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1. Cauchy => $[a,b]$ - complete metric space

BANACH fixed-point theorem

$f:[a,b]\to[a,b]$ - contraction

then f has a unique fixed point $x^*\in[a,b]$, $(f(x^*)=x^*)$

and for each $x_0\in[a,b]$ we have $f^n(x_0)\xrightarrow{n\to\infty}x^*$

Proof: $x_0\in[a,b]$; let $x_n:=f^n(x_0)$

$$d(x_{n+1},x_n)=d(f(x_n),f(x_{n-1}))\le\alpha\cdot d(x_n,x_{n-1})$$
$$=\alpha\cdot d(f(x_{n-1}),f(x_{n-2}))\le\alpha^2\cdot d(x_{n-1},x_{n-2})$$
$$\dots\le\alpha^n\cdot d(x_1,x_0)$$ proof by induction

for $n>m$ (2 indexes): $d(x_n,x_m)\le d(x_n,x_{n-1})+\dots d(x_{m+1},x_m)$

$$\le(\alpha^{n-1}+\alpha^{n-2}+\dots+\alpha^m)\cdot d(x_1,x_0)$$
$$=\alpha^m\cdot\underbrace{\sum_{k=0}^{n-1-m}\alpha^k}_{\le\sum_{k=0}^{\infty}\alpha^k=\frac{1}{1-\alpha}}\cdot(x_1,x_0)$$
$$\le\frac{\alpha^m}{1\ \alpha}\cdot d(x_1,x_0)$$

$\Rightarrow(x_n)\to$ Cauchy $\left(d(x_n,x_m)\xrightarrow{n,m\to\infty}0\right)$

completeness

$\Rightarrow(x_n)\to$ unique limit $x^*\in[a,b]$

$$f(x^*)=f(\lim_{n\to\infty}x_n)=\lim_{n\to\infty}f(x_n)=\lim_{n\to\infty}x_{n+1}=x^*$$

contraction is continuous

2. $x_{n+2} = \alpha x_{n+1} + (1-\alpha) x_n$

$x_n = y^n$

$y^{n+2} = \alpha y^{n+1} + (1-\alpha) y^n \,|: y^n$

$y^2 = \alpha y + (1-\alpha)$

$y^2 - \alpha y - (1-\alpha) = 0$

$$y = \frac{\alpha + \sqrt{\alpha^2 + 4(1-\alpha)}}{2}$$

$\alpha^2 + 4(1-\alpha) = (2-\alpha)^2$

$y_1 = 1$

$y_2 = \alpha - 1$

let A, B - constants

$x_n = A \cdot 1 + B(\alpha-1)^n$

$\alpha \in (0,1)$

$\Rightarrow \lim_{n \to \infty} (\alpha-1)^n = 0$

$$\lim_{n \to \infty} x_n = A \Rightarrow \text{convergent}$$

$$x_1 = A + B(\alpha - 1)$$

$$x_2 = A + B(\alpha - 1)^2$$

$$x_2 - x_1 = B[(\alpha-1)^2 - (\alpha - 1)] = B(\alpha-1)(\alpha-2)$$

$$B = \frac{x_2 - x_1}{(1-\alpha)(2-\alpha)}$$

$$A = x_1 - B(\alpha - 1) = x_1 + \frac{x_2 - x_1}{2-\alpha}$$

$$\Rightarrow \lim_{n \to \infty} x_n = x_1 + \frac{x_2 - x_1}{2 - \alpha}$$

3. set of limit points equal to $[0,1] \Rightarrow$ dense in $[0,1]$

example : $x_n = \sin(n) \bmod 1$

$\sin(n)$ - dense in $[-1,1]$

$\Rightarrow \sin \bmod (1)$ - dense in $[0,1]$

$\forall a \in [0,1]$ and $\varepsilon > 0, \exists\, n$ s.t $|x_n - a| < \varepsilon$