TITLE

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1 Some categorial flavour to algebraic notions

1.1 Definition (Projective and injective objects)**:**
$P \in Ob(\mathsf{A})$ is called projective iff for every epimorphism $B \twoheadrightarrow A$ and every morphism $P \to A$ there is a morphism $P \to B$ making the triangle commutative.
Dually $I \in Ob(\mathsf{A})$ is called injective iff for every monomorphism $A \hookrightarrow B$ and every morphism $A \to I$ there is a morphism $B \to I$ making the triangle commutative.

$$
\begin{tikzcd}
 & P \arrow[dl, dotted] \arrow[d] & \\
B \arrow[r, two heads] & A \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
 & I & \\
B \arrow[ur, dotted] & A \arrow[l, hook'] \arrow[u] & 0 \arrow[l]
\end{tikzcd}
$$

1.2: In both cases, the morphisms need not be unique and in many cases they aren't.

2 Some homological algebra

2.1 Definition (Chain complexes)**:**
Let A be an additive category. A chain complex (A_*, ∂) is a pair consisting of a graded object $A_* \in \mathsf{A}^{\mathbb{N}}$ and a morphism $\partial : A \to A$ of degree -1, i.e. $\partial_n : A_n \to A_{n-1}$, such that $\partial \circ \partial = 0$.
The category of chain complexes is denoted $Ch(\mathsf{A})$.

2.2 Definition (Homology)**:**
Let A be an abelian category and $A_* \in Ch(\mathsf{A})$ a chain complex. Then its homology is defined to be the graded object $H_n := \underbrace{\ker(\partial_n)}_{=:Z_n} / \underbrace{\mathrm{im}(\partial_{n+1})}_{=:B_n}$.

2.1 Mapping cone

2.3 Definition:
Let $f : (A_*, \partial^A) \to (B_*, \partial^B)$ be a chain-map. The mapping cone $C(f) = (C(f)_*, \partial^{C(f)})$ is the chain complex given by

$$C(f)_n := A_{n-1} \oplus B_n \quad \text{and} \quad \partial_n^{C(f)} := \begin{pmatrix} -\partial_{n-1}^A & 0 \\ -f_{n-1} & \partial_n^B \end{pmatrix}$$

2.4 Lemma (Mapping cones vs. quasi-isomorphisms)**:**
Let $f : (A_*, \partial^A) \to (B_*, \partial^B)$ be a chain-map.

a.) $0 \to B \overset{i}{\hookrightarrow} C(f) \xrightarrow{q} A[-1] \to 0$ is a short exact sequence of chain complexes.

b.) The induced long exact sequence in homology

$$\cdots \to H_{n+1}(B) \xrightarrow{i_*} H_{n+1}(C(f)) \xrightarrow{q_*} \underbrace{H_{n+1}(A[-1])}_{=H_n(A)} \xrightarrow{\delta} H_n(B) \to H_n(C(f)) \to \cdots$$

has f_* as connecting morphism δ.

c.) f quasi-isomorphism $\iff$ $C(f)$ is acylic.

d.) TFAE:

i.) $H_*(f) = 0$

ii.) $i_* : H_*(B) \to H_*(C(f))$ is mono.

iii.) $0 \to H_*(B) \xrightarrow{i_*} H(C(f)) \xrightarrow{q_*} H_{*-1}(A) \to 0$ is a short exact sequence.

iv.) $q_* : H_*(C(f)) \to H_{*-1}(A)$ is epi.

2.5 Lemma (Mapping cones vs. chain homotopy)**:**
Let $f : (A_*, \partial^A) \to (B_*, \partial^B)$ be a chain-map.

a.) f is a homotopy-equivalence $\iff$ $C(f)$ is contractible.

b.) TFAE:

i.) f is null-homotopic.

ii.) f factors through $A \hookrightarrow C(\mathrm{id}_A)$.

iii.) f factors through a contractible complex.

iv.) The short exact sequence $0 \to B \hookrightarrow C(f) \to A[-1] \to 0$ splits.

Proof. a. A map

$$H := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \begin{matrix} A_{n-1} \\ \oplus \\ B_n \end{matrix} \to \begin{matrix} A_n \\ \oplus \\ B_{n+1} \end{matrix}$$

is a homotopy $\mathrm{id}_{C(f)} \simeq 0$ iff $H\partial^{C(f)} + \partial^{C(f)}H = \mathrm{id}$, that is iff

$$-\begin{pmatrix} \alpha\partial + \beta f + \partial\alpha & -\beta\partial + \partial\beta \\ \gamma\partial + \delta f + f\alpha - \partial\gamma & -\delta\partial + f\beta - \partial\delta \end{pmatrix} = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{pmatrix}$$

i.e. iff $\beta : B \to A$ is a chain-map, $-\alpha$ is a homotopy $\mathrm{id} \simeq (-\beta)f$, δ is homotopy $\mathrm{id} \simeq f(-\beta)$ and γ is some map satisfying the last equation.
This already proves one direction: If $C(f) \simeq 0$, then $f(-\beta) \simeq \mathrm{id}$ and $(-\beta)f \simeq \mathrm{id}$ so that $A \simeq B$.

Conversely, if $f(-\beta) \simeq \mathrm{id}$ and $(-\beta)f \simeq \mathrm{id}$ via homotopies δ and $-\alpha$ respectively, then setting $\gamma := 0$ for the moment, we instead get a homotopy $\tilde{H}$ of 0 with $\psi := \begin{pmatrix} \mathrm{id} & 0 \\ \delta f + f\alpha & \mathrm{id} \end{pmatrix}$ which is obviously an isomorphism on the level of modules. ψ is in fact a chain map:

$$\partial^{C(f)}\psi - \psi\partial^{C(f)} = \begin{pmatrix} 0 & 0 \\ \partial\delta f + \partial f\alpha + \delta f\partial + f\alpha\partial & 0 \end{pmatrix}$$

This is zero because

$$\partial\delta f + \delta f\partial = \partial\delta f + \delta\partial f = (-\mathrm{id} + f\beta)f$$

and

$$\partial f\alpha + f\alpha\partial = f\partial\alpha + f\alpha\partial = f(\mathrm{id} - \beta f)$$

Therefore $H := \psi^{-1}\tilde{H}$ is a homotopy $0 \simeq \mathrm{id}_{C(f)}$ so that $C(f)$ is contractible as claimed.

b. i. $\Longleftarrow$ ii. $\Longleftarrow$ iii. is trivial. We show i. $\Longleftrightarrow$ iii.: f factorises over the inclusion $A \hookrightarrow C(\mathrm{id}_A), a \mapsto (0, a)$ iff there is a chain-map

$$(h_{n-1}, f_n) : \begin{matrix} A_{n-1} \\ \oplus \\ A_n \end{matrix} \to B_n$$

And for a family $(h_n : A_n \to B_{n+1})$ to induce such a chain map $C(\mathrm{id}_A) \to B$ is equivalent to $\partial^B h = -h\partial^A - f$, i.e. to h being a homotopy $f \simeq 0$.
i. $\Longleftrightarrow$ iv.: $B \hookrightarrow C(f)$ splits iff there is a chain map

$$(r, \mathrm{id}) : \begin{matrix} A_{n-1} \\ \oplus \\ B_n \end{matrix} \to B_n$$

And for a family $(r_n : A_n \to B_{n+1})$ to induce such a chain map $C(f) \to B$ is equivalent to $\partial r = -r\partial - f$, i.e. to r being a homotopy $f \simeq 0$. □

2.6 Lemma (Universal properties of cones)**:**
Let $f : A_* \to B_*$ be a chain map. Then

$$\mathrm{Hom}_{Ch}(X_*, Cone(f)) = \left\{ \begin{pmatrix} \gamma \\ h \end{pmatrix} \,\middle|\, X \xrightarrow{\gamma} A[-1], X \xrightarrow{h} B, f[1] \circ \gamma \overset{h}{\simeq} 0 \right\}$$

$$\mathrm{Hom}_{Ch}(Cone(f), Y_*) = \left\{ (h, \beta) \,\middle|\, A[-1] \xrightarrow{h} Y, B \xrightarrow{\beta} Y, \beta \circ f \overset{h}{\simeq} 0 \right\}$$

2.2 Replacing objects by projective / injective resolutions

2.7 Lemma:
Chain maps between projectives / acyclic complexes are unique up to homotopy:

a.) Homology: If $C_* \in Ch^-(\mathsf{A})$ is acyclic and $P_* \in Ch^-(Proj(\mathsf{A}))$ all morphisms $P_* \to C_*$ are null-homotopic.

b.) Cohomology: If $C^* \in Ch^+(\mathsf{A})$ is acyclic and $I^* \in Ch^+(Inj(\mathsf{A}))$ all morphisms $C^* \to I^*$ are null-homotopic.

Proof. Let $\alpha : P_* \to C_*$ be a chain map. Inductively we construct a homotopy $h : P_* \to C_*[1]$ to the zero map.

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "\alpha_2"] \arrow[dl, dotted, "h_2"'] & P_1 \arrow[r] \arrow[d, "\alpha_1"] \arrow[dl, dotted, "h_1"'] & P_0 \arrow[r] \arrow[d, "\alpha_0"] \arrow[dl, dotted, "h_0"'] & 0 \\
\cdots \arrow[r] & C_2 \arrow[r] & C_1 \arrow[r] & C_0 \arrow[r] & 0
\end{tikzcd}$$

We begin with setting $h_n := 0$ for all $n < 0$. First step is to construct h_0. Since C is exact, $C_1 \to C_0$ is epi so that α_0 lifts to some $h_0 : P_0 \to C_1$ by projectivity, so that $\partial_1 h_0 + \partial 0 = \alpha_0$ is satisfied.
If $h_0, \ldots, h_{n-1}$ are already known and a partial homotopy, then

$$\begin{aligned}
\partial_n \alpha_n &= \alpha_{n-1} \partial_n \\
&= (\partial_n h_{n-1} + h_{n-2} \partial_{n-1}) \partial_n \\
&= \partial_n h_{n-1} \partial_n
\end{aligned}$$

So that $\partial(\alpha_n - h_{n-1}\partial_n) = 0$. Therefore $\alpha_n - h_{n-1}\partial_n$ maps into $Z_n(C)$ which equals $B_n(C) = \mathrm{im}(\partial_{n+1})$ by exactness. By projectivity, we can find h_n such that

$$\alpha_n - h_{n-1}\partial_n = \partial_{n+1} h_n$$

is satisfied which proves the lemma. $\square$

2.8 Corollary (Fundamental lemma of homological algebra)**:**
"Objects can be replaced by their projective or injective resolutions"

a.) Homology: Assume that A has enough projectives and that a projective resolution has been fixed for every object.

Any $f : A \to B$ extends to a chain map between the augmented complexes

$$\begin{tikzcd} P_*(A) \arrow[r] \arrow[d, dashed, "\phi"] & A \arrow[r] \arrow[d, "f"] & 0 \\ P_*(B) \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

ϕ is unique up to homotopy.

In particular: $\mathsf{A} \xrightarrow{P_*} K^-(Proj(\mathsf{A}))$ is a well-defined functor with $H_0 \circ P_* \cong \mathrm{id}_{\mathsf{A}}$.

b.) Cohomology: Assume that A has enough injectives and that a injective resolution has been fixed for every object.

Any $f : A \to B$ extends to a chain map between the augmented complexes

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & I^*(A) \arrow[d, dashed, "\phi"] \\ 0 \arrow[r] & B \arrow[r] & I^*(B) \end{tikzcd}$$

ϕ is unique up to homotopy.

In particular: $\mathsf{A} \xrightarrow{I^*} K^+(Inj(\mathsf{A}))$ is a well-defined functor with $H_0 \circ I^* \cong \mathrm{id}_{\mathsf{A}}$.

As a consequence, projective and injective resolutions are unique up to homotopy equivalence.

Proof. Uniqueness up to homotopy follows from the lemma. We only have to show existence. Again, we work inductively:

$$\begin{tikzcd} \cdots \arrow[r] & P_2(A) \arrow[r] \arrow[d, dotted, "\phi_2"] & P_1(A) \arrow[r] \arrow[d, dotted, "\phi_1"] & P_0(A) \arrow[r] \arrow[d, dotted, "\phi_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\ \cdots \arrow[r] & P_2(B) \arrow[r] & P_1(B) \arrow[r] & P_0(B) \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

We set $P_{-1}(A) := A$, $\phi_{-1} := f$, and $P_{-1}(B) := B$ for notational convenience. If ϕ_{n-1} is already constructed, then

$$\begin{tikzcd} P_n(A) \arrow[r, "\partial"] & P_{n-1}(A) \arrow[d, "\phi_{n-1}"] & \\ & P_{n-1}(B) \arrow[r, "\partial"] & P_{n-2}(B) \end{tikzcd} = \begin{tikzcd} P_n(A) \arrow[r, "\partial"] & P_{n-1}(A) \arrow[r, "\partial"] & P_{n-2}(A) \arrow[d, "\phi_{n-2}"] \\ & & P_{n-2}(B) \end{tikzcd} = 0$$

Therefore $\phi_{n-1} \circ \partial_n : P_n(A) \to P_{n-1}(B)$ maps into $Z_{n-1}(P_*(B))$ which equals $B_{n-1}(P_*(B)) = \mathrm{im}(\partial_n)$ by exactness. By projectivity, we get a lift $\phi_n : P_n(A) \to P_n(B)$. □

2.9 Lemma (Horseshoe lemma)**:**
"P_* and I^* are exact"

a.) Homology: Every diagram

$$\begin{array}{ccccccccc}
 & & P_*(A) & & & & P_*(C) & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}$$

with exact first row and projective resolutions in the columns can be extended with some projective resolution $P_*(B) \to B \to 0$ to a diagram

$$\begin{array}{ccccccccc}
0 & \to & P_*(A) & \to & P_*(B) & \to & P_*(C) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

in which all rows are exact.

b.) Cohomology: Every diagram

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & I^*(A) & & & & I^*(C) & &
\end{array}$$

with exact first row and injective resolutions in the columns can be extended with some injective resolution $0 \to B \to I^*(B)$ to a diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I^*(A) & \to & I^*(B) & \to & I^*(C) & \to & 0
\end{array}$$

in which all rows are exact.

Proof. Set $A_{-1} := A$ and $A_n := P_n(A)$, $C_{-1} := C$ and $C_n := P_n(C)$ as well as $B_{-1} := B$. Then define $P_n(B) := B_n := A_n \oplus C_n$.
For the vertical maps consider

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r] \arrow[d] & A_n \oplus C_n \arrow[r] \arrow[d, dotted, "g"] & C_n \arrow[r] \arrow[d] \arrow[dll, dotted, "f"'] \arrow[dl, dotted, "h"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{n-2} \arrow[r] & B_{n-2} \arrow[r] & C_{n-2} \arrow[r] & 0
\end{tikzcd}$$

We define $g : A_n \oplus C_n \to B_{n-1}$ separately on the two components. Define $g : A_n \oplus 0 \to B_{n-1}$ to be the composition $A_n \to A_{n-1} \to B_{n-1}$.
The map $g : 0 \oplus C_n \to B_{n-1}$ we choose in two steps. First choose $h : C_n \to B_{n-1}$ to make the triangle on the right side commute. Then

$$\begin{array}{ccc}
0 \oplus C_n & & \\
\downarrow h & & \\
B_{n-1} & & \\
\downarrow & & \\
B_{n-2} & \to & C_{n-2}
\end{array}
\quad = \quad
\begin{array}{ccc}
0 \oplus C_n & \xrightarrow{=} & C_n \\
 & & \downarrow \\
 & & C_{n-1} \\
 & & \downarrow \\
 & & C_{n-2}
\end{array}
\quad = 0$$

d.h. $\partial h(C_n) \subseteq A_{n-2}$ because the $(n-2)$th row is exact and of course $\partial\partial h = 0$ so that $\partial h(C_n) \subseteq Z_{n-2}(A_*) = B_{n-2}(A_*)$ by exactness of A_*. Using projectivity once again, we can lift ∂h to $f : C_n \to A_{n-1}$ and finally define $g : 0 \oplus C_n \to B_{n-1}$ as $h - f$. Note that $\overline{g(c_n)} = \partial c_n$ still holds because $\operatorname{im}(f) \subseteq \ker(B_{n-1} \to C_{n-1})$.

This ensures $\partial g = 0$ which proves that the middle column is a(n incomplete) complex. We still have to show exactness. So let $b_{n-1} \in B_{n-1}$ with $\partial b_{n-1} = 0$.
Then its image $c_{n-1} = \overline{b_{n-1}}$ also satisfies $\partial c_{n-1} = 0$ so that a c_n exists with $c_{n-1} = \partial c_n$ by exactness of C_*. Then $\overline{b_{n-1} - g(0 \oplus c_n)} = c_{n-1} - \partial c_n = 0$ so that $b_{n-1} - g(0 \oplus c_n) \in \ker(B_{n-1} \to C_{n-1})$ which is $\operatorname{im}(A_{n-1} \to B_{n-1})$ by exactness of the $(n-1)$th row so that $b_{n-1} - g(0 \oplus c_n) = a_{n-1}$. Then $0 = 0 - 0 = \partial b_{n-1} - \partial g(0 \oplus c_n) = \partial a_{n-1}$ so that $a_{n-1} = \partial a_n = g(a_n \oplus 0)$. That shows $b_{n-1} = g(a_n \oplus c_n)$. □

2.3 Replacing complexes by projective / injective resolutions

2.10 Corollary:
"Complexes can be replaces by double complexes of projectives/injectives"

a.) Homology: For every $K_* \in Ch(\mathsf{A})$ exists a commutative double complex $P_{*,*}$ and maps $P_{n,*} \to K_n$ such that $P_{*,n} \to K_n \to 0$ is a projective resolution.

b.) Cohomology: For every $K^* \in Ch(\mathsf{A})$ exists a commutative double complex $I^{*,*}$ and maps $K^n \to I^{n,*} \to K^n$ such that $0 \to K^n \to I^{n,*} \to K^n \to 0$ is an injective resolution.

Proof. Consider the short exact sequences

$$0 \to Z_n \to K_n \to B_{n-1} \to 0$$

and choose projective resolutions $P''_{n,*} \to Z_n \to 0$ and $P'_{n,*} \to B_n \to 0$. Apply the horseshoe lemma to obtain a projective resolution $P_{n,*} \to K_n \to 0$ fitting in the exact sequence.
Now apply the fundamental lemma of homological algebra to get a chain-map $P'_{n,*} \to P''_{n-1,*}$ that extends the canonical map $B_{n-1} \to Z_{n-1}$ and let $P_{n,*} \to P_{n-1,*}$ be the composition $P_{n,*} \twoheadrightarrow P'_{n,*} \to P''_{n-1,*} \hookrightarrow P_{n-1,*}$. Since $P' \to P \to P''$ are short exact sequences, we obtain a commutative double complex in this way. □

2.11 Lemma:
"Projective / injective resolutions of complexes exist"

a.) Homology: For any bounded above complex $K_* \in Ch^-(\mathsf{A})$ there is a $P_* \in Ch^-(Proj(\mathsf{A}))$ and a quasi-isomorphism $P_* \to K_*$.

P_* can be chosen such that the quasi-isomorphism is termwise epi: $P_n \twoheadrightarrow K_n$.

b.) Cohomology: For any bounded below complex $K^* \in Ch^+(\mathsf{A})$ there is a $I^* \in Ch^+(Inj(\mathsf{A}))$ and a quasi-isomorphism $K^* \to I^*$.

I_* can be chosen such that the quasi-isomorphism is termwise mono: $K_n \hookrightarrow I_n$.

Proof. Take the total complex of $P_{*,*}$ in the previous statement. □

2.12 Lemma:
"Fundamental lemma of homological algebra upgraded to complexes"
Let $A_*, Q_* \in Ch^-(\mathsf{A}), P_* \in Ch^-(Proj(\mathsf{A}))$ be quasi-isomorphic, say $Q_* \underset{\sim}{\xrightarrow{\alpha}} A_*$ and $P_* \underset{\sim}{\xrightarrow{\beta}} A_*$.
If $Q_n \xrightarrow{\alpha} A_n$ is termwise epi, then there exists a chain-map $P_* \xrightarrow{\gamma} Q_*$ such that $\alpha \circ \gamma = \beta$.
If α is arbitrary, there exists a γ s.t. $\alpha \circ \gamma \simeq \beta$.
Any two chain-maps with $\alpha \circ \gamma_1 \simeq \beta \simeq \alpha \circ \gamma_2$ are homotopic.

2.13: If A is concentrated in a single degree, then $Q_* \to A_0 \to 0$ is just an acyclic complex and the statement reduces to the fundamental lemma of homological algebra. In this sense this statement is a generalisation of the fundamental lemma from A to $D^-(\mathsf{A})$.

Homological version, seems harder?? Assume that a partial chain map $\gamma_0, \ldots, \gamma_{n-1}$ is already constructed. We want to construct the missing arrow in the commutative diagram

$$\begin{tikzcd}
P_n \arrow[rr] \arrow[d, dotted] \arrow[ddd, bend right] & & P_{n-1} \arrow[d] \arrow[ddd, bend left] \\
Q_n \arrow[rr] \arrow[dr] \arrow[dd, two heads] & & Q_{n-1} \arrow[dd, two heads] \\
& F \arrow[ur] \arrow[dl] & \\
A_n \arrow[rr] & & A_{n-1}
\end{tikzcd}$$

We set

$$F := A_n \times_{A_{n-1}} Z(Q_{n-1}) = \{\ (a_n, q_{n-1}) \in A_n \times Q_{n-1} \mid \partial q_{n-1} = 0 \wedge \partial a_n = \alpha(q_{n-1})\ \}$$

First we prove that the map $Q \xrightarrow{(\alpha,\partial)} F$ is epi. Let $(a_n, q_{n-1}) \in F$ be arbitrary.
Then $q_{n-1} \in Z_{n-1}(Q)$ so that the homology class is well-defined. Then $\alpha_*[q_{n-1}]_{H_{n-1}(Q)} = [\alpha(q_{n-1})]_{H_{n-1}(A)} = [\partial a_n] = 0$. Since α is injective on homology, this means $[q_{n-1}] = 0$, i.e. $q_{n-1} = \partial q_n'$ for some $q_n' \in Q_n$.
Then $\partial a_n = \alpha(q_{n-1}) = \alpha\partial(q_n') = \partial\alpha(q_n')$ so that $a_n - \alpha(q_n') \in Z_n(A)$. Since α is surjective on homology, there is a $z_n \in Z_n(Q)$ such that $[\alpha(z_n)] = [a_n - \alpha(q_n')]$, i.e. there exists a a_{n+1} such that $\alpha(z_n) = a_n - \alpha(q_n') + \partial a_{n+1}$.

Now choose an preimage $q_{n+1} \in Q_{n+1}$ of a_{n+1} and set $q_n := z_n + q'_n - \partial q_{n+1}$. This is the preimage of (a_n, q_{n-1}):

$$\alpha(q_n) = \underbrace{\alpha(z_n) + \alpha(q'_n)}_{=a_n+b_n} - \alpha(\partial q_{n+1}) = a_n + b_n - \partial\alpha(q_{n+1}) = a_n$$

$$\partial(q_n) = \underbrace{\partial z_n}_{=0} + \underbrace{\partial q'_n}_{=q_{n-1}} + 0$$

Since we now know that $Q_n \to F$ is epi, we can lift the morphism $(\beta_n, \gamma_{n-1}\partial) : P_n \to F$ to a morphism $\gamma_n : P_n \to Q_n$. By construction it makes the diagram commute so that it is a partial chain map.

b. If α is not term-wise epi Meep

c. For uniqueness observe that $\alpha \circ (\gamma_1 - \gamma_2) \simeq 0$ so that there is a chain-map $P_*[-1] \xrightarrow{(h, \gamma_1 - \gamma_2)} Cone(\alpha), p \mapsto (h(p), (\gamma_1 - \gamma_2)(p))$. Since α is a quasi-isomorphism, $Cone(\alpha)$ is acyclic so that any such map is null homotopic. In particular $\gamma_1 - \gamma_2 = quotient \circ \widehat{\gamma} \simeq 0$. □

2.14 Corollary (Resolution functor)**:**
Fixing a projective resolution of every complex, $P_* : D^-(\mathsf{A}) \to K^-(Proj(\mathsf{A}))$ is a pseudo-inverse to the localisation functor.

Proof. We have to show $\mathrm{Hom}_K(P_*(A_*), P_*(B_*)) = \mathrm{Hom}_D(A_*, B_*)$.
Morphisms $A \underset{\gamma}{\to} B$ in D^- are roofs $A \to M \underset{\sim}{\leftarrow} B$.

$$\begin{tikzcd}
P_*(A) \arrow[d] \arrow[rr, dotted, bend left] & M & P_*(B) \arrow[d] \\
A \arrow[ur] \arrow[rr, dashed] & & B \arrow[ul]
\end{tikzcd}$$

Since $P_*(A)$ is termwise projective and $P_*(B) \underset{\sim}{\to} B \underset{\sim}{\to} M$ is a quasi-isomorphism, we can complete the triangle of $P_*(A) \to M$ and $M \underset{\sim}{\leftarrow} P_*(B)$ with a chain-map $\gamma : P_*(A) \to P_*(B)$ making the diagram commute up to homotopy. □

3 Derived functors I

3.1 (The Problem)**:** Given a right-exact functor $F : \mathsf{A} \to \mathsf{B}$, and exact sequence

$$0 \to A \to B \to C \to 0$$

gives a exact sequence

$$F(A) \to F(B) \to F(C) \to 0$$

We want to find functors L_nF and natural transformations δ_n (natural w.r.t. the short exact sequence) such that this sequence extends to a long exact sequence

$$\cdots \to L_2F(C) \xrightarrow{\delta_2} L_1F(A) \to L_1F(B) \to L_1F(C) \xrightarrow{\delta_1} \underbrace{F(A)}_{=L_0F(A)} \to \underbrace{F(B)}_{L_0F(B)} \to \underbrace{F(C)}_{=0} \to 0$$

And similarly for left-exact functors.
Of course, we want the universal solution to this problem.

3.2 Definition (δ-functors)**:**
A family $(F_n, \delta_n)_{n\in\mathbb{N}}$ of functors $\mathsf{A} \xrightarrow{F_n} \mathsf{B}$ and natural transformations $F_n(C) \xrightarrow{\delta_n} F_{n-1}(A)$ for every short exact sequence $0 \to A \to B \to C \to 0$ that transforms such short exact sequences into long exact sequences as above is called a (homological) δ-functor.
A morphism $F \xrightarrow{t} G$ of δ-functors is a family (t_n) of natural transformations $F_n \xrightarrow{t_n} G_n$ which induces morphisms between the long exact sequences, i.e. $t_{n-1}\delta_n^F = \delta_n^G t_n$.
Cohomological δ-Functors (F^n, d^n) are analogously defined.

3.3 Definition (Universal δ-functors)**:**
A δ-functor (F_n, δ_n) is called the universal δ-functor if for every (G_n, δ_n) and every $G_0 \xrightarrow{t_0} F_0$ there exists a unique morphism $G \xrightarrow{t} F$ of δ-functors extending t_0.

3.4 Definition (Derived functors)**:**
Let $F : \mathsf{A} \to \mathsf{B}$ be right-exact. A δ-functor (L_nF, δ_n) together with an isomorphism $L_0F \xrightarrow{\tau} F$ is called the left derived functor of F if (LF, τ) is a final object in the category of all δ-functor-with-isomorphisms.
It is in other words a representation of the functor $\{\,\delta\text{-functors}\,\} \to \mathsf{Set}, (G_n, \delta_n) \mapsto \mathrm{Nat}(G_0, F)$ such that the universal element $\tau \in Nat(F_0, F)$ is an iso.

Similarly right derived functor of a left exact F is defined as an initial object in the appropriate category of δ-functors with isomorphism $F \underset{\cong}{\xrightarrow{\tau}} F_0$, i.e. a representation of the functor $(G^n, d^n) \mapsto \mathrm{Nat}(F, G_0)$.

3.5: In particular: Derived functors are exactly those universal δ-functors that agree with F in degree 0.

3.6 Lemma (Recognising universal δ-functors)**:**
Let (F_n, δ_n) be a δ-functor.

a.) Homology: If $F_n(P) = 0$ for all $n \geq 1$ and all $P \in Proj(\mathsf{A})$, then F is a universal homological δ-functor.

b.) Cohomology: Assume $F^n(I) = 0$ for all $n \geq 1$ and all $I \in Inj(\mathsf{A})$, then F is a universal cohomological δ-functor.

Proof. Let $(\tilde{F}_n, \tilde{\delta}_n)$ be another δ-functor and assume that unique transformations $t_0, \dots, t_{n-1}$ have already been constructed. Fix $C \in \mathsf{A}$ and choose a short exact $0 \to K \xrightarrow{j} P \xrightarrow{q} C \to 0$ with P projective. Then

$$\begin{tikzcd}
\cdots \arrow[r] & \tilde{F}_n(P) \arrow[r] \arrow[d, dotted] & \tilde{F}_n(C) \arrow[r, "\tilde{\delta_n}"] \arrow[d, dotted] & \tilde{F}_{n-1}(K) \arrow[r] \arrow[d, "t_{n-1}"] & \tilde{F}_{n-1}(P) \arrow[r] \arrow[d, "t_{n-1}"] & \cdots \\
\cdots \arrow[r] & \underbrace{F_n(P)}_{=0} \arrow[r] & F_n(C) \arrow[r, "\delta_n"] & F_{n-1}(K) \arrow[r] & F_{n-1}(P) \arrow[r] & \cdots
\end{tikzcd}$$

It follows that $F_n(C) \xrightarrow[\cong]{\delta_n} \ker(F_{n-1}(j))$ and since t_{n-1} is natural, there is a unique $t_n : \tilde{F}_n(C) \to F_n(C)$ that makes the square commute. This t_n does not depend on the choice of K and P by Schanuel's lemma.

Naturality of t_n follows from a simple diagram chase using naturality of δ_n and $\tilde{\delta_n}$, naturality of t_{n-1} and that $F_n(A) \to F_{n-1}(K)$ is mono.

It remains to show that t_n commutes with the deltas for an arbitrary short exact $0 \to A \to B \to C \to 0$. This also follows from a simple diagram chase. $\square$

3.7 Theorem (Derived functors exist.)**:**
Let $F : \mathsf{A} \to \mathsf{B}$ be right / left exact functor.

a.) Homology:

- i.) If A has enough projectives, then F has a left derived functor.
- ii.) $L_i F(P) = 0$ for all projectives P and all $i > 0$.
- iii.) Deriving is a functor $L_i : \mathsf{Fun}_{\text{r.e.}}(\mathsf{A}, \mathsf{B}) \to \mathsf{Fun}_{\text{add}}(\mathsf{A}, \mathsf{B})$.

b.) Cohomology:

- i.) If A has enough injectives, then F has a right derived functor.
- ii.) $R^i F(I) = 0$ for all injectives I and all $i > 0$.
- iii.) Deriving is a functor $R^i : \mathsf{Fun}_{\text{l.e.}}(\mathsf{A}, \mathsf{B}) \to \mathsf{Fun}_{\text{add}}(\mathsf{A}, \mathsf{B})$.

Proof. Existence: Define $L_n F := H_n \circ P_*$. Note that this does not depend on the choice of the projective resolutions P_* because all choices are homotopy equivalent and homology forgets homotopy. Note that $L_i F(P) = 0$ for P projective and $i > 0$ because $0 \to P \xrightarrow{\text{id}} P \to 0$ is a projective resolution of P.

Horseshoe lemma implies that every short exact sequence

$$0 \to A \to B \to C \to 0$$

lifts to exact sequence up to homotopy $0 \to P_*(A) \to P_*(B) \to P_*(C) \to 0$ which implies a long exact sequence in homology. Therefore $LF = (L_i F, \delta_i)$ is a δ-functor. It extends P because $P_*(A) \to A \to 0$ is a projective resolution such that $H_0(P_*(A)) \cong A$ naturally.

We still have to show universality. Let $((\tilde{F}_n, \tilde{\delta}_n), \tilde{\tau})$ be another δ-functor extending F. The above lemma shows that there is a unique morphism of δ-functors $t : \tilde{F} \to F$ which extends $t_0 := \tau^{-1} \circ \tilde{\tau}$.

The lemma also proves that every natural transformation $F \to G$ between right exact functors extends to $LF \to LG$ since $L_i G(P) = 0$. □

3.1 Computing derived functors via acyclic resolutions

3.8 Definition (F-acyclic objects)**:**
An object $Q \in \mathsf{A}$ is called F-acyclic if

a.) Homology: $L_n F(Q) = 0$

b.) Cohomology: $R^n F(Q) = 0$

holds for all $n \geq 1$.

3.9: $Proj(A) \subseteq Acycl(F)$ for all F. For some F (like $\mathrm{Hom}(A, -)$ equality may hold), but depending on F, the class of acyclics may be bigger then the projectives (or injectives). For example all $Proj(A{-}\mathsf{Mod}) \subsetneq Flat(A{-}\mathsf{Mod}) \subseteq Acycl(M \otimes -)$.

3.10 Theorem:
Let $F : \mathsf{A} \to \mathsf{B}$ be right / left exact.

a.) Homology: Let $Q_* \to A \to 0$ be a resolution of A by F-acyclic objects. Then $L_n F(A) \cong H_n(F(Q_*))$ via a unique isomorphism.

More precisely: Choose a projective resolution $P_*(A) \to A \to 0$. Then the unique-up-to-homotopy chain map $P_*(A) \xrightarrow{\gamma} Q_*$ induces a unique isomorphism $L_n F(A) = H_n(F(P_*(A))) \xrightarrow{H_n(F\gamma)} H_n(F(Q_*))$.

3.11 Lemma:
The class of F-acyclics has the following properties:

a.) Homology: Assume A has enough projectives. Then

i.) Every $A \in \mathsf{A}$ has a covering $Q \twoheadrightarrow A \to 0$ for some acyclic Q.

ii.) It is closed under direct sums.

iii.) If in an exact sequence $0 \to A \to B \to C \to 0$ both B and C are acyclic, then A is two.

iv.) If in an exact sequence $0 \to A \to B \to C \to 0$ the object C is acyclic, then $0 \to FA \to FB \to FC \to 0$ is also exact.

Proof. a. Projectives are always acyclic.
b. follows because $L_n F$ is additive.
c. and d. follow from the long exact sequence. □

3.12 Lemma:
Let $Q_* \in Ch^{\pm}(Acyc)$ be a complex of F-acyclics. If Q_* is exact, then FQ_* is also exact.

Proof. Let K_n be the kernels / images of the boundary maps so that we get a diagram

$$\begin{tikzcd}
 & & K_2 \arrow[dr, hook] & & & & & \\
\cdots \arrow[r] & Q_3 \arrow[ur, two heads] \arrow[rr] & & Q_2 \arrow[rr] \arrow[dr, two heads] & & Q_1 \arrow[r, two heads] & Q_0 \arrow[r] & 0 \\
K_3 \arrow[ur, hook] & & & & K_1 \arrow[ur, hook] & & &
\end{tikzcd}$$

where the diagonals are short exact sequences. First observation: Per induction all K_n are acyclic, because the Q_n are.
The transformed sequence

$$\begin{tikzcd}
 & & FK_2 \arrow[dr] & & & & & \\
\cdots \arrow[r] & FQ_3 \arrow[ur, two heads] \arrow[rr] & & FQ_2 \arrow[rr] \arrow[dr, two heads] & & FQ_1 \arrow[r, two heads] & FQ_0 \arrow[r] & 0 \\
FK_3 \arrow[ur] & & & & FK_1 \arrow[ur] & & &
\end{tikzcd}$$

is exact iff the diagonals are exact again, i.e. if $FK_n \to FQ_n$ is mono. This follows from exactness of $K_n \hookrightarrow Q_n \twoheadrightarrow K_{n-1}$ and K_{n-1} being acyclic. $\square$

Proof of the main theorem. Let $P_* \to A$ be a projective resolution, $Q_* \to A$ an acylic resolution and $\gamma : P_* \to Q_*$ be a chain-map extending $A \xrightarrow{\mathrm{id}} A$ along those resolutions. γ is a quasi-isomorphism because both resolutions compute have homology $A[0]$.
Therefore $Cone(\gamma)$ is exact. Termwise it is direct sum of projective and F-acyclic objects and therefore an exact complex of F-acyclic objects. Thus $Cone(F\gamma) = F(Cone(\gamma))$ is also exact. Therefore $F\gamma$ is a quasi-isomorphism. $\square$

4 Examples

4.1 Example (Snake lemma)**:**
Taking kernels is a left-exact functor $\mathsf{A}^{\{*\to *\}} \to \mathsf{Ab}$. Its right derived functor is the cokernel in degree 1 and zero further up.
Dually taking cokernels is right-exact and its left derived functor is the kernel in degree 1 and zero everywhere else.
This is a manifestation of the snake lemma.

4.2 Example (Sheaf (co)homology):
Sheaf cohomology $H^*(X, \mathcal{F})$ is the right derived functor of the global section functor $\Gamma : Sh(X) \to \mathsf{Ab}$.

4.3 Example (DeRham cohomology):
$H^*_{\mathrm{dR}}(M)$ is Sheaf cohomology of the sheaf $\underline{\mathbb{R}}_M$ of locally constant functions $M \to \mathbb{R}$. This uses that

$$0 \to \underline{\mathbb{R}}_M \hookrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \to 0$$

is a resolution of $\underline{\mathbb{R}}_M$ by fine sheafs and that fine sheafs are Γ-acyclic.

4.4 Example (Singular cohomology):
$H^*_{\mathrm{sing}}(X; G)$ is sheaf cohomology of the sheaf $\underline{G}_X \in Sh(X)$ of locally constant G-valued functions if X is paracompact.

4.5 Example (Étale cohomology):
Étale cohomology is the Sheaf cohomology for sheafs on the étale site, i.e. the right derived functor of global sections $\mathsf{Sh}_{et}(X) \to \mathsf{Ab}$.

4.6 Example (Ext and Tor):
$\mathrm{Ext}^i_A(M, N)$ is right derived of $\mathrm{Hom}_A(M, -) : A{-}\mathsf{Mod} \to \mathsf{Ab}$ as well as left derived of $\mathrm{Hom}_A(-, N) : \mathsf{Mod}{-}A \to \mathsf{Ab}^{op}$.
$\mathrm{Tor}^A_i(M, N)$ is left derived of both $M \otimes_A - : A{-}\mathsf{Mod} \to \mathsf{Ab}$ and $- \otimes_A N : \mathsf{Mod}{-}A \to \mathsf{Ab}$.

4.7 Example (Group (co)homology):
$H_*(G, M)$ is the left derived functor of the functor of coinvariants $k \otimes_{kG} -$, i.e. it is $Tor^{kG}_*(k, M)$.
$H^*_k(G, -)$ is the right derived functor of the functor of fixed points $(-)^G = \mathrm{Hom}_{kG}(k, -)$, i.e. it is $Ext^*_{kG}(k, M)$.

4.8 Example (Hochschild (co)homology):
Let $A^e := A \otimes_k A^{op}$ be the enveloping algebra of the k-algebra A.
$HH_n(A, M) := Tor^{A^e}_n(A, M)$, i.e. it is the left derived functor of the functors of coinvariant $M/[A, M] = A \otimes_{A^e} M : (A, A){-}\mathsf{Bimod} \to \mathsf{Ab}$.
$HH^n(A, M) := Ext^N_{A^e}(A, M)$, i.e. the right derived functor of invariants $Z(M) := \mathrm{Hom}_{A^e}(A, M) : (A, A){-}\mathsf{Bimod} \to \mathsf{Ab}$.

4.9 Example (Lie-algebra (co)homology):
$H_n(\mathfrak{g}, M) := Tor^{U(\mathfrak{g})}_n(k, M)$, i.e. left derived of taking coinvariants.
$H^n(\mathfrak{g}, M) := Ext^n_{U(\mathfrak{g})}(k, M)$, i.e. right derived of taking invariants.

5 Derived functors II: Total derived functors

5.1 Definition:
Let $p^?_{\mathsf{A}} : K^?(\mathsf{A}) \to D^?(\mathsf{A})$ be the projection functor from the homotopy category onto the derived category.
The total left/right derived functor of $F : \mathsf{A} \to \mathsf{B}$ is the "best approximation" of $K^\pm(F) : K^\pm(\mathsf{A}) \to K^\pm(\mathsf{B})$ on the level of derived categories, i.e. it fits into the diagram

$$\begin{tikzcd} K^\pm(\mathsf{A}) \arrow[r, "K^\pm(F)"] \arrow[d, "p^\pm_{\mathsf{A}}"'] & K^\pm(\mathsf{B}) \arrow[d, "p^\pm_{\mathsf{B}}"] \\ D^\pm(\mathsf{A}) \arrow[r, dotted, "RF", "LF"'] & D^\pm(\mathsf{B}) \end{tikzcd}$$

5.2: In this situation LF / RF is also both the left and right Kan-extension of $Q_{\mathsf{B}} \circ K(F)$ along the localisation p_{A}. Concretely: LF fits into a diagram

$$\begin{tikzcd} K^-(\mathsf{A}) \arrow[rr, "p^-_{\mathsf{B}} \circ K(F)"] \arrow[dr, "p^\pm_{\mathsf{A}}"'] & \arrow[d, phantom, "\Uparrow f"] & D^-(\mathsf{B}) \\ & D^-(\mathsf{A}) \arrow[ur, bend right=60, "G"'] & \end{tikzcd} = \begin{tikzcd} K^-(\mathsf{A}) \arrow[rr, "p^\pm_{\mathsf{B}} \circ K(F)"] \arrow[dr, "p^-_{\mathsf{A}}"'] & \arrow[d, phantom, "\Uparrow \cong"] & D^-(\mathsf{B}) \\ & D^-(\mathsf{A}) \arrow[ur, dotted, "LF"] \arrow[ur, bend right=60, "G"'] \arrow[ur, phantom, bend right=30, "\exists!"'] & \end{tikzcd}$$

which is commutative up to natural isomorphism $p_{\mathsf{B}} \circ K(F) \underset{\cong}{\to} LF \circ p_{\mathsf{A}}$ such that for every other functor $D^-(\mathsf{A}) \xrightarrow{G} D^-(\mathsf{B})$ every natural transformation $G \circ p_{\mathsf{A}} \underset{\cong}{\xrightarrow{f}} p_{\mathsf{B}} \circ K(F)$ factors uniquely through this iso.
Therefore some authors *define* LF of *any* additive functor F as the right Kan extension $Ran_{p^-_{\mathsf{A}}}(p^-_{\mathsf{B}} \circ K(F))$ and RF as the left Kan extension $Lan_{p^+_{\mathsf{A}}}(p^+_{\mathsf{B}} \circ K(F))$. In this situation however, LF and RF do in general not extend F.

5.3 Theorem (Total derived functors exist)**:**
Total derived functors of right / left exact functors exist if A has enough projectives / injectives.

Proof. Choose a resolution functor $D^\pm(\mathsf{A}) \to K^\pm(\mathsf{A})$ and pre-compose with $p^\pm_{\mathsf{B}} \circ K^\pm(F)$. □

5.4: Note that we do not need projective / injective resolutions, F-acyclic resolutions are fine too because we have already proven that $F(P_*(A))$ is quasi-isomorphic to $F(Q_*)$ if Q_* is any resolution by F-acyclic objects.