Operation: $*: A \times A \to A$

$\forall x, y \in A \Rightarrow x * y \in A$

Groupoid: $(A, *)$

Subgroupoid: groupoid + associativity

$\forall x, y, z \in A: (x*y)*z = x*(y*z)$

Monoid: subgroupoid + identity element (neutral element)

$\exists e \in A, \forall x \in A: x*e = e*x = x$

Group: Monoid + all elements have a symmetric

$\forall x \in A, \exists x' \in A: x*x' = x'*x = e$

Abelian group: group + comutativity

$\forall x, y \in A: x*y = y*x$

Subgroup: $S \leq (A,*)$ - group

$$\left.\begin{array}{l}\text{such that } S \subseteq A \\ (S,*)\text{-group}\end{array}\right\} \Rightarrow$$

always inherits commutativity + asociativity

$\boxed{1}$ $\mathbb{N}$ - natural $+, \cancel{-}, \cdot, \cancel{:}$

$\mathbb{Z}$ - integers $+, -, \cdot, \cancel{:}$

$\mathbb{Q}$ - rational $+, -, \cdot, \cancel{:} \Rightarrow \mathbb{Q}^* - :$

$$\left.\begin{array}{ll}\mathbb{R} \text{ - real} & +, -, \cdot, \cancel{:} \\ \mathbb{C} \text{ - complex} & +, -, \cdot, \cancel{:}\end{array}\right\} \Rightarrow \mathbb{R}^*, \mathbb{C}^* - :$$

$\boxed{3}$ $\cancel{(\mathbb{N},+)}$, $\cancel{(\mathbb{N},\cdot)}$

$(\mathbb{Z},+)$, $\cancel{(\mathbb{Z},\cdot)}$

$(\mathbb{Q},+)$, $\cancel{(\mathbb{Q},\cdot)}$ $(\mathbb{Q}^*,\cdot)$

$(\mathbb{R},+)$, $\cancel{(\mathbb{R},\cdot)}$ $(\mathbb{R}^*,\cdot)$

$(\mathbb{C},+)$, $\cancel{(\mathbb{C},\cdot)}$ $(\mathbb{C}^*,\cdot)$

$\boxed{4}$ $*$ operation defined on $\mathbb{R}$

$x*y = x+y+xy, \forall x,y \in \mathbb{R}$

(i) $(\mathbb{R},*)$ commutative

(ii) $[-1,+\infty)$ stable part of $(\mathbb{R},*)$

(i) $(x*y)*z = x*(y*z) \Rightarrow (x+y+xy)*z = x*(y+z+yz)$

$\Rightarrow x+y+xy+z+xz+yz+xyz$

$= x+y+z+yz+xy+xz+xyz$

$x*y = y*x \Rightarrow x+y+xy = y+x+yx$

$e*x = x*e = x$

$x+e+xe = e+x+ex = x$

$x+e+xe = x$

$x+e(x+1) = x$

$e(x+1) = 0$

$e = 0 \in \mathbb{R}$

(ii) $\forall x,y \in [-1,\infty)$

$x*y = x+y+xy \in [-1,\infty)$

$= (x+1)(y+1)-1 \in [-1,\infty)$

$(x+1), \forall x \in (-1,\infty)$

1) $(x+1) \in [0, \infty) \geq 0$

2) $(y+1) \in [0, \infty) \geq 0$

$1+2) \Rightarrow (x+1)(y+1)-1 \geq -1$; $(x+1)(y+1)-1 \in [-1, \infty)$

$\boxed{5}$ $*$ op. def. $\mathbb{N}$

$x * y = \gcd(x, y)$

i) $(\mathbb{N}, *)$ comm monoid

ii) $D_n = \{x \in \mathbb{N} \mid x \mid n\}, n \in \mathbb{N}^*$

D_n is a stable part

$(D_n, *)$ comm. monoid

iii) Fill in the table (D_6)

i) $x * y = \gcd(x, y) = \gcd(y, x)$

$(x*y)*z = \gcd(\gcd(x, y), z) = \gcd(x, \gcd(y, z))$

$x * e = e * x = x$

$\gcd(x, e) = x$

$\Rightarrow e = 0 \in \mathbb{N}$

ii) $\forall x, y \in D_n$

$x * y \in D_n$

$\Rightarrow x \mid n$ and $y \mid n$

$x * y = \gcd(x, y) = a$

$a \mid x$ and $a \mid y \Rightarrow a \mid n$

$\Rightarrow a \in D_n$

$x * y$ ass+comm.

$x * e = e * x = x$

$\gcd(x, e) = x$

$\Rightarrow e = n \in D_n$

iii)

$*$	1	2	3	6
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
6	1	2	3	6

! $\boxed{6}$ Det the finite stable subsets of $(\mathbb{Z}, \cdot)$ (in partial)

Let H is a st. subset of $(\mathbb{Z}, \cdot)$

$\Rightarrow \exists\, x \in H$ such that $x \cdot x \in H \Rightarrow \exists\, x \in H$ such that $x^2 \in H$

$\exists\, x \in H$ such that $x \cdot x^2 \in H \Rightarrow \exists\, x \in H$ such that $x^3 \in H$

$\exists\, x \in H$ such that $\exists\, n \in \mathbb{N}^*: x^n \in H$

H - finite set

$\Rightarrow \exists\, i, j \in \mathbb{N}^*, i \leq j$ such that $x^i = x^j \in H \Rightarrow$

$\Rightarrow x \in \{-1, 0, 1\}$

$\Rightarrow H$ can be $\{1\}, \{0\}, \cancel{\{-1\}}, \{-1, 1\}, \{0, 1\}, \cancel{\{0, -1\}}, \{1, 0, -1\}, \emptyset$

$\boxed{7}$ $(G, \cdot)$ group

(i) G is Abelian $\Leftrightarrow \forall\, x, y \in G$, $(xy)^2 = x^2 y^2$

(ii) If $x^2 = 1, \forall\, x \in G \Rightarrow G$ is Abelian

(i) G is Abelian $\Leftrightarrow \forall\, x, y \in G: x \cdot y = y \cdot x$

$(xy)^2 = (x\, y)(x\, y)$

$\Leftrightarrow (xy)^2 = x\, x\, y\, y = x^2 y^2$

(ii) Hint: $x^2 = x\, x = 1$ - id. element

$\Rightarrow x = x^{-1}$

Homework: 2, 8, 7(ii)

2. $A = \{a_1, a_2, a_3\}$