SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

Signals and Systems (Lab)

Lab 4: The Continuous-Time Fourier Transform

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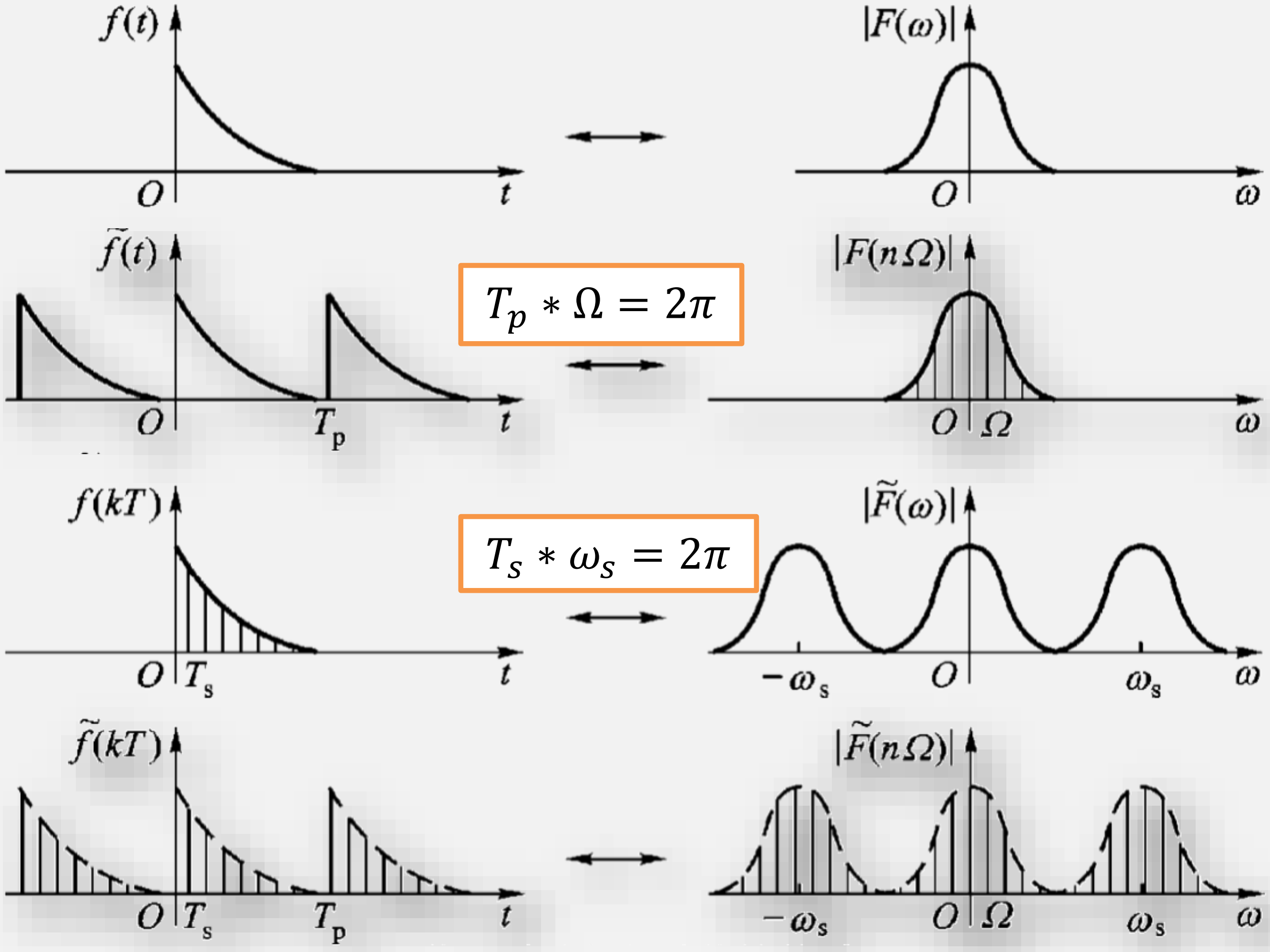
Review

- ✓ 1. How to calculate the output of DT LTI system in **frequency domain**
- ✓ 2. How to calculate the output of CT LTI system
- ✓ 3. How to calculate the ***DTFS*** of signal via Matlab

FT

DFT

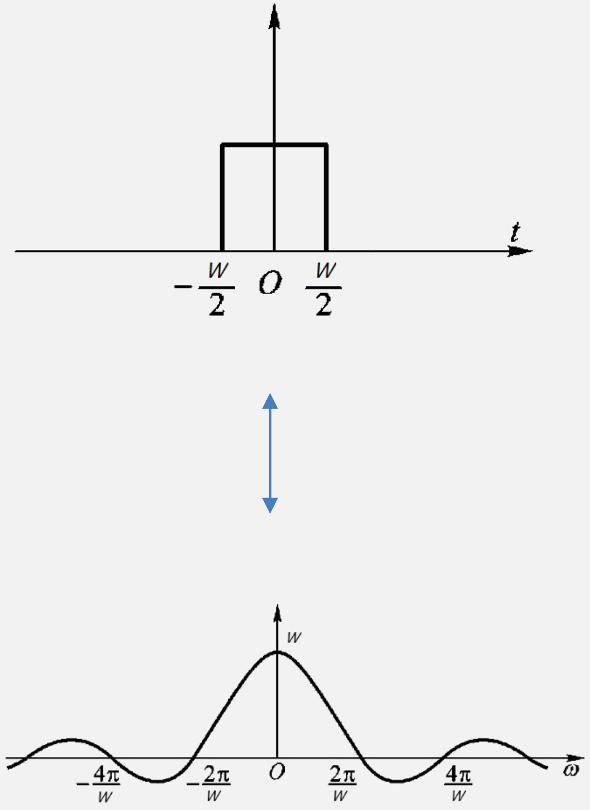


$\boxed{T_p * \Omega = 2\pi}$

$\boxed{T_s * \omega_s = 2\pi}$

Overview

- **CT Fourier Transform (CTFT):**
  - How to calculate via Matlab?
- **Frequency Response:**
  - How to calculate via Matlab ?
  - How to convert to impulse response
- **Application of CTFT:**
  - Analysis of amplitude modulation (AM)



CT Fourier Transform

- Definition of CT Fourier transform:

$$\mathbf{X}(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

  - Function of continuous frequency ω
  - Represent the "spectrum" of a signal

Use fft() to calculate the CTFT

CT Fourier Transform: $\mathbf{X}(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$

$$\Updownarrow$$

DT Fourier Series: $a_k = \frac{1}{N}\sum_{n=0}^{N-1} \mathrm{x[n]}e^{-jk\left(\frac{2\pi}{N}\right)n}$

Let $t = n\tau$

CT Fourier Transform:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$= \boxed{\lim_{\tau \to 0} \tau \sum_{n=-\infty}^{\infty} x(n\tau)e^{-j\omega\tau n}}$$

$$\Updownarrow$$

DT Fourier Series:

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\left(\frac{2\pi}{N}\right)n}$$

- Suppose the dominant region of x(t) is in [0,T], then we can use the following approximation

$$\boxed{X(j\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \approx \int_0^T x(t)e^{-j\omega t}dt \approx \sum_{n=0}^{N-1} x(n\tau)e^{-j\omega n\tau}\tau}$$

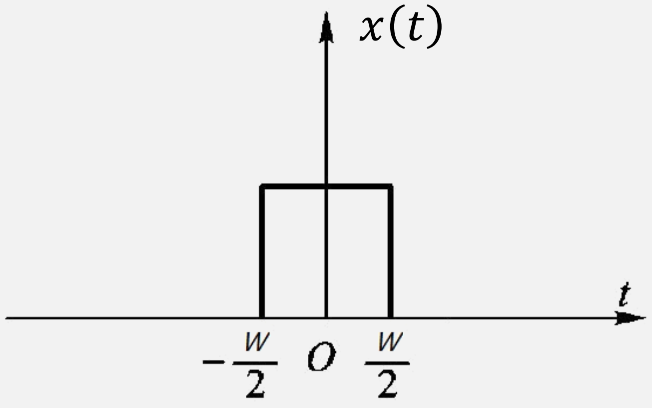
- It could be observed that

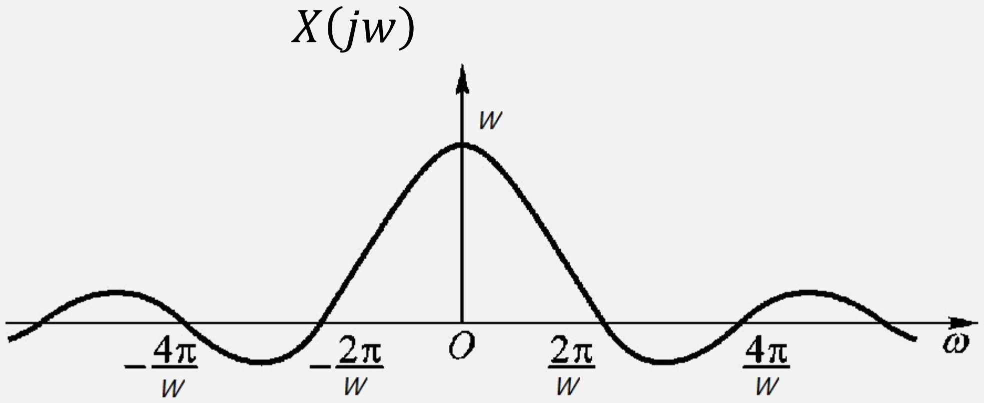
$$\boxed{X\left(j\frac{2\pi k}{N\tau}\right)\approx \boxed{\tau \sum_{n=0}^{N-1} x(n\tau)e^{-j\frac{2\pi kn}{N}}}}$$

- Slice into N intervals, each with length τ=T/N
- $k=\frac{1-N}{2},\dots,-1,0,1,\dots,\frac{N-1}{2}$
- fft() of [$x(0)$, $x(\tau)$, $x(2\tau)$, …, $x((N-1)\tau)$]

- **Conclusion**: we can use fft() to calculate CTFT approximately, which could reduce the computation complexity

Example-sinc function



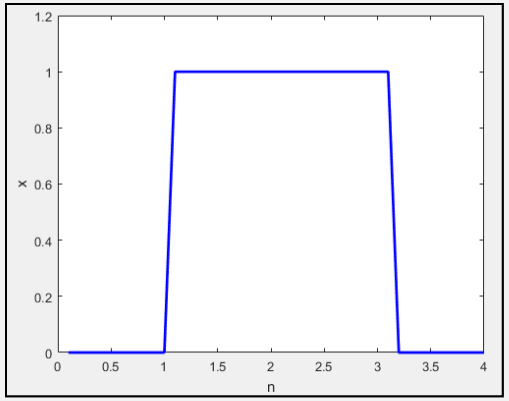


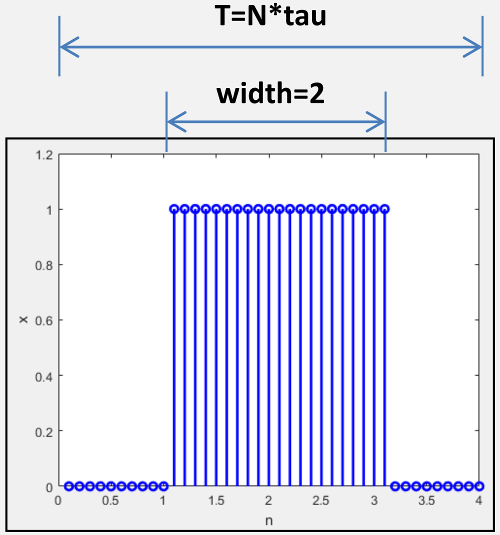
$$x(t) = \begin{cases} 1, t < \left|\frac{W}{2}\right| \\ 0, t > \left|\frac{W}{2}\right| \end{cases}$$

$$X(jw) = \left[\frac{\sin\left(\frac{\omega W}{2}\right)}{\frac{\omega W}{2}}\right] = WSinc\left(\frac{\omega W}{2}\right)$$

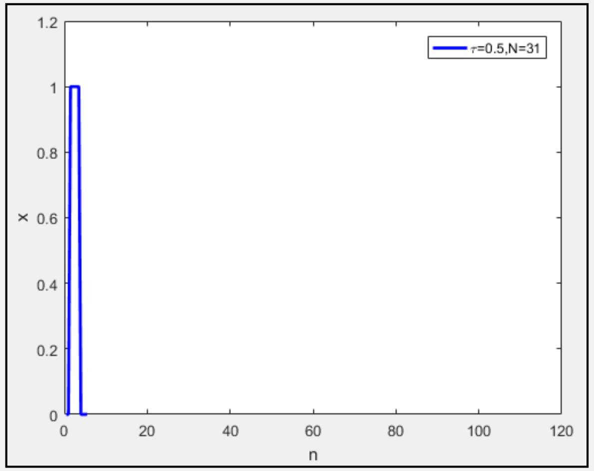
```
tau=0.5;
N=201; % N should be odd number,
N*tau=T>3
x=[zeros(1,1/tau),ones(1,2/tau+1),zeros(
1,N-3/tau-1)];
figure(1)
plot([1:N]*tau,x,‘r-','LineWidth',2)
xlabel('n');ylabel('x');
axis([0 4 0 1.2])
legend('\tau=0.5,N=201')

y=fftshift(fft(x));
X=tau*y;
lb = (1-N)*pi/N/tau;
ub = (N-1)*pi/N/tau;
step = 2*pi/N/tau;
figure(2)
plot(lb:step:ub, abs(X),‘r-','LineWidth',2);
xlabel('\omega');ylabel('abs(X)');
legend('\tau=0.5,N=201')
```

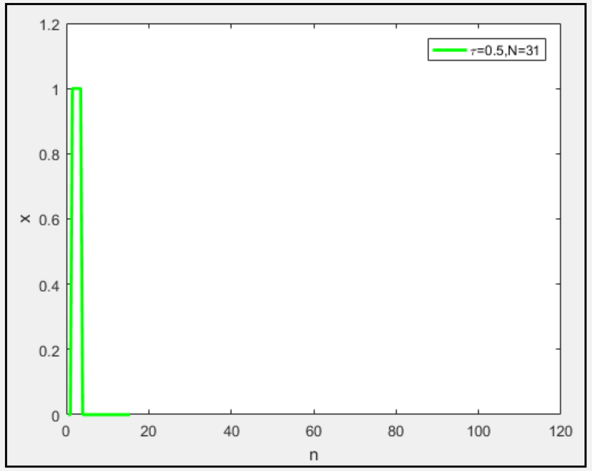
- **Single rectangular wave with width=2**
- **T should be larger than 3**

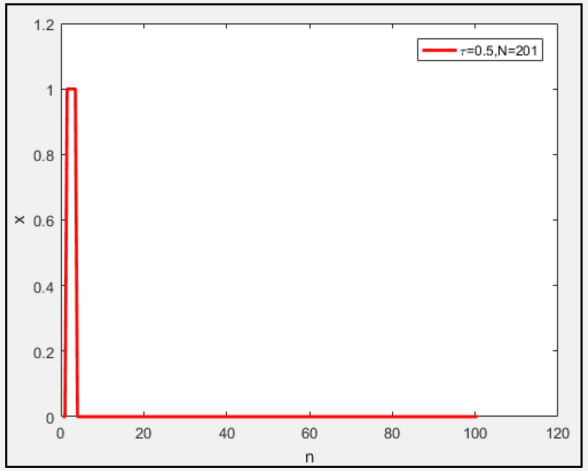




What happen if we change tau and N? Why?


τ=0.5,N=31
x
n

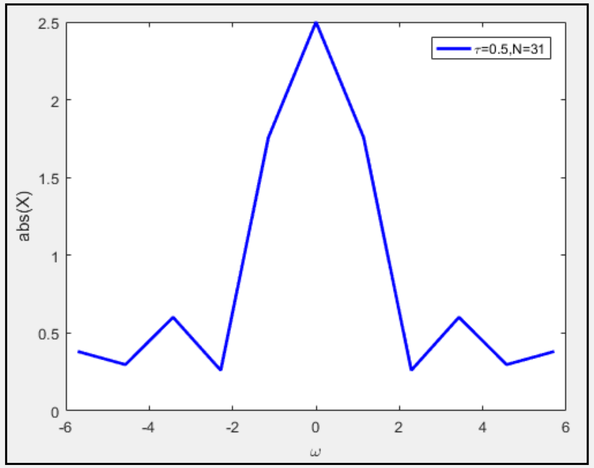

τ=0.5,N=31
x
n

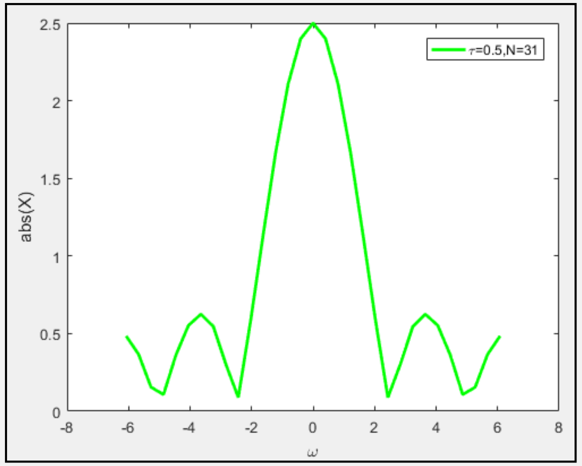

τ=0.5,N=201
x
n

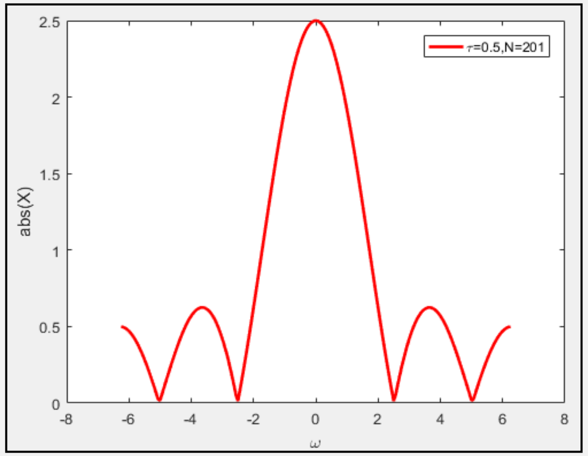
N=11

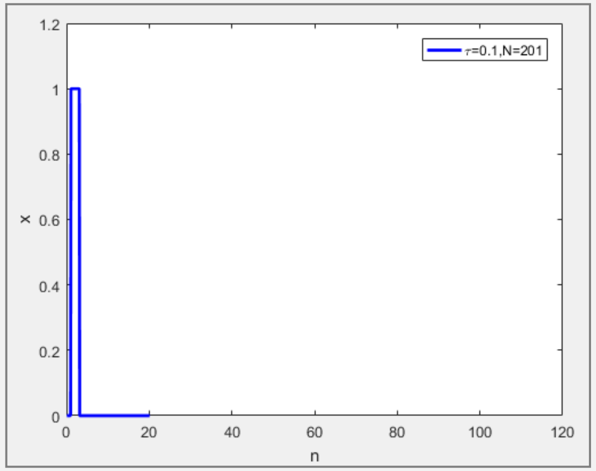
N=31

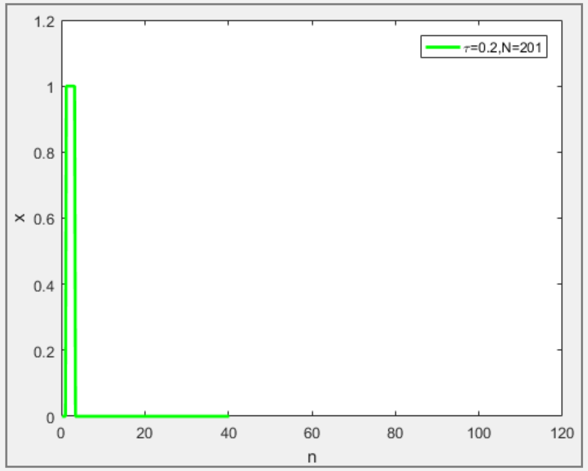
N=201

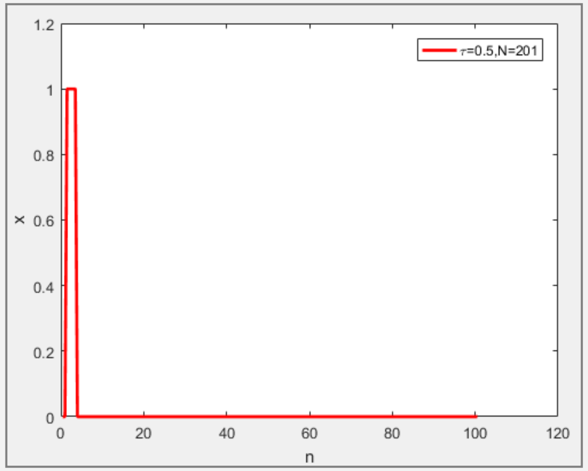

τ=0.5,N=31
abs(X)
ω


τ=0.5,N=31
abs(X)
ω


τ=0.5,N=201
abs(X)
ω



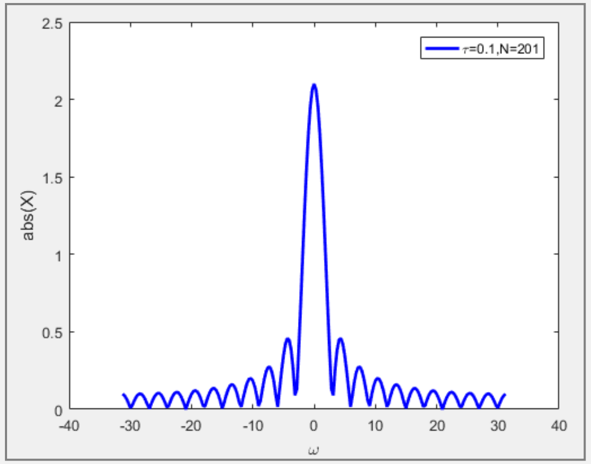


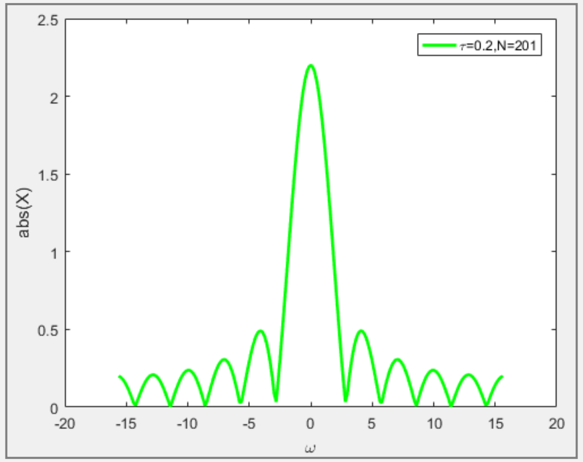


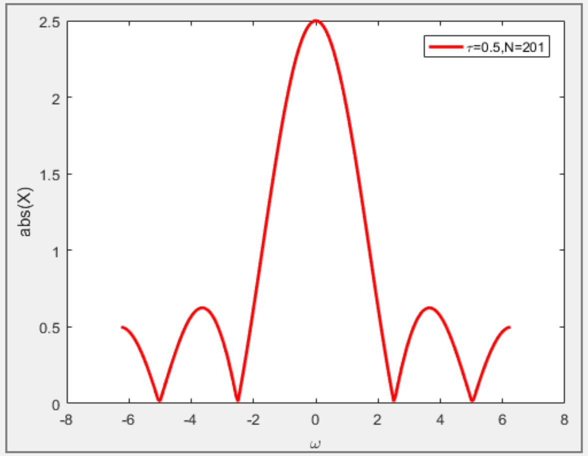
$\tau = 0.1$

$\tau = 0.2$

$\tau = 0.5$







What's Frequency Response ?

- **Frequency response**
  - A function of frequency
  - System gain in frequency domain

$$e^{j\omega_0 t} \Longrightarrow \boxed{\text{CT LTI System: H(jw)}} \Longrightarrow H(j\omega_0)e^{j\omega_0 t}$$

- **Math definition**
  - Fourier transform of impulse response

$$\boxed{H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt = Y(j\omega)/X(j\omega)}$$

Differential Equation and Frequency Response

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$H(j\omega) = \frac{b_M (j\omega)^M + b_{M-1}(j\omega)^{M-1} + \cdots + b_1(j\omega) + b_0}{a_N (j\omega)^N + a_{N-1}(j\omega)^{N-1} + \cdots + a_1(j\omega) + a_0}$$

Proof:

$$\mathcal{F}\left[\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right] = \sum_{k=0}^{N} \mathcal{F}\left[a_k \frac{d^k y(t)}{dt^k}\right] = \sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega)$$

$$\mathcal{F}\left[\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right] = \sum_{k=0}^{M} \mathcal{F}\left[b_k \frac{d^k x(t)}{dt^k}\right] = \sum_{k=0}^{M} b_k (j\omega)^k \boldsymbol{X}(j\omega)$$

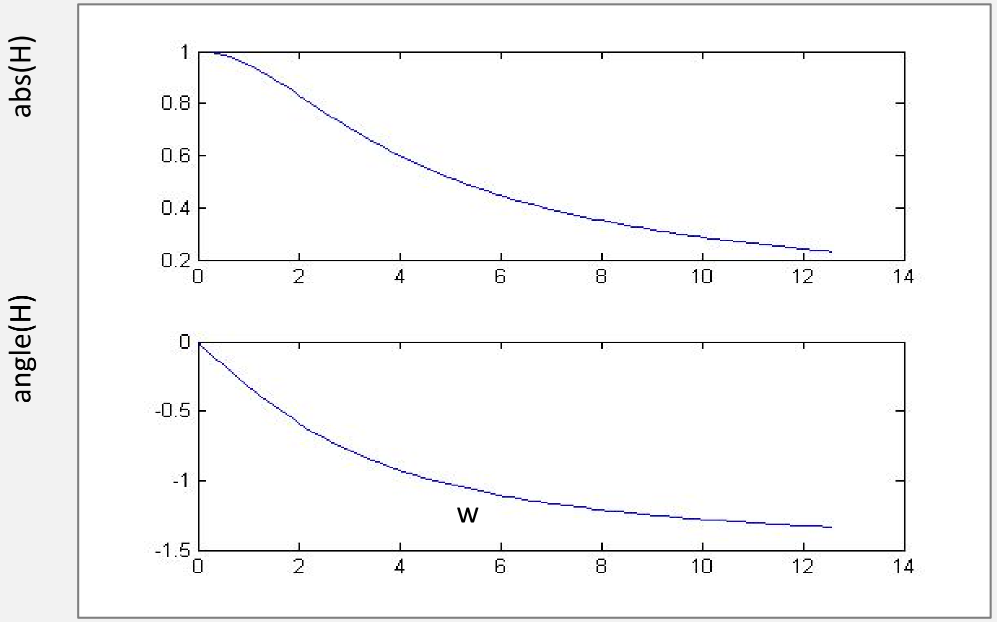
Matlab Function: freqs()

- **Description:** generate the frequency response of LTI system
- **Syntax:** freqs(b,a,w)
- **Example:**

High-pass or Low-pass?

$$\frac{dy(t)}{dt} + 3y(t) = 3x(t)$$

```
a=[1 3];
b=3;
w=linspace(0,4*pi);
H=freqs(b,a,w);
subplot(2,1,1), plot(w,abs(H));
subplot(2,1,2),
plot(w,angle(H));
```



Frequency Response and Impulse Response

$$\boxed{\text{Frequency Response}} \xLeftrightarrow{\text{Fourier Transform}} \boxed{\text{Impulse Response}}$$

- **Transform pair:**
  - $e^{-at}u(t) \quad Re\{a\} > 0 \quad \Leftrightarrow \quad \frac{1}{j\omega+a}$
- **Example:**
  - $\frac{3}{j\omega+3} \Leftrightarrow \quad 3e^{-3t}u(t)$
- **How about general fraction?**

$$H(j\omega) = \frac{b_M(j\omega)^M + b_{M-1}(j\omega)^{M-1} + \cdots + b_1(j\omega) \quad + b_0}{a_N(j\omega)^N + a_{N-1}(j\omega)^{N-1} + \cdots + a_1(j\omega) \quad + a_0}$$

Partial Fraction Expansion

- **Partial Fraction Expansion (No identical poles):**

$$H(j\omega) = \frac{b_M(j\omega + z_1)(j\omega + z_2)\ldots(j\omega + z_M)}{a_N(j\omega + p_1)(j\omega + p_2)\ldots(j\omega + p_N)}$$
$$= \frac{A_1}{j\omega + p_1} + \frac{A_2}{j\omega + p_2} + \cdots + \frac{A_N}{j\omega + p_N}$$

- **Partial Fraction Expansion (with identical poles):**

$$H(j\omega) = \frac{b_M(j\omega + z_1)(j\omega + z_2)\ldots(j\omega + z_M)}{a_N(j\omega + p_1)^{k_1}(j\omega + p_2)^{k_2}\ldots(j\omega + p_n)^{k_n}} =$$
$$= \frac{A_{1,1}}{(j\omega + p_1)^{k_1}} + \frac{A_{1,2}}{(j\omega + p_1)^{k_1-1}} + \cdots \frac{A_{1,k_1}}{(j\omega + p_1)}$$
$$+ \cdots +$$
$$\frac{A_{n,1}}{(j\omega + p_n)^{k_n}} + \frac{A_{n,2}}{(j\omega + p_n)^{k_n-1}} + \cdots \frac{A_{n,k_n}}{(j\omega + p_n)}$$

Partial Fraction Expansion

Therefore, we should know the FT of $\boxed{\dfrac{1}{(j\omega + a)^k}}$

We already know: $e^{-at}u(t) \quad Re\{a\} > 0 \quad \Leftrightarrow \quad \dfrac{1}{j\omega + a}$

According to the FT property: $tx(t) \quad \Leftrightarrow \quad j\dfrac{d}{d\omega}X(j\omega)$

We have: $\boxed{te^{-at}u(t) \quad Re\{a\} > 0 \quad \Leftrightarrow \quad \dfrac{1}{(j\omega + a)^2}}$

…

Matlab Function: residue()

- Convert between partial fraction expansion and polynomial coefficients
- [r,p,k] = residue(b,a) ;   [r,p] = residue(b,a)

$$\frac{b(s)}{a(s)} = \frac{b_1 s^m + b_2 s^{m-1} + b_3 s^{m-2} + \cdots + b_{m+1}}{a_1 s^n + a_2 s^{n-1} + a_3 s^{n-2} + \cdots + a_{n+1}}$$

$$\frac{b(s)}{a(s)} = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} + \cdots + \frac{r_n}{s - p_n} + k(s)$$

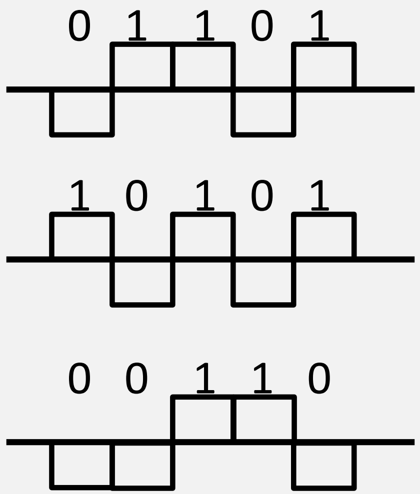
- Example：

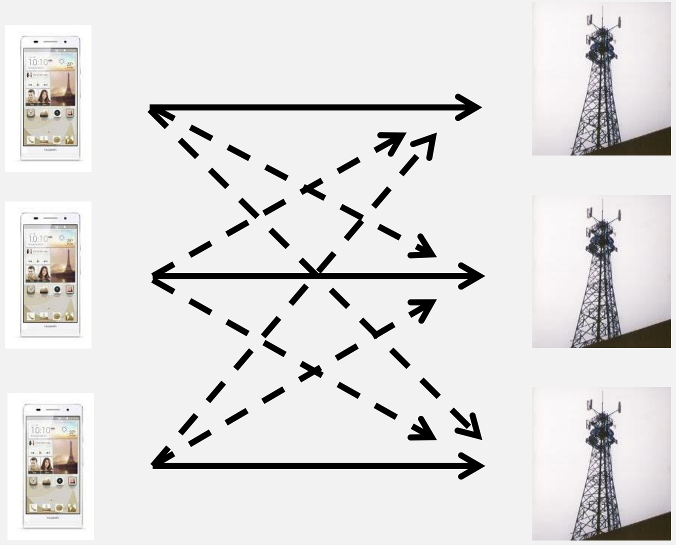
$$\frac{b(s)}{a(s)} = \frac{5s^3 + 3s^2 - 2s + 7}{-4s^3 + 8s + 3}$$

$\Longleftrightarrow$

```
b = [ 5 3 -2 7];
a = [-4 0 8 3];
[r, p, k] = residue(b,a);
```

What’s Modulation

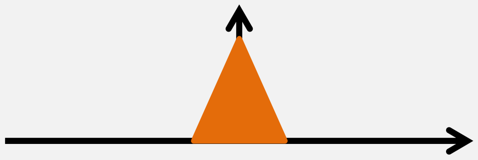




- Each phone wants to deliver information to its base station
- Therefore, there is cross-talk in the wireless channel
- How can we solve this issue?

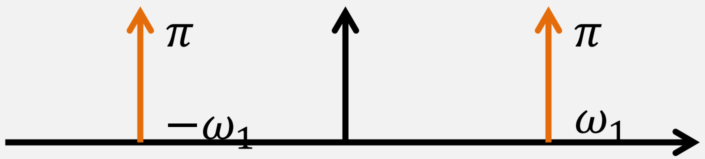
Amplitude Modulation

- Signal: m(t)



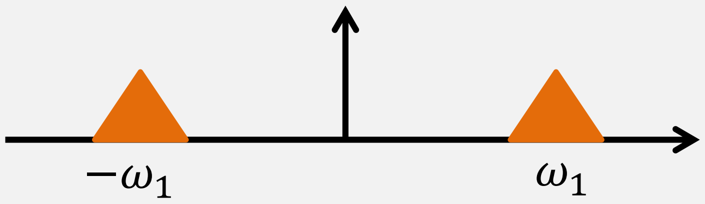
Spectrum of m(t)

- Carrier: $\cos(\omega_1 t) \Leftrightarrow \pi[\delta(\omega - \omega_1) + \delta(\omega + \omega_1)]$



Spectrum of carrier

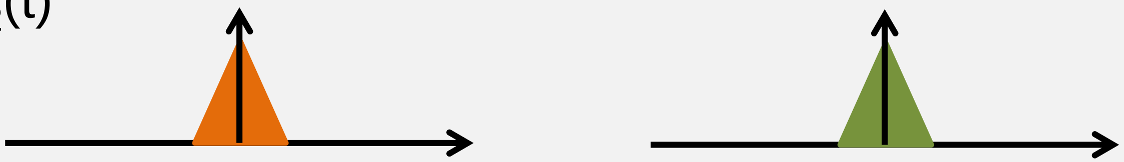
- Amplitude Modulation: $y(t) = m(t) \times \cos(\omega_1 t)$



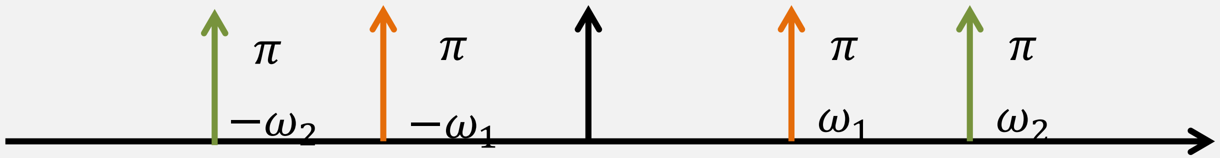
Shift by ω_1 and Scale by 1/2

No information loss

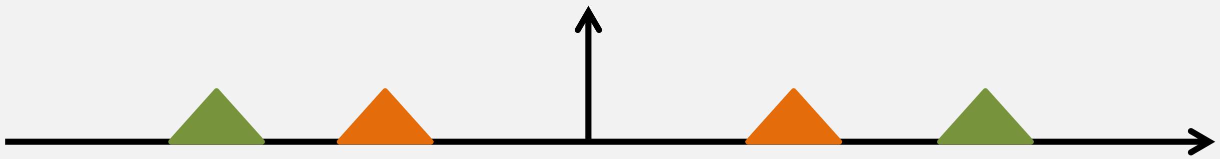
- Signals: $m_1(t)$ and $m_2(t)$



- Carriers: $\cos(\omega_1 t)$ and $\cos(\omega_2 t)$



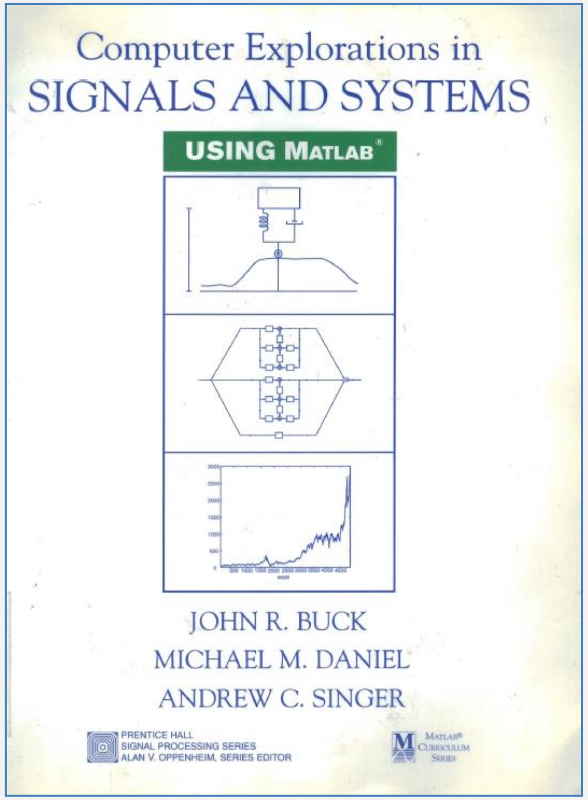
- Amplitude Modulation: $y(t) = m_1(t) \times \cos(\omega_1 t) + m_2(t) \times \cos(\omega_2 t)$



- **Signals are distinguished in frequency domain ！**

Lab Assignments

- Read tutorial 4.1 by yourself
- 4.2、4.5 & 4.6



- **Question ?**