Some stuff about orbital graphs

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0.1 Definition:
Let $G \curvearrowright \Omega$ be any group action on a set Ω. An orbit of the induced action of G on $\Omega^2 = \Omega \times \Omega$ is called an orbital of the action $G \curvearrowright \Omega$.
If Γ is any orbital, then the directed graph with vertex set Ω and edge set Γ is called an orbital graph of the action.

0.2 Definition:
Let $G \curvearrowright \Omega$ be a transitive action and $\omega \in \Omega$. A G_ω-orbit of this action is called a suborbit. The sizes of the suborbits are called the subdegrees of the action. By transitivity, the subdegrees are independent of $\omega \in \Omega$.

Convention:
For everything that follows, fix a non-empty, finite set Ω and a permutation group $G \leq Sym(\Omega)$. In other words, from now on we consider only faithful permutation actions of G.

1 Orbital graphs vs. suborbits and double cosets

1.1 Theorem:
Let $G \curvearrowright \Omega$ be transitive, $\omega \in \Omega$ a fixed element and $H := G_\omega$ its stabiliser.
There are inclusion-preserving bijections between the following sets

a.) G-invariant subsets $\Gamma \subseteq \Omega \times \Omega$.

b.) H-invariant subsets $\Delta \subseteq \Omega$

c.) Subsets $D \subseteq G$ invariant under left- and right-multiplication by H.

given in the following dictionary

$\Gamma \subseteq \Omega \times \Omega$	$\Delta \subseteq \Omega$	$D \subseteq G$
Γ	$\Gamma(\omega) := \{\, \alpha \mid (\alpha, \omega) \in \Gamma \,\}$	$\{\, y \in G \mid (\omega, {}^{y}\omega) \in \Gamma \,\}$
$\{\, ({}^{g}\alpha, {}^{g}\omega) \mid \alpha \in \Delta, g \in G \,\}$	Δ	$\{\, y \in G \mid {}^{y}\omega \in \Delta \,\}$
$\{\, ({}^{g_0}\omega, {}^{g_1}\omega) \mid H g_0^{-1} g_1 H \subseteq D \,\}$	${}^{D}\omega$	D

In particular the minimal non-empty elements of these posets, namely the orbitals, the suborbitals and the H-H-double cosets respectively, are mapped bijectively onto each other.
Moreover, these bijections translate the following properties:

$\Gamma \subseteq \Omega \times \Omega$	$\Delta \subseteq \Omega$	$D \subseteq G$
$\{\, (\alpha, \alpha) \mid \alpha \in \Omega \,\}$	$\{\, \omega \,\}$	H
Γ^{op}	Δ^*	D^{-1}
$\lvert\Gamma\rvert / \lvert\Omega\rvert$	$\lvert\Delta\rvert$	$\lvert D\rvert / \lvert H\rvert$
$\Gamma \circ \Gamma'$	$\Delta \circ_\omega \Delta'$	DD'

where

$$\Delta^* := \{\, {}^{g^{-1}}\omega \mid {}^{g}\omega \in \Delta \,\}$$

$$\Delta \circ_\omega \Delta' := \{\, \alpha \in \Omega \mid \exists g \in G, \beta \in \Delta' : {}^{g}\alpha \in \Delta \wedge {}^{g}\beta = \omega \,\}$$

Proof. □

1.2 Corollary:
Let $G \curvearrowright \Omega$ be transitive, let $\Gamma \subseteq \Omega^2$ be any orbital, and let HyH be its associated double coset.

a.) Connected components of (Ω, Γ) are automatically strongly connected.

b.) The connected components of (Ω, Γ) are exactly the U-orbits on Ω, where $U := \langle H, y \rangle$.

c.) (Ω, Γ) is connected iff $\langle H, y \rangle = G$.

d.) G acts primitively iff all non-diagonal orbital graphs are connected.

Proof. a. If that were not the case, there would be a connected component $\emptyset \neq C \subseteq \Omega$ which decomposes further $C = X_0 \sqcup \ldots \sqcup X_n$ into strongly connected components such that only edges from X_i into X_j exist where $i < j$ but not the other way around. Pick any $x_0 \in X_0$, $x_k \in X_k$. Since G is transitive, there would be a $g \in G$ such that ${}^{g}x_0 = x_k$. In particular ${}^{g}C = C$, since C is connected component. Hence $\langle g \rangle$ acts as graph automorphisms on C and must permute the strongly connected components. But that means it must map X_0 to X_k which is impossible because the former only only has in-coming edges, while the latter only has out-going edges.

Now identify Ω with G/H and Γ with $\Gamma_y = \left\{ (g_0H, g_1H) \;\middle|\; Hg_0^{-1}g_1H = HyH \right\}$ as above. Set $U := \langle H, y\rangle$. Note that $U = H \cup HyH \cup HyHyH \cup \ldots$ because the order of y is finite.

b. Now xH and $x'H$ are connected by a directed path iff there exists a sequence $x = x_0, x_1, \ldots, x_k = x'$ such that $(x_{i-1}H, x_iH) \in \Gamma_y$, i.e. $x_{i-1}^{-1}x_i \in HyH$.
In particular: If xH and $x'H$ are connected by a directed path, then $x_{i-1}U = x_iU$ for all $i \in \{1, \ldots, k\}$. Therefore $xU = x_0U = x_kU = x'U$.
Conversely: If $xU = x'U$, then there exists an element $h_0yh_1 \cdots yh_k \in U$ with $h_i \in H$ such that $x' = x(h_0yh_1 \cdots yh_k)$. Now we can define $x_i := x \cdot (h_0yh_1 \cdots yh_i)$ for $i \in \{0, \ldots, k\}$ and have found a sequence connecting $xH = x_0H$ and $x'H = x_kH$ in the orbital graph.

c. follows directly from b.

d. follows directly from c. and the fact that G acts primitively on Ω iff H is a maximal subgroup. $\square$

1.3 Remark: This lemma allows for easy identification of at least one block system for the action of G on Ω, namely the connected components of (Ω, Γ). They coincide with the sets ${}^U\omega$.
Moreover: ${}^U\omega$ is the smallest possible block containing both ω and ${}^y\omega$.

2 Orbital graphs vs. representation theory

2.1 Definition:
Now let $V := V$ be the $\mathbb{K}$-vector space with basis Ω. This vector space is naturally a $\mathbb{K}G$-module by extending the action of G on the basis elements linearly to the whole space.
We will identify $\mathrm{End}_{\mathbb{K}}(V)$ with the space $\mathbb{K}^{\Omega\times\Omega}$ of matrices indexed by $\Omega \times \Omega$. We will also identify $Sym(\Omega)$ with the group of permutation matrices.

2.2 Theorem:
$\mathrm{End}_{\mathbb{K}G}(V)$ has a natural $\mathbb{K}$-basis $\{ X_\Gamma \mid \Gamma \subseteq \Omega^2 \text{ orbital} \}$ defined as

$$(X_\Gamma)_{\alpha\beta} := \begin{cases} 1 & \text{if } (\alpha,\beta) \in \Gamma \\ 0 & \text{otherwise} \end{cases}$$

The structure constants w.r.t. this basis, i.e. the numbers d_{ij}^k such that

$$X_{\Gamma_i} \cdot X_{\Gamma_j} = \sum_k d_{ij}^k X_{\Gamma_k},$$

are given by $d_{ij}^k := |\{ \beta \in \Omega \mid (\alpha,\beta) \in \Gamma_i \wedge (\beta,\gamma) \in \Gamma_j \}|$ where (α,γ) is any element of Γ_k.

2.3 Remark: In other words: X_Γ is the adjacency matrix of the orbital graph (Ω, Γ). Note that the right hand side in the definition of d_{ij}^k really is independent of the choice of the element $(\alpha, \gamma) \in \Gamma_k$, because G acts transitively on Γ_k.
Also note that multiplication is connected to composition via

$$X_{\Gamma_i} \cdot X_{\Gamma_j} \in \operatorname{span}_{\mathbb{K}} \{ X_\Gamma \mid \Gamma \subseteq \Gamma_i \circ \Gamma_j \}$$

Proof. Writing out the defining condition

$$X \in \operatorname{End}_{\mathbb{K}G}(V) \iff \forall g \in G : gXg^{-1} = X$$

in components shows that every $\mathbb{K}G$-linear endomorphism is indeed a linear combination of the X_Γ. The X_Γ are obviously linearly independent and therefore a basis.
The structure constants similarly follow by writing out the definition of matrix multiplication in this case. □

2.4 Definition:
The 2-closure of G is defined as the largest subgroup $\widehat{G} \subseteq Sym(\Omega)$ that has the same orbits as G on Ω^2, i.e.

$$\widehat{G} := \left\{ \pi \in Sym(\Omega) \;\middle|\; \forall \Gamma \in \Omega^2/G : \pi(\Gamma) = \Gamma \right\}$$

G is called 2-closed iff $G = \widehat{G}$ holds.

2.5 Remark: One can rephrase this definition by saying that the 2-closure of G is the largest subgroup $H \leq Sym(\Omega)$ that still satisfies $End_{\mathbb{K}G}(V) = End_{\mathbb{K}H}(V)$

2.6 Lemma (2-closure in terms of endomorphism algebras)**:**
$\widehat{G} = Sym(\Omega) \cap C(\operatorname{End}_{\mathbb{K}G}(V))$.

Proof. Let $\widehat{G}$ be the 2-closure of G. By definition $\pi \in \widehat{G}$ if and only if $\pi X_\Gamma \pi^{-1} = X_\Gamma$ for all $\Gamma \in \Omega^2/G$. In other words π is in the 2-closure iff it is a permutation matrix and an element of the centraliser of the endomorphism ring of the $\mathbb{K}G$-module V. This proves the first equation. □

2.7 Lemma (2-closure in terms of linear algebra)**:**
$\widehat{G} = Sym(\Omega) \cap \operatorname{span}_{\mathbb{K}}(G)$.
In particular, G is 2-closed if no permutation matrix outside of G is a linear combination of elements of G.

Proof. Observe that $\operatorname{End}_{\mathbb{K}G}(V)$ is by definition the centraliser algebra of the subalgebra $\operatorname{span}_{\mathbb{K}}(G) \subseteq \mathbb{K}^{\Omega \times \Omega}$.
V is a faithful $\mathbb{K}G$-module and $\mathbb{K}G$ is a symmetric algebra. Therefore V has the double centraliser property so that $C(\operatorname{End}_{\mathbb{K}G}(V)) = C(C(\operatorname{span}_{\mathbb{K}}(G))) = \operatorname{span}_{\mathbb{K}}(G)$. □

2.8 Theorem (2-closure in terms of invariant subspaces)**:**
Let $G \leq Sym(\Omega)$ be a permutation group and assume $\mathbb{K} = \mathbb{C}$. Then

$$\widehat{G} = \left\{ \pi \in Sym(\Omega) \;\middle|\; \forall U \leq \mathbb{C}^{\Omega} : U \ G\text{-invariant} \implies U \ \pi\text{-invariant} \right\}.$$

Proof. We consider the standard scalar product on V defined by declaring Ω to be an orthonormal basis so that V becomes a finite-dimensional Hilbert space.
Then all permutation matrices are unitary. In particular, $\mathrm{span}_{\mathbb{C}}(G) \subseteq \mathbb{C}^{\Omega\times\Omega}$ is closed under taking adjoints and its centraliser $\mathrm{End}_{\mathbb{C}G}(V)$ is also closed under taking adjoints. Both are therefore C^*-algebras. In particular, both are isomorphic to a direct product of matrix rings. It is a consequence of the spectral theorem that $\prod_i \mathbb{C}^{n_i\times n_i}$ is spanned by all the self-adjoint idempotents it contains.

Self-adjoint idempotent matrices correspond bijectively to subspaces by identifying U with the orthogonal projection p_U onto U. A subspace U is g-invariant if g centralises p_U.
Therefore

$$\mathrm{End}_{\mathbb{C}G}(V) = \mathrm{span}_{\mathbb{C}} \left\{ p_U \;\middle|\; U \leq \mathbb{C}^G \ G\text{-invariant} \right\}$$

and

$$\widehat{G} = Sym(\Omega) \cap C(\mathrm{End}_{\mathbb{C}G}(V)) = Sym(\Omega) \cap \bigcap_{\substack{U\leq V\\ G\text{-invariant}}} C(p_U)$$

which proves the claim. $\square$

2.9 Definition:
A permutation group $G \leq Sym(\Omega)$ is reconstructible from $\mathcal{X} \subseteq \mathrm{End}_{\mathbb{K}G}(V)$ if

$$G = Sym(\Omega) \cap \bigcap_{X\in\mathcal{X}} C(X).$$

Similarly, we define that G is ...

- ... orbital-graph-reconstructible if G is reconstructible from $\left\{ X_\Gamma \;\middle|\; \Gamma \in \Omega^2/G \right\}$,
- ... strongly orbital-graph-reconstructible from $\Gamma \in \Omega^2/G$ iff it is reconstructible from X_Γ alone,
- ... absolutely orbital-graph-reconstructible iff it is strongly orbital-graph-reconstructible from any non-diagonal orbital $\Gamma \in \Omega^2/G$.
- ... subspace-reconstructible from $\mathcal{U}$, a set of G-invariant subspaces of V, if G is reconstructible from $\{ p_U \mid U \in \mathcal{U} \}$.
- ... subspace-reconstructible over $\mathbb{K}$ if G is reconstructible from the set of all G-invariant subspaces of $\mathbb{K}^{\Omega}$.

- ... strongly subspace-reconstructible from $U \leq V$ if G is reconstructible from U alone,
- ... absolutely subspace-reconstructible over $\mathbb{K}$ if G is strongly subspace-reconstructible from any minimal, non-zero, G-invariant $U \leq \mathbb{K}^\Omega$ which is not $\mathrm{span}_{\mathbb{K}}\{(1,1,\ldots,1)\}$.

2.10 Corollary:
$G \leq Sym(\Omega)$ is 2-closed iff it is orbital-graph reconstructible iff it is subspace-reconstructible over $\mathbb{C}$.

Proof. The first equivalence follows from the fact that X_Γ is a basis of $\mathrm{End}_{\mathbb{C}G}(V)$. The second follows from theorem 2.8. □

2.11 Example:
A regular permutation group is always 2-closed.
This is because a regular G-set is isomorphic to G itself endowed with left multiplication. The orbitals of this action are given by $\Gamma_h := \{(x,y) \in G^2 \mid x^{-1}y = h\}$ for $h \in G$ and one can readily verify that the only permutations fixing all the orbitals are the left multiplication maps themselves.

2.12 Lemma (Subspace reconstructibility is sufficient)**:**
Let $\mathbb{K} = \mathbb{C}$ and $X \in \mathrm{End}_{\mathbb{C}G}(V)$ be arbitrary.
Then G is reconstructible from X iff it is subspace-reconstructible from

$$\{\mathrm{Eig}_\lambda(\mathfrak{Re}(X)), \mathrm{Eig}_\lambda(\mathfrak{Im}(X)) \mid \lambda \in \mathbb{R}\}.$$

Proof. Permutation matrices are unitary. Therefore $g \in Sym(\Omega)$ centralises X iff it centralises X^*.
$\mathfrak{Re}(X) = \frac{1}{2}(X + X^*)$ and $\mathfrak{Im}(X) = \frac{1}{2i}(X - X^*)$ are self-adjoint matrices with $X = \mathfrak{Re}(X) + i\,\mathfrak{Im}(X)$ and for a self-adjoint matrices Y the spectral theorem shows

$$Y = \sum_{\lambda \in \mathbb{R}} \lambda e_\lambda$$

where $e_\lambda = p_{\mathrm{Eig}_\lambda(Y)}$ is the orthogonal projection onto the λ-eigenspace. Moreover e_λ is a polynomial of Y by Lagrange-interpolation.
Therefore if $g \in GL(V)$ commutes with Y it must commute with all e_λ and vice versa. Thus

$$C(X) = C(X, X^*) = C(\mathfrak{Re}(X), \mathfrak{Im}(X)) = \bigcap_{\lambda \in \mathbb{R}} C(p_{\mathrm{Eig}_\lambda(\mathfrak{Re}(X))}) \cap C(p_{\mathrm{Eig}_\lambda(\mathfrak{Im}(X))})$$

which proves the lemma. □

2.13 Remark: The concept of subspace reconstructibility also makes sense if we replace $Sym(n)$ by some other finite subgroup of $U_n(\mathbb{C})$, for example the subgroup of monomial matrices with m-th roots of unity as entries. This is the complex reflection group called $G(n,1,m)$.