# TITLE

# Johannes Hahn

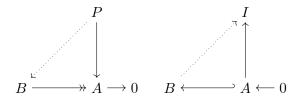
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# 1 Some categorial flavour to algebraic notions

### 1.1 **Definition** (Projective and injective objects):

 $P \in Ob(A)$  is called projective iff for every epimorphism  $B \twoheadrightarrow A$  and every morphism  $P \to A$  there is a morphism  $P \to B$  making the triangle commutative.

Dually  $I \in Ob(A)$  is called injective iff for every monomorphism  $A \hookrightarrow B$  and every morphism  $A \to I$  there is a morphism  $B \to I$  making the triangle commutative.



1.2: In both cases, the morphisms need not be unique and in many cases they aren't.

# 2 Some homological algebra

# **2.1 Definition** (Chain complexes):

Let A be an additive category. A chain complex  $(A_*, \partial)$  is a pair consisting of a graded object  $A_* \in A^{\mathbb{N}}$  and a morphism  $\partial: A \to A$  of degree -1, i.e.  $\partial_n: A_n \to A_{n-1}$ , such that  $\partial \circ \partial = 0$ .

The category of chain complexes is denoted Ch(A).

# **2.2 Definition** (Homology):

Let A be an abelian category and  $A_* \in Ch(A)$  a chain complex. Then its homology is defined to be the graded object  $H_n := \underbrace{\ker(\partial_n)}_{=:Z_n} / \underbrace{\operatorname{im}(\partial_{n+1})}_{=:B_n}$ .

# 2.1 Mapping cone

### 2.3 Definition:

Let  $f:(A_*,\partial^A)\to (B_*,\partial^B)$  be a chain-map. The mapping cone  $C(f)=(C(f)_*,\partial^{C(f)})$  is the chain complex given by

$$C(f)_n := A_{n-1} \oplus B_n$$
 and  $\partial_n^{C(f)} := \begin{pmatrix} -\partial_{n-1}^A & 0 \\ -f_{n-1} & \partial_n^B \end{pmatrix}$ 

# 2.4 Lemma (Mapping cones vs. quasi-isomorphisms):

Let  $f: (A_*, \partial^A) \to (B_*, \partial^B)$  be a chain-map.

- a.)  $0 \to B \stackrel{i}{\hookrightarrow} C(f) \stackrel{q}{-\!\!\!-\!\!\!-\!\!\!-} A[-1] \to 0$  is a short exact sequence of chain complexes.
- b.) The induced long exact sequence in homology

$$\cdots \to H_{n+1}(B) \xrightarrow{i_*} H_{n+1}(C(f)) \xrightarrow{q_*} \underbrace{H_{n+1}(A[-1])}_{=H_n(A)} \xrightarrow{\delta} H_n(B) \to H_n(C(f)) \to \cdots$$

has  $f_*$  as connecting morphism  $\delta$ .

- c.) f quasi-isomorphism  $\iff C(f)$  is acylic.
- d.) TFAE:
  - i.)  $H_*(f) = 0$
  - ii.)  $i_*: H_*(B) \to H_*(C(f))$  is mono.
  - iii.)  $0 \to H_*(B) \xrightarrow{i_*} H(C(f)) \xrightarrow{q_*} H_{*-1}(A) \to 0$  is a short exact sequence.
  - iv.)  $q_*: H_*(C(f)) \to H_{*-1}(A)$  is epi.

# 2.5 Lemma (Mapping cones vs. chain homotopy):

Let  $f: (A_*, \partial^{\dot{A}}) \to (B_*, \partial^B)$  be a chain-map.

- a.) f is a homotopy-equivalence  $\iff C(f)$  is contractible.
- b.) TFAE:
  - i.) f is null-homotopic.
  - ii.) f factors through  $A \hookrightarrow C(\mathrm{id}_A)$ .
  - iii.) f factors through a contractible complex.
  - iv.) The short exact sequence  $0 \to B \hookrightarrow C(f) \to A[-1] \to 0$  splits.

Proof. a. A map

$$H := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \begin{array}{ccc} A_{n-1} & A_n \\ \oplus & \to & \oplus \\ B_n & B_{n+1} \end{array}$$

is a homotopy  $\mathrm{id}_{C(f)} \simeq 0$  iff  $H\partial^{C(f)} + \partial^{C(f)}H = \mathrm{id}$ , that is iff

$$-\begin{pmatrix} \alpha \partial + \beta f + \partial \alpha & -\beta \partial + \partial \beta \\ \gamma \partial + \delta f + f \alpha - \partial \gamma & -\delta \partial + f \beta - \partial \delta \end{pmatrix} = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{pmatrix}$$

i.e. iff  $\beta: B \to A$  is a chain-map,  $-\alpha$  is a homotopy id  $\simeq (-\beta)f$ ,  $\delta$  is homotopy id  $\simeq f(-\beta)$  and  $\gamma$  is some map satisfying the last equation.

This already proves one direction: If  $C(f) \simeq 0$ , then  $f(-\beta) \simeq \operatorname{id}$  and  $(-\beta)f \simeq \operatorname{id}$  so that  $A \simeq B$ .

Conversely, if  $f(-\beta) \simeq \operatorname{id}$  and  $(-\beta)f \simeq \operatorname{id}$  via homotopies  $\delta$  and  $-\alpha$  respectively, then setting  $\gamma := 0$  for the moment, we instead get a homotopy  $\tilde{H}$  of 0 with  $\psi := \begin{pmatrix} \operatorname{id} & 0 \\ \delta f + f\alpha & \operatorname{id} \end{pmatrix}$  which is obviously an isomorphism on the level of modules.  $\psi$  is in fact a chain map:

$$\partial^{C(f)}\psi - \psi \partial^{C(f)} = \begin{pmatrix} 0 & 0 \\ \partial \delta f + \partial f \alpha + \delta f \partial + f \alpha \partial & 0 \end{pmatrix}$$

This is zero because

$$\partial \delta f + \delta f \partial = \partial \delta f + \delta \partial f = (-\operatorname{id} + f \beta) f$$

and

$$\partial f\alpha + f\alpha\partial = f\partial\alpha + f\alpha\partial = f(\mathrm{id} - \beta f)$$

Therefore  $H := \psi^{-1}\tilde{H}$  is a homotopy  $0 \simeq \mathrm{id}_{C(f)}$  so that C(f) is contractible as claimed. b. i.  $\iff$  iii. is trivial. We show i.  $\iff$  iii.: f factorises over the inclusion  $A \hookrightarrow C(\mathrm{id}_A), a \mapsto (0, a)$  iff there is a chain-map

$$(h_{n-1}, f_n): \begin{array}{c} A_{n-1} \\ \oplus \\ A_n \end{array} \to B_n$$

And for a family  $(h_n : A_n \to B_{n+1})$  to induce such a chain map  $C(\mathrm{id}_A) \to B$  is equivalent to  $\partial^B h = -h\partial^A - f$ , i.e. to h being a homotopy  $f \simeq 0$ . i.  $\iff$  iv.:  $B \hookrightarrow C(f)$  splits iff there is a chain map

$$(r, \mathrm{id}): \begin{array}{c} A_{n-1} \\ \oplus \\ B_n \end{array} \to B_n$$

And for a family  $(r_n : A_n \to B_{n+1})$  to induce such a chain map  $C(f) \to B$  is equivalent to  $\partial r = -r\partial - f$ , i.e. to r being a homotopy  $f \simeq 0$ .

2.6 Lemma (Universal properties of cones):

Let  $f: A_* \to B_*$  be a chain map. Then

$$\operatorname{Hom}_{Ch}(X_*, Cone(f)) = \left\{ \begin{pmatrix} \gamma \\ h \end{pmatrix} \middle| X \xrightarrow{\gamma} A[-1], X \xrightarrow{h} B, f[1] \circ \gamma \stackrel{h}{\simeq} 0 \right\}$$

$$\operatorname{Hom}_{Ch}(Cone(f), Y_*) = \left\{ (h, \beta) \middle| A[-1] \xrightarrow{h} Y, B \xrightarrow{\beta} Y, \beta \circ f \stackrel{h}{\simeq} 0 \right\}$$

## 2.2 Replacing objects by projective / injective resolutions

# 2.7 Lemma:

Chain maps between projectives / acyclic complexes are unique up to homotopy:

- a.) Homology: If  $C_* \in Ch^-(A)$  is acyclic and  $P_* \in Ch^-(Proj(A))$  all morphisms  $P_* \to C_*$  are null-homotopic.
- b.) Cohomology: If  $C^* \in Ch^+(A)$  is acyclic and  $I^* \in Ch^+(Inj(A))$  all morphisms  $C^* \to I^*$  are null-homotopic.

*Proof.* Let  $\alpha: P_* \to C_*$  be a chain map. Inductively we construct a homotopy  $h: P_* \to C_*[1]$  to the zero map.

We begin with setting  $h_n := 0$  for all n < 0. First step is to construct  $h_0$ . Since C is exact,  $C_1 \to C_0$  is epi so that  $\alpha_0$  lifts to some  $h_0 : P_0 \to C_1$  by projectivity, so that  $\partial_1 h_0 + \partial_0 = \alpha_0$  is satisfied.

If  $h_0, \ldots, h_{n-1}$  are already known and a partial homotopy, then

$$\begin{aligned} \partial_n \alpha_n &= \alpha_{n-1} \partial_n \\ &= (\partial_n h_{n-1} + h_{n-2} \partial_{n-1}) \partial_n \\ &= \partial_n h_{n-1} \partial_n \end{aligned}$$

So that  $\partial(\alpha_n - h_{n-1}\partial_n) = 0$ . Therefore  $\alpha_n - h_{n-1}\partial_n$  maps into  $Z_n(C)$  which equals  $B_n(C) = \operatorname{im}(\partial_{n+1})$  by exactness. By projectivity, we can find  $h_n$  such that

$$\alpha_n - h_{n-1}\partial_n = \partial_{n+1}h_n$$

is satisfied which proves the lemma.

2.8 Corollary (Fundamental lemma of homological algebra):

"Objects can be replaced by their projective or injective resolutions"

a.) Homology: Assume that A has enough projectives and that a projective resolution has been fixed for every object.

Any  $f: A \to B$  extends to a chain map between the augmented complexes

$$P_*(A) \longrightarrow A \longrightarrow 0$$

$$\downarrow \phi \qquad \qquad \downarrow f$$

$$\downarrow \phi \qquad \qquad \downarrow \phi$$

$$P_*(B) \longrightarrow B \longrightarrow 0$$

 $\phi$  is unique up to homotopy.

In particular:  $A \xrightarrow{P_*} K^-(Proj(A))$  is a well-defined functor with  $H_0 \circ P_* \cong id_A$ .

b.) Cohomology: Assume that A has enough injectives and that a injective resolution has been fixed for every object.

Any  $f: A \to B$  extends to a chain map between the augmented complexes

$$0 \longrightarrow A \longrightarrow I^*(A)$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad$$

 $\phi$  is unique up to homotopy.

In particular: A  $\xrightarrow{I^*} K^+(Inj(A))$  is a well-defined functor with  $H_0 \circ I^* \cong id_A$ .

As a consequence, projective and injective resolutions are unique up to homotopy equivalence.

*Proof.* Uniqueness up to homotopy follows from the lemma. We only have to show existence. Again, we work inductively:

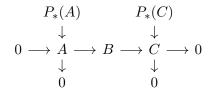
We set  $P_{-1}(A) := A$ ,  $\phi_{-1} := f$ , and  $P_{-1}(B) := B$  for notational convenience. If  $\phi_{n-1}$  is already constructed, then

Therefore  $\phi_{n-1} \circ \partial_n : P_n(A) \to P_{n-1}(B)$  maps into  $Z_{n-1}(P_*(B))$  which equals  $B_{n-1}(P_*(B)) = \operatorname{im}(\partial_n)$  by exactness. By projectivity, we get a lift  $\phi_n : P_n(A) \to P_n(B)$ .

### 2.9 Lemma (Horseshoe lemma):

" $P_*$  and  $I^*$  are exact"

a.) Homology: Every diagram



with exact first row and projective resolutions in the columns can be extended with some projective resolution  $P_*(B) \to B \to 0$  to a diagram

in which all rows are exact.

b.) Cohomology: Every diagram

$$0 \longrightarrow 0 \qquad \downarrow \qquad \downarrow \qquad \downarrow \qquad 0$$

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \qquad I^*(A) \qquad I^*(C)$$

with exact first row and injective resolutions in the columns can be extended with some injective resolution  $0 \to B \to I^*(B)$  to a diagram

$$0 \longrightarrow 0 \longrightarrow 0$$

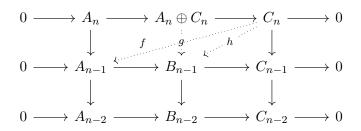
$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$0 \rightarrow I^*(A) \rightarrow I^*(B) \rightarrow I^*(C) \rightarrow 0$$

in which all rows are exact.

*Proof.* Set  $A_{-1} := A$  and  $A_n := P_n(A), C_{-1} := C$  and  $C_n := P_n(C)$  as well as  $B_{-1} := B$ . Then define  $P_n(B) := B_n := A_n \oplus C_n$ . For the vertical maps consider



We define  $g: A_n \oplus C_n \to B_{n-1}$  separately on the two components. Define  $g: A_n \oplus 0 \to 0$  $B_{n-1}$  to be the composition  $A_n \to A_{n-1} \to B_{n-1}$ .

The map  $g:0\oplus C_n\to B_{n-1}$  we choose in two steps. First choose  $h:C_n\to B_{n-1}$  to make the triangle on the right side commute. Then

$$\begin{array}{ccc}
0 \oplus C_n & 0 \oplus C_n \stackrel{\equiv}{\to} C_n \\
\downarrow^h & \downarrow \\
B_{n-1} & = & C_{n-1} = 0 \\
\downarrow^h & \downarrow^h & \downarrow^h \\
B_{n-2} \to C_{n-2} & C_{n-2}
\end{array}$$

d.h.  $\partial h(C_n) \subseteq A_{n-2}$  because the (n-2)th row is exact and of course  $\partial \partial h = 0$  so that  $\partial h(C_n) \subseteq Z_{n-2}(A_*) = B_{n-2}(A_*)$  by exactness of  $A_*$ . Using projectivity once again, we can lift  $\partial h$  to  $f: C_n \to A_{n-1}$  and finally define  $g: 0 \oplus C_n \to B_{n-1}$  as h-f. Note that  $g(c_n) = \partial c_n$  still holds because  $\operatorname{im}(f) \subseteq \ker(B_{n-1} \to C_{n-1})$ .

This ensures  $\partial g = 0$  which proves that the middle column is a(n incomplete) complex. We still have to show exactness. So let  $b_{n-1} \in B_{n-1}$  with  $\partial b_{n-1} = 0$ .

Then its image  $c_{n-1} = \overline{b_{n-1}}$  also satisfies  $\partial c_{n-1} = 0$  so that a  $c_n$  exists with  $c_{n-1} = \partial c_n$  by exactness of  $C_*$ . Then  $\overline{b_{n-1} - g(0 \oplus c_n)} = c_{n-1} - \partial c_n = 0$  so that  $b_{n-1} - g(0 \oplus c_n) \in \ker(B_{n-1} \to C_{n-1})$  which is  $\operatorname{im}(A_{n-1} \to B_{n-1})$  by exactness of the (n-1)th row so that  $b_{n-1} - g(0 \oplus c_n) = a_{n-1}$ . Then  $0 = 0 - 0 = \partial b_{n-1} - \partial g(0 \oplus c_n) = \partial a_{n-1}$  so that  $a_{n-1} = \partial a_n = g(a_n \oplus 0)$ . That shows  $b_{n-1} = g(a_n \oplus c_n)$ .

## 2.3 Replacing complexes by projective / injective resolutions

### 2.10 Corollary:

"Complexes can be replaces by double complexes of projectives/injectives"

- a.) Homology: For every  $K_* \in Ch(\mathsf{A})$  exists a commutative double complex  $P_{*,*}$  and maps  $P_{n,*} \to K_n$  such that  $P_{*,n} \to K_n \to 0$  is a projective resolution.
- b.) Cohomology: For every  $K^* \in Ch(A)$  exists a commutative double complex  $I^{*,*}$  and maps  $K^n \to I^{n,*} \to K^n$  such that  $0 \to K^n \to I^{n,*} \to K^n \to 0$  is an injective resolution.

*Proof.* Consider the short exact sequences

$$0 \to Z_n \to K_n \to B_{n-1} \to 0$$

and choose projective resolutions  $P''_{n,*} \to Z_n \to 0$  and  $P'_{n,*} \to B_n \to 0$ . Apply the horseshoe lemma to obtain a projective resolution  $P_{n,*} \to K_n \to 0$  fitting in the exact sequence.

Now apply the fundamental lemma of homological algebra to get a chain-map  $P'_{n,*} \to P''_{n-1,*}$  that extends the canonical map  $B_{n-1} \to Z_{n-1}$  and let  $P_{n,*} \to P_{n-1,*}$  be the composition  $P_{n,*} \to P'_{n,*} \to P''_{n-1,*} \hookrightarrow P_{n-1,*}$ . Since  $P' \to P \to P''$  are short exact sequences, we obtain a commutative double complex in this way.

#### **2.11** Lemma:

"Projective / injective resolutions of complexes exist"

- a.) Homology: For any bounded above complex  $K_* \in Ch^-(A)$  there is a  $P_* \in Ch^-(Proj(A))$  and a quasi-isomorphism  $P_* \to K_*$ .
  - $P_*$  can be chosen such that the quasi-
- isomorphism is termwise epi:  $P_n woheadrightarrow K_n$ .
- b.) Cohomology: For any bounded below complex  $K^* \in Ch^+(A)$  there is a  $I^* \in Ch^+(Inj(A))$  and a quasi-

isomorphism  $K^* \to I^*$ .

 $I_*$  can be chosen such that the

quasi-isomorphism is termwise mono:  $K_n \hookrightarrow I_n$ .

*Proof.* Take the total complex of  $P_{*,*}$  in the previous statement.

### **2.12** Lemma:

"Fundamental lemma of homological algebra upgraded to complexes"

Let  $A = O \in Ch^{-}(\Lambda)$   $P \in Ch^{-}(Proi(\Lambda))$  be quasi isomorphic, say O

Let  $A_*, Q_* \in Ch^-(A), P_* \in Ch^-(Proj(A))$  be quasi-isomorphic, say  $Q_* \xrightarrow{\alpha} A_*$  and  $P_* \xrightarrow{\beta} A_*$ .

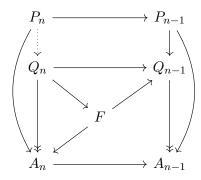
If  $Q_n \xrightarrow{\alpha} A_n$  is termwise epi, then there exists a chain-map  $P_* \xrightarrow{\gamma} Q_*$  such that  $\alpha \circ \gamma = \beta$ .

If  $\alpha$  is arbitrary, there exists a  $\gamma$  s.t.  $\alpha \circ \gamma \simeq \beta$ .

Any two chain-maps with  $\alpha \circ \gamma_1 \simeq \beta \simeq \alpha \circ \gamma_2$  are homotopic.

**2.13:** If A is concentrated in a single degree, then  $Q_* \to A_0 \to 0$  is just an acyclic complex and the statement reduces to the fundamental lemma of homological algebra. In this sense this statement is a generalisation of the fundamental lemma from A to  $D^-(A)$ .

Homological version, seems harder?? Assume that a partial chain map  $\gamma_0, \ldots, \gamma_{n-1}$  is already constructed. We want to construct the missing arrow in the commutative diagram



We set

$$F := A_n \times_{A_{n-1}} Z(Q_{n-1}) = \{ (a_n, q_{n-1}) \in A_n \times Q_{n-1} \mid \partial q_{n-1} = 0 \land \partial a_n = \alpha(q_{n-1}) \}$$

First we prove that the map  $Q \xrightarrow{(\alpha,\partial)} F$  is epi. Let  $(a_n,q_{n-1}) \in F$  be arbitrary. Then  $q \in Z$   $_{\alpha}(Q)$  so that the homology class is well defined. Then  $\alpha$   $[a_{n-1}]_{\alpha}$ 

Then  $q_{n-1} \in Z_{n-1}(Q)$  so that the homology class is well-defined. Then  $\alpha_*[q_{n-1}]_{H_{n-1}(Q)} = [\alpha(q_{n-1})]_{H_{n-1}(A)} = [\partial a_n] = 0$ . Since  $\alpha$  is injective on homology, this means  $[q_{n-1}] = 0$ , i.e.  $q_{n-1} = \partial q'_n$  for some  $q'_n \in Q_n$ .

Then  $\partial a_n = \alpha(q_{n-1}) = \alpha \partial(q'_n) = \partial \alpha(q'_n)$  so that  $a_n - \alpha(q'_n) \in Z_n(A)$ . Since  $\alpha$  is surjective on homology, there is a  $z_n \in Z_n(Q)$  such that  $[\alpha(z_n)] = [a_n - \alpha(q'_n)]$ , i.e. there exists a  $a_{n+1}$  such that  $\alpha(z_n) = a_n - \alpha(q'_n) + \partial a_{n+1}$ .

Now choose an preimage  $q_{n+1} \in Q_{n+1}$  of  $a_{n+1}$  and set  $q_n := z_n + q'_n - \partial q_{n+1}$ . This is the preimage of  $(a_n, q_{n-1})$ :

$$\alpha(q_n) = \underbrace{\alpha(z_n) + \alpha(q'_n)}_{=a_n + b_n} - \alpha(\partial q_{n+1}) = a_n + b_n - \partial \alpha(q_{n+1}) = a_n$$

$$\partial(q_n) = \underbrace{\partial z_n}_{=0} + \underbrace{\partial q'_n}_{=q_{n-1}} + 0$$

Since we now know that  $Q_n \to F$  is epi, we can lift the morphism  $(\beta_n, \gamma_{n-1}\partial) : P_n \to F$  to a morphism  $\gamma_n : P_n \to Q_n$ . By construction it makes the diagram commute so that it is a partial chain map.

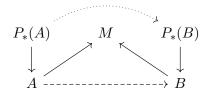
b. If  $\alpha$  is not term-wise epi Meep

c. For uniqueness observe that  $\alpha \circ (\gamma_1 - \gamma_2) \simeq 0$  so that there is a chain-map  $P_*[-1] \xrightarrow{(h,\gamma_1 - \gamma_2)} Cone(\alpha), p \mapsto (h(p), (\gamma_1 - \gamma_2)(p))$ . Since  $\alpha$  is a quasi-isomorphism,  $Cone(\alpha)$  is acyclic so that any such map is null homotopic. In particular  $\gamma_1 - \gamma_2 = quotient \circ \widehat{\gamma} \simeq 0$ .

### **2.14 Corollary** (Resolution functor):

Fixing a projective resolution of every complex,  $P_*: D^-(A) \to K^-(Proj(A))$  is a pseudo-inverse to the localisation functor.

*Proof.* We have to show  $\operatorname{Hom}_K(P_*(A_*), P_*(B_*)) = \operatorname{Hom}_D(A_*, B_*)$ . Morphisms  $A \to B$  in  $D^-$  are roofs  $A \to M \leftarrow B$ .



Since  $P_*(A)$  is termwise projective and  $P_*(B) \to B \to M$  is a quasi-isomorphism, we can complete the triangle of  $P_*(A) \to M$  and  $M \leftarrow P_*(B)$  with a chain-map  $\gamma : P_*(A) \to P_*(B)$  making the diagram commute up to homotopy.

# 3 Derived functors I

**3.1** (The Problem): Given a right-exact functor  $F: A \to B$ , and exact sequence

$$0 \to A \to B \to C \to 0$$

gives a exact sequence

$$F(A) \to F(B) \to F(C) \to 0$$

We want to find functors  $L_nF$  and natural transformations  $\delta_n$  (natural w.r.t. the short exact sequence) such that this sequence extends to a long exact sequence

$$\cdots \to L_2F(C) \xrightarrow{\delta_2} L_1F(A) \to L_1F(B) \to L_1F(C) \xrightarrow{\delta_1} \underbrace{F(A)}_{=L_0F(A)} \to \underbrace{F(B)}_{L_0F(B)} \to \underbrace{F(C)}_{=0} \to 0$$

And similarly for left-exact functors.

Of course, we want the universal solution to this problem.

# **3.2 Definition** ( $\delta$ -functors):

A family  $(F_n, \delta_n)_{n \in \mathbb{N}}$  of functors  $A \xrightarrow{F_n} B$  and natural transformations  $F_n(C) \xrightarrow{\delta_n} F_{n-1}(A)$  for every short exact sequence  $0 \to A \to B \to C \to 0$  that transforms such short exact sequences into long exact sequences as above is called a (homological)  $\delta$ -functor.

A morphism  $F \xrightarrow{t} G$  of  $\delta$ -functors is a family  $(t_n)$  of natural transformations  $F_n \xrightarrow{t_n} G_n$  which induces morphisms between the long exact sequences, i.e.  $t_{n-1}\delta_n^F = \delta_n^G t_n$ . Cohomological  $\delta$ -Functors  $(F^n, d^n)$  are analogously defined.

### 3.3 **Definition** (Universal $\delta$ -functors):

A  $\delta$ -functor  $(F_n, \delta_n)$  is called the universal  $\delta$ -functor if for every  $(G_n, \delta_n)$  and every  $G_0 \xrightarrow{t_0} F_0$  there exists a unique morphism  $G \xrightarrow{t} F$  of  $\delta$ -functors extending  $t_0$ .

### **3.4 Definition** (Derived functors):

Let  $F: A \to B$  be right-exact. A  $\delta$ -functor  $(L_n F, \delta_n)$  together with an isomorphism  $L_0 F \xrightarrow{\tau} F$  is called the left derived functor of F if  $(LF, \tau)$  is a final object in the category of all  $\delta$ -functor-with-isomorphisms.

It is in other words a representation of the functor  $\{\delta$ -functors  $\} \to \mathsf{Set}, (G_n, \delta_n) \mapsto \mathsf{Nat}(G_0, F)$  such that the universal element  $\tau \in Nat(F_0, F)$  is an iso.

Similarly right derived functor of a left exact F is defined as an initial object in the appropriate category of  $\delta$ -functors with isomorphism  $F \xrightarrow{\tau} F_0$ , i.e. a representation of the functor  $(G^n, d^n) \mapsto \operatorname{Nat}(F, G_0)$ .

**3.5:** In particular: Derived functors are exactly those universal  $\delta$ -functors that agree with F in degree 0.

# **3.6 Lemma** (Recognising universal $\delta$ -functors): Let $(F_n, \delta_n)$ be a $\delta$ -functor.

- a.) Homology: If  $F_n(P) = 0$  for all  $n \ge 1$  and all  $P \in Proj(A)$ , then F is a universal homological  $\delta$ -functor.
- b.) Cohomology: Assume  $F^n(I) = 0$  for all  $n \ge 1$  and all  $I \in Inj(A)$ , then F is a universal cohomological  $\delta$ -functor.

*Proof.* Let  $(\tilde{F}_n, \tilde{\delta}_n)$  be another  $\delta$ -functor and assume that unique transformations  $t_0, \ldots, t_{n-1}$  have already been constructed. Fix  $C \in A$  and choose a short exact  $0 \to K \xrightarrow{j} P \xrightarrow{q} C \to 0$  with P projective. Then

$$\cdots \longrightarrow \tilde{F}_n(P) \longrightarrow \tilde{F}_n(C) \xrightarrow{\tilde{\delta_n}} \tilde{F}_{n-1}(K) \longrightarrow \tilde{F}_{n-1}(P) \longrightarrow \cdots$$

$$\downarrow \qquad \qquad \downarrow t_{n-1} \qquad \downarrow t_{n-1}$$

$$\cdots \longrightarrow \underbrace{F_n(P)}_{=0} \longrightarrow F_n(C) \xrightarrow{\delta_n} F_{n-1}(K) \longrightarrow F_{n-1}(P) \longrightarrow \cdots$$

It follows that  $F_n(C) \xrightarrow{\delta_n} \ker(F_{n-1}(j))$  and since  $t_{n-1}$  is natural, there is a unique  $t_n : \tilde{F}_n(C) \to F_n(C)$  that makes the square commute. This  $t_n$  does not depend on the choice of K and P by Schanuel's lemma.

Naturality of  $t_n$  follows from a simple diagram chase using naturality of  $\delta_n$  and  $\tilde{\delta_n}$ , naturality of  $t_{n-1}$  and that  $F_n(A) \to F_{n-1}(K)$  is mono.

It remains to show that  $t_n$  commutes with the deltas for an arbitrary short exact  $0 \to A \to B \to C \to 0$ . This also follows from a simple diagram chase.

## 3.7 Theorem (Derived functors exist.):

Let  $F: A \to B$  be right / left exact functor.

- a.) Homology:
  - i.) If A has enough projectives, then F has a left derived functor.
  - ii.)  $L_i F(P) = 0$  for all projectives P and all i > 0.
  - iii.) Deriving is a functor  $L_i$ :  $\operatorname{\mathsf{Fun}}_{\operatorname{r.e.}}(\mathsf{A},\mathsf{B}) \to \operatorname{\mathsf{Fun}}_{\operatorname{add}}(\mathsf{A},\mathsf{B}).$
- b.) Cohomology:
  - i.) If A has enough injectives, then F has a right derived functor.
  - ii.)  $R^i F(I) = 0$  for all injectives I and all i > 0.
  - iii.) Deriving is a functor  $R^i$  $\mathsf{Fun}_{\mathrm{l.e.}}(\mathsf{A},\mathsf{B}) \to \mathsf{Fun}_{\mathrm{add}}(\mathsf{A},\mathsf{B}).$

*Proof.* Existence: Define  $L_nF := H_n \circ P_*$ . Note that this does not depend on the choice of the projective resolutions  $P_*$  because all choices are homotopy equivalent and homology forgets homotopy. Note that  $L_iF(P) = 0$  for P projective and i > 0 because  $0 \to P \xrightarrow{\mathrm{id}} P \to 0$  is a projective resolution of P.

Horseshoe lemma implies that every short exact sequence

$$0 \to A \to B \to C \to 0$$

lifts to exact sequence up to homotopy  $0 \to P_*(A) \to P_*(B) \to P_*(C) \to 0$  which implies a long exact sequence in homology. Therefore  $LF = (L_iF, \delta_i)$  is a  $\delta$ -functor. It extends P because  $P_*(A) \to A \to 0$  is a projective resolution such that  $H_0(P_*(A)) \cong A$  naturally.

We still have to show universality. Let  $((\tilde{F}_n, \tilde{\delta}_n), \tilde{\tau})$  be another  $\delta$ -functor extending F. The above lemma shows that there is a unique morphism of  $\delta$ -functors  $t : \tilde{F} \to F$  which extends  $t_0 := \tau^{-1} \circ \tilde{\tau}$ .

The lemma also proves that every natural transformation  $F \to G$  between right exact functors extends to  $LF \to LG$  since  $L_iG(P) = 0$ .

# 3.1 Computing derived functors via acyclic resolutions

## **3.8 Definition** (F-acyclic objects):

An object  $Q \in A$  is called F-acyclic if

- a.) Homology:  $L_n F(Q) = 0$
- b.) Cohomology:  $R^n F(Q) = 0$

holds for all  $n \geq 1$ .

**3.9:**  $Proj(A) \subseteq Acycl(F)$  for all F. For some F (like Hom(A, -) equality may hold), but depending on F, the class of acyclics may be bigger then the projectives (or injectives). For example all  $Proj(A-Mod) \subseteq Flat(A-Mod) \subseteq Acycl(M \otimes -)$ .

### 3.10 Theorem:

Let  $F: A \to B$  be right / left exact.

a.) Homology: Let  $Q_* \to A \to 0$  be a resolution of A by F-acyclic objects. Then  $L_nF(A) \cong H_n(F(Q_*))$  via a unique isomorphism.

More precisely: Choose a projective resolution  $P_*(A) \to A \to 0$ . Then the unique-up-to-homotopy chain map  $P_*(A) \xrightarrow{\gamma} Q_*$  induces a unique isomorphism  $L_n F(A) = H_n(F(P_*(A))) \xrightarrow{H_n(F\gamma)} H_n(F(Q_*))$ .

### **3.11 Lemma:**

The class of F-acyclics has the following properties:

- a.) Homology: Assume A has enough projectives. Then
  - i.) Every  $A \in A$  has a covering  $Q \twoheadrightarrow A \to 0$  for some acyclic Q.
  - ii.) It is closed under direct sums.
  - iii.) If in an exact sequence  $0 \to A \to B \to C \to 0$  both B and C are acyclic, then A is two.
  - iv.) If in an exact sequence  $0 \to A \to B \to C \to 0$  the object C is acyclic, then  $0 \to FA \to FB \to FC \to 0$  is also exact.

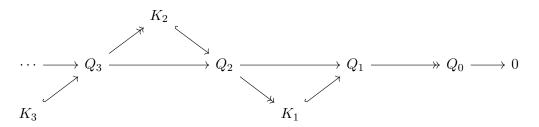
*Proof.* a. Projectives are always acyclic.

- b. follows because  $L_nF$  is additive.
- c. and d. follow from the long exact sequence.

### 3.12 Lemma:

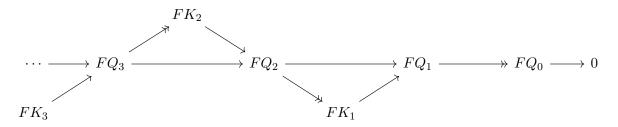
Let  $Q_* \in Ch^{\pm}(Acyc)$  be a complex of F-acyclics. If  $Q_*$  is exact, then  $FQ_*$  is also exact.

*Proof.* Let  $K_n$  be the kernels / images of the boundary maps so that we get a diagram



where the diagonals are short exact sequences. First observation: Per induction all  $K_n$  are acyclic, because the  $Q_n$  are.

The transformed sequence



is exact iff the diagonals are exact again, i.e. if  $FK_n \to FQ_n$  is mono. This follows from exactness of  $K_n \hookrightarrow Q_n \twoheadrightarrow K_{n-1}$  and  $K_{n-1}$  being acyclic.

Proof of the main theorem. Let  $P_* \to A$  be a projective resolution,  $Q_* \to A$  an acylic resolution and  $\gamma: P_* \to Q_*$  be a chain-map extending  $A \xrightarrow{\mathrm{id}} A$  along those resolutions.  $\gamma$  is a quasi-isomorphism because both resolutions compute have homology A[0]. Therefore  $Cone(\gamma)$  is exact. Termwise it is direct sum of projective and F-acyclic objects and therefore an exact complex of F-acyclic objects. Thus  $Cone(F\gamma) = F(Cone(\gamma))$  is also exact. Therefore  $F\gamma$  is a quasi-isomorphism.

# 4 Examples

### 4.1 Example (Snake lemma):

Taking kernels is a left-exact functor  $A^{\{*\to *\}} \to Ab$ . Its right derived functor is the cokernel in degree 1 and zero further up.

Dually taking cokernels is right-exact and its left derived functor is the kernel in degree 1 and zero everywhere else.

This is a manifestation of the snake lemma.

### 4.2 Example (Sheaf (co)homology):

Sheaf cohomology  $H^*(X, \mathcal{F})$  is the right derived functor of the global section functor  $\Gamma: Sh(X) \to \mathsf{Ab}$ .

## **4.3 Example** (DeRham cohomology):

 $H^*_{\mathrm{dR}}(M)$  is Sheaf cohomology of the sheaf  $\underline{\mathbb{R}}_M$  of locally constant functions  $M \to \mathbb{R}$ . This uses that

$$0 \to \underline{\mathbb{R}}_M \hookrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \to 0$$

is a resolution of  $\mathbb{R}_M$  by fine sheafs and that fine sheafs are  $\Gamma$ -acyclic.

# **4.4 Example** (Singular cohomology):

 $H^*_{\text{sing}}(X;G)$  is sheaf cohomology of the sheaf  $\underline{G}_X \in Sh(X)$  of locally constant G-valued functions if X is paracompact.

## **4.5 Example** (Étale cohomology):

Étale cohomology is the Sheaf cohomology for sheafs on the étale site, i.e. the right derived functor of global sections  $\mathsf{Sh}_{et}(X) \to \mathsf{Ab}$ .

### **4.6 Example** (Ext and Tor):

 $\operatorname{Ext}_A^i(M,N)$  is right derived of  $\operatorname{Hom}_A(M,-):A-\operatorname{\mathsf{Mod}}\to\operatorname{\mathsf{Ab}}$  as well as left derived of  $\operatorname{Hom}_A(-,N):\operatorname{\mathsf{Mod}}-A\to\operatorname{\mathsf{Ab}}^{op}$ .

 $\operatorname{Tor}_i^A(M,N)$  is left derived of both  $M\otimes_A -: A\operatorname{\mathsf{-Mod}} \to \operatorname{\mathsf{Ab}}$  and  $-\otimes_A N:\operatorname{\mathsf{Mod}} -A\to\operatorname{\mathsf{Ab}}$ .

### 4.7 Example (Group (co)homology):

 $H_*(G,M)$  is the left derived functor of the functor of coinvariants  $k \otimes_{kG} -$ , i.e. it is  $Tor_*^{kG}(k,M)$ .

 $H_k^*(G,-)$  is the right derived functor of the functor of fixed points  $(-)^G = \operatorname{Hom}_{kG}(k,-)$ , i.e. it is  $\operatorname{Ext}_{kG}^*(k,M)$ .

### 4.8 Example (Hochschild (co)homology):

Let  $A^e := A \otimes_k A^{op}$  be the enveloping algebra of the k-algebra A.

 $HH_n(A,M) := Tor_n^{A^e}(A,M)$ , i.e. it is the left derived functor of the functors of coinvariant  $M/[A,M] = A \otimes_{A^e} M : (A,A) - \mathsf{Bimod} \to \mathsf{Ab}$ .

 $HH^n(A,M) := Ext^N_{A^e}(A,M)$ , i.e. the right derived functor of invariants  $Z(M) := \operatorname{Hom}_{A^e}(A,M) : (A,A) - \operatorname{Bimod} \to \operatorname{Ab}$ .

# 4.9 Example (Lie-algebra (co)homology):

 $H_n(\mathfrak{g},M):=Tor_n^{U(\mathfrak{g})}(k,M)$ , i.e. left derived of taking coinvariants.

 $H^n(\mathfrak{g},M):=Ext^n_{U(\mathfrak{g})}(k,M)$ , i.e. right derived of taking invariants.

# 5 Derived functors II: Total derived functors

### 5.1 Definition:

Let  $p_A^2: K^2(A) \to D^2(A)$  be the projection functor from the homotopy category onto the derived category.

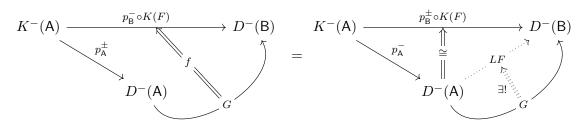
The total left/right derived functor of  $F: A \to B$  is the "best approximation" of  $K^{\pm}(F): K^{\pm}(A) \to K^{\pm}(B)$  on the level of derived categories, i.e. it fits into the diagram

$$K^{\pm}(\mathsf{A}) \xrightarrow{K^{\pm}(F)} K^{\pm}(\mathsf{B})$$

$$\downarrow^{p_{\mathsf{A}}^{\pm}} \qquad \qquad \downarrow^{p_{\mathsf{B}}^{\pm}}$$

$$D^{\pm}(\mathsf{A}) \xrightarrow{RF} D^{\pm}(\mathsf{B})$$

**5.2:** In this situation LF / RF is also both the left and right Kan-extension of  $Q_B \circ K(F)$  along the localisation  $p_A$ . Concretely: LF fits into a diagram



which is commutative up to natural isomorphism  $p_{\mathsf{B}} \circ K(F) \xrightarrow{\cong} LF \circ p_{\mathsf{A}}$  such that for every other functor  $D^{-}(\mathsf{A}) \xrightarrow{G} D^{-}(\mathsf{B})$  every natural transformation  $G \circ p_{\mathsf{A}} \xrightarrow{f} p_{\mathsf{B}} \circ K(F)$  factors uniquely through this iso.

Therefore some authors define LF of any additive functor F as the right Kan extension  $Ran_{p_{\mathsf{A}}^-}(p_{\mathsf{B}}^-\circ K(F))$  and RF as the left Kan extension  $Lan_{p_{\mathsf{A}}^+}(p_{\mathsf{B}}^+\circ K(F))$ . In this situation however, LF and RF do in general not extend F.

### **5.3 Theorem** (Total derived functors exist):

Total derived functors of right / left exact functors exist if A has enough projectives / injectives.

*Proof.* Choose a resolution functor  $D^{\pm}(\mathsf{A}) \to K^{\pm}(\mathsf{A})$  and pre-compose with  $p_{\mathsf{B}}^{\pm} \circ K^{\pm}(F)$ .

**5.4:** Note that we do not need projective / injective resolutions, F-acyclic resolutions are fine too because we have already proven that  $F(P_*(A))$  is quasi-isomorphic to  $F(Q_*)$  if  $Q_*$  is any resolution by F-acyclic objects.