Cucuteanu Tudor-Mircea 912

1. any union of open sets is open

let A_1, A_2 open sets

let $(a_1, b_1) \in A_1$

$(a_2, b_2) \in A_2$

$A_1 \cup A_2 = (a_1, b_1) \cup (a_2, b_2)$

$\forall a_1, b_1 \in A_1$ and $a_2, b_2 \in A_2$, $\exists \inf(A_1 \cup A_2)$

① such that $\inf(A_1 \cup A_2) \in \{a_1, a_2, b_1, b_2\}$

② $\exists \sup(A_1 \cup A_2)$ such that

$\sup(A_1 \cup A_2) \in \{a_1, a_2, b_1, a_2\} / \inf(A_1 \cup A_2)$

A_1 - open set $\Rightarrow \nexists \max(A_1), \min(A_1)$

A_2 - open set $\Rightarrow \nexists \max(A_2), \min(A_2)$

③ $\Rightarrow \nexists \max(A_1 \cup A_2), \min(A_1 \cup A_2)$

① + ② + ③ $\Rightarrow$

$\Rightarrow A_1 \cup A_2 = (\min(a_1, a_2, b_1, b_2); \max(a_1, a_2, b_1, b_2))$

any intersection of closed sets is closed

let $A_1 = [a_1, b_1]$

$A_2 = [a_2, b_2]$

$\sup(A_1) = b_1$, since closed $\Rightarrow \max(A_1) =$
$= \sup(A_1) = b_1$

$\inf(A_1) = a_1$, since closed $\Rightarrow \min(A_1) =$
$= \inf(A_1) = a_1$

$\Leftarrow$ $\sup(A_2) = \max(A_2) = b_2$

$\inf(A_2) = \min(A_2) = a_2$

let $B = A_1 \cap A_2$

$B \subseteq A_1 \cup A_2$

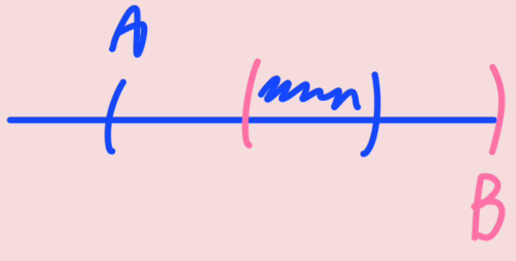
$\exists x \in B \Rightarrow x \in A_1 \cup A_2$

$\Rightarrow \min(B) \in A_1 \cup A_2$

$\max(B) \in A_1 \cup A_2$

$\Rightarrow$ B-closed for any A_1, A_2-closed

any finite intersection of open sets is open



let $A_1 = (a_1, b_1)$

let $A_2 = (a_2, b_2)$

let $B = A_1 \cap A_2$

A_1, A_2 open $\Rightarrow \nexists \max, \min(A_1, A_2)$

$$\Rightarrow \left.\begin{aligned} A_1 &= (\inf(A_1), \sup(A_1)) \\ A_2 &= (\inf(A_2), \sup(A_2)) \end{aligned}\right\} \Rightarrow$$

$\Rightarrow A_1 \cap A_2 = (\max(\inf(A_1), \inf(A_2)); \min(\sup(A_1), \sup(A_2))$

(1)

$\Rightarrow P(A_1) = \{ \{\varnothing\}, \{a\}, \{b\}, \dots \}$

$a_1, b_1 \notin P(A_1)$

$a_2, b_2 \notin P(A_2)$ (2)

(1)+(2) => any finite intersection of open sets is open

Any finite union of closed sets is closed

$$\left.\begin{aligned} &\text{let } A_1 = [a_1, a_2] \\ &\text{let } A_2 = [b_1, b_2] \end{aligned}\right\} \Rightarrow B = A_1 \cup A_2$$

$$B = [\underbrace{\min(a_1, a_2, b_1, b_2)}_{\in B}, \underbrace{\max(a_1, a_2, b_1, b_2)}_{\in B}]$$

example of intersection of open sets that is not open:

let $A = (-1, 3)$, $A = \{ x \mid x \in \mathbb{R} \}$

$A \cap \mathbb{N} = \{0, 1, 2\} = [0, 2]$

example of union of closed sets that is not closed

let $A = [\frac{1}{n}, 1-\frac{1}{n}]$

$$\bigcup_{n=2}^{\infty} [\frac{1}{n}; 1-\frac{1}{n}] = (0,1)$$

$$\lim_{n\to\infty} \frac{1}{n} = 0 \qquad \lim_{n\to\infty} 1-\frac{1}{n} = 1$$

2. a) $S_\alpha = \{\{n\alpha\} \mid n \in \mathbb{N}\}$

$\{n\alpha\} = n\alpha - [n\alpha]$

$0 \leq \{n\alpha\} < 1$

$\forall \varepsilon > 0$ and $x \in [0,1]$, $\exists n$ s.t $|\{n\alpha - x\}| < \varepsilon$

$\Rightarrow S_\alpha$ dense in $[0,1]$

b) Dense in $\mathbb{R}$

$T = \{\{n\alpha\} + m \mid n, m \in \mathbb{Z}\}$

$\{n\alpha\}$ - dense in $[0,1]$

$\forall x \in \mathbb{R}, \varepsilon > 0$

$|\{n\alpha\} - (x - m)| < \varepsilon, \forall m \in \mathbb{N}$

if $m, n \in \mathbb{Z} \Rightarrow$

$$\Rightarrow \left.\begin{array}{l} \{n\alpha\} \in [0,1) \\ m \in (-\infty, \infty) \cap \mathbb{Z} \end{array}\right\} \Rightarrow m + \{n\alpha\} \in \mathbb{R}$$

$\forall n, m \in \mathbb{Z}$

$cl(\{\{n\alpha\} + m \mid n, m \in \mathbb{Z}\}) = \mathbb{R}$

$\{\{n\alpha\} + m \mid n, m \in \mathbb{Z}\} \Leftrightarrow \mathbb{R}$

$\Rightarrow$ T dense in $\mathbb{R}$