CUCUTEANU TUDOR-MIRCEA

1) $\lim_{n \to \infty} \frac{n+1}{2n+3} = \frac{1}{2}$

let $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $|x_n - \frac{1}{2}| < \varepsilon$

$$\left| \frac{n+1}{2n+3} - \frac{1}{2} \right| < \varepsilon$$

$$\left| \frac{2(n+1) - (2n+3)}{2(2n+3)} \right| < \varepsilon$$

$$\left| \frac{-1}{2(2n+3)} \right| = \frac{1}{2(2n+3)}$$

$$\frac{1}{2(2n+3)} < \varepsilon$$

$-1)$ $-1)$

$$2(2n+3) > \frac{1}{\varepsilon}$$

$$4n + 6 > \frac{1}{\varepsilon}$$

$$\boxed{n > \frac{1}{4\varepsilon} - \frac{6}{4}}$$

take $N = \left[\frac{1}{4\varepsilon} - \frac{6}{4}\right] + 1$

$n \geq N$

② a) $\lim\limits_{n\to\infty} (1+2+\ldots+n)^{1/n}$; let $f(n) = (1+2+\ldots+n)^{1/n}$

$$\lim_{n\to\infty} f(n) = \lim_{n\to\infty}\left[\frac{n(n+1)}{2}\right]^{1/n}$$

$$= \lim_{n\to\infty}\left(\frac{n^2+n}{2}\right)^{1/n}$$

$$\overset{\infty^0}{=} e^{\lim\limits_{n\to\infty} \ln\left(\frac{n^2+n}{2}\right)^{1/n}}$$

$$= e^{\lim\limits_{n\to\infty} \frac{1}{n}\ln\left(\frac{n^2+n}{2}\right)}$$

$$= e^{\lim\limits_{n\to\infty} \frac{\ln\left(\frac{n^2+n}{2}\right)}{n}}$$

$$\overset{L'H}{=} e^{\lim\limits_{n\to\infty} \frac{2n+1}{n^2+n}}$$

$$\overset{L'H}{=} e^{\lim\limits_{n\to\infty} \frac{4}{n+1}} = e^0 = 1$$

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b)

$$\left(\frac{\ln(n+1)}{\ln n}\right)^n = L$$

$$\ln L = n \ln\left(\frac{\ln(n+1)}{\ln n}\right)$$

$$\ln\left(\frac{(n+1)}{\ln n}\right) = \ln(\ln(n+1)) - \ln(\ln n)$$

$$L = n(\ln(\ln(n+1)) - \ln(\ln n))$$

$$a_n = \ln(\ln(n+1)) - \ln(\ln n))$$

$b_n = n \to \lim_{n\to\infty} b_n = \infty$

b_n - strictly increasing

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}$$

$a_{n+1} - a_n =$

$= [\ln(\ln(n+2)) - \ln(\ln(n+1))] - [\ln(\ln(n+1)) - \ln(\ln n)] =$

$= \ln(\ln(n+2)) - 2\ln(\ln(n+1)) + \ln(\ln n)$

$b_{n+1} - b_n = 1$

S.C

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim_{n\to\infty} [\ln(\ln(n+2)) - 2\ln(\ln(n+1)) + \ln(\ln n)]$$

$= 0$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{\ln(n+1)}{\ln n}\right)^n = e^0 = 1$$

3. $x_n = \frac{\sin(n)}{n}$

$-1 \le \sin(n) \le 1$

$-\frac{1}{n} \le \frac{\sin(n)}{n} \le \frac{1}{n}$ - bounded

$$\lim_{n\to\infty} -\frac{1}{n} \leq \lim_{n\to\infty} \frac{\sin(n)}{n} \leq \lim_{n\to\infty} \frac{1}{n}$$

$$0$$

$$\Rightarrow \lim_{n\to\infty} x_n = 0$$

x_n [not monotonic
converges to 0

4. $x_n = 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n} - \ln n$

$x_{n+1} = 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}+\frac{1}{n+1} - \ln(n+1)$

$x_{n+1} \leq x_n$

$$x_{n+1} - x_n = \frac{1}{n+1} - \ln(n+1) + \ln n \leq 0$$

$$\ln(n) - \ln(n+1) \leq -\frac{1}{n+1}$$

$$-\ln\left(1+\frac{1}{n}\right) \leq -\frac{1}{n+1}$$

$$\ln\left(1+\frac{1}{n}\right) \geq \frac{1}{n+1} \text{ TRUE}$$

$\Rightarrow x_n$ - decreasing (1)

x_n - harmonic sum

$$1+\frac{1}{2}+\dots+\frac{1}{n} \leq 1+\ln n$$

$$\left(1+\frac{1}{2}+\dots+\frac{1}{n}\right) - \ln n \geq 1$$

$\Rightarrow x_n$ - bounded below ②

① + ② $\Rightarrow x_n$ - convergent

2. c) $\lim\limits_{n\to\infty} \dfrac{n^n}{1+2^2+3^3+\ldots n^n} =$

$$\lim_{n\to\infty} \frac{n^n \cdot 1}{n^n\Big(1+\underbrace{\frac{1}{n^n}\big(1+2^2+3^3+\ldots+(n-1)^{n-1}\big)}_{0}\Big)} = \lim_{n\to\infty} \frac{n^n}{n^n} = 1$$