Notes

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CT Fourier Series Pairs

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{j2\pi kt/T}$$

Harmonically related

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt$$

Periodicity Properties of DT Complex Exponentials

- For DT complex exponentials, signal are periodic only when

$$\omega_0 N = k \cdot 2\pi, \qquad k = 0, \pm 1, \pm 2, \cdots$$

$$\boxed{e^{j\omega_0 n} = e^{j\omega_0 (n+N)} \rightarrow e^{j\omega_0 N} = 1 \rightarrow \omega_0 N = k \cdot 2\pi}$$

- For DT complex exponentials, signals with frequencies ω_0 and ω_0+k•2π are identical. $\boxed{e^{j(\omega_0 + k\cdot 2\pi)n} = e^{j\omega_0 n} \cdot e^{jk\cdot 2\pi n} = e^{j\omega_0 n}}$
  - We need only consider a frequency interval of length 2π, and on most cases, we use the interval: $0 \le \omega_0 < 2\pi$, or $-\pi \le \omega_0 < \pi$

Review

- $e^{j\omega_0 n}$ does ***not*** have a continually increasing rate of oscillation as ω_0 is increased in magnitude.

  low-frequency (slowly varying): ω_0 near 0, 2π, …, or 2k • π

  high-frequency (rapid variation): ω_0 near ± π, …, or (2k+1) • π

$$e^{j(2k+1)\pi n} = e^{j\pi n} = (e^{j\pi})^n = (-1)^n$$

DT Fourier Series Representation

Arbitrary periodic DT signal with period N can be written as

$$x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

$\sum_{k=<N>}$ = Sum over *any* N consecutive values of k

— This is a *finite* series

$\{a_k\}$ - Fourier (series) coefficients

Frequency component: $\frac{2k\pi}{N}$ $k = 0,1,2, \ldots, N-1 \; or \; 1,2, \ldots, N$

Why?

Existence

Any DT periodic signal has a Fourier series representation

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \qquad \forall n$$

$$\Downarrow$$

n=0 $\quad x[0] = \sum_{k=<N>} a_k$

n=1 $\quad x[1] = \sum_{k=<N>} a_k e^{jk\omega_0}$

n=2 $\quad x[2] = \sum_{k=<N>} a_k e^{j2k\omega_0}$

.

. $\quad \vdots$

n=N-1 $\quad x[N-1] = \sum_{k=<N>} a_k e^{j(N-1)k\omega_0}$

N equations for N unknowns, a_0, a_1, ... a_{N-1}

How to calculate a_k

- **Define inner product as**

$$< e^{jk\omega_0 n} \cdot e^{jm\omega_0 n} > \; = \; \frac{1}{N}\sum_{n=0}^{N-1} e^{jk\omega_0 n} e^{-jm\omega_0 n}$$

- **We have**

$$< e^{jk\omega_0 n} \cdot e^{jm\omega_0 n} > = \;\; \mathbf{1}\,(\mathbf{k} = \mathbf{m} + Nk')$$

$$< e^{jk\omega_0 n} \cdot e^{jm\omega_0 n} > = \;\; 0\,(Otherwise)$$

- $\{e^{jk\omega_0 n} | k =< N >\}$ **is similar to basis of vector space**

- **So**

$$< x[n] \cdot e^{jk\omega_0 n} > \; = < \sum_{m=0}^{N-1} a_m e^{jm\omega_0 n} \cdot e^{jk\omega_0 n} >$$

$$= \sum_{m=0}^{N-1} a_m < e^{jm\omega_0 n} \cdot e^{jk\omega_0 n} > \; = a_k$$

- **Hence,**

$$a_k = < x[n] \cdot e^{jk\omega_0 n} > \; = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\omega_0 n}$$

DT Fourier Series Pair $\left(\omega_o = \frac{2\pi}{N}\right)$

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \qquad \text{(Synthesis equation)}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n} \qquad \text{(Analysis equation)}$$

Different from CT Fourier series

Cont.

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \qquad \text{(Synthesis equation)}$$

- a_k can be defined for all integers k, and we have $\boxed{a_{k+N} = a_k}$

$$x[n] = a_0 e^{\frac{j0\times 2\pi}{N}n} + a_1 e^{\frac{j1\times 2\pi}{N}n} + \cdots + a_{N-1} e^{\frac{j(N-1)\times 2\pi}{N}n}$$

$$x[n] = a_1 e^{\frac{j1\times 2\pi}{N}n} + \cdots + a_{N-1} e^{\frac{j(N-1)\times 2\pi}{N}n} + a_N e^{\frac{jN\times 2\pi}{N}n}$$

  - a_k is periodic w.r.t. k
  - CT is different

Example #1: Sum of a pair of sinusoids

$$x[n] = \cos(\pi n / 8) + \cos(\pi n / 4 + \pi / 4)$$

— periodic with period $N = ?$

$$x[n] = \frac{1}{2}\left[e^{j\omega_0 n} + e^{-j\omega_0 n}\right] + \frac{1}{2}\left[e^{j\pi/4} e^{j2\omega_0 n} + e^{-j\pi/4} e^{-j2\omega_0 n}\right]$$

$$\Downarrow$$

$a_0 = 0$

$a_1 = 1/2$

$a_{-1} = 1/2$

$a_2 = e^{j\pi/4}/2$

$a_{-2} = e^{-j\pi/4}/2$

$a_3 = 0$

$a_{-3} = 0$

$\vdots$

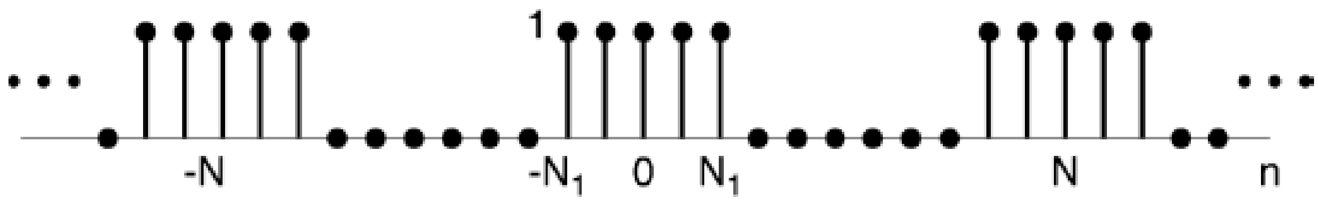
$a_{15} = a_{-1+16} = a_{-1} = 1/2$

$a_{66} = a_{2+4\times 16} = a_2 = e^{j\pi/4}/2$

$$\boxed{\begin{aligned} \cos(x) &= \mathrm{Re}(e^{jx}) = \frac{1}{2}(e^{jx} + e^{-jx}) \\ \sin(x) &= \mathrm{Im}(e^{jx}) = \frac{1}{2}(e^{jx} - e^{-jx}) \end{aligned}}$$

Example 3.12 DT Square wave

Period=?



$$a_0 = \frac{1}{N}\sum_{n=-N_1}^{N_1} x[n] = \frac{(2N_1+1)}{N} = a_N = a_{-N} = a_{6N} = \cdots$$

For $k \neq$ multiple of N:

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk\omega_0 n} \overset{n=m-N_1}{=} \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk\omega_0(m-N_1)}$$

$$= \frac{1}{N} e^{jk\omega_0 N_1} \sum_{m=0}^{2N_1} (e^{-jk\omega_0})^m = \frac{1}{N} e^{jk\omega_0 N_1} \frac{1-e^{-jk\omega_0(2N_1+1)}}{1-e^{-jk\omega_0}}$$

$$= \frac{1}{N}\frac{\sin\left[k(N_1+1/2)\omega_0\right]}{\sin(k\omega_0/2)} = \frac{1}{N}\frac{\sin\left[2\pi k(N_1+1/2)/N\right]}{\sin(\pi k/N)}$$

DT Square wave (continued)

$$a_k = \frac{1}{N}\frac{\sin\left[2\pi k(N_1 + 1/2)/N\right]}{\sin(\pi k / N)}$$

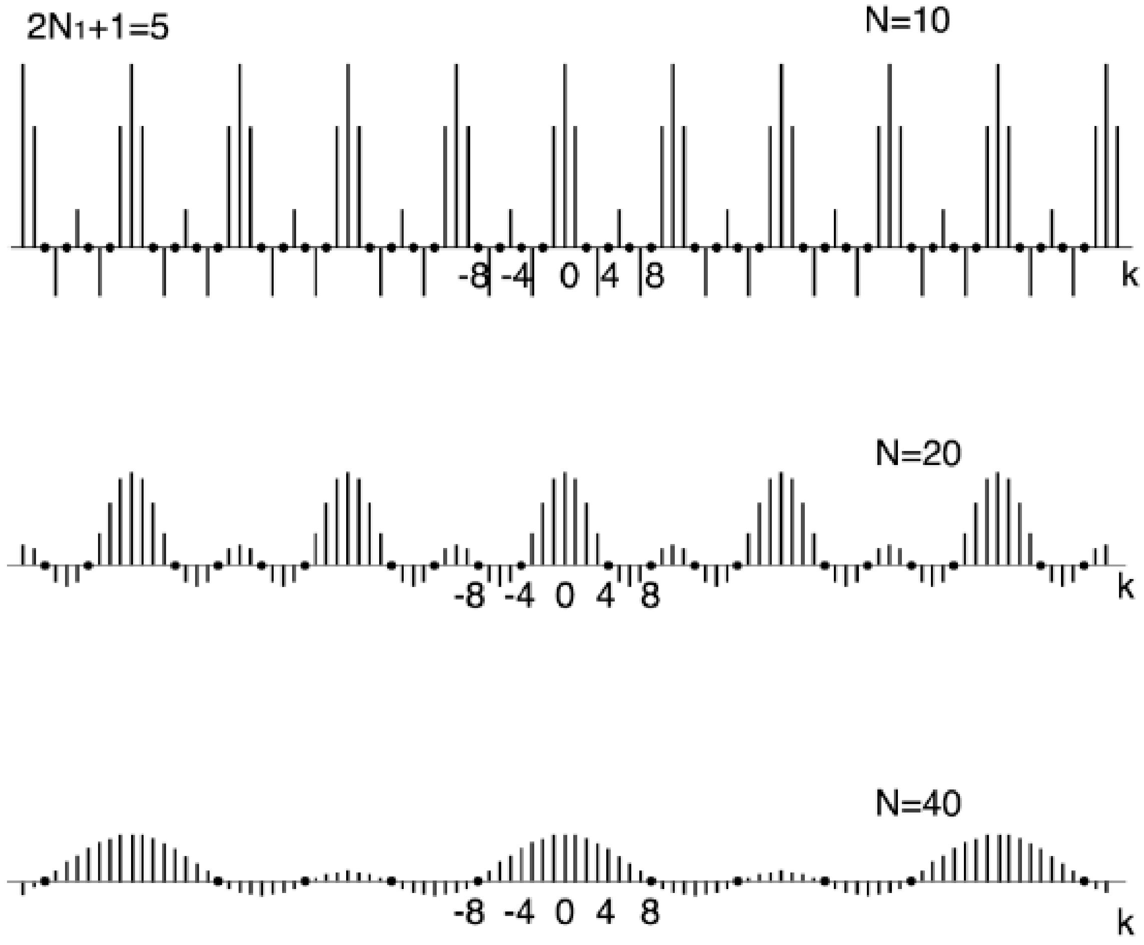
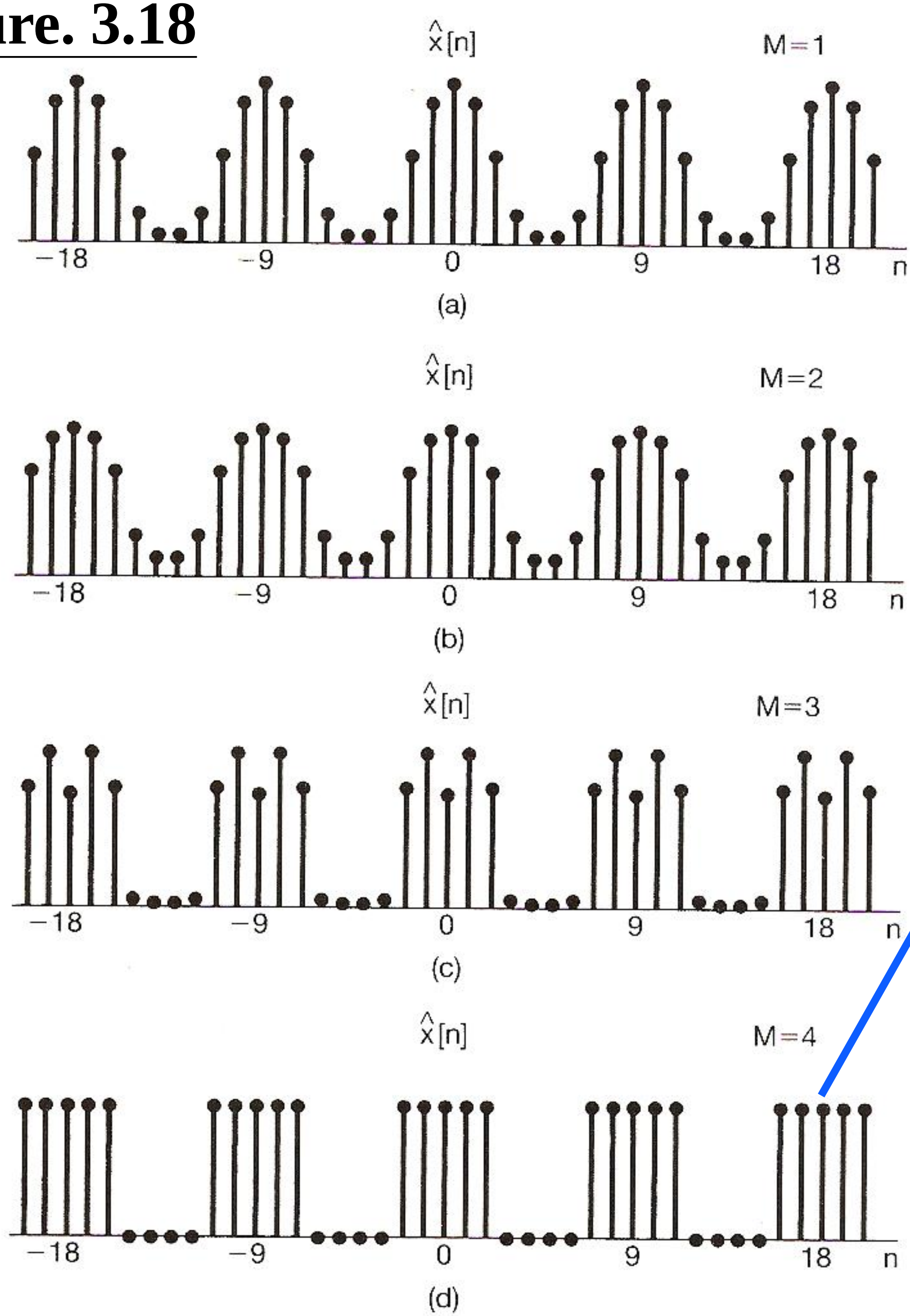


Figure. 3.18

N=9, 2N$_1$+1=5



1) The same as original DT square wave

2) No Gibbs phenomenon, and no discontinuity

Figure 3.18 Partial sums of eqs. (3.106) and (3.107) for the periodic square wave of Figure 3.16 with $N = 9$ and $2N_1 + 1 = 5$: (a) $M = 1$; (b) $M = 2$; (c) $M = 3$; (d) $M = 4$.

DT Fourier Series - Properties

- Strong similarities between the properties of DT and CT Fourier series [Comparing Table 3.2 to Table 3.1.]

Example: $x[n] \longleftrightarrow a_k$

$e^{jM\omega_0 n}x[n] \longleftrightarrow b_k = ?$

$$x[n]e^{jM\omega_0 n} = \sum_{r=<N>} a_r e^{jr\omega_0 n} e^{jM\omega_0 n}$$

$$\overset{k=r+M}{=} \sum_{k=<N>} a_{k-M} e^{jk\omega_0 n}$$

$$\Downarrow$$

$$b_k = a_{k-M}$$

Frequency shift

$jM\omega_0, \rightarrow j(k\text{-}M)\omega_o$

Two Important Properties

- **Periodic convolution:**

  Suppose x and y are two periodic signals with common period N, the periodic convolution between x and y is defined as

$$x[n] \circledast y[n] = \sum_{k=<N>} x[k]y[n-k]$$

- **Suppose $x[n] \rightarrow a_k$ and $y[n] \rightarrow b_k$, then**

$$x[n] \circledast y[n] \rightarrow \mathrm{N}a_k b_k \quad \text{and} \quad x[n]y[n] \rightarrow a_k \circledast b_k$$

- **Parseval's Relation**

$$\frac{1}{N} \sum_{n=<N>} |x[n]|^2 = \sum_{k=<N>} |a_k|^2$$

Frequency Behavior of LTI Systems

System Functions *H*(*s*) or *H*(*z*)

CT:

$$x(t) \xrightarrow{e^{st}} \boxed{h(t)} \xrightarrow{H(s)e^{st}} y(t)$$

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

$$x(t) = \sum a_k e^{s_k t} \longrightarrow y(t) = \sum H(s_k)a_k e^{s_k t}$$

DT:

$$x[n] \xrightarrow{z^n} \boxed{h[n]} \xrightarrow{H(z)z^n} y[n]$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

$$x[n] = \sum a_k z_k^n \longrightarrow y[n] = \sum H(z_k)a_k z_k^n$$

Frequency Response of an LTI System

$$(s = j\omega)$$

$$e^{j\omega t} \longrightarrow \boxed{H(j\omega)} \longrightarrow H(j\omega)e^{j\omega t}$$

CT Frequency response: $\boxed{H(j\omega) = \int_{-\infty}^{+\infty} h(t)e^{-j\omega t}dt}$

$$e^{j\omega n} \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow H(e^{j\omega})e^{j\omega n}$$

$$(z = e^{j\omega})$$

DT Frequency response: $\boxed{H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n]e^{-j\omega n}}$ Periodic

Fourier Series and LTI Systems

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} H(jk\omega_0) a_k e^{jk\omega_0 t}$$

$$a_k \longrightarrow \underbrace{H(jk\omega_0)}_{\text{"gain"}} a_k$$

$$H(j\omega)$$

$$H(jk\omega_o) = |H(jk\omega_o)|\, e^{j\angle H(jk\omega_o)},$$

includes both amplitude & phase

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} H(e^{jk\omega_o}) a_k e^{jk\omega_0 n}$$

$$a_k \longrightarrow \underbrace{H(e^{jk\omega_o})}_{\text{"gain"}} a_k$$

$$H(e^{j\omega})$$

$$H(e^{jk\omega_o}) = |H(e^{jk\omega_o})|\, e^{j\angle H(e^{jk\omega_o})},$$

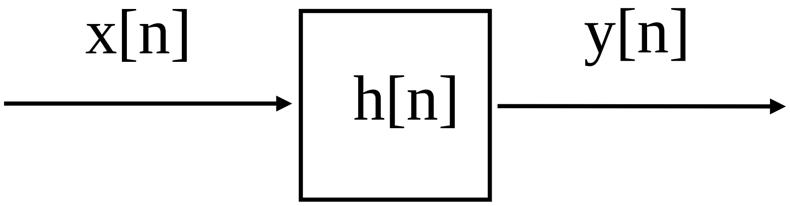
includes both amplitude & phase

The effect of the LTI system is to modify each a_k through multiplication by the value of the frequency response at the corresponding frequency.

Example 3.17

$$h[n]=\alpha^{n}u[n] \quad , \qquad |\alpha|<1$$

$$x[n]=\cos\left(\frac{2\pi n}{N}\right)=\frac{1}{2}e^{j(\frac{2\pi}{N})n}+\frac{1}{2}e^{-j(\frac{2\pi}{N})n}$$



$$H\left(e^{j\omega}\right)=\sum_{n=0}^{\infty}\alpha^{n}e^{-j\omega n}=\frac{1}{1-\alpha e^{-j\omega}}$$

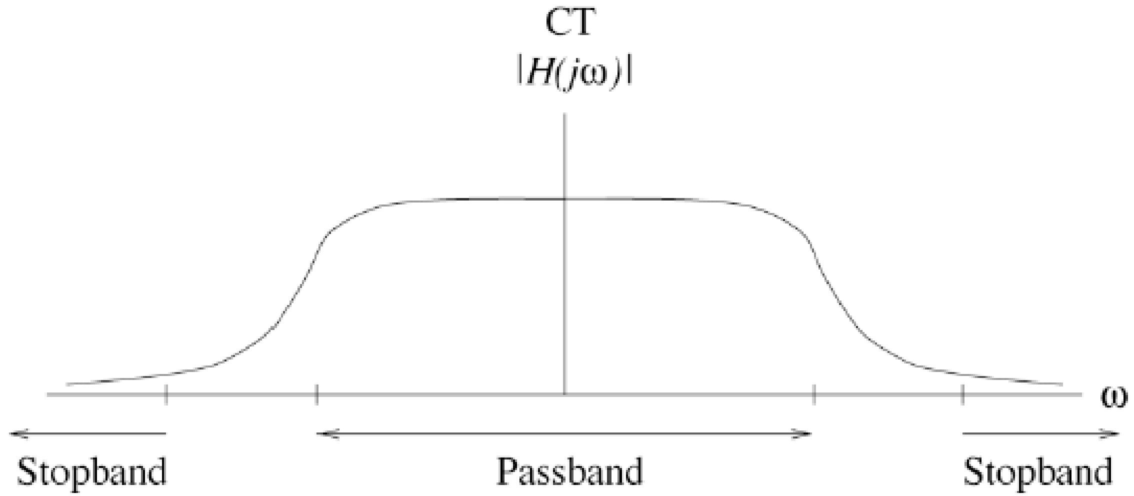
$$y[n]=\frac{1}{2}H\left(e^{j\frac{2\pi}{N}}\right)e^{j(\frac{2\pi}{N})n}+\frac{1}{2}H\left(e^{-j\frac{2\pi}{N}}\right)e^{-j(\frac{2\pi}{N})n}$$

$$=r\cos\left(\frac{2\pi n}{N}+\theta\right)$$

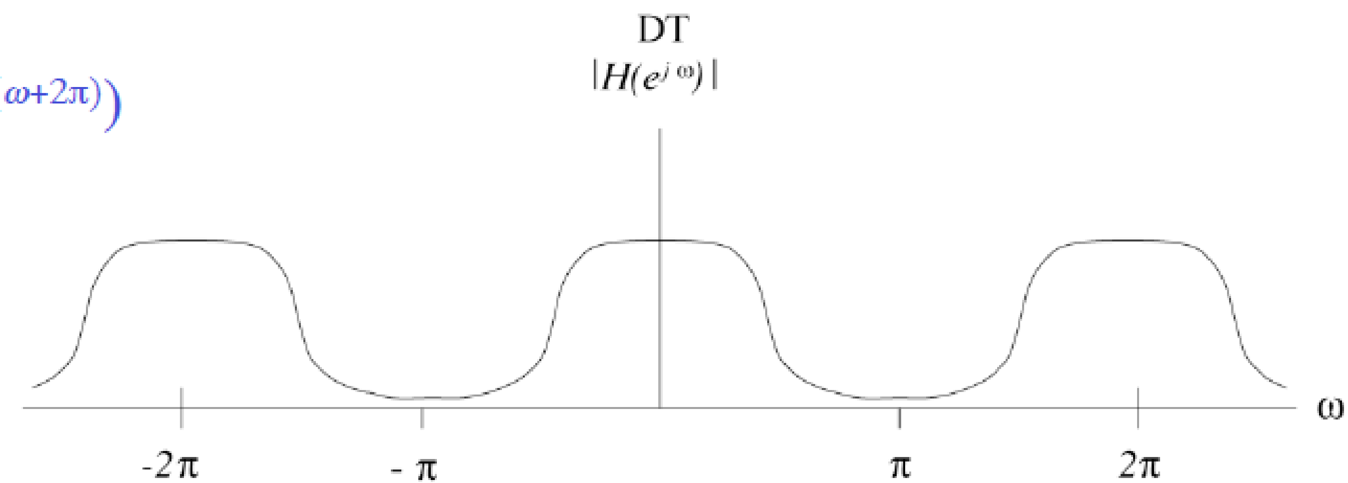
where $\quad re^{j\theta}=\dfrac{1}{1-\alpha e^{-j\frac{2\pi}{N}}}$

Lowpass Filter

Lowpass Filters:
Only show
amplitude here.

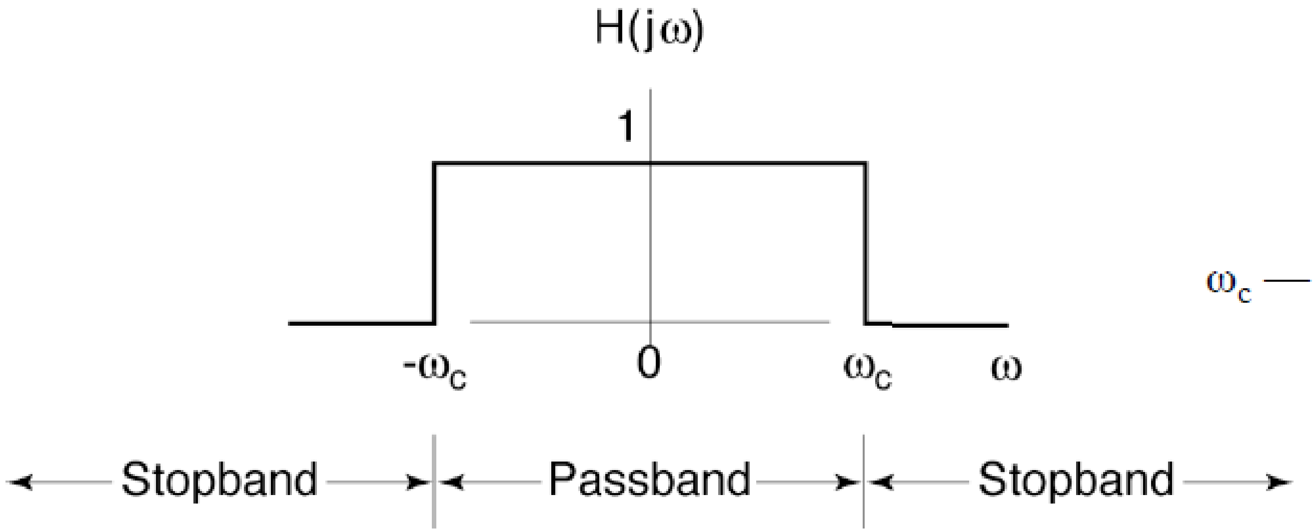


Note for DT:
$H(e^{j\omega}) = H(e^{j(\omega+2\pi)})$



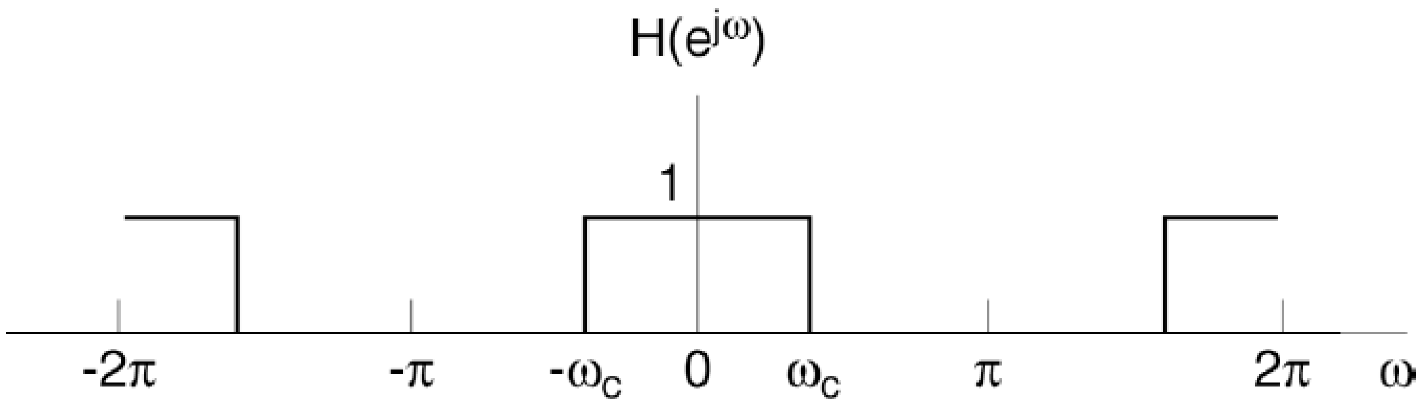
Ideal Lowpass Filter

CT



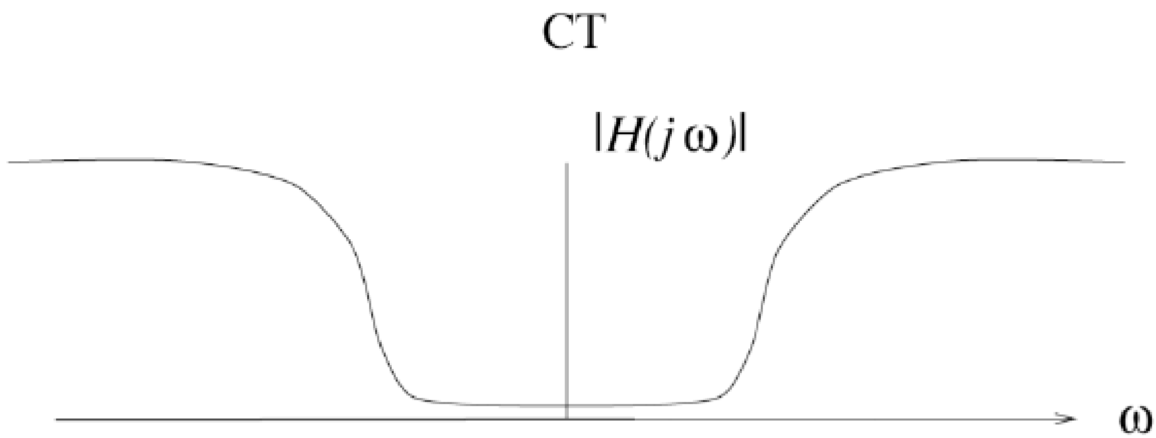
ω_c — cutoff frequency

DT



Note: $|H| = 1$ and $\angle H = 0$ for the ideal filters in the passbands, no need for the phase plot.

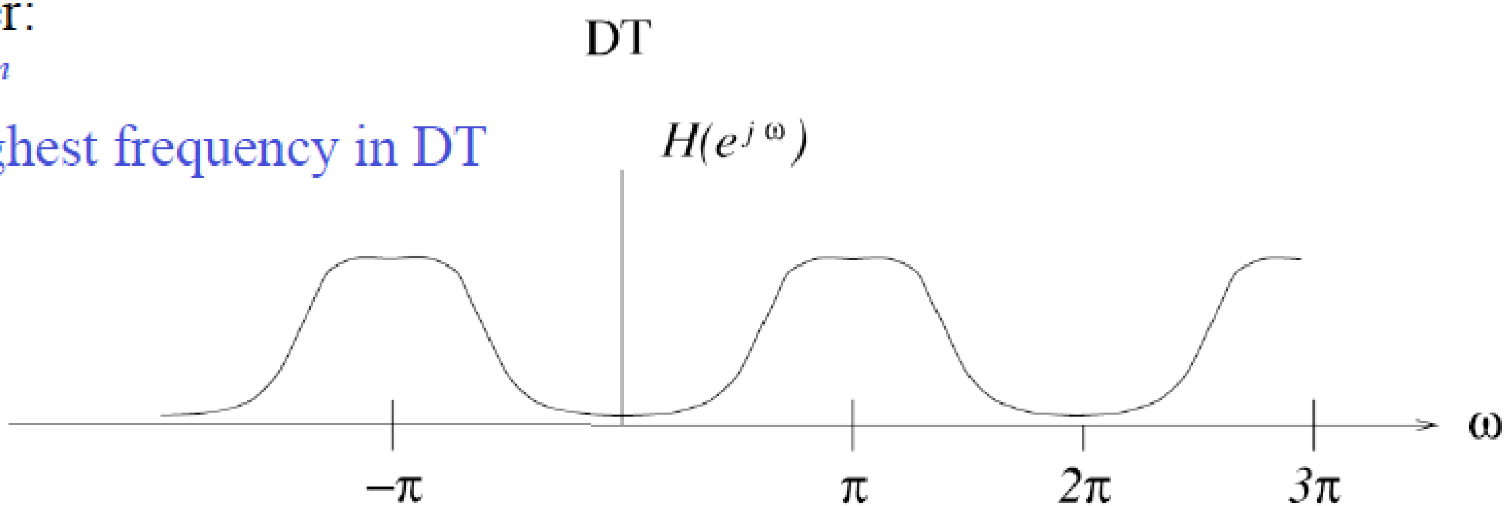
Highpass Filters



Remember:

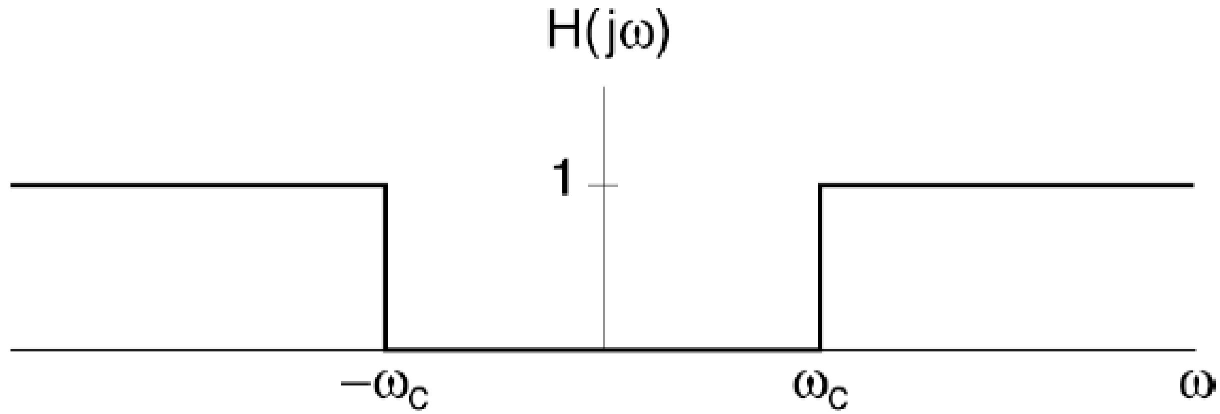
$(-1)^n = e^{j\pi n}$

— π = highest frequency in DT

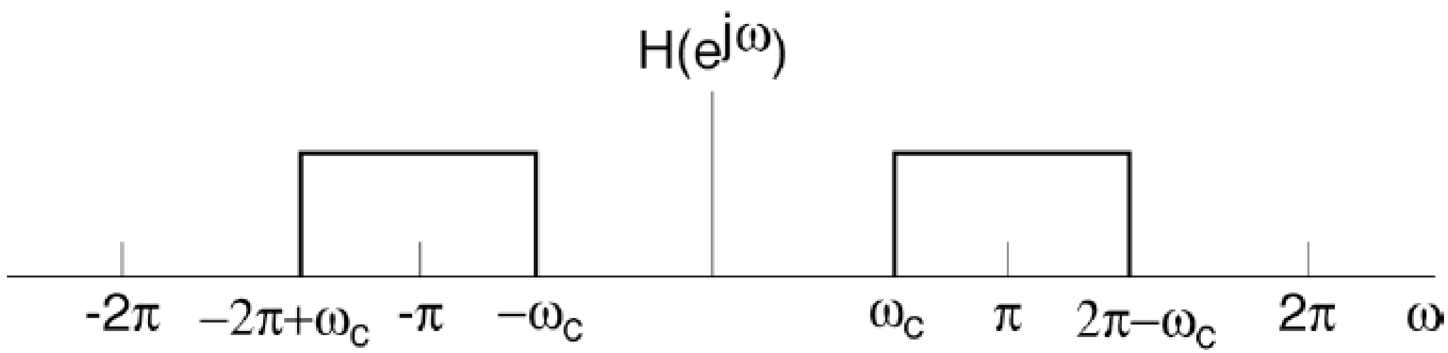


Ideal Highpass Filter

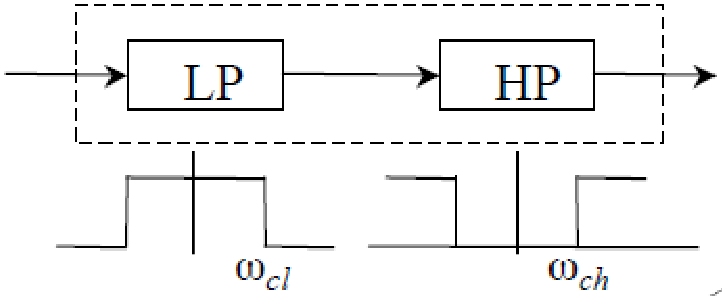
CT



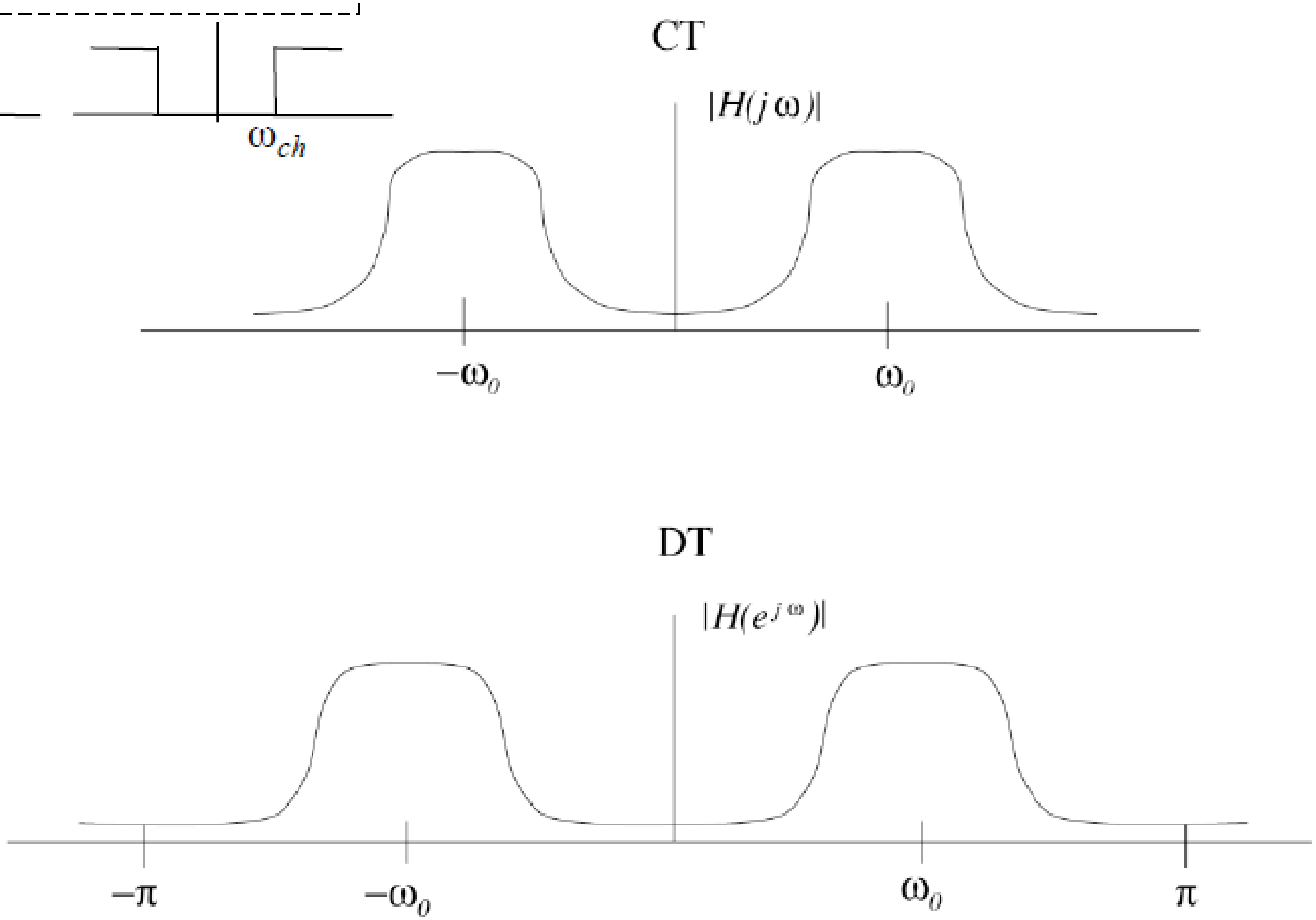
DT



Bandpass Filters

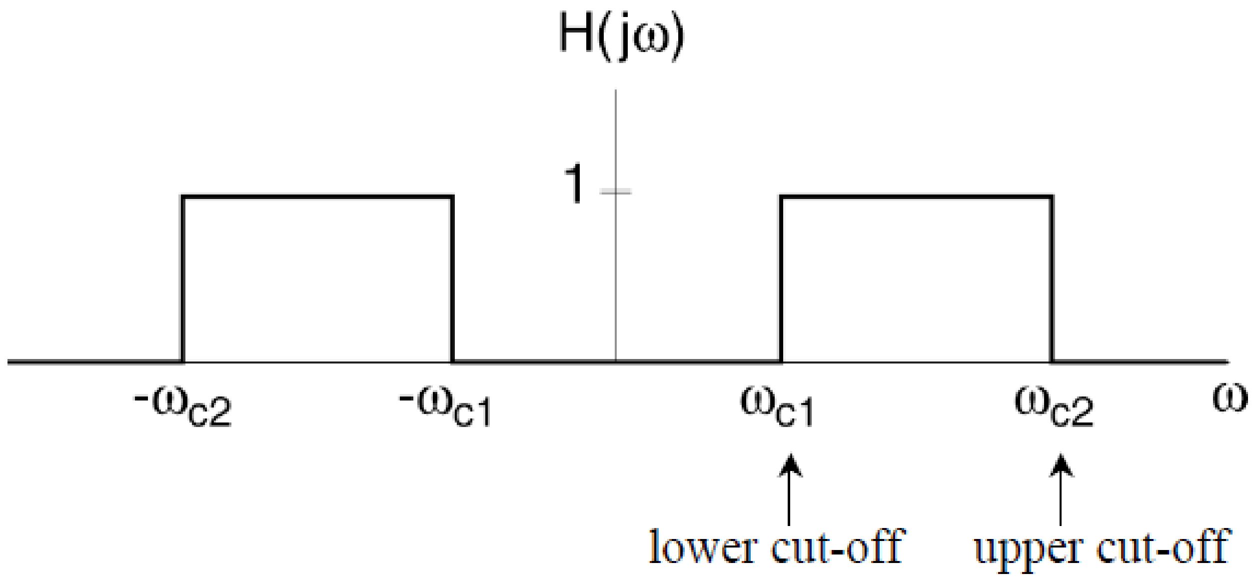


BPF if $\omega_{cl} > \omega_{ch}$

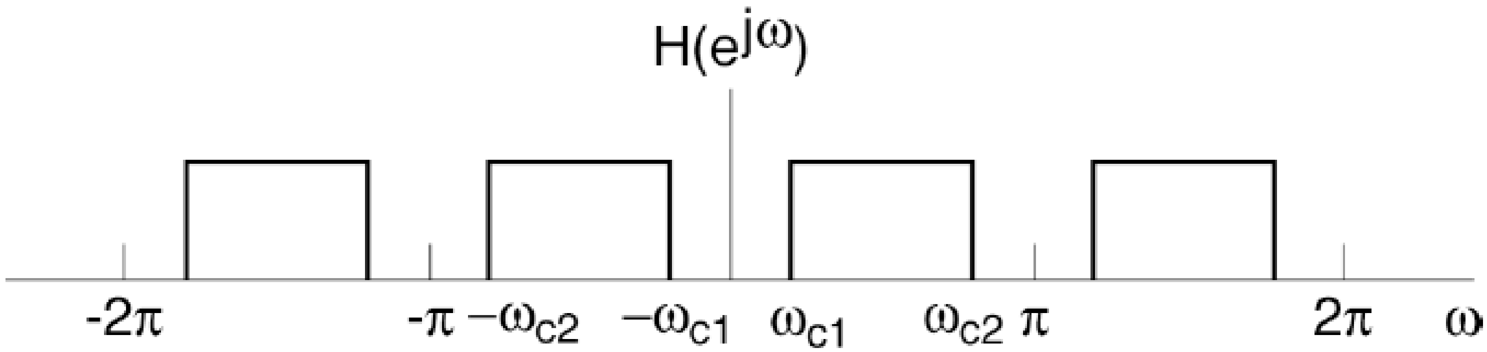


Ideal Bandpass Filter

CT



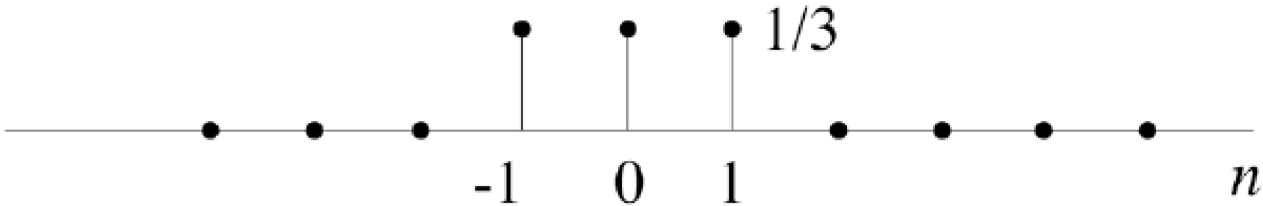
DT



Example: DT Averager/Smoother

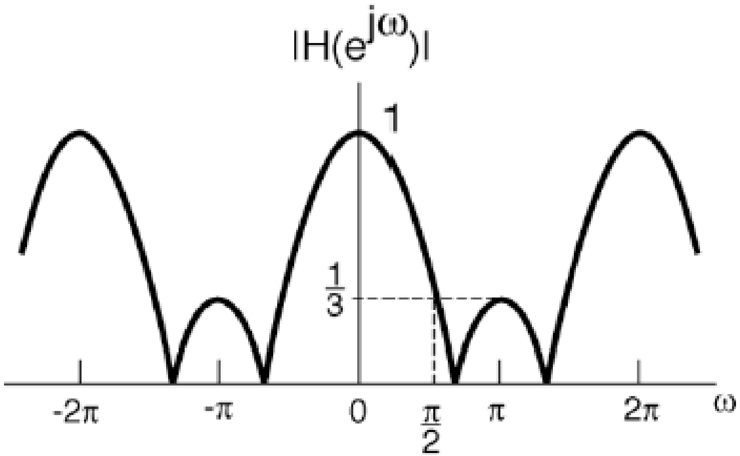
$$y[n] = \frac{1}{3}\{x[n-1] + x[n] + x[n+1]\}$$

$$h[n] = \frac{1}{3}\{\delta[n-1] + \delta[n] + \delta[n+1]\}$$



Frequency response:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = \frac{1}{3}[e^{-j\omega} + 1 + e^{j\omega}] = \frac{1}{3} + \frac{2}{3}\cos\omega$$



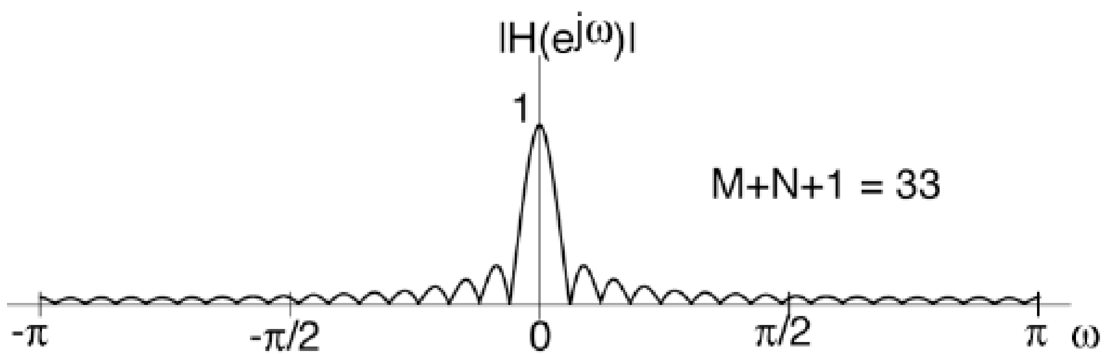
A LPF

Example: Nonrecursive DT (FIR) filters

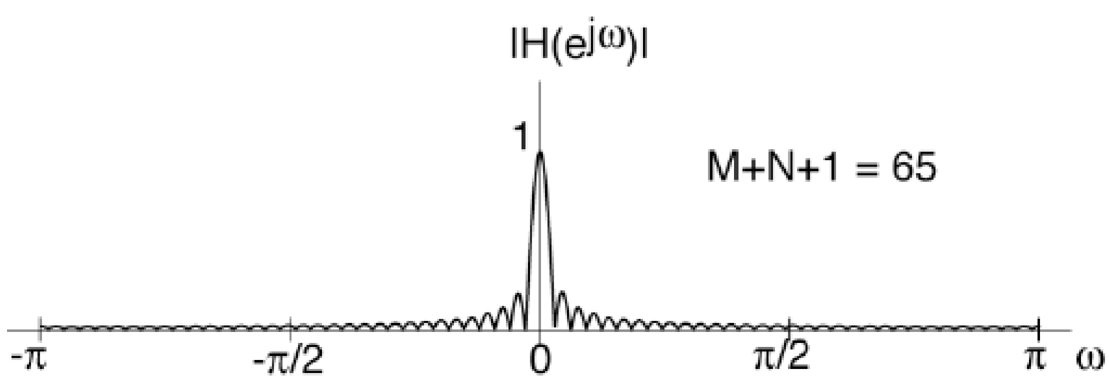
$$y[n] = \frac{1}{N+M+1}\sum_{k=-N}^{M} x[n-k] \longrightarrow h[n] = \frac{1}{N+M+1}\sum_{k=-N}^{M} \delta[n-k]$$

Frequency response:

$$H(e^{j\omega}) = \frac{1}{N+M+1}\sum_{k=-N}^{M} e^{-jk\omega} = \frac{1}{N+M+1} e^{j\omega(N-M)/2} \frac{\sin[\omega(M+N+1)/2]}{\sin(\omega/2)}$$



Rolls off at lower ω as M+N+1 increases



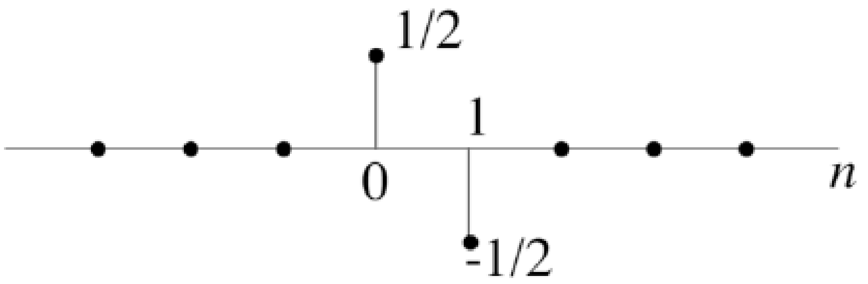
Example

Simple DT "Edge" Detector

— DT 2-points "differentiator"

$$y[n] = \frac{1}{2}[x[n] - x[n-1]]$$

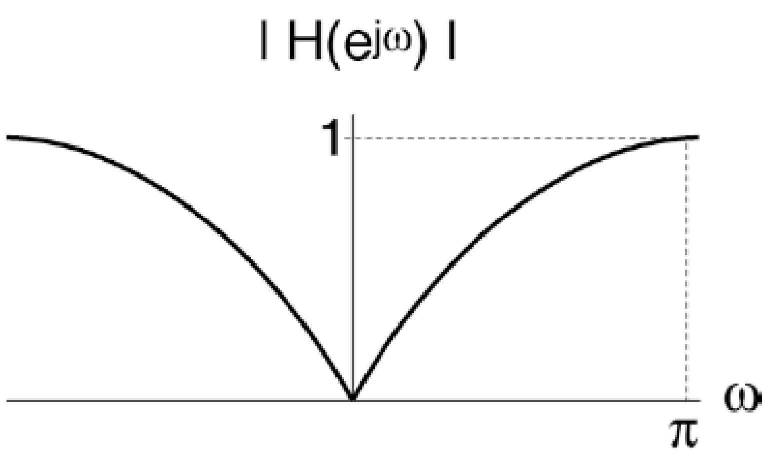
$$h[n] = \frac{1}{2}[\delta[n] - \delta[n-1]]$$



Frequency response:

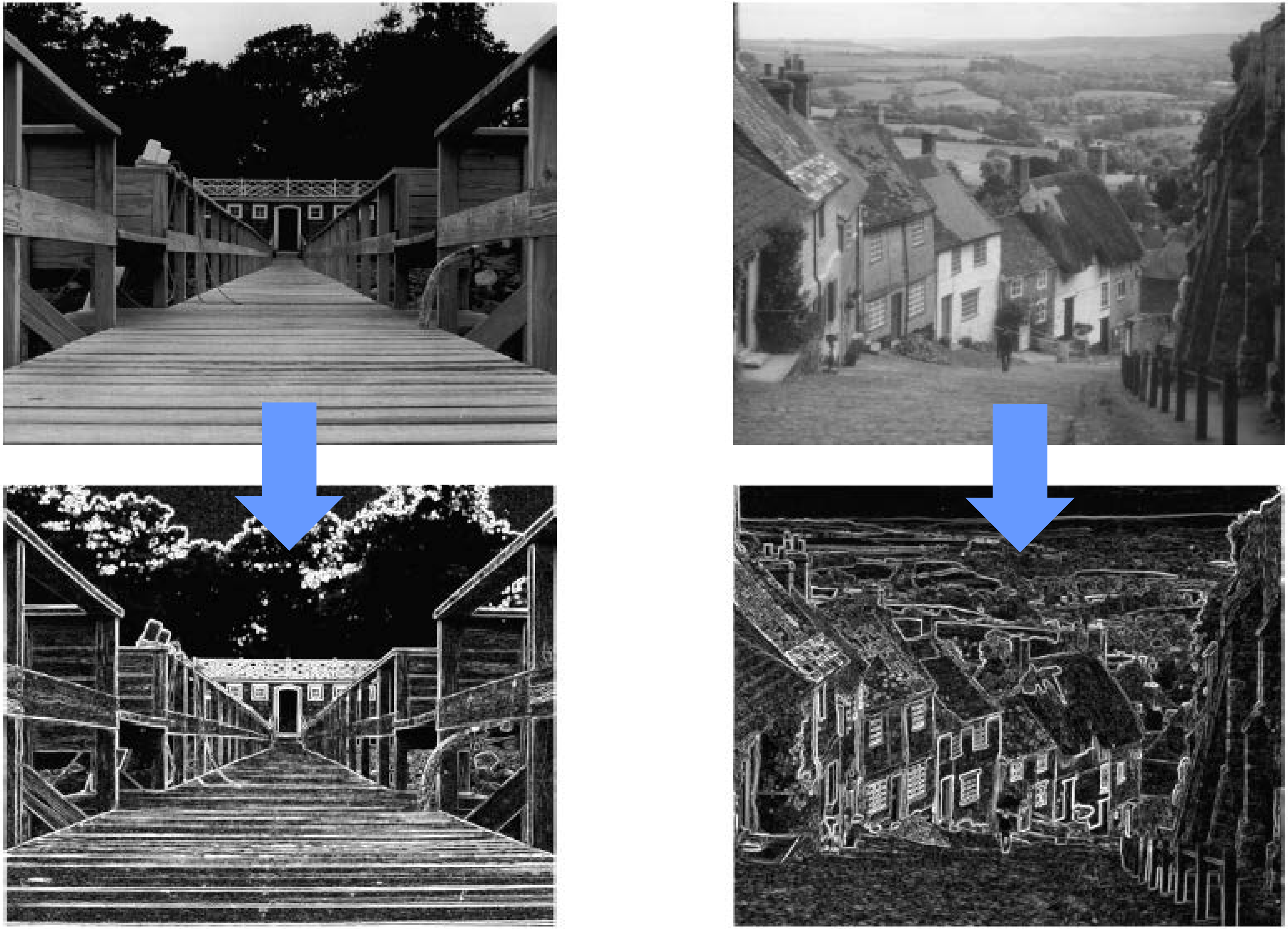
$$H(e^{j\omega}) = \frac{1}{2}(1 - e^{-j\omega}) = je^{j\omega/2}\sin(\omega/2)$$

$$|H(e^{j\omega})| = |\sin(\omega/2)|$$



Amplifies high-frequency components

Edge enhancement using DT differentiator



Summary

- DT Fourier Series pair
  - Understand the difference between CT and DT
- Frequency response
  - How to determine frequency response?
- Filtering