Broué's conjecture in a special case

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Convention:
Let G be a finite group, p a prime, $(\mathbb{K}, \mathcal{O}, \mathbb{F})$ a p-modular system with $\mathbb{K}$ and $\mathbb{F}$ large enough (e.g. algebraically closed).

1 What was Broué's conjecture again?

1.1 Definition (Derived category)**:**
$D^?(\mathsf{A}) := Q^{-1}K^?(\mathsf{A})$ (with $? \in \{\text{unbounded}, +, -, b\}$ where Q is the class of quasi-isomorphisms, i.e.

$$Q := \{\, f \in \mathrm{Mor}(K) \mid H(f)\,\text{isomorphism}\,\}$$

1.2 Conjecture (Abelian Defect Group Conjecture (Broué, Rickard))**:**
Let G be a finite group, $B \in Bl(G)$ a p-block of G, $D \leq G$ its defect group and $b \in Bl(N_G(D))$ the Brauer correspondent of B.
If D is abelian, then

$$D^b(B) \cong D^b(b)$$

as triangulated categories.

2 Modular representation theory of A_5

2.1 Theorem (Ordinary character table of A_5)**:**
The character table of A_5 in characteristic zero is

C	1	$(12)(34)$	(123)	(12345)	(13524)
$ord(x)$	1	2	3	5	5
$C_G(x)$	A_5	$C_2 \times C_2$	C_3	C_5	C_5
$\lvert C\rvert$	1	15	20	12	12
χ_1	1	1	1	1	1
χ_2	3	-1	0	α	$\overline{\alpha}$
χ_3	3	-1	0	$\overline{\alpha}$	α
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

with $\alpha := \frac{1+\sqrt{5}}{2}$.

2.2 Theorem (2-modular representation theory of A_5)**:**
The 2-modular Brauer character table of A_5 is

	1	(123)	(12345)	(13524)
ϕ_1	1	1	1	1
ϕ_2	2	-1	$\alpha-1$	$\overline{\alpha}-1$
ϕ_3	2	-1	$\overline{\alpha}-1$	$\alpha-1$
ϕ_4	4	-2	-1	-1

$$D = \begin{pmatrix} 1 & 1 & 1 & . & 1 \\ . & 1 & . & . & 1 \\ . & . & 1 & . & 1 \\ . & . & . & 1 & . \end{pmatrix} \qquad C = \begin{pmatrix} 4 & 2 & 2 & . \\ 2 & 2 & 1 & . \\ 2 & 1 & 2 & . \\ . & . & . & 1 \end{pmatrix}$$

In particular there are two 2-blocks:

- The principal block: $\mathrm{IBr}(B_0) = \{\,\phi_1, \phi_2, \phi_3\,\}$, $\mathrm{Irr}(B_0) = \{\,\chi_1, \chi_2, \chi_3, \chi_5\,\}$, $D = C_2 \times C_2$, $N_G(D) = A_4 = D \rtimes C_3$.
- One block of defect zero: $\mathrm{IBr}(B_1) = \{\,\phi_4\,\}$, $\mathrm{Irr}(B_1) = \{\,\chi_4\,\}$

2.3 Theorem (3-modular representation theory of A_5)**:**
The 3-modular character table of A_5 is

C	1	$(12)(34)$	(12345)	(13524)
ϕ_1	1	1	1	1
ϕ_2	3	-1	α	$\overline{\alpha}$
ϕ_3	3	-1	$\overline{\alpha}$	α
ϕ_4	4	1	-1	-1

$$D = \begin{pmatrix} 1 & . & . & . & 1 \\ . & 1 & . & . & . \\ . & . & 1 & . & . \\ . & . & . & 1 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 2 & & & 1 \\ & 1 & & \\ & & 1 & \\ 1 & & & 2 \end{pmatrix}$$

In particular there are three 3-blocks:

- The principal block: $\mathrm{IBr}(B_0) = \{\,\phi_1, \phi_4\,\}$, $\mathrm{Irr}(B_0) = \{\,\chi_1, \chi_4, \chi_5\,\}$, $D = C_3$, $N_G(D) = C_3 \rtimes C_2$.

- Two blocks of defect zero: $\mathrm{IBr}(B_1) = \{ \phi_2 \}$, $\mathrm{Irr}(B_1) = \{ \chi_2 \}$, $\mathrm{IBr}(B_2) = \{ \phi_3 \}$, $\mathrm{Irr}(B_2) = \{ \phi_3 \}$

2.4 Theorem (5-modular representation theory of A_5)**:**
The 5-modular character table of A_5 is

C	1	(12)(34)	(123)
ϕ_1	1	1	1
ϕ_2	3	-1	0
ϕ_3	5	1	-1

$$D = \begin{pmatrix} 1 & . & . & 1 & . \\ . & 1 & 1 & 1 & . \\ . & . & . & . & 1 \end{pmatrix} \quad C = \begin{pmatrix} 2 & 1 & \\ 1 & 3 & \\ & & 1 \end{pmatrix}$$

In particular there are two 3-blocks:

- The principal block: $\mathrm{IBr}(B_0) = \{ \phi_1, \phi_2 \}$, $\mathrm{Irr}(B_0) = \{ \chi_1, \chi_2, \chi_3, \chi_4 \}$, $D = C_5$, $N_G(D) = C_5 \rtimes C_2$.
- One block of defect zero: $\mathrm{IBr}(B_1) = \{ \phi_5 \}$, $\mathrm{Irr}(B_1) = \{ \chi_5 \}$

3 Step minus one: Defect zero

3.1: ADGC is trivially true for defect zero, because $N_G(1) = G$ and $B = b$ in this case.

3.2 Theorem:
All blocks of defect zero are matrix rings.

Proof. Standard theorem shows that blocks of $\mathbb{F}G$ of defect zero are simply matrix rings $\mathbb{F}^{a\times a}$.
Sketch: Defect zero $\overset{V\leq D}{\Longrightarrow}$ All vertices trivial $\Longrightarrow$ all simple modules of this block are projective $\Longrightarrow$ all modules are projective $\Longrightarrow$ B is semisimple $\overset{Wedderburn}{\Longrightarrow}$ $B \cong \mathbb{F}^{\dim(S)\times\dim(S)}$ because B is indecomposable. $\square$

4 Step 0: Different equivalences

4.1 Theorem (Morita)**:**
Let A and B be two k-algebras. TFAE:

a.) $A-\mathsf{Mod} \cong B-\mathsf{Mod}$.

b.) $A-\mathsf{proj} \cong B-\mathsf{proj}$.

c.) There ex. bimodules ${}_AM_B$ and ${}_BN_A$ s.t.

$${}_AM_B \otimes {}_BN_A \cong {}_AA_A \quad \text{and} \quad {}_BN_A \otimes {}_AM_B \cong {}_BB_B$$

In this case M and N determine each other via $N \cong \mathrm{Hom}_{\mathsf{Mod}-B}(M, B)$ and $M \cong \mathrm{Hom}_{\mathsf{Mod}-A}(N, A)$. Moreover $A \cong \mathrm{End}_{\mathsf{Mod}-B}(M)$ and $B \cong \mathrm{End}_{\mathsf{Mod}-A}(N)$.

d.) There ex. a progenerator, i.e. a module M s.t.

i.) M is f.g. projective and

ii.) M is a generators of $A-\mathsf{Mod}$, i.e. every module X is a quotient of $\bigoplus_{i\in I} P$ for some sufficiently large I.

which satisfies $B \cong \mathrm{End}(M)$.

Proof. a. $\Longrightarrow$ b. because finite generation and projectivity can be categorically defined. b. $\Longrightarrow$ c. If an equivalence $A-\mathsf{proj} \underset{G}{\overset{F}{\rightleftarrows}} B-\mathsf{proj}$ is given, then $M := G({}_AA)$ and $N := F({}_BB)$ are the desired bimodules. In fact $F = N \otimes -$ and $G = M \otimes -$ because F and G are additive.
c. $\Longrightarrow$ a. Conversely if M, N are given, then $F := N \otimes -$ and $G := M \otimes -$ are pseudo-inverse functors. $\square$

4.2: $\mathrm{Hom}(M_B, B_B)$ is isomorphic to $M^\vee = \mathrm{Hom}_k(M,k)$ if B is a symmetric k-algebra. Similarly $\mathrm{Hom}(N_A, A_A) \cong N^\vee$ if A is symmetric.

4.3 Theorem (Rickard, Keller, ...)**:**
Let A and B be two k-algebras which are f.g. projective over k. TFAE:

a.) $D^b(A) \cong D^b(B)$ as triangulated categories.

b.) $K^b(A-\mathsf{proj}) \cong K^b(B-\mathsf{proj})$ as triangulated categories.

c.) There exist P of A-B-bimodules and Q of B-A-bimodules s.t.

$$P \otimes_B^L Q \sim A \quad \text{and} \quad Q \otimes_A^L P \sim B$$

In this case P and Q determine each other via $Q = \mathrm{Hom}_{D^b(\mathsf{Mod}-B)}(P,B)$ and $P = \mathrm{Hom}_{D^b(\mathsf{Mod}-A)}(Q,A)$. Moreover $A \cong \mathrm{End}_{D^b(B)}(P)$ and $B \cong \mathrm{End}_{D^b(A)}(Q)$.

d.) There exists a tilting complex, i.e. a bounded complex P of A-modules s.t.

i.) P consists of f.g. projective A-modules

ii.) $\mathrm{add}(P)$, the smallest full subcategory which contains P and is closed under taking direct sums and direct summands, generates $K^b(A-\mathsf{proj})$ as a triangulated category.

iii.) P is "rigid":

$$\forall n \neq 0 : \mathrm{Hom}(P, P[n]) = 0$$

such that $B \cong \mathrm{End}({}_AP)$

4.4: Now it is not (known to be) true that every equivalence $F : D^b(A) \to D^b(B)$ is actually isomorphic to some $Q \otimes^L -$ as it is in the Morita case. If so, F is called "standard".

4.5 Definition (Rickard)**:**
Let G, H be two finite groups with a common (fixed) p-subgroup D.
A splendid tilting complex for two blocks $B \in Bl(\mathbb{F}G)$ and $C \in Bl(\mathbb{F}H)$ is a complex B-b-bimodules as above such that additionally the following hold:

a.) Homotopy instead of quasi-isomorphism:

$$P \otimes_b P^\vee \simeq B \quad \text{and} \quad P^\vee \otimes_B P \simeq b$$

b.) Each term of P is a p-permutation module of $G \times H$ and is projective relative to $\Delta(D)$.

4.6 Theorem (Rickard)**:**
Let A be a self-injective $\mathbb{F}$-algebra. The canonical functor $A{-}\mathsf{mod} \to D^b(A)$ which maps a module M to the complex $\cdots \to 0 \to M \to 0 \to \cdots$ with M concentrated in 0 induces an equivalence

$$\underbrace{A{-}\mathsf{mod}/A{-}\mathsf{proj}}_{=A{-}\underline{\mathsf{mod}}} \to D^b(A)/K^b(A{-}\mathsf{proj})$$

of triangulated categories.

4.7 Corollary:
Let A and B be finite-dimensional, self-injective $\mathbb{F}$-algebras. If they are derived equivalent, they are also stably equivalent. In fact, there is a stable equivalence of Morita type.

4.8 Theorem (Okuyama, Rickard, ...)**:**
Let A and B be symmetric k-algebras over a field.
If $\mathcal{F} : A{-}\mathsf{mod} \to B{-}\mathsf{mod}$ is an exact functor which induces a stable equivalence, $\mathrm{Irr}(A) = \{ S_1, \ldots, S_n \}$ are and $\mathcal{X} = \{ X_1, \ldots, X_n \}$ are objects in $D^b(B)$ s.t.

a.) $\mathrm{Hom}(X_i, X_j[m]) = 0$ for all $m < 0$ and all i, j.

b.) $\mathrm{Hom}(X_i, X_j) = \begin{cases} k & i = j \\ 0 & i \neq j \end{cases}$

c.) $\mathcal{X}$ generates $D^b(B)$ as triangulated category.

and such that X_i is stably isomorphic to $\mathcal{F}(S_i)$ for all i (i.e. isomorphic in $D^b(A)/K^b(A{-}\mathsf{proj})$), then $\mathcal{F}$ also induces a derived equivalence.

4.9: The proof is based on a theorem by Linckelmann that a stable equivalence of Morita type (i.e. induced by a tensor-functor) between indecomposable, finite-dimensional, self-injective K-algebras which also maps simples to simples is a Morita equivalence.

5 Step one: Cyclic defect

5.1 Theorem:
Blocks with cyclic defect group D are Brauer tree algebras with $e = |\mathrm{IBr}(B)|$ edges and multiplicity $\mu = \frac{|D|-1}{e}$.

5.2 Theorem (Rickard)**:**
ADGC holds for blocks of cyclic defect. In fact, all Brauer tree algebras with the same number of edges and the same multiplicity are derived equivalent.

6 Step two: Klein four defect

6.1 Theorem (2-modular representation theory of A_4)**:**
The 2-modular Brauer character table of A_4 is

	1	(123)	(132)
ψ_1	1	1	1
ψ_2	1	ζ_3	ζ_3^2
ψ_3	1	ζ_3^2	ζ_3

$$D = \begin{pmatrix} 1 & . & . & 1 \\ . & 1 & . & 1 \\ . & . & 1 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

In particular there is only one 2-block.

6.2 Example:
ADGC holds for A_5.

Proof. $p = 3$ and $p = 5$ are already done because cyclic defect.
Only other case is the principal 2-block. We set $D := V_4 \in Syl_2(G)$, $H := N_G(D) = A_4$. Since D is a TI subgroup of A_5, Green correspondence gives a stable equivalence of Morita type

$$\{\, M \in \mathbb{F}H\text{--}\underline{\mathsf{mod}} \mid vx(M) \leq D \,\} \underset{\mathrm{Res}_H^G}{\overset{\mathrm{Ind}_H^G}{\rightleftarrows}} \{\, M \in \mathbb{F}G\text{--}\underline{\mathsf{mod}} \mid vx(M) \leq D \,\}$$

which restricts to a stable equivalence of Morita type

$$b\text{--}\underline{\mathsf{mod}} \underset{\mathrm{Res}_H^G}{\overset{\mathrm{Ind}_H^G}{\rightleftarrows}} B\text{--}\underline{\mathsf{mod}}$$

The restriction of the three simple B-modules ϕ_1, ϕ_2, ϕ_3 are

$$Y_1 := \psi_1 \quad Y_2 = \begin{matrix} \psi_2 \\ \psi_3 \end{matrix} \quad Y_3 = \begin{matrix} \psi_3 \\ \psi_2 \end{matrix}$$

which can be verified by explicit calculations. Simple constituents can be seen from the character tables. Which one is the socle and which the head of Y_i can be calculated by looking at explicit modules.

Then $X_1 := Y_1$ is already simple. Furthermore

$$\Omega Y_2 = \begin{matrix} \psi_1 \\ \psi_2 \end{matrix} \qquad \Omega Y_3 = \begin{matrix} \psi_1 \\ \psi_3 \end{matrix}$$

Then we can set $X_2 := \Omega Y_2[1]$ and $X_3 := \Omega Y_3[1]$. These three generate $D^b(b)$ because $\psi_1 = X_1$ is already in $\mathcal{X}$, ψ_2 and ψ_3 are kernels of $X_i \twoheadrightarrow X_1$. It is also easy to see that $\dim_k \operatorname{Hom}(X_i, X_j) = \delta_{ij}$ and that $\operatorname{Hom}(X_i, X_j[m]) = 0$ for $m < 0$.
Okuyama's method therefore upgrades the stable equivalence to a derived equivalence.

$\square$

6.3: In fact, one can do the same with the cyclic defect groups instead of using the heavy guns.