TITLE

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1 Some categorial flavour to algebraic notions

1.1 Definition (Projective and injective objects)**:**
$P \in Ob(\mathsf{A})$ is called projective iff for every epimorphism $B \twoheadrightarrow A$ and every morphism $P \to A$ there is a morphism $P \to B$ making the triangle commutative.
Dually $I \in Ob(\mathsf{A})$ is called injective iff for every monomorphism $A \hookrightarrow B$ and every morphism $A \to I$ there is a morphism $B \to I$ making the triangle commutative.

$$\begin{tikzcd} & P \arrow[dl, dotted] \arrow[d] & \\ B \arrow[r, two heads] & A \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} & I & \\ B \arrow[ur, dotted] & A \arrow[l, hook'] \arrow[u] & 0 \arrow[l] \end{tikzcd}$$

1.2: In both cases, the morphisms need not be unique and in many cases they aren't.

2 Some homological algebra

2.1 Definition (Chain complexes)**:**
Let A be an additive category. A chain complex (A_*, ∂) is a pair consisting of a graded object $A_* \in \mathsf{A}^{\mathbb{N}}$ and a morphism $\partial : A \to A$ of degree -1, i.e. $\partial_n : A_n \to A_{n-1}$, such that $\partial \circ \partial = 0$.
The category of chain complexes is denoted $Ch(\mathsf{A})$.

2.2 Definition (Homology)**:**
Let A be an abelian category and $A_* \in Ch(\mathsf{A})$ a chain complex. Then its homology is defined to be the graded object $H_n := \underbrace{\ker(\partial_n)}_{=:Z_n} / \underbrace{\mathrm{im}(\partial_{n+1})}_{=:B_n}$.

2.1 Mapping cone

2.3 Definition:
Let $f : (A_*, \partial^A) \to (B_*, \partial^B)$ be a chain-map. The mapping cone $C(f) = (C(f)_*, \partial^{C(f)})$ is the chain complex given by

$$C(f)_n := A_{n-1} \oplus B_n \quad \text{and} \quad \partial_n^{C(f)} := \begin{pmatrix} -\partial_{n-1}^A & 0 \\ -f_{n-1} & \partial_n^B \end{pmatrix}$$

2.4 Lemma (Mapping cones vs. quasi-isomorphisms)**:**
Let $f : (A_*, \partial^A) \to (B_*, \partial^B)$ be a chain-map.

a.) $0 \to B \overset{i}{\hookrightarrow} C(f) \xrightarrow{q} A[-1] \to 0$ is a short exact sequence of chain complexes.

b.) The induced long exact sequence in homology

$$\cdots \to H_{n+1}(B) \xrightarrow{i_*} H_{n+1}(C(f)) \xrightarrow{q_*} \underbrace{H_{n+1}(A[-1])}_{=H_n(A)} \xrightarrow{\delta} H_n(B) \to H_n(C(f)) \to \cdots$$

has f_* as connecting morphism δ.

c.) f quasi-isomorphism $\iff$ $C(f)$ is acylic.

2.5 Lemma (Mapping cones vs. chain homotopy)**:**
Let $f : (A_*, \partial^A) \to (B_*, \partial^B)$ be a chain-map.

a.) f is a homotopy-equivalence $\iff$ $C(f)$ is contractible.

b.) TFAE:

i.) f is null-homotopic.

ii.) f factorises through $A \hookrightarrow C(\mathrm{id}_A)$.

iii.) f factorises through a contractible complex.

iv.) The short exact sequence $0 \to B \hookrightarrow C(f) \to A \to 0$ splits.

Proof. a. A map

$$H := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \begin{matrix} A_{n-1} \\ \oplus \\ B_n \end{matrix} \to \begin{matrix} A_n \\ \oplus \\ B_{n+1} \end{matrix}$$

is a homotopy $\mathrm{id}_{C(f)} \simeq 0$ iff $H\partial^{C(f)} + \partial^{C(f)}H = \mathrm{id}$, that is iff

$$-\begin{pmatrix} \alpha\partial + \beta f + \partial\alpha & -\beta\partial + \partial\beta \\ \gamma\partial + \delta f + f\alpha - \partial\gamma & -\delta\partial + f\beta - \partial\delta \end{pmatrix} = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{pmatrix}$$

i.e. iff $\beta : B \to A$ is a chain-map, $-\alpha$ is a homotopy $\mathrm{id} \simeq (-\beta)f$, δ is homotopy $\mathrm{id} \simeq f(-\beta)$ and γ is some map satisfying the last equation.
This already proves one direction: If $C(f) \simeq 0$, then $f(-\beta) \simeq \mathrm{id}$ and $(-\beta)f \simeq \mathrm{id}$ so that $A \simeq B$.

Conversely, if $f(-\beta) \simeq \mathrm{id}$ and $(-\beta)f \simeq \mathrm{id}$ via homotopies δ and $-\alpha$ respectively, then setting $\gamma := 0$ for the moment, we instead get a homotopy $\tilde{H}$ of 0 with $\psi := \begin{pmatrix} \mathrm{id} & 0 \\ \delta f + f\alpha & \mathrm{id} \end{pmatrix}$ which is obviously an isomorphism on the level of modules. ψ is in fact a chain map:

$$\partial^{C(f)}\psi - \psi\partial^{C(f)} = \begin{pmatrix} 0 & 0 \\ \partial\delta f + \partial f\alpha + \delta f\partial + f\alpha\partial & 0 \end{pmatrix}$$

This is zero because

$$\partial\delta f + \delta f\partial = \partial\delta f + \delta\partial f = (-\mathrm{id} + f\beta)f$$

and

$$\partial f\alpha + f\alpha\partial = f\partial\alpha + f\alpha\partial = f(\mathrm{id} - \beta f)$$

Therefore $H := \psi^{-1}\tilde{H}$ is a homotopy $0 \simeq \mathrm{id}_{C(f)}$ so that $C(f)$ is contractible as claimed.

b. i. $\Longleftarrow$ ii. $\Longleftarrow$ iii. is trivial. We show i. $\Longleftrightarrow$ iii.: f factorises over the inclusion $A \hookrightarrow C(\mathrm{id}_A), a \mapsto (0, a)$ iff there is a chain-map

$$(h_{n-1}, f_n) : \begin{matrix} A_{n-1} \\ \oplus \\ A_n \end{matrix} \to B_n$$

And for a family $(h_n : A_n \to B_{n+1})$ to induce such a chain map $C(\mathrm{id}_A) \to B$ is equivalent to $\partial^B h = -h\partial^A - f$, i.e. to h being a homotopy $f \simeq 0$.
i. $\Longleftrightarrow$ iv.: $B \hookrightarrow C(f)$ splits iff there is a chain map

$$(r, \mathrm{id}) : \begin{matrix} A_{n-1} \\ \oplus \\ B_n \end{matrix} \to B_n$$

And for a family $(r_n : A_n \to B_{n+1})$ to induce such a chain map $C(f) \to B$ is equivalent to $\partial r = -r\partial - f$, i.e. to r being a homotopy $f \simeq 0$. □

2.2 Replacing objects and complexes by projective or injective resolutions

2.6 Lemma:
Chain maps between projectives / acyclic complexes are unique up to homotopy:

a.) Homology: If $C_* \in Ch^-(\mathsf{A})$ is acyclic and $P_* \in Ch^-(Proj(\mathsf{A}))$ all morphisms $P_* \to C_*$ are null-homotopic.

b.) Cohomology: If $C^* \in Ch^+(\mathsf{A})$ is acyclic and $I^* \in Ch^+(Inj(\mathsf{A}))$ all morphisms $C^* \to I^*$ are null-homotopic.

Proof. Let $\alpha, \beta : P_* \to C_*$ be two chain maps. Inductively we construct a homotopy $h : P_* \to C_*[1]$ between the two.

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "\alpha_2-\beta_2"] \arrow[dl, dotted, "h_2"'] & P_1 \arrow[r] \arrow[d, "\alpha_1-\beta_1"] \arrow[dl, dotted, "h_1"'] & P_0 \arrow[r] \arrow[d, "\alpha_0-\beta_0"] \arrow[dl, dotted, "h_0"'] & 0 \\
\cdots \arrow[r] & C_2 \arrow[r] & C_1 \arrow[r] & C_0 \arrow[r] & 0
\end{tikzcd}$$

We begin with setting $h_n := 0$ for all $n < 0$. First step is to construct h_0. Since C is exact, $C_1 \to C_0$ is epi so that $\alpha_0 - \beta_0$ lifts to some $h_0 : P_0 \to C_1$ by projectivity, so that $\partial_1 h_0 + \partial 0 = \alpha_0 - \beta_0$ is satisfied.

If $h_0, \ldots, h_{n-1}$ are already known and a partial homotopy, then

$$\begin{aligned}
\partial_n(\alpha_n - \beta_n) &= (\alpha_{n-1} - \beta_{n-1})\partial_n \\
&= (\partial_n h_{n-1} + h_{n-2}\partial_{n-1})\partial_n \\
&= \partial_n h_{n-1} \partial_n
\end{aligned}$$

So that $\partial(\alpha_n - \beta_n - h_{n-1}\partial_n) = 0$. Therefore $\alpha_n - \beta_n - h_{n-1}\partial_n$ maps into $Z_n(C)$ which equals $B_n(C) = \mathrm{im}(\partial_{n+1})$ by exactness. By projectivity, we can find h_n so that

$$\alpha_n - \beta_n - h_{n-1}\partial_n = \partial_{n+1} h_n$$

is satisfied which proves the lemma. □

2.7 Corollary (Fundamental lemma of homological algebra)**:**
"Objects can be replaced by their projective or injective resolutions"

a.) Homology: Assume that A has enough projectives and that a projective resolution has been fixed for every object.

Any $f : A \to B$ extends to a chain map between the augmented complexes

$$\begin{tikzcd}
P_*(A) \arrow[r] \arrow[d, dashed, "\phi"] & A \arrow[r] \arrow[d, "f"] & 0 \\
P_*(B) \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

ϕ is unique up to homotopy.

In particular: $\mathsf{A} \xrightarrow{P_*} K^-(Proj(\mathsf{A}))$ is a well-defined functor with $H_0 \circ P_* \cong \mathrm{id}_{\mathsf{A}}$.

b.) Cohomology: Assume that A has enough injectives and that a injective resolution has been fixed for every object.

Any $f : A \to B$ extends to a chain map between the augmented complexes

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & I^*(A) \arrow[d, dashed, "\phi"] \\
0 \arrow[r] & B \arrow[r] & I^*(B)
\end{tikzcd}$$

ϕ is unique up to homotopy.

In particular: $\mathsf{A} \xrightarrow{I^*} K^+(Inj(\mathsf{A}))$ is a well-defined functor with $H_0 \circ I^* \cong \mathrm{id}_{\mathsf{A}}$.

As a consequence, projective and injective resolutions are unique up to homotopy equivalence.

Proof. Uniqueness up to homotopy follows from the lemma. We only have to show existence. Again, we work inductively:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2(A) \arrow[r] \arrow[d, dotted, "\phi_2"] & P_1(A) \arrow[r] \arrow[d, dotted, "\phi_1"] & P_0(A) \arrow[r] \arrow[d, dotted, "\phi_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & P_2(B) \arrow[r] & P_1(B) \arrow[r] & P_0(B) \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

We set $P_{-1}(A) := A$, $\phi_{-1} := f$, and $P_{-1}(B) := B$ for notational convenience. If ϕ_{n-1} is already constructed, then

$$\begin{tikzcd}
P_n(A) \arrow[r, "\partial"] & P_{n-1}(A) \arrow[d, "\phi_{n-1}"] \\
& P_{n-1}(B) \arrow[r, "\partial"] & P_{n-2}(B)
\end{tikzcd}
=
\begin{tikzcd}
P_n(A) \arrow[r, "\partial"] & P_{n-1}(A) \arrow[r, "\partial"] & P_{n-2}(A) \arrow[d, "\phi_{n-2}"] \\
& & P_{n-2}(B)
\end{tikzcd} = 0$$

Therefore $\phi_{n-1} \circ \partial_n : P_n(A) \to P_{n-1}(B)$ maps into $Z_{n-1}(P_*(B))$ which equals $B_{n-1}(P_*(B)) = \mathrm{im}(\partial_n)$ by exactness. By projectivity, we get a lift $\phi_n : P_n(A) \to P_n(B)$. $\square$

2.8 Lemma (Horseshoe lemma)**:**
"P_* and I^* are exact"

a.) Homology: Every diagram

$$\begin{tikzcd}
& P_*(A) \arrow[d] & & P_*(C) \arrow[d] \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] & C \arrow[r] \arrow[d] & 0 \\
& 0 & & 0
\end{tikzcd}$$

with exact first row and projective resolutions in the columns can be extended with some projective resolution $P_*(B) \to B \to 0$ to a diagram

$$\begin{tikzcd}
0 \arrow[r] & P_*(A) \arrow[r] \arrow[d] & P_*(B) \arrow[r] \arrow[d] & P_*(C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0
\end{tikzcd}$$

in which all rows are exact.

b.) Cohomology: Every diagram

$$\begin{tikzcd}
& 0 \arrow[d] & & 0 \arrow[d] \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] & C \arrow[r] \arrow[d] & 0 \\
& I^*(A) & & I^*(C)
\end{tikzcd}$$

with exact first row and injective resolutions in the columns can be extended with some injective resolution $0 \to B \to I^*(B)$ to a diagram

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^*(A) \arrow[r] & I^*(B) \arrow[r] & I^*(C) \arrow[r] & 0
\end{tikzcd}$$

in which all rows are exact.

Proof. Set $A_{-1} := A$ and $A_n := P_n(A)$, $C_{-1} := C$ and $C_n := P_n(C)$ as well as $B_{-1} := B$. Then define $P_n(B) := B_n := A_n \oplus C_n$.
For the vertical maps consider

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r] \arrow[d] & A_n \oplus C_n \arrow[r] \arrow[d, dotted, "g"] & C_n \arrow[r] \arrow[d] \arrow[dll, dotted, "f"'] \arrow[dl, dotted, "h"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{n-2} \arrow[r] & B_{n-2} \arrow[r] & C_{n-2} \arrow[r] & 0
\end{tikzcd}$$

We define $g : A_n \oplus C_n \to B_{n-1}$ separately on the two components. Define $g : A_n \oplus 0 \to B_{n-1}$ to be the composition $A_n \to A_{n-1} \to B_{n-1}$.
The map $g : 0 \oplus C_n \to B_{n-1}$ we choose in two steps. First choose $h : C_n \to B_{n-1}$ to make the triangle on the right side commute. Then

$$\begin{array}{ccc}
\begin{array}{l}
0 \oplus C_n \\
\ \downarrow h \\
B_{n-1} \\
\ \downarrow \\
B_{n-2} \to C_{n-2}
\end{array}
& = &
\begin{array}{r}
0 \oplus C_n \xrightarrow{=} C_n \\
\downarrow \\
C_{n-1} \\
\downarrow \\
C_{n-2}
\end{array}
= 0
\end{array}$$

d.h. $\partial h(C_n) \subseteq A_{n-2}$ because the $(n-2)$th row is exact and of course $\partial\partial h = 0$ so that $\partial h(C_n) \subseteq Z_{n-2}(A_*) = B_{n-2}(A_*)$ by exactness of A_*. Using projectivity once again, we can lift ∂h to $f : C_n \to A_{n-1}$ and finally define $g : 0 \oplus C_n \to B_{n-1}$ as $h - f$. Note that $\overline{g(c_n)} = \partial c_n$ still holds because $\mathrm{im}(f) \subseteq \ker(B_{n-1} \to C_{n-1})$.

This ensures $\partial g = 0$ which proves that the middle column is a(n incomplete) complex. We still have to show exactness. So let $b_{n-1} \in B_{n-1}$ with $\partial b_{n-1} = 0$.
Then its image $c_{n-1} = \overline{b_{n-1}}$ also satisfies $\partial c_{n-1} = 0$ so that a c_n exists with $c_{n-1} = \partial c_n$ by exactness of C_*. Then $\overline{b_{n-1} - g(0 \oplus c_n)} = c_{n-1} - \partial c_n = 0$ so that $b_{n-1} - g(0 \oplus c_n) \in \ker(B_{n-1} \to C_{n-1})$ which is $\mathrm{im}(A_{n-1} \to B_{n-1})$ by exactness of the $(n-1)$th row so that $b_{n-1} - g(0 \oplus c_n) = a_{n-1}$. Then $0 = 0 - 0 = \partial b_{n-1} - \partial g(0 \oplus c_n) = \partial a_{n-1}$ so that $a_{n-1} = \partial a_n = g(a_n \oplus 0)$. That shows $b_{n-1} = g(a_n \oplus c_n)$. □

2.9 Corollary:
"Complexes can be replaces by double complexes of projectives/injectives"

a.) Homology: For every $K_* \in Ch(\mathsf{A})$ exists a commutative double complex $P_{*,*}$ and maps $P_{n,*} \to K_n$ such that $P_{*,n} \to K_n \to 0$ is a projective resolution.

b.) Cohomology: For every $K^* \in Ch(\mathsf{A})$

exists a commutative double complex $I^{*,*}$ and maps $K^n \to I^{n,*} \to K^n$ such that $0 \to K^n \to I^{n,*} \to K^n \to 0$ is an injective resolution.

Proof. Consider the short exact sequences

$$0 \to Z_n \to K_n \to B_{n-1} \to 0$$

and choose projective resolutions $P''_{n,*} \to Z_n \to 0$ and $P'_{n,*} \to B_n \to 0$. Apply the horseshoe lemma to obtain a projective resolution $P_{n,*} \to K_n \to 0$ fitting in the exact sequence.
Now apply the fundamental lemma of homological algebra to get a chain-map $P'_{n,*} \to P''_{n-1,*}$ that extends the canonical map $B_{n-1} \to Z_{n-1}$ and let $P_{n,*} \to P_{n-1,*}$ be the composition $P_{n,*} \twoheadrightarrow P'_{n,*} \to P''_{n-1,*} \hookrightarrow P_{n-1,*}$. Since $P' \to P \to P''$ are short exact sequences, we obtain a commutative double complex in this way. □

2.10 Theorem:
"Complexes can be replaced by projective / injective resolutions"

a.) Homology: For any bounded above complex $K_* \in Ch^-(\mathsf{A})$ there is a $P_* \in Ch^-(Proj(\mathsf{A}))$ and a quasi-isomorphism $P_* \to K_*$.

b.) Cohomology: For any bounded below complex $K^* \in Ch^+(\mathsf{A})$ there is a $I^* \in Ch^+(Inj(\mathsf{A}))$ and a quasi-isomorphism $K^* \to I^*$.

Proof. Take the total complex of $P_{*,*}$ in the previous statement. □

3 Derived functors

3.1 (The Problem)**:** Given a right-exact functor $F : \mathsf{A} \to \mathsf{B}$, and exact sequence

$$0 \to A \to B \to C \to 0$$

gives a exact sequence

$$F(A) \to F(B) \to F(C) \to 0$$

We want to find functors L_nF and natural transformations δ_n (natural w.r.t. the short exact sequence) such that this sequence extends to a long exact sequence

$$\cdots \to L_2F(C) \xrightarrow{\delta_2} L_1F(A) \to L_1F(B) \to L_1F(C) \xrightarrow{\delta_1} \underbrace{F(A)}_{=L_0F(A)} \to \underbrace{F(B)}_{L_0F(B)} \to \underbrace{F(C)}_{=0} \to 0$$

And similarly for left-exact functors.
Of course, we want the universal solution to this problem.

3.2 Definition (δ-Functors & Derived functors)**:**
A family $(F_n, \delta_n)_{n\in\mathbb{N}}$ of functors $\mathsf{A} \xrightarrow{F_n} \mathsf{B}$ and natural transformations $F_n(C) \xrightarrow{\delta_n} F_{n-1}(A)$ for every short exact sequence $0 \to A \to B \to C \to 0$ that transforms such short exact sequences into long exact sequences as above is called a (homological) δ-functor.
A morphism of δ-functors is a family (t_n) of natural transformations $F_n \xrightarrow{t_n} G_n$ which induces morphisms between the long exact sequences, i.e. $t_n\delta_n^F = \delta_n^G t_n$.
Cohomological δ-Functors (F^n, δ^n) are analogously defined for left-exact functors.

3.3 Definition (Universal δ-functors and derived functors)**:**
Let $F : \mathsf{A} \to \mathsf{B}$ be right-exact. A δ-functor (F_n, δ_n) is called the universal δ-functor extending F or left derived functor of F if there is an iso $F_0 \underset{\cong}{\xrightarrow{\tau}} F$ and $((F_n, \delta_n), \tau)$ is the final object in the category of all such δ-functors.
Similarly the right derived functor of a left exact F is defined as an initial object in the appropriate category of cohomological δ-functors with fixed iso $F \underset{\cong}{\xrightarrow{\tau}} F^0$.

3.4 Lemma (Extending natural transformations)**:**
Let (F_n, δ_n), $(\tilde{F}, \tilde{\delta}_n)$ be two δ-functors $\mathsf{A} \to \mathsf{B}$.

a.) Homology: Assume $F_n(P) = 0$ for all $n \leq 1$ and all $P \in Proj(\mathsf{A})$.

Then every $\tilde{F} \xrightarrow{t_0} F$ extends uniquely to a family of natural transformations $\tilde{F}_n \xrightarrow{t_n} F_n$ which induces a morphism between the long exact sequences.

b.) Cohomology: Assume $F^n(I) = 0$ for all $n \leq 1$ and all $I \in Inj(\mathsf{A})$.

Then every $F \xrightarrow{t_0} \tilde{F}$ extends uniquely to a family of natural transformations $F^n \xrightarrow{t^n} \tilde{F}^n$ which induces a morphism between the long exact sequences.

Proof. Assume that unique transformations $t_0, \ldots, t_{n-1}$ have already been constructed. Fix $C \in \mathsf{A}$ and choose a short exact $0 \to K \xrightarrow{j} P \xrightarrow{q} C \to 0$ with P projective. Then

$$\begin{tikzcd}
\cdots \arrow[r] & \tilde{F}_n(P) \arrow[r] \arrow[d, dotted] & \tilde{F}_n(C) \arrow[r, "\tilde{\delta}_n"] \arrow[d, dotted] & \tilde{F}_{n-1}(K) \arrow[r] \arrow[d, "t_{n-1}"] & \tilde{F}_{n-1}(P) \arrow[r] \arrow[d, "t_{n-1}"] & \cdots \\
\cdots \arrow[r] & \underbrace{F_n(P)}_{=0} \arrow[r] & F_n(C) \arrow[r, "\delta_n"] & F_{n-1}(K) \arrow[r] & F_{n-1}(P) \arrow[r] & \cdots
\end{tikzcd}$$

It follows that $F_n(C) \underset{\cong}{\xrightarrow{\delta_n}} \ker(F_{n-1}(j))$ and since t_{n-1} is natural, there is a unique $t_n : \tilde{F}_n(C) \to F_n(C)$ that makes the square commute. This t_n does not depend on the choice of K and P by Schanuel's lemma.
Naturality of t_n follows from a simple diagram chase using naturality of δ_n and $\tilde{\delta_n}$, naturality of t_{n-1} and that $F_n(A) \to F_{n-1}(K)$ is mono.
It remains to show that t_n commutes with the deltas for an arbitrary short exact $0 \to A \to B \to C \to 0$. This also follows from a simple diagram chase. $\square$

3.5 Theorem (Derived functors exist.)**:**
Let $F : \mathsf{A} \to \mathsf{B}$ be right / left exact functor.

a.) Homology:

i.) If A has enough projectives, then F has a left derived functor.

ii.) $L_i F(P) = 0$ for all projectives P and all $i > 0$.

iii.) Deriving is a functor L_i : $\mathsf{Fun}_{\text{r.e.}}(\mathsf{A}, \mathsf{B}) \to \mathsf{Fun}_{\text{add}}(\mathsf{A}, \mathsf{B})$.

b.) Cohomology:

i.) If A has enough injectives, then F has a right derived functor.

ii.) $R^i F(I) = 0$ for all injectives I and all $i > 0$.

iii.) Deriving is a functor R^i : $\mathsf{Fun}_{\text{l.e.}}(\mathsf{A}, \mathsf{B}) \to \mathsf{Fun}_{\text{add}}(\mathsf{A}, \mathsf{B})$.

Proof. Existence: Define $L_n F := H_n \circ P_*$. Note that this does not depend on the choice of the projective resolutions P_* because all choices are homotopy equivalent and homology forgets homotopy. Note that $L_i F(P) = 0$ for P projective and $i > 0$ because $0 \to P \xrightarrow{\text{id}} P \to 0$ is a projective resolution of P.

Horseshoe lemma implies that every short exact sequence

$$0 \to A \to B \to C \to 0$$

lifts to exact sequence $0 \to P_*(A) \to P_*(B) \to P_*(C) \to 0$ which implies a long exact sequence in homology. Therefore $LF = (L_i F, \delta_i)$ is a δ-functor extending F.

We still have to show universality. Let $((\tilde{F}_n, \tilde{\delta}_n), \tilde{\tau})$ be another δ-functor extending F. The above lemma shows that there is a unique morphism of δ-functors $t : \tilde{F} \to F$ which extends $t_0 := \tau^{-1} \circ \tilde{\tau}$.
It also proves functoriality of deriving. $\square$

3.6: Projective / injective resolutions work for all functors to construct derived functors. With a bit more care one can make also resolutions work that consist of more general "F-acyclic" objects.

4 Examples

4.1 Example (Snake lemma)**:**
Taking kernels is a left-exact functor $\mathsf{A}^{*\to *} \to \mathsf{Ab}$. Its right derived functor is the cokernel in degree 1 and zero further up.
Dually taking cokernels is right-exact and its left derived functor is the kernel in degree 1 and zero everywhere else.
This is a manifestation of the snake lemma.

4.2 Example (Sheaf (co)homology)**:**
Sheaf cohomology $H^*(X, \mathcal{F})$ is the right derived functor of the global section functor $\Gamma : Sh(X) \to \mathsf{Ab}$.

4.3 Example (DeRham cohomology):
$H^*_{\mathrm{dR}}(M)$ is Sheaf cohomology of the sheaf $\Omega^0(M)$ of locally constant functions $M \to \mathbb{R}$.

4.4 Example (Singular cohomology):
$H^*_{\mathrm{sing}}(X;G)$ is sheaf cohomology of the sheaf $\underline{G}_X \in Sh(X)$ of locally constant G-valued functions.

4.5 Example (Étale cohomology):
Étale cohomology is the Sheaf cohomology for sheafs on the étale site, i.e. the right derived functor of global sections $\mathsf{Sh}_{et}(X) \to \mathsf{Ab}$.

4.6 Example (Ext and Tor):
$\mathrm{Ext}^i_A(M,N) := (R^i \operatorname{Hom}_A(M,-))(N)$.
$\mathrm{Tor}^A_i(M,N) := (L_i M \otimes_A -)(N) = (L_i - \otimes_A N)(M)$.

4.7 Example (Group (co)homology):
$H_*(G,M)$ is the left derived functor of the functor of coinvariants $k \otimes_{kG} -$, i.e. it is $Tor^{kG}_*(k,M)$.
$H^*_k(G,-)$ is the right derived functor of the functor of fixed points $(-)^G = \operatorname{Hom}_{kG}(k,-)$, i.e. it is $Ext^*_{kG}(k,M)$.

4.8 Example (Hochschild (co)homology):
Let $A^e := A \otimes_k A^{op}$ be the enveloping algebra of the k-algebra A.
$HH_n(A,M) := Tor^{A^e}_n(A,M)$, i.e. it is the left derived functor of the functors of coinvariant $M/[A,M] = A \otimes_{A^e} M : (A,A)\mathrm{-}\mathsf{Bimod} \to \mathsf{Ab}$.
$HH^n(A,M) := Ext^N_{A^e}(A,M)$, i.e. the right derived functor of invariants $Z(M) := \operatorname{Hom}_{A^e}(A,M) : (A,A)\mathrm{-}\mathsf{Bimod} \to \mathsf{Ab}$.

4.9 Example (Lie-algebra (co)homology):
$H_n(\mathfrak{g},M) := Tor^{U(\mathfrak{g})}_n(k,M)$, i.e. left derived of taking coinvariants.
$H^n(\mathfrak{g},M) := Ext^n_{U(\mathfrak{g})}(k,M)$, i.e. right derived of taking invariants.

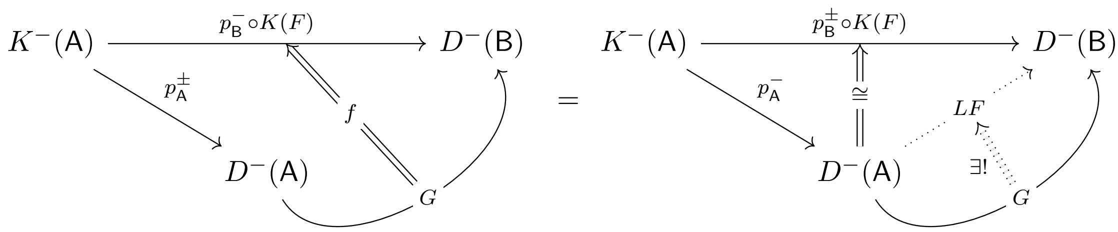
5 The derived category and total derived functors

5.1 Theorem:
Let $p^?_{\mathsf{A}} : K^?(\mathsf{A}) \to D^?(\mathsf{A})$ be the projection functor from the homotopy category onto the derived category.

The total left/right derived functor of $F : \mathsf{A} \to \mathsf{B}$ is the "best approximation" of $K^{\pm}(F) : K^{\pm}(\mathsf{A}) \to K^{\pm}(\mathsf{B})$ on the level of derived categories, i.e. it fits into the diagram

$$\begin{tikzcd} K^{\pm}(\mathsf{A}) \arrow[r, "K(F)"] \arrow[d, "p^{\pm}_{\mathsf{A}}"'] & K^{\pm}(\mathsf{B}) \arrow[d, "p^{\pm}_{\mathsf{B}}"] \\ D^{\pm}(\mathsf{A}) \arrow[r, dotted, "RF", "LF"'] & D^{\pm}(\mathsf{B}) \end{tikzcd}$$

5.2: In this situation LF / RF is also both the left and right Kan-extension of $Q_{\mathsf{B}} \circ K(F)$ along the localisation p_{A}. Concretely: LF fits into a diagram



which is commutative up to natural isomorphism $p_{\mathsf{B}} \circ K(F) \underset{\cong}{\to} LF \circ p_{\mathsf{A}}$ such that for every other functor $D^-(\mathsf{A}) \xrightarrow{G} D^-(\mathsf{B})$ every natural transformation $G \circ p_{\mathsf{A}} \underset{\cong}{\xrightarrow{f}} p_{\mathsf{B}} \circ K(F)$ factors uniquely through this iso.
Therefore some authors *define* LF of *any* additive functor F as the right Kan extension $Ran_{p^-_{\mathsf{A}}}(p^-_{\mathsf{B}} \circ K(F))$ and RF as the left Kan extension $Lan_{p^+_{\mathsf{A}}}(p^+_{\mathsf{B}} \circ K(F))$. In this situation however, LF and RF do in general not extend F.