Self-injective serial algebras II

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Convention:
Let K be a field of characteristic $p \in \mathbb{P} \cup \{0\}$ and A be a finite-dimensional K-algebra.

1 Last week on Dragonball...

1.1 Lemma and definition (Lemma 11.3.1)**:**
Let M be a finite-dimensional A-module. TFAE:

a.) M is uniserial, i.e. it has exactly one composition series.

b.) $M > J(A)M > J(A)^2M > \ldots > 0$ is a composition series.

c.) Every submodule of M is of the form $J(A)^sM$ for some $s \in \mathbb{N}$.

d.) The submodules of M are totally ordered.

e.) $M^* := \mathrm{Hom}_K(M, K)$ is uniserial.

In particular: If M is uniserial, then all submodules and all quotients of M are uniserial.

1.2: In particular the radical and socle series of M coincide:

$$rad^k(M) = J(A)^kM = soc_{l-k}(M)$$

where $l = l(M)$ is the length of M.

1.3 Definition:
A is called serial if all of it finite-dimensional, indecomposable modules are uniserial.

1.4 Theorem (11.3.4)**:**
Let A be a finite-dimensional, non-simple, serial, indecomposable, self-injective K-algebra. Furthermore let $S_1, \ldots, S_n$ be a full set of representatives of isomorphism classes of simple A-modules and let $P_1, \ldots, P_n$ be the corresponding projective covers.
Then:

a.) There is a unique n-cycle $\pi \in Sym(n)$ such that

$$J(A)P_i/J(A)^2P_i \cong S_{\pi(i)}$$

b.) The P_i all have the same composition length q and the composition factors in

$$P_i > J(A)P_i > J(A)^2P_i > \ldots > J(A)^{q-1}P_i > 0$$

are (in this order) $S_i, S_{\pi(i)}, S_{\pi^2(i)}, \ldots, S_{\pi^{q-1}(i)}$.

c.) For all i, there is a short exact sequence

$$0 \to S_{\pi^q(i)} \to P_{\pi(i)} \to J(A)P_i \to 0$$

d.) If A is symmetric, then $n \mid q-1$.

1.5: Non-simple + indecomposable implies that all the P_i have length > 1: If $l(P_i) = 1$, then $P_i = S_i$ is simple and lies in a single-element block so that A must necessarily be equal to that block because it is indecomposable. In particular there is only one simple module, namely P_i and ${}_AA$ is a sum of copies of P_i and thus semisimple. Being indecomposable, it must be simple by Wedderburn's theorem.

1.6: After a suitable reindexing, $\pi = (0, 1, 2, 3, ..., n-1)$ and the $PIMs$ have the form

$$\begin{array}{ccccc} P_0 & P_1 & P_2 & \cdots & P_{n-1} \\ \hline \begin{pmatrix} 0 \\ 1 \\ 2 \\ \vdots \\ q-1 \end{pmatrix} & \begin{pmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ q \end{pmatrix} & \begin{pmatrix} 2 \\ 3 \\ 4 \\ \vdots \\ q+1 \end{pmatrix} & \cdots & \end{array}$$

where all numbers are to be read modulo n.

2 And now for the conclusion

2.1 Corollary (11.3.5)**:**
Let A be serial, self-injective, non-simple and indecomposable. Let π and q be as before. Then

$$\Omega^2(S_i) = S_{\pi^q(i)}$$

for all simple modules S_i.

Proof. By definition $\Omega(S_i) = \ker(P_i \twoheadrightarrow S_i) = J(A)P_i$ and by the theorem

$$0 \to S_{\pi^q(i)} \to P_{\pi(i)} \to J(A)P_i \to 0$$

so that $\Omega(\Omega(S_i)) = \Omega(J(A)P_i) = S_{\pi^q(i)}$. □

2.1 Recognising serial algebras

2.2 Proposition (11.3.6)**:**
Let A be a finite-dimensional K-algebra. Then A is serial iff every indecomposable projective A-module and every indecomposable injective A-module is uniserial.

Proof. One direction is trivial by definition. For the other direction let $0 \neq U \in A-\mathsf{mod}$ be f.g., indecomposable. We have to show that U is uniserial. Let $V \leq U$ be uniserial submodule of maximal dimension. We will prove $V = U$.
Since all simple modules are uniserial $V \neq 0$. Let I be the injective envelope of V so that in particular $soc(I) = soc(V)$ is simple. Then I is indecomposable and thus uniserial by assumption. Then $\alpha : V \hookrightarrow I$ extends by injectivity of I to a homomorphism $\widehat{\alpha} : U \to I$. We set $X := \ker(\widehat{\alpha})$. By construction $X \cap V = \ker(\widehat{\alpha}_{|V}) = \ker(\alpha) = 0$.
Since U/X is isomorphic to a submodule of I, it is uniserial. Therefore its radical quotient is simple. Let P be the projective cover of U/X. Because its radical quotient is simple, P is indecomposable and uniserial by assumption. Since P is projective, we can lift the quotient $P \to U/X$ to a map $\beta : P \to U$ such that $U = \operatorname{im}(\beta) + X = \operatorname{im}(\beta) + \ker(\widehat{\alpha})$. Now

$$\begin{aligned}
\dim(V) &= \dim(\alpha(V)) && \text{because } \alpha \text{ is injective} \\
&\leq \dim(\widehat{\alpha}(U)) && \text{because } \alpha = \widehat{\alpha}_{|V} \\
&\leq \dim(\operatorname{im}(\beta)) && \text{because } U = \operatorname{im}(\beta) + \ker(\widehat{\alpha})
\end{aligned}$$

and $\operatorname{im}(\beta)$ is a uniserial submodule of U (because it is a quotient of P). Because V is a uniserial submodule of maximal dimension, all inequalities are in fact equalities. Therefore, in particular $(V + X)/X = \alpha(V) = \widehat{\alpha}(U) = U/X$ and thus $V \oplus X = U$. Since U is indecomposable and $V \neq 0$, we finally find $X = 0$. □

2.3 Lemma (11.3.7)**:**
Let A be a self-injective, finite-dimensional K-algebra. TFAE:

a.) $J(A)e/J(A)^2e$ is either zero or simple for all primitive idempotents $e \in A$.

b.) A is serial.

Proof. a. $\Longrightarrow$ b. is known.
b. $\Longrightarrow$ a.: By the previous proposition, it is enough to prove that all PIMs $P = Ae$ are uniserial, i.e.

$$\forall s \in \mathbb{N} : J(A)^s e/J(A)^{s+1}e \in \operatorname{Irr}(A) \cup \{ 0 \}$$

For $s = 0$ this is by definition, for $s = 1$ this is by assumption. We proceed by induction. If $J(A)^s e = 0$, there is nothing to prove. Otherwise, the projective cover of $J(A)e$ is the projective cover of he simple module $J(A)e/J(A)^2e$, say $Af \twoheadrightarrow J(A)e$ for some primitive idempotent f. Therefore $J(A)^{s+1}e = J(A)^s \cdot (J(A)e)$ is a quotient of $J(A)^s Af = J(A)^s f$. By induction assumption, $J(A)^{s+1}e/J(A)^s e \cong J(A)^s f/J(A)^{s-1}f$ is simple or zero. □

2.4 Lemma:
Let A be a symmetric K-algebra and $J \trianglelefteq A$ a two-sided ideal. Then

$$RAnn(J) = LAnn(J)$$

Proof. Let τ be a symmetrising form.

$$\begin{aligned}
x \in LAnn(J) &\iff \forall z \in J : xz = 0 \\
&\iff \forall y \in J \forall a \in A : xay = 0 \\
&\iff \forall y \in J \forall a, b \in A : \tau(bxay) = 0 \\
&\iff \forall y \in J \forall a, b \in A : \tau(aybx) = 0 \\
&\iff \forall y \in J \forall b \in A : ybx = 0 \\
&\iff \forall z \in J : zx = 0 \\
&\iff x \in RAnn(J)
\end{aligned}$$

□

2.5 Lemma (11.3.8)**:**
Let A be a symmetric, indecomposable K-algebra. TFAE:

a.) There is a $t \in A$ such that $J(A) = At$.

b.) There is a $t \in A$ such that $J(A) = tA$.

c.) A is serial and all PIMs occur with the same multiplicity in A.

In this case one can chosen to same t in a. and b.

Proof. a. $\iff$ b. We prove only one direction. So assume $J(A) = At$. First we prove that $tA \to At, ta \mapsto at$ is a well-defined K-linear map. Namely if $ta = 0$, then $J(A)a = Ata = 0$ so that $a \in RAnn(J(A)) \overset{A\text{ symm}}{=} LAnn(J(A))$. Thus $at \in aJ(A) = 0$ which proves well-definedness.
Obviously our map is surjective so that $\dim_K(At) \leq \dim_K(tA)$. But $At = J(A)$ by assumption and $tA \subseteq J(A)$ because $t \in J(A)$. Therefore equalities hold.

a.+b. $\implies$ c. Let $1 = \sum e_i$ be a decomposition into pairwise orthogonal, primitive idempotents. Then

$$\bigoplus_{i=1}^{n} J(A)e_i = J(A) = At = \sum_{i=1}^{n} Ae_i t$$

We claim that the sum on the RHS is direct. Namely let $a_i \in Ae_i$ such that $\sum_i a_i t = 0$. Then $\sum_i a_i \in LAnn(t) = LAnn(tA) = LAnn(J(A)) \overset{A\text{ symm}}{=} RAnn(J(A))$ so that $t \sum_i a_i = 0$ and therefore $ta_i = t(\sum_i a_i)e_i = 0$. This in turn means $a_i \in RAnn(t) = RAnn(At) = RAnn(J(A)) \overset{A\text{ symm}}{=} LAnn(J(A))$ so that $a_i t = 0$ which proves that the sum is direct.

Moreover $J(A)e_i$ is indecomposable because $soc(J(A)e_i) = soc(Ae_i)$ is simple (A is self-injective). Furthermore Ae_it is indecomposable, because it is a quotient of Ae_i and therefore has a simple head. The Krull-Schmidt theorem implies that there is a permutation $\pi \in Sym(n)$ such that $J(A)e_i = Ae_{\pi(i)}t$.
Since Ae_j has a simple head and $J(A)e_i$ is a quotient of one of these, $J(A)e_i/J(A)^2e_i$ is either zero or simple. By the previous lemma, A is serial.

If $Ae_i \cong Ae_j$, then $J(A)e_i \cong J(A)e_j$ as well. Because A is serial, the second layer in the radical filtration of these is $S_{\pi(i)}$ and $S_{\pi(j)}$, i.e. the head of $Ae_{\pi(i)}t$ (which is a quotient of $Ae_{\pi(i)}$). Therefore $Ae_{\pi(i)} \cong Ae_{\pi(j)}$ so that π is compatible with isomorphism classes. By the main theorem, π permutes the isomorphism classes transitively. This proves that all PIMs occur with the same multiplicity.

c. $\Longrightarrow$ a. Let m be the common multiplicity of the PIMs. Then ${}_AA = (\bigoplus_{i=1}^n Ae_i)^m = (Ae)^m$ for some pairwise orthogonal, non-isomorphic idempotents e_i and $e := e_1 + \ldots + e_n$. Then $A/J(A) \cong J(A)/J(A)^2$ as left modules by the main theorem, because every simple occurs exactly m times in both semisimple modules. Choose $t + J(A)^2$ as the image of $1 + J(A)$ under this isomorphism. Then $At + J(A)^2 = J(A)$ so that $At = J(A)$ by Nakayama. □

2.2 Specific serial algebras

2.6 Theorem (11.3.2)**:**
Let K be an algebraically closed field of characteristic p, P a cyclic p-group and $E \leq \mathrm{Aut}(P)$ a p'-subgroup. Then

a.) $K[P \rtimes E]$ is symmetric, indecomposable and serial.

b.) $K[P \rtimes E]$ has exactly $|E|$ simples, all of which are one-dimensional. All PIMs have dimension $|P|$ and occur with multiplicity one. In particular $K[P \rtimes E]$ is split basic.

Proof. a. Group algebras are always symmetric. It is a well-known fact that all Block idempotents of $K[N_G(P)]$ lie in $K[C_G(P)]^{N_G(G)}$. Here $G = N_G(P)$ and $C_{P\rtimes E}(P) = Z(P) = P$. Furthermore $K[P]$ is local so that the only non-zero idempotent is the identity. Thus $K[G]$ only has one block idempotent and is therefore indecomposable.
The epi $P \rtimes E \twoheadrightarrow E$ induces an surjective morphism $\phi : K[P \rtimes E] \to K[E]$. Because E is a p'-group and K has char. p, this is a semisimple quotient so that $J(A) \subseteq \ker(\phi)$. In fact $\ker(\phi) = \langle g - 1 | g \in P\rangle = J(K[P])A = AJ(K[P])$ so that $\ker(\phi)$ is a nilpotent ideal and therefore $J(A) = \ker(\phi)$. Moreover $\ker(\phi) = AJ(K[P]) = AK[P](y-1) = A(y-1)$ where $P = \langle y\rangle$. By the previous lemma, A is serial.

b. It is a well known fact that $O_p(G)$ acts trivial on all simple $K[G]$-modules. In this case this means that all simple $K[G]$-modules are also simple $K[E]$-modules. Since E is a p'-group and K is algebraically closed, the Wedderburn decomposition of $K[E]$ is $K \times \ldots \times K = K^{|E|}$. In other words, all simple modules are one-dimensional and there are exactly $|E|$ of them.

All PIMs have the same length, i.e. the same dimension q, and occur with the same multiplicity m. Since $K[G] \twoheadrightarrow K[E]$ has $J(A)$ as kernel, we can lift a orthogonal decomposition into primitive idempotents from $K[E]$ to $K[G]$. There are $|E|$ many such idempotents. Therefore $m = 1$. Thus $|E||P| = |G| = \dim(K[G]) = \sum_{i=1}^{|E|} \dim(Ae_i) = |E|q$ so that $\dim(Ae_i) = |P|$. $\square$

2.3 Classification of serial algebras

2.7 Lemma:
Let A be a K-algebra and $B \subseteq A$ a subalgebra such that $B + J(A)^2 = A$. Then $\forall r \geq 2 : B + J(A)^r = A$. In particular, if A is finite-dimensional, then $B = A$.

Proof. Fix elements $t_1, \ldots, t_k \in B$ such that their images form a generating set of $J(A)/J(A)^2$ as a K-vectorspace. Then inductively $\operatorname{span}_K \{ t_{i_1} \cdots t_{i_r} \mid 1 \leq i_1, \ldots, i_r \leq k \} = J(A)^r/J(A)^{r+1}$ for all $r \geq 1$. All those products are in B so that $B + J(A)^{r+1}$ contains $J(A)^r$ for all $r \geq 1$. Therefore

$$B + J(A)^{r+1} = B + J(A)^r = \ldots = B + J(A)^2$$

as claimed. If A is finite-dimensional, then there is some $r \gg 0$ with $J(A)^r = 0$. $\square$

2.8 Theorem (11.3.9, Serial algebras are Nakayama algebras)**:**
Let A be a finite-dimensional, indecomposable, non-simple, self-injective, serial K-algebra. Let $n := |\mathrm{Irr}(A)|$ be the number of isomorphism classes of simples and let q be the common length of the PIMs.
The following holds:

a.) The Ext-quiver of A is a (oriented) n-cycle, namely just the cycle π.

b.) Let I be the ideal of KQ spanned by all paths of length $\geq q$. If A is split basic, then $A \cong KQ/I$.

In particular, the isomorphism type of A is uniquely determined by n and q.

Proof. a. is clear because by the main theorem $Ext^1(S_i, S_j) \neq 0 \iff \exists s.e.s. : 0 \to S_j \to X \to S_i \to 0$ non-split $\iff e_j J(A)/J(A)^2 e_i \neq 0 \iff j = \pi(i)$.

b. Choose primitive idempotents $1 = e_1 + \cdots + e_n$ and elements $t_{ij} \in e_i J(A) e_j$ with $t_{\pi(i)i} \neq 0$. These induce an K-algebra homomorphism $\phi : KQ \to A$.
Since the t_{ij} span $J(A)/J(A)^2$ (remember that $e_i J(A)/J(A)^2 e_j$ is zero- or one-dimensional for all i, j), $\mathrm{im}(\phi)$ is a subalgebra with $\mathrm{im}(\phi) + J(A)^2 = A$ so that ϕ is surjective by the above lemma.
Since $J(A)^q = \sum_i J(A)^q e_i = 0$, the ideal I is contained in the kernel of ϕ. Since A is split basic, each of the n PIMs occurs exactly once in ${}_A A$ and their composition length equals their dimension (because split basic $\implies$ all simples are one-dimensional). Thus A is exactly nq-dimensional. KQ/I is also nq-dimensional. Therefore ϕ induces the desired isomorphism. $\square$

2.9: This already characterises all finite-dimensional, serial, self-injective, and *split* algebras up to Morita equivalence.

2.10 Proposition (11.3.10, upgrade to the non-split case)**:**
Let A be a finite-dimensional, self-injective, serial K-algebra.
If U is finite-dimensional and indecomposable, $S := soc(U)$, $T := U/rad(U)$. Then

$$\begin{cases} \mathrm{End}(S) & \overset{\cong}{\longleftarrow} & \mathrm{End}(U)/J(\mathrm{End}(U)) & \overset{\cong}{\longrightarrow} & \mathrm{End}(T) \\ f_{|soc(U)} & \leftarrow & f & \mapsto & \overline{f} \end{cases}$$

In particular, if A is indecomposable, then all simples have isomorphic skewfields as endomorphism algebras.

Proof. Socle and radical are characteristic submodules so that there are canonical morphism $\phi : \mathrm{End}(U) \to \mathrm{End}(S)$ and $\psi : \mathrm{End}(U) \to \mathrm{End}(T)$.
U is indecomposable so that $\mathrm{End}(U)$ is local. On the other hand, $\mathrm{End}(S)$ and $\mathrm{End}(T)$ are skew fields. Therefore the two morphisms must induce the claimed isomorphisms. □

2.11: By using valued quivers, one can tweak the previous theorem to characterise all non-split serial, self-injective algebras as well.