Chapter 6
Time & Frequency Characterization of Signals and Systems

Outline

1. Magnitude and phase of Fourier transform
2. Group delay
3. Non-ideal low-pass filter

1. Magnitude & Phase

$$y(t) = A\cos(\omega t + \theta)$$

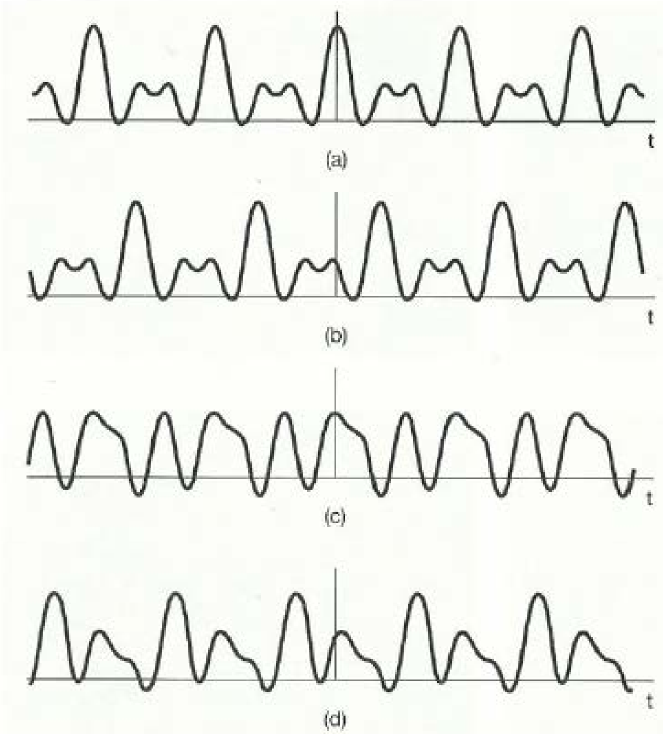
$$y(t) = \sum_i A_i \cos(\omega_i t + \theta_i)$$

Magnitude & Phase (Cont.)

- Magnitude of spectrum, $|X(j\omega)|$ or $|X(e^{j\omega})|$, determines the energy of frequency component
- Phase of spectrum affects the shape of time-domain signal

$$x(t) = 1 + \frac{1}{2}\cos(2\pi t + \phi_1) + \cos(4\pi t + \phi_2) + \frac{2}{3}\cos(6\pi t + \phi_3).$$

Figure 6.1



$\phi_1 = \phi_2 = \phi_3 = 0$;

$\phi_1 = 4$ rad, $\phi_2 = 8$ rad, $\phi_3 = 12$ rad;

$\phi_1 = 6$ rad, $\phi_2 = -2.7$ rad, $\phi_3 = 0.93$ rad;

$\phi_1 = 1.2$ rad, $\phi_2 = 4.1$ rad, $\phi_3 = -7.02$ rad.

Example 1: Magnitude & Phase of Image

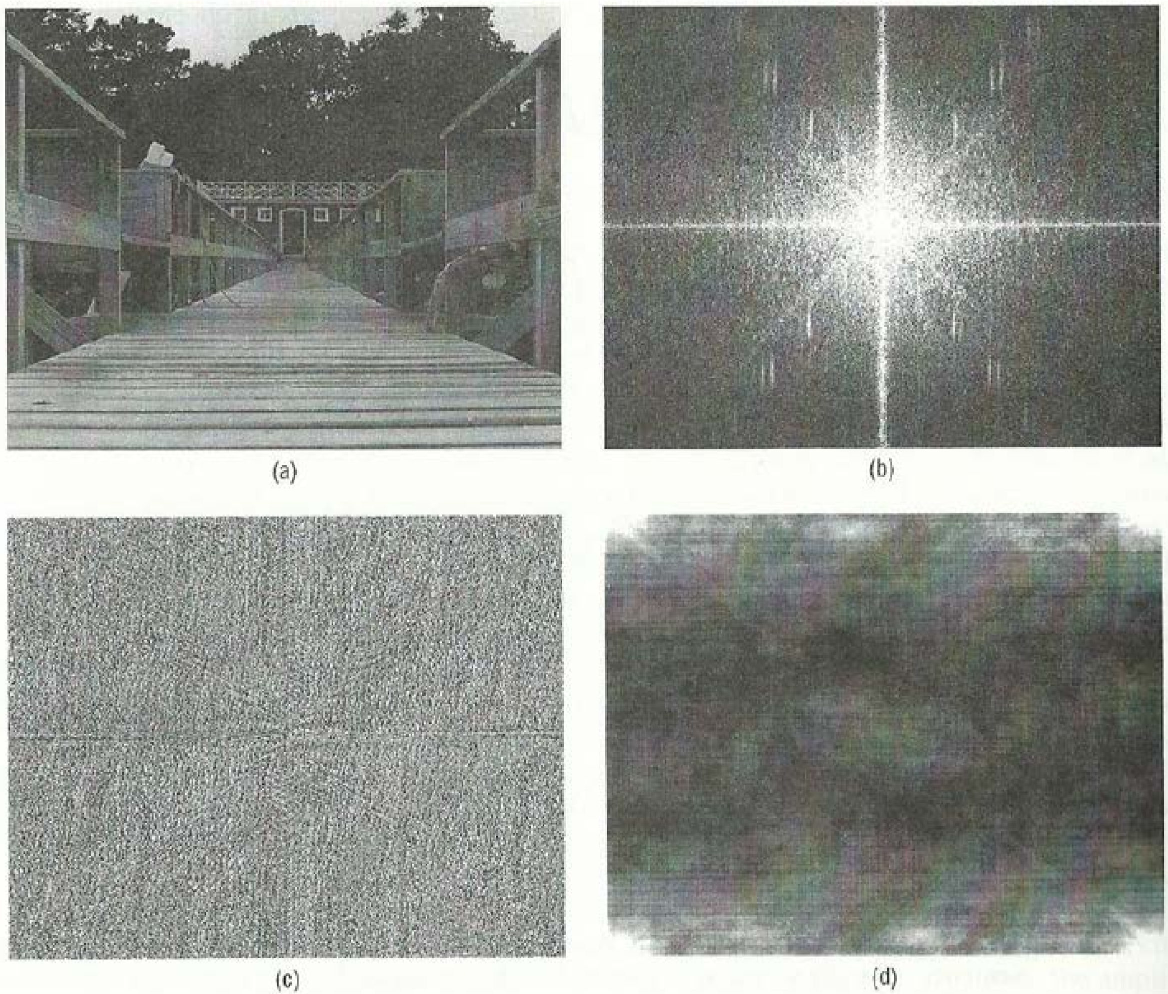


Figure 6.2 (a) Original image; (b) Magnitude; (c) Phase; (d) Set phase to zero

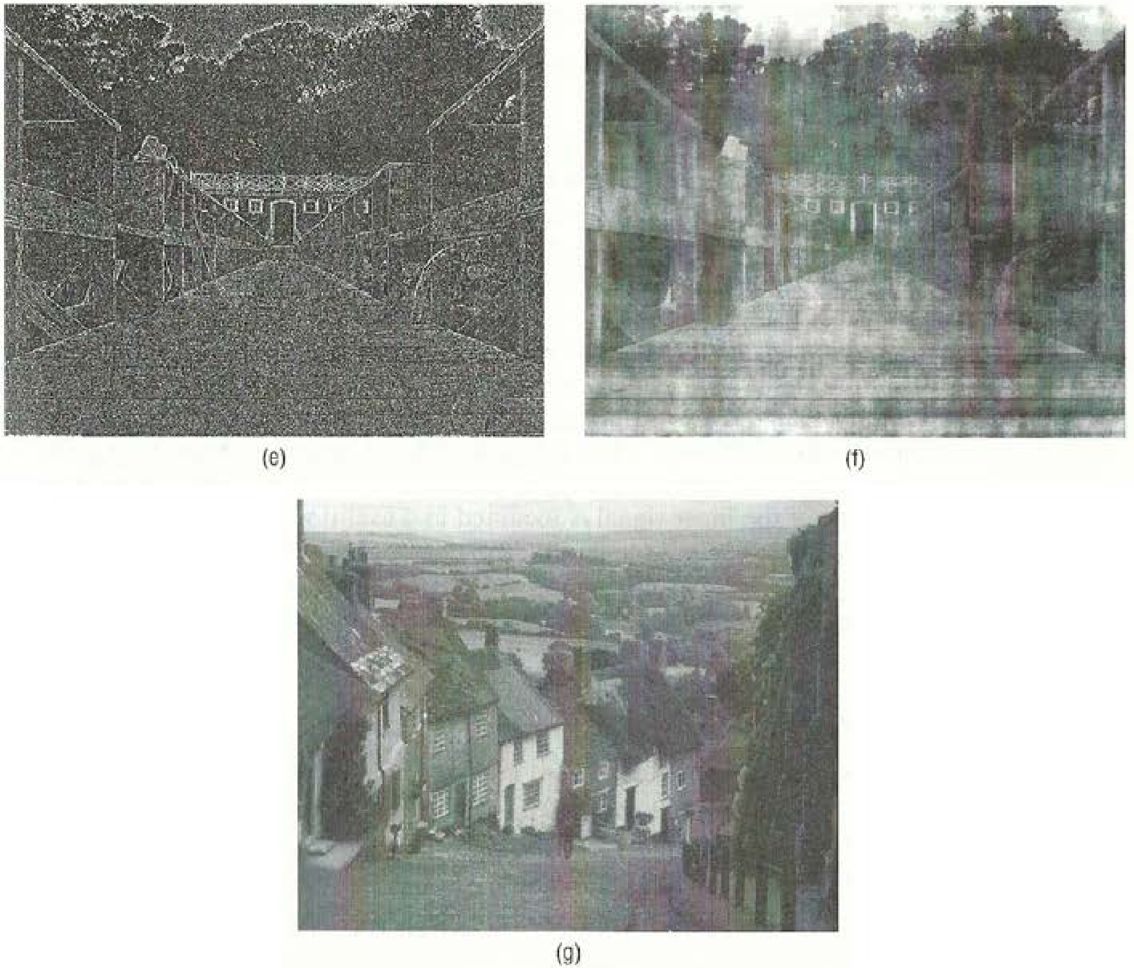


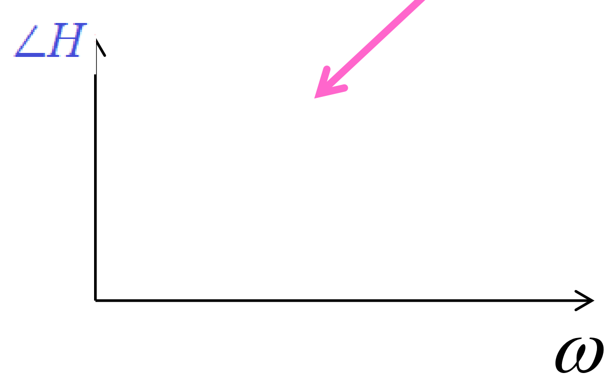
Figure 6.2 (e) Set magnitude to 1; (f) Original phase + (g)'s magnitude

2. LTI Systems: Linear & Nonlinear Phase

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega) = H(j\omega)X(j\omega)$$

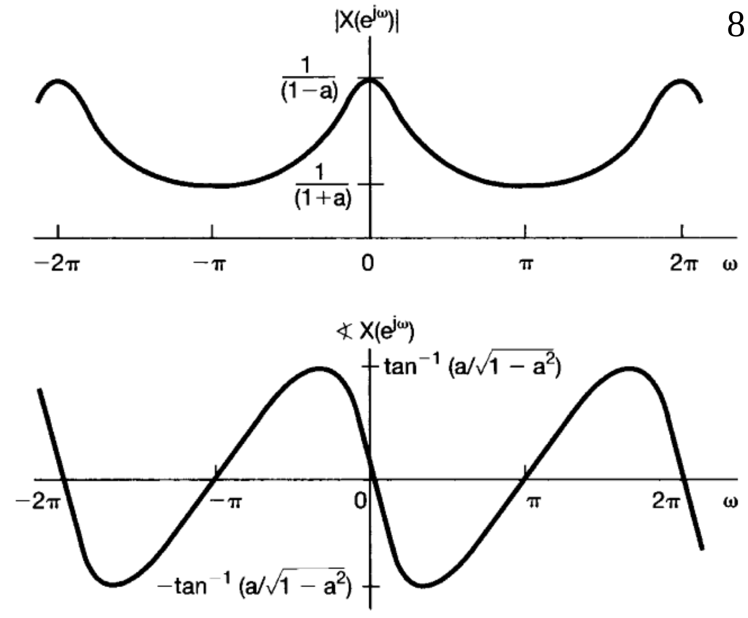
$$X(e^{j\omega}) \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

Linear?



Example 5.1

a>0



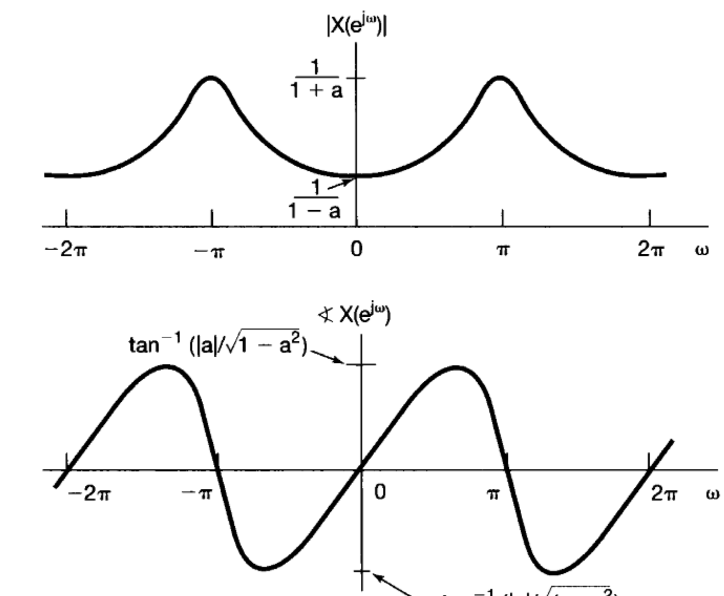
(a)

$$x[n] = a^n u[n], \qquad |a| < 1.$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^n u[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}.$$

a<0

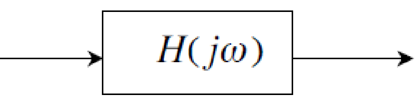


(b)

Group Delay

- Linear phase response leads to delay

$$X(j\omega)H(j\omega) = X(j\omega)e^{-j\omega t_0} \longleftrightarrow x(t - t_0)$$



$$H(j\omega) = e^{-j\omega t_0}$$

$$|H(j\omega)| = 1, \quad \sphericalangle H(j\omega) = -\omega t_0$$

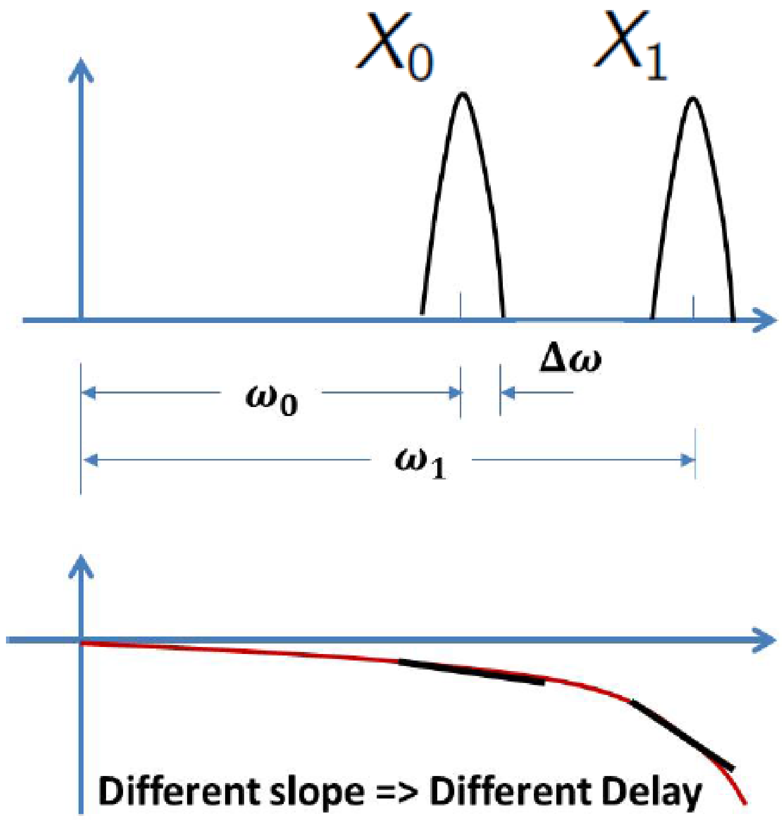
$$y(t) = x(t - t_0)$$

Cont.

- Systems with linear phase characteristics have the particularly simple interpretation as time shifts.
- The phase slope tells us the size of the time shift.
  - If $\sphericalangle H(j\omega) = -\omega t_0$, the system imparts a time shift of $-t_o$

Cont.

- Non-linear phase response leads to distortion: $e^{-j\omega^2 t_0}$
- Narrow-band signal: phase response can be approximated as linear



Cont.

Suppose two narrow-band signals X_0 and X_1 are delivered into a system $H(j\omega) = e^{-j\omega^2 t_0}$:

$$-j\omega^2 t_0 \approx -2j\omega_0 t_0 \omega + j\omega_0^2 t_0$$

$$\begin{aligned} X_0(j\omega)H(j\omega) &= X_0(j\omega)e^{-j\omega^2 t_0} \\ &\approx X_0(j\omega)e^{-2j\omega_0 t_0 \omega} e^{j\omega_0^2 t_0} \\ &\leftrightarrow e^{j\omega_0^2 t_0} x_0(t - 2\omega_0 t_0) \end{aligned}$$

$$X_1(j\omega)H(j\omega) \leftrightarrow e^{j\omega_1^2 t_0} x_1(t - 2\omega_1 t_0)$$

Group delay: different frequency components have different delay

$\Rightarrow$ waveform distortion

Example

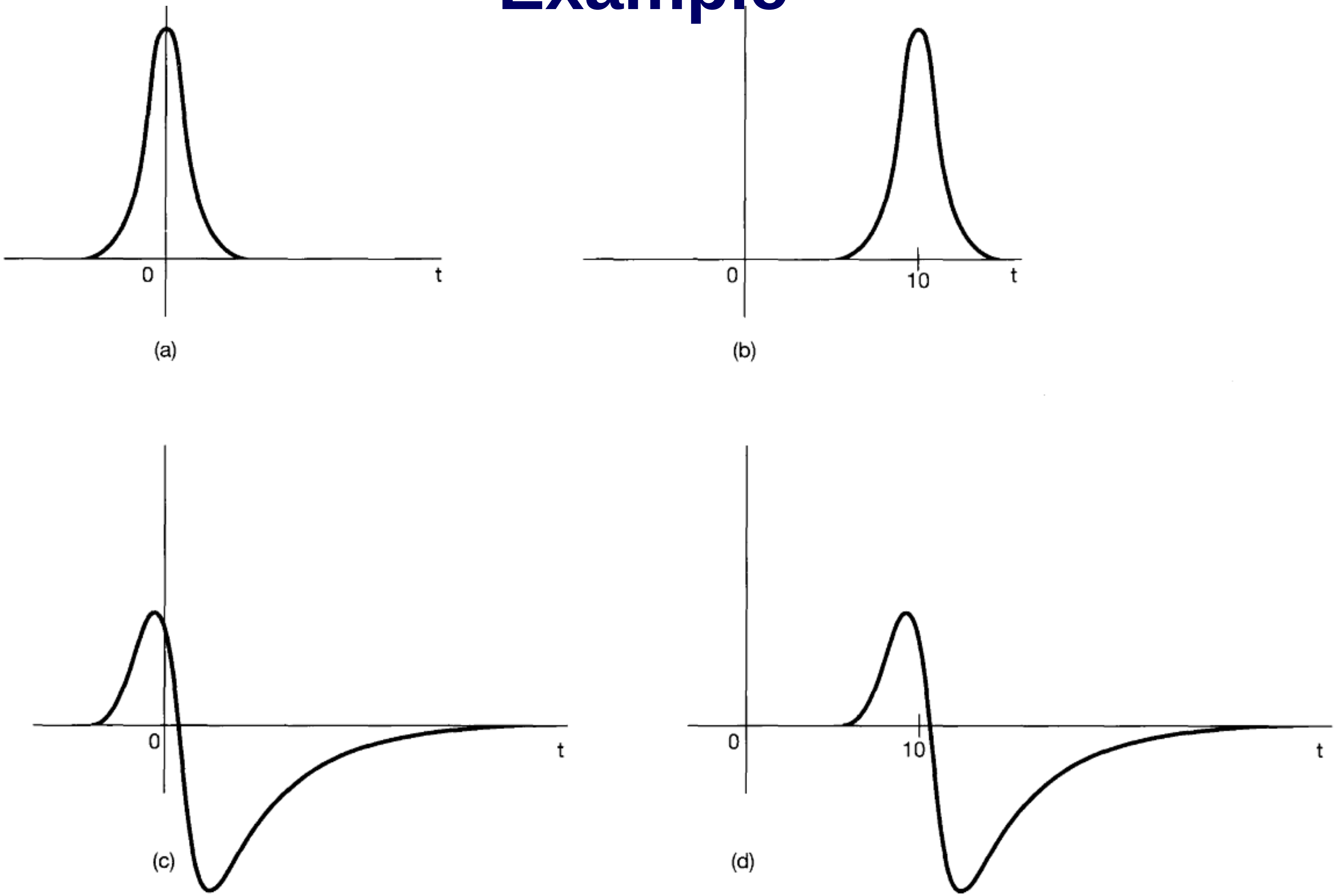


Figure 6.3 (a) Continuous-time signal that is applied as the input to several systems for which the frequency response has unity magnitude; (b) response for a system with linear phase; (c) response for a system with nonlinear phase; and (d) response for a system with phase equal to the nonlinear phase of the system in part (c) plus a linear phase term.

Bode plot

Plots of $20\log_{10}|H(j\omega)|$ and $\sphericalangle H(j\omega)$ versus $\log_{10}(\omega)$

in units of $20\log_{10}$

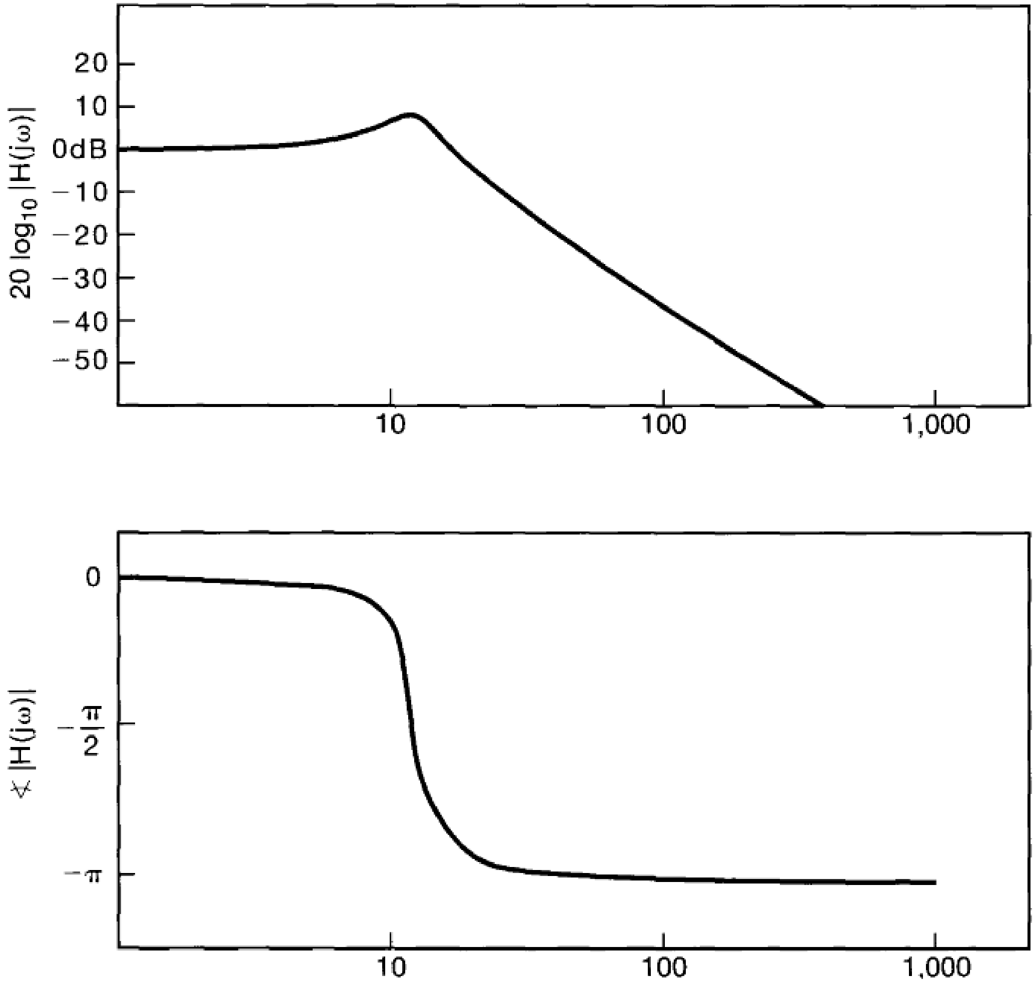


Figure 6.8 A typical Bode plot. (Note that ω is plotted using a logarithmic scale.)

```
b = fir1(80,0.5,kaiser(81,8));
freqz(b,1);
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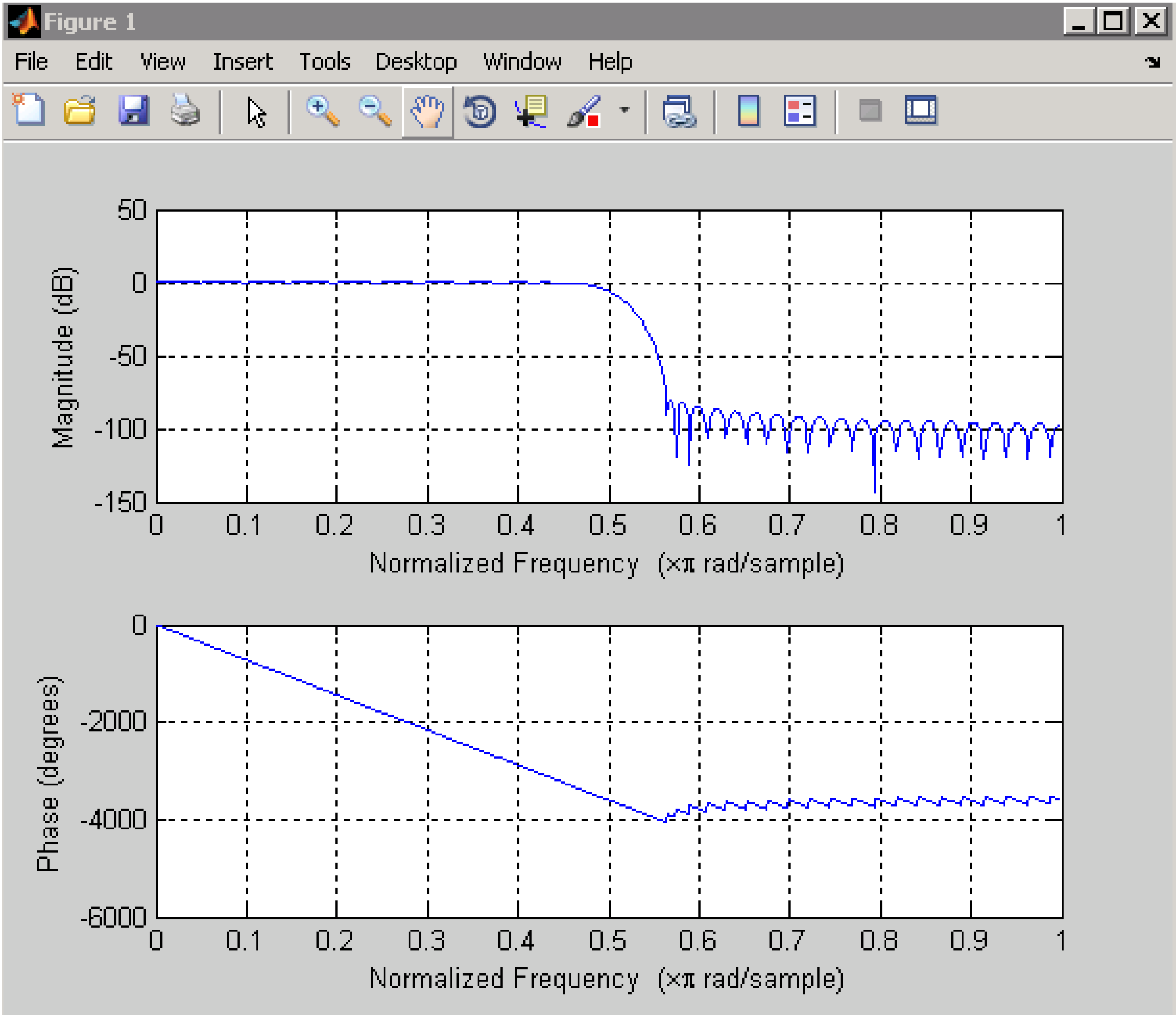
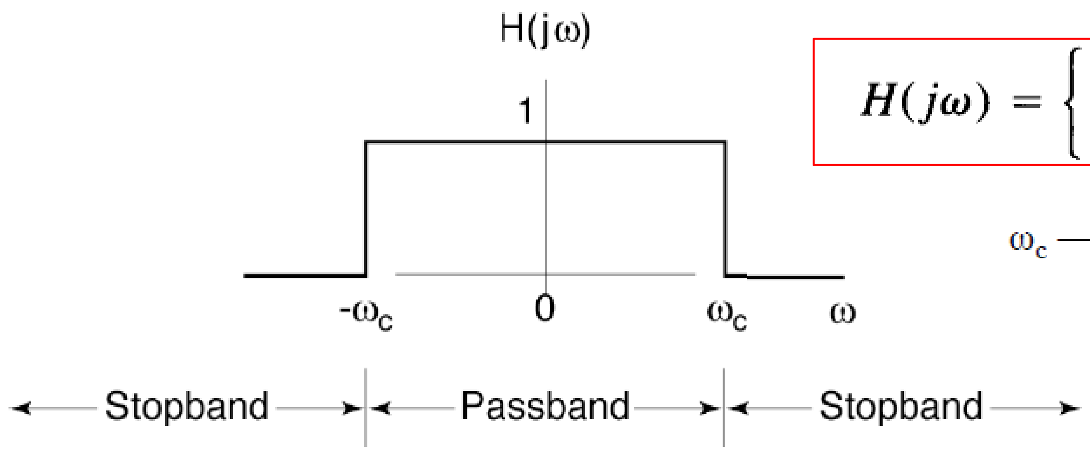


3. Ideal Low-pass Filter

- Ideal low-pass filter: non-causal, sharp edge, implementation issue

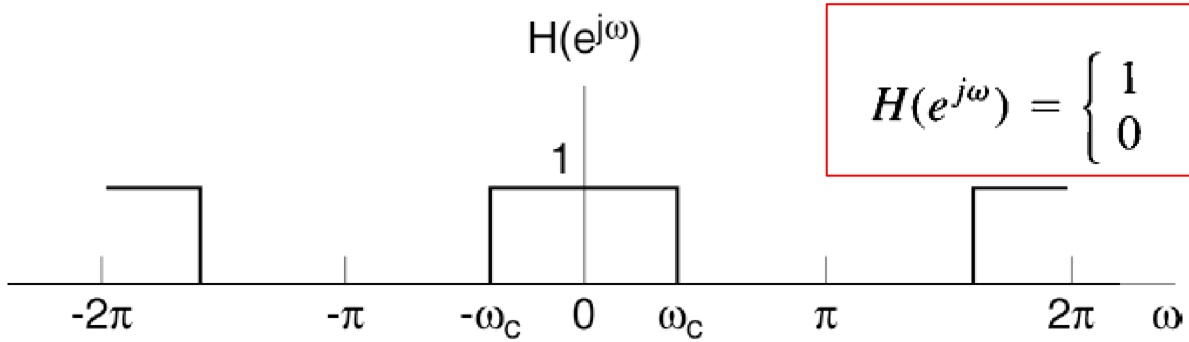
CT



$$H(j\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

ω_c — cutoff frequency

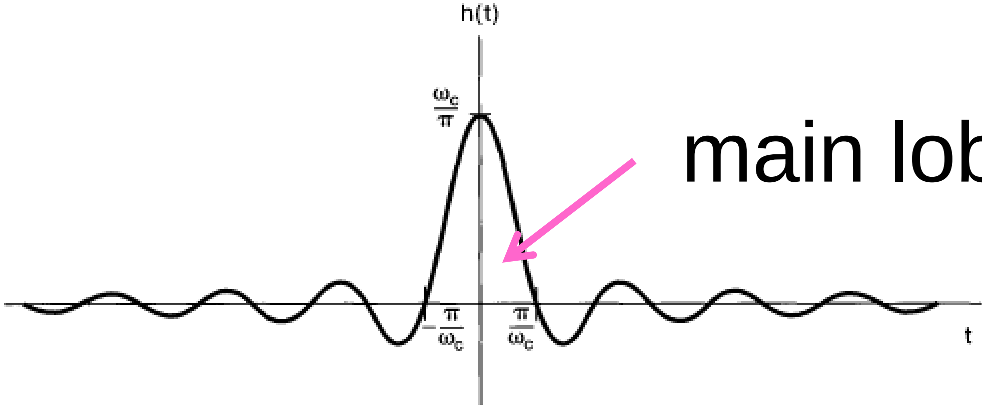
DT



$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$$

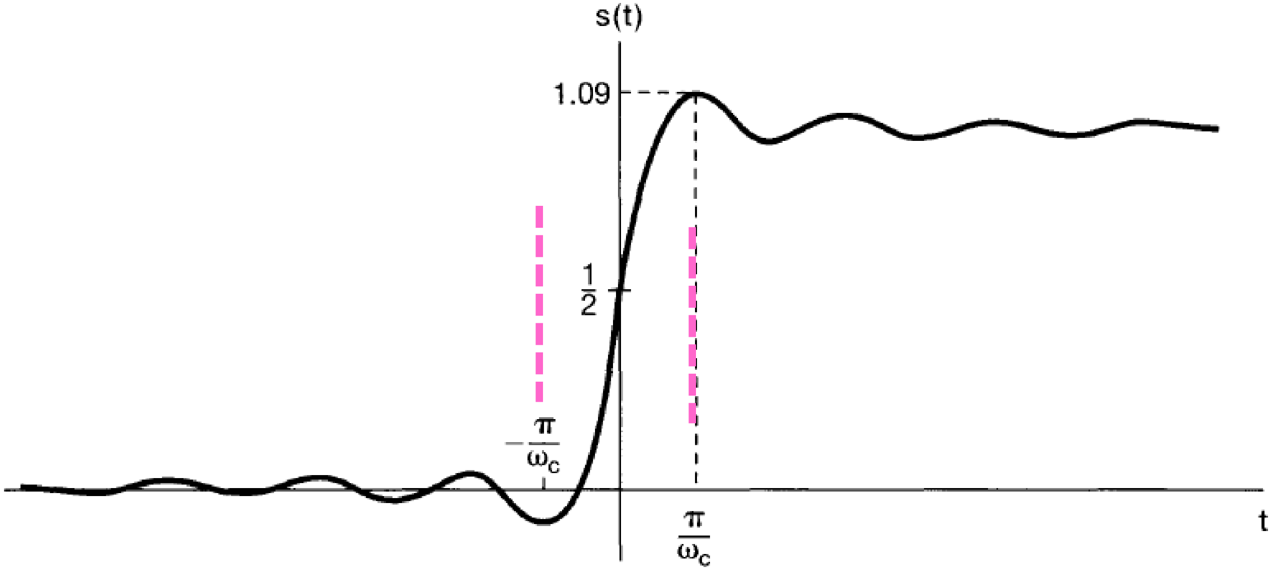
Note: $|H| = 1$ and $\angle H = 0$ for the ideal filters in the passbands, no need for the phase plot.

Step response of low-pass filter

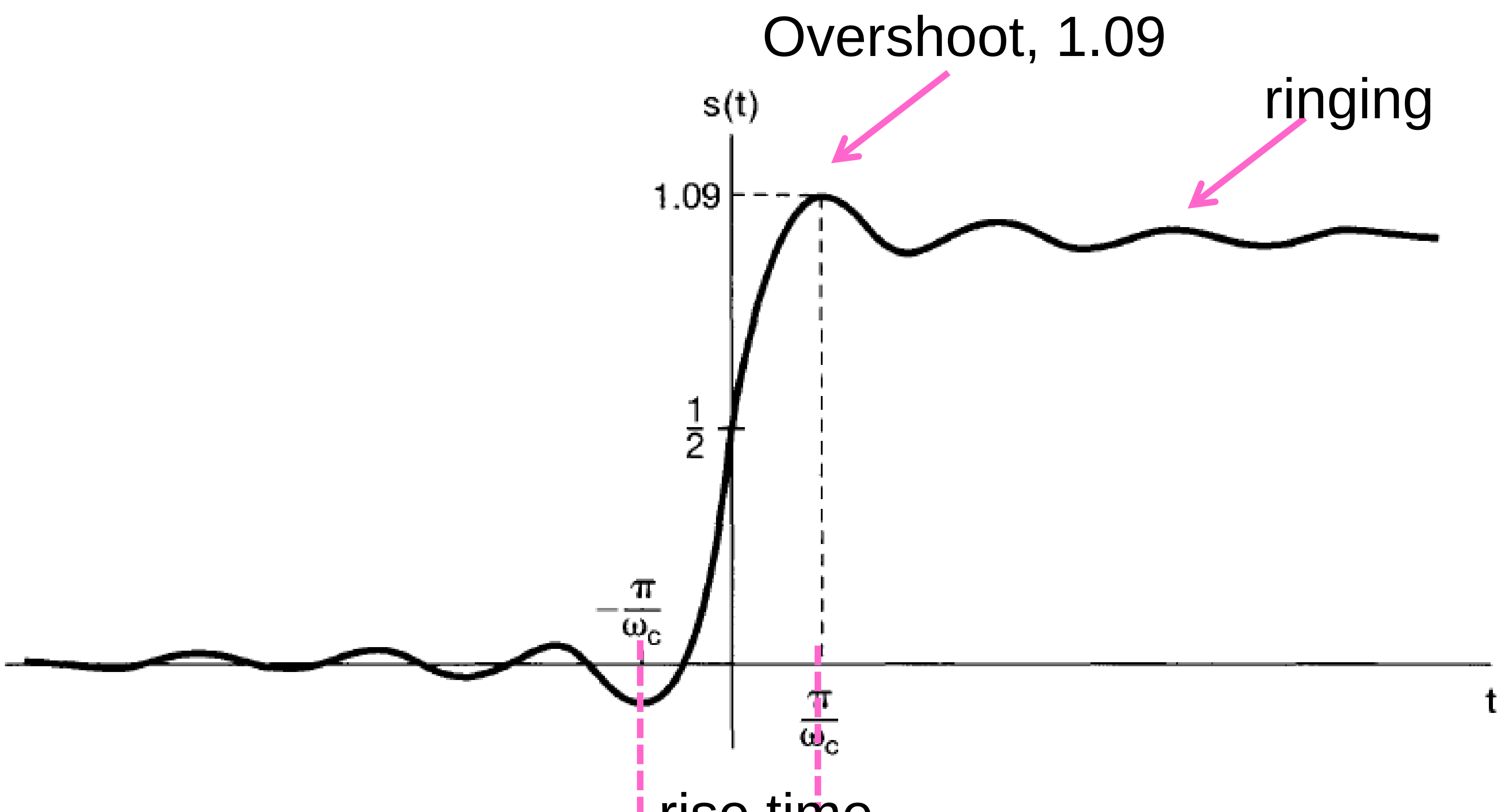


main lobe

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau,$$



Cont.



rise time,
inversely related to the bandwidth

Non-ideal Low-pass Filter

- Non-ideal filter: passband (with tolerant ripple) $\rightarrow$ transition band $\rightarrow$ stopband

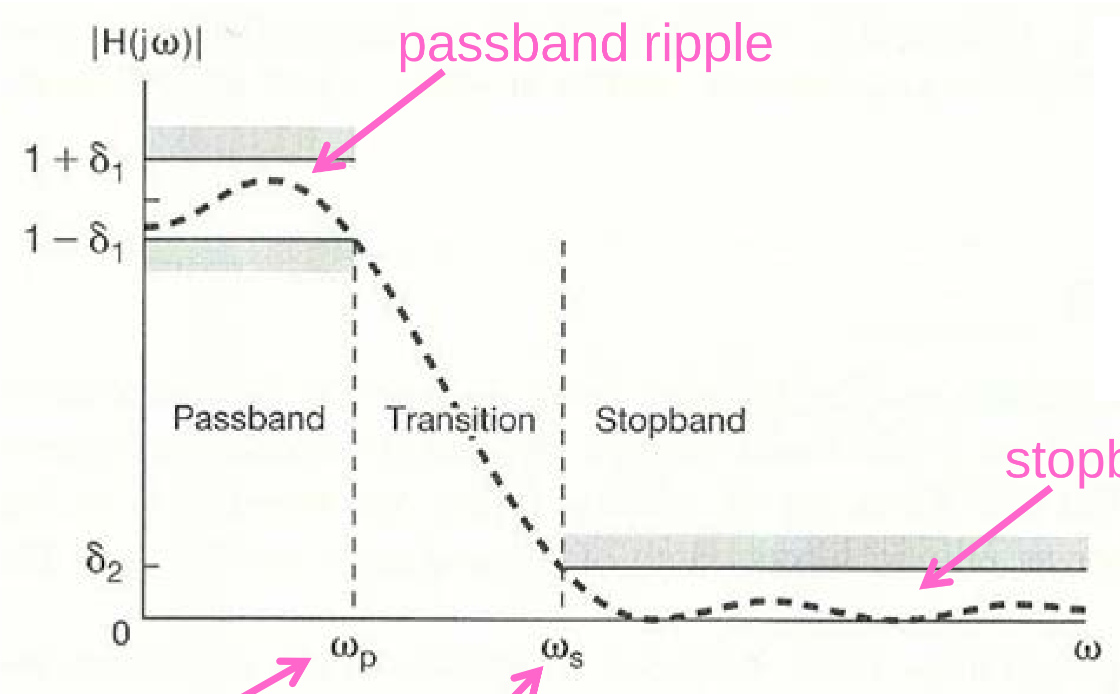


Figure 6.16 Tolerances for the magnitude characteristic of a lowpass filter. The allowable passband ripple is δ_1 and stopband ripple is δ_2. The dashed curve illustrates one possible frequency response that stays within the tolerable limits.

Cont.

- Step response: rise time, overshot, ringing frequency, setting time

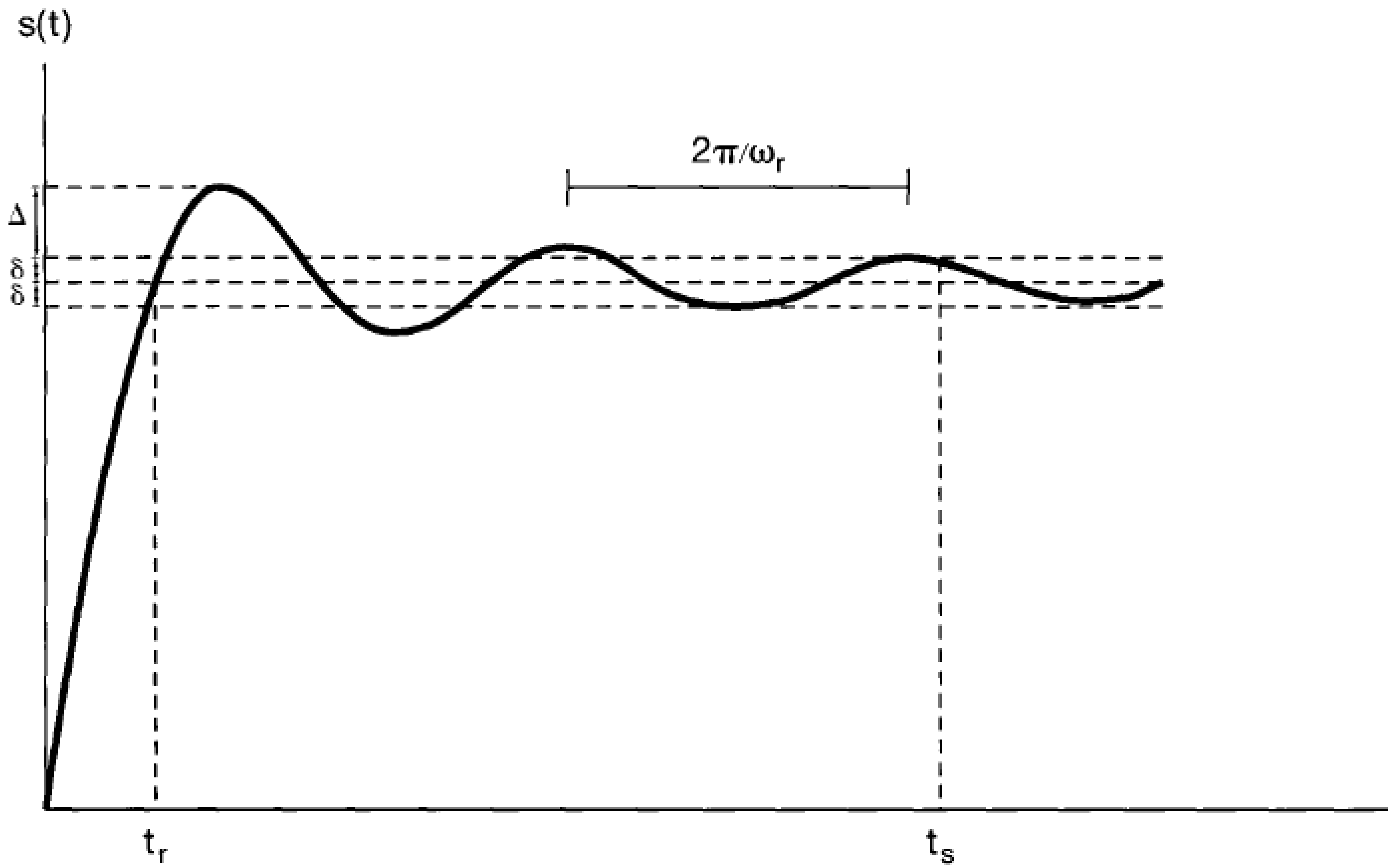
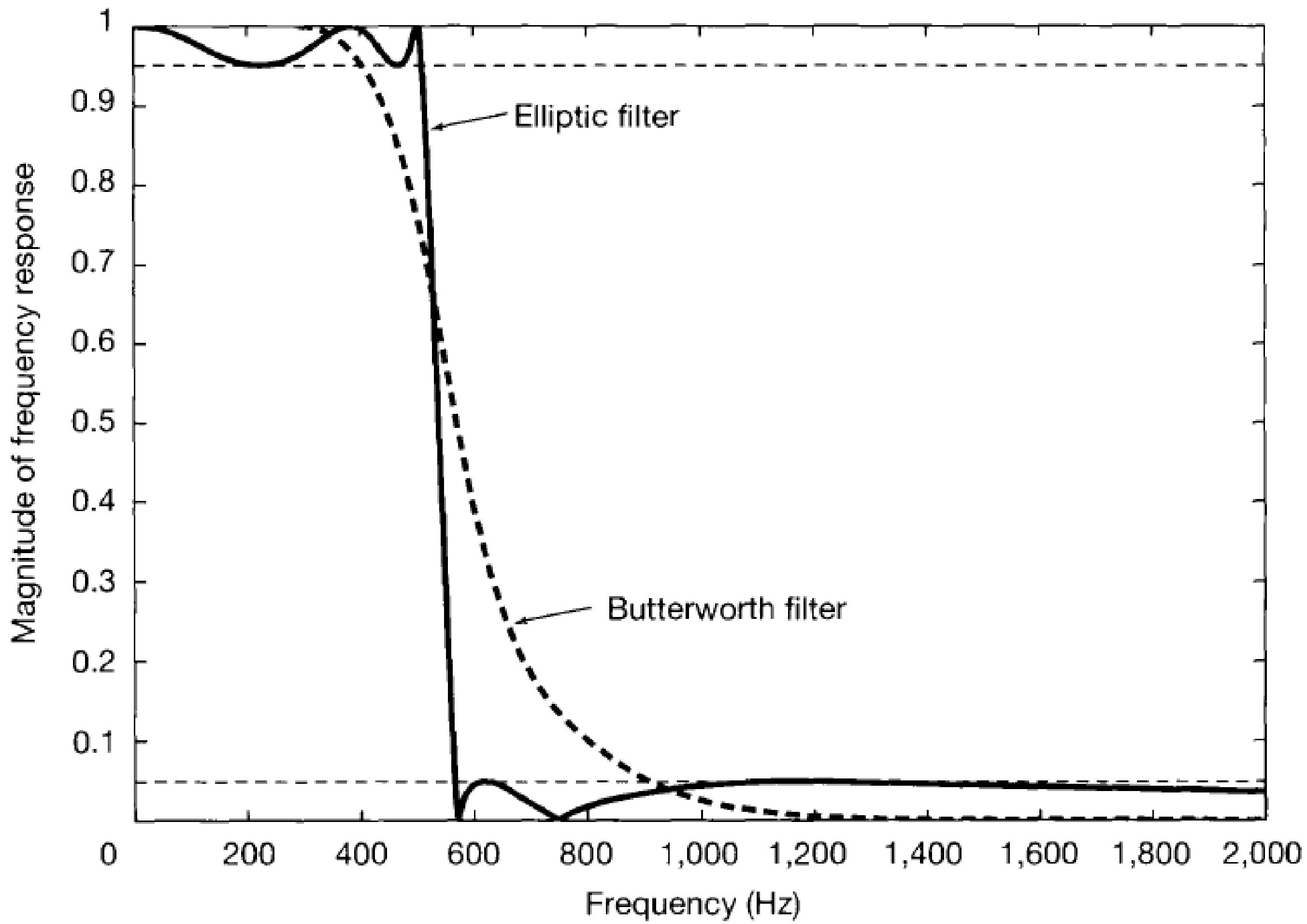


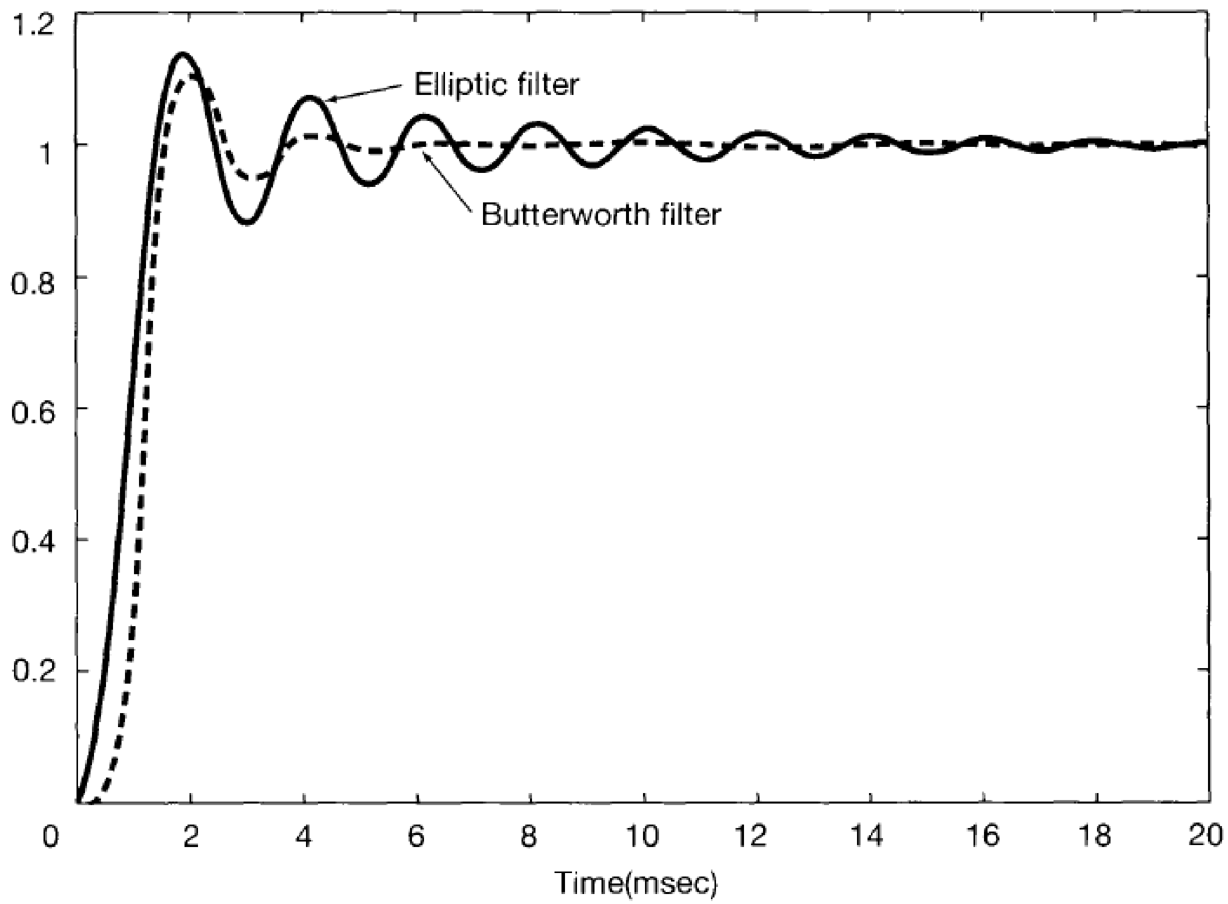
Figure 6.17 Step response of a continuous-time lowpass filter, indicating the rise time t_r, overshoot Δ, ringing frequency ω_r, and settling time t_s—i.e., the time at which the step response settles to within $\pm\delta$ of its final value.

Example:
Elliptic Filter vs. Butterworth Filter



- Transition band: Butterworth filter > Elliptic filter

Cont. – Step response

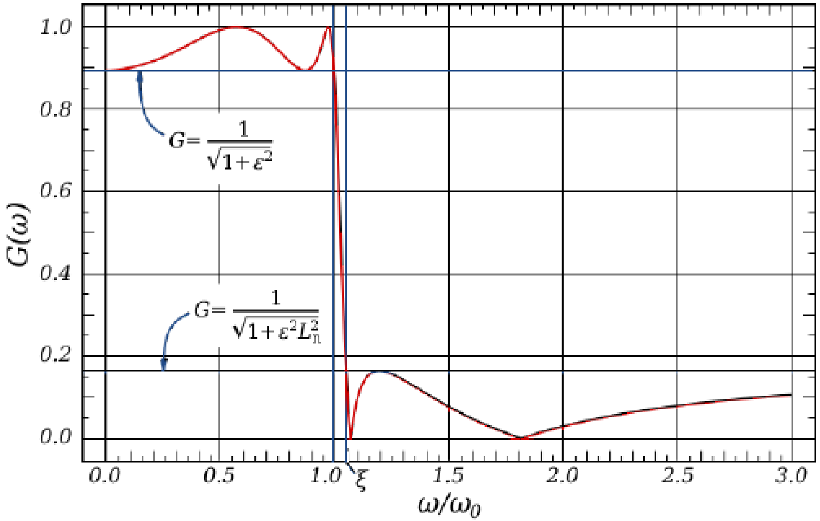


- The ringing in the elliptic filter's step response is more prominent than for the Butterworth step response.

Elliptic Filter

$$G_n(j\omega) = \frac{1}{\sqrt{1 + \epsilon^2 R_n^2(\xi, \omega/\omega_0)}}$$

- where R_n is the nth-order elliptic rational function, ω_0 is the cut-off frequency, ϵ is the ripple factor, ξ is the selectivity factor
- "No other filter of equal order can have a faster transition in gain between the passband and the stopband, for the given values of ripple"



Butterworth Filter

$$G_n(j\omega) = \sqrt{\frac{1}{1+\omega^{2n}}}$$

- "have as flat a frequency response as possible in the passband"
- Maximally flat magnitude filter