Homework and Tutorial

Assignments

- 4.14, 4.25, 4.31, 4.33, 4.35

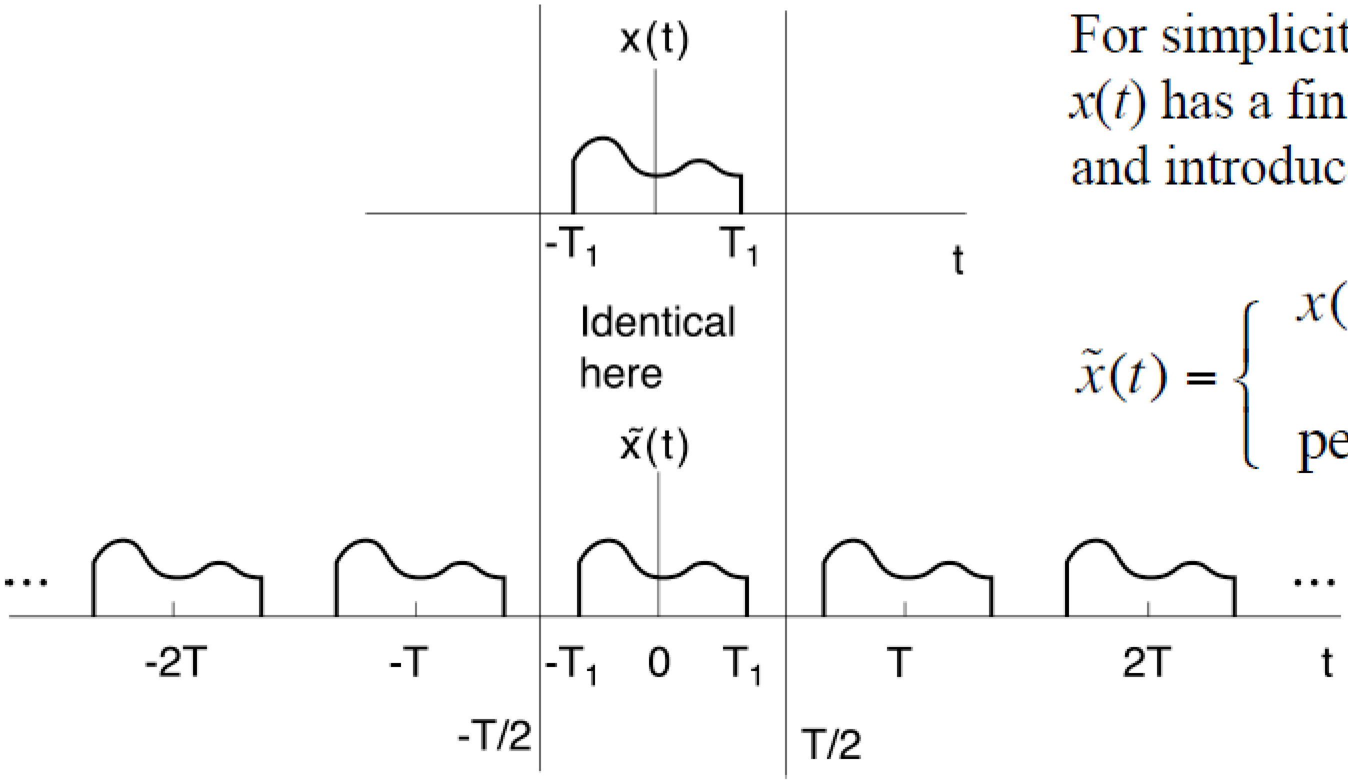
Tutorial problems

- Basic Problems wish Answers 4.10, 4.16
- Basic Problems 4.26, 4.30

Chapter 4
The Continuous-Time Fourier Transform

(cont.)

So, on the derivation of FT ...

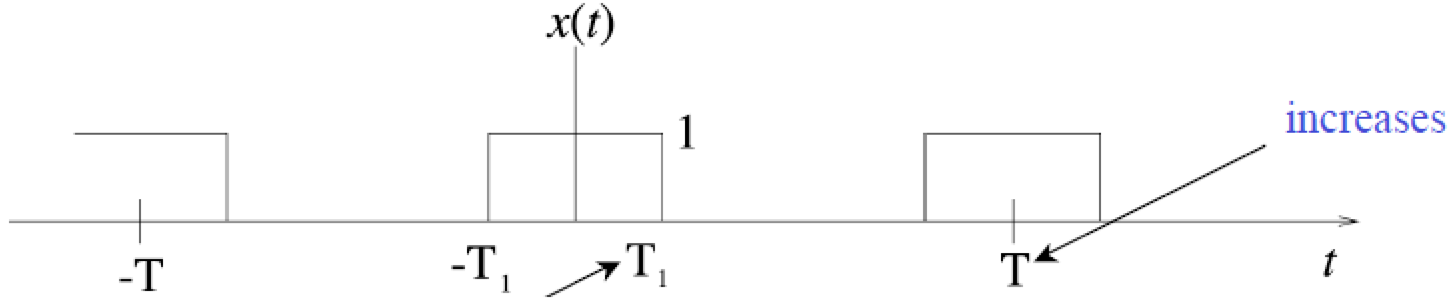


For simplicity, assume $x(t)$ has a finite duration, and introduce a periodic $\tilde{x}(t)$

$$\tilde{x}(t) = \begin{cases} x(t) & \frac{-T}{2} < t < \frac{T}{2} \\ \text{periodic} & |t| > \frac{T}{2} \end{cases}$$

$$\text{As } T \to \infty, \qquad x(t) = \tilde{x}(t) \ \text{ for all } t$$

Review
Motivating Examples: Square wave



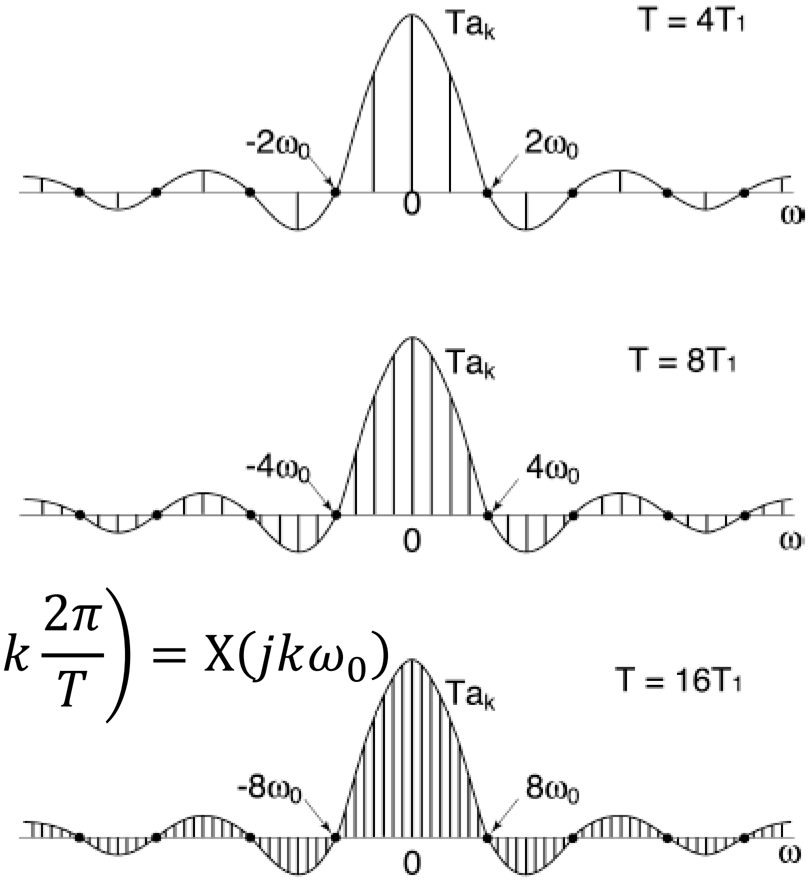
$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$$

$\Downarrow$

$$Ta_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0}$$

(independent of T)

$$Let\ X(j\omega) = \frac{2sin\omega T_1}{\omega}, \mathrm{Ta_k} = \mathrm{X}\left(jk\frac{2\pi}{T}\right) = \mathrm{X}(jk\omega_0)$$



$$(\omega_0 = \frac{2\pi}{T})$$

Become denser in ω as T increases

The CT Fourier Transform Pair

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$ — FT

Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$$ — Inverse FT

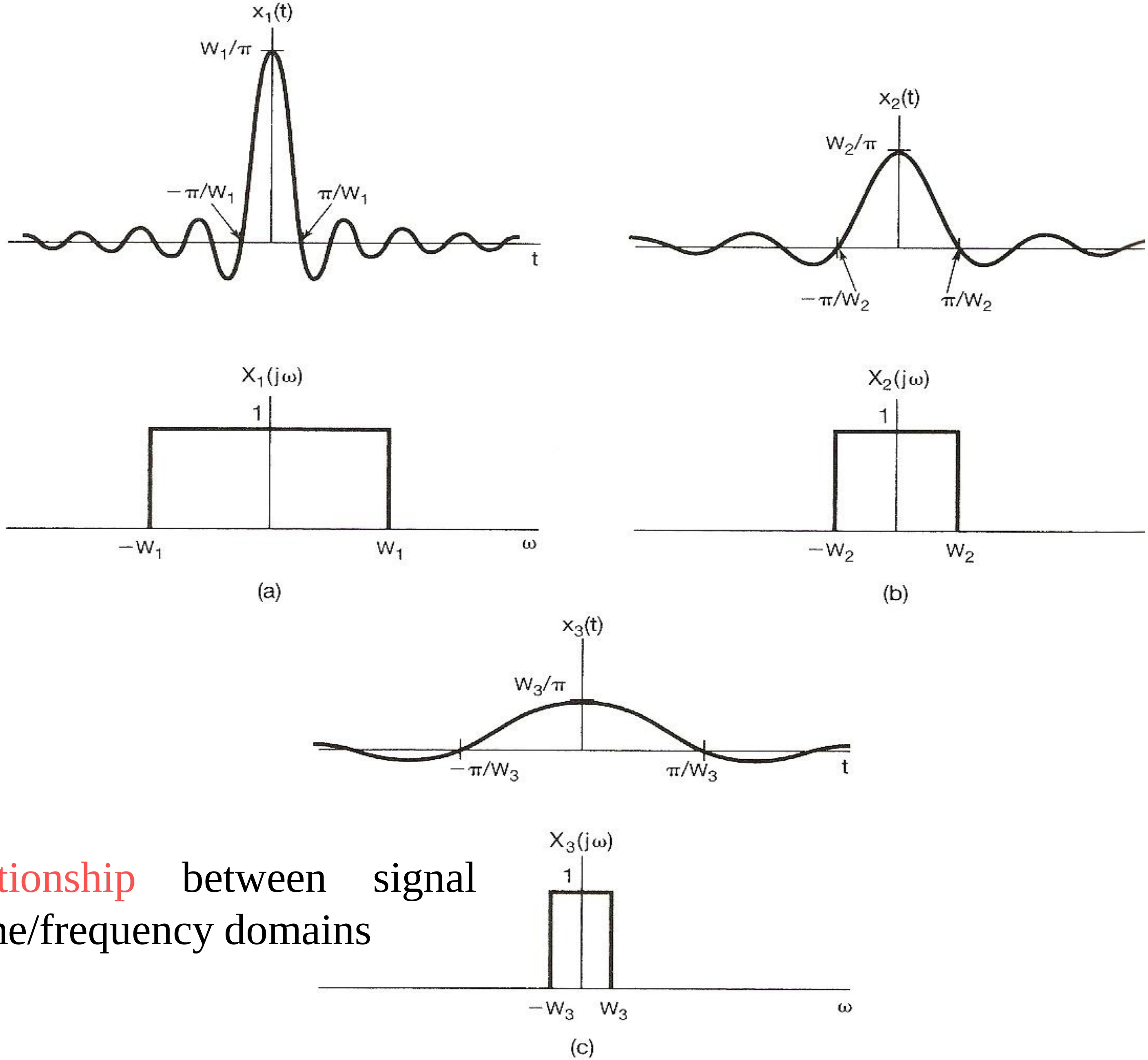
Inverse Fourier Transform

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

$$\mathcal{F}(x(t)) = X(j\omega)$$

$$x(t) = \mathcal{F}^{-1}(X(j\omega))$$

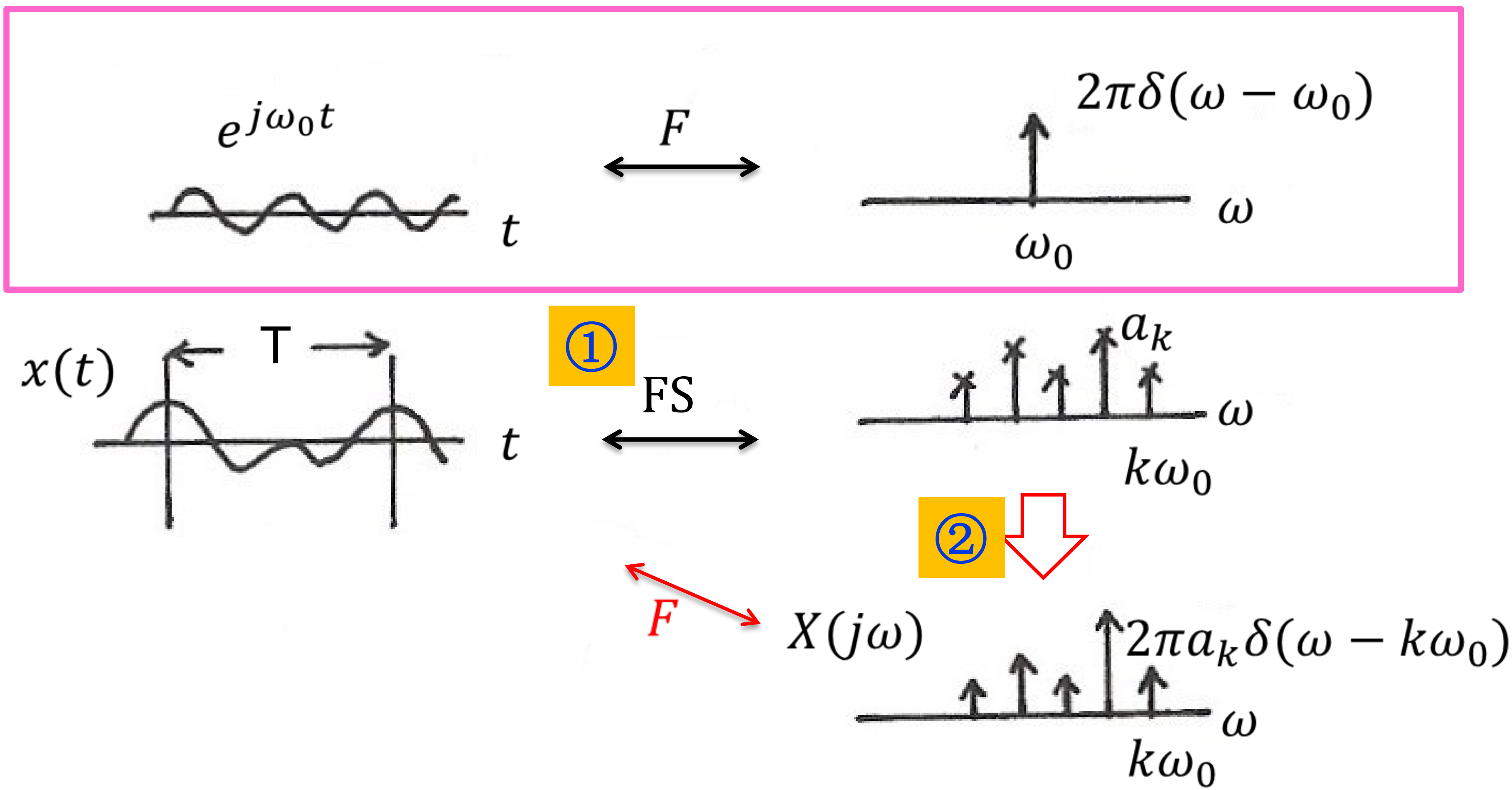
Review



Inverse relationship between signal "width" in time/frequency domains

Figure 4.11 Fourier transform pair of Figure 4.9 for several different values of W.

Fourier Transform for Periodic Signals – Unified Framework



Outline

- Interesting properties of CTFT
  - Real signal
  - Convolution property – LTI systems
  - Multiplication property – Modulation & Sampling

- Linear-constant-coefficient differential equation of LTI systems

$x(t) \longleftrightarrow X(j\omega)$

CTFT Properties (cont.)

- Time reversal $x(-t) \longleftrightarrow X(-j\omega)$

- Conjugate Symmetry $x(t)$ real $\longleftrightarrow X(-j\omega) = X^*(j\omega)$

$|X(-j\omega)| = |X(j\omega)|$ *Even* Or $\mathrm{Re}\{X(-j\omega)\} = \mathrm{Re}\{X(j\omega)\}$ *Even*

$\angle X(-j\omega) = -\angle X(j\omega)$ *Odd* $\mathrm{Im}\{X(-j\omega)\} = -\mathrm{Im}\{X(j\omega)\}$ *Odd*

$\Downarrow$

a) $x(t)$ real and even $x(t) = x(-t) = x^*(t)$

$\Rightarrow X(j\omega) = X(-j\omega) = X^*(j\omega)$ — Real & even

b) $x(t)$ real and odd $x(t) = -x(-t) = x^*(t)$

$\Rightarrow X(j\omega) = -X(-j\omega) = -X^*(j\omega)$ — Purely imaginary & odd

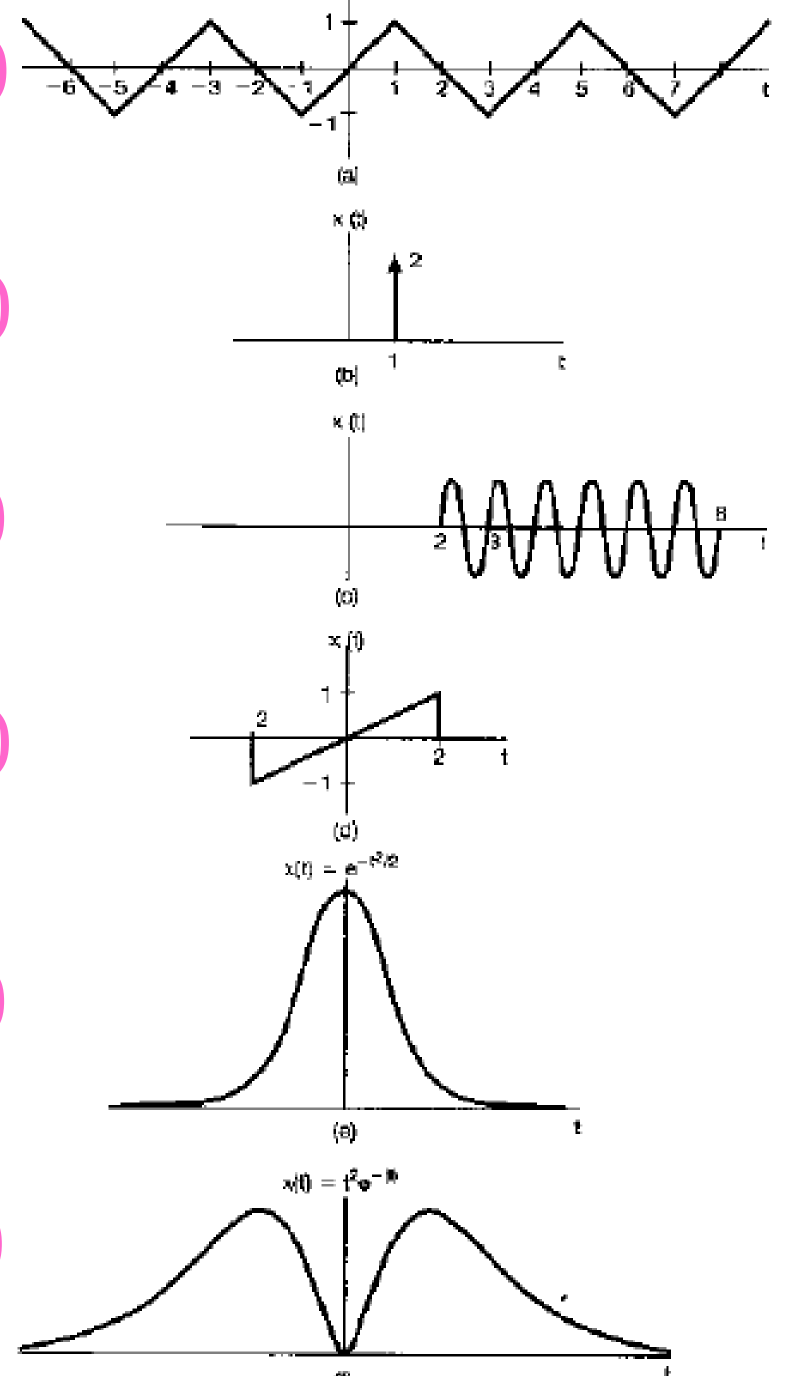
c) $X(j\omega) = \mathrm{Re}\{X(j\omega)\} + j\,\mathrm{Im}\{X(j\omega)\}$

For real $x(t) = Ev\{x(t)\} + Od\{x(t)\}$

Problem 4.24 (a)

- Determine which, if any, of the real signals in (a)-(f) have Fourier transforms that satisfy each of the following condition:
  - $\mathrm{Re}\{X(j\omega)\}=0$
  - $\mathrm{Im}\{X(j\omega)\}=0$
  - There exists a real a such that $e^{ja\omega}X(j\omega)$ is real
  - $\int_{-\infty}^{\infty} X(j\omega)d\omega=0$
  - $\int_{-\infty}^{\infty} \omega X(j\omega)d\omega=0$
  - $X(j\omega)$ is periodic

(a) (b) (c) (d) (e) (f)



Impulse & Frequency Responses

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} H(jk\omega_0) a_k e^{jk\omega_0 t}$$

$$\boxed{H(j\omega) = \int_{-\infty}^{+\infty} h(t) e^{-j\omega t} dt}$$

$$\boxed{\text{Impulse response} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \text{Frequency response}}$$

Remark:
Not every LTI system has frequency response; but according to Dirichlet conditions, most of stable LTI systems have.

Convolution Property

$$\boxed{\begin{array}{c} y(t) = h(t) * x(t) \longleftrightarrow Y(j\omega) = H(j\omega)X(j\omega) \\ \text{where} \quad h(t) \longleftrightarrow H(j\omega) \quad x(t) \longleftrightarrow X(j\omega) \end{array}}$$

Basically a consequence of the eigenfunction property

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega \;\Longrightarrow\; x(t) = \int_{-\infty}^{+\infty} \underbrace{\left(\frac{1}{2\pi}X(j\omega)d\omega\right)}_{\text{coefficient}} e^{j\omega t}$$

$$a\,e^{j\omega t} \xrightarrow{\quad} \boxed{h(t)} \xrightarrow{\quad} H(j\omega)a\,e^{j\omega t}$$

$$\Downarrow \text{ superposition}$$

$$y(t) = \int_{-\infty}^{+\infty} \underbrace{\left(H(j\omega)\cdot\frac{1}{2\pi}X(j\omega)d\omega\right)}_{\text{New coefficient}} e^{j\omega t}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \underbrace{H(j\omega)X(j\omega)}_{Y(j\omega)}e^{j\omega t}\,d\omega$$

Frequency Response Examples

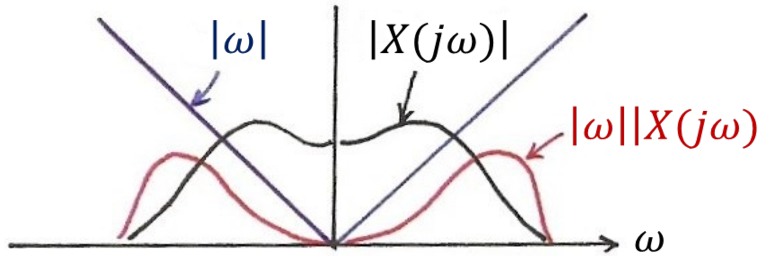
Example 4.16 A differentiator

$$y(t) = \frac{dx(t)}{dt} \qquad \text{— an LTI system}$$

From differentiation property $\Rightarrow$ $\frac{d}{dt} \xleftrightarrow{FT} j\omega$

$$\Downarrow$$

$$H(j\omega) = j\omega$$



1) Amplifies high frequencies (enhances sharp edges)

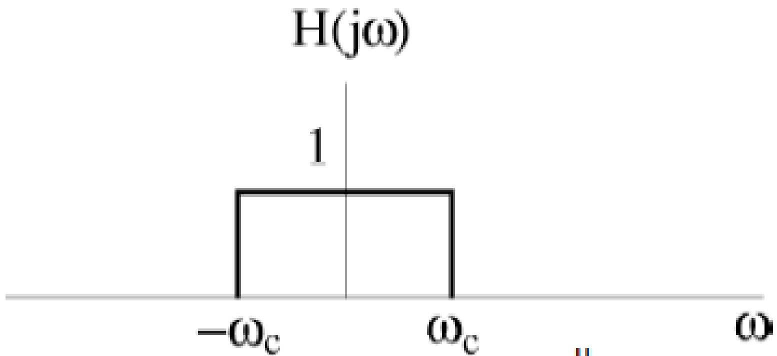
2) $+\pi/2$ phase shift ($j = e^{j\pi/2}$)

Larger at high ω_o phase shift

$$\frac{d}{dt}\sin\omega_0 t = \omega_0 \cos\omega_0 t = \omega_0 \sin(\omega_0 t + \pi/2)$$

$$\frac{d}{dt}\cos\omega_0 t = -\omega_0 \sin\omega_0 t = \omega_0 \cos(\omega_0 t + \pi/2)$$

Example 4.18 : Impulse Response of an *Ideal* Lowpass Filter



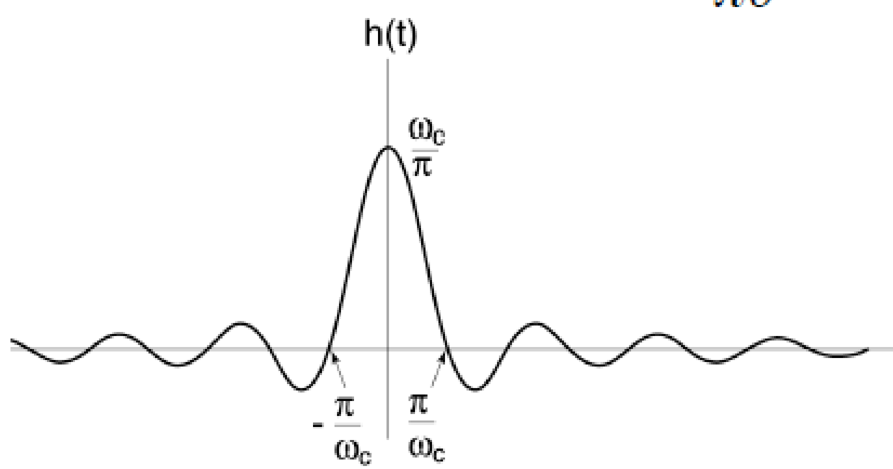
$$\Downarrow$$

$$\boxed{h(t) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t}\,d\omega = \frac{\sin\omega_c t}{\pi t}} = \frac{\omega_c}{\pi}\operatorname{sin}c\left(\frac{\omega_c t}{\pi}\right)$$

Questions:

1) Is this a causal system?
2) What is $h(0)$?

$$\sin c(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$$

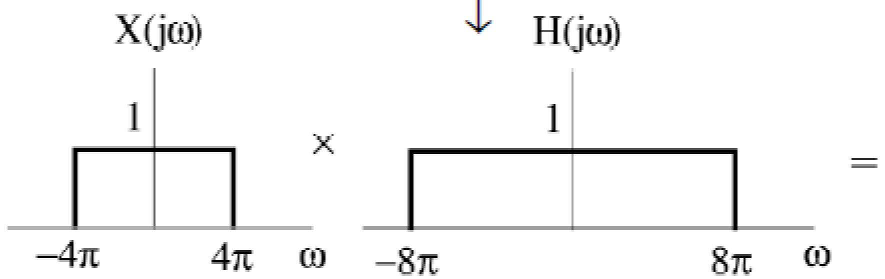


Convolution Calculation Examples

Example 4.20

$$\underbrace{\frac{\sin 4\pi t}{\pi t}}_{x(t)} * \underbrace{\frac{\sin 8\pi t}{\pi t}}_{h(t)} = ?$$

$$\updownarrow$$



Example 4.19

$$h(t) = e^{-t}u(t) \;,\; x(t) = e^{-2t}u(t)$$

$$y(t) = h(t) * x(t) = ?$$

$$\Downarrow$$

$$Y(j\omega) = H(j\omega)X(j\omega) = \frac{1}{(1+j\omega)} \cdot \frac{1}{(2+j\omega)}$$

$$\Downarrow \boxed{\text{Partial fraction expansion}}$$

$$Y(j\omega) = \frac{1}{1+j\omega} - \frac{1}{2+j\omega}$$

$a=1$ $a=2$

$$\Downarrow \text{ inverse } FT$$

$$y(t) = [e^{-t} - e^{-2t}]u(t)$$

Multiplication Property

Since *FT* is highly symmetric,

$$x(t) = \mathfrak{F}^{-1}\{X(j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} d\omega, \quad X(j\omega) = \mathfrak{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$$

thus if $\quad x(t) * y(t) \longleftrightarrow X(j\omega)\cdot Y(j\omega)$

then the other way around is also true

$$\boxed{\begin{aligned} x(t)\cdot y(t) &\longleftrightarrow \frac{1}{2\pi} X(j\omega) * Y(j\omega) \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta) Y(j(\omega-\theta))\, d\theta \end{aligned}}$$

$\frac{1}{2\pi}$ — A consequence of *Duality*

Example: Frequency Shifting

$$e^{j\omega_0 t}x(t) \xleftrightarrow{F} X\left(j\left(\omega-\omega_0\right)\right)$$

$$e^{j\omega_0 t}\cdot x(t) \leftrightarrow \frac{1}{2\pi}\left[2\pi\delta(\omega-\omega_0) * X(j\omega)\right]$$
$$= X(j(\omega-\omega_0))$$

① ② ③

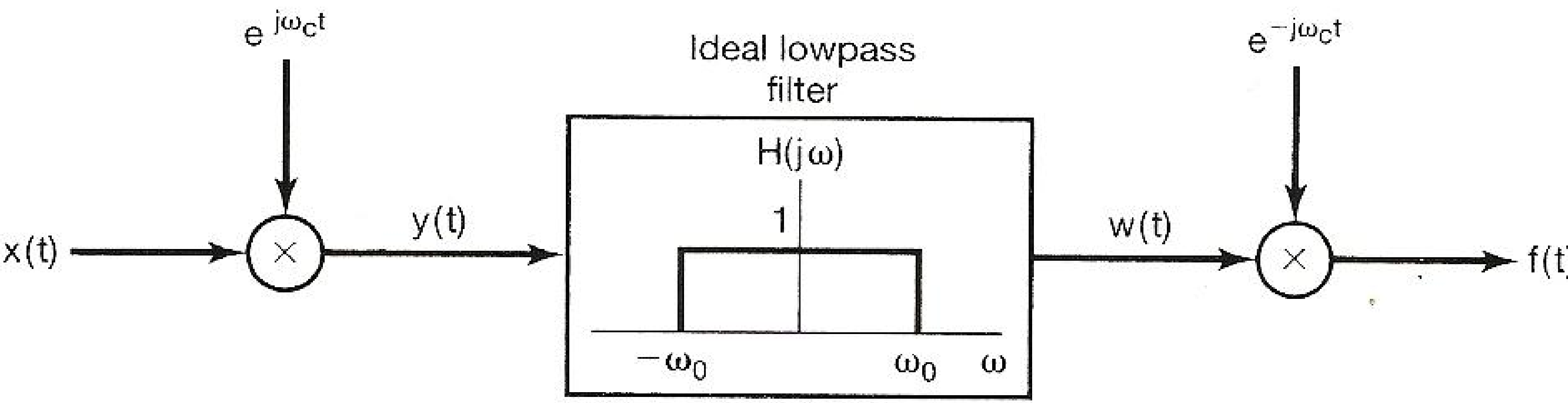


Figure 4.26 Implementation of a bandpass filter using amplitude modulation with a complex exponential carrier.

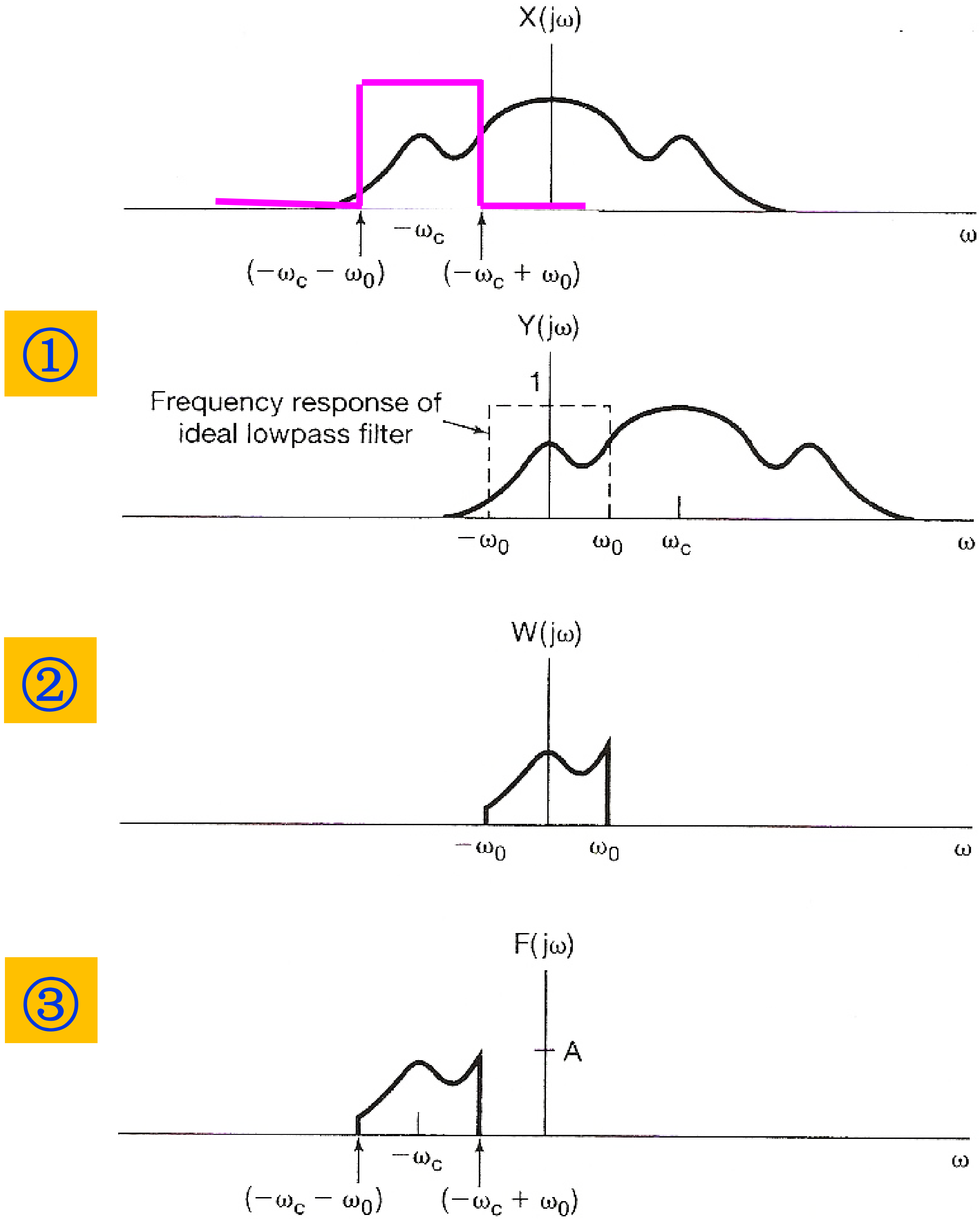


Figure 4.27 Spectra of the signals in the system of Figure 4.26.

Example: Amplitude Modulation

Example 4.21

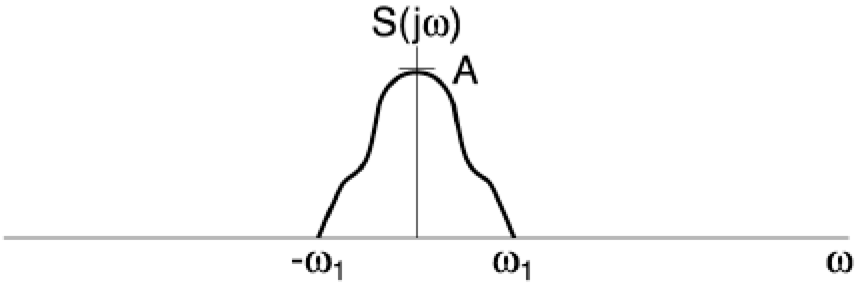
$$r(t) = s(t) \cdot p(t) \longleftrightarrow R(j\omega) = \frac{1}{2\pi}\left[S(j\omega) * P(j\omega)\right]$$

For $p(t) = \cos\omega_0 t \longleftrightarrow P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$

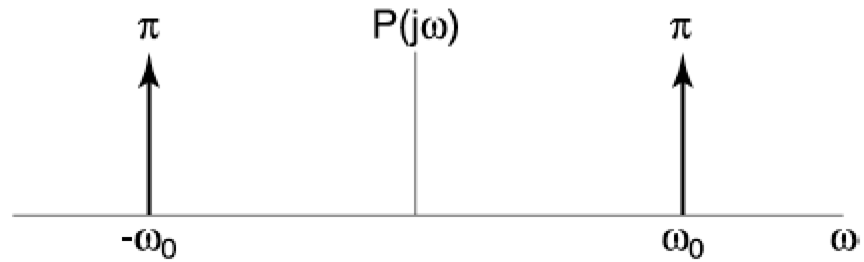
$$R(j\omega) = \frac{1}{2}S(j(\omega - \omega_0)) + \frac{1}{2}S(j(\omega + \omega_0))$$

(cont.)

ω_1 : bandwidth

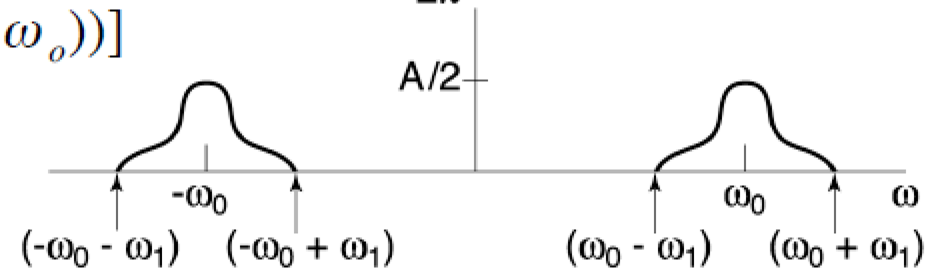


$r(t) = s(t) \cdot \cos(\omega_o t)$
Amplitude modulation (*AM*)



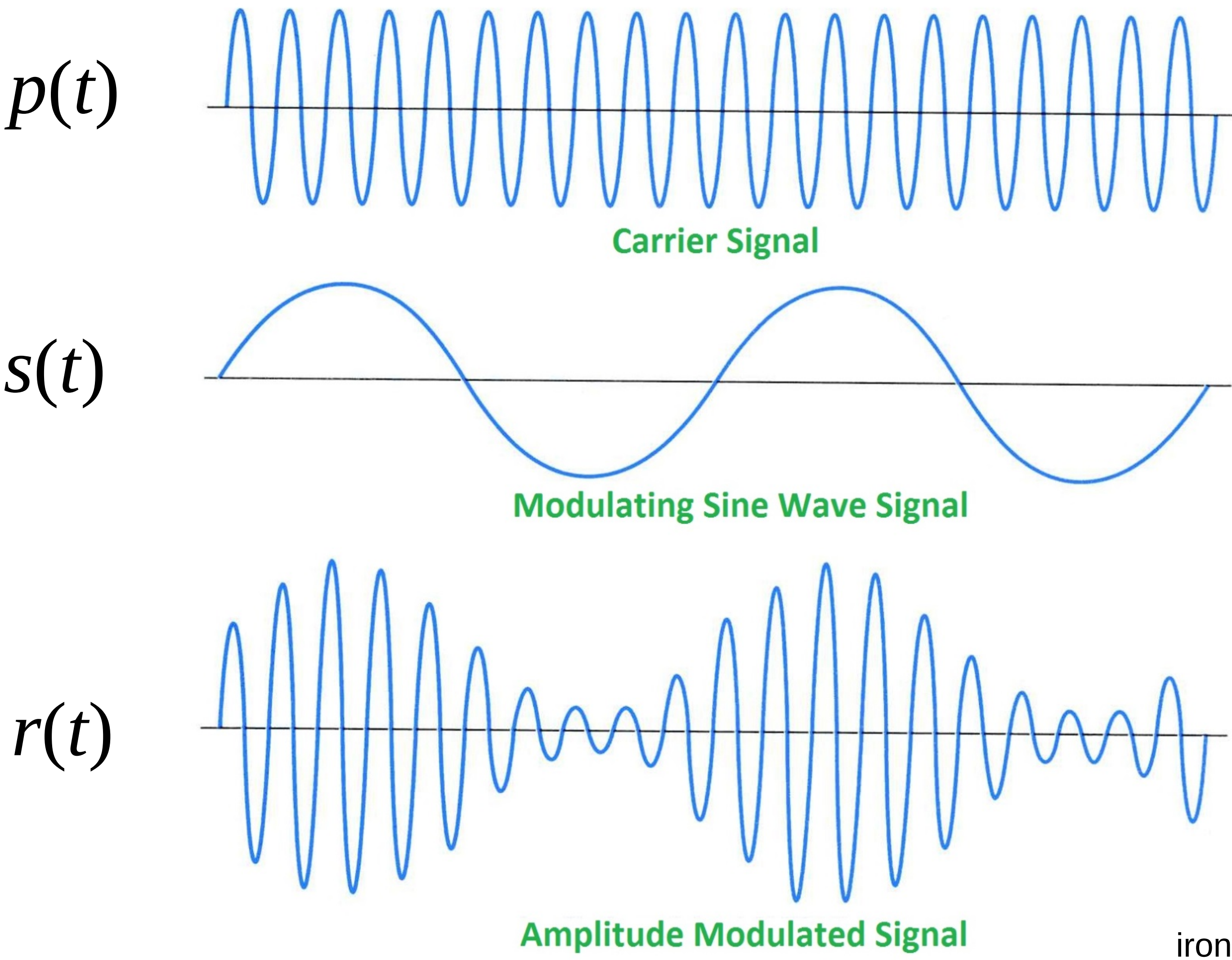
$$R(j\omega) = \frac{1}{2}[S(j(\omega - \omega_o)) + S(j(\omega + \omega_o))]$$

$$R(j\omega) = \frac{1}{2\pi}[S(j\omega) * P(j\omega)]$$



Drawn assume
$\omega_o - \omega_1 > 0$
i.e. $\omega_o > \omega_1$

(cont.)



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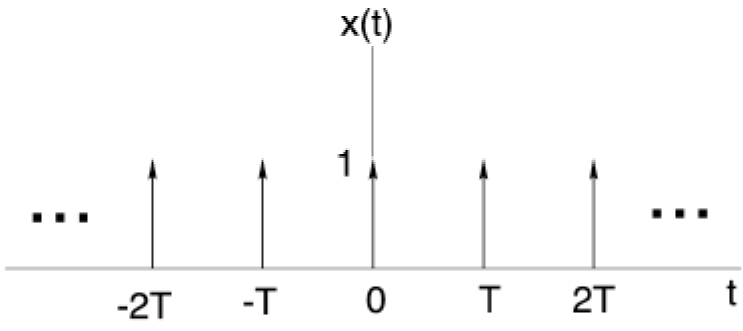
Review

Example 4.8

$$x(t)=\sum_{n=-\infty}^{+\infty}\delta(t-nT)$$ — sampling function



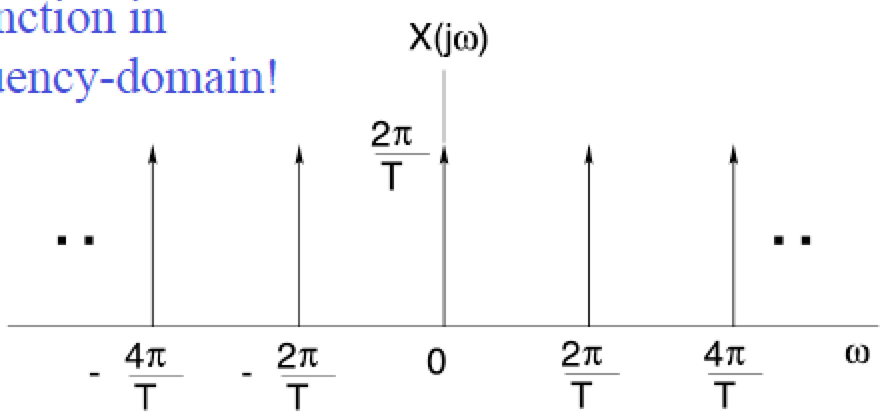
$$x(t)\xleftrightarrow{FS}a_k=\frac{1}{T}\int_{-T/2}^{T/2}x(t)e^{-jk\omega_o t}dt=\frac{1}{T}$$

$$\Downarrow\ x(t)=\sum_{k=-\infty}^{+\infty}a_k e^{jk\omega_o t}=\sum_{k=-\infty}^{+\infty}\frac{1}{T}e^{jk\omega_o t}$$

$$X(j\omega)=\sum_{k=-\infty}^{+\infty}\frac{2\pi}{T}\delta(\omega-\underbrace{\frac{k2\pi}{T}}_{k\omega_o})$$

$\omega_s=2\pi/T$: sampling frequency

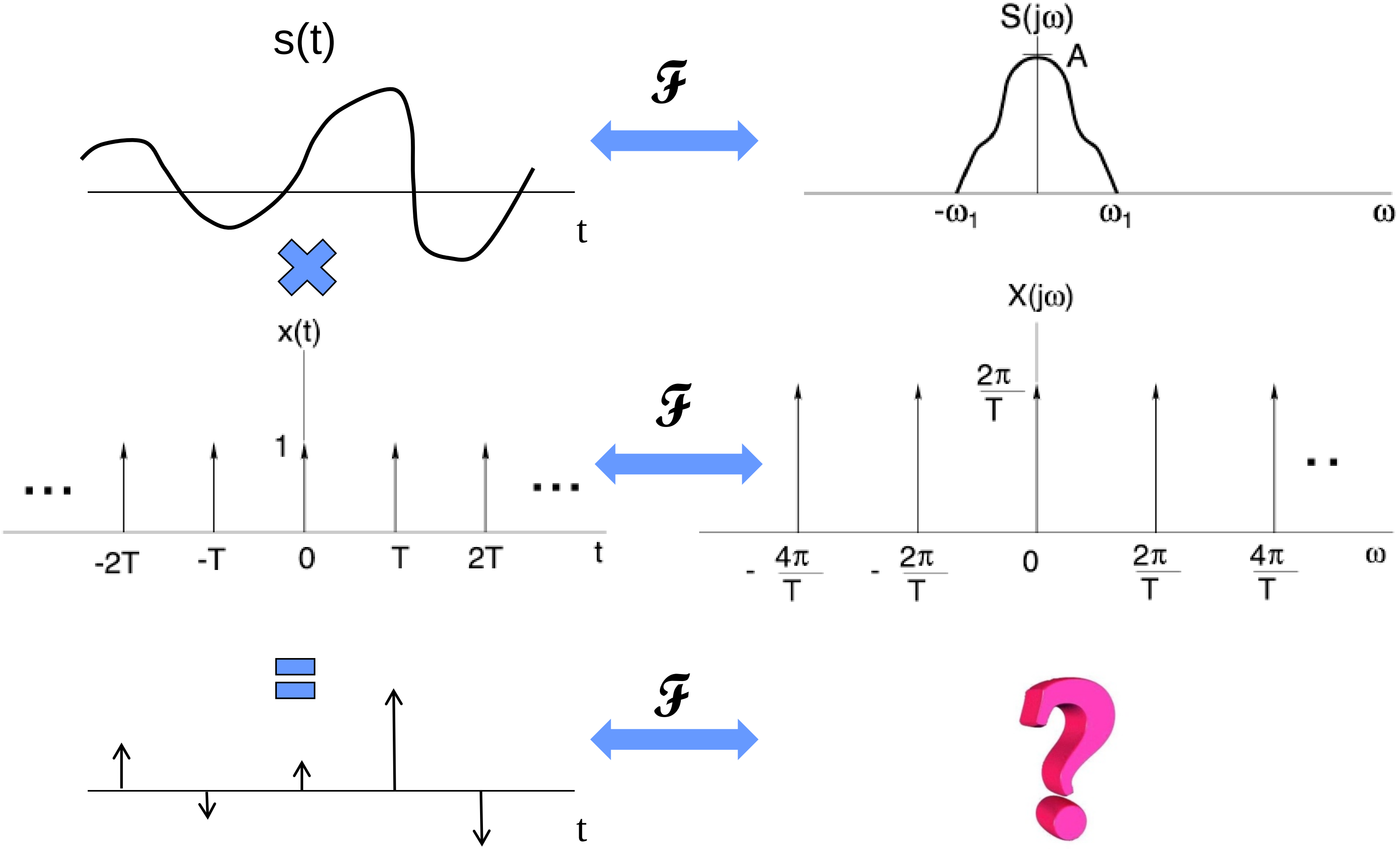
Same function in the frequency-domain!



Note in this case, periodic in both time domain (with a period T) and frequency domain (with a period $2\pi/T$)

Example: Sampling

- **Sample a continuous signal**



LTI Systems by LCCDE

(Linear-constant-coefficient differential equations)

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Using the Differentiation Property

$$\frac{d^k x(t)}{dt^k} \longleftrightarrow (j\omega)^k X(j\omega)$$

$\Downarrow$ Transform both sides of the equation

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

$$\Downarrow$$

$$Y(j\omega) = \underbrace{\left[\frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}\right]}_{H(j\omega)} X(j\omega)$$

$$\boxed{H(j\omega) = \left[\frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}\right]}$$

Partial Fraction Expansion

Partial Fraction Expansion (No identical poles):

$$H(j\omega) = \frac{b_M(j\omega + z_1)(j\omega + z_2)\ldots(j\omega + z_M)}{a_N(j\omega + p_1)(j\omega + p_2)\ldots(j\omega + p_N)}$$
$$= \frac{A_1}{j\omega + p_1} + \frac{A_2}{j\omega + p_2} + \cdots + \frac{A_N}{j\omega + p_N}$$

Partial Fraction Expansion (with identical poles):

$$H(j\omega) = \frac{b_M(j\omega + z_1)(j\omega + z_2)\ldots(j\omega + z_M)}{a_N(j\omega + p_1)^{k_1}(j\omega + p_2)^{k_2}\ldots(j\omega + p_n)^{k_n}} =$$
$$= \frac{A_{1,1}}{(j\omega + p_1)^{k_1}} + \frac{A_{1,2}}{(j\omega + p_1)^{k_1-1}} + \cdots \frac{A_{1,k_1}}{(j\omega + p_1)}$$
$$+ \ldots +$$
$$\frac{A_{n,1}}{(j\omega + p_n)^{k_n}} + \frac{A_{n,2}}{(j\omega + p_n)^{k_n-1}} + \cdots \frac{A_{n,k_n}}{(j\omega + p_n)}$$

Problem 4.34

- A stable LTI system S has the frequency response

$$H(j\omega)=\frac{j\omega+4}{6-\omega^2+5j\omega}$$

  - Determine a differential equation relating the input $x(t)$ and output $y(t)$ of S
  - Determine the impulse response $h(t)$ of S
  - What is the output of S when the input is $x(t)=e^{-4t}u(t)-te^{-4t}u(t)$

①

$$\frac{Y(j\omega)}{X(j\omega)}=H(j\omega)=\frac{j\omega+4}{6-\omega^2+5j\omega}$$

$$Y(j\omega)\cdot(6-\omega^2+5j\omega)=X(j\omega)\cdot(j\omega+4)$$

$$\Updownarrow$$

$$\frac{d^2y(t)}{dt^2}+5\frac{dy(t)}{dt}+6y(t)=\frac{dx(t)}{dt}+4x(t)$$

②

$$H(j\omega)=\frac{j\omega+4}{(2+j\omega)(3+j\omega)}=\frac{A}{2+j\omega}-\frac{B}{3+j\omega}=\frac{2}{2+j\omega}-\frac{1}{3+j\omega}$$

$$\because e^{-at}u(t)\longleftrightarrow\frac{1}{a+j\omega}\qquad\therefore h(t)=2e^{-2t}u(t)-e^{-3t}u(t)$$

③

$$\because te^{-at}u(t)\longleftrightarrow\frac{1}{(a+j\omega)^2}$$

$$\therefore X(j\omega)=\frac{1}{4+j\omega}-\frac{1}{(4+j\omega)^2}$$

$$\therefore Y(j\omega)=H(j\omega)\cdot X(j\omega)=\frac{1}{(4+j\omega)(2+j\omega)}=\frac{A}{4+j\omega}-\frac{B}{2+j\omega}$$

Table 4.1 Properties of the Fourier Transform

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega - \theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \Re e\{X(j\omega)\} = \Re e\{X(-j\omega)\} \\ \Im m\{X(j\omega)\} = -\Im m\{X(-j\omega)\} \\ \|X(j\omega)\| = \|X(-j\omega)\| \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\Re e\{X(j\omega)\}$ $j\Im m\{X(j\omega)\}$

4.3.7 Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

Table 4.2
Basic Fourier Transform Pairs

$$e^{-at}u(t) \longleftrightarrow \frac{1}{a+j\omega}$$

$$te^{-at}u(t) \longleftrightarrow \frac{1}{(a+j\omega)^2}$$

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1,\ a_k = 0,\ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t+T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2\sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—