Moduli and Algebraic Stack

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Abstract

This notes is taken during the seminar on moduli and algebraic stacks.
Main references: [Ols16], [Vis04]

The lectures notes were taken using overleaf. All errors are mine own.

If you have comments or suggestions, which are very welcomed, please email me at $\boxed{\textit{zhengze.xin@stonybrook.edu}}$.

Contents

1 Overview(Incomplete)

What do we mean when we talk about "moduli"? This could be a vague concept with various meanings. In general it presents a family of objects up to certain equivalent relations. In this sense we will study *moduli problems, moduli spaces, moduli groupoids, moduli stacks* and so on. We encounter moduli spaces in different settings, say, flat connections, Higgs bundles, branched covers (Hurwitz moduli), stable sheaves, curves with fixed invariants... Here we will concentrate on the 'moduli spaces' in algebraic geometry.

The first example in algebraic geometry could be the Picard group. The generalized Picard groups classify the line bundles of a fixed degree. From Abel-Jacobian theory we know it's an abelian variety. Another example comes from Grothendieck's theorem on classification of the vector bundles on $\mathbb{P}^1$. The vector bundles are given by $\mathcal{O}(a_1) \oplus \oplus \mathcal{O}(a_r)$, where $\sum_1^r a_i = d$. So the moduli set is the increasing tuples of integers with sum of d.

However, not all moduli spaces are 'good' (some aren't even varieties). If the moduli spaces are as good as varieties, then we are able to talk about their dimension, smoothness, properness and so on. For instance, the dimension of Deligne-Mumford Stacks is well-studied, and this would be one of the crucial objects in the following lectures.

A point in moduli space represents a space (resp, vector bundle, variety...). How do we recover this space? This leads to a difficult part in this topic. Let's say we have moduli space $\mathcal{M}$ which parametrize a family of varieties. What we want is to find an universal family $U \to \mathcal{M}$ such that the fiber at point $[X] \in \mathcal{M}$ is exactly X. The notion of 'algebraic stacks' tells us when we can find an universal family.

2 Organizing talk

In the first section we give an overview and outline of the materials we will go through in this seminar. Please have a look on Jarod Alper's Notes on Stacks and Moduli for a better overview.

The starting point is considering the moduli problems. Roughly speaking, the moduli problems study families of algebraic objects.

Definition 2.1. Let S be a base scheme. Assume S is noetherian and of finite type over base field k.

A moduli functor is a contravariant functor from the category of S-scheme to the category of sets that associates to an S-scheme X the equivalence classes of families of geometric objects which parametrized by X.

Example 2.2.

- (Moduli functor of smooth curves) Fix genus g, we can define the functor

$$F_{M_g} : \mathcal{S}ch \to \mathcal{S}et, \quad S \mapsto \{\text{families of smooth curves } \mathcal{C} \to S \text{ with genus } g\},$$

  where by family of smooth curves we mean a smooth proper morphism $\mathcal{C} \to S$ such that every fiber $\mathcal{C}_s$ is connected curve with genus g.

- (Moduli functor of vector bundles on a curve) Fix a smooth connected projective curve C, and integers $r \geq 0, d$. Then for scheme S, we can define family of vector bundles over S, by assigning a vector bundle $\mathcal{E}$ on $C \times S$, where the fibers of $\mathcal{E}$ over $s \in S$ (as a vector bundle over $C_{k(s)}$) has rank r and degree d. The moduli functor of vector bundles on C of rank r and degree d is then given by

$$F_{M_{C,r,d}} : \mathcal{S}ch \to \mathcal{S}et, \quad S \mapsto \left\{ \begin{array}{c} \text{families of vector bundles } \mathcal{E} \text{ on } C \times S \\ \text{with rank } r \text{ and degree } d \end{array} \right\}$$

- (Moduli functor of orbits) This example is about taking quotients. Consider an algebraic group G acting on scheme X. We can consider the functor

$$F_{G,X} : \mathcal{S}ch \to \mathcal{S}et, \quad S \mapsto X(S)/G(S),$$

  where the quotient set is defined via GIT.

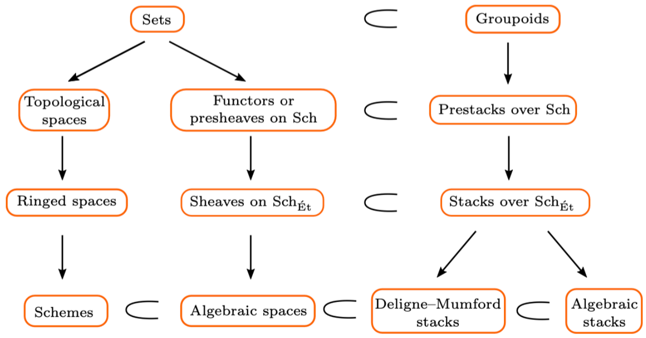

Figure 1: An algebraic-geometric enrichment of sets and groupoids

2.1 Representability

Definition 2.3. We say that a functor $F : \mathcal{S}ch \to \mathcal{S}et$ is representable by a scheme X if there's an isomorphism of functors $F \cong h_X$.

Representability will play a crucial role in the theory.

A typical example of the representability comes from algebraic topology.

Example 2.4. Classifying space

Let G be a topological group and let $\mathcal{CW}_\bullet$ be the homotopy category of pointed connected CW-complexes. A *classifying space* for G is a space BG such that for all $X \in CW_\bullet$, there exists a bijection between the pointed principal G-bundles on X and the set $[X, BG]$.

The existence of BG is guaranteed by the Brown's representability theorem.

Lemma 2.5. *Let $F : h\mathcal{CW}_\bullet \to h\mathcal{S}et_\bullet$ be a contravariant functor from the homotopy category of pointed connected CW-complexes to pointed sets, which satisfies the following two properties.*

- *Given any collection $\{X_\alpha\}$ of elements in $\mathcal{CW}_\bullet$, $F(\vee_\alpha X_)) \to \prod_\alpha F(X_\alpha)$ is an isomorphism.*
- *$F(Y \cup_X Z) \to F(Y) \times_{F(X)} F(Z)$ is a surjection for any two cofibrations $X \to Y$ and $X \to Z$.*

Then F is representable.

And we have a universal family $EG \to BG$.

Example 2.6. The Grassmannian functor $Gr(k, n)$ is representable by a projective scheme over $\mathbb{Z}$. The same is ture for Hilb and Quot functors.

This is illustrated by the following theorem, which is saying that $Gr(1, n)$ is representable by $\mathbb{P}_{\mathbb{Z}}^{n-1}$.

Theorem 2.7. *(Moduli functor of projective spaces) There's a functorial bijection*

$$Mor(S, \mathbb{P}_{\mathbb{Z}}^n) \cong \{\textit{globally generated line bundles up to isomorphism}\}$$

How can we determine a moduli functor is representable or not? The following lemma reveals that the non-trivial automorphisms form the obstruction to representability.

To see this, first consider the sheaves represented by schemes. Let's say $F = \mathrm{Mor}(-, X) : \mathcal{S}ch \to \mathcal{S}et$, then as a presheaf in the big Zariski site, it must be a sheaf.

However glueing can fail. For instance, we take F by sending scheme S to the set of its globally sections. Then working with $\mathbb{P}^1$ and its open cover U_1, U_2 we see that $F(\mathbb{P}^1)$ cannot be obtained from $F(U_0)$ and $F(U_1)$ because there're nontrivial transition maps between U_0 and U_1. In this case F is not representable. We can see that representability is not always satisfied. A necessary condition is that the functor can be "glued". For instance, assume that $F = \mathrm{Mor}(-, X)$ is represented by scheme X. Then for scheme S, F defines a presheaf on S by sending $U \subset S$ to $F(U) = \mathrm{Mor}(S, X)$. It's in fact a sheaf in big Zariski site over S. Thus failure to be a sheaf results in the failure to be representable.

Definition 2.8. Let $F : \mathcal{C}^{op} \to \mathcal{S}et$ be a moduli functor representable by a scheme X via an isomorphism $\alpha : F \to h_X$. The universal family of F is the object $\xi \in F(X)$ corresponding under α to the identity $id_X \in h_X(X) = \mathrm{Mor}(X, X)$. Another way of saying this, the pair $(X, \xi \in F(X))$ has the property that, for each object $U \in \mathcal{C}$ and each $\sigma \in F(U)$, there's an unique arrow $f : U \to X$ such that the pullback of ξ is σ.

Proposition 2.9. *The functor F is representable iff it has an universal family. For instance, if F has an universal object (X, ξ), then F is represented by X.*

Example 2.10. The universal family of the moduli functor of projective spac e is the line bundle $\mathcal{O}(1)$ together with the sections $x_0, \cdots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$.

2.2 Moduli stacks

The failure of representability of moduli functors is the motivation for introducing the moduli stack.

Since the representability can be determined by the existence of the universal family, we need to additionally specify how families pull back under morphisms. The non-trivial pullback can be thought as the obstruction to the representability.

This is telling us we need a more precise discussion on pullbacks $\{f^*\}$. At first glimpse, we should expect the compatibility for compositions, i.e. there should be an isomorphism of functors $\mu_{f,g} : (f^* \circ g^*) \cong (g \circ f)^*$. This illustrates a more general setting on the target of moduli functors–we should not only consider the category of sets. This leads to the notion of category of groupoids.

We should consider moduli functor as follows:

$$F : \mathcal{S}ch \to \mathcal{G}roupoids, \quad S \mapsto Fam_S.$$

Let's consider a massive category $\mathcal{X}$ encoding all the groupoids Fam_S.

Definition 2.11. A prestack over $\mathcal{S}ch$ is a category $\mathcal{X}$ together with functor $p : \mathcal{X} \to \mathcal{S}ch$, such that pullback exists and pullbacks are universal, which we will define later.

Given a scheme S, $\mathcal{X}(S)$ will be a fibered category $\mathcal{S} = Fam_S$.

Example 2.12. We define the moduli prestack of smooth curves as the category $\mathcal{M}_g$ of families of smooth curves $\mathcal{C} \to S$ together with the functor $p : \mathcal{M}_g \to \mathcal{S}ch$, where $p((\mathcal{C} \to S)) \mapsto S$.

Also we can define moduli prestack of vector bundles $\mathcal{B}un_{r,d}(C)$ where C is a fixed smooth connected and projective curve.

Stacks to prestacks is just as sheaves to presheaves. The definition is postponed.

2.3 Fibered category

Definition 2.13. Fixed a category $\mathcal{C}$. A fibered category over $\mathcal{C}$ is a category $\mathcal{F}$ over $\mathcal{C}$, such that given an arrow $f : U \to V$ in $\mathcal{C}$ and an object $\eta \in \mathcal{F}$ mapping to V, there's a cartesian arrow $\phi : \xi \to \eta$ with $p_{\mathcal{F}}\phi = f$.

Example 2.14.

- Let $\mathcal{C}$ be a category with fibered products. Let $\mathcal{F}$ be the category of arrows in $\mathcal{C}$. Then $\mathcal{F}$ is a fibered category over $\mathcal{C}$. Where p_F is given by sending $(X \to Y)$ to Y.
- Consider the category of sheaves $\mathcal{S}h$ on a site. Let $\mathcal{F}$ be the category whose objects are $\mathcal{S}h(X)$. Then the functor p_F sending $\mathcal{S}h(X)$ to scheme X is a fibered category. The pullback is given by the pullback of sheaves, $f^* : \mathcal{S}h(Y) \to \mathcal{S}h(X)$.

Now we introduce a baby example for glueing in fibered category.

Example 2.15. (Cont is a *stack* over Top)

Let $\mathcal{C} = \mathcal{T}op$ be the category of topological spaces and $\mathcal{F} = \mathcal{C}ont$ be the category of continuous maps. Then $\mathcal{F}$ is a fibered category over $\mathcal{C}$ via functor $p_{\mathcal{F}}$ sending each morphism to its target.

2.4 Algebraic stack

Even if a moduli functor is represented by some moduli space, it might not be a scheme. To impose algebraic geometry, we need criterions of determining whether a stack is algebraic or not.

Definition 2.16. We say that a stack $\mathcal{X} \to \mathcal{S}ch$ is algebraic if the diagonal morphism is representable and there's a smooth surjective morphism $f : X \to \mathcal{X}$ where X is a scheme.

If we replace 'smooth' by 'étale' (stronger condition), then we obtain Deligne-Mumford stack.

Remark. The requirement of diagonal being representable is due to technical reasons in algebraic spaces.

Example 2.17.

- (Moduli stack of smooth curves).

2.5 Descent

Theorem 2.18. *(Descent for quasi-coherent sheaves) Let $X \to Y$ be a faithfully flat morphism of schemes such that it's either quasi-compact or locally of finite presentation. Then we have the categorical equivalence:*

$$\{Qcoh(Y)\} \cong \left\{ \begin{array}{c} \textit{Descent datum } (\mathcal{F} \in Qcoh(X), \alpha : p_1^*\mathcal{F} \to p_2^*\mathcal{F}) \\ \textit{satisfies cocycle condition} \end{array} \right\}$$

where $p_i : X \times_Y X \to X$ are the projections.

The descent of morphisms shows that the representable functor $F = \mathrm{Mor}(-, Z)$ is a sheaf in the fppf(resp. fpqc) topology.

Theorem 2.19. *(Descent for morphisms) Let $f : X \to Y$ be a faithfully flat morphism of schemes that is either quasi-compact or locally of finite presentation. If*

$g : X \to Z$ is any morphism such that $p_1 \circ g = p_2 \circ g$ on $X \times_Y X$, then there exists an unique morphism $h : Y \to Z$ which make the following diagram commutative.

$$\begin{tikzcd} X \times_Y X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \arrow[r, "f"] \arrow[dr, "g"] & Y \arrow[d, dashed, "h"] \\ & & Z \end{tikzcd}.$$

2.6 The geometry of moduli spaces

In general there's no hope to make the moduli space quasi-compact or bounded, because it usually contains too much information. For example, $\mathcal{B}un_{r,d}(C)$ is not 'bounded ' for any curve C.

However, it's possible to appoximate the moduli space via a 'bounded' space. We can expect a substack $\mathcal{B}un^{ss}_{r,d}(C)$ which is an algebraic stack of finite type/$\mathbb{C}$ by only considering the 'semi-stable' objects.

Meanwhile, we can expect the compactification of the moduli space by tolerating more objects, say, singularities.

Example 2.20. Consider the moduli space of smooth curves with genus 1 $\mathcal{M}_1$ (smooth elliptic curves). The family of elliptic curves $y^2z = x(x-z)(x-\lambda z)$ is parametrized by λ. When $\lambda \neq 0$ or 1, the elliptic curve given by the equation is smooth. The resulting moduli space of smooth elliptic curves is open, however we can find it's quasi-compact enlargement containing $\lambda = 0, 1$ by admitting nodal singularities.

Now we give an outline of finding projective approximation of the moduli functor:

Goal: For a stack $\mathcal{X}$, we want to find an enlargement $\mathcal{X} \subset \mathcal{M}$, s.t. $\mathcal{X} = \mathcal{M}^{ss}$ as the substack of $\mathcal{M}$ by considering semi-stable objects, and $\mathcal{M}^{ss}$ can be approximated by an proper (or projective) space.

1. (Algebraicity): Find $\mathcal{M}$ which is an algebraic stack, locally of finite type, by appling Artin's criteria and deformation theory.

2. (Openness of semi-stability) Discuss semi-stability which satisfies open condition. i.e, $\mathcal{M}^s s \subset \mathcal{M}$ is an open substack.

3. (Boundness) Prove $\mathcal{M}^{ss}$ is of finite type.

4. (Existence of coarse/good moduli space) Find a coarse or good moduli space $\mathcal{M}^{ss} \to M^{ss}$ where M^{ss} is an seperated algebraic space.

5. (Semi-stable reduction) Prove that $\mathcal{M}$ is universally closed, and M^{ss} is a proper algebraic space as a corollary.

6. (Projectivity) Find a tautological line bundle on $\mathcal{M}^{ss}$, and it descends to an ample line bundle on M^{ss}.

3 Grothendieck topology

Definition 3.1. We define Grothendieck topology as follows: Let $\mathcal{C}$ be a category. A Grothendieck topology τ on $\mathcal{C}$ is given by a set of coverings $\mathrm{Cov}(\mathcal{C})$ so that the following conditions are satisfied:

1. (Identity) If $V \to U$ is an isomorphism. Then $\{V \to U\}$ is a covering.

2. (Base change) If $\{U_i \to U\}$ is a covering, and $V \to U$ is any arrow, then base change $\{U_i \times_U V \to V\}$ is a covering.

3. (Composition) If $\{U_i \to U\}$ is a covering, and $\{U_{ij} \to U_i\}$ are coverings. Then the composition $\{U_{ij} \to U\}$ is a covering.

We call the category together with the Grothendieck topology $(\mathcal{C}, \mathrm{Cov}(\mathcal{C})$ a site.

Definition 3.2. A presheaf on a category $\mathcal{C}$ is a functor $F : \mathcal{C}^{op} \to \mathcal{S}et$.

Definition 3.3. Let $(\mathcal{C}, \mathrm{Cov}(\mathcal{C}))$ be a site and $\mathcal{F}$ is a presheaf.

1. We say $\mathcal{F}$ is separated if $\mathcal{F}(U) \to \prod \mathcal{F}(U_i)$ is injective for every covering $\{U_i \to U\}$.

2. We say a separated presheaf $\mathcal{F}$ is a sheaf if the sequence

$$\mathcal{F}(U) \longrightarrow \prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_i \times_U U_j)$$

   is exact.

A topos is a category $\mathcal{T}$ which is equivalent to the category of sheaves of sets on a site.

Remark. Different sites can give the same topos.

Example 3.4. Fixed a base scheme X. Let $\mathcal{C}$ to be the category of X-schemes. We require the coverings to be *jointly surjective*, i.e. for each covering $\{U_i \to U\}$, $\coprod_i U_i \to U$ is surjective.

1. Small Zariski site X_{zar}. Given by the natural Zariski topology, where $\mathrm{Cov}(\mathcal{C})$ is given by the inclusion of open subschemes.

2. Big Zariski site X_{Zar}. The morphisms in $\mathrm{Cov}(\mathcal{C})$ are given by open embeddings.

3. Small étale site $X_{ét}$. For each covering $\{U_i \to U\}$ we require $U_i \to X$ and $U \to X$ are étale morphisms (So that $U_i \to U$ is also étale for each i).

4. Big étale site $X_{Ét}$. The morphisms in $\mathrm{Cov}(\mathcal{C})$ are étale morphisms.

5. fppf site. The morphisms in $\mathrm{Cov}(\mathcal{C})$ are flat and locally of finite presentation.

6. Lisse-étale site. The morphisms in $\mathrm{Cov}(\mathcal{C})$ are étale and $U_i \to X$ and $U \to X$ are smooth.

Remark. The 'small' and 'big' differ by the topology, howerver the cohomology on big and small étale site are the same (The inclusion of the full subcategory $\mathcal{S}_\tau \to (\mathcal{S}ch/S)_\tau$ preserve the injective objects, and the global section functors are the same in both sites. Thus the derived functors, which computes the cohomology, are the same). When talking about the sheaves associated to schemes, small site is not a good choice, see this answer on mathoverflow.

The definition of fpqc topology is more subtle. Be careful, simply take 'faithfully flat and quasi-compact' doesn't give you the right definition! Instead, the coverings in fpqc topology satisfy a weaker condition compared with being 'quasi-compact'.

Remark. 1. There could be set-theoretical difficulties. See Stacks Lemma 34.9.14.

2. If we only take 'faithfully flat and quasi-compact' morphisms, then we are not able to make comparison with Zariski site. See remarks in FGA explained.

Definition 3.5. We say that a surjective morphism of schemes $f : X \to Y$ is a fpqc morphism if it satisfies the following equivalent conditions:

1. Every quasi-compact open subsets of Y is the image of a quasi-compact open subset of X.

2. There exists a covering $\{V_i \to Y\}$ by affine open subschemes such that each V_i is the image of a quasi-compact open subset of X.

3. Given a point $x \in X$, there exists an open neighborhood U of x in X, such that the image $f(U)$ is open in Y, and the restriction $f|_U$ is quasi-compact.

4. Given a point $x \in X$, there exists a quasi-compact open neighborhood U of x in X, such that the image $f(U)$ is open and affine in Y.

Proposition 3.6. *1. The composition of fpqc morphisms is fpqc.*

2. Being fpqc is local on the target.

3. An open faithfully flat morphism is fpqc

4. A faithfully flat morphism that is locally of finite presentation is fpqc

5. fpqc is preserved by base change.

6. If $f : X \to Y$ is fpqc, then $U \subset Y$ is open iff $f^{-1}(U)$ is open.

Definition 3.7. Let S be a base scheme. The fpqc topology on $\mathcal{S}ch/S$ is the topology in which the coverings $\{U_i \to U\}$ such that $\coprod U_i \to U$ is fpqc.

Remark. In general, we can define Grothendieck topologies on any coherent category. See Lurie's lecture notes.

Now we come to the main theorem of today's talk.

Theorem 3.8. *(Grothendieck) Any representable functor on $(\mathcal{S}ch/S)$ is a sheaf in the fpqc topology.*

Fact. fpqc topology is coarser than canonical topology. [AF22]

Corollary 3.9. *(After introducing the fibered category and stacks) The fibered category of quasi-coherent sheaves is a stack on the fpqc site.*

Definition 3.10. We say a Grothendieck topology defining a site is subcanonical if all representable presheaves on this site are sheaves.

Remark. 1. In Olsson's book, he only proves a weak version(in fppf site) of the Grothendieck's theorem as above.

2. One can ask whether the category of sheaves is equivalent to the category of representable functors. This generalizes to the Giraud's axioms, in the setting of ∞-categories. Roughly speaking, The Giraud's axioms help us to determine when a category is a topos. See [Lur09] 6.1.1.

The method of proving the main theorem is by reducing to the standard covering(where $f : U \to V$ is a faithfully flat morphism between affine schemes), which is a corollary of descent of modules.

Lemma 3.11. *Let S be a base scheme. $\mathcal{F} : (\mathcal{S}ch/S)^{op} \to \mathcal{S}et$ is a presheaf. Then if $\mathcal{F}$ satisfies the following two conditions:*

1. *$\mathcal{F}$ is a sheaf on Big Zariski site.*

2. *For any standard fpqc covering, i.e. for $V \to U$ faithfully flat of affine S-schemes, the following sequence is exact:*

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(V) \rightrightarrows \mathcal{F}(U \times_V U) \ .$$

Then $\mathcal{F}$ is a sheaf in fpqc topology.

Proof.

- **Step 1: reduction to a single morphism.** Let $\{U_i \to U\}$ be a fpqc covering over S, and let $V = \coprod U_i$. Then $V \to U$ is fpqc. Since F is a Zariski sheaf, thus $\mathcal{F}(V) \to \prod \mathcal{F}(U_i)$ is an isomorphism. Then we have a commutative diagram:

$$\begin{array}{ccccc} \mathcal{F}(U) & \longrightarrow & \mathcal{F}(V) & \longrightarrow & \mathcal{F}(V \times_U V) \\ \Downarrow & & \downarrow & & \downarrow \\ \mathcal{F}(U) & \longrightarrow & \prod \mathcal{F}(U_i) & \longrightarrow & \prod \mathcal{F}(U_i \times_U U_j) \end{array} ,$$

  where the columns are bijective. Thus to prove the bottom is exact, it suffices to prove the top is exact. So we reduce to the case of single covering.

- **Step 2: Prove that $\mathcal{F}$ is separated.** Let $f : V \to U$ be a fpqc morphism. By definition we can take an open covering $\{V_i\}$ of V by open quasi-compact subsets, and $U_i = f(V_i)$ open and affine. Let $V_i = \cup_a V_{ia}$, then we have

$$\begin{array}{cccc} \mathcal{F}(U) & \longrightarrow & \mathcal{F}(V) & ? \\ \downarrow & & \downarrow & \\ \prod \mathcal{F}(U_i) & \longrightarrow & \prod \mathcal{F}(V_{ia}) & inj \\ & & & \\ inj & & inj & \end{array} ,$$

  where the columns and the second row are injective since $\mathcal{F}$ is a zariski sheaf. So $\mathcal{F}(U) \to \mathcal{F}(V)$ is injective.

- **Step 3: When V is quasi-compact and U is affine.** Let $\{V_i \to V\}$ be a finite covering of V by open affine subschemes. Then $\{V_i \to U\}$ is a finite fpqc covering by affines schemes. By Step 2 the sequence

$$\mathcal{F}(U) \longrightarrow \prod \mathcal{F}(V_i) \rightrightarrows \prod \mathcal{F}(V_i \times_U V_j)$$

is exact. Assume $b \in \mathcal{F}(V)$ s.t. $p_1^* b = p_2^* b \in \mathcal{F}(V \times_U V)$, let b_i be the restriction of b on V_i, and they satisfy $p_1^* b_i = p_2^* b_j \in \mathcal{F}(V_i \times_U V_j)$. Then by the exact sequence above, we can find $a \in \mathcal{F}(U)$ whose pullback is b_i when restrict to $\mathcal{F}(V_i)$. Thus $f^* a = b$ since $\mathcal{F}$ is a zariski sheaf.

- **Step 4: When U is affine**. By definition of fpqc we can find $\{V_i\}$ a covering of V by quasi-compact open subschemes, such that $V_i \to U$ is surjective for all i. Let $b \in \mathcal{F}(V)$ as in Step 3, $b_i \in \mathcal{F}(V_i)$ be the restriction. Since $V_i \to V$ is fpqc, thus by Step 3 we can find $a_i \in \mathcal{F}(U)$ such that its pullback in V_i is b_i. Now we can prove $a_i = a_j$. Notice that $V_i \cup V_j \to U$ is also fpqc, and $V_i \cup V_j$ is quasi-compact, thus by step 3 we can find a_{ij} whose pullback is the retriction of b on $V_i \cup V_j$, and thus restrict to b_i and b_j. Thus $a_{ij} = a_i = a_j$.

- **Step 5: The genearl case**. Let $\{U_i\}$ be an covering of U by open affine subschemes and V_i be the inverse image of U_i. Then we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{F}(V) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathcal{F}(V \times_U V) \arrow[d] & ? \\
\prod \mathcal{F}(U_i) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \prod \mathcal{F}(V_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \prod \mathcal{F}(V_i \times_U {}_i V_i) & \textit{exact} \\
\prod \mathcal{F}(U_i \cap U_j) \arrow[r] & \prod \mathcal{F}(V_i \cap V_j) & & \textit{injective} \\
\textit{exact} & \textit{exact} & &
\end{tikzcd},$$

The columns and the second row are exact by previous steps. And the bottom row is injective since $\mathcal{F}$ is separated. The result follows from diagram chasing. □

To finish the proof of the main theorem we need to check two conditions for $\mathcal{F} = h_X$ where X is a S-scheme. It suffice to prove for $S = \operatorname{Spec} \mathbb{Z}$ by the following lemma:

Lemma 3.12. *If $\mathcal{C}$ is a subcanonical site and $S \in \mathrm{Ob}(\mathcal{C})$, then $\mathcal{C}/S$ is also subcanonical.*

The first condition is a basic exercise.

Exercise. h_X is a Zariski sheaf.

Next we check the second condition.

First we assume X is affine. In this case the result is given by descent of modules:

Lemma 3.13. *Let $f : A \to B$ be a faithfully flat morphism and M is an A module, then the sequence:*

$$M \longrightarrow M \otimes_A B \rightrightarrows M \otimes_A B \otimes_A B$$

is exact.

Proof. Suffice to check the exactness after tensoring B, which follows from the existence of sections. □

Now assume X is not affine, then let $X = \cup X_i$ as a union of affine open subschemes. To check the condition in this case, we first check set-theoretically and then scheme-theoretically by reducing to standard case.

First we show that h_X is separated. That is, to show $f, g : U \to X$ are equal if their pullback to $V \to X$ are equal. Since $V \to U$ is surjective, we see that f, g coincide set-theoretically. Then by taking inverse image we pass to the standard case $V_i \to U_i \to X_i$, thus $f|_{U_i}, g|_{U_i}$ coincide scheme-theoretically for every U_i, thus $f = g$ as desired.

To check exactness, we need to prove that if $g : V \to X$ equalizes two projections $V \times_U V \to V$ then it factors through U. Set-theoretical part follows from the following lemma,

Lemma 3.14. *Let $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ be morphisms of schemes. If $x_1 \in X_1$ and $x_2 \in X_2$ are points with the same image in Y, then there exists $z \in X_1 \times_Y X_2$ such that $p_1(z) = x_1$ and $p_2(z) = x_2$.*

Thus $g : V \to X$ factors through $f : U \to X$ set-theoretically. By taking inverse image $U_i = f^{-1}X_i$ and $V_i = g^{-1}X_i$ we pass to the standard case,

$$V_i \times_U V_i \rightrightarrows V_i \longrightarrow U_i \longrightarrow X_i \ .$$

Then $g|_{V_i}$ factors through a morphism of schemes $f_i : U_i \to X_i$. Meanwhile f_i and f_j coincide on $U_i \cap U_j$ since h_X is separated. Thus f_i glued together to give the desire factorization $V \to U \to X$. We complish the proof.

4 Descent Theory

[Man]

5 Fibered category

6 Algebraic spaces

References

[AF22] Yves André and Luisa Fiorot. On the canonical, fpqc, and finite topologies on affine schemes. the state of the art. *ANNALI SCUOLA NORMALE SUPERIORE-CLASSE DI SCIENZE*, pages 81–114, 2022.

[Lur09] Jacob Lurie. *Higher topos theory*. Princeton University Press, 2009.

[Man] Risn Mangan. On the uniquity of simplicial objects. *Available on-line at https://algant.eu/documents/theses/mangan.pdf*.

[Ols16] Martin Olsson. *Algebraic spaces and stacks*, volume 62. American Mathematical Soc., 2016.

[Vis04] Angelo Vistoli. Notes on grothendieck topologies, fibered categories and descent theory. *arXiv preprint math/0412512*, 2004.