Some stuff about orbital graphs

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0.1 Definition:
Let $G \leq Sym(\Omega)$ be a permutation group. Then an orbit of G on $\Omega \times \Omega$ is called an orbital of the action $G \curvearrowright \Omega$.
If $\Gamma \subseteq \Omega \times \Omega$ is an orbital, the directed graph with vertex set Ω and edge set Γ is called an orbital graph.

0.2 Definition:
Let $G \curvearrowright \Omega$ be a transitive action. A G_ω-orbit of this action is called a suborbit. The sizes of the suborbits are called the subdegrees of G.

1 Orbital graph, suborbits and double cosets

1.1 Theorem:
Let $G \curvearrowright \Omega$ be transitive, $\omega \in \Omega$ any element and $H := G_\omega$ its stabiliser.
There are inclusion-preserving bijections between the following sets

a.) G-invariant subsets $\Gamma \subseteq \Omega \times \Omega$.

b.) H-invariant subsets $\Delta \subseteq \Omega$

c.) Subsets $D \subseteq G$ invariant under left- and right-multiplication by H.

given in the following dictionary

$\Gamma \subseteq \Omega \times \Omega$	$\Delta \subseteq \Omega$	$D \subseteq G$
Γ	$\Gamma(\omega) := \{\, \alpha \mid (\alpha, \omega) \in \Gamma \,\}$	$\{\, y \in G \mid (\omega, {}^y\omega) \in \Gamma \,\}$
$\{\, ({}^g\alpha, {}^g\omega) \mid \alpha \in \Delta, g \in G \,\}$	Δ	$\{\, y \in G \mid {}^y\omega \in \Delta \,\}$
$\{\, ({}^{g_0}\omega, {}^{g_1}\omega) \mid Hg_0^{-1}g_1H \subseteq D \,\}$	${}^D\omega$	D

In particular the minimal non-empty elements of these posets, namely the orbitals, the suborbitals and the H-H-double cosets respectively, are mapped bijectively onto each other.

Moreover, these bijections translate the following properties:

$\Gamma \subseteq \Omega \times \Omega$	$\Delta \subseteq \Omega$	$D \subseteq G$
$\{ (\alpha, \alpha) \mid \alpha \in \Omega \}$	$\{ \omega \}$	H
Γ^{op}	Δ^*	D^{-1}
$\lvert\Gamma\rvert / \lvert\Omega\rvert$	$\lvert\Delta\rvert$	$\lvert D\rvert / \lvert H\rvert$
$\Gamma \circ \Gamma'$	$\Delta \circ_\omega \Delta'$	DD'

where

$$\Delta^* := \{ {}^{g^{-1}}\omega \mid {}^g\omega \in \Delta \}$$

$$\Delta \circ_\omega \Delta' := \{ \alpha \mid \exists g \in G, \beta \in \Delta' : {}^g\alpha \in \Delta \wedge {}^g\beta = \omega \}$$

Proof. □

1.2 Corollary:
Let $G \curvearrowright \Omega$ be transitive.

a.) The connected components of the orbital graph (Ω, Γ) with associated double coset HyH are exactly the U-orbits, where $U := \langle H, y \rangle$.

b.) (Ω, Γ) is connected iff $\langle H, y \rangle = G$.

c.) G acts primitively iff all non-trivial orbital graphs are connected.

Proof. Identify Ω with G/H and Γ with Γ_y as above. Set $U := \langle H, y \rangle$. Note that $U = H \cup HyH \cup HyHyH \cup \ldots$ because the order of y is finite.

a. Now xH and $x'H$ are connected iff there exists a sequence $x = x_0, x_1, \ldots, x_k = x'$ such that $(x_{i-1}H, x_iH) \in \Gamma_y$, i.e. $x_{i-1}^{-1}x_i \in HyH$.
In particular: If xH and $x'H$ are connected, then $x_{i-1}U = x_iU$ for all $i \in \{ 1, \ldots, k \}$. Therefore $xU = x_0U = x_kU = x'U$.
Conversely: If $xU = x'U$, then there exists an element $h_0yh_1 \cdots yh_k \in U$ with $h_i \in H$ such that $x' = x(h_0yh_1 \cdots yh_k)$. Now we can define $x_i := x \cdot (h_0yh_1 \cdots yh_i)$ for $i \in \{ 0, \ldots, k \}$ and have found a sequence connecting $xH = x_0H$ and $x'H$ in the orbital graph.

b. follows directly from a.

c. follows directly from the fact that G acts primitively on Ω iff H is a maximal subgroup.
□

1.3 Remark: This lemma allows for easy identification of at least one block system for the action of G on Ω, namely the connected components of (Ω, Γ). They coincide with the sets ${}^U\omega$.
Moreover: ${}^U\omega$ is the smallest possible block containing both ω and ${}^y\omega$.

2 Orbital graphs and representation theory

2.1 Definition:
Let Ω be a finite G-set. Then define $\mathbb{K}^\Omega$ is defined as the $\mathbb{K}$-vector space with basis Ω. This vector space is naturally a $\mathbb{K}G$-module by extending the action of G on the basis elements linearly to the whole space.
We also define $Perm(\Omega) \leq GL(\mathbb{K}^\Omega)$ to be the set of all permutation matrices acting on $\mathbb{K}^\Omega$.
The representation $G \to Perm(n)$ will be denoted by ρ_Ω.

2.2 Remark: We mainly introduce $Perm(\Omega)$ to have a clean notational distinction between a permutation and its associated permutation matrix.

2.3 Theorem:
$\mathrm{End}_{\mathbb{K}G}(\mathbb{K}^\Omega)$ has a natural $\mathbb{K}$-basis $\{\, X_\Gamma \mid \Gamma \subseteq \Omega^2 \text{ orbital} \,\}$ defined as

$$(X_\gamma)_{ij} := \begin{cases} 1 & \text{if } (i,j) \in \Gamma \\ 0 & \text{otherwise} \end{cases}$$

The structure constants w.r.t. this basis, i.e. the numbers d_{ij}^k such that

$$X_{\Gamma_i} \cdot X_{\Gamma_j} = \sum_k d_{ij}^k X_{\Gamma_k},$$

are given by $d_{ij}^k := |\{\, \beta \in \Omega \mid (\alpha,\beta) \in \Gamma_i \wedge (\beta,\gamma) \in \Gamma_j \,\}|$ where (α,γ) is any element of Γ_k.

2.4 Remark: In other words: X_Γ is the adjacency matrix of the orbital graph (Ω,Γ).
Note that the right hand side in the definition of d_{ij}^k really is independent of the choice of the element $(\alpha,\gamma) \in \Gamma_k$, because G acts transitively on Γ_k.
Also note that multiplication is connected to composition via

$$X_{\Gamma_i} \cdot X_{\Gamma_j} \in \mathrm{span}_{\mathbb{K}} \{\, X_\Gamma \mid \Gamma \subseteq \Gamma_i \circ \Gamma_j \,\}$$

Proof. Writing out the defining condition

$$X \in \mathrm{End}_{\mathbb{K}G}(\mathbb{K}^\Omega) \iff \forall g \in G : gXg^{-1} = X$$

in components shows that every $\mathbb{K}G$-linear endomorphism is a linear combination of the X_Γ. They are obviously linear independent and therefore a basis.
The structure constants similarly follow by writing out the definition of matrix multiplication in this case. □

2.5 Remark: $\mathrm{End}_{\mathbb{C}G}(\mathbb{K}^\Omega)$ is by definition the centraliser algebra of the subalgebra $\mathrm{span}_{\mathbb{K}}(\rho_\Omega(G)) \subseteq \mathbb{K}^{\Omega\times\Omega}$.
If $\mathbb{K}$ has characteristic zero, $\mathbb{K}G$ is semisimple and therefore all modules have the double centraliser property so that $C(\mathrm{End}_{\mathbb{K}G}(\mathbb{K}^\Omega)) = \mathrm{span}_{\mathbb{K}}(\rho_\Omega(G))$.

2.6 Definition:
The 2-closure of a permutation group $G \leq Sym(\Omega)$ is defined as the largest subgroup $\widehat{G} \subseteq Sym(\Omega)$ that has the same orbits as G on Ω^2, i.e.

$$\widehat{G} := \left\{ \pi \in Sym(\Omega) \;\middle|\; \forall \Gamma \in \Omega^2/G : \pi(\Gamma) = \Gamma \right\}$$

G is called 2-closed iff $G = \widehat{G}$ holds.

2.7 Lemma (2-closure in terms of endomorphism algebras)**:**
The 2-closure of G is the largest subgroup $H \leq Sym(\Omega)$ that still satisfies $End_{\mathbb{K}G}(\mathbb{K}^\Omega) = End_{\mathbb{K}H}(\mathbb{K}^\Omega)$, i.e.

$$\widehat{G} = Perm(\Omega) \cap C(\mathrm{End}_{\mathbb{K}G}(\mathbb{K}^\Omega)).$$

In particular, G is 2-closed if no permutation matrix outside G commutes with all endomorphisms of $\mathbb{K}^\Omega$.

Proof. Let $\widehat{G}$ be the 2-closure of G. By definition $\pi \in \widehat{G}$ if and only if $\pi X_\Gamma \pi^{-1} = X_\Gamma$ for all $\Gamma \in \Omega^2/G$. In other words π is in the 2-closure iff it is a permutation and an element of the centraliser of the endomorphism ring of the $\mathbb{K}G$-module $\mathbb{K}^\Omega$. □

2.8 Theorem (2-closure in terms of invariant subspaces)**:**
$\widehat{G} = \left\{ \gamma \in Perm(\Omega) \;\middle|\; \forall U \leq \mathbb{C}^\Omega : U \text{ } G\text{-invariant} \implies U \text{ } \gamma\text{-invariant} \right\}$

Proof. We consider the standard scalar product on $\mathbb{C}^\Omega$ defined by declaring Ω to be an orthonormal basis.
Then all permutation matrices are unitary. In particular, $\mathrm{span}_\mathbb{C}(\rho_\Omega(G))$ is closed under taking adjoints and its centraliser $\mathrm{End}_{\mathbb{C}G}(\mathbb{C}^\Omega)$ is also closed under taking adjoints. Both are therefore C^*-algebras. In particular, both are isomorphic to a direct product of matrix rings. It is a consequence of the spectral theorem that $\prod_i \mathbb{C}^{n_i \times n_i}$ is spanned by all the self-adjoint idempotents it contains.

Self-adjoint idempotent matrices correspond bijectively to subspaces by identifying U with the orthogonal projection p_U onto U. A subspace U is g-invariant if g centralises p_U.
Therefore

$$\mathrm{End}_{\mathbb{C}G}(\mathbb{C}^\Omega) = \mathrm{span}_\mathbb{C} \left\{ p_U \;\middle|\; U \leq \mathbb{C}^G \text{ } G\text{-invariant} \right\}$$

and

$$\widehat{G} = Perm(\Omega) \cap C(\mathrm{End}_{\mathbb{C}G}(\mathbb{C}^\Omega)) = Perm(\Omega) \cap \bigcap_{\substack{U \leq \mathbb{C}^\Omega \\ G\text{-invariant}}} C(p_U)$$

which proves the claim. □

2.9 Definition:
A permutation group $G \leq Sym(\Omega)$ is

- reconstructible from $\mathcal{X} \subseteq \mathrm{End}_{\mathbb{K}G}(\mathbb{K}^\Omega)$ if $G = Perm(\Omega) \cap \bigcap_{X\in\mathcal{X}} C(X)$,
- orbital-graph-reconstructible if G is reconstructible from $\{ X_\Gamma \mid \Gamma \in \Omega^2/G \}$,
- strongly orbital-graph-reconstructible from $\Gamma \in \Omega^2/G$ iff it is reconstructible from X_Γ alone,
- absolutely orbital-graph-reconstructible iff it is strongly orbital-graph-reconstrucible from any non-diagonal $\Gamma \in \Omega^2$.
- subspace-reconstructible from a set $\mathcal{U}$ of G-invariant subspaces of $\mathbb{K}^\Omega$ if G is reconstructible from $\{ p_U \mid U \in \mathcal{U} \}$.
- strongly subspace-reconstructible from $U \leq \mathbb{K}^\Omega$ if G is reconstructible from p_U alone,
- absolutely subspace-reconstructible over $\mathbb{K}$ if G is strongly subspace-reconstructible from any p_U where $U \leq \mathbb{K}^G$ is a minimal, non-zero G-invariant subspace which is not $\mathrm{span}_{\mathbb{K}} \{ (1, 1, \ldots, 1) \}$.

2.10 Corollary:
$G \leq Sym(\Omega)$ is 2-closed iff it is orbital-graph reconstructible iff it is subspace-reconstructible over $\mathbb{C}$.

Proof. The first equivalence follows from the fact that X_Γ is a basis of $\mathrm{End}_{\mathbb{C}G}(\mathbb{C}^\Omega)$. The second follows from theorem 2.8. □

2.11 Example:
A regular permutation group is always 2-closed.
This is because a regular G-set is isomorphic to G itself endowed with left multiplication. The orbitals of this action are given by $\Gamma_h := \{ (x, y) \in G^2 \mid x^{-1}y = h \}$ for $h \in G$ and one can readily verify that the only permutations fixing all the orbitals are the left multiplication maps themselves.

2.12 Lemma:
Let Γ be an orbital. Then G is strongly orbital-graph-reconstructible from Γ iff if strongly orbital-graph-reconstructible from Γ^{op} iff it is subspace-reconstrucible from

$$\{ \mathrm{Eig}_\lambda(\mathfrak{Re}(X_\Gamma)), \mathrm{Eig}_\lambda(\mathfrak{Im}(X_\Gamma)) \mid \lambda \in \mathbb{R} \} .$$

Proof. Permutation matrices are unitary. Therefore $g \in Perm(n)$ centralises X iff it centralises X^*. The adjoint of X_Γ is exactly $X_{\Gamma^{op}}$ as we have previously observed.
$\mathfrak{Re}(X) = \frac{1}{2}(X + X^*)$ and $\mathfrak{Im}(X) = \frac{1}{2i}(X - X^*)$ are self-adjoint matrices with $X = \mathfrak{Re}(X) + i\,\mathfrak{Im}(X)$ and for a self-adjoint matrices Y the spectral theorem shows

$$Y = \sum_{\lambda \in \mathbb{R}} \lambda e_\lambda$$

where $e_\lambda = p_{\mathrm{Eig}_\lambda(Y)}$ is the orthogonal projection onto the λ-eigenspace. Moreover e_λ is a polynomial of Y by Lagrange-interpolation.
Therefore if $g \in GL(\mathbb{K}^\Omega)$ commutes with Y it must commute with all e_λ and vice versa. Thus

$$C(X_\Gamma) = C(X_\Gamma, X_\Gamma^*) = C(\mathfrak{Re}(X_\Gamma), \mathfrak{Im}(X_\Gamma)) = \bigcap_\lambda C(p_{\mathrm{Eig}_\lambda(\mathfrak{Re}(X_\Gamma))}) \cap C(p_{\mathrm{Eig}_\lambda(\mathfrak{Im}(X_\Gamma))})$$

which proves the lemma. □

2.13 Remark: The concept of subspace reconstructibility also makes sense if we replace $Perm(\Omega)$ by some other finite subgroup, for example the subgroup of monomial matrices.