

Homological algebra for derived functors and categories

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1 Some categorial flavour to algebraic notions

1.1 Definition (Projective and injective objects):

$P \in Ob(\mathbf{A})$ is called projective iff for every epimorphism $B \twoheadrightarrow A$ and every morphism $P \rightarrow A$ there is a morphism $P \rightarrow B$ making the triangle commutative.

Dually $I \in Ob(\mathbf{A})$ is called injective iff for every monomorphism $A \hookrightarrow B$ and every morphism $A \rightarrow I$ there is a morphism $B \rightarrow I$ making the triangle commutative.

$$\begin{array}{ccc}
 & P & \\
 & \downarrow & \\
 B & \twoheadrightarrow A & \rightarrow 0
 \end{array}
 \quad
 \begin{array}{ccc}
 & I & \\
 & \uparrow & \\
 B & \hookrightarrow A & \leftarrow 0
 \end{array}$$

The full subcategory of all projectives / injectives is denoted by $Proj(\mathbf{A})$ / $Inj(\mathbf{A})$.

The category \mathbf{A} is said to have enough projectives / injectives if every $A \in \mathbf{A}$ is a quotient / subobject of some projective / injective object.

1.2: In both cases, the morphisms whose existence is required are usually highly non-unique.

2 Some homological algebra

2.1 Definition (Chain complexes):

Let \mathbf{A} be an additive category. A chain complex (A_*, ∂) is a pair consisting of a graded object $A_* \in \mathbf{A}^{\mathbb{Z}}$ and a morphism $\partial : A \rightarrow A$ of degree -1 , i.e. $\partial_n : A_n \rightarrow A_{n-1}$, such that $\partial \circ \partial = 0$.

Dually cochain complex consists of a graded object A^* and morphisms $d^n : A^n \rightarrow A^{n+1}$ such that $d \circ d = 0$.

2.2: One can switch between chain and cochain complexes by setting $A_n := A^{-n}$ and vice versa.

2.3 Definition:

The category of chain complexes is denoted $Ch(\mathbf{A})$.

The full subcategory of all chain complexes with $A_n = 0$ for $n \ll 0$ ($n \gg 0$) is denoted $Ch^-(\mathbf{A})$ and $Ch^+(\mathbf{A})$ respectively.

2.4 Lemma:

Let $\mathbf{A} \in \mathbf{Cat}$ be additive.

- a.) $Ch(\mathbf{A})$ is an additive category too.
- b.) If \mathbf{A} is abelian, then $Ch(\mathbf{A})$ is an abelian category too. Kernels and cokernels are computed termwise.

2.5 Definition (Homology):

Let \mathbf{A} be an abelian category and $A_* \in Ch(\mathbf{A})$ a chain complex. Then its homology is defined to be the graded object $H_n(A) := \underbrace{\ker(\partial_n)}_{=: Z_n} / \underbrace{\operatorname{im}(\partial_{n+1})}_{=: B_n}$.

Similarly we define cohomology of a cochain complex.

2.6 Definition:

Two morphisms $f, g : A_* \rightarrow B_*$ between chain complexes are homotopic iff there exists $h : A_* \rightarrow B_{*+1}$ such that

$$f - g = \partial^B h + h \partial^A$$

Notation $f \simeq g$.

2.7 Lemma: a.) \simeq is an equivalence relation on $\operatorname{Hom}(A_*, B_*)$.

- b.) \simeq is compatible with addition and composition of morphisms.

2.8 Definition (Homotopy category):

Let $\mathbf{A} \in \mathbf{Cat}$ be additive. Then $K(\mathbf{A})$ is the category with $Ob(K(\mathbf{A})) := Ob(Ch(\mathbf{A}))$ and $\operatorname{Hom}_{K(\mathbf{A})}(X, Y) := \operatorname{Hom}_{Ch(\mathbf{A})}(X, Y) / \simeq$.

Similarly we define $K^\pm(\mathbf{A})$.

2.9 Definition (Homotopy equivalences & Quasi-isomorphisms):

Isomorphisms in the homotopy category are called homotopy equivalences, denoted $A \simeq B$.

A chain map $f : A_* \rightarrow B_*$ that induces isomorphisms $H(A_*) \rightarrow H(B_*)$ is called a quasi-isomorphism, denoted $A \sim B$.

2.1 Double Mapping cylinder and mapping cone

2.10 Definition (Double mapping cylinder):

Let $A \xleftarrow{f} X \xrightarrow{g} B$ be two chain maps. Then the double mapping cylinder $DMC(f, g)$ is defined as:

$$DMC(f, g)_n := A_n \oplus X_{n-1} \oplus B_n \quad \text{and} \quad \partial^{DMC} = \begin{pmatrix} \partial^A & f & \\ & -\partial^X & \\ & g & \partial^B \end{pmatrix}$$

2.11 Definition (Mapping cone):

$$Cone(g) := DMC(0 \leftarrow A \xrightarrow{g} B).$$

2.12: Conversely one can define $DMC(f, g)$ as the pullback of the two projections $Cone(f) \rightarrow X[-1] \leftarrow Cone(g)$.

2.13: Observe that $Cone(0 \rightarrow A) = A$ while $Cone(A \rightarrow 0) = A[-1]$.

2.14 Theorem (Universal mapping properties):

With the notation from the definition the following hold:

a.) Out-going morphisms:

$$\text{Hom}_{Ch(\mathbf{A})}(DMC(f, g), Z) = \{ (\alpha, \xi, \beta) \mid A \xrightarrow{\alpha} Z, B \xrightarrow{\beta} Z, \alpha \circ f + \beta \circ g \xrightarrow{\xi} 0 \}$$

In particular

$$\text{Hom}_{Ch(\mathbf{A})}(Cone(f), Z) = \{ (\alpha, \xi) \mid A \xrightarrow{\alpha} Z, \alpha \circ f \xrightarrow{\xi} 0 \}$$

b.) In-coming morphisms:

$$\text{Hom}_{Ch(\mathbf{A})}(W, DMC(f, g)[+1]) = \{ \begin{pmatrix} \alpha \\ \xi \\ \beta \end{pmatrix} \mid W \xrightarrow{\xi} X, f \circ \xi \xrightarrow{\alpha} 0 \wedge g \circ \xi \xrightarrow{\beta} 0 \}$$

In particular

$$\text{Hom}_{Ch(\mathbf{A})}(W, Cone(f)[+1]) = \{ \begin{pmatrix} \alpha \\ \xi \end{pmatrix} \mid W \xrightarrow{\xi} X, f \circ \xi \xrightarrow{\alpha} 0 \}$$

Proof. Obvious calculations. □

2.15 Theorem (Functoriality): a.) Diagrams of the form

$$\begin{array}{ccccc} A & \xleftarrow{f} & X & \xrightarrow{g} & B \\ \downarrow \alpha & & \downarrow \chi & & \downarrow \beta \\ A' & \xleftarrow{f'} & X' & \xrightarrow{g'} & B' \end{array}$$

which commute up to homotopy correspond to morphisms between the double mapping cylinders: If $\alpha f - f' \chi \stackrel{h_A}{\simeq} 0$ and $\beta g - \chi g' \stackrel{h_B}{\simeq} 0$ then

$$\begin{pmatrix} \alpha & h_A & 0 \\ 0 & \chi & 0 \\ 0 & h_B & \beta \end{pmatrix} : DMC(f, g) \rightarrow DMC(f', g')$$

is a chain-map and vice-versa.

b.) Given a diagram as in a. it extends to a commutative up to homotopy diagram

$$\begin{array}{ccccccc} X & \xrightarrow{g} & B & \xrightarrow{i_B} & Cone(g) & \xrightarrow{q_{X[-1]}} & X[-1] \\ \downarrow \chi & \xRightarrow{h} & \downarrow \beta & & \downarrow & & \downarrow \chi[-1] \\ X' & \xrightarrow{g'} & B' & \xrightarrow{i_{B'}} & Cone(g') & \xrightarrow{q_{X'[-1]}} & X'[-1] \end{array}$$

where the arrow between cones is $\begin{pmatrix} \chi & 0 \\ h & \beta \end{pmatrix}$. In particular $Cone(f)$ is a homotopy-invariant of f .

2.16 Theorem (Mapping cylinders and cones vs. homotopy I):

Let $A \xleftarrow{f} X \xrightarrow{g} B$ be two chain-maps.

a.) TFAE:

- i.) f factors through g up to homotopy, i.e. $f \simeq \beta g$ for some $\beta : B \rightarrow A$.
- ii.) $0 \rightarrow A \rightarrow DMC(f, g) \rightarrow Cone(g) \rightarrow 0$ splits.

b.) TFAE:

- i.) f is null-homotopic.
- ii.) f factors through $A \hookrightarrow Cone(id_A)$.
- iii.) f factors through some contractible complex.
- iv.) The short exact sequence $0 \rightarrow A \hookrightarrow Cone(f) \rightarrow X[-1] \rightarrow 0$ splits.

Proof. a. Follows immediately from the universal property for out-going morphisms $DMC(f, g) \rightarrow A$, because $\beta g \stackrel{h}{\simeq} f \iff (-\beta, h, 1) \in \text{Hom}(DMC(f, g), A)$.

b. The universal property for out-going morphisms immediately proves i. \iff ii. and ii \implies iii. \implies i. is trivially true. i. \iff iv. follows from a. with $B = 0$. Note that $Cone(X \rightarrow 0) = X[-1]$.

Alternatively: A splitting $X \rightarrow Cone(f)[+1]$ is a chain map $s = (\sigma, id)$ which is equivalent to $f \circ id_X \stackrel{\sigma}{\simeq} 0$ by the universal property for in-coming morphisms. \square

2.17 Lemma (Modifying homotopies to obtain additional equations):

If $A \xrightleftharpoons[\beta]{f} B$ are chain maps with $\text{id}_B - f\beta \xrightarrow{\tilde{\delta}} 0$ and $\text{id}_A - \beta f \xrightarrow{-\alpha} 0$ for some maps $\tilde{\delta} : B_* \rightarrow B_{*+1}$ and $\alpha : A_* \rightarrow A_{*+1}$, then

a.) $\psi := \begin{pmatrix} \text{id} & 0 \\ \tilde{\delta}f + f\alpha & \text{id} \end{pmatrix} : \begin{matrix} A_n \\ B_{n+1} \end{matrix} \rightarrow \begin{matrix} A_n \\ B_{n+1} \end{matrix}$ is a chain-map automorphism of $\text{Cone}(f)[+1]$.

b.) There exist another homotopy $\delta : B_* \rightarrow B_{*+1}$ such that $\text{id} - f\beta \xrightarrow{\delta} 0$ and additionally a map $\gamma : A_* \rightarrow B_{*+2}$ such that $f\alpha + \delta f = \gamma\partial - \partial\gamma$.

Proof. a. $\psi = \partial^C \tilde{H} + \tilde{H} \partial^C$ for $\tilde{H} = \begin{pmatrix} \alpha & \beta \\ 0 & \tilde{\delta} \end{pmatrix}$ and every map of the form $\partial h + h\partial$ is a chain-map, because $\partial\psi = 0 + \partial(h\partial) = (\partial h)\partial + 0 = \psi\partial$. It is obviously an automorphism because it is lower triangular with isomorphisms on the diagonal.

b. $H := \psi^{-1} \tilde{H} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ now satisfies $\partial H + H\partial = \text{id}$. Checking the components (2,1) and (2,2) we find the two necessary equations. \square

2.18 Theorem (Mapping cylinders and cones vs. homotopy II):

Let $A \xleftarrow{f} X \xrightarrow{g} B$ be two chain-maps.

a.) TFAE:

- i.) A is a deformation retract of $DMC(f, g)$.
- i.) B is a deformation retract of $DMC(f, g)$.
- ii.) i_A is a homotopy-equivalence.
- ii.) i_B is a homotopy-equivalence.
- iii.) g is a homotopy-equivalence.
- iii.) f is a homotopy-equivalence.

b.) In particular the following are also equivalent:

- i.) $\text{Cone}(g)$ is contractible.
- i.) $\text{Cone}(f)$ is contractible.
- ii.) g is a homotopy-equivalence.
- ii.) f is a homotopy-equivalence.

Proof. a. \implies b. is trivially true. For b. \implies c. let $\phi = (\phi_A, \phi_X, \phi_B) : DMC(f, g) \rightarrow A$ be a homotopy inverse so that $\phi_A = \phi \circ i_A \simeq \text{id}_A$ and $i_A \circ \phi \simeq \text{id}$, say with

$$H = \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \zeta \\ \eta & \theta & \kappa \end{pmatrix} : \begin{matrix} A_n \\ X_{n-1} \\ B_n \end{matrix} \rightarrow \begin{matrix} A_{n+1} \\ X_n \\ B_{n+1} \end{matrix}$$

Writing the equation $\partial H + H\partial = \text{id} - i_A \circ \phi$ in components, gives us nine individual equations:

$$1 - \phi_A = \partial\alpha + f\delta + \alpha\partial \quad (1,1)$$

$$-\phi_X = \partial\beta + f\varepsilon + \alpha f - \beta\partial + \gamma g \quad (1,2)$$

$$-\phi_B = \partial\gamma + f\zeta + \gamma\partial \quad (1,3)$$

$$0 = -\partial\delta + \delta\partial \quad (2,1)$$

$$1 = -\partial\varepsilon + \delta f - \varepsilon\partial + \zeta g \quad (2,2)$$

$$0 = -\partial\zeta + \zeta\partial \quad (2,3)$$

$$0 = g\delta + \partial\eta + \eta\partial \quad (3,1)$$

$$0 = g\varepsilon + \partial\theta + \eta f - \theta\partial + \kappa g \quad (3,2)$$

$$1 = g\zeta + \partial\kappa + \kappa\partial \quad (3,3)$$

The equations (2,1) and (2,3) show that $\delta : A \rightarrow X$ and $\zeta : B \rightarrow X$ are chain-maps.

The equations (3,3) shows $1 \simeq g\zeta$ and (2,2) shows $1 - \delta f \simeq \zeta g$ which is almost what we want.

If we look at (1,1) we find that $1 - \phi_A \stackrel{\alpha}{\simeq} f\delta$ but ϕ_A is already homotopic to id_A by assumption so that $f\delta \simeq 0$. This in turn proves that $(\delta f)^2 \simeq 0$ so that $1 - \delta f$ is invertible in the homotopy category. Now we are in the situation that $[g][\zeta] = [\text{id}_B]$ and $[\zeta][g] = [1 - \delta f] \in \text{Aut}_{K(A)}(X)$. In this situation $[g]$ and $[\zeta]$ are automatically inverse to each other which is what we wanted to show.

For the direction $c. \implies a.$ we choose a homotopy inverse ζ of g and two homotopies $1 - \zeta g \stackrel{-\varepsilon}{\simeq} 0$ and $1 - g\zeta \stackrel{\kappa'}{\simeq} 0$. This already makes (2,3) and (3,3) true. If we choose $\delta := 0$ we also get (2,1) and (2,2) for free.

Choosing $\eta := 0$ also gives us the truth of (3,1). Then with lemma 2.17 we can replace κ' by another homotopy κ (which also makes (3,3) true) and get a θ such that (3,2) becomes true.

We define $\phi_A := 1, \alpha := 0$ so that (1,1) becomes true. We define $\phi_B := -f\zeta, \gamma := 0$ so that (1,3) becomes true and $\phi_X := -f\varepsilon, \beta := 0$ so that (1,2) becomes true.

Now all nine equations are satisfied so that $\text{id} - i_A \circ \phi \stackrel{H}{\simeq} 0$. Also by construction $\phi \circ i_A = 1$ which proves a. \square

2.19 Lemma:

With the notation from the definition the following diagram commutes up to homotopy:

$$\begin{array}{ccccccc} X & \xrightarrow{-g} & B & \xrightarrow{i_B} & \text{Cone}(g) & \xrightarrow{q_{X[-1]}} & X[-1] \\ \downarrow f & & \downarrow i_B & & \parallel & & \downarrow f \\ A & \xrightarrow{i_A} & \text{DMC}(f, g) & \xrightarrow{q_{Cg}} & \text{Cone}(g) & \xrightarrow{f \circ q_{X[-1]}} & A[-1] \end{array}$$

Proof. The middle and right square commute as is, the left hand square commutes up to homotopy because

$$i_A \circ f - i_B \circ (-g) = \begin{pmatrix} f \\ 0 \\ g \end{pmatrix} = \begin{pmatrix} \partial^A & f & \\ & -\partial^X & \\ g & & \partial^B \end{pmatrix} \begin{pmatrix} 0 \\ \text{id} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \text{id} \\ 0 \end{pmatrix} \partial^X \quad \square$$

2.20 Theorem (Mapping cylinders and cones vs. homology):

Let $A \xleftarrow{f} X \xrightarrow{g} B$ be two chain-maps. Then the following hold:

a.) Exact sequences: The inclusion and projection maps in the following sequences are chain-maps:

- i.) $0 \rightarrow A \xrightarrow{i_A} DMC(f, g) \xrightarrow{q_{X[-1]} \oplus q_B} Cone(g) \rightarrow 0$
- ii.) $0 \rightarrow B \xrightarrow{i_B} DMC(f, g) \xrightarrow{q_A \oplus q_{X[-1]}} Cone(f) \rightarrow 0$
- iii.) $0 \rightarrow A \oplus B \xrightarrow{i_A \oplus i_B} DMC(f, g) \xrightarrow{q_{X[-1]}} X[-1] \rightarrow 0.$

The connecting morphisms in the long exact homology sequences are

- i.) $H_{n+1}(Cone(g)) \xrightarrow{f \circ q_X} H_n(A)$
- ii.) $H_{n+1}(Cone(f)) \xrightarrow{g \circ q_X} H_n(B)$
- iii.) $H_{n+1}(X[-1]) = H_n(X) \xrightarrow{f \oplus g} H_n(A) \oplus H_n(B) = H_n(A \oplus B)$

respectively.

b.) TFAE:

- i.) i_A is a quasi-isomorphism.
- ii.) $Cone(g)$ is acyclic.
- iii.) g is a quasi-isomorphism.
- i.) i_B is a quasi-isomorphism.
- ii.) $Cone(f)$ is acyclic.
- iii.) f is a quasi-isomorphism.

c.) TFAE:

- i.) $H_*(f) = 0$
- ii.) $i_* : H_*(A) \rightarrow H_*(Cone(f))$ is mono.
- iii.) $0 \rightarrow H_*(A) \xrightarrow{i_*} H_*(Cone(f)) \xrightarrow{q_*} H_{*-1}(X) \rightarrow 0$ is exact.
- iv.) $q_* : H_*(Cone(f)) \rightarrow H_{*-1}(X)$ is epi.

Proof. a. is mostly a consequence of the definition and the explicit construction of the connecting homomorphism.

b. follows from a. by looking at the long exact homology sequences induced by those short exact sequences. By the above lemma, we get a commutative diagram of two of

those sequences:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & H_n(X) & \xrightarrow{-g} & H_n(B) & \xrightarrow{i_B} & H_n(\text{Cone}(g)) \xrightarrow{q_{X[-1]}} H_n(X[-1]) \longrightarrow \cdots \\
& & \downarrow f & & \downarrow i_B & & \parallel \\
\cdots & \longrightarrow & H_n(A) & \xrightarrow{i_A} & H_n(\text{DMC}(f, g)) & \xrightarrow{q_{Cg}} & H_n(\text{Cone}(g)) \xrightarrow{f \circ q_{X[-1]}} H_n(A[-1]) \longrightarrow \cdots
\end{array}$$

These two long exact sequences each prove one equivalence: $H_*(g) \text{ iso} \iff H_*(\text{Cone}(g)) = 0$ and $H_*(i_A) \text{ iso} \iff H_*(\text{Cone}(g)) = 0$. \square

2.2 Replacing objects by projective / injective resolutions

2.21 Lemma:

Chain maps between projectives / acyclic complexes are unique up to homotopy:

- a.) Homology: If $C_* \in Ch^-(A)$ is acyclic and $P_* \in Ch^-(\text{Proj}(A))$ all morphisms $P_* \rightarrow C_*$ are null-homotopic. b.) Cohomology: If $C^* \in Ch^+(A)$ is acyclic and $I^* \in Ch^+(\text{Inj}(A))$ all morphisms $C^* \rightarrow I^*$ are null-homotopic.

Proof. Let $\alpha : P_* \rightarrow C_*$ be a chain map. Inductively we construct a homotopy $h : P_* \rightarrow C_*[1]$ between α and the zero map.

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 \longrightarrow 0 \\
& & \downarrow \alpha_2 & & \downarrow \alpha_1 & & \downarrow \alpha_0 \\
& \nearrow h_2 & & \nearrow h_1 & & \nearrow h_0 & \\
\cdots & \longrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & C_0 \longrightarrow 0
\end{array}$$

We begin with setting $h_n := 0$ for all $n < 0$. First step is to construct h_0 . Since C is exact, $C_1 \rightarrow C_0$ is epi so that α_0 lifts to some $h_0 : P_0 \rightarrow C_1$ by projectivity, so that $\partial_1 h_0 + \partial_0 = \alpha_0$ is satisfied.

If h_0, \dots, h_{n-1} are already known and a partial homotopy, then

$$\begin{aligned}
\partial_n \alpha_n &= \alpha_{n-1} \partial_n \\
&= (\partial_n h_{n-1} + h_{n-2} \partial_{n-1}) \partial_n \\
&= \partial_n h_{n-1} \partial_n
\end{aligned}$$

So that $\partial(\alpha_n - h_{n-1} \partial_n) = 0$. Therefore $\alpha_n - h_{n-1} \partial_n$ maps into $Z_n(C)$ which equals $B_n(C) = \text{im}(\partial_{n+1})$ by exactness. By projectivity, we can find h_n such that

$$\alpha_n - h_{n-1} \partial_n = \partial_{n+1} h_n$$

is satisfied which proves the lemma. \square

2.22 Corollary (Fundamental lemma of homological algebra):

“Objects can be replaced by their projective or injective resolutions”

a.) Homology: Assume that \mathbf{A} has enough projectives and that a projective resolution has been fixed for every object.

Any $f : A \rightarrow B$ extends to a chain map between the augmented complexes

$$\begin{array}{ccccccc} P_*(A) & \longrightarrow & A & \longrightarrow & 0 \\ \downarrow \phi & & \downarrow f & & \\ P_*(B) & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

ϕ is unique up to homotopy.

In particular: $\mathbf{A} \xrightarrow{P_*} K^-(\text{Proj}(\mathbf{A}))$ is a well-defined functor with $H_0 \circ P_* \cong \text{id}_{\mathbf{A}}$.

b.) Cohomology: Assume that \mathbf{A} has enough injectives and that an injective resolution has been fixed for every object.

Any $f : A \rightarrow B$ extends to a chain map between the augmented complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & I^*(A) \\ & & \downarrow f & & \downarrow \phi \\ 0 & \longrightarrow & B & \longrightarrow & I^*(B) \end{array}$$

ϕ is unique up to homotopy.

In particular: $\mathbf{A} \xrightarrow{I^*} K^+(\text{Inj}(\mathbf{A}))$ is a well-defined functor with $H_0 \circ I^* \cong \text{id}_{\mathbf{A}}$.

As a consequence, projective and injective resolutions are unique up to homotopy equivalence.

Proof. Uniqueness up to homotopy follows from the lemma. We only have to show existence. Again, we work inductively:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & P_2(A) & \longrightarrow & P_1(A) & \longrightarrow & P_0(A) & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow \phi_2 & & \downarrow \phi_1 & & \downarrow \phi_0 & & \downarrow f & & \\ \cdots & \longrightarrow & P_2(B) & \longrightarrow & P_1(B) & \longrightarrow & P_0(B) & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

We set $P_{-1}(A) := A$, $\phi_{-1} := f$, and $P_{-1}(B) := B$ for notational convenience. If ϕ_{n-1} is already constructed, then

$$\begin{array}{ccc} P_n(A) \xrightarrow{\partial} P_{n-1}(A) & & P_n(A) \xrightarrow{\partial} P_{n-1}(A) \xrightarrow{\partial} P_{n-2}(A) \\ \downarrow \phi_{n-1} & = & \downarrow \phi_{n-2} = 0 \\ P_{n-1}(B) \xrightarrow{\partial} P_{n-2}(B) & & P_{n-2}(B) \end{array}$$

Therefore $\phi_{n-1} \circ \partial_n : P_n(A) \rightarrow P_{n-1}(B)$ maps into $Z_{n-1}(P_*(B))$ which equals $B_{n-1}(P_*(B)) = \text{im}(\partial_n)$ by exactness. By projectivity, we get a lift $\phi_n : P_n(A) \rightarrow P_n(B)$. \square

2.23 Lemma (Horseshoe lemma):

“ P_* and I^* are exact”

a.) Homology: Every diagram

$$\begin{array}{ccccccc}
 & P_*(A) & & P_*(C) & & & \\
 & \downarrow & & \downarrow & & & \\
 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

with exact bottom row and projective resolutions in the columns can be extended with some projective resolution $P_*(B) \rightarrow B \rightarrow 0$ to a diagram

$$\begin{array}{ccccccc}
 0 & \rightarrow & P_*(A) & \rightarrow & P_*(B) & \rightarrow & P_*(C) \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

in which all rows are exact.

b.) Cohomology: Every diagram

$$\begin{array}{ccccccc}
 & 0 & & 0 & & & \\
 & \downarrow & & \downarrow & & & \\
 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & I^*(A) & & I^*(C) & &
 \end{array}$$

with exact top row and injective resolutions in the columns can be extended with some injective resolution $0 \rightarrow B \rightarrow I^*(B)$ to a diagram

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & I^*(A) & \rightarrow & I^*(B) & \rightarrow & I^*(C) \rightarrow 0
 \end{array}$$

in which all rows are exact.

Proof. Set $A_{-1} := A$ and $A_n := P_n(A)$, $C_{-1} := C$ and $C_n := P_n(C)$ as well as $B_{-1} := B$. Then define $P_n(B) := B_n := A_n \oplus C_n$.

For the vertical maps consider

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A_n & \longrightarrow & A_n \oplus C_n & \longrightarrow & C_n \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A_{n-1} & \xrightarrow{f} & B_{n-1} & \longrightarrow & C_{n-1} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A_{n-2} & \longrightarrow & B_{n-2} & \longrightarrow & C_{n-2} \longrightarrow 0
 \end{array}$$

$\begin{array}{c} \nearrow f \\ \nearrow g \\ \nearrow h \end{array}$

We define $g : A_n \oplus C_n \rightarrow B_{n-1}$ separately on the two components. Define $g : A_n \oplus 0 \rightarrow B_{n-1}$ to be the composition $A_n \rightarrow A_{n-1} \rightarrow B_{n-1}$.

The map $g : 0 \oplus C_n \rightarrow B_{n-1}$ we choose in two steps. First choose $h : C_n \rightarrow B_{n-1}$ to make the triangle on the right side commute. Then

$$\begin{array}{ccc}
 0 \oplus C_n & & 0 \oplus C_n \xrightarrow{\cong} C_n \\
 \downarrow h & & \downarrow \\
 B_{n-1} & = & C_{n-1} = 0 \\
 \downarrow & & \downarrow \\
 B_{n-2} \rightarrow C_{n-2} & & C_{n-2}
 \end{array}$$

d.h. $\partial h(C_n) \subseteq A_{n-2}$ because the $(n-2)$ th row is exact and of course $\partial \partial h = 0$ so that $\partial h(C_n) \subseteq Z_{n-2}(A_*) = B_{n-2}(A_*)$ by exactness of A_* . Using projectivity once again, we can lift ∂h to $f : C_n \rightarrow A_{n-1}$ and finally define $g : 0 \oplus C_n \rightarrow B_{n-1}$ as $h - f$. Note that $g(c_n) = \partial c_n$ still holds because $\text{im}(f) \subseteq \ker(B_{n-1} \rightarrow C_{n-1})$.

This ensures $\partial g = 0$ which proves that the middle column is a(n incomplete) complex. We still have to show exactness. So let $b_{n-1} \in B_{n-1}$ with $\partial b_{n-1} = 0$. Then its image $c_{n-1} = \overline{b_{n-1}}$ also satisfies $\partial c_{n-1} = 0$ so that a c_n exists with $c_{n-1} = \partial c_n$ by exactness of C_* . Then $\overline{b_{n-1} - g(0 \oplus c_n)} = c_{n-1} - \partial c_n = 0$ so that $b_{n-1} - g(0 \oplus c_n) \in \ker(B_{n-1} \rightarrow C_{n-1})$ which is $\text{im}(A_{n-1} \rightarrow B_{n-1})$ by exactness of the $(n-1)$ th row so that $b_{n-1} - g(0 \oplus c_n) = a_{n-1}$. Then $0 = 0 - 0 = \partial b_{n-1} - \partial g(0 \oplus c_n) = \partial a_{n-1}$ so that $a_{n-1} = \partial a_n = g(a_n \oplus 0)$. That shows $b_{n-1} = g(a_n \oplus c_n)$. \square

2.3 Replacing complexes by projective / injective resolutions

2.24 Corollary (Cartan-Eilenberg-resolution):

“Complexes have resolutions by double complexes of projectives/injectives”

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|--|--|
| <p>a.) Homology: For every $K_* \in Ch(A)$ exists a commutative double complex $P_{*,*} \in Ch^2(Proj(A))$ and maps $P_{n,*} \rightarrow K_n$ such that</p> <ul style="list-style-type: none"> i.) $P_{n,*} \rightarrow K_n \rightarrow 0$ ii.) $Z(P_{n,*}) \rightarrow Z_n(K) \rightarrow 0$ iii.) $B(P_{n,*}) \rightarrow B_n(K) \rightarrow 0$ iv.) $H(P_{n,*}) \rightarrow H_n(K) \rightarrow 0$ <p>are projective resolutions.</p> | <p>b.) Cohomology: For every $K^* \in Ch(A)$ exists a commutative double complex $I^{*,*} \in Ch^2(Inj(A))$ and maps $K^n \rightarrow I^{n,*}$ such that</p> <ul style="list-style-type: none"> i.) $0 \rightarrow K^n \rightarrow I^{n,*}$ ii.) $0 \rightarrow Z^n(K) \rightarrow Z(I^{n,*})$ iii.) $0 \rightarrow B^n(K) \rightarrow B(I^{n,*})$ iv.) $0 \rightarrow H^n(K) \rightarrow H(I^{n,*})$ <p>are injective resolutions.</p> |
|--|--|

Proof. Consider the short exact sequences

$$0 \rightarrow B_n(K) \rightarrow Z_n(K) \rightarrow H_n(K) \rightarrow 0$$

$$0 \rightarrow Z_n(K) \rightarrow K_n \xrightarrow{\partial} B_{n-1}(K) \rightarrow 0$$

and choose projective resolutions $P'_{n,*} \rightarrow B_n \rightarrow 0$ and $P''_{n,*} \rightarrow H_n \rightarrow 0$. Apply the horseshoe lemma to the first short exact sequence obtain a projective resolution $P'''_{n,*} \rightarrow Z_n \rightarrow 0$ fitting in the exact sequence and apply it again to the second short exact sequence obtain $P_{n,*} \rightarrow K_n \rightarrow 0$.

Now let $P_{n,*} \rightarrow P_{n-1,*}$ be the composition $P_{n,*} \twoheadrightarrow P'_{n,*} \hookrightarrow P'''_{n-1,*} \hookrightarrow P_{n-1,*}$. Since $P''' \rightarrow P \rightarrow P'$ are short exact sequences, we obtain a commutative double complex in this way.

By construction $Z(P_{n,*}) = P'''_{n,*}$, $B(P_{n,*}) = P'_{n,*}$ and $H(P_{n,*}) = P''_{n,*}$. \square

2.25 Lemma:

“Projective / injective resolutions of complexes exist”

- a.) Homology: For any bounded above complex $K_* \in Ch^-(A)$ there is a $P_* \in Ch^-(Proj(A))$ and a quasi-isomorphism $P_* \rightarrow K_*$.
 P_* can be chosen such that the quasi-isomorphism is termwise epi: $P_n \twoheadrightarrow K_n$.
- b.) Cohomology: For any bounded below complex $K^* \in Ch^+(A)$ there is a $I^* \in Ch^+(Inj(A))$ and a quasi-isomorphism $K^* \rightarrow I^*$.
 I_* can be chosen such that the quasi-isomorphism is termwise mono: $K_n \hookrightarrow I_n$.

Proof. Take the total complex of $P_{*,*}$ in the previous statement. \square

2.26 Lemma:

Homology: Let $M_* \in Ch^-(A), P_* \in Ch^-(Proj(A))$ and $M_* \xrightarrow{f} P_*$ termwise epi. If f is a quasi-isomorphism, then it splits.

Cohomology: Let $M^* \in Ch^+(A), I^* \in Ch^+(Inj(A))$ and $I^* \xrightarrow{f} M^*$ termwise mono. If f is a quasi-isomorphism, then it splits.

Proof. Let $K := \ker(f) \leq M$. We assume that sections s_0, \dots, s_{n-1} have already been chosen such that $\partial s_i = s_{i-1} \partial$ already holds. Then we can write $M_i = P_i \oplus K_i$ and assume s_i is the inclusion and $\partial_i^M = \begin{pmatrix} \partial_i^P & \\ & \partial_i^K \end{pmatrix}$ for all $i < n$.

Since P_n is projective and $M_n \rightarrow P_n$ is epi, we assume from the start $M_n = P_n \oplus K_n$, i.e. we first chose an arbitrary section. We now have to find a section $s_n = \begin{pmatrix} \text{id} \\ \sigma \end{pmatrix} : P_n \rightarrow \begin{pmatrix} P_n \\ K_n \end{pmatrix}$ which also commutes with ∂ .

We write $\partial^M : P_* \oplus K_* \rightarrow P_{*-1} \oplus K_{*-1}$ as $\begin{pmatrix} \partial^P & \\ \delta & \partial^K \end{pmatrix}$. Note that

$$0 = \partial_{n-1}^M \partial_n^M = \begin{pmatrix} \partial_{n-1}^P & \\ & \partial_{n-1}^K \end{pmatrix} \begin{pmatrix} \partial_n^P & \\ \delta & \partial_n^K \end{pmatrix} = \begin{pmatrix} \partial_n - 1^P \partial_n^P & \\ \partial_{n-1}^K \delta_n & \partial_{n-1}^K \partial_n^K \end{pmatrix}$$

so that $\partial_{n-1}^K \delta = 0$, i.e. $P_n \xrightarrow{\delta_n} Z_{n-1}(K)$. Now we use that f is a quasi-isomorphism so that $K = \ker(f)$ is acyclic. Therefore $Z_{n-1}(K) = B_{n-1}(K) = \text{im}(\partial_n^K)$.

By projectivity there exists $\sigma : P_n \rightarrow K_n$ with $\delta = \partial_n^K(-\sigma)$. With this definition s_n commutes with ∂ :

$$\partial_n^M s_n = \begin{pmatrix} \partial_n^P & \\ \delta & \partial_n^K \end{pmatrix} \begin{pmatrix} \text{id} \\ \sigma \end{pmatrix} = \begin{pmatrix} \partial_n^P & \\ -\partial_n^K \sigma & \partial_n^K \end{pmatrix} \begin{pmatrix} \text{id} \\ \sigma \end{pmatrix} = \begin{pmatrix} \partial_n^P & \\ 0 & \end{pmatrix} = \underbrace{\begin{pmatrix} \text{id} \\ 0 \end{pmatrix}}_{=s_{n-1}} \partial_n^P \quad \square$$

2.27 Lemma ("Fundamental lemma of homological algebra upgraded to complexes"):

- a.) Homology: Let $A_*, Q_* \in Ch^-(A)$ be quasi-isomorphic, say $Q_* \xrightarrow[\sim]{\alpha} A_*$. Furthermore let $P_* \in Ch^-(Proj(A))$ and $P_* \xrightarrow{\beta} A_*$ be arbitrary.
- b.) Cohomology: Let $A^*, Q^* \in Ch^+(A)$ be quasi-isomorphic, say $A_* \xrightarrow[\sim]{\alpha} Q_*$. Furthermore let $I^* \in Ch^+(Inj(A))$ and $A^* \xrightarrow{\beta} I^*$ be arbitrary.
- i.) If α is termwise epi, then there exists a chain-map $P_* \xrightarrow{\gamma} Q_*$ such that $\alpha \circ \gamma = \beta$.
- ii.) If α is arbitrary, there exists a γ such that $\alpha \circ \gamma \simeq \beta$.
- iii.) Any two chain-maps with $\alpha \circ \gamma_1 \simeq \beta \simeq \alpha \circ \gamma_2$ are homotopic.
- i.) If α is termwise mono, then there exists a chain-map $Q^* \xrightarrow{\gamma} I^*$ such that $\gamma \circ \alpha = \beta$.
- ii.) If α is arbitrary, there exists a γ such that $\gamma \circ \alpha \simeq \beta$.
- iii.) Any two chain-maps with $\gamma_1 \circ \alpha \simeq \beta \simeq \gamma_2 \circ \alpha$ are homotopic.

2.28: If A is concentrated in a single degree, then $Q_* \rightarrow A_0 \rightarrow 0$ is just an acyclic complex and the statement reduces to the fundamental lemma of homological algebra. In this sense this statement is a generalisation of the fundamental lemma from A to $D^\pm(A)$.

Homological version, seems harder?? Assume that a partial chain map $\gamma_0, \dots, \gamma_{n-1}$ is already constructed. We want to construct the missing arrow in the commutative diagram

$$\begin{array}{ccc}
 P_n & \xrightarrow{\quad} & P_{n-1} \\
 \downarrow \scriptstyle \text{dotted} & & \downarrow \\
 Q_n & \xrightarrow{\quad} & Q_{n-1} \\
 \downarrow & \searrow & \downarrow \\
 & F & \\
 \downarrow & \swarrow & \downarrow \\
 A_n & \xrightarrow{\quad} & A_{n-1}
 \end{array}$$

We set

$$F := A_n \times_{A_{n-1}} Z(Q_{n-1}) = \{ (a_n, q_{n-1}) \in A_n \times Q_{n-1} \mid \partial q_{n-1} = 0 \wedge \partial a_n = \alpha(q_{n-1}) \}$$

First we prove that the map $Q \xrightarrow{(\alpha, \partial)} F$ is epi. Let $(a_n, q_{n-1}) \in F$ be arbitrary.

Then $q_{n-1} \in Z_{n-1}(Q)$ so that the homology class is well-defined. Then $\alpha_*[q_{n-1}]_{H_{n-1}(Q)} = [\alpha(q_{n-1})]_{H_{n-1}(A)} = [\partial a_n] = 0$. Since α is injective on homology, this means $[q_{n-1}] = 0$, i.e. $q_{n-1} = \partial q'_n$ for some $q'_n \in Q_n$.

Then $\partial a_n = \alpha(q_{n-1}) = \alpha \partial(q'_n) = \partial \alpha(q'_n)$ so that $a_n - \alpha(q'_n) \in Z_n(A)$. Since α is surjective on homology, there is a $z_n \in Z_n(Q)$ such that $[\alpha(z_n)] = [a_n - \alpha(q'_n)]$, i.e. there exists a a_{n+1} such that $\alpha(z_n) = a_n - \alpha(q'_n) + \partial a_{n+1}$.

Now choose an preimage $q_{n+1} \in Q_{n+1}$ of a_{n+1} and set $q_n := z_n + q'_n - \partial q_{n+1}$. This is the preimage of (a_n, q_{n-1}) :

$$\alpha(q_n) = \underbrace{\alpha(z_n) + \alpha(q'_n)}_{=a_n+b_n} - \alpha(\partial q_{n+1}) = a_n + b_n - \partial\alpha(q_{n+1}) = a_n$$

$$\partial(q_n) = \underbrace{\partial z_n}_{=0} + \underbrace{\partial q'_n}_{=q_{n-1}} + 0$$

Since we now know that $Q_n \rightarrow F$ is epi, we can lift the morphism $(\beta_n, \gamma_{n-1}\partial) : P_n \rightarrow F$ to a morphism $\gamma_n : P_n \rightarrow Q_n$. By construction it makes the diagram commute so that it is a partial chain map.

b. If α is not term-wise epi, we can replace Q by the double mapping *cocone* $D := DMC^{op}(Q \xrightarrow{\alpha} A \xleftarrow{\beta} P)$, i.e. the double mapping cone in the opposite category. It comes with two epimorphisms $P \xleftarrow{q_P} D \xrightarrow{p_Q} Q$ such that

$$\begin{array}{ccc} D & \xrightarrow{q_P} & P \\ \downarrow q_Q & & \downarrow \beta \\ Q & \xrightarrow[\sim]{\alpha} & A \end{array}$$

anticommutes up to homotopy. Furthermore q_P is a quasi-isomorphism because α is one. Since q_P is epi and a quasi-isomorphism, there is a splitting of $P \leftarrow D$ by lemma 2.26 which corresponds to β factoring through α .

c. For uniqueness observe that $\alpha \circ (\gamma_1 - \gamma_2) \simeq 0$ so that there is a chain-map

$$\hat{\gamma} : P_*[-1] \xrightarrow{(h, \gamma_1 - \gamma_2)} Cone(\alpha), p \mapsto (h(p), (\gamma_1 - \gamma_2)(p))$$

by the universal mapping property of cones. Since α is a quasi-isomorphism, $Cone(\alpha)$ is acyclic so that any such map is null homotopic. In particular $\gamma_1 - \gamma_2 = quotient \circ \hat{\gamma} \simeq 0$. \square

3 The derived category

4 Derived functors I: δ -functors

4.1 (The Problem): Given abelian categories \mathbf{A} and \mathbf{B} and a right-exact functor $F : \mathbf{A} \rightarrow \mathbf{B}$, and exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

gives a exact sequence

$$F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow 0$$

We want to find functors $L_n F$ and natural transformations δ_n (natural w.r.t. the short exact sequence) such that this sequence extends to a long exact sequence

$$\cdots \rightarrow L_2 F(C) \xrightarrow{\delta_2} L_1 F(A) \rightarrow L_1 F(B) \rightarrow L_1 F(C) \xrightarrow{\delta_1} \underbrace{F(A)}_{=L_0 F(A)} \rightarrow \underbrace{F(B)}_{L_0 F(B)} \rightarrow \underbrace{F(C)}_{=0} \rightarrow 0$$

And similarly for left-exact functors.

Of course, we want the universal solution to this problem.

4.2 Definition (δ -functors):

A family $F = (F_n, \delta_n)_{n \in \mathbb{N}}$ of functors $\mathbf{A} \xrightarrow{F_n} \mathbf{B}$ and natural transformations $F_n(C) \xrightarrow{\delta_n} F_{n-1}(A)$ for every short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ that transforms such short exact sequences into long exact sequences as above is called a (homological) δ -functor.

A morphism $F \xrightarrow{t} G$ of δ -functors is a family (t_n) of natural transformations $F_n \xrightarrow{t_n} G_n$ which induces a morphism between the long exact sequences, i.e. $t_{n-1} \delta_n^F = \delta_n^G t_n$.

Cohomological δ -Functors (F^n, d^n) are analogously defined.

4.3 Definition (Universal δ -functors):

A homological δ -functor (F_n, δ_n) is called the universal δ -functor if for every (G_n, δ_n) and every $G_0 \xrightarrow{t_0} F_0$ there exists a unique morphism $G \xrightarrow{t} F$ of δ -functors extending t_0 .

Similarly a cohomological δ -functor is one where every morphism $F^0 \xrightarrow{t^0} G^0$ extends uniquely to a morphism $F \xrightarrow{t} G$.

4.4 Definition (Derived functors):

Let $F : \mathbf{A} \rightarrow \mathbf{B}$ be right-exact. A δ -functor $(L_n F, \delta_n)$ together with an isomorphism $L_0 F \xrightarrow{\tau} F$ is called the left derived functor of F if $(L F, \tau)$ is a final object in the category of all δ -functor-with-isomorphisms.

It is in other words a representation of the functor $\{\delta\text{-functors}\} \rightarrow \mathbf{Set}, (G_n, \delta_n) \mapsto \text{Nat}(G_0, F)$ such that the universal element $\tau \in \text{Nat}(F_0, F)$ is an iso.

Similarly right derived functor $R F$ of a left exact F is defined as an initial object in the appropriate category of δ -functors with isomorphism $F \xrightarrow[\cong]{\tau} R^0 F$, i.e. a representation of the functor $(G^n, d^n) \mapsto \text{Nat}(F, G^0)$ such that the universal element is an isomorphism.

4.5 Lemma (Recognising universal δ -functors):

Let (F_n, δ_n) be a δ -functor.

- a.) Homology: If \mathbf{A} has enough projectives and $F_n(P) = 0$ for all $n \geq 1$ and all $P \in \text{Proj}(\mathbf{A})$, then F is a universal homological δ -functor.
- b.) Cohomology: If \mathbf{A} has enough injectives and $F_n(I) = 0$ for all $n \geq 1$ and all $I \in \text{Inj}(\mathbf{A})$, then F is a universal cohomological δ -functor.

tives and $F^n(I) = 0$ for all $n \geq 1$ and all $I \in \text{Inj}(\mathbf{A})$, then F is a universal cohomological δ -functor.

Proof. Let $(\tilde{F}_n, \tilde{\delta}_n)$ be another δ -functor and assume that unique transformations t_0, \dots, t_{n-1} have already been constructed. Fix $A \in \mathbf{A}$ and choose a short exact $0 \rightarrow K \xrightarrow{j} P \xrightarrow{q} A \rightarrow 0$ with P projective. Then

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \tilde{F}_n(P) & \longrightarrow & \tilde{F}_n(A) & \xrightarrow{\tilde{\delta}_n} & \tilde{F}_{n-1}(K) \longrightarrow \tilde{F}_{n-1}(P) \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow t_{n-1} \\ \cdots & \longrightarrow & \underbrace{F_n(P)}_{=0} & \longrightarrow & F_n(A) & \xrightarrow{\delta_n} & F_{n-1}(K) \longrightarrow F_{n-1}(P) \longrightarrow \cdots \end{array}$$

It follows that $F_n(A) \xrightarrow{\delta_n} \ker(F_{n-1}(j))$ and since t_{n-1} is natural, there is a unique $t_n : \tilde{F}_n(A) \rightarrow F_n(A)$ that makes the square commute. This t_n does not depend on the choice of K and P by Schanuel's lemma.

Naturality of t_n follows from a simple diagram chase using naturality of δ_n and $\tilde{\delta}_n$, naturality of t_{n-1} and that $F_n(A) \rightarrow F_{n-1}(K)$ is mono.

It remains to show that t_n commutes with the deltas for an arbitrary short exact $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$. This also follows from a simple diagram chase. \square

4.6 Theorem (Derived functors exist):

Let $F : \mathbf{A} \rightarrow \mathbf{B}$ be additive.

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| <p>a.) Homology: Let F be right-exact.</p> <p>i.) If \mathbf{A} has enough projectives, then F has a left derived functor.</p> <p>ii.) $L_i F(P) = 0$ for all projectives P and all $i \geq 1$.</p> <p>iii.) Deriving is a functor $L_i : \text{Fun}_{\text{r.e.}}(\mathbf{A}, \mathbf{B}) \rightarrow \text{Fun}_{\text{add}}(\mathbf{A}, \mathbf{B})$.</p> | <p>b.) Cohomology: Let F be left-exact.</p> <p>i.) If \mathbf{A} has enough injectives, then F has a right derived functor.</p> <p>ii.) $R^i F(I) = 0$ for all injectives I and all $i \geq 1$.</p> <p>iii.) Deriving is a functor $R^i : \text{Fun}_{\text{l.e.}}(\mathbf{A}, \mathbf{B}) \rightarrow \text{Fun}_{\text{add}}(\mathbf{A}, \mathbf{B})$.</p> |
|---|---|

Proof. Existence: Define

$$L_n F := \mathbf{A} \xrightarrow{P_*} K^-(\mathbf{A}) \xrightarrow{K^-(F)} K^-(\mathbf{B}) \xrightarrow{H_n} \mathbf{B}$$

Note that this does not depend on the choice of the projective resolutions P_* because all choices are homotopy equivalent and homology forgets homotopy. Note that $L_i F(P) = 0$ for P projective and $i > 0$ because $0 \rightarrow P \xrightarrow{\text{id}} P \rightarrow 0$ is a projective resolution of P .

Horseshoe lemma implies that every short exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

lifts to exact sequence up to homotopy $0 \rightarrow P_*(A) \rightarrow P_*(B) \rightarrow P_*(C) \rightarrow 0$ which is termwise split. Thus $0 \rightarrow F(P_*(A)) \rightarrow F(P_*(B)) \rightarrow F(P_*(C)) \rightarrow 0$ is also exact. That implies a long exact sequence in homology with a natural connecting morphisms from the snake lemma. Therefore $LF = (L_i F, \delta_i)$ is a δ -functor. It extends P because $P_*(A) \rightarrow A \rightarrow 0$ is a projective resolution and F is right exact so that $FP_1 \rightarrow FP_0 \rightarrow FA \rightarrow 0$ is still exact so that $H_0(F(P_*(A))) \cong A$ naturally.

We still have to show universality. Let $(\tilde{F}_n, \tilde{\delta}_n)$ be another δ -functor and $t_0 : \tilde{F} \rightarrow F_0$. The above lemma shows that there is a unique morphism of δ -functors $t : \tilde{F} \rightarrow F$ which extends t_0 .

The lemma also proves that every natural transformation $F \rightarrow G$ between right exact functors extends to $LF \rightarrow LG$ since $L_i G(P) = 0$. \square

4.1 Computing derived functors via acyclic resolutions

4.7 Definition (F -acyclic objects):

An object $Q \in \mathbf{A}$ is called F -acyclic if

- a.) Homology: $L_n F(Q) = 0$
- b.) Cohomology: $R^n F(Q) = 0$

holds for all $n \geq 1$.

4.8: $Proj(\mathbf{A}) \subseteq Acyc(F)$ for all right-exact F and $Inj(\mathbf{A}) \subseteq Acyc(F)$ for all left exact F . For some F (like $\text{Hom}(A, -)$) equality may hold, but depending on F , the class of acyclics may be bigger then the class of projectives (or injectives). For example in general $Proj(\mathbf{A-Mod}) \subsetneq Flat(\mathbf{A-Mod}) \subseteq Acyc(M \otimes -)$.

We want to show that complexes of F -acyclic objects are just as good to compute derived functors as projectives / injectives are.

4.9 Theorem:

Let $F : \mathbf{A} \rightarrow \mathbf{B}$ be additive.

- a.) Homology: Assume F is right-exact and \mathbf{A} has enough projectives. If $Q_* \rightarrow A \rightarrow 0$ is a resolution of A by F -acyclic objects, then $L_n F(A) \cong H_n(F(Q_*))$.

up-to-homotopy chain-map $P_* \xrightarrow{\gamma} Q_*$ induces an isomorphism

$$L_n F(A) = H_n(F(P_*)) \xrightarrow[\cong]{H_n(F\gamma)} H_n(F(Q_*))$$

More precisely: Given any projective resolution $P_* \rightarrow A \rightarrow 0$, the unique-

- b.) Cohomology: Assume F is left-exact and \mathbf{A} has enough injectives. If $0 \rightarrow A \rightarrow Q^*$ is a resolution of A by

F -acyclic objects, then $R^n F(A) \cong H^n(F(Q^*))$.

up-to-homotopy chain-map $Q^* \xrightarrow{\gamma} I^*$ induces an isomorphism

More precisely: Given any injective resolution $0 \rightarrow A \rightarrow I^*$, the unique- $H^n(F(Q^*)) \xrightarrow[\cong]{H^n(F\gamma)} H^n(F(I^*)) = R^n F(A)$

The proof needs to small bit of work.

4.10 Lemma:

The class of F -acyclics has the following properties:

- | | |
|--|--|
| <p>a.) Homology: Assume F is right-exact and \mathbf{A} has enough projectives. Then</p> <ul style="list-style-type: none"> i.) Every $A \in \mathbf{A}$ is a quotient $Q \twoheadrightarrow A$ for some acyclic Q. ii.) It is closed under direct sums and direct summands. iii.) If in an exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ both B and C are acyclic, then A is too. iv.) If in an exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ the object C is acyclic, then $0 \rightarrow FA \rightarrow FB \rightarrow FC \rightarrow 0$ is also exact. | <p>b.) Cohomology: Assume F is left-exact and \mathbf{A} has enough injectives. Then</p> <ul style="list-style-type: none"> i.) Every $A \in \mathbf{A}$ is a subobject $A \hookrightarrow Q$ for some acyclic Q. ii.) It is closed under direct sums and direct summands. iii.) If in an exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ both A and B are acyclic, then C is too. iv.) If in an exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ the object A is acyclic, then $0 \rightarrow FA \rightarrow FB \rightarrow FC \rightarrow 0$ is also exact. |
|--|--|

Proof. i. Projectives are always acyclic.

ii. follows because $L_n F$ is additive.

iii. and iv. follow from the long exact sequence. □

4.11 Lemma:

Let $F : \mathbf{A} \rightarrow \mathbf{B}$ be additive.

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| <p>a.) Homology: Let F be right exact and $Q_* \in Ch^-(Ac(F))$ be a complex of F-acyclic objects.</p> | <p>b.) Cohomology: Let F be left exact and $Q^* \in Ch^+(Ac(F))$ be a complex of F-acyclic objects.</p> |
|---|--|

If Q is exact, then FQ is also exact.

Proof. Let K_n be the kernels / images of the boundary maps so that we get a diagram

$$\begin{array}{ccccccc}
 & & & K_2 & & & K_0 \\
 & & \nearrow & \searrow & & \nearrow & \rightrightarrows \\
 \cdots & \longrightarrow & Q_3 & \longrightarrow & Q_2 & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & 0 \\
 & \nearrow & & & \searrow & \nearrow & & & & & \\
 & K_3 & & & & K_1 & & & & &
 \end{array}$$

where the diagonals are short exact sequences. First observation: By induction all K_n are F -acyclic, because the Q_n are.

The transformed sequence

$$\begin{array}{ccccccc}
 & & & FK_2 & & & FK_0 \\
 & & \nearrow & \searrow & & \nearrow & \rightrightarrows \\
 \cdots & \longrightarrow & FQ_3 & \longrightarrow & FQ_2 & \longrightarrow & FQ_1 & \longrightarrow & FQ_0 & \longrightarrow & 0 \\
 & \nearrow & & & \searrow & \nearrow & & & & & \\
 & FK_3 & & & & FK_1 & & & & &
 \end{array}$$

is exact iff the diagonals are exact again, i.e. if $FK_n \rightarrow FQ_n$ is mono. This follows from exactness of $K_n \hookrightarrow Q_n \rightarrow K_{n-1}$ and $L_1F(K_{n-1}) = 0$. \square

4.12 Corollary:

“ F maps quasi-isomorphisms between complexes of acyclic objects to quasi-isomorphisms”

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| <p>a.) Homology: Let F be right exact and $Q_*, \tilde{Q}_* \in Ch^-(Acyc(F))$ be complexes of F-acyclic objects.</p> | <p>b.) Cohomology: Let F be left exact and $Q^*, \tilde{Q}^* \in Ch^+(Acyc(F))$ be complexes of F-acyclic objects.</p> |
|--|---|

If $Q \xrightarrow[\sim]{\alpha} \tilde{Q}$ is a quasi-isomorphism, then $FQ \xrightarrow[\sim]{F\alpha} F\tilde{Q}$ is a quasi-isomorphism too.

Proof. α being a quasi-isomorphism implies that $Cone(\alpha)$ is exact. This is also a complex of F -acyclic objects. Hence $F(Cone(\alpha)) = Cone(F\alpha)$ is exact by the lemma. Therefore $F\alpha$ is a quasi-isomorphism. \square

Proof of the main theorem. Let $P_* \rightarrow A$ be a projective resolution, $Q_* \rightarrow A$ an acyclic resolution and $\gamma : P_* \rightarrow Q_*$ be a chain-map extending $A \xrightarrow{id} A$ along those resolutions. γ is a quasi-isomorphism because both resolutions have homology $H_n = \begin{cases} A & n = 0 \\ 0 & \text{otherwise} \end{cases}$. Therefore $F\gamma$ is a quasi-isomorphism. \square

5 Examples

5.1 Example (Snake lemma):

Taking kernels is a left-exact functor $\mathbf{A}^{\{*\rightarrow*\}} \rightarrow \mathbf{Ab}$. Its right derived functor is the cokernel in degree 1 and zero further up.

Dually taking cokernels is right-exact and its left derived functor is the kernel in degree 1 and zero everywhere else.

This is a manifestation of the snake lemma.

5.2 Example (Sheaf (co)homology):

Sheaf cohomology $H^*(X, \mathcal{F})$ is the right derived functor of the global section functor $\Gamma : Sh(X) \rightarrow \mathbf{Ab}$.

5.3 Example (DeRham cohomology):

$H_{\text{dR}}^*(M)$ is Sheaf cohomology of the sheaf $\underline{\mathbb{R}}_M$ of locally constant functions $M \rightarrow \mathbb{R}$.

This uses that

$$0 \rightarrow \underline{\mathbb{R}}_M \hookrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n(M) \rightarrow 0$$

is a resolution of $\underline{\mathbb{R}}_M$ by fine sheafs and that fine sheafs are Γ -acyclic.

5.4 Example (Singular cohomology):

$H_{\text{sing}}^*(X; G)$ is sheaf cohomology of the sheaf $\underline{G}_X \in Sh(X)$ of locally constant G -valued functions if X is paracompact.

5.5 Example (Étale cohomology):

Étale cohomology is the Sheaf cohomology for sheafs on the étale site, i.e. the right derived functor of global sections $\Gamma : \mathbf{Sh}_{\text{et}}(X) \rightarrow \mathbf{Ab}$.

5.6 Example (Ext and Tor):

$\text{Ext}_A^i(M, N)$ is right derived of $\text{Hom}_A(M, -) : A\text{-Mod} \rightarrow \mathbf{Ab}$ as well as left derived of $\text{Hom}_A(-, N) : \mathbf{Mod}\text{-}A \rightarrow \mathbf{Ab}^{\text{op}}$.

$\text{Tor}_i^A(M, N)$ is left derived of both $M \otimes_A - : A\text{-Mod} \rightarrow \mathbf{Ab}$ and $- \otimes_A N : \mathbf{Mod}\text{-}A \rightarrow \mathbf{Ab}$.

It is also the left derived of $- \otimes - : \mathbf{Mod}\text{-}A \times A\text{-Mod} \rightarrow \mathbf{Ab}$!

5.7 Example (Group (co)homology):

$H_*(G, M)$ is the left derived functor of the functor of coinvariants $(-)_G = k \otimes_{kG} -$, i.e. it is $\text{Tor}_*^{kG}(k, M)$.

$H_k^*(G, -)$ is the right derived functor of the functor of fixed points $(-)^G = \text{Hom}_{kG}(k, -)$, i.e. it is $\text{Ext}_{kG}^*(k, M)$.

5.8 Example (Hochschild (co)homology):

Let $A^e := A \otimes_k A^{op}$ be the enveloping algebra of the k -algebra A .

$HH_n(A, M) := Tor_n^{A^e}(A, M)$, i.e. it is the left derived functor of the functors of coinvariant $M/[A, M] = A \otimes_{A^e} M : (A, A)\text{-Bimod} \rightarrow \mathbf{Ab}$.

$HH^n(A, M) := Ext_{A^e}^n(A, M)$, i.e. the right derived functor of the functor of invariants

$Z(M) := \text{Hom}_{A^e}(A, M) : (A, A)\text{-Bimod} \rightarrow \mathbf{Ab}$.

5.9 Example (Lie-algebra (co)homology):

$H_n(\mathfrak{g}, M) := Tor_n^{U(\mathfrak{g})}(k, M)$, i.e. left derived of taking coinvariants.

$H^n(\mathfrak{g}, M) := Ext_{U(\mathfrak{g})}^n(k, M)$, i.e. right derived of taking invariants.

6 Derived functors II: Total derived functors

6.1 Motivation

6.1: Instead of looking at homology alone, we should look at chain complexes up to some notion of equivalence, i.e. we should retain more of the information about the boundary morphisms ∂ than just their homology groups.

The reason for this lies in things like Whitehead's theorem:

6.2 Theorem (Whitehead's theorem):

Let X, Y be two simply connected CW-complexes. Then X is homotopy equivalent to Y iff there exists a quasi-isomorphism $C_*(X) \rightarrow C_*(Y)$.

For this theorem it is not sufficient to just have $H_*(X) \cong H_*(Y)$. There must be a chain map inducing this isomorphism. In other words there are spaces, even manifolds, with $H_*(X) \cong H_*(Y)$ and $\pi_1(X) = \pi_1(Y) = 1$ such that $X \not\cong Y$. The isomorphism in homology is "accidental" in a sense, it does not come from a chain-map.

In the sense of Whitehead's theorem the object $C_*(X)$ up to chain-isomorphism is enough to determine homotopy type, but $H_*(X)$ is not.

Also note that $C_*(X)$ is enough to determine the cohomology $H^*(X)$ simply by dualising $H^*(X) = H(\text{Hom}(C_*(X), \mathbb{Z}))$ while $H_*(X)$ alone is not sufficient since $H^*(X) \not\cong \text{Hom}(H_*(X), \mathbb{Z})$ in general.

6.3: On the other hand, going from homology to $K(A)$, i.e. to view everything up to homotopy, is not good enough too, because several complexes which we use to compute (co)homologies (say F -acyclic-resolutions and projective resolutions) are not homotopy equivalent even though for (co)homological purposes they should be the same, because they are (uniquely / naturally) quasi-isomorphic.

The derived category combines the best of both worlds by retaining the chain complexes and morphisms between them, but formally inverting all quasi-isomorphisms.

6.2 The derived category

6.4 Definition (Derived category):

Let \mathbf{A} be an additive category. Then $D(\mathbf{A})$ is defined as the localisation of $K(\mathbf{A})$ at quasi-isomorphisms, i.e. it is the universal functor $K(\mathbf{A}) \rightarrow D(\mathbf{A})$ such that

- a.) it turns (homotopy classes consisting of) quasi-isomorphisms into isomorphisms
- b.) Every other functor $K(\mathbf{A}) \rightarrow D$ with this property factors uniquely through $D(\mathbf{A})$.

$D^\pm(\mathbf{A})$ and $D^b(\mathbf{A})$ are defined analogously.

6.5 Theorem:

Morphisms $A \rightarrow B$ in $D(\mathbf{A})$ can be described as equivalence classes of roofs

$$\begin{array}{ccc} & X & \\ f \nearrow & & \nwarrow \alpha \\ A & & B \end{array}$$

with $B \xrightarrow[\sim]{\alpha} X$ a quasi-isomorphism and $A \xrightarrow{f} B$ a morphism in $K(\mathbf{A})$. We denote such a roof by $\alpha^{-1}f$.

Two such roofs $\alpha^{-1}f, \beta^{-1}g$ are defined as equivalent if there is a diagram of the form

$$\begin{array}{ccccc} & & X & & \\ & f \nearrow & \uparrow & \nwarrow \sim & \\ A & \longrightarrow & Z & \longleftarrow \sim & B \\ & g \searrow & \downarrow & \swarrow \sim & \\ & & Y & & \end{array}$$

which commutes up to homotopy.

Composition of two roofs $\alpha^{-1}f : A \rightarrow B, \beta^{-1}g : B \rightarrow C$ is defined by the double mapping cylinder:

$$\begin{array}{ccccc} & & DMC(\alpha, \beta) & & \\ & i_X \nearrow & & \nwarrow i_Y & \\ & X & & Y & \\ f \nearrow & \nwarrow \alpha & & g \nearrow & \nwarrow \beta \\ A & & B & & C \end{array}$$

6.6: Note that $Y \xrightarrow{i_Y} DMC(\alpha, \beta)$ is a quasi-isomorphism because α is a quasi-isomorphism. Also note that $i_X\alpha + i_Y\beta \simeq 0$ so that $-i_X^{-1}i_Y \equiv \alpha\beta^{-1}$ holds in $D(\mathbf{A})$.

The sum of two parallel roofs $\alpha^{-1}f, \beta^{-1}g : A \rightarrow B$ can therefore be realised as $\alpha^{-1}f + \beta^{-1}g \equiv (i_X\alpha)^{-1}(i_Xf - i_Yg)$ where $X \xrightarrow{i_X} DMC(\alpha, \beta) \xleftarrow{i_Y} Y$ are the canonical inclusions.

6.7: In general $D(\mathbf{A})$ will not be locally small if \mathbf{A} is not a small category.
However: If \mathbf{A} has enough projectives / injectives or is small itself, then $D^-(\mathbf{A}) / D^+(\mathbf{A})$ is guaranteed to be locally small.

6.8 Theorem:

Let $A \rightarrow fB$ be a chain-map.

- a.) f Iso. in $D(\mathbf{A}) \iff f$ quasi-isomorphism. In other words: Not more than needed gets inverted.
- b.) TFAE:
 - i.) $f = 0$ in $D(\mathbf{A})$.
 - ii.) There exists a quasi-isomorphism $B \xrightarrow{\beta} X$ s.t. $\beta \circ f \simeq 0$.
 - iii.) $0 \rightarrow B \xrightarrow{i_B} \text{Cone}(f) \xrightarrow{q} A[-1] \rightarrow 0$ is a split exact sequence in $D(\mathbf{A})$.

6.9 Lemma (Resolution functors):

Let \mathbf{A} be a small abelian category.

- a.) Assume \mathbf{A} has enough projectives and a projective resolution $P_*(A) \xrightarrow{\sim} A_*$ has been fixed for every complex.
Then $P_* : D^-(\mathbf{A}) \rightarrow K^-(\text{Proj}(\mathbf{A}))$ is a well-defined functor which is a right inverse to the localisation functor.
- b.) Assume \mathbf{A} has enough injectives and an injective resolution $A_* \xrightarrow{\sim} I^*(A)$ has been fixed for every complex.
Then $I^* : D^+(\mathbf{A}) \rightarrow K^+(\text{Inj}(\mathbf{A}))$ is a well-defined functor which is a right inverse to the localisation functor.

Proof. We have to show $\text{Hom}_K(P_*(A_*), P_*(B_*)) = \text{Hom}_D(A_*, B_*)$.

Morphisms $A \xrightarrow{\gamma} B$ in D^- are roofs $A \rightarrow M \xleftarrow{\sim} B$.

$$\begin{array}{ccccc}
P_*(A) & \xrightarrow{\quad \gamma \quad} & P_*(B) & & \\
\downarrow \sim & & \downarrow \sim & & \\
A & \xrightarrow{\quad \quad} M & \xleftarrow{\quad \sim \quad} & B & \\
& \dashrightarrow & & &
\end{array}$$

Since $P_*(A)$ is termwise projective and $P_*(B) \xrightarrow{\sim} B \xrightarrow{\sim} M$ is a quasi-isomorphism, we can complete the triangle of $P_*(A) \rightarrow M$ and $M \xleftarrow{\sim} P_*(B)$ with a unique-up-to-homotopy chain-map $\gamma : P_*(A) \rightarrow P_*(B)$ making the diagram commute up to homotopy. \square

6.3 Total derived functors

6.10: Given the way we constructed derived functors, we already worked with an object $LF(A) \in D^-(\mathbf{B})$, namely the complex $F(P_*(A))$ (which was also quasi-isomorphic to $F(Q_*)$ for any resolution of A by F -acyclic objects).

Simply by not taking homology at the end, we already have a functor $\mathbf{A} \rightarrow D^-(\mathbf{B})$. We will now extend this functor to the much nicer functor $LF : D^-(\mathbf{A}) \rightarrow D^-(\mathbf{B})$, called the total derived functor.

Note that \mathbf{A} embeds into $D^-(\mathbf{A})$ by identifying every $A \in \mathbf{A}$ with the complex $A[0] := \cdots \rightarrow 0 \rightarrow A \rightarrow 0 \rightarrow \cdots$ having A in degree 0. In this sense this really is an extension of the functor to a larger category.

6.11 Definition:

Let $p_A^\pm : K^\pm(\mathbf{A}) \rightarrow D^\pm(\mathbf{A})$ be the projection functor from the homotopy category onto the derived category. And let $F : \mathbf{A} \rightarrow \mathbf{B}$ be additive.

The total derived functor of $F : \mathbf{A} \rightarrow \mathbf{B}$ is the “best approximation” of $K^\pm(F) : K^\pm(\mathbf{A}) \rightarrow K^\pm(\mathbf{B})$ on the level of derived categories, i.e. it fits into the commutative (up to natural isomorphism) diagram of functors

$$\begin{array}{ccc} K^\pm(\mathbf{A}) & \xrightarrow{K^\pm(F)} & K^\pm(\mathbf{B}) \\ p_A^\pm \downarrow & & \downarrow p_B^\pm \\ D^\pm(\mathbf{A}) & \xrightarrow[LF]{RF} & D^\pm(\mathbf{B}) \end{array}$$

6.12: In this situation LF / RF is a right / left Kan-extension of $p_B \circ K(F)$ along the localisation p_A . Concretely: LF fits into a diagram

$$\begin{array}{ccc} K^-(\mathbf{A}) & \xrightarrow{p_B^+ \circ K(F)} & D^-(\mathbf{B}) \\ & \searrow p_A^- & \downarrow \cong \nearrow LF \\ & D^-(\mathbf{A}) & \end{array}$$

together with a natural transformation $LF \circ p_A \rightarrow p_B \circ K(F)$ (which happens to be an isomorphism in this case) such that for every other functor $D^-(\mathbf{A}) \xrightarrow{G} D^-(\mathbf{B})$ any given natural transformation $G \circ p_A \xrightarrow{f} p_B \circ K(F)$ factors uniquely through $LF \circ p_A$.

$$\begin{array}{ccc} K^-(\mathbf{A}) & \xrightarrow{p_B^- \circ K(F)} & D^-(\mathbf{B}) \\ & \searrow p_A^- & \downarrow \cong \nearrow f \\ & D^-(\mathbf{A}) & \xrightarrow{G} \end{array} = \begin{array}{ccc} K^-(\mathbf{A}) & \xrightarrow{p_B^- \circ K(F)} & D^-(\mathbf{B}) \\ & \searrow p_A^- & \downarrow \cong \nearrow LF \\ & D^-(\mathbf{A}) & \xrightarrow{G} \end{array}$$

(Note: In the right diagram, there is a dashed arrow from $D^-(\mathbf{A})$ to $D^-(\mathbf{B})$ labeled LF and a curved arrow from $D^-(\mathbf{A})$ to $D^-(\mathbf{B})$ labeled G . A small triangle with $\exists!$ indicates the unique factorization.)

Therefore some authors *define* LF of *any* additive functor F as the right Kan extension $Ran_{p_A^-}(p_B^- \circ K(F))$ and RF as the left Kan extension $Lan_{p_A^+}(p_B^+ \circ K(F))$. In this situation

however, even if they exist, LF and RF do in general not extend F if F is not right / left exact.

6.13 Theorem (Total derived functors exist):

Let $F : \mathbf{A} \rightarrow \mathbf{B}$ be additive.

- a.) Homology: If F is right exact and \mathbf{A} has enough projectives, LF exists.
- b.) Cohomology: If F is left exact and \mathbf{A} has enough injectives, then RF exists.

Proof. Choose a resolution functor $D^\pm(\mathbf{A}) \rightarrow K^\pm(\mathbf{A})$ and compose with $p_{\mathbf{B}}^\pm \circ K^\pm(F)$. \square

6.14: Note that we do not need projective / injective resolutions, F -acyclic resolutions are fine too because we have already proven that $F(P_*(A))$ is quasi-isomorphic to $F(Q_*)$ if Q_* is any resolution by F -acyclic objects.