question How to prove it?

I create the grid as the following



This gives $\sinh(N+\frac{1}{2})x$. Since

$$\begin{aligned} \sinh Nx + \sinh(N+1)x &= 2\cosh\tfrac{x}{2}\sinh(N+\tfrac{1}{2})x \\ \sinh x &= 2\cosh\tfrac{x}{2}\sinh\tfrac{x}{2} \end{aligned}$$

$\boxed{\text{Question}}$ Do we have any analogy for cosh?

Tableaux (A type). In A type, the half sum of positive roots (to suit GL_n) is

$$\rho = (n-1, n-2, \ldots, 1, 0)$$

The Weyl character formula

$$\chi(L(\lambda)) = \frac{\sum(-1)^w wX^{\lambda+\rho}}{\sum(-1)^w wX^{\rho}} = \frac{\det(X_i^{\lambda_j+n-j})}{\prod_{i<j}(X_i - X_j)}$$

here $X_i = e^{x_i}$.

So,

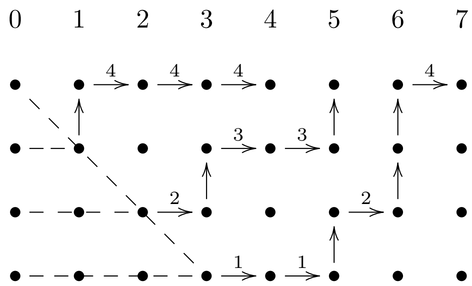
For A type (Littlewood).

$$1 < 2 < 3 < \cdots < n$$

(1) entries weakly increase across rows from left to right;
(2) entries strickly increase down columns from top to bottom.

1	1	2	4
2	3	3	
4	4	4	

$$n = 4$$
$$\lambda = (4, 3, 3, 0)$$
$$\lambda + \rho = (7, 5, 4, 0)$$



Tableaux (C type). C type is relatively easier.

$$\rho = \big(n, n-1, \ldots, 1\big)$$

The Weyl character formula

$$\chi(L(\lambda)) = \frac{\sum(-1)^w wX^{\lambda+\rho}}{\sum(-1)^w wX^{\rho}} = \frac{\det(\sinh(\lambda_j + n + 1 - j)x_i)}{\prod_{i<j}(\cosh x_i - \cosh x_j)\prod_i \sinh x_i}$$

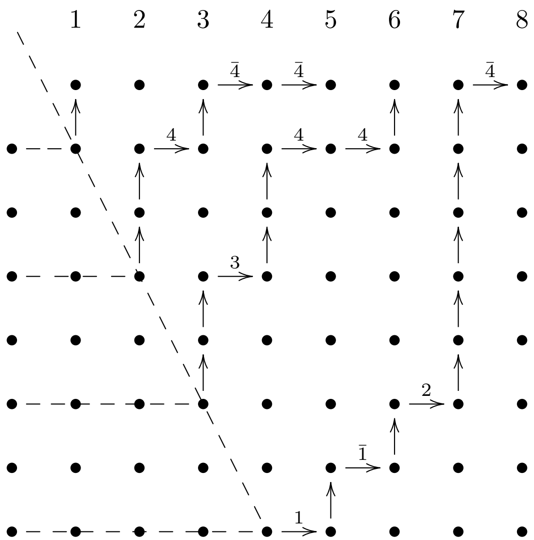
For C type (King).

$$1 < \bar{1} < 2 < \bar{2} < \cdots < n < \bar{n}$$

(1) entries weakly increase across rows from left to right;
(2) entries strickly increase down columns from top to bottom;
(3) neither k nor $\bar{k}$ appear lower than the k-th row.

1	$\bar{1}$	2	$\bar{4}$
$\bar{3}$	4	4	
4	$\bar{4}$	$\bar{4}$	

$n = 4$
$\lambda = (4, 3, 3, 0)$
$\lambda + \rho = (8, 6, 5, 1)$



Tableaux (B type). In B type, we have to have 1/2. $\rho = \left(k - \frac{1}{2}, k - \frac{1}{2} - 1, \ldots, \frac{1}{2}\right)$. In this case, $\lambda_i \in \frac{1}{2}\mathbb{Z}$ also makes sense (the spinor representations), but we omit.

The Weyl character formula

$$\chi(L(\lambda)) = \frac{\sum (-1)^w w X^{\lambda+\rho}}{\sum (-1)^w w X^{\rho}} = \frac{\det(\sinh(\lambda_j + n + \frac{1}{2} - j)x_i)}{\prod_{i<j}(\cosh x_i - \cosh x_j)\prod_i \sinh(x_i/2)}$$

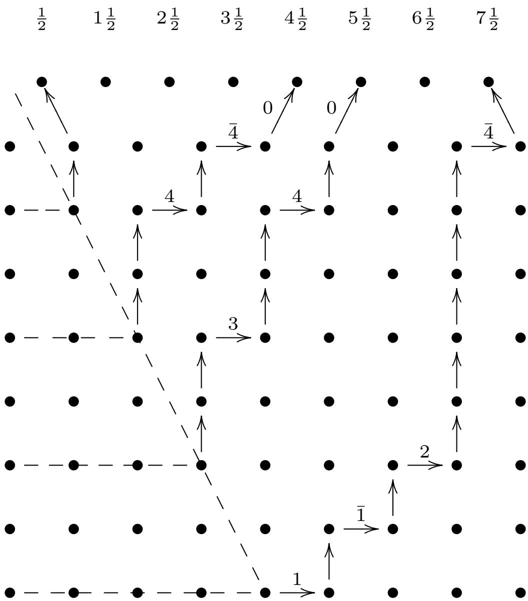
For B type (Sundaram).

$$1 < \bar{1} < 2 < \bar{2} < \cdots < n < \bar{n} < 0$$

(1) entries weakly increase across rows from left to right;
(2) entries strickly increase down columns from top to bottom;
(3) neither k nor $\bar{k}$ appear lower than the k-th row;
(4) in any row 0 appears at most once.

1	$\bar{1}$	2	$\bar{4}$
$\bar{3}$	4	0	
4	$\bar{4}$	0	

$$n = 4$$
$$\lambda = (4, 3, 3, 0)$$
$$\lambda + \rho = (7\tfrac{1}{2}, 5\tfrac{1}{2}, 4\tfrac{1}{2}, \tfrac{1}{2})$$



In any case

$$\chi(L(\lambda)) = \sum_{\text{tableaux} T} X^T$$

$1, \ldots, n$	$e^{x_1}, \ldots, e^{x_n}$
$\bar{1}, \ldots, \bar{n}$	$e^{-x_1}, \ldots, e^{-x_n}$
0	1

where $X_i = e^{x_i}$ as usual.

Type D. In type D, we have $\rho = (n-1, \ldots, 1, 0)$.

$$\chi(L(\lambda)) = \frac{\det(\cosh(\lambda_j + n - j)X_i) + \det(\sinh(\lambda_j + n - j)X_i)}{\prod_{i<j}(\cosh x_i - \cosh x_j)}.$$

$\boxed{\text{Question}}$ Can one create a model for type D? (well, firstly, let us also think about cosh, it is relative).

$\boxed{\text{Question}}$ Is there RSK algorithm, or LR description of the decomposition of product of them?

$\boxed{\text{Question}}$ This is also a description of O_{2n}, which is harder than D $(= SO_{2n})$.

$\boxed{\text{Question}}$ What is the Littelmann path model?

References:

`https://arxiv.org/pdf/1607.06982.pdf`;

`https://arxiv.org/pdf/1710.00638.pdf`.

A more general question.

$\boxed{\text{Question}}$ Note that all the grid I created above (well, half of each of them is created by me) has the following property that there is a set of polynomials $\{f_n(x)\}$ (in on variable) and point $\{q_n\}$; with a set of variables $\{x_k\}$ and points $\{p_k\}$ such that

$$f_n(x_k) = \sum \text{weights of all paths } p_k \to q_n.$$

It should not very easy to create.

Maybe one can try more classic polynomials. For example, the second Chebyshev polynomial is what I created for BC type.

2 Schubert Polynomials

We should concern on the cohomology of $H_T^*(G/B)$. On the contrast of A type, the Grothendieck polynomials are less relative to Schubert polynomials in general.

I would like to say that the classic type should admit a stable choice, but I never see “infinite isotropic/sympletic flags”.

Schubert Polynomials (Type A). Define the Schubert polynomial for $w \in \mathfrak{S}_\infty$

$$\mathfrak{S}_w(x,t) \in \mathbb{Z}[x_1,\ldots,t_1,\ldots]$$

with the following properties

$$\begin{cases} \mathfrak{S}_w(x,ux) = 0 & \text{whenever } u < w \\ \partial_i \mathfrak{S}_w(x,t) = \begin{cases} \mathfrak{S}_{ws_i}(x,t), & \ell(ws_i) = \ell(w)-1, \\ 0, & \ell(ws_i) = \ell(w)+1. \end{cases} \\ \mathfrak{S}_{\mathrm{id}}(x,t) = 1 \end{cases}$$

Here the Demazure operator

$$\partial_i f(x,y) = \frac{f(x,y) - f(s_i x, y)}{x_i - x_{i+1}}.$$

This uniquely determine $\mathfrak{S}_w(x,y)$ by the Atyiah–Borel localization theorem.

Note that ∂_i satisfies the braid relation, so we can define the Nil-Hecke algebra NH. So that ∂_w makes sense for $w \in \mathfrak{S}_\infty$.

It is easy to see that

$$\partial_{w_0^n} f = \frac{\sum_{w\in\mathfrak{S}_n} (-1)^w f}{\prod_{i<j}(x_i - x_j)}.$$

Some basic example

$$\begin{aligned} \mathfrak{S}_{s_i} &= x_1 + \cdots + x_i - (t_1 + \cdots + t_i), \\ \mathfrak{S}_{w_0^n} &= \textstyle\prod_{i+j\le n}(x_i - t_j). \end{aligned}$$

where w_0 is the longest word in $\mathfrak{S}_n$. If w is a Grassmannian permutation (that is, a shuffle), then it is a Schur polynomial.

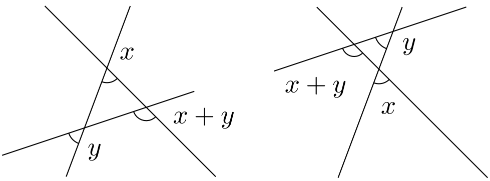
Remark. I would like to say that the Schur polynomials appear both in representation theory and geometry is a coincidence that $BGL_n = Gr(n, \infty)$ and $x \mapsto e^x$ is $\mathfrak{S}_n$-equivariant.

Pipe Dreams (Type A). The coefficient of ∂_w of

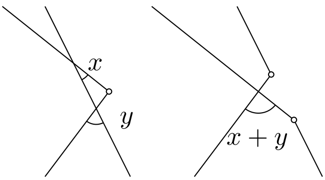
$$\mathfrak{S}(x,y) = \begin{array}{ccc} (1+(x_1-y_{n-1})\partial_{n-1}) & \cdots & (1+(x_1-y_1)\partial_1) \\ & \ddots & \vdots \\ & & (1+(x_{n-1}-y_1)\partial_{n-1}) \end{array}$$

is the double Schubert polynomial $\mathfrak{S}_w(x,y)$. Here ∂_i commutes with all x_i's and y_i's. See this and this.

The key to prove this is the Yang–Baxter equation, for $h_i(x) = 1 + x\partial_i$

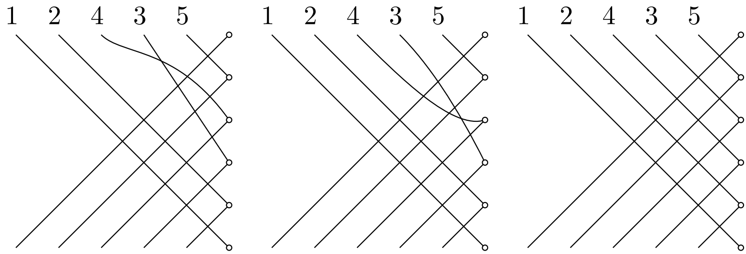


$$\begin{cases} h_i(x)h_j(y) = h_j(y)h_i(x) & |i-j| \geq 2 \\ h_i(x)h_{i+1}(x+y)h_i(y) = h_{i+1}(y)h_i(x+y)h_{i+1}(x). \end{cases}$$



$$h_i(x)h_i(y) = h_i(x+y)$$

So

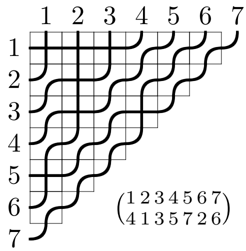


Show that

$$\mathfrak{S}(x,y)\partial_i = \frac{\mathfrak{S}(x,y) - \mathfrak{S}(s_i x, y)}{x_i - x_{i+1}}.$$

Pipe dream is to fill $\square$ by $+$ or ⌟⌜, such that no pair of pipes crossing more than twice.

$$j \cdots \boxed{+} = x_i - y_j \qquad j \cdots \boxed{⌟⌜} = 1$$



Generalized Schubert Polynomials. In the case of BC type, we have one more reflection s_0. Define

$$\begin{aligned}
\partial_i f(x,t) &= \frac{f(x,t) - f(s_0 x, t)}{\alpha_0} \\
&= \begin{cases} \dfrac{f(x_1, x_2, \ldots, t_1, \ldots) - f(-x_1, x_2, \ldots, t_1, \ldots)}{-x_1} & \text{For type B} \\[2ex] \dfrac{f(x_1, x_2, \ldots, t_1, \ldots) - f(-x_1, x_2, \ldots, t_1, \ldots)}{-2x_1} & \text{For type C} \end{cases}
\end{aligned}$$

The $-$ looks weird, maybe $-x_1 = 0 - x_1$ would help understand.

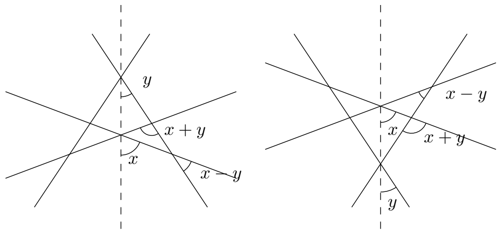
Remark. In standard book of Lie algebras, the standard notation of x_i is x_{n-i+1} here. PS: The standard notation is a bad notation, it is not stable.

Then ∂_i's also satisfy the braid relations (for its Coxeter datum). So ∂_w makes sense for BC_∞.

It is easy to see that

$$\partial_{w_0^n} f = \begin{cases} \dfrac{\sum_{w \in BC_n} (-1)^w f}{\prod_{i<j}(x_i^2 - x_j^2)\prod_i(-x_i)} & \text{For type B} \\ \dfrac{\sum_{w \in BC_n} (-1)^w f}{\prod_{i<j}(x_i^2 - x_j^2)\prod_i(-2x_i)} & \text{For type C} \end{cases}$$

Denote $h_i(x) = 1 + x\partial_i$. Then we have



$$\begin{aligned} &h_0(y)h_1(x+y)h_0(x)h_1(x-y) \\ &\quad = h_1(x-y)h_0(x)h_1(x+y)h_0(y) \end{aligned}$$

See this.

Define the Schubert polynomials for $w \in BC_\infty$, for type X

$$\mathfrak{S}_w^X(x,t) \in \mathbb{Z}[x_1, \ldots, t_1, \ldots]$$

with the following properties

$$\begin{cases} \mathfrak{S}_w^X(x, ux) = 0 & \text{whenever } u < w \\ \partial_i \mathfrak{S}_w(x,t) = \begin{cases} \mathfrak{S}_{ws_i}^X(x,t), & \ell(ws_i) = \ell(w) - 1, \\ 0, & \ell(ws_i) = \ell(w) + 1. \end{cases} \\ \mathfrak{S}_{\mathrm{id}}^X(x,t) = 1 \end{cases}$$

Actually, $\mathfrak{S}_w^X(x, wx) = \delta_{w=\mathrm{id}}$ characterize them.

In type B the coefficients are not always integers. In type C it is always integers, and

$$\mathfrak{S}_w^B(x,y) = 2^{-\#\bullet}\mathfrak{S}_w^C(x,y).$$

For type D, it is similar, but I do not bother to present them here.

$\boxed{\text{Question}}$ What is the choice satisfying above? Is this the one in Smirnov, and Tutubalina's paper? Is this the one in Kirillov's paper? It exists by geometric reason. But I did not see the assertion.

But I want to remark that

$$\partial_{w_0^n} f = \frac{\sum_{w \in BC_n} (-1)^w f}{\prod(\text{all positive roots})},$$

which would inspire a good choice. [We computed!]

We only need to find polynomials $\mathfrak{S}_{w_0^n}(x, y)$, such that $\mathfrak{S}_{w_0^n}(x, wx) = 0$ for $w < w_0^n$, and $\partial_{w_0^n w_0^{n+1}} \mathfrak{S}_{w_0^{n+1}}(x, y)$. Note that $\deg \mathfrak{S}_{w_0^n} = \ell(w_0^n) = n^2$.

(Maybe firstly find some good choice such that $\partial_{w_0^n} f = 1$)

Cauchy Identities. In my paper, I find a formula to prove Cauchy identities.

$$(-1)^{w_0} \partial_{w_0} \big(F(w_0 x) G(x)\big) = \sum_{w \in W} \partial_{w w_0} F(x) \cdot (-1)^w \partial_w G(x).$$

But it requires $\mathfrak{S}_w^X(x, ux) = 0$.

We need the Leibniz rule

$$\partial_i(fg) = \frac{fg - s_i f s_i g}{\alpha} = \frac{fg - s_i fg}{\alpha} + \frac{s_i fg - s_i f s_i g}{\alpha} = (\partial_i f) g + (s_i f)(\partial_i g),$$

We can assume

$$(-1)^{w_0} \partial_{w_0} \big(F(w_0 x) G(x)\big) = \sum_{w \in W} c_w(x) \cdot (-1)^w \partial_w G(x).$$

Since whenever we apply ∂_i, the left hand side will vanish, we have

$$\sum_{w \in W} \big(\partial_i c_w(x)\big) \cdot (-1)^w \partial_w G(x) + \sum_{w \in W} c_w(s_i x) \cdot (-1)^w \partial_i \partial_w G(x) = 0.$$

In other word, $\partial_i c_w(x) = \begin{cases} c_{s_i w}(s_i x), & s_i w < w, \\ 0, & s_i w > w. \end{cases}$. But $\partial_i c_w(x)$ is fixed by s_i, so

$$\partial_i c_w(x) = \begin{cases} c_{s_i w}(x), & \ell(s_i w) = \ell(w) - 1, \\ 0, & \ell(s_i w) = \ell(w) + 1. \end{cases}.$$

It is clear that $c_{w_0}(x) = F(x)$. So the lemma follows from induction.

We can apply this lemma to show the Cauchy identities

$$\begin{aligned} (-1)^{w_0} \partial^y_{w_0} \big(\mathfrak{S}_{w_0}(w_0 y, x) \mathfrak{S}_w(y, t)\big) &= \sum_u (-1)^u \big(\partial^y_{u w_0} \mathfrak{S}_{w_0}(y, x)\big) \cdot \partial^y_u \mathfrak{S}_w(y, t) \\ &= \sum_{w = wu^{-1} \odot u} (-1)^u \mathfrak{S}_{u^{-1}}(y, x) \cdot \mathfrak{S}_{wu^{-1}}(y, t) \\ &= \sum_{w = u \odot v} (-1)^v \mathfrak{S}_{v^{-1}}(y, x) \cdot \mathfrak{S}_u(y, t). \end{aligned}$$

Since after applying ∂_{w_0}, any polynomial becomes symmetric, so the right hand side is symmetric in y.

Let us take $w_0 \in BC_n$ for n big enough, then the right hand side will not appear y_n. But it is symmetric in $y_1, \dots, y_n$, the sum actually does not relate to y. So in particular,

$$\begin{aligned} & \sum_{w = u \odot v} (-1)^v \mathfrak{S}_{v^{-1}}(y, x) \cdot \mathfrak{S}_u(y, t) \\ = & \sum_{w = u \odot v} (-1)^v \mathfrak{S}_{v^{-1}}(x, x) \cdot \mathfrak{S}_u(x, t) = \mathfrak{S}_w(x, t), \\ = & \sum_{w = u \odot v} (-1)^v \mathfrak{S}_{v^{-1}}(t, x) \cdot \mathfrak{S}_u(t, t) = (-1)^w \mathfrak{S}_w(t, x). \end{aligned}$$

Since $\mathfrak{S}_w(x, x) = \begin{cases} 1, & w = 1 \\ 0, & w \neq 1. \end{cases}$. So we proved

$$\mathfrak{S}_w(x, t) = \sum_{w = u \odot v} \mathfrak{S}_v(x, y) \mathfrak{S}_u(y, t)$$

Taking in the second identity to above, finally we get

$$\mathfrak{S}_w(x, t) = \sum_{w = u \odot v} \mathfrak{S}_v(x, y) \cdot \mathfrak{S}_u(y, t).$$

The proof is complete.

$\boxed{\text{Question}}$ I have not yet generalize it to Grothendieck polynomials.

$\boxed{\text{Question}}$ What is the relation of BCD type and the A type?

$\boxed{\text{Question}}$ What is the polynomial for generalized Grassmannian (for maximal parabolic)? Usually, the maximal parabolic is relatively easier to be understood.

$\boxed{\text{Question}}$ Do we have bumpless pipe dream for type BCD?

$\boxed{\text{Question}}$ What is for Grothendieck polynomial? (Well, this is easy, simply exchanging each $+$ by $+_\beta$, but I did not see too much people do research in this direction)

$\boxed{\text{Question}}$ In type B, we can take more general $s_0(x) = \frac{-x}{\beta x+1}$, the $+_\beta$ inverse. Then there might be more combinatorics.

References:

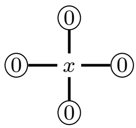
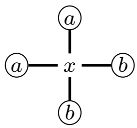
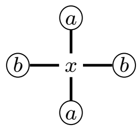
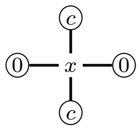
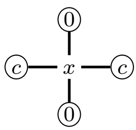
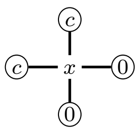
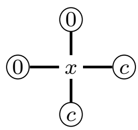
`https://arxiv.org/pdf/1504.01469.pdf` (it is $\mathfrak{S}_w(x,-y)$ to use suit the convention of the Grothendieck polynomials)

`https://arxiv.org/pdf/2009.14120.pdf` (relates to Stanley functions etc.)

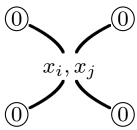
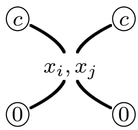
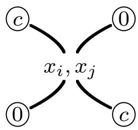
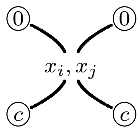
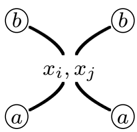
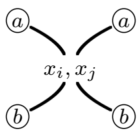
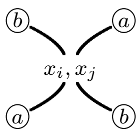
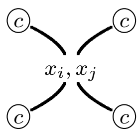
Appendix

The vertices model for Bumpless pipe dream of Grothendieck polynomials by Buciumas, and Scrimshaw in `https://arxiv.org/pdf/2007.04533.pdf`

The vertices ($0 < a < b$, $0 < c$)

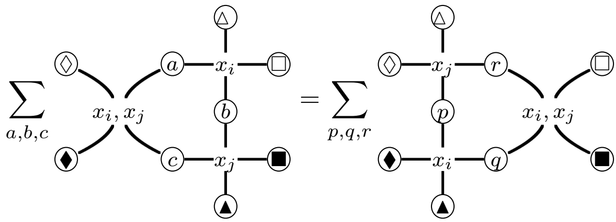
			
			
$\beta(x \oplus y)$	1	1	1
			
			others
1	$1 + \beta(x \oplus y)$	1	0

The R-matrices ($0 < a < b$, $0 < c$)

			
$1 + \beta x_i$	$1 + \beta x_i$	$\beta(x_j - x_i)$	$1 + \beta x_j$
			
$1 + \beta x_i$	$1 + \beta x_j$	$\beta(x_j - x_i)$	$1 + \beta x_i$

For the others, the weights are all zero.

For fixed $\lozenge, \blacklozenge, \triangle, \blacktriangle, \square, \blacksquare \in \mathbb{Z}_{\geq 0}$,



It is interesting that this is checked by computer in the original paper.

Let us omit the sum and the fixed boundary conditions.