Solutions to Additional Problems

4.16. Find the FT representations for the following periodic signals: Sketch the magnitude and phase spectra.
(a) $x(t) = 2\cos(\pi t) + \sin(2\pi t)$

$$
\begin{aligned}
x(t) &= e^{j\pi t} + e^{-j\pi t} + \frac{1}{2j}e^{j2\pi t} - \frac{1}{2j}e^{-j2\pi t} \\
\omega_o &= lcm(\pi, 2\pi) = \pi \\
x[1] &= x[-1] = 1 \\
x[2] &= -x[-2] = \frac{1}{2j} \\
X(j\omega) &= 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k\omega_o) \\
X(j\omega) &= 2\left[\pi\delta(\omega-\pi) + \pi\delta(\omega+\pi)\right] + \frac{1}{j}\left[\pi\delta(\omega-2\pi) - \pi\delta(\omega+2\pi)\right]
\end{aligned}
$$

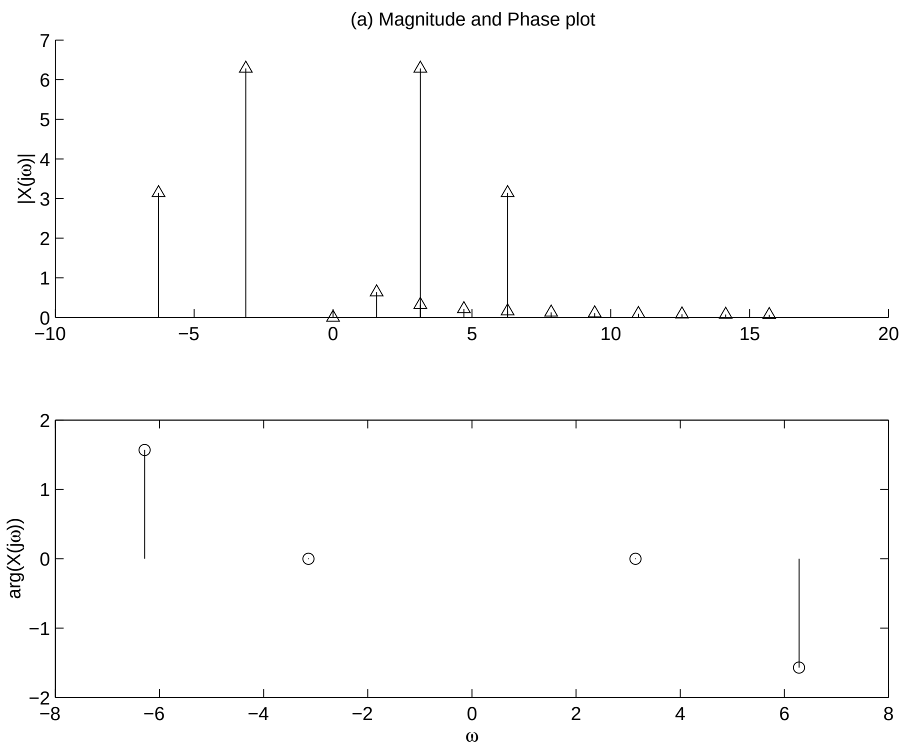


Figure P4.16. (a) Magnitude and Phase plot

(b) $x(t) = \sum_{k=0}^{4} \frac{(-1)^k}{k+1}\cos((2k+1)\pi t)$

$$
x(t) = \frac{1}{2}\sum_{k=0}^{4} \frac{(-1)^k}{k+1}\left[e^{j(2k+1)\pi t} + e^{-j(2k+1)\pi t}\right]
$$

$$X(j\omega) = \pi \sum_{k=0}^{4} \frac{(-1)^k}{k+1} \left[\delta(\omega - (2k+1)\pi) + \delta(\omega + (2k+1)\pi) \right]$$

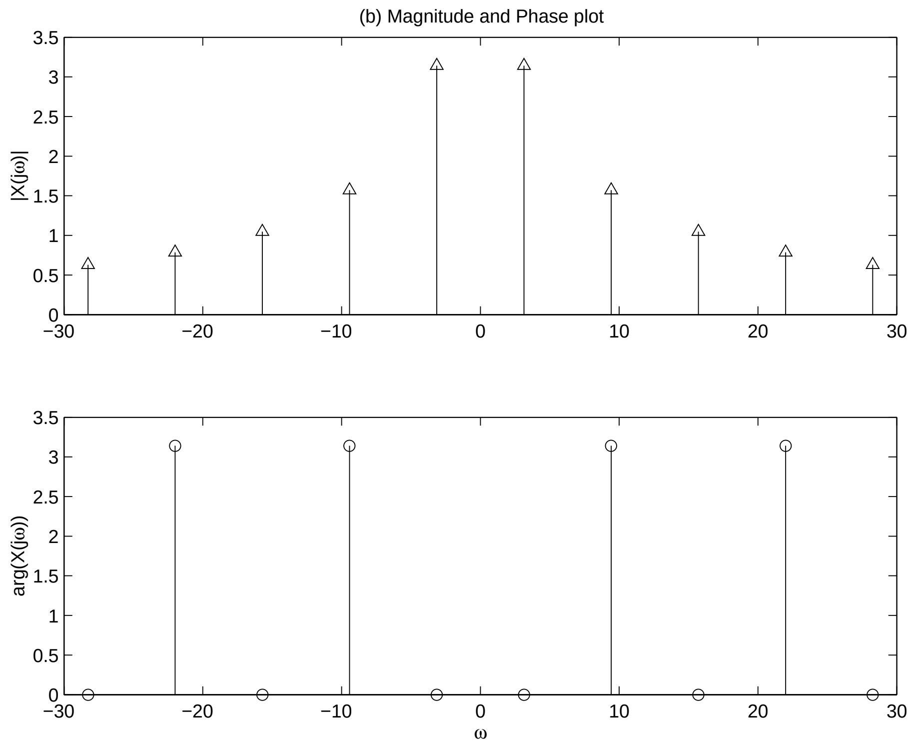


Figure P4.16. (b) Magnitude and Phase plot

(c) $x(t)$ as depicted in Fig. P4.16 (a).

$$x(t) = \begin{cases} 1 & |t| \leq 1 \\ 0 & \text{otherwise} \end{cases} + \begin{cases} 2 & |t| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} \left[\frac{2\sin(k\frac{2\pi}{3})}{k} + \frac{4\sin(k\frac{\pi}{3})}{k} \right] \delta(\omega - k\frac{2\pi}{3})$$

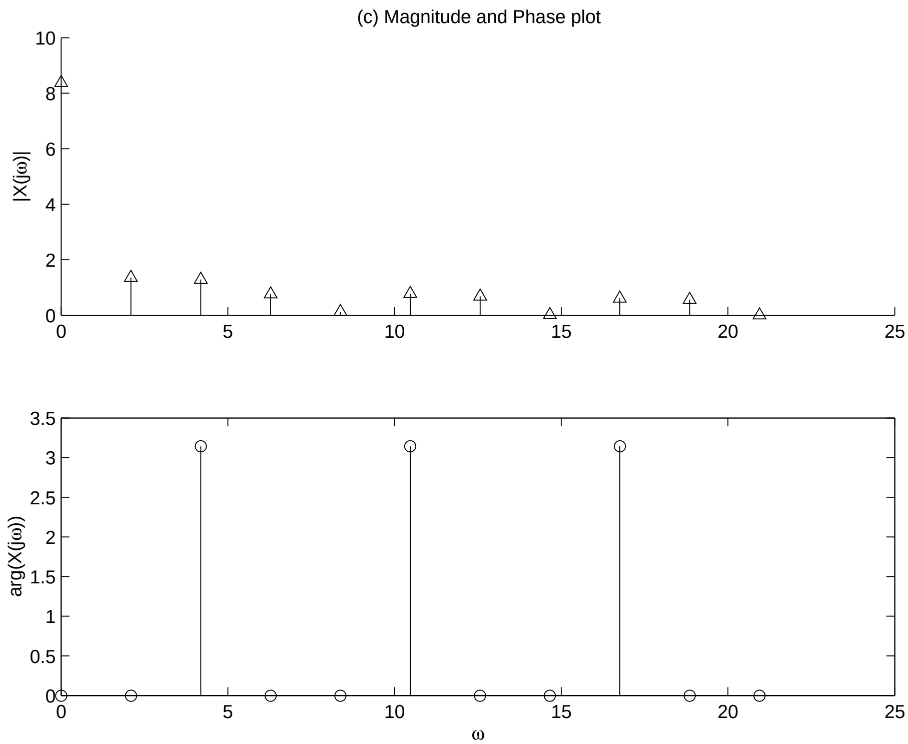


Figure P4.16. (c) Magnitude and Phase plot

(d) $x(t)$ as depicted in Fig. P4.16 (b).

$$
\begin{aligned}
T = 4 \qquad & \omega_o = \frac{\pi}{2} \\
X[k] &= \frac{1}{4}\int_{-2}^{2} 2te^{-j\frac{\pi}{2}kt}dt \\
&= \begin{cases} 0 & k = 0 \\ \frac{2j\cos(\pi k)}{\pi k} & k \neq 0 \end{cases} \\
X(j\omega) &= 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - \frac{\pi}{2}k)
\end{aligned}
$$

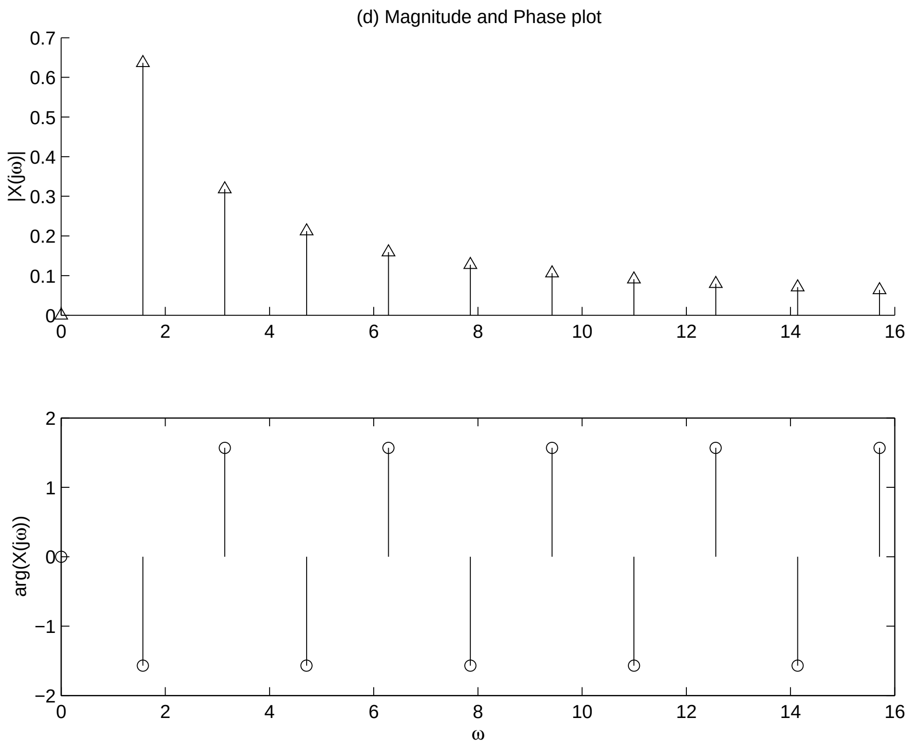


Figure P4.16. (d) Magnitude and Phase plot

4.17. Find the DTFT representations for the following periodic signals: Sketch the magnitude and phase spectra.

(a) $x[n] = \cos(\frac{\pi}{8}n) + \sin(\frac{\pi}{5}n)$

$$
\begin{aligned}
x[n] &= \frac{1}{2}\left[e^{j\frac{\pi}{8}n} + e^{-j\frac{\pi}{8}n}\right] + \frac{1}{2j}\left[e^{j\frac{\pi}{5}n} - e^{-j\frac{\pi}{5}n}\right] \\
\Omega_o &= lcm(\frac{\pi}{8}, \frac{\pi}{5}) = \frac{\pi}{40} \\
X[5] &= X[-5] = \frac{1}{2} \\
X[8] &= -X[-8] = \frac{1}{2j} \\
X(e^{j\Omega}) &= 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\Omega - k\Omega_o) \\
X(e^{j\Omega}) &= \pi\left[\delta(\Omega - \frac{\pi}{8}) + \delta(\Omega + \frac{\pi}{8})\right] + \frac{\pi}{j}\left[\delta(\Omega - \frac{\pi}{5}) - \delta(\Omega + \frac{\pi}{5})\right]
\end{aligned}
$$

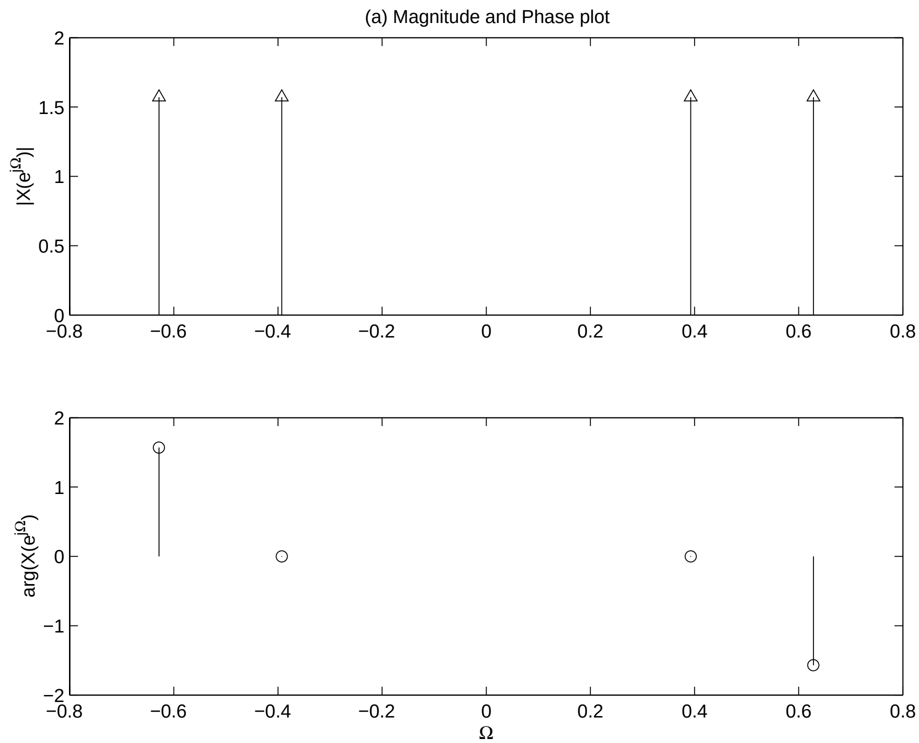


Figure P4.17. (a) Magnitude and Phase response

(b) $x[n] = 1 + \sum_{m=-\infty}^{\infty} \cos(\frac{\pi}{4}m)\delta[n-m]$

$$
\begin{aligned}
N = 8 \qquad & \Omega_o = \frac{\pi}{4} \\
x[n] &= 1 + \sum_{m=-\infty}^{\infty} \cos(\frac{\pi}{4}m)\delta[n-m] \\
&= 1 + \cos(\frac{\pi}{4}n) \\
X[k] &= \frac{1}{8}\sum_{n=-4}^{3} x[n]e^{-jk\frac{\pi}{4}n} \\
& \text{For one period of } X[k], k \in [-4, 3] \\
X[-4] &= 0 \\
X[-3] &= \frac{1-2^{-0.5}}{8}e^{jk\frac{3\pi}{4}} \\
X[-2] &= \frac{1}{8}e^{jk\frac{2\pi}{4}} \\
X[-1] &= \frac{1+2^{-0.5}}{8}e^{jk\frac{\pi}{4}} \\
X[0] &= \frac{2}{8} \\
X[1] &= \frac{1+2^{-0.5}}{8}e^{-jk\frac{\pi}{4}} \\
X[2] &= \frac{1}{8}e^{-jk\frac{2\pi}{4}}
\end{aligned}
$$

$$
\begin{aligned}
X[3] &= \frac{1-2^{-0.5}}{8}e^{-jk\frac{3\pi}{4}} \\
X(e^{j\Omega}) &= 2\pi\sum_{k=-\infty}^{\infty}X[k]\delta(\Omega-k\Omega_o) \\
&= \pi\left[\frac{(1-2^{-0.5})}{4}\delta(\Omega+\frac{3\pi}{4})+\frac{1}{4}\delta(\Omega+\frac{\pi}{2})+\frac{(1+2^{-0.5})}{4}\delta(\Omega+\frac{\pi}{4})+\frac{1}{4}\delta(2\Omega)\right]+ \\
&\quad \pi\left[\frac{(1+2^{-0.5})}{4}\delta(\Omega-\frac{\pi}{4})+\frac{1}{4}\delta(\Omega-\frac{\pi}{2})+\frac{(1-2^{-0.5})}{4}\delta(\Omega-\frac{3\pi}{4})\right]
\end{aligned}
$$

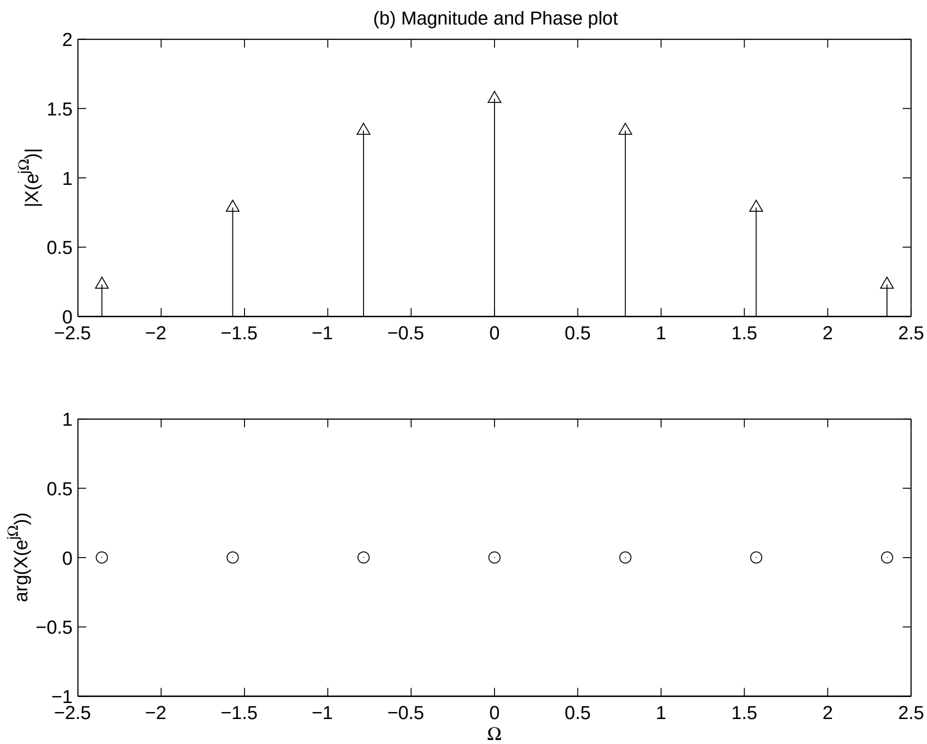


Figure P4.17. (b) Magnitude and Phase response

(c) $x[n]$ as depicted in Fig. P4.17 (a).

$$
N = 8 \qquad \Omega_o = \frac{\pi}{4}
$$

$$
\begin{aligned}
X[k] &= \frac{\sin(k\frac{5\pi}{8})}{8\sin(\frac{\pi}{8}k)} \\
X[0] &= \frac{5}{8} \\
X(e^{j\Omega}) &= 2\pi\sum_{k=-\infty}^{\infty}X[k]\delta(\Omega-k\frac{\pi}{4})
\end{aligned}
$$

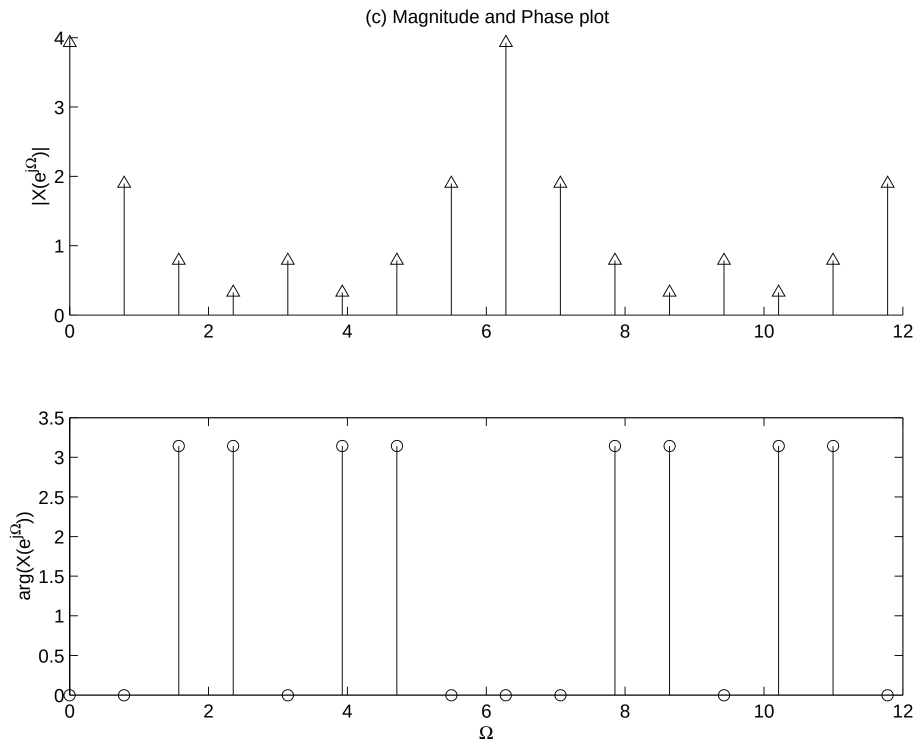


Figure P4.17. (c) Magnitude and Phase response

(d) $x[n]$ as depicted in Fig. P4.17 (b).

$$N = 7 \qquad \Omega_o = \frac{2\pi}{7}$$

$$\begin{aligned} X[k] &= \frac{1}{7}\left(1 - e^{jk\frac{2\pi}{7}} - e^{-jk\frac{2\pi}{7}}\right) \\ &= \frac{1}{7}\left(1 - 2\cos(k\frac{2\pi}{7})\right) \\ X(e^{j\Omega}) &= 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\Omega - k\frac{2\pi}{7}) \end{aligned}$$

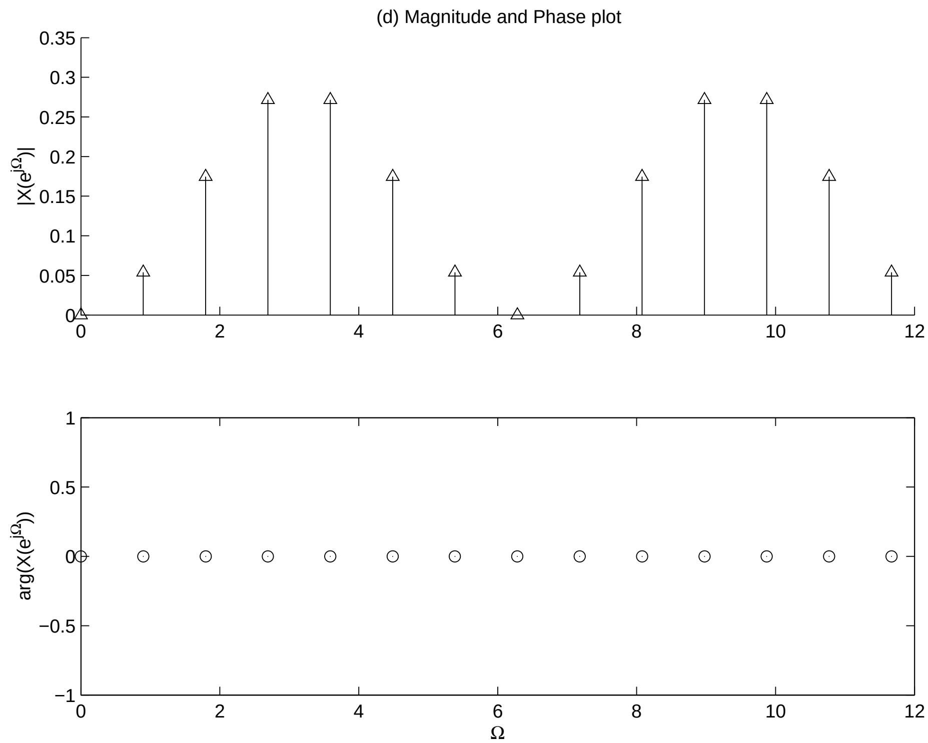


Figure P4.17. (d) Magnitude and Phase response

(e) $x[n]$ as depicted in Fig. P4.17 (c).

$$N = 4 \qquad \Omega_o = \frac{\pi}{2}$$

$$\begin{aligned}
X[k] &= \frac{1}{4}\left(1 + e^{-jk\frac{\pi}{2}} - e^{-jk\pi} - e^{-jk\frac{3\pi}{2}}\right) \\
&= \frac{1}{4}(1-(-1)^k) - \frac{j}{2}\sin(k\frac{\pi}{2}) \\
X(e^{j\Omega}) &= 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\Omega - k\frac{\pi}{2})
\end{aligned}$$

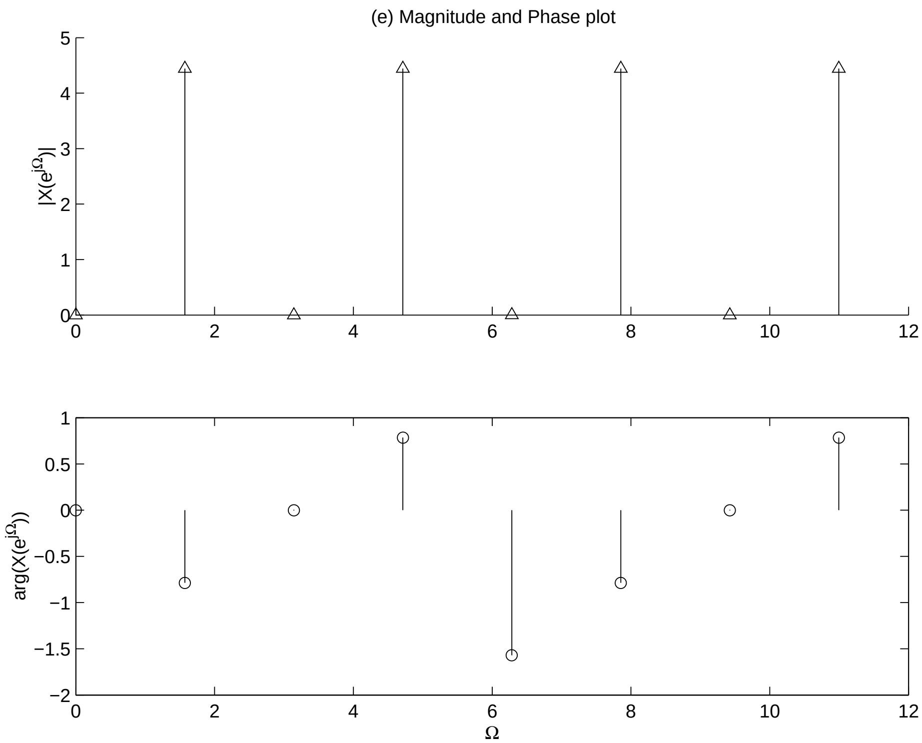


Figure P4.17. (e) Magnitude and Phase response

4.18. An LTI system has the impulse response

$$h(t) = 2\frac{\sin(2\pi t)}{\pi t}\cos(7\pi t)$$

Use the FT to determine the system output for the following inputs, $x(t)$.

$$\text{Let } a(t) = \frac{\sin(2\pi t)}{\pi t} \quad \xleftrightarrow{FT} \quad A(j\omega) = \begin{cases} 1 & |\omega| < 2\pi \\ 0 & \text{otherwise} \end{cases}$$

$$h(t) = 2a(t)\cos(7\pi t) \quad \xleftrightarrow{FT} \quad H(j\omega) = A(j(\omega - 7\pi)) + A(j(\omega + 7\pi))$$

(a) $x(t) = \cos(2\pi t) + \sin(6\pi t)$

$$\begin{aligned}
X(j\omega) &= 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k\omega_o) \\
X(j\omega) &= \pi\delta(\omega - 2\pi) + \pi\delta(\omega + 2\pi) + \frac{\pi}{j}\pi\delta(\omega - 6\pi) - \frac{\pi}{j}\pi\delta(\omega + 6\pi) \\
Y(j\omega) &= X(j\omega)H(j\omega) \\
&= \frac{\pi}{j}\delta(\omega - 6\pi) - \frac{\pi}{j}\delta(\omega + 6\pi) \\
y(t) &= \sin(6\pi t)
\end{aligned}$$

(b) $x(t) = \sum_{m=-\infty}^{\infty}(-1)^m\delta(t-m)$

$$\begin{aligned}
X[k] &= \frac{1}{2}(1-e^{-jk\pi}) = \frac{1}{2}(1-(-1)^k) \\
&= \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases} \\
X(j\omega) &= 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - l2\pi) \\
Y(j\omega) &= X(j\omega)H(j\omega) \\
&= 2\pi \sum_{l=3}^{4} [\delta(\omega - l2\pi) + \delta(\omega + l2\pi)] \\
y(t) &= 2\cos(6\pi t) + 2\cos(8\pi t)
\end{aligned}$$

(c) $x(t)$ as depicted in Fig. P4.18 (a).

$$\begin{aligned}
T = 1 & \qquad \omega_o = 2\pi \\
X(j\omega) &= 2\pi \sum_{k=-\infty}^{\infty} \left(\frac{\sin(k\frac{\pi}{4})}{k\pi}(1-e^{-jk\pi}) \right) \delta(\omega - k2\pi) \\
Y(j\omega) &= X(j\omega)H(j\omega) \\
&= 2\pi \left[\frac{\sin(3\frac{\pi}{4})}{3\pi}(1-e^{-j3\pi})\delta(\omega - 6\pi) + \frac{\sin(-3\frac{\pi}{4})}{-3\pi}(1-e^{j3\pi})\delta(\omega + 6\pi) \right] \\
&= \frac{4\sin(3\frac{\pi}{4})}{3}\delta(\omega - 6\pi) + \frac{4\sin(3\frac{\pi}{4})}{3}\delta(\omega + 6\pi) \\
y(t) &= \frac{4\sin(\frac{3\pi}{4})}{3\pi}\cos(6\pi t)
\end{aligned}$$

(d) $x(t)$ as depicted in Fig. P4.18 (b).

$$\begin{aligned}
X[k] &= 4\int_{-\frac{1}{8}}^{\frac{1}{8}} 16te^{-jk8\pi t}dt \\
&= \frac{2j\cos(\pi k)}{\pi k} \\
X(j\omega) &= 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k8\pi) \\
Y(j\omega) &= -4j\delta(\omega - 8\pi) + 4j\delta(\omega + 8\pi) \\
y[n] &= -\frac{4}{\pi}\sin(8\pi n)
\end{aligned}$$

(e) $x(t)$ as depicted in Fig. P4.18 (c).

$$T = 1 \qquad \omega_o = 2\pi$$

$$
\begin{aligned}
X[k] &= \int_0^1 e^{-t}e^{-jk2\pi t}dt \\
&= \frac{1-e^{-(1+jk2\pi)}}{1+jk2\pi} \\
&= \frac{1-e^{-1}}{1+jk2\pi} \\
X(j\omega) &= 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k2\pi) \\
Y(j\omega) &= X(j\omega)H(j\omega) \\
&= 2\pi\left(\frac{1-e^{-1}}{1+j6\pi}\delta(\omega-6\pi)+\frac{1-e^{-1}}{1+j8\pi}\delta(\omega-8\pi)+\frac{1-e^{-1}}{1-j6\pi}\delta(\omega+6\pi)+\frac{1-e^{-1}}{1-j8\pi}\delta(\omega+8\pi)\right) \\
Y(j\omega) &= (1-e^{-1})\left[\frac{e^{j6\pi t}}{1+j6\pi}+\frac{e^{j8\pi t}}{1+j8\pi}+\frac{e^{-j6\pi t}}{1-j6\pi}+\frac{e^{-j8\pi t}}{1-j8\pi}\right] \\
y(t) &= 2(1-e^{-1})\left[\mathrm{Re}\left\{\frac{e^{j6\pi t}}{1+j6\pi}+\frac{e^{j8\pi t}}{1+j8\pi}\right\}\right] \\
y(t) &= 2(1-e^{-1})\left[r_1\cos(6\pi t+\phi_1)+r_2\cos(8\pi t+\phi_2)\right]
\end{aligned}
$$

Where

$$
\begin{aligned}
r_1 &= \sqrt{1+(6\pi)^2} \\
\phi_1 &= -\tan^{-1}(6\pi) \\
r_2 &= \sqrt{1+(8\pi)^2} \\
\phi_1 &= -\tan^{-1}(8\pi)
\end{aligned}
$$

4.19. We may design a dc power supply by cascading a full-wave rectifier and an RC circuit as depicted in Fig. P4.19. The full wave rectifier output is given by

$$z(t) = |x(t)|$$

Let $H(j\omega) = \frac{Y(j\omega)}{Z(j\omega)}$ be the frequency response of the RC circuit as shown by

$$H(j\omega) = \frac{1}{j\omega RC + 1}$$

Suppose the input is $x(t) = \cos(120\pi t)$.
(a) Find the FT representation for $z(t)$.

$$\omega_o = 240\pi \qquad T = \frac{1}{120}$$

$$
\begin{aligned}
Z[k] &= 120\int_{-\frac{1}{240}}^{\frac{1}{240}} \frac{1}{2}\left(e^{j120\pi t}+e^{-j120\pi t}\right)e^{-jk240\pi t}dt \\
&= \frac{2(-1)^k}{\pi(1-4k^2)} \\
Z(j\omega) &= 4\sum_{k=-\infty}^{\infty}\frac{(-1)^k}{\pi(1-4k^2)}\delta(\omega-k240\pi)
\end{aligned}
$$

(b) Find the FT representation for $y(t)$.

$$\begin{aligned}
H(j\omega) &= \frac{Y(j\omega)}{Z(j\omega)} \\
\text{In the time domain:} & \\
z(t) - y(t) = RC\frac{d}{dt}y(t) &\xleftrightarrow{FT} Z(j\omega) = (1 + j\omega RC)Y(j\omega) \\
H(j\omega) &= \frac{1}{1 + j\omega RC} \\
Y(j\omega) &= Z(j\omega)H(j\omega) \\
&= 4\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{\pi(1-4k^2)} \left(\frac{1}{1 + jk240\pi RC}\right)\delta(\omega - k240\pi)
\end{aligned}$$

(c) Find the range for the time constant RC such that the first harmonic of the ripple in $y(t)$ is less than 1% of the average value.
The ripple results from the exponential terms. Let $\tau = RC$.

$$\begin{aligned}
\text{Use first harmonic only:} & \\
Y(j\omega) &\approx \frac{4}{\pi}\left[\delta(\omega) + \frac{1}{3}\left(\frac{\delta(\omega - 240\pi)}{1 + j240\pi RC} + \frac{\delta(\omega + 240\pi)}{1 + j240\pi RC}\right)\right] \\
y(t) &= \frac{2}{\pi^2} + \frac{2}{3\pi^2}\left[\frac{e^{j240\pi t}}{1 + j240\pi RC} + \frac{e^{-j240\pi t}}{1 - j240\pi RC}\right] \\
|\text{ripple}| &= \frac{2}{3\pi^2}\left[\frac{2}{\sqrt{1 + (240\pi\tau)^2}}\right] < 0.01\left(\frac{2}{\pi^2}\right) \\
240\pi\tau &> 66.659 \\
\tau &> 0.0884s
\end{aligned}$$

4.20. Consider the system depicted in Fig. P4.20 (a). The FT of the input signal is depicted in Fig. 4.20 (b). Let $z(t) \xleftrightarrow{FT} Z(j\omega)$ and $y(t) \xleftrightarrow{FT} Y(j\omega)$. Sketch $Z(j\omega)$ and $Y(j\omega)$ for the following cases.
(a) $w(t) = \cos(5\pi t)$ and $h(t) = \frac{\sin(6\pi t)}{\pi t}$

$$\begin{aligned}
Z(j\omega) &= \frac{1}{2\pi}X(j\omega) * W(j\omega) \\
W(j\omega) &= \pi\left(\delta(\omega - 5\pi) + \delta(\omega + 5\pi)\right) \\
Z(j\omega) &= \frac{1}{2}\left(X(j(\omega - 5\pi)) + X(j(\omega + 5\pi))\right) \\
H(j\omega) &= \begin{cases} 1 & |\omega| < 6\pi \\ 0 & \text{otherwise} \end{cases} \\
C(j\omega) &= H(j\omega)Z(j\omega) = Z(j\omega) \\
Y(j\omega) &= \frac{1}{2\pi}C(j\omega) * \left[\pi\left(\delta(\omega - 5\pi) + \delta(\omega + 5\pi)\right)\right] \\
Y(j\omega) &= \frac{1}{2}\left[Z(j(\omega - 5\pi)) + Z(j(\omega + 5\pi))\right]
\end{aligned}$$

$$= \frac{1}{4}\left[X(j(\omega-10\pi)) + 2X(j\omega) + X(j(\omega+10\pi))\right]$$

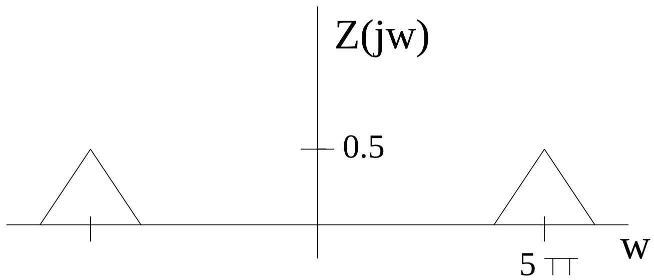


Figure P4.20. (a) Sketch of $Z(j\omega)$

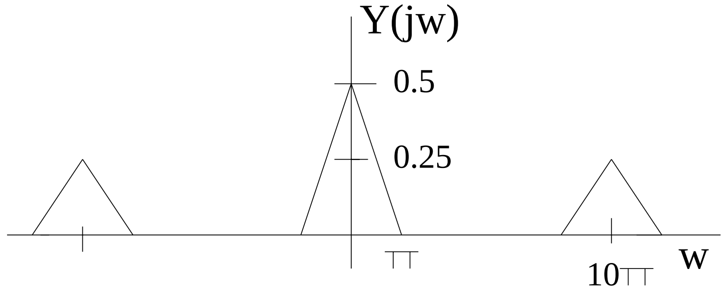


Figure P4.20. (a) Sketch of $Y(j\omega)$

(b) $w(t) = \cos(5\pi t)$ and $h(t) = \frac{\sin(5\pi t)}{\pi t}$

$$
\begin{aligned}
Z(j\omega) &= \frac{1}{2}\left[X(j(\omega-5\pi)) + X(j(\omega+5\pi))\right] \\
H(j\omega) &= \begin{cases} 1 & |\omega| < 5\pi \\ 0 & \text{otherwise} \end{cases} \\
C(j\omega) &= H(j\omega)Z(j\omega) \\
Y(j\omega) &= \frac{1}{2\pi}C(j\omega) * \left[\pi\left(\delta(\omega-5\pi) + \delta(\omega+5\pi)\right)\right] \\
Y(j\omega) &= \frac{1}{2}\left[C(j(\omega-5\pi)) + C(j(\omega+5\pi))\right]
\end{aligned}
$$

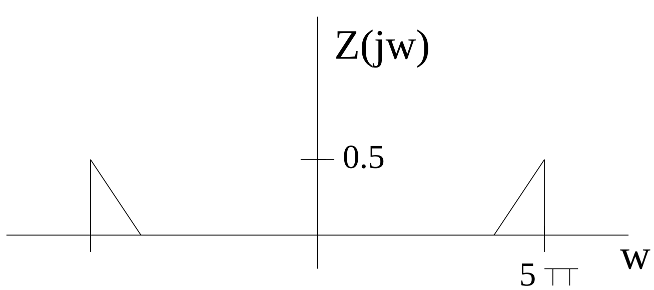


Figure P4.20. (b) Sketch of $Z(j\omega)$

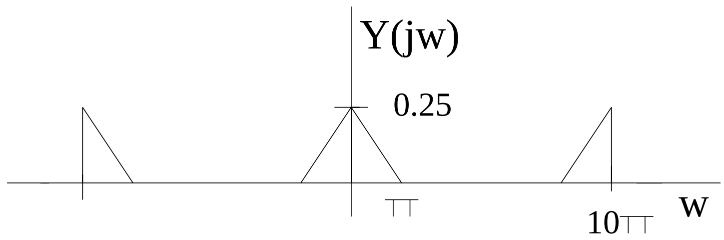


Figure P4.20. (b) Sketch of $Y(j\omega)$

(c) $w(t)$ depicted in Fig. P4.20 (c) and $h(t) = \frac{\sin(2\pi t)}{\pi t}\cos(5\pi t)$
$T = 2,\ \omega_o = \pi,\ T_o = \frac{1}{2}$

$$\begin{aligned}
W(j\omega) &= \sum_{k=-\infty}^{\infty} \frac{2\sin(k\frac{\pi}{2})}{k}\delta(\omega - k\pi) \\
Z(j\omega) &= \frac{1}{2\pi}X(j\omega) * W(j\omega) \\
&= \sum_{k=-\infty}^{\infty} \frac{\sin(k\frac{\pi}{2})}{k\pi}X(j(\omega - k\pi)) \\
C(j\omega) &= H(j\omega)Z(j\omega) \\
&= \sum_{k=3}^{7} \frac{\sin(k\frac{\pi}{2})}{k\pi}X(j(\omega - k\pi)) + \sum_{k=-3}^{-7} \frac{\sin(k\frac{\pi}{2})}{k\pi}X(j(\omega - k\pi)) \\
Y(j\omega) &= \frac{1}{2}\left[C(j(\omega - 5\pi)) + C(j(\omega + 5\pi))\right]
\end{aligned}$$

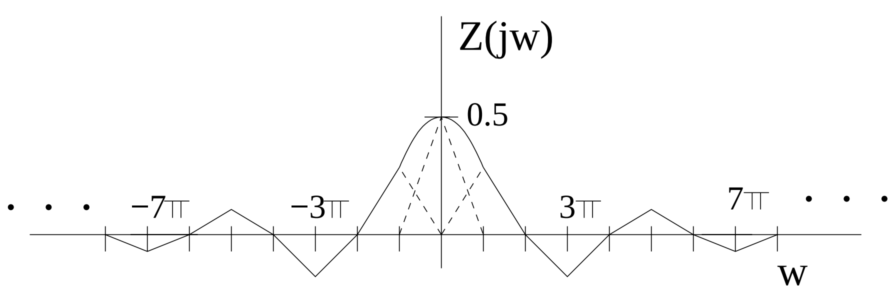


Figure P4.20. (c) Sketch of $Z(j\omega)$

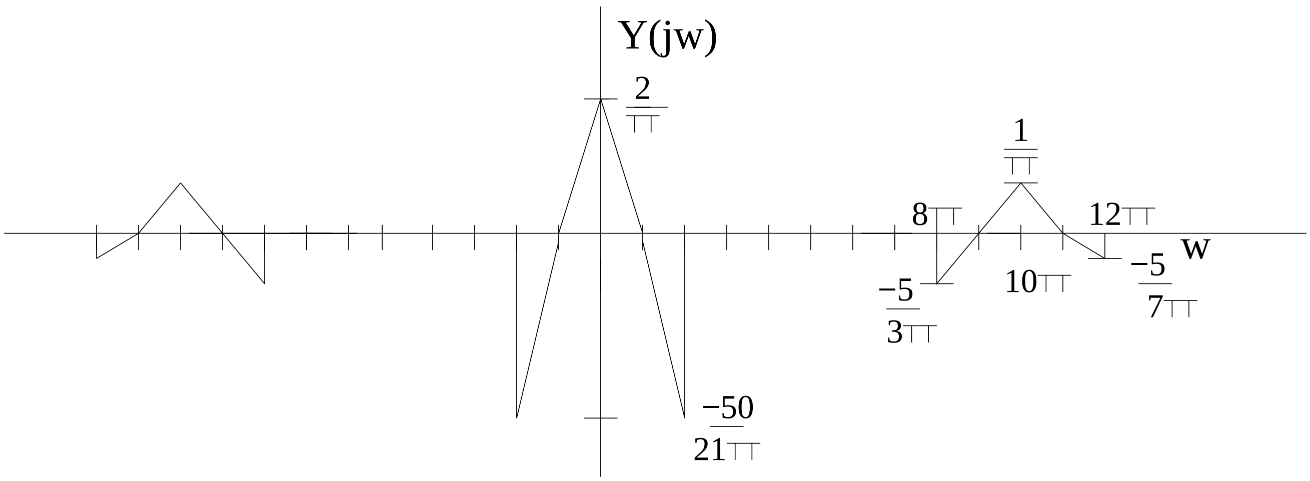


Figure P4.20. (c) Sketch of $Y(j\omega)$

4.21. Consider the system depicted in Fig. P4.21. The impulse response $h(t)$ is given by

$$h(t) = \frac{\sin(11\pi t)}{\pi t}$$

and we have

$$x(t) = \sum_{k=1}^{\infty} \frac{1}{k^2} \cos(k5\pi t)$$

$$g(t) = \sum_{k=1}^{10} \cos(k8\pi t)$$

Use the FT to determine $y(t)$.

$$\begin{aligned}
y(t) &= [(x(t) * h(t))(g(t) * h(t))] * h(t) \\
&= [x_h(t) g_h(t)] * h(t) \\
&= m(t) * h(t)
\end{aligned}$$

$$h(t) = \frac{\sin(11\pi t)}{\pi t} \quad \xleftrightarrow{FT} \quad H(j\omega) = \begin{cases} 1 & |\omega| \leq 11\pi \\ 0 & \text{otherwise} \end{cases}$$

$$x(t) = \sum_{k=1}^{\infty} \frac{1}{k^2} \cos(k5\pi t) \quad \xleftrightarrow{FT} \quad X(j\omega) = \pi \sum_{k=1}^{\infty} \frac{1}{k^2} [\delta(\omega - 5k\pi) + \delta(\omega + 5k\pi)]$$

$$g(t) = \sum_{k=1}^{10} \cos(k8\pi t) \quad = \quad \pi \sum_{k=1}^{10} [\delta(\omega - 8k\pi) + \delta(\omega + 8k\pi)]$$

$$\begin{aligned}
X_h(j\omega) &= X(j\omega) H(j\omega) \\
&= \pi \sum_{k=1}^{2} \frac{1}{k^2} [\delta(\omega - 5k\pi) + \delta(\omega + 5k\pi)]
\end{aligned}$$

$$
\begin{aligned}
G_h(j\omega) &= G(j\omega)H(j\omega) \\
&= \pi\delta(\omega - 8\pi) + \pi\delta(\omega - 8\pi) \\
M(j\omega) &= \frac{1}{2\pi} X_h(j\omega) * G_h(j\omega) \\
&= \frac{1}{2}\left[X_h(j(\omega - 8\pi)) + X_h(j(\omega + 8\pi))\right] \\
&= \pi \sum_{k=1}^{2} \frac{1}{k^2}\left[(\delta(\omega - 8\pi - 5k\pi) + \delta(\omega - 8\pi + 5k\pi)) + (\delta(\omega + 8\pi - 5k\pi) + \delta(\omega + 8\pi + 5k\pi))\right] \\
Y(j\omega) &= M(j\omega)H(j\omega) \\
&= \frac{\pi}{2}\left[\delta(\omega + 3\pi) + \delta(\omega - 3\pi)\right] + \frac{\pi}{8}\left[\delta(\omega - 2\pi) + \delta(\omega + 2\pi)\right] \\
y(t) &= \frac{1}{2}\cos(3\pi t) + \frac{1}{8}\cos(2\pi t)
\end{aligned}
$$

4.22. The input to a discrete-time system is given by

$$
x[n] = \cos(\frac{\pi}{8}n) + \sin(\frac{3\pi}{4}n)
$$

Use the DTFT to find the output of the system, $y[n]$, for the following impulse responses $h[n]$, first note that

$$
X(e^{j\Omega}) = \pi\left[\delta(\Omega - \frac{\pi}{8}) + \delta(\Omega + \frac{\pi}{8})\right] + \frac{\pi}{j}\left[\delta(\Omega - \frac{3\pi}{4}) - \delta(\Omega + \frac{3\pi}{4})\right]
$$

(a) $h[n] = \frac{\sin(\frac{\pi}{4}n)}{\pi n}$

$$
\begin{aligned}
H(e^{j\Omega}) &= \begin{cases} 1 & |\Omega| \leq \frac{\pi}{4} \\ 0 & \frac{\pi}{4} \leq |\Omega| < \pi, 2\pi \text{ periodic.} \end{cases} \\
Y(e^{j\Omega}) &= H(e^{j\Omega})X(e^{j\Omega}) \\
&= \pi\left[\delta(\Omega - \frac{\pi}{8}) + \delta(\Omega + \frac{\pi}{8})\right] \\
y[n] &= \cos(\frac{\pi}{8}n)
\end{aligned}
$$

(b) $h[n] = (-1)^n \frac{\sin(\frac{\pi}{2}n)}{\pi n}$

$$
\begin{aligned}
h[n] &= e^{j\pi n}\frac{\sin(\frac{\pi}{2}n)}{\pi n} \\
H(e^{j\Omega}) &= \begin{cases} 1 & |\Omega - \pi| \leq \frac{\pi}{4} \\ 0 & \frac{\pi}{4} \leq |\Omega - \pi| < \pi, 2\pi \text{ periodic.} \end{cases} \\
Y(e^{j\Omega}) &= H(e^{j\Omega})X(e^{j\Omega})
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\pi}{j}\left[\delta(\Omega - \frac{3\pi}{4}) + \delta(\Omega + \frac{3\pi}{4})\right] \\
y[n] &= \sin(\frac{3\pi}{4}n)
\end{aligned}
$$

(c) $h[n] = \cos(\frac{\pi}{2}n)\frac{\sin(\frac{\pi}{5}n)}{\pi n}$

$$
\begin{aligned}
h[n] &= \cos(\frac{\pi}{2}n)\frac{\sin(\frac{\pi}{5}n)}{\pi n} \\
H(e^{j\Omega}) &= \begin{cases} \frac{1}{2} & |\Omega - \frac{\pi}{2}| \le \frac{\pi}{5} \\ 0 & \frac{\pi}{5} \le |\Omega - \frac{\pi}{2}| < \pi \end{cases} + \begin{cases} \frac{1}{2} & |\Omega + \frac{\pi}{2}| \le \frac{\pi}{5} \\ 0 & \frac{\pi}{5} \le |\Omega + \frac{\pi}{2}| < \pi, \quad 2\pi \text{ periodic.} \end{cases} \\
Y(e^{j\Omega}) &= H(e^{j\Omega})X(e^{j\Omega}) \\
&= 0 \\
y[n] &= 0
\end{aligned}
$$

4.23. Consider the discrete-time system depicted in Fig. P4.23. Assume $h[n] = \frac{\sin(\frac{\pi}{2}n)}{\pi n}$. Use the DTFT to determine the output, $y[n]$ for the following cases: Also sketch $G(e^{j\Omega})$, the DTFT of $g[n]$.

$$
\begin{aligned}
y[n] &= (x[n]w[n]) * h[n] \\
&= g[n] * h[n] \\
h[n] = \frac{\sin(\frac{\pi}{2}n)}{\pi n} &\stackrel{FT}{\longleftrightarrow} H(e^{j\Omega}) = \begin{cases} 1 & |\Omega| \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \le |\Omega| < \pi \end{cases} \\
& \quad H(e^{j\Omega}) \text{ is } 2\pi \text{ periodic.}
\end{aligned}
$$

(a) $x[n] = \frac{\sin(\frac{\pi}{4}n)}{\pi n}, \quad w[n] = (-1)^n$

$$
\begin{aligned}
x[n] = \frac{\sin(\frac{\pi}{4}n)}{\pi n} &\stackrel{DTFT}{\longleftrightarrow} X(e^{j\Omega}) = \begin{cases} 1 & |\Omega| \le \frac{\pi}{4} \\ 0 & \frac{\pi}{4} \le |\Omega| < \pi \end{cases} \\
w[n] = e^{j\pi n} &\stackrel{DTFT}{\longleftrightarrow} W(e^{j\Omega}) = 2\pi\delta(\Omega - \pi) \\
G(e^{j\Omega}) &= \frac{1}{2\pi}X(e^{j\Omega}) * W(e^{j\Omega}) \\
&= \begin{cases} 1 & |\Omega - \pi| \le \frac{\pi}{4} \\ 0 & \frac{\pi}{4} \le |\Omega - \pi| < \pi \end{cases} \\
g[n] &= e^{j\pi n}\frac{\sin(\frac{\pi}{4}n)}{\pi n} \\
Y(e^{j\Omega}) &= G(e^{j\Omega})H(e^{j\Omega}) \\
&= 0 \\
y[n] &= 0
\end{aligned}
$$

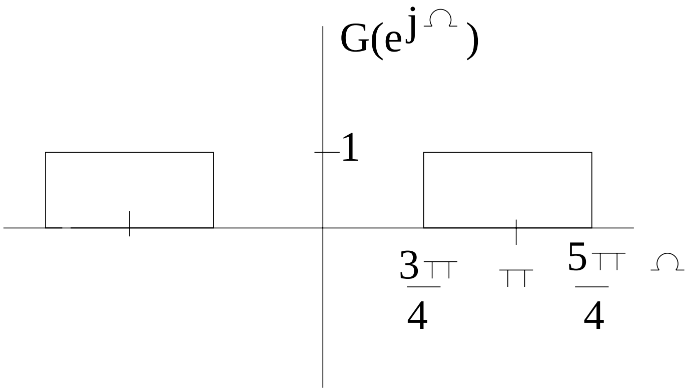


Figure P4.23. (a) The DTFT of $g[n]$

(b) $x[n] = \delta[n] - \frac{\sin(\frac{\pi}{4}n)}{\pi n}, \quad w[n] = (-1)^n$

$$\begin{aligned}
x[n] = \delta[n] - \frac{\sin(\frac{\pi}{4}n)}{\pi n} & \quad \xleftrightarrow{DTFT} \quad X(e^{j\Omega}) = \begin{cases} 0 & |\Omega| \le \frac{\pi}{4} \\ 1 & \frac{\pi}{4} \le |\Omega| < \pi \end{cases} \\
w[n] = e^{j\pi n} & \quad \xleftrightarrow{DTFT} \quad W(e^{j\Omega}) = 2\pi\delta(\Omega - \pi) \\
G(e^{j\Omega}) & \quad = \quad \frac{1}{2\pi} X(e^{j\Omega}) * W(e^{j\Omega}) \\
& \quad = \quad \begin{cases} 0 & |\Omega - \pi| \le \frac{3\pi}{4} \\ 1 & \frac{3\pi}{4} \le |\Omega - \pi| < \pi \end{cases} \\
g[n] & \quad = \quad \frac{\sin(\frac{3\pi}{4}n)}{\pi n} \\
Y(e^{j\Omega}) & \quad = \quad G(e^{j\Omega}) H(e^{j\Omega}) \\
& \quad = \quad H(e^{j\Omega}) \\
y[n] & \quad = \quad \frac{\sin(\frac{\pi}{2}n)}{\pi n}
\end{aligned}$$

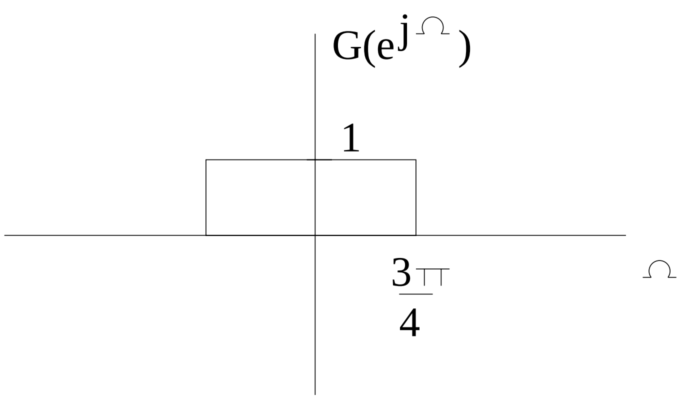


Figure P4.23. (b) The DTFT of $g[n]$

(c) $x[n] = \frac{\sin(\frac{\pi}{2}n)}{\pi n}, \quad w[n] = \cos(\frac{\pi}{2}n)$

$$
\begin{aligned}
W(e^{j\Omega}) &= \pi\left[\delta(\Omega-\frac{\pi}{2})+\delta(\Omega+\frac{\pi}{2})\right],\ 2\pi \text{ periodic}\\
G(e^{j\Omega}) &= \frac{1}{2\pi}X(e^{j\Omega})*W(e^{j\Omega})\\
&= \begin{cases} \frac{1}{2} & |\Omega-\frac{\pi}{2}|\le\frac{\pi}{2}\\ 0 & \frac{\pi}{2}\le|\Omega-\frac{\pi}{2}|<\pi \end{cases} + \begin{cases} \frac{1}{2} & |\Omega+\frac{\pi}{2}|\le\frac{\pi}{2}\\ 0 & \frac{\pi}{2}\le|\Omega+\frac{\pi}{2}|<\pi \end{cases}\\
g[n] &= \frac{1}{2}\frac{\sin(\frac{\pi}{2}n)}{\pi n}\left(e^{j\frac{\pi}{2}n}+e^{-j\frac{\pi}{2}n}\right)\\
&= \frac{\sin(\frac{\pi}{2}n)}{\pi n}\cos(\frac{\pi}{2}n)\\
&= \frac{\sin(\pi n)}{2\pi n}\\
&= \frac{1}{2}\delta(n)\\
y[n] &= g[n]*h[n]\\
&= \frac{1}{2}h[n]\\
&= \frac{\sin(\pi n)}{2\pi n}
\end{aligned}
$$

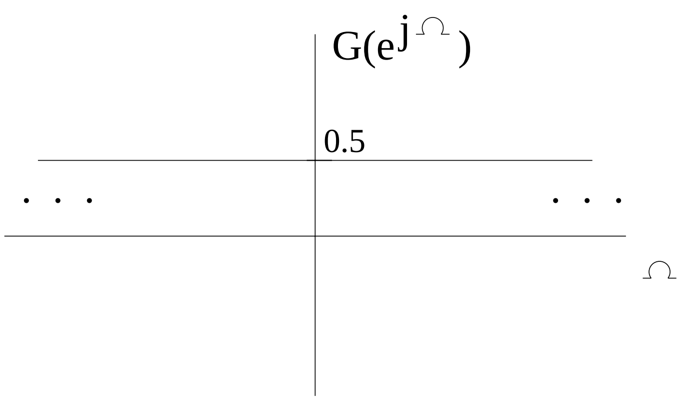


Figure P4.23. (c) The DTFT of $g[n]$

(d) $x[n]=1+\sin(\frac{\pi}{16}n)+2\cos(\frac{3\pi}{4}n), \quad w[n]=\cos(\frac{3\pi}{8}n)$

$$
\begin{aligned}
X(e^{j\Omega}) &= 2\pi\delta(\Omega)+\frac{\pi}{j}\left[[\delta(\Omega-\frac{\pi}{16})+\delta(\Omega+\frac{\pi}{16})\right]+2\pi\left[\delta(\Omega-\frac{3\pi}{4})+\delta(\Omega+\frac{3\pi}{4})\right],\ 2\pi \text{ periodic.}\\
W(e^{j\Omega}) &= \pi\left[\delta(\Omega-\frac{3\pi}{8})+\delta(\Omega+\frac{3\pi}{8})\right],\ 2\pi \text{ periodic}\\
G(e^{j\Omega}) &= \frac{1}{2\pi}X(e^{j\Omega})*W(e^{j\Omega})\\
&= \frac{1}{2\pi}\left[X(e^{j(\Omega-\frac{3\pi}{8})})+X(e^{j(\Omega+\frac{3\pi}{8})})\right]\\
&= \frac{1}{2\pi}\left[2\pi\delta(\Omega-\frac{3\pi}{8})+\frac{\pi}{j}\left(\delta(\Omega-\frac{7\pi}{16})-\delta(\Omega-\frac{5\pi}{16})\right)+2\pi\left(\delta(\Omega-\frac{9\pi}{8})+\delta(\Omega+\frac{3\pi}{8})\right)\right]\\
&\quad+\frac{1}{2\pi}\left[2\pi\delta(\Omega+\frac{3\pi}{8})+\frac{\pi}{j}\left(\delta(\Omega+\frac{5\pi}{16})-\delta(\Omega+\frac{7\pi}{16})\right)+2\pi\left(\delta(\Omega-\frac{3\pi}{8})+\delta(\Omega+\frac{9\pi}{8})\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
g[n] &= \frac{2}{\pi}\cos(\frac{3\pi}{8}n) + \frac{1}{2\pi}\sin(\frac{7\pi}{16}n) - \frac{1}{2\pi}\sin(\frac{5\pi}{16}n) + \frac{1}{\pi}\cos(\frac{9\pi}{8}n) \\
Y(e^{j\Omega}) &= G(e^{j\Omega})H(e^{j\Omega}) \\
y[n] &= \frac{2}{\pi}\cos(\frac{3\pi}{8}n) + \frac{1}{2\pi}\sin(\frac{7\pi}{16}n) - \frac{1}{2\pi}\sin(\frac{5\pi}{16}n)
\end{aligned}
$$

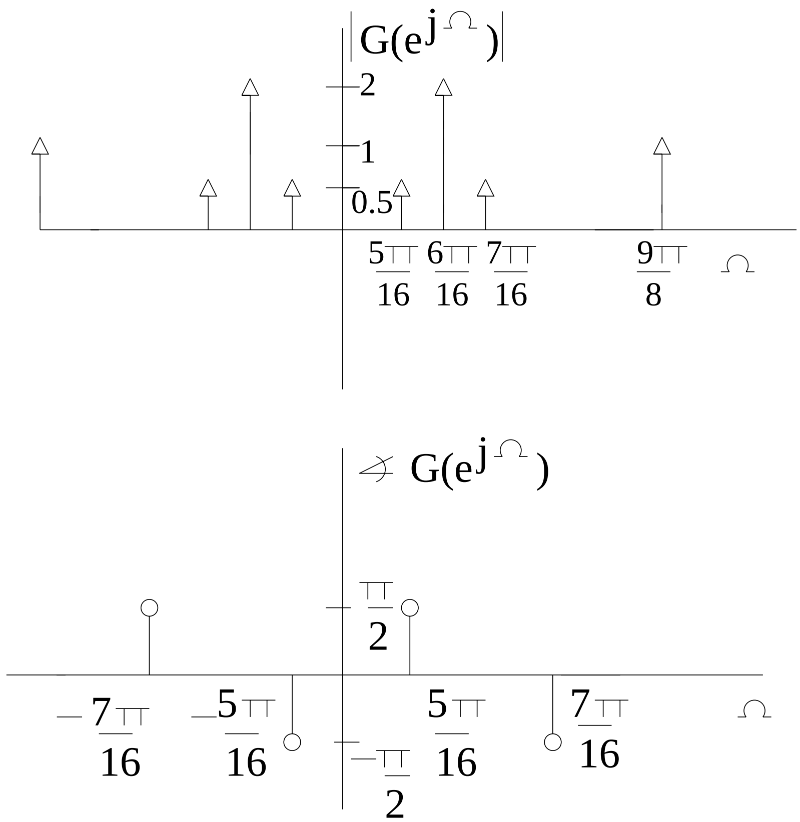


Figure P4.23. (d) The DTFT of $g[n]$

4.24. Determine and sketch the FT representation, $X_\delta(j\omega)$, for the following discrete-time signals with the sampling interval T_s as given:

$$
\begin{aligned}
X_\delta(j\omega) &= \sum_{n=-\infty}^{\infty} e^{-j\omega T_s n} \\
&= X(e^{j\Omega})\big|_{\Omega=\omega T_s}
\end{aligned}
$$

(a) $x[n] = \frac{\sin(\frac{\pi}{3}n)}{\pi n}, \quad T_s = 2$

$$
X(e^{j\Omega}) = \begin{cases} 1 & |\Omega| \le \frac{\pi}{3} \\ 0 & \frac{\pi}{3} \le |\Omega| < \pi \end{cases}
$$

$$X_\delta(j\omega) = \begin{cases} 1 & |2\omega| \le \frac{\pi}{3} \\ 0 & \frac{\pi}{3} \le |2\omega| < \pi \end{cases}$$

$$= \begin{cases} 1 & |\omega| \le \frac{\pi}{6} \\ 0 & \frac{\pi}{6} \le |\omega| < \frac{\pi}{6} \end{cases}$$

$X_\delta(j\omega)$ is π periodic

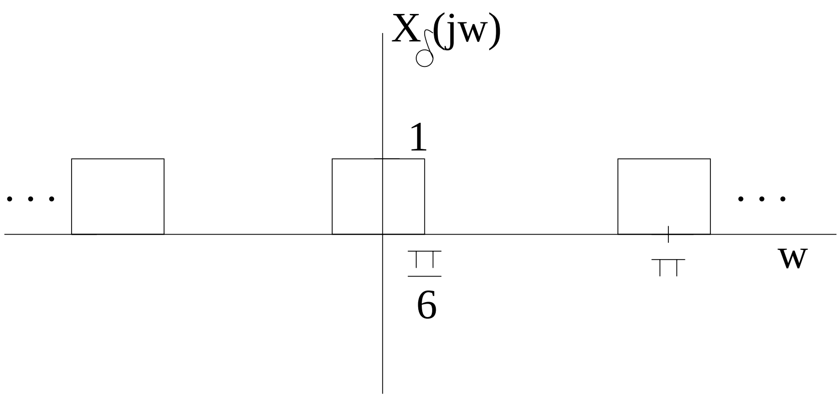


Figure P4.24. (a) FT of $X_\delta(j\omega)$

(b) $x[n] = \frac{\sin(\frac{\pi}{3}n)}{\pi n}, \quad T_s = \frac{1}{4}$

$$X(e^{j\Omega}) = \begin{cases} 1 & |\Omega| \le \frac{\pi}{3} \\ 0 & \frac{\pi}{3} \le |\Omega| < \pi \end{cases}$$

$$X_\delta(j\omega) = \begin{cases} 1 & |\frac{1}{4}\omega| \le \frac{\pi}{3} \\ 0 & \frac{\pi}{3} \le |\frac{1}{4}\omega| < \pi \end{cases}$$

$$= \begin{cases} 1 & |\omega| \le \frac{4\pi}{3} \\ 0 & \frac{4\pi}{3} \le |\omega| < 4\pi \end{cases}$$

$X_\delta(j\omega)$ is 8π periodic

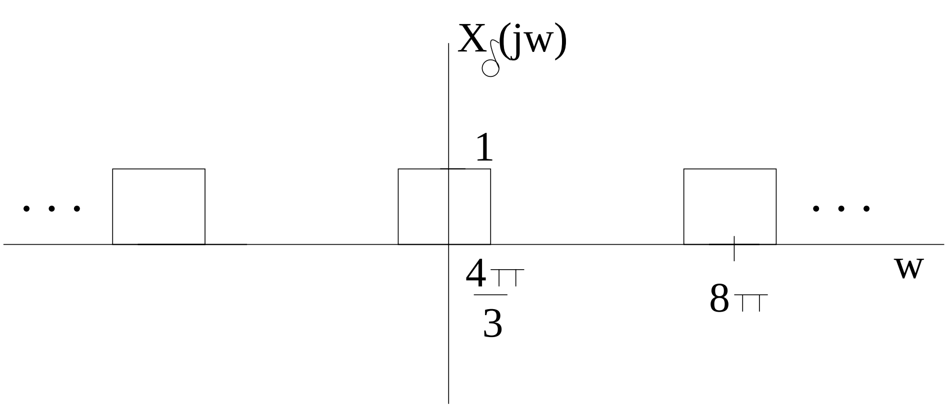


Figure P4.24. (b) FT of $X_\delta(j\omega)$

(c) $x[n] = \cos(\frac{\pi}{2}n)\frac{\sin(\frac{\pi}{4}n)}{\pi n}, \quad T_s = 2$

$$
\begin{aligned}
X(e^{j\Omega}) &= \begin{cases} \frac{1}{2} & |\Omega - \frac{\pi}{2}| \le \frac{\pi}{4} \\ 0 & \frac{\pi}{4} \le |\Omega - \frac{\pi}{2}| < \pi \end{cases} + \begin{cases} \frac{1}{2} & |\Omega + \frac{\pi}{2}| \le \frac{\pi}{4} \\ 0 & \frac{\pi}{4} \le |\Omega + \frac{\pi}{2}| < \pi \end{cases} \\
X_\delta(j\omega) &= \begin{cases} \frac{1}{2} & |2\omega - \frac{\pi}{2}| \le \frac{\pi}{3} \\ 0 & \frac{\pi}{3} \le |2\omega - \frac{\pi}{2}| < \pi \end{cases} + \begin{cases} \frac{1}{2} & |2\omega + \frac{\pi}{2}| \le \frac{\pi}{3} \\ 0 & \frac{\pi}{3} \le |2\omega + \frac{\pi}{2}| < \pi \end{cases} \\
&= \begin{cases} \frac{1}{2} & \frac{\pi}{8} < \omega < \frac{3\pi}{8} \\ 0 & \text{otherwise} \end{cases} + \begin{cases} \frac{1}{2} & -\frac{3\pi}{8} < \omega < -\frac{\pi}{8} \\ 0 & \text{otherwise} \end{cases} \\
& X_\delta(j\omega) \text{ is } \pi \text{ periodic}
\end{aligned}
$$

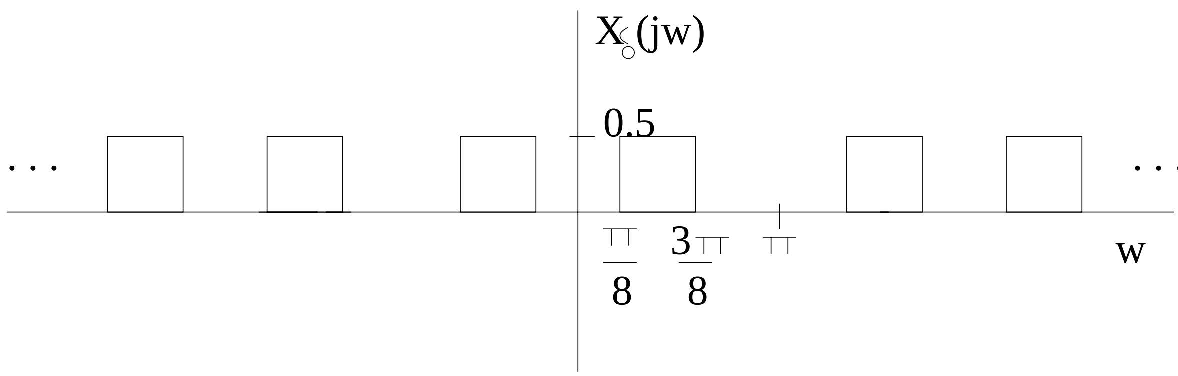


Figure P4.24. (c) FT of $X_\delta(j\omega)$

(d) $x[n]$ depicted in Fig. P4.17 (a) with $T_s = 4$.

$$
\begin{aligned}
\text{DTFS: } N = 8 \qquad & \Omega_o = \frac{\pi}{4} \\
X[k] &= \frac{\sin(k\frac{5\pi}{8})}{8\sin(k\frac{\pi}{8})}, \; k \in [-3, 4] \\
\text{DTFT: } X(e^{j\Omega}) &= 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\Omega - k\frac{\pi}{4}) \\
\text{FT: } X_\delta(j\omega) &= \frac{\pi}{4} \sum_{k=-\infty}^{\infty} \frac{\sin(k\frac{5\pi}{8})}{8\sin(k\frac{\pi}{8})}\delta(4\omega - k\frac{\pi}{4}) \\
&= \frac{\pi}{16} \sum_{k=-\infty}^{\infty} \frac{\sin(k\frac{5\pi}{8})}{8\sin(k\frac{\pi}{8})}\delta(\omega - k\frac{\pi}{16}) \\
& X_\delta(j\omega) \text{ is } \frac{\pi}{2} \text{ periodic}
\end{aligned}
$$

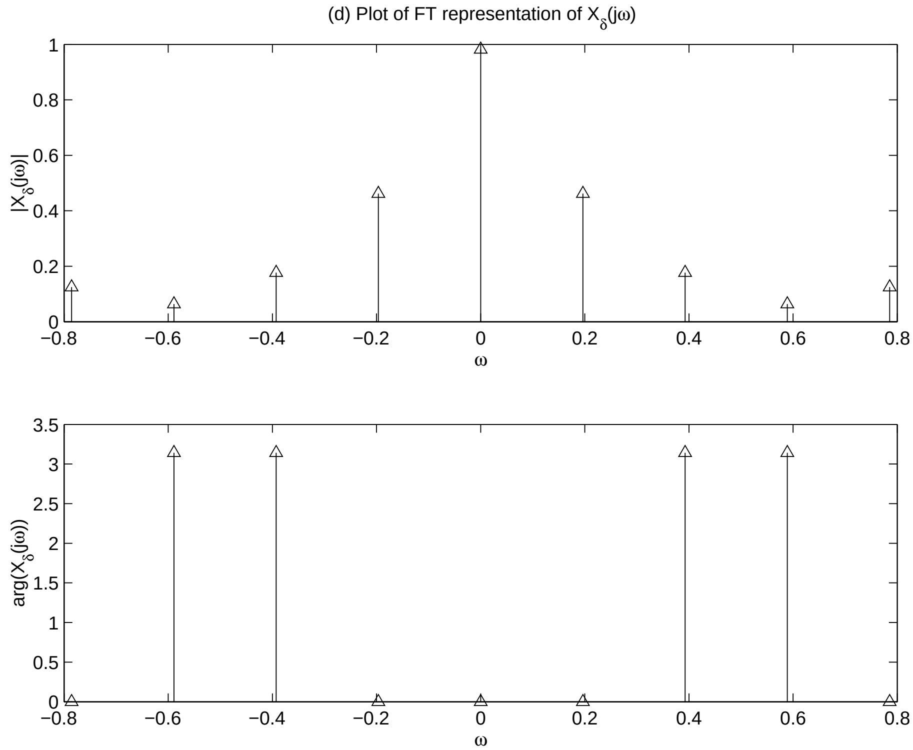


Figure P4.24. (d) FT of $X_\delta(j\omega)$

(e) $x[n] = \sum_{p=-\infty}^{\infty} \delta[n-4p], \quad T_s = \frac{1}{8}$

$$\text{DTFS: } N = 4 \qquad \Omega_o = \frac{\pi}{2}$$

$$\begin{aligned}
X[k] &= \frac{1}{4},\ k \in [0,3] \\
\text{DTFT: } X(e^{j\Omega}) &= \frac{\pi}{2} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\frac{\pi}{2}) \\
\text{FT: } X_\delta(j\omega) &= \frac{\pi}{2} \sum_{k=-\infty}^{\infty} \delta(\frac{1}{8}\omega - k\frac{\pi}{2}) \\
&= 4\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k4\pi) \\
& \quad X_\delta(j\omega) \text{ is } 4\pi \text{ periodic}
\end{aligned}$$

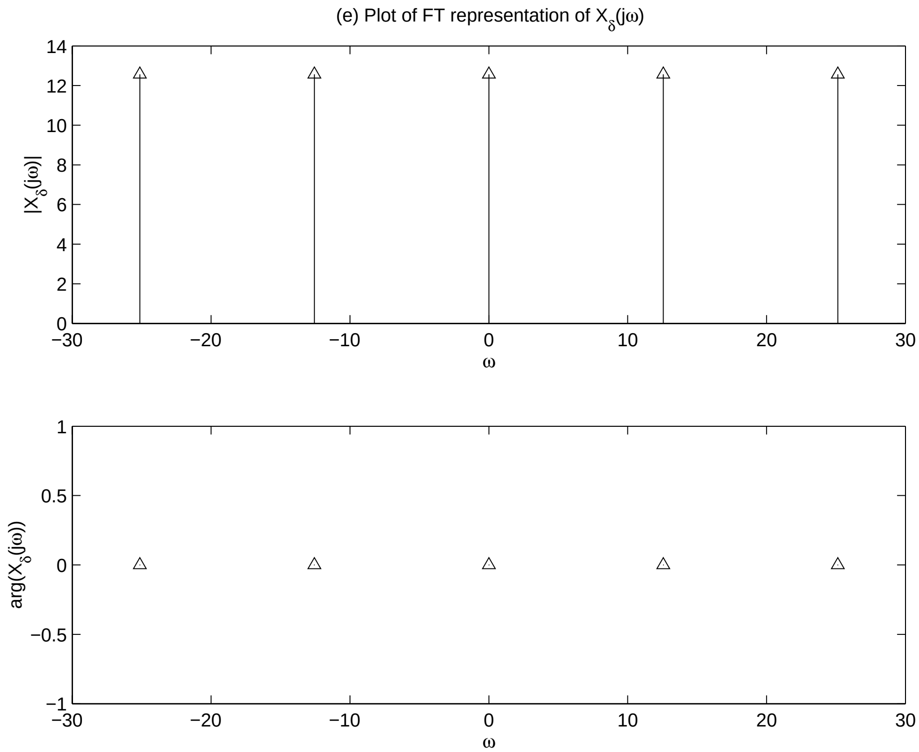


Figure P4.24. (e) FT of $X_\delta(j\omega)$

4.25. Consider sampling the signal $x(t) = \frac{1}{\pi t}\sin(2\pi t)$.

(a) Sketch the FT of the sampled signal for the following sampling intervals:

(i) $T_s = \frac{1}{8}$

(ii) $T_s = \frac{1}{3}$

(iii) $T_s = \frac{1}{2}$

(iv) $T_s = \frac{2}{3}$

In part (iv), aliasing occurs. The signals overlap and add, which can be seen in the following figure.

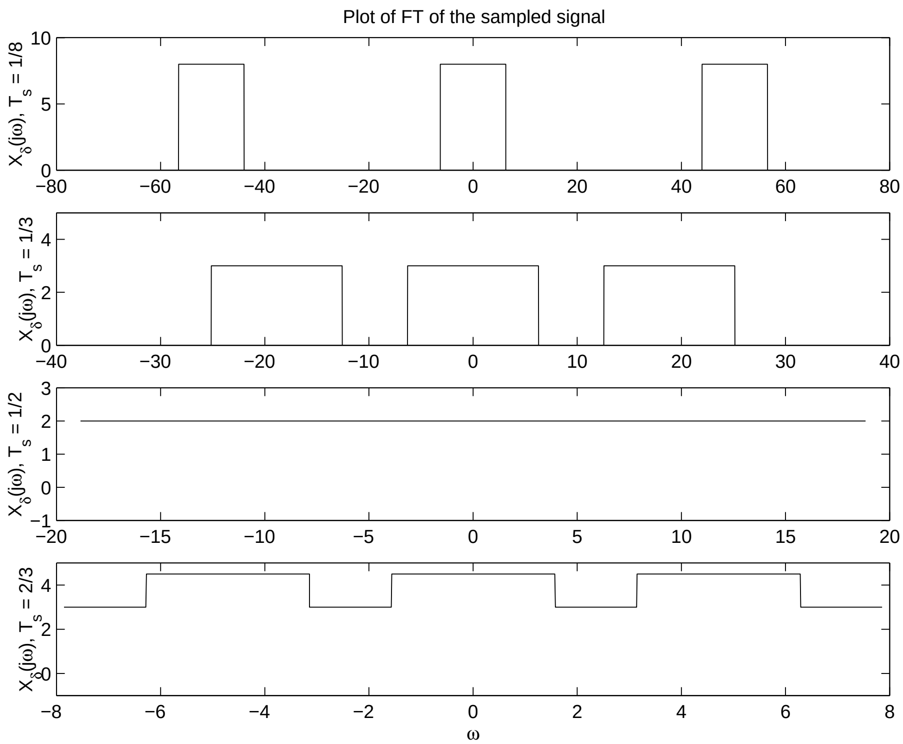


Figure P4.25. (a) Sketch of $X_\delta(j\omega)$

(b) Let $x[n] = x(nT_s)$. Sketch the DTFT of $x[n]$, $X(e^{j\Omega})$, for each of the sampling intervals given in (a).

$$x[n] = x(nT_s) = \frac{1}{\pi n T_s}\sin(2\pi n T_s) \quad \overset{DTFT}{\longleftrightarrow} \quad \tilde{X}(e^{j\Omega})$$

$$\tilde{X}(e^{j\Omega}) \quad = \quad \begin{cases} \frac{1}{T_s} & |\Omega| \le T_s\pi \\ 0 & T_s\pi \le |\Omega| < \pi \end{cases} \quad 2\pi \text{ periodic}$$

Notice that the difference between the figure in (a) and (b) is that the 'ω' axis has been scaled by the sampling rate.

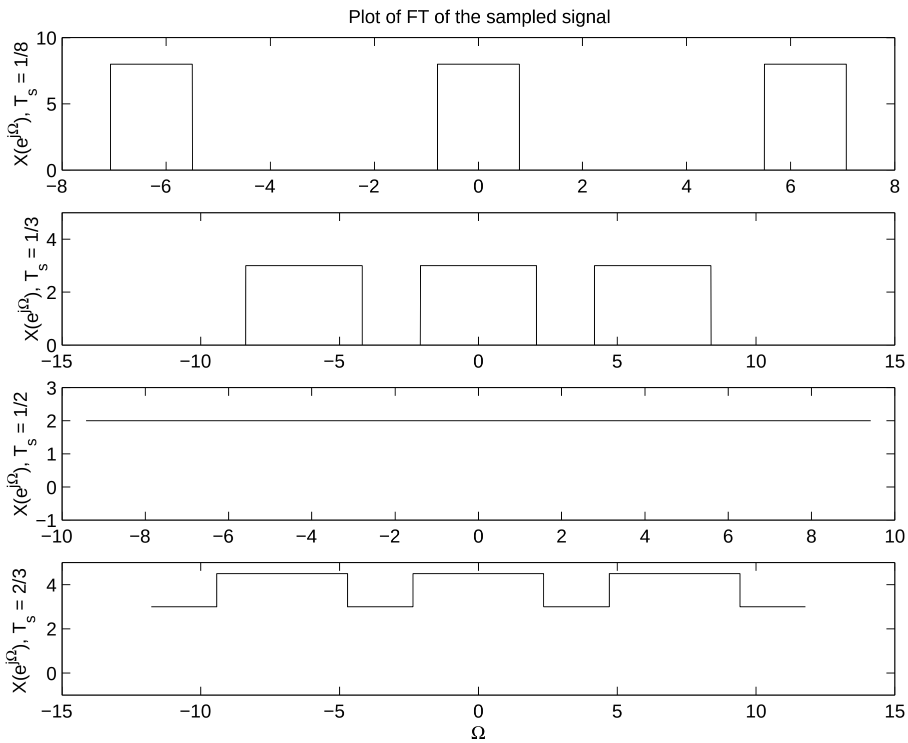


Figure P4.25. (b) Sketch of $X(e^{j\Omega})$

4.26. The continuous-time signal $x(t)$ with FT as depicted in Fig. P4.26 is sampled. Identify in each case if aliasing occurs.

(a) Sketch the FT of the sampled signal for the following sampling intervals:

(i) $T_s = \frac{1}{14}$

No aliasing occurs.

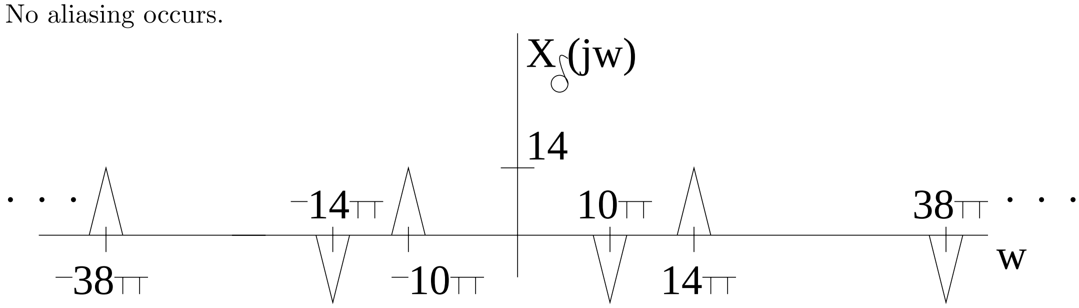


Figure P4.26. (i) FT of the sampled signal

(ii) $T_s = \frac{1}{7}$

Since $T_s > \frac{1}{11}$, aliasing occurs.

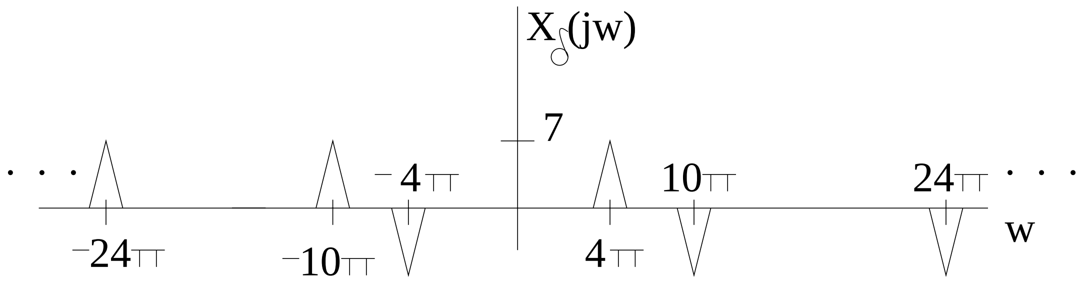


Figure P4.26. (ii) FT of the sampled signal

(iii) $T_s = \frac{1}{5}$
Since $T_s > \frac{1}{11}$, aliasing occurs.

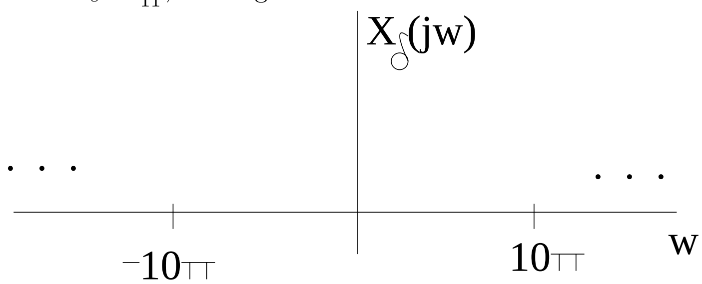


Figure P4.26. (iii) FT of the sampled signal

(b) Let $x[n] = x(nT_s)$. Sketch the DTFT of $x[n]$, $X(e^{j\Omega})$, for each of the sampling intervals given in (a).
The DTFT simple scales the 'x' axis by the sampling rate.
(i) $T_s = \frac{1}{14}$

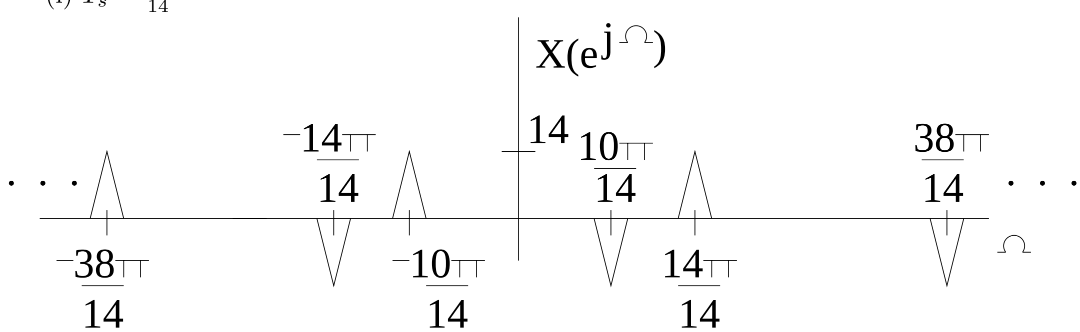


Figure P4.26. (i) DTFT of $x[n]$

(ii) $T_s = \frac{1}{7}$

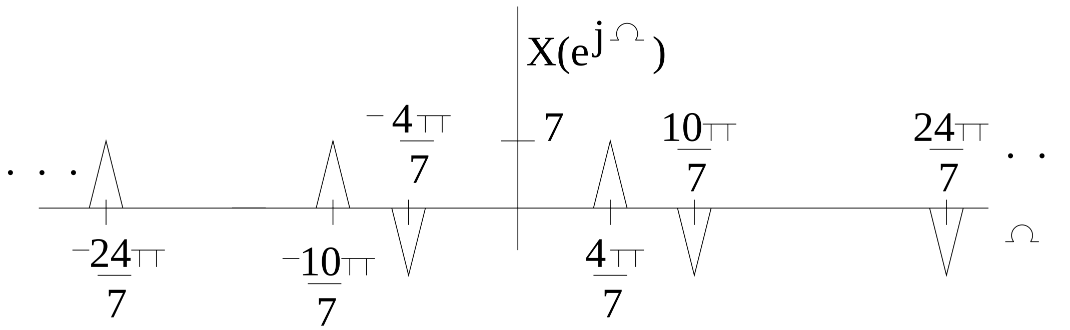


Figure P4.26. (ii) DTFT of $x[n]$

(iii) $T_s = \frac{1}{5}$

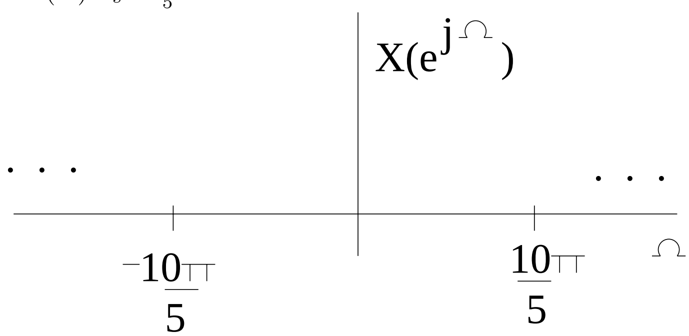


Figure P4.26. (iii) DTFT of $x[n]$

4.27. Consider subsampling the signal $x[n] = \frac{\sin(\frac{\pi}{6}n)}{\pi n}$ so that $y[n] = x[qn]$. Sketch $Y(e^{j\Omega})$ for the following choices of q:

$$
\begin{aligned}
X(e^{j\Omega}) &= \begin{cases} 1 & |\omega| \leq \frac{\pi}{6} \\ 0 & \frac{\pi}{6} \leq |\omega| < \pi \quad 2\pi\text{periodic} \end{cases} \\
q[n] &= x[qn] \\
Y(e^{j\Omega}) &= \frac{1}{q}\sum_{m=0}^{q-1} X\left(e^{j\frac{1}{q}(\Omega - m2\pi)}\right)
\end{aligned}
$$

(a) $q = 2$

$$Y(e^{j\Omega}) = \frac{1}{2}\sum_{m=0}^{1} X\left(e^{j\frac{1}{2}(\Omega - m2\pi)}\right)$$

(b) $q = 4$

$$Y(e^{j\Omega}) = \frac{1}{4}\sum_{m=0}^{3} X\left(e^{j\frac{1}{4}(\Omega - m2\pi)}\right)$$

(c) $q = 8$

$$Y(e^{j\Omega}) = \frac{1}{8}\sum_{m=0}^{7} X\left(e^{j\frac{1}{8}(\Omega - m2\pi)}\right)$$

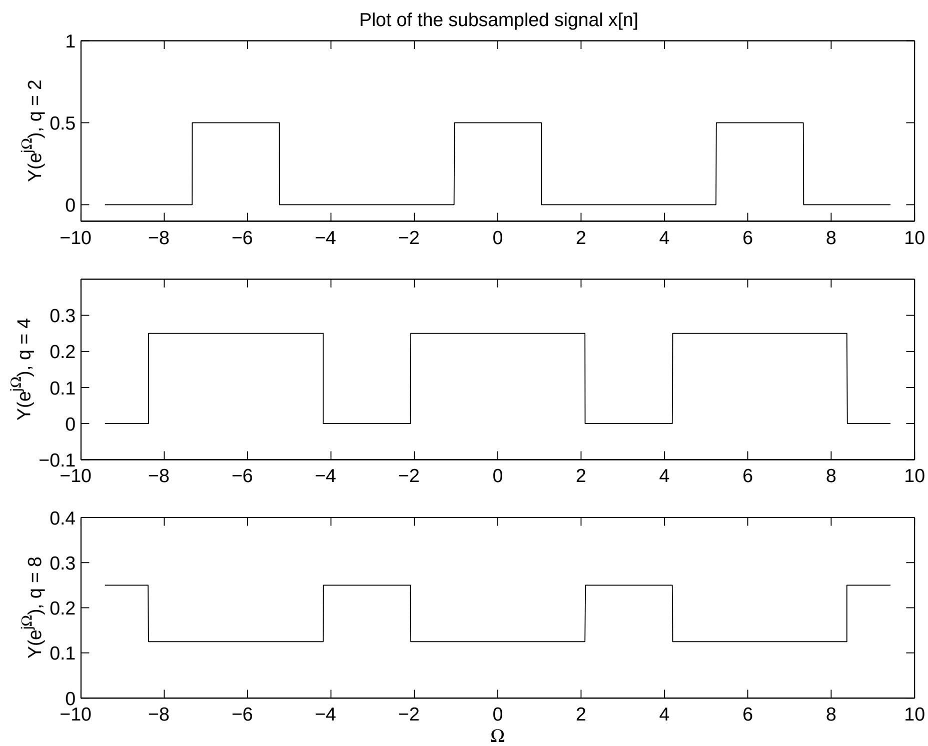


Figure P4.27. Sketch of $Y(e^{j\Omega})$

4.28. The discrete-time signal $x[n]$ with DTFT depicted in Fig. P4.28 is subsampled to obtain $y[n] = x[qn]$. Sketch $Y(e^{j\Omega})$ for the following choices of q:
(a) $q = 3$
(b) $q = 4$
(c) $q = 8$

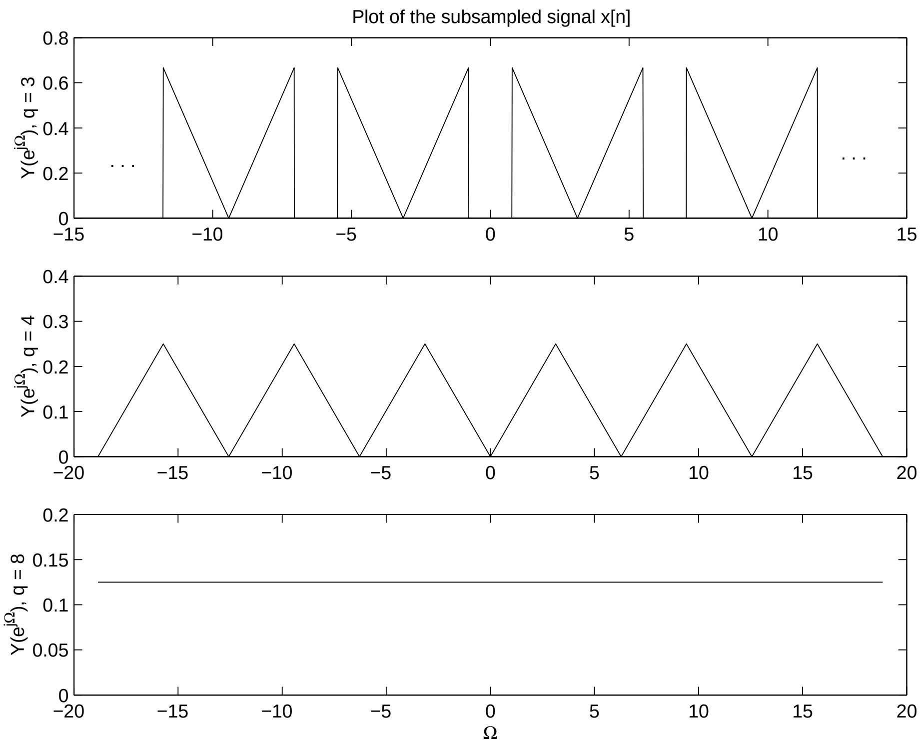


Figure P4.28. Sketch of $Y(e^{j\Omega})$

4.29. For each of the following signals sampled with sampling interval T_s, determine the bounds on T_s that gaurantee there will be no aliasing.
(a) $x(t) = \frac{1}{t}\sin 3\pi t + \cos(2\pi t)$

$$\begin{aligned}
\frac{1}{t}\sin(3\pi t) &\xleftrightarrow{FT} \begin{cases} \frac{1}{\pi} & |\omega| \leq 3\pi \\ 0 & \text{otherwise} \end{cases} \\
\cos(2\pi t) &\xleftrightarrow{FT} \pi\delta(\Omega - 2\pi) + \pi\delta(\Omega + 2\pi) \\
\omega_{max} &= 3\pi \\
T &< \frac{\pi}{\omega_{max}} \\
T &< \frac{1}{3}
\end{aligned}$$

(b) $x(t) = \cos(12\pi t)\frac{\sin(\pi t)}{2t}$

$$\begin{aligned}
X(j\omega) &= \begin{cases} \frac{1}{4\pi} & |\omega - 12\pi| \leq \pi \\ 0 & \text{otherwise} \end{cases} + \begin{cases} \frac{1}{4\pi} & |\omega + 12\pi| \leq \pi \\ 0 & \text{otherwise} \end{cases} \\
\omega_{max} &= 13\pi \\
T &< \frac{\pi}{\omega_{max}}
\end{aligned}$$

$$T < \frac{1}{13}$$

(c) $x(t) = e^{-6t}u(t) * \frac{\sin(Wt)}{\pi t}$

$$\begin{aligned}
X(j\omega) &= \frac{1}{6+j\omega}\left[u(\omega+W) - u(\omega-W)\right] \\
\omega_{max} &= W \\
T &< \frac{\pi}{\omega_{max}} \\
T &< \frac{\pi}{W}
\end{aligned}$$

(d) $x(t) = w(t)z(t)$, where the FTs $W(j\omega)$ and $Z(j\omega)$ are depicted in Fig. P4.29.

$$\begin{aligned}
X(j\omega) &= \frac{1}{2\pi}W(j\omega) * G(j\omega) \\
\omega_{max} &= 4\pi + w_a \\
T &< \frac{\pi}{\omega_{max}} \\
T &< \frac{\pi}{4\pi + w_a}
\end{aligned}$$

4.30. Consider the system depicted in Fig. P4.30. Assume $|X(j\omega)| = 0$ for $|\omega| > \omega_m$. Find the largest value of T such that $x(t)$ can be reconstructed from $y(t)$. Determine a system that will perform the reconstruction for this maximum value of T.

For reconstruction, we need to have $w_s > 2w_{max}$, or $T < \frac{\pi}{\omega_{max}}$. A finite duty cycle results in distortion.

$$\begin{aligned}
W[k] &= \frac{\sin(\frac{\pi}{2}k)}{k\pi}e^{-j\frac{\pi}{2}k} \\
W(j\omega) &= 2\pi\sum_{k=-\infty}^{\infty} W[k]\delta(\omega - k\frac{2\pi}{T}) \\
\text{After multiplication:} & \\
Y(j\omega) &= \sum_{k=-\infty}^{\infty}\frac{\sin(\frac{\pi}{2}k)}{k\pi}e^{-j\frac{\pi}{2}k}X(j(\omega - k\frac{2\pi}{T})) \\
\text{To reconstruct:} & \\
H_r(j\omega)Y(j\omega) &= X(j\omega), \quad |\omega| < \omega_{max}, \quad \frac{2\pi}{T} > 2\omega_{max} \\
k = 0 & \\
H_r(j\omega)\frac{1}{2}X(j\omega) &= X(j\omega) \\
H_r(j\omega) &= \begin{cases} 2 & |\omega| < \omega_{max} \\ \text{don't care} & \omega_{max} < |\omega| < \frac{2\pi}{T} - \omega_{max} \\ 0 & |\omega| > \frac{2\pi}{T} - \omega_{max} \end{cases}
\end{aligned}$$

4.31. Let $|X(j\omega)| = 0$ for $|\omega| > \omega_m$. Form the signal $y(t) = x(t)[\cos(3\pi t) + \sin(10\pi t)]$. Determine the maximum value of ω_m for which $x(t)$ can be reconstructed from $y(t)$ and specify a system that that will perform the reconstruction.

$$Y(j\omega) = \frac{1}{2}\left[X(j(\omega - 2\pi)) + X(j(\omega + 2\pi)) - jX(j(\omega - 10\pi)) + jX(j(\omega + 10\pi))\right]$$

$x(t)$ can be reconstructed from $y(t)$ if there is no overlap amoung the four shifted versions of $X(j\omega)$, yet $x(t)$ can still be reconstructed when overlap occurs, provided that there is at least one shifted $X(j\omega)$ that is not contaminated.

$$\omega_m \max = \frac{10\pi - 4\pi}{2} = 4\pi$$

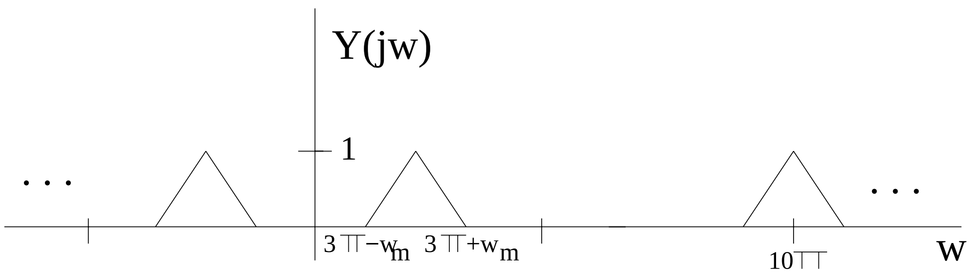


Figure P4.31. $Y(j\omega)$

We require $10\pi - \omega_m > 3\pi + \omega_m$, thus $\omega_m < \frac{7\pi}{2}$. To recover the signal, bandpassfilter with passband $6.5\pi \leq \omega \leq 13.5\pi$ and multiply with $2\sin(10\pi t)$ to retrieve $x(t)$.

4.32. A reconstruction system consists of a zero-order hold followed by a continuous-time anti-imaging filter with frequency response $H_c(j\omega)$. The original signal $x(t)$ is bandlimited to ω_m, that is, $X(j\omega) = 0$ for $|\omega| > \omega_m$ and is sampled with a sampling interval of T_s. Determine the constraints on the magnitude response of the anti-imaging filter so that the overall magnitude response of this reconstruction system is between 0.99 and 1.01 in the signal passband and less than 10^{-4} to the images of the signal spectrum for the following values:

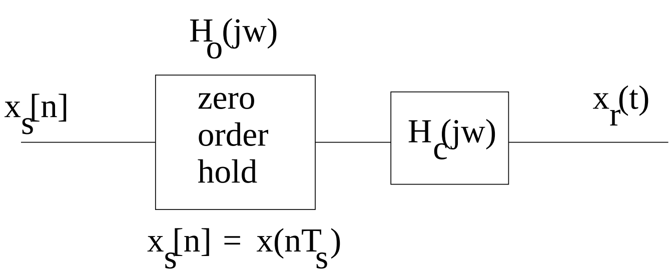


Figure P4.32. Reconstruction system.

(1) $0.99 < |H_o(j\omega)||H_c(j\omega)| < 1.01, \; -\omega_m \leq \omega \leq \omega_m$

Thus:

$$(i) \quad |H_c(j\omega)| > \frac{0.99\omega}{2\sin(\omega\frac{T_s}{2})}$$

$$(ii)\ |H_c(j\omega)| \quad < \quad \frac{1.01\omega}{2\sin(\omega\frac{T_s}{2})}$$

The passband constraint for each case is:

$\frac{0.99\omega}{2\sin(\omega\frac{T_s}{2})} < |H_c(j\omega)| < \frac{1.01\omega}{2\sin(\omega\frac{T_s}{2})}$

The stopband constraint is $|H_o(j\omega)||H_c(j\omega)| < 10^{-4}$, at the worst case $\omega = \frac{2\pi}{T_s} - \omega_m$.

$$|H_c(j\omega)| \quad < \quad \left|\frac{10^{-4}\omega}{2\sin(\omega\frac{T_s}{2})}\right|$$

(a) $\omega_m = 10\pi, \quad T_s = 0.1$

$$\begin{aligned}
\omega &= \frac{2\pi}{T_s} - \omega_m \\
&= \frac{2\pi}{0.1} - 10\pi = 10\pi \\
|H_c(j10\pi)| &< \left|\frac{10^{-4}10\pi}{2\sin(10\pi(\frac{0.1}{2}))}\right| \\
&= \left|5\pi(10^{-4})\right| = 0.001571
\end{aligned}$$

(b) $\omega_m = 10\pi, \quad T_s = 0.05$

$$\begin{aligned}
\omega &= \frac{2\pi}{0.05} - 10\pi = 30\pi \\
|H_c(j30\pi)| &< \left|\frac{30\pi(10^{-4})}{2\sin(30\pi(\frac{0.05}{2}))}\right| \\
&= \left|\frac{30\pi}{\sqrt{2}}(10^{-4})\right| = 0.006664
\end{aligned}$$

(c) $\omega_m = 10\pi, \quad T_s = 0.02$

$$\begin{aligned}
\omega &= \frac{2\pi}{0.02} - 10\pi = 90\pi \\
|H_c(j90\pi)| &< \left|\frac{10^{-4}90\pi}{2\sin(90\pi(\frac{0.02}{2}))}\right| \\
&= \left|\frac{90\pi(10^{-4})}{2\sin(0.9\pi)}\right| = 0.04575
\end{aligned}$$

(d) $\omega_m = 2\pi, \quad T_s = 0.05$

$$\omega = \frac{2\pi}{0.05} - 10\pi = 30\pi$$

$$
\begin{aligned}
|H_c(j30\pi)| &< \left|\frac{30\pi(10^{-4})}{2\sin(30\pi(\frac{0.05}{2}))}\right| \\
&= \left|\frac{10^{-4}30\pi}{2\sin(\frac{3}{4}\pi)}\right| = 0.006664
\end{aligned}
$$

4.33. The zero-order hold produces a stairstep approximation to the sampled signal $x(t)$ from samples $x[n] = x(nT_s)$. A device termed a first-order hold linearly interpolates between the samples $x[n]$ and thus produces a smoother approximation to $x(t)$. The output of the first- order hold may be described as

$$
x_1(t) = \sum_{n=-\infty}^{\infty} x[n]h_1(t - nT_s)
$$

where $h_1(t)$ is the triangular pulse shown in Fig. P4.33 (a). The relationship between $x[n]$ and $x_1(t)$ is depicted in Fig. P4.33 (b).
(a) Identify the distortions introduced by the first-order hold and compare them to those introduced by the zero-order hold. *Hint:* $h_1(t) = h_o(t) * h_o(t)$.

$$
\begin{aligned}
x_1(t) &= \sum_{n=-\infty}^{\infty} x[n]h_1(t - nT_s) \\
&= h_o(t) * h_o(t) * \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s) \\
\text{Thus:} & \\
X_1(j\omega) &= H_o(j\omega)H_o(j\omega)X_\Delta(j\omega) \\
\text{Which implies:} & \\
H_1(j\omega) &= e^{-j\omega T_s}\frac{4\sin^2(\omega\frac{T_s}{2})}{\omega^2}
\end{aligned}
$$

Distortions:
(1) A linear phase shift corresponding to a time delay of T_s seconds (a unit of sampling time).
(2) $\sin^2(.)$ term introduces more distortion to the portion of $X_\delta(j\omega)$, especially the higher frequency part is severely attenuated compared to the low frequency part which falls within the mainlobe, between $-\omega_m$ and ω_m.
(3) Distorted and attenuated versions of $X(j\omega)$ still remain at the nonzero multiples of ω_m, yet it is lower than the case of the zero order hold.

(b) Consider a reconstruction system consisting of a first-order hold followed by an anti-imaging filter with frequency response $H_c(j\omega)$. Find $H_c(j\omega)$ so that perfect reconstruction is obtained.

$$
\begin{aligned}
X_\Delta(j\omega)H_1(j\omega)H_c(j\omega) &= X(j\omega) \\
H_c(j\omega) &= \frac{e^{j\omega T_s}\omega^2}{4\sin^2(\omega\frac{T_s}{2})}T_s H_{LPF}(j\omega) \\
&\quad \text{where } H_{LPF}(j\omega) \text{ is an ideal low pass filter.}
\end{aligned}
$$

$$H_c(j\omega) = \begin{cases} \frac{e^{j\omega T_s}\omega^2}{4\sin^2(\omega\frac{T_s}{2})}T_s & |\omega| \leq \omega_m \\ \text{don't care} & \omega_m \leq |\omega| < \frac{2\pi}{T_s} - \omega_m \\ 0 & |\omega| > \frac{2\pi}{T_s} - \omega_m \end{cases}$$

Assuming $X(j\omega) = 0$ for $|\omega| > \omega_m$

(c) Determine the constraints on $|H_c(j\omega)|$ so that the overall magnitude response of this reconstruction system is between 0.99 and 1.01 in the signal passband and less than 10^{-4} to the images of the signal spectrum for the following values. Assume $x(t)$ is bandlimited to 12π, that is, $X(j\omega) = 0$ for $|\omega| > 12\pi$. Constraints:

(1) In the pass band:

$$0.99 < \quad |H_1(j\omega)||H_c(j\omega)| \quad < 1.01$$

$$\frac{0.99\omega^2}{4\sin^2(\omega\frac{T_s}{2})} < \quad |H_c(j\omega)| \quad < \frac{1.01\omega^2}{4\sin^2(\omega\frac{T_s}{2})}$$

(2) In the image region: $\omega = \frac{2\pi}{T_s} - \omega_m$

$$\begin{aligned} |H_1(j\omega)||H_c(j\omega)| &< 10^{-4} \\ |H_c(j\omega)| &< \frac{10^{-4}\omega^2}{4\sin^2(\omega\frac{T_s}{2})} \end{aligned}$$

(i) $T_s = .05$

$$\begin{aligned} \omega &= \frac{2\pi}{T_s} - \omega_m \\ &= \frac{2\pi}{0.05} - 12\pi = 28\pi \\ |H_c(j\omega)| &< \left| \frac{10^{-4}(28\pi)^2}{4\sin^2(28\pi(\frac{0.05}{2}))} \right| \\ &\approx 0.2956 \end{aligned}$$

$$\begin{aligned} \frac{0.99\omega^2}{4\sin^2(\omega\frac{T_s}{2})} < \quad |H_c(j\omega)| \quad &< \frac{1.01\omega^2}{4\sin^2(\omega\frac{T_s}{2})} \\ 2926.01 < \quad |H_c(j\omega)| \quad &< 2985.12 \end{aligned}$$

(ii) $T_s = .02$

$$\begin{aligned}\omega &= \frac{2\pi}{0.02} - 12\pi = 88\pi \\ |H_c(j\omega)| &< \left|\frac{10^{-4}(88\pi)^2}{4\sin^2(88\pi(\frac{0.02}{2}))}\right| \\ &\approx 14.1\end{aligned}$$

$$\begin{aligned}\frac{0.99\omega^2}{4\sin^2(\omega\frac{T_s}{2})} < \quad & |H_c(j\omega)| \quad < \frac{1.01\omega^2}{4\sin^2(\omega\frac{T_s}{2})} \\ 139589 < \quad & |H_c(j\omega)| \quad < 142409\end{aligned}$$

4.34. Determine the maximum factor q by which $x[n]$ with DTFT $X(e^{j\Omega})$ depicted in Fig. P4.34 can be decimated without aliasing. Sketch the DTFT of the sequence that results when $x[n]$ is decimated by this amount.
Looking at the following equation:

$$Y(e^{j\Omega}) = \frac{1}{q}\sum_{m=0}^{q-1} X\left(e^{j\frac{1}{q}(\Omega - m2\pi)}\right)$$

For the bandlimited signal, overlap starts when:

$$\begin{aligned}2qW &> 2\pi \\ \text{Thus:} & \\ q_{max} &= \frac{\pi}{W} = 3 \\ \text{After decimation:} & \\ Y(e^{j\Omega}) &= \frac{1}{3}\sum_{m=0}^{2} X\left(e^{j\frac{1}{3}(\Omega - m2\pi)}\right)\end{aligned}$$

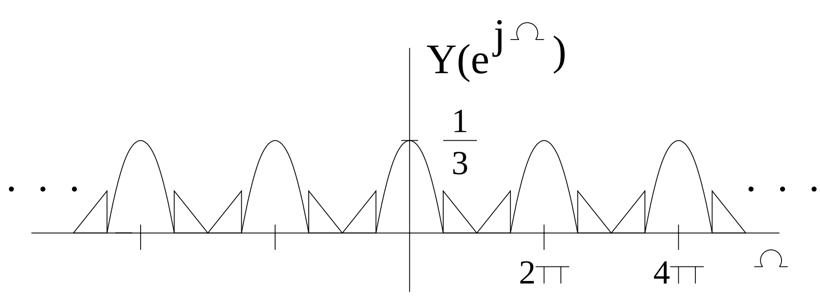


Figure P4.34. Sketch of the DTFT

4.35. A discrete-time system for processing continuous-time signals is shown in Fig. P4.35. Sketch the magnitude of the frequency response of an equivalent continuous-time system for the following cases:

$$|H_T(j\omega)| \quad = \quad |H_a(j\omega)|\frac{1}{T_s}|H(e^{j\omega T_s})|\left|\frac{2\sin(\omega\frac{T_s}{2})}{\omega}\right||H_c(j\omega)|$$

(a) $\Omega_1 = \frac{\pi}{4},\;\; W_c = 20\pi$

$$\omega_{max} \quad = \quad \min(10\pi,\; \frac{\pi}{4}(20), 20\pi) = 5\pi$$

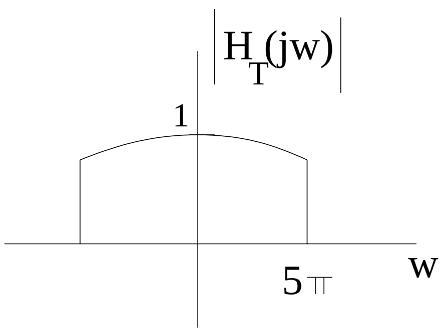


Figure P4.35. (a) Magnitude of the frequency response

(b) $\Omega_1 = \frac{3\pi}{4},\;\; W_c = 20\pi$

$$\omega_{max} \quad = \quad \min(10\pi,\; \frac{3\pi}{4}(20), 20\pi) = 10\pi$$

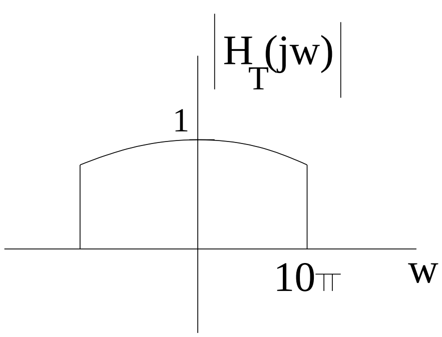


Figure P4.35. (b) Magnitude of the frequency response

(c) $\Omega_1 = \frac{\pi}{4},\;\; W_c = 2\pi$

$$\omega_{max} \quad = \quad \min(10\pi,\ \frac{\pi}{4}(20), 2\pi) = 2\pi$$

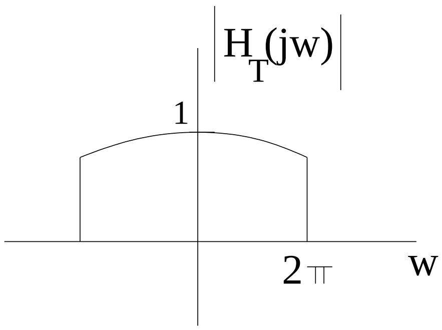


Figure P4.35. (c) Magnitude of the frequency response

4.36. Let $X(e^{j\Omega}) = \frac{\sin(\frac{11\Omega}{2})}{\sin(\frac{\Omega}{2})}$ and define $\tilde{X}[k] = X(e^{jk\Omega_o})$. Find and sketch $\tilde{x}[n]$ where $\tilde{x}[n] \xleftrightarrow{DTFS;\,\Omega_o} \tilde{X}[k]$ for the following values of Ω_o:

$$\tilde{X}[k] \quad = \quad X(e^{jk\Omega_o})$$

$$\tilde{X}[k] = \frac{\sin(\frac{11k\Omega_o}{2})}{\sin(\frac{k\Omega_o}{2})} \quad \xleftrightarrow{DTFS;\,\Omega_o} \quad \tilde{x}[n] = \begin{cases} N & |n| \le 5 \\ 0 & 5 < |n| < \frac{N}{2}, \quad N \text{ periodic} \end{cases}$$

(a) $\Omega_o = \frac{2\pi}{15}$, $N = 15$

$$\tilde{x}[n] \quad = \quad \begin{cases} 15 & |n| \le 5 \\ 0 & 5 < |n| < 7, \quad 15 \text{ periodic} \end{cases}$$

(b) $\Omega_o = \frac{\pi}{10}$, $N = 20$

$$\tilde{x}[n] \quad = \quad \begin{cases} 20 & |n| \le 5 \\ 0 & 5 < |n| < 10, \quad 20 \text{ periodic} \end{cases}$$

(c) $\Omega_o = \frac{\pi}{3}$, $N = 6$
Overlap occurs

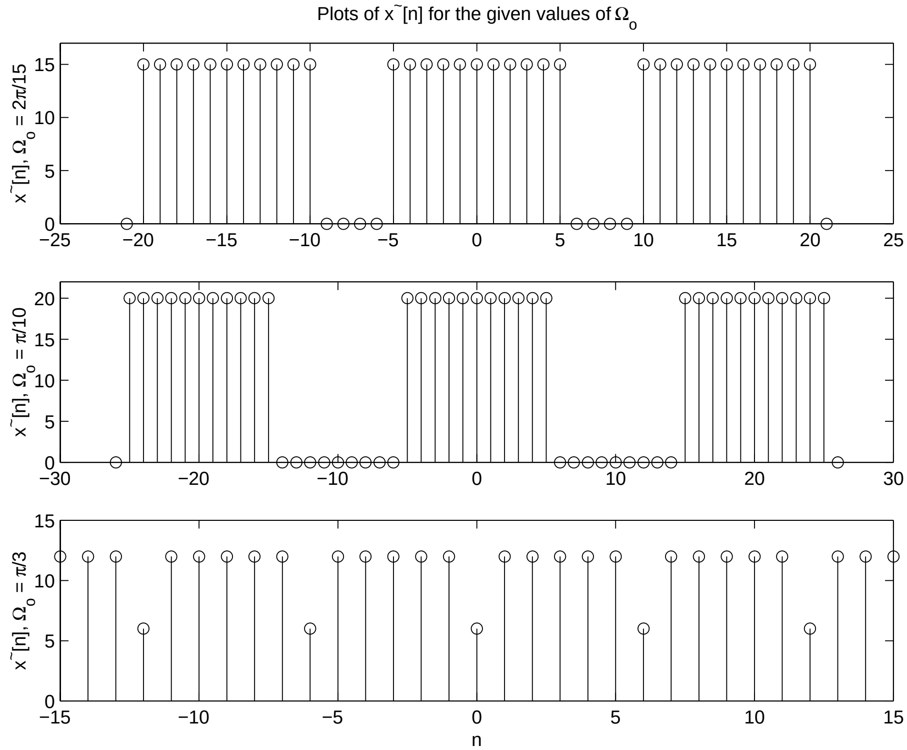


Figure P4.36. Sketch of $\tilde{x}[n]$

4.37. Let $X(j\omega) = \frac{\sin(2\omega)}{\omega}$ and define $\tilde{X}[k] = X(jk\omega_o)$. Find and sketch $\tilde{x}(t)$ where $\tilde{x}(t) \xleftrightarrow{FS;\,\omega_o} \tilde{X}[k]$ for the following values of ω_o:

$$
\begin{aligned}
x(t) &= \begin{cases} \frac{1}{2} & |t| \leq 2 \\ 0 & \text{otherwise} \end{cases} \\
\tilde{X}[k] = X(jk\omega_o) &= \frac{\sin(2k\omega_o)}{k\omega_o} \\
\tilde{x}(t) &= T \sum_{m=-\infty}^{\infty} x(t - mT)
\end{aligned}
$$

(a) $\omega_o = \frac{\pi}{8}$

$$
\begin{aligned}
\tilde{X}[k] &= \frac{\sin(\frac{\pi}{4}k)}{\frac{\pi}{8}k} \\
T_s &= 2 \\
T &= 16 \\
\tilde{x}(t) &= \begin{cases} 8 & |t| < 2 \\ 0 & 2 < |t| < 8, \quad 16 \;\; \text{periodic} \end{cases}
\end{aligned}
$$

(b) $\omega_o = \frac{\pi}{4}$

$$\begin{aligned}
\tilde{X}[k] &= \frac{\sin(\frac{\pi}{2}k)}{\frac{\pi}{4}k} \\
T_s &= 2 \\
T &= 8 \\
\tilde{x}(t) &= \begin{cases} 4 & |t| < 2 \\ 0 & 2 < |t| < 4, \quad 8 \text{ periodic} \end{cases}
\end{aligned}$$

(c)$\omega_o = \frac{\pi}{2}$

$$\begin{aligned}
\tilde{X}[k] &= \frac{\sin(\pi k)}{\frac{\pi}{2}k} \\
&= 2\delta[k] \\
\tilde{x}(t) &= 2
\end{aligned}$$

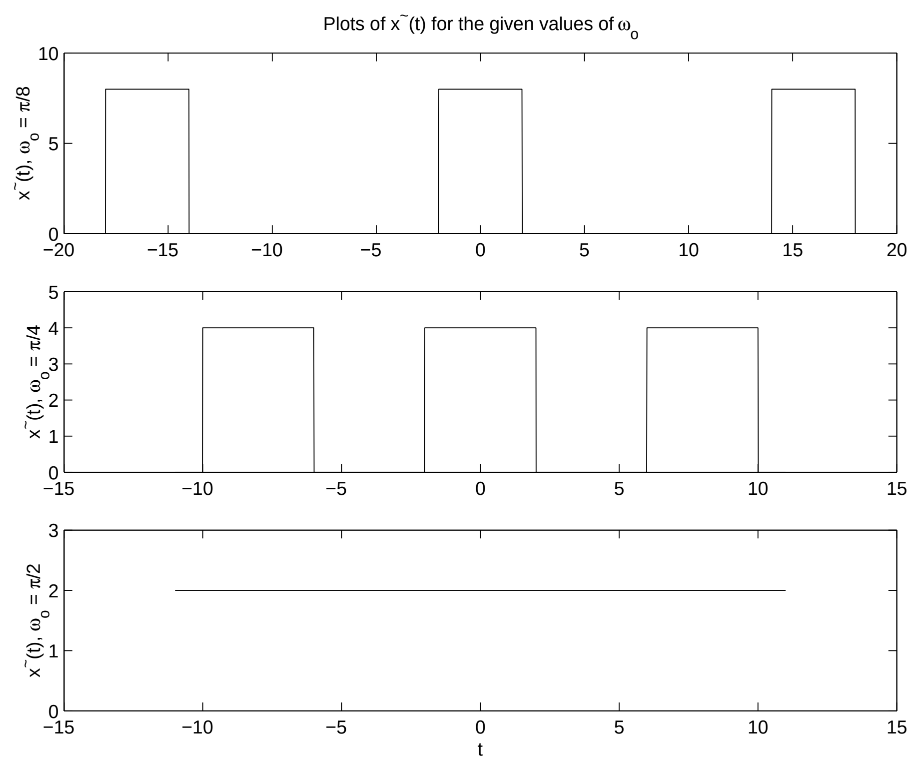


Figure P4.37. Sketch of $\tilde{x}(t)$

4.38. A signal $x(t)$ is sampled at intervals of $T_s = 0.01$ s. One hundred samples are collected and a 200-point DTFS is taken in an attempt to approximate $X(j\omega)$. Assume $|X(j\omega)| \approx 0$ for $|\omega| > 120\pi$ rad/s. Determine the frequency range $-\omega_a < \omega < \omega_a$ over which the DTFS offers a reasonable approximation to

$X(j\omega)$, the effective resolution of this approximation, ω_r, and the frequency interval between each DTFS coefficient, $\Delta\omega$.

$$\begin{array}{rl} x(t) & T_s = 0.01 \\ 100 \text{ samples} & M = 100 \end{array}$$

Use $N = 200$ DTFS to approximate $X(j\omega)$, $|X(j\omega)| \approx 0,\ |\omega| > 120\pi$, $\omega_m = 120\pi$

$$\begin{aligned} T_s &< \frac{2\pi}{\omega_m + \omega_a} \\ \omega_a &< \frac{2\pi}{T_s} - \omega_m \end{aligned}$$

Therefore:

$$\begin{aligned} \omega_a &< 80\pi \\ MT_s &> \frac{2\pi}{\omega_r} \end{aligned}$$

Therefore:

$$\begin{aligned} \omega_r &> 2\pi \\ N &> \frac{\omega_s}{\Delta\omega} \\ \Delta\omega &> \frac{\omega_s}{N} \\ \Delta\omega &= \frac{2\pi}{NT_s} \end{aligned}$$

Therefore:

$$\Delta\omega > \pi$$

4.39. A signal $x(t)$ is sampled at intervals of $T_s = 0.1$ s. Assume $|X(j\omega)| \approx 0$ for $|\omega| > 12\pi$ rad/s. Determine the frequency range $-\omega_a < \omega < \omega_a$ over which the DTFS offers a reasonable approximation to $X(j\omega)$, the minimum number of samples required to obtain an effective resolution $\omega_r = 0.01\pi$ rad/s, and the length of the DTFS required so the frequency interval between DTFS coefficients is $\Delta\omega = 0.001\pi$ rad/s.

$$\begin{aligned} T_s &< \frac{2\pi}{\omega_m + \omega_a} \\ \omega_m &= 12\pi \\ T_s &= 0.1 \\ \omega_a &< 8\pi \end{aligned}$$

The frequency range $|\omega| < 8\pi$ provides a reasonable approximation to the FT.

$$M \geq \frac{\omega_s}{\omega_r}$$

$$
\begin{aligned}
\omega_s &= \frac{2\pi}{T_s} = 20\pi \\
\omega_r &= 0.01\pi \\
M &\geq 2000
\end{aligned}
$$

$M = 2000$ samples is sufficient for the given resolution.

$$
\begin{aligned}
N &\geq \frac{\omega_s}{\Delta\omega} \\
\Delta\omega &= 0.001\pi \\
N &\geq 20,000
\end{aligned}
$$

The required length of the DTFS is $N = 20,000$.

4.40. Let $x(t) = a\sin(\omega_o t)$ be sampled at intervals of $T_s = 0.1$ s. Assume 100 samples of $x(t)$, $x[n] = x(nT_s), n = 0, 1, \ldots 99$, are available. We use the DTFS of $x[n]$ to approximate the FT of $x(t)$ and wish to determine a from the DTFS coefficient of largest magnitude. The samples $x[n]$ are zero-padded to length N before taking the DTFS. Determine the minimum value of N for the following values of ω_o: Determine which DTFS coefficient has the largest magnitude in each case.

Choose $\Delta\omega$ so that ω_o is an integer multiple of $\Delta\omega, (\omega_o = p\Delta\omega)$, where p is an integer, and set $N = M = 100$. Using these two conditions results in the DTFS sampling $W_\delta(j(\omega - \omega_o))$ at the peak of the mainlobe and at all of the zero crossings. Consequently,

$$
Y[k] = \begin{cases} a & k = p \\ 0 & \text{otherwise on } 0 \leq k \leq N-1 \end{cases}
$$

(a) $\omega_o = 3.2\pi$

$$
\begin{aligned}
N &= \frac{\omega_s}{\Delta\omega} \\
&= \frac{20\pi p}{\omega_o} \\
&= \frac{20\pi p}{3.2\pi} \\
&= 25 \\
p &= 4 \\
& \quad \text{Since } p \text{ and } N \text{ have to be integers}
\end{aligned}
$$

$$
Y[k] = \begin{cases} a & k = 4 \\ 0 & \text{otherwise on } 0 \leq k \leq 24 \end{cases}
$$

(b) $\omega_o = 3.1\pi$

$$\begin{aligned} N &= \frac{\omega_s}{\Delta\omega} \\ &= \frac{20\pi p}{\omega_o} \\ &= \frac{20\pi p}{3.1\pi} \\ &= 200 \\ p &= 31 \\ & \text{Since } p \text{ and } N \text{ have to be integers} \end{aligned}$$

$$Y[k] = \begin{cases} a & k = 31 \\ 0 & \text{otherwise on } 0 \le k \le 199 \end{cases}$$

(c) $\omega_o = 3.15\pi$

$$\begin{aligned} N &= \frac{\omega_s}{\Delta\omega} \\ &= \frac{20\pi p}{\omega_o} \\ &= \frac{20\pi p}{3.15\pi} \\ &= 400 \\ p &= 63 \\ & \text{Since } p \text{ and } N \text{ have to be integers} \end{aligned}$$

$$Y[k] = \begin{cases} a & k = 63 \\ 0 & \text{otherwise on } 0 \le k \le 399 \end{cases}$$

Solutions to Advanced Problems

4.41. A continuous-time signal lies in the frequency band $|\omega| < 5\pi$. This signal is contaminated by a large sinusoidal signal of frequency 120π. The contaminated signal is sampled at a sampling rate of $\omega_s = 13\pi$.

(a) After sampling, at what frequency does the sinusoidal intefering signal appear?

$X(j\omega)$ is bandlimited to 5π

$$\begin{aligned} s(t) &= x(t) + A\sin(120\pi t) \\ s[n] = s(nT_s) &= x[n] + A\sin(\frac{240\pi}{13}n) \end{aligned}$$

$$
\begin{aligned}
&= x[n] + A\sin(9(2\pi)n + \frac{6\pi}{13}n) \\
&= x[n] + A\sin(\frac{6\pi}{13}n) \\
\Omega \sin &= \frac{6\pi}{13} \\
\omega &= \frac{\Omega}{T_s} = \left(\frac{6\pi}{13}\right)\left(\frac{13}{2}\right) = 3\pi
\end{aligned}
$$

The sinusoid appears at $\omega = 3\pi$ rads/sec in $S_\delta(j\omega)$.

(b) The contaminated signal is passed through an anti-aliasing filter consisting of the RC circuit depicted in Fig. P4.41. Find the value of the time constant RC required so that the contaminating sinusoid is attenuated by a factor of 1000 prior to sampling.

Before the sampling, $s(t)$ is passed through a LPF.

$$
\begin{aligned}
T(j\omega) &= \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \\
&= \frac{1}{1 + j\omega RC} \\
&= \frac{1}{1 + j\omega\tau} \\
|T(j\omega)| &= \left.\frac{1}{\sqrt{1+\omega^2\tau^2}}\right|_{\omega=120\pi} \\
&= \frac{1}{1000} \\
\tau &= 2.65\ s
\end{aligned}
$$

(c) Sketch the magnitude response in dB that the anti-aliasing filter presents to the signal of interest for the value of RC identified in (b).

$$
\begin{aligned}
T(j\omega) &= \frac{1}{1 + j\omega 2.65} \\
|T(j\omega)| &= \frac{1}{\sqrt{1+\omega^2(2.65)^2}}
\end{aligned}
$$

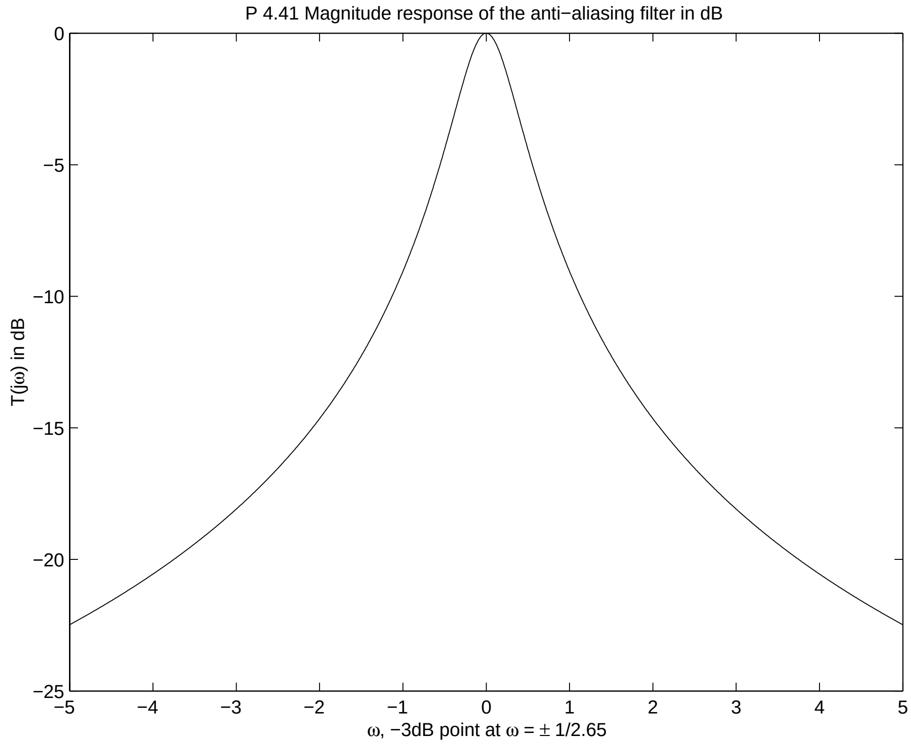


Figure P4.41. Sketch of the magnitude response

4.42. This problem derives the frequency-domain relationship for subsampling given in Eq.(4.27). Use Eq. (4.17) to represent $x[n]$ as the impulse-sampled continuous-time signal with sampling interval T_s, and thus write

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s)$$

Suppose $x[n]$ are the samples of a continuous-time signal $x(t)$, obtained at integer multiples of T_s. That is, $x[n] = x(nT_s)$. Let $x(t) \xleftrightarrow{FT} X(j\omega)$. Define the subsampled signal $y[n] = x[qn]$ so that $y[n] = x(nqT_s)$ is also expressed as samples of $x(t)$.

(a) Apply Eq. (4.23) to express $X_\delta(j\omega)$ as a function of $X(j\omega)$. Show that

$$Y_\delta(j\omega) = \frac{1}{qT_s}\sum_{k=-\infty}^{\infty} X(j(\omega - \frac{k}{q}\omega_s))$$

Since $y[n]$ is formed by using every qth sample of $x[n]$, the effective sampling rate is $T_s' = qT_s$

$$y_\delta(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t - nT_s') \quad \xleftrightarrow{FT} \quad Y_\delta(j\omega) = \frac{1}{T_s'}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s'))$$

Substituting $T_s' = qT_s,$ and $\omega_s' = \dfrac{\omega_s}{q}$ yields:

$$Y_\delta(j\omega) \quad = \quad \frac{1}{qT_s}\sum_{k=-\infty}^{\infty} X(j(\omega - \frac{k}{q}\omega_s))$$

(b) The goal is to express $Y_\delta(j\omega)$ as a function of $X_\delta(j\omega)$ so that $Y(e^{j\Omega})$ can be expressed in terms of $X(e^{j\Omega})$. To this end, write $\frac{k}{q}$ in $Y_\delta(j\omega)$ as the proper fraction

$$\frac{k}{q} = l + \frac{m}{q}$$

where l is the integer portion of $\frac{k}{q}$ and m is the remainder. Show that we may thus rewrite $Y_\delta(j\omega)$ as

$$Y_\delta(j\omega) = \frac{1}{q}\sum_{m=0}^{q-1}\left\{\frac{1}{T_s}\sum_{l=-\infty}^{\infty} X(j(\omega - l\omega_s - \frac{m}{q}\omega_s))\right\}$$

Letting k to range from $-\infty$ to ∞ corresponds to having l range from $-\infty$ to ∞ and m from 0 to $q-1$, which permits us to rewrite $Y_\delta(j\omega)$ as :

$$Y_\delta(j\omega) \quad = \quad \frac{1}{q}\sum_{m=0}^{q-1}\left\{\frac{1}{T_s}\sum_{l=-\infty}^{\infty} X(j(\omega - l\omega_s - \frac{m}{q}\omega_s))\right\}$$

Next show that

$$Y_\delta(j\omega) = \frac{1}{q}\sum_{m=0}^{q-1} X_\delta(j(\omega - \frac{m}{q}\omega_s))$$

Recognizing that the term in braces corresponds to $X_\delta(j(\omega - \frac{m}{q}\omega_s)$, allows us to rewrite the equation as the following double sum:

$$Y_\delta(j\omega) \quad = \quad \frac{1}{q}\sum_{m=0}^{q-1} X_\delta(j(\omega - \frac{m}{q}\omega_s))$$

(c) Now we convert from the FT representation back to the DTFT in order to express $Y(e^{j\Omega})$ as a function of $X(e^{j\Omega})$. The sampling interval associated with $Y_\delta(j\omega)$ is qT_s. Using the relationship $\Omega = \omega q T_s$ in

$$Y(e^{j\Omega}) = Y_\delta(j\omega)|_{\omega=\frac{\Omega}{qT_s}}$$

show that

$$Y(e^{j\Omega}) = \frac{1}{q}\sum_{m=0}^{q-1} X_\delta\left(\frac{j}{T_s}\left(\frac{\Omega}{q} - \frac{m}{q}2\pi\right)\right)$$

Substituting $\omega = \frac{\Omega}{qT_s}$ yields

$$Y(e^{j\Omega}) \quad = \quad \frac{1}{q}\sum_{m=0}^{q-1} X_\delta\left(\frac{j}{T_s}\left(\frac{\Omega}{q} - \frac{m}{q}2\pi\right)\right)$$

(d) Lastly, use $X(e^{j\Omega}) = X_\delta(j\frac{\Omega}{T_s})$ to obtain

$$Y(e^{j\Omega}) = \frac{1}{q}\sum_{m=0}^{q-1} X\left(e^{j\frac{1}{q}(\Omega - m2\pi)}\right)$$

The sampling interval associated with $X_\delta(j\omega)$ is T_s, so $X(e^{j\Omega}) = X_\delta(j\frac{\Omega}{T_s})$. Hence we may substitute $X\left(e^{j(\frac{\Omega}{q} - m\frac{2\pi}{q})}\right)$ for $X_\delta\left(\frac{j}{T_s}(\frac{\Omega}{q} - m\frac{2\pi}{q})\right)$ and obtain

$$Y(e^{j\Omega}) \quad = \quad \frac{1}{q}\sum_{m=0}^{q-1} X\left(e^{j\frac{1}{q}(\Omega - m2\pi)}\right)$$

4.43. A bandlimited signal $x(t)$ satisfies $|X(j\omega)| = 0$ for $|\omega| < \omega_1$ and $|\omega| > \omega_2$. Assume $\omega_1 > \omega_2 - \omega_1$. In this case we can sample $x(t)$ at a rate less than that indicated by the sampling interval and still perform perfect reconstruction by using a bandpass reconstruction filter $H_r(j\omega)$. Let $x[n] = x(nT_s)$. Determine the maximum sampling interval T_s such that $x(t)$ can be perfectly reconstructed from $x[n]$. Sketch the frequency response of the reconstruction filter required for this case.

We can tolerate aliasing as long as there is no overlap on $\omega_1 \leq |\omega| \leq \omega_2$
We require:

$$\begin{aligned}
\omega_s - \omega_2 &\geq -\omega_1 \\
\omega_s &\geq \omega_2 - \omega_1 \\
\text{Implies:} & \\
T_s &\leq \frac{2\pi}{\omega_2 - \omega_1} \\
H_r(j\omega) &= \begin{cases} T_s & \omega_1 \leq |\omega| \leq \omega_2 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

4.44. Suppose a periodic signal $x(t)$ has FS coefficients

$$X[k] = \begin{cases} (\frac{3}{4})^k, & |k| \leq 4 \\ 0, & \text{otherwise} \end{cases}$$

The period of this signal is $T = 1$.
(a) Determine the minimum sampling interval for this signal that will prevent aliasing.

$$\begin{aligned}
X(j\omega) &= 2\pi \sum_{k=-4}^{4} (\frac{3}{4})^k \delta(\omega - k2\pi) \\
\omega_m &= 8\pi \\
\frac{2\pi}{T_s} &> 2(8\pi) \\
\min T_s &= \frac{1}{8}
\end{aligned}$$

(b) The constraints of the sampling theorem can be relaxed somewhat in the case of periodic signals if we allow the reconstructed signal to be a time-scaled version of the original. Suppose we choose a sampling interval $T_s = \frac{20}{19}$ and use a reconstruction filter

$$H_r(j\omega) = \begin{cases} 1, & |\omega| < \pi \\ 0, & \text{otherwise} \end{cases}$$

Show that the reconstructed signal is a time-scaled version of $x(t)$ and identify the scaling factor.

$$\begin{aligned}
T_s &= \frac{20}{19} \\
X_\delta(j\omega) &= \frac{19}{20} \sum_{l=-\infty}^{\infty} X(j(\omega - l1.9\pi))
\end{aligned}$$

Aliasing produces a " frequency scaled" replica of $X(j\omega)$ centered at zero. The scaling is by a factor of 20 from $\omega_o = 2\pi$ to $\omega_o' = 0.1\pi$. Applying the LPF, $|\omega| < \pi$ gives $x(\frac{t}{20})$, and $x_{reconstructed}(t) = \frac{19}{20}x(\frac{t}{20})$

(c) Find the constraints on the sampling interval T_s so that use of $H_r(j\omega)$ in (b) results in the reconstruction filter being a time-scaled version of $x(t)$ and determine the relationship between the scaling factor and T_s.

The choice of T_s is so that no aliasing occurs.

$$\begin{aligned}
(1) \qquad \frac{2\pi}{T_s} &< 2\pi \\
T_s &> 1 \text{ period of the original signal.} \\
(2) \qquad \left(2\pi - \frac{2\pi}{T_s}\right)4 &< \frac{1}{2}\frac{2\pi}{T_s} \\
T_s &< \frac{9}{8} \\
1 < \quad T_s &< \frac{9}{8}
\end{aligned}$$

4.45. In this problem we reconstruct a signal $x(t)$ from its samples $x[n] = x(nT_s)$ using pulses of width less than T_s followed by an anti-imaging filter with frequency response $H_c(j\omega)$. Specifically, we apply

$$x_p(t) = \sum_{n=-\infty}^{\infty} x[n]h_p(t - nT_s)$$

to the anti-imaging filter, where $h_p(t)$ is a pulse of width T_o as depicted in Fig. P4.45 (a). An example of $x_p(t)$ is depicted in Fig. P4.45 (b). Determine the constraints on $|H_c(j\omega)|$ so that the overall magnitude response of this reconstruction system is between 0.99 and 1.01 in the signal passband and less than 10^{-4} to the images of the signal spectrum for the following values with $x(t)$ bandlimited to 10π, that is, $X(j\omega) = 0$ for $|\omega| > 10\pi$:

$$\begin{aligned}
X_p(j\omega) &= H_p(j\omega)X_\Delta(j\omega) \\
H_p(j\omega) &= \frac{2\sin(\omega\frac{T_o}{2})}{T_o\omega}e^{-j\omega\frac{T_o}{2}}
\end{aligned}$$

Constraints:

$$\begin{aligned}
&\text{(1) Passband} \\
0.99 < |H_p(j\omega)||H_c(j\omega)| &< 1.01, \text{ using } \omega_{max} = 10\pi \\
\frac{0.99T_o\omega}{2\sin(\omega\frac{T_o}{2})} < |H_c(j\omega)| &< \frac{1.01T_o\omega}{2\sin(\omega\frac{T_o}{2})} \\
&\text{(2) In the image location} \\
|H_p(j\omega)| &< \frac{10^{-4}T_o\omega}{2\sin(\omega\frac{T_o}{2})}
\end{aligned}$$

$$\text{where} \qquad \omega = \frac{2\pi}{T_s} - 10\pi$$

(a) $T_s = 0.08, \;\; T_o = 0.04$

$$\begin{array}{rcl}
\text{(1) Passband} & & \\
1.1533 < & |H_c(j\omega)| < 1.1766 & \\
\text{(2) In the image location} & & \\
\omega & = & 15\pi \\
|H_c(j15\pi)| & < & 1.165 \times 10^{-4}
\end{array}$$

(b) $T_s = 0.08, \;\; T_o = 0.02$

$$\begin{array}{rcl}
\text{(1) Passband} & & \\
1.0276 < & |H_c(j\omega)| < 1.0484 & \\
\text{(2) In the image location} & & \\
\omega & = & 15\pi \\
|H_c(j15\pi)| & < & 1.038 \times 10^{-4}
\end{array}$$

(c) $T_s = 0.04, \;\; T_o = 0.02$

$$\begin{array}{rcl}
\text{(1) Passband} & & \\
1.3081 < & |H_c(j\omega)| < 1.3345 & \\
\text{(2) In the image location} & & \\
\omega & = & 40\pi \\
|H_c(j40\pi)| & < & 1.321 \times 10^{-4}
\end{array}$$

(d) $T_s = 0.04, \;\; T_o = 0.01$

$$\begin{array}{rcl}
\text{(1) Passband} & & \\
1.0583 < & |H_c(j\omega)| < 1.0796 & \\
\text{(2) In the image location} & & \\
\omega & = & 40\pi \\
|H_c(j40\pi)| & < & 1.0690 \times 10^{-4}
\end{array}$$

4.46. A non-ideal sampling operation obtains $x[n]$ from $x(t)$ as

$$x[n] = \int_{(n-1)T_s}^{nT_s} x(t)dt$$

(a) Show that this can be written as ideal sampling of a filtered signal $y(t) = x(t) * h(t)$, that is, $x[n] = y(nT_s)$, and find $h(t)$.

$$\begin{aligned}
y(t) &= \int_{t-T_s}^{T_s} x(\tau)d\tau \text{ by inspection} \\
&= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
&= x(t) * h(t) \\
\text{choose } h(t) &= \begin{cases} 1 & 0 \le t \le T_s \\ 0 & \text{otherwise} \end{cases} \\
h(t-\tau) &= \begin{cases} 1 & t-T_s \le \tau \le T_s \\ 0 & \text{otherwise} \end{cases} \\
h(t) &= u(t) - u(t-T_s)
\end{aligned}$$

(b) Express the FT of $x[n]$ in terms of $X(j\omega)$, $H(j\omega)$, and T_s.

$$\begin{aligned}
Y(j\omega) &= X(j\omega)H(j\omega) \\
y(nT_s) &\xleftrightarrow{FT} \frac{1}{T_s}\sum_{k=-\infty}^{\infty} Y(j(\omega - k\frac{2\pi}{T_s})) \\
FT\{x[n]\} &= \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\omega - k\frac{2\pi}{T_s}))H(j(\omega - k\frac{2\pi}{T_s}))
\end{aligned}$$

(c) Assume that $x(t)$ is bandlimited to the frequency range $|\omega| < \frac{3\pi}{4T_s}$. Determine the frequency response of a discrete-time system that will correct the distortion in $x[n]$ introduced by non-ideal sampling.

$x(t)$ is bandlimited to:

$$|\omega| < \frac{3\pi}{4T_s} < \frac{2\pi}{T_s}$$

We can use:

$$\begin{aligned}
H_r(j\omega) &= \begin{cases} \frac{T_s}{H(j\omega)} & |\omega| \le \frac{3\pi}{4T_s} \\ 0 & \text{otherwise} \end{cases} \\
h(t) &= \begin{cases} 1 & 0 \le t \le T_s \\ 0 & \text{otherwise} \end{cases} \\
h(t+\frac{T_s}{2}) &\xleftrightarrow{FT} \frac{2\sin(\omega\frac{T_s}{2})}{\omega} \\
H(j\omega) &= \frac{2\sin(\omega\frac{T_s}{2})}{\omega} e^{-j\omega\frac{T_s}{2}} \\
H_r(j\omega) &= \begin{cases} \frac{\omega T_s e^{j\omega\frac{T_s}{2}}}{2\sin(\omega\frac{T_s}{2})} & |\omega| \le \frac{3\pi}{4T_s} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

4.47. The system depicted in Fig. P4.47 (a) converts a continuous-time signal $x(t)$ to a discrete-time signal $y[n]$. We have

$$H(e^{j\Omega}) = \begin{cases} 1, & |\Omega| < \frac{\pi}{4} \\ 0, & \text{otherwise} \end{cases}$$

Find the sampling frequency $\omega_s = \frac{2\pi}{T_s}$ and the constraints on the anti-aliasing filter frequency response $H_a(j\omega)$ so that an input signal with FT $X(j\omega)$ shown in Fig. P4.47 (b) results in the output signal with DTFT $Y(e^{j\Omega})$.

$$\begin{aligned} X_a(j\omega) &= X(j\omega)H_a(j\omega) \\ X_{a\delta}(j\omega) &= \frac{1}{T_s}\sum_k X_a(j(\omega - k\frac{2\pi}{T_s})) \end{aligned}$$

To discard the high frequency of $X(j\omega)$, and anticipate $\frac{1}{T_s}$, use:

$$H_a(j\omega) = \begin{cases} T_s & |\omega| \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

Given $Y(e^{j\Omega})$, we can conclude that $T_s = \frac{1}{4}$, since $Y(j\omega)$ is

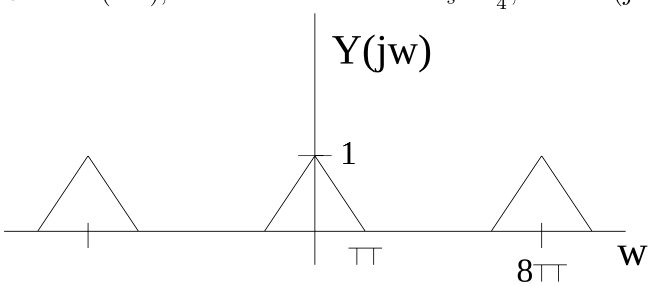


Figure P4.47. Graph of $Y(j\omega)$

Also, the bandwidth of $x(t)$ should not change, therefore:

$$\begin{aligned} \omega_s &= 8\pi \\ H_a(j\omega) &= \begin{cases} \frac{1}{4} & |\omega| \leq \pi \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

4.48. The discrete-time signal $x[n]$ with DTFT $X(e^{j\Omega})$ shown in Fig. P4.48 (a) is decimated by first passing $x[n]$ through the filter with frequency response $H(e^{j\Omega})$ shown in Fig. P4.48 (b) and then subsampling by the factor q. For the following values of q and W, determine the minimum value of Ω_p and maximum value of Ω_s so that the subsampling operation does not change the shape of the portion of $X(e^{j\Omega})$ on $|\Omega| < W$.

$$Y(e^{j\Omega}) = \frac{1}{q}\sum_{m=0}^{q-1} X\left(e^{j\frac{1}{q}(\Omega - m2\pi)}\right)$$

From the following figure, one can see to preserve the shape within $|\Omega| < W$, we need:

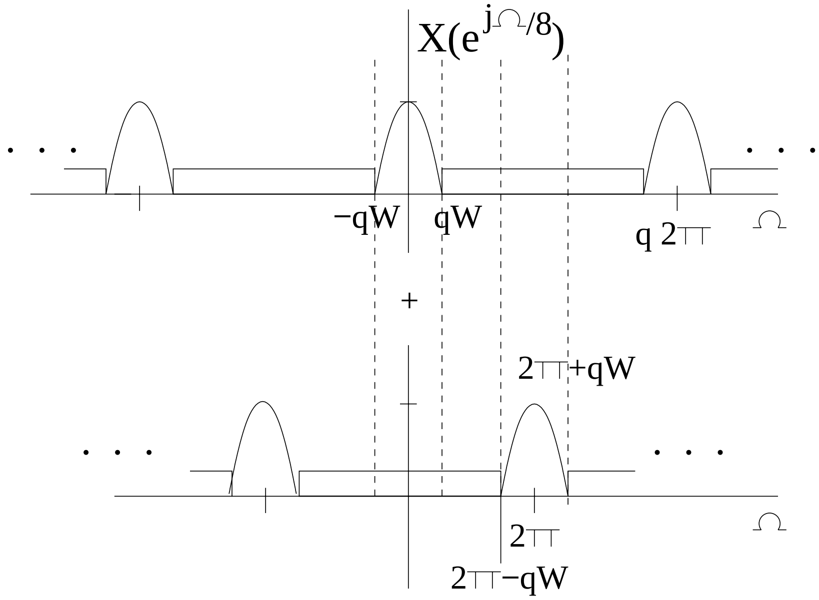


Figure P4.48. Figure showing the necessary constraints to preserve the signal

$$\begin{aligned}\min \Omega_p &= \frac{qW}{q} = W \\ \max \Omega_s &= \frac{2\pi - qW}{q} = \frac{2\pi}{q} - W\end{aligned}$$

(a) $q = 2, \ \ W = \frac{\pi}{3}$

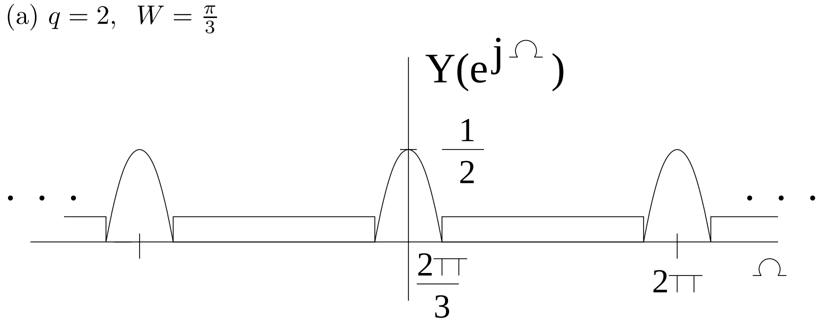


Figure P4.48. (a) Sketch of the DTFT

(b) $q = 2, \ \ W = \frac{\pi}{4}$

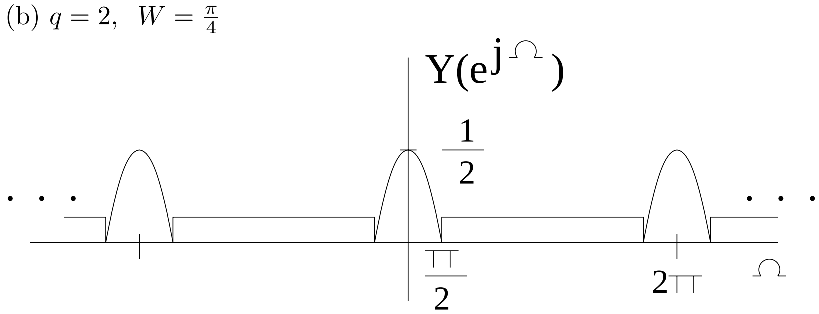


Figure P4.48. (b) Sketch of the DTFT

(c) $q = 3, \;\; W = \frac{\pi}{4}$
In each case sketch the DTFT of the subsampled signal.

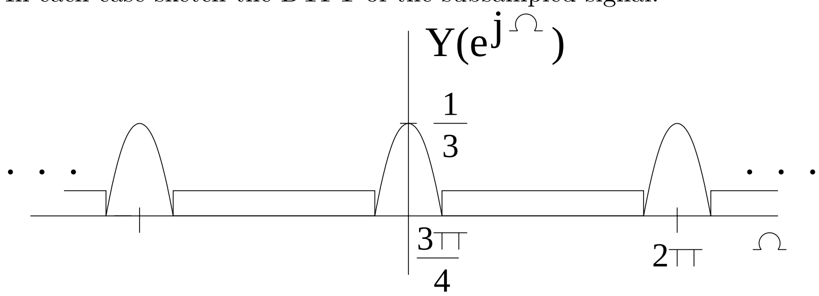


Figure P4.48. (c) Sketch of the DTFT

4.49. A signal $x[n]$ is interpolated by the factor q by first inserting $q-1$ zeros between each sample and next passing the zero-stuffed sequence through a filter with frequency response $H(e^{j\Omega})$ depicted in Fig. P4.48 (b). The DTFT of $x[n]$ is depicted in Fig. P4.49. Determine the minimum value of Ω_p and maximum value of Ω_s so that ideal interpolation is obtained for the following cases. In each case sketch the DTFT of the interpolated signal.

$$X_z(e^{j\Omega}) \;\; = \;\; X(e^{j\Omega q})$$

For ideal interpolation,

$$\begin{aligned} \min \Omega_p &= \frac{W}{q} \\ \max \Omega_s &= 2\pi - \frac{W}{q} \end{aligned}$$

(a) $q = 2, \;\; W = \frac{\pi}{2}$

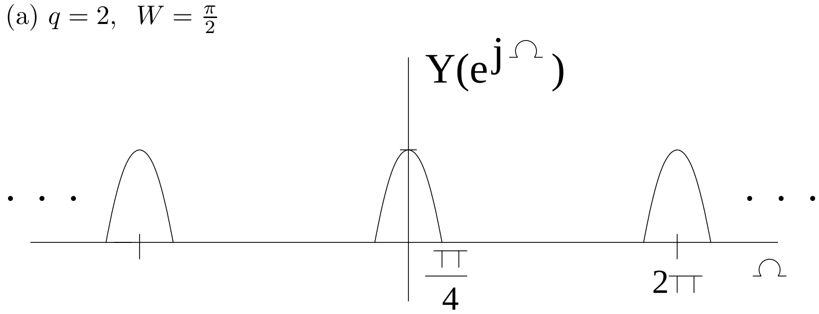


Figure P4.49. (a) Sketch of the DTFT of the interpolated signal

(b) $q = 2, \;\; W = \frac{3\pi}{4}$

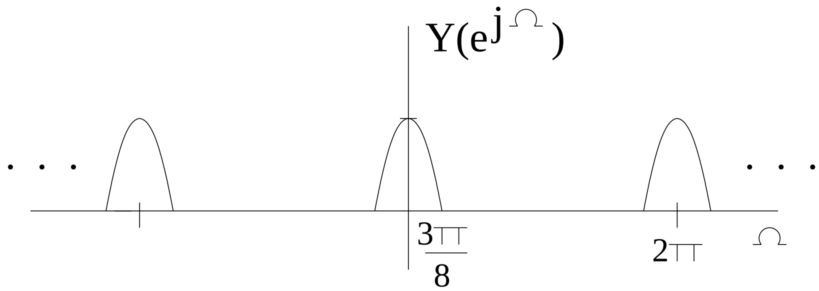


Figure P4.49. (b) Sketch of the DTFT of the interpolated signal

(c) $q = 3, \ \ W = \frac{3\pi}{4}$

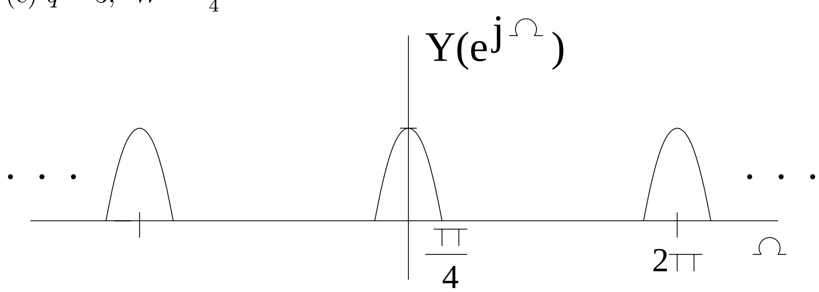


Figure P4.49. (c) Sketch of the DTFT of the interpolated signal

4.50. Consider interpolating a signal $x[n]$ by repeating each value q times as depicted in Fig. P4.50. That is, we define $x_o[n] = x[\text{floor}(\frac{n}{q})]$ where floor(z) is the integer less than or equal to z. Letting $x_z[n]$ be derived from $x[n]$ by inserting $q-1$ zeros between each value of $x[n]$, that is,

$$x_z[n] = \begin{cases} x[\frac{n}{q}], & \frac{n}{q} \text{ integer} \\ 0, & \text{otherwise} \end{cases}$$

We may then write $x_o[n] = x_z[n] * h_o[n]$, where $h_o[n]$ is:

$$h_o[n] = \begin{cases} 1, & 0 \le n \le q-1 \\ 0, & \text{otherwise} \end{cases}$$

Note that this is the discrete-time analog of the zero-order hold. The interpolation process is completed by passing $x_o[n]$ through a filter with frequency response $H(e^{j\Omega})$.

$$X_z(e^{j\Omega}) \quad = \quad X(e^{j\Omega q})$$

(a) Express $X_o(e^{j\Omega})$ in terms of $X(e^{j\Omega})$ and $H_o(e^{j\Omega})$. Sketch $|X_o(e^{j\Omega})|$ if $x[n] = \frac{\sin(\frac{3\pi}{4}n)}{\pi n}$.

$$X_o(e^{j\Omega}) \quad = \quad X(e^{j\Omega q})H_o(e^{j\Omega})$$

$$x[n] = \frac{\sin(\frac{3\pi}{4}n)}{\pi n} \quad \xleftrightarrow{DTFT} \quad X(e^{j\Omega}) = \begin{cases} 1 & |\Omega| < \frac{3\pi}{4} \\ 0 & \frac{3\pi}{4} \le |\Omega| < \pi, \quad 2\pi \ \text{ periodic} \end{cases}$$

$$|X_o(e^{j\Omega})| \quad = \quad |X(e^{j\Omega q})| \left| \frac{\sin(\Omega \frac{q}{2})}{\sin(\frac{\Omega}{2})} \right|$$

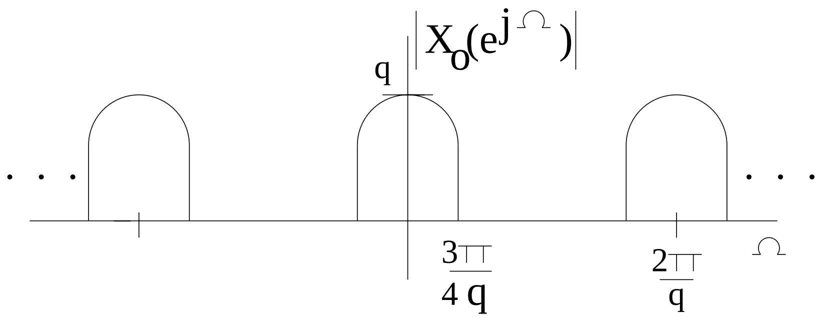


Figure P4.50. (a) Sketch of $|X_o(e^{j\Omega})|$

(b) Assume $X(e^{j\Omega})$ is as shown in Fig. P4.49. Specify the constraints on $H(e^{j\Omega})$ so that ideal interpolation is obtained for the following cases.

For ideal interpolation, discard components other than those centered at multiples of 2π. Also, some correction is needed to correct for magnitude and phase distortion.

$$H(e^{j\Omega}) \quad = \quad \begin{cases} \frac{\sin(\frac{\Omega}{2})}{\sin(\Omega\frac{q}{2})} e^{j\Omega\frac{q}{2}} & |\Omega| < \frac{W}{q} \\ 0 & \frac{W}{q} \leq |\Omega| < 2\pi - \frac{W}{q}, \quad 2\pi \quad \text{periodic} \end{cases}$$

(i) $q = 2, \;\; W = \frac{3\pi}{4}$

$$H(e^{j\Omega}) \quad = \quad \begin{cases} \frac{\sin(\frac{\Omega}{2})e^{j\Omega}}{\sin(\Omega)} & |\Omega| < \frac{3\pi}{8} \\ 0 & \frac{3\pi}{8} \leq |\Omega| < \frac{13\pi}{8}, \quad 2\pi \quad \text{periodic} \end{cases}$$

(ii) $q = 4, \;\; W = \frac{3\pi}{4}$

$$H(e^{j\Omega}) \quad = \quad \begin{cases} \frac{\sin(\frac{\Omega}{2})}{\sin(2\Omega)} e^{j2\Omega} & |\Omega| < \frac{3\pi}{16} \\ 0 & \frac{3\pi}{16} \leq |\Omega| < \frac{29\pi}{16}, \quad 2\pi \quad \text{periodic} \end{cases}$$

4.51. The system shown in Fig. P4.51 is used to implement a bandpass filter. The discrete-time filter $H(e^{j\Omega})$ has frequency response on $-\pi < \Omega \leq \pi$

$$H(e^{j\Omega}) = \begin{cases} 1, & \Omega_a \leq |\Omega| \leq \Omega_b \\ 0, & \text{otherwise} \end{cases}$$

Find the sampling interval T_s, Ω_a, Ω_b, W_1, W_2, W_3, and W_4, so that the equivalent continuous-time frequency response $G(j\omega)$ satisfies

$$0.9 < |G(j\omega)| < 1.1, \quad \text{for } 100\pi < \omega < 200\pi$$

$$G(j\omega) = 0 \quad \text{elsewhere}$$

In solving this problem, choose W_1 and W_3 as small as possible and choose T_s, W_2 and W_4 as large as possible.

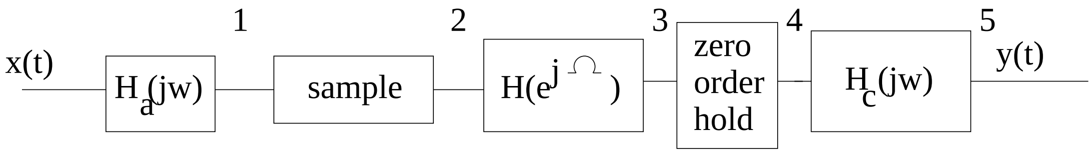


Figure P4.51. Graph of the system

(3) Passband:

$$\begin{aligned}
100\pi < \omega &< 200\pi \\
\text{Thus} & \\
\Omega_a &= 100\pi T_s \\
\Omega_b &= 200\pi T_s \\
(4)\ |H_o(j\omega)| &= \left|\frac{2\sin(\omega\frac{T_s}{2})}{\omega}\right| \\
\text{at } \omega = 100\pi & \\
\frac{2\sin(50\pi T_s)}{100\pi T_s} &< 1.1 \\
\text{at } \omega = 200\pi & \\
\frac{2\sin(100\pi T_s)}{200\pi T_s} &> 0.9 \\
\text{implies:} & \\
T_s(100\pi) &< 0.785 \\
\max T_s &= 0.0025 \\
(5)\ \ \min W_3 &= 200\pi \\
\max W_4 &= \frac{2\pi}{T_s} - 200\pi = 600\pi \\
(3)\ \Omega_a &= 0.25\pi \\
\Omega_b &= 0.5\pi \\
(1) \text{ and } (2) & \\
\min W_1 &= 200\pi \\
\max W_2 &= \frac{1}{2}\frac{2\pi}{T_s} = 400\pi, \ \text{No overlap.}
\end{aligned}$$

4.52. A time-domain interpretation of the interpolation procedure described in Fig. 4.50 (a) is derived in this problem. Let $h_i[n] \xleftrightarrow{DTFT} H_i(e^{j\Omega})$ be an ideal low-pass filter with transition band of zero width.

That is

$$H_i(e^{j\Omega}) = \begin{cases} q, & |\Omega| < \frac{\pi}{q} \\ 0, & \frac{\pi}{q} < |\Omega| < \pi \end{cases}$$

(a) Substitute for $h_i[n]$ in the convolution sum

$$x_i[n] = \sum_{k=-\infty}^{\infty} x_z[k] * h_i[n-k]$$

$$\begin{aligned} h_i[n] &= q\frac{\sin\left(\frac{\pi}{q}n\right)}{\pi n} \\ x_i[n] &= \sum_{k=-\infty}^{\infty} x_z[k]\frac{q\sin\left(\frac{\pi}{q}(n-k)\right)}{\pi(n-k)} \end{aligned}$$

(b) The zero-insertion procedure implies $x_z[k] = 0$ unless $k = qm$ where m is integer. Rewrite $x_i[n]$ using only the non-zero terms in the sum as a sum over m and substitute $x[m] = x_z[qm]$ to obtain the following expression for ideal discrete-time interpolation:

$$x_i[n] = \sum_{m=-\infty}^{\infty} x[m]\frac{q\sin\left(\frac{\pi}{q}(n-qm)\right)}{\pi(n-qm)}$$

Substituting $k = qm$ yields:

$$\begin{aligned} x_i[n] &= \sum_{m=-\infty}^{\infty} x_z[qm]\frac{q\sin\left(\frac{\pi}{q}(n-qm)\right)}{\pi(n-qm)} \\ \text{Now use } x_z[qm] &= x[m]. \end{aligned}$$

4.53. The continuous-time representation for a periodic discrete-time signal $x[n] \xleftrightarrow{DTFS;\, \frac{2\pi}{N}} X[k]$ is periodic and thus has a FS representation. This FS representation is a function of the DTFS coefficients $X[k]$, as we show in this problem. The result establishes the relationship between the FS and DTFS representations. Let $x[n]$ have period N and let $x_\delta(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s)$.

$$\begin{aligned} x[n] &= x[n+N] \\ x_\delta(t) &= \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT_s) \end{aligned}$$

(a) Show $x_\delta(t)$ is periodic and find the period, T.

$$x_\delta(t+T) = \sum_{n=-\infty}^{\infty} x[n]\delta(t+T-nT_s)$$

$$\begin{aligned}
\text{Now use} \qquad & x[n] = x[n-N] \text{ to rewrite} \\
x_\delta(t+T) &= \sum_{n=-\infty}^{\infty} x[n-N]\delta(t+T-nT_s) \\
\text{let } k = n-N & \\
x_\delta(t+T) &= \sum_{k=-\infty}^{\infty} x[k]\delta(t-kT_s+(T-NT_s))
\end{aligned}$$

It is clear that if $T = NT_s$, then $x_\delta(t+T) = x_\delta(t)$.
Therefore, $x_\delta(t)$ is periodic with $T = NT_s$.

(b) Begin with the definition of the FS coefficients

$$X_\delta[k] = \frac{1}{T}\int_0^T x_\delta(t)e^{-jk\omega_o t}dt.$$

Substitute for T, ω_o, and one period of $x_\delta(t)$ to show

$$X_\delta[k] = \frac{1}{T_s}X[k].$$

$$\begin{aligned}
X_\delta[k] &= \frac{1}{T}\int_0^T x_\delta(t)e^{-jk\omega_o t}dt \\
&= \frac{1}{T}\int_0^T \sum_{n=0}^{N-1} x[n]\delta(t-nT_s)e^{-jk\omega_o t}dt \\
&= \frac{1}{NT_s}\int_0^T \sum_{n=0}^{N-1} x[n]\delta(t-nT_s)e^{-jk\omega_o t}dt \\
&= \frac{1}{NT_s}\sum_{n=0}^{N-1} x[n]\int_0^T \delta(t-nT_s)e^{-jk\omega_o t}dt \\
&= \frac{1}{NT_s}\sum_{n=0}^{N-1} x[n]e^{-jk\omega_o nT_s} \\
&= \frac{1}{T_s}\left(\frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\frac{2\pi}{N}n}\right) \\
&= \frac{1}{T_s}X[k]
\end{aligned}$$

4.54. The fast algorithm for evaluating the DTFS (FFT) may be used to develop a computationally efficient algorithm for determining the output of a discrete-time system with a finite-length impulse reponse. Instead of directly computing the convolution sum, the DTFS is used to compute the output by performing multiplication in the frequency domain. This requires that we develop a correspondence between the periodic convolution implemented by the DTFS and the linear convolution associated with the system output, the goal of this problem. Let $h[n]$ be an impulse response of length M so that $h[n] = 0$ for $n < 0, n \geq M$. The system output $y[n]$ is related to the input via the convolution sum

$$y[n] = \sum_{k=0}^{M-1} h[k]x[n-k]$$

(a) Consider the N-point periodic convolution of $h[n]$ with N consecutive values of the input sequence $x[n]$ and assume $N > M$. Let $\tilde{x}[n]$ and $\tilde{h}[n]$ be N periodic versions of $x[n]$ and $h[n]$, respectively

$$\begin{aligned}\tilde{x}[n] &= x[n], \quad \text{for } 0 \le n \le N-1 \\ \tilde{x}[n+mN] &= \tilde{x}[n], \quad \text{for all integer } m,\ 0 \le n \le N-1\end{aligned}$$

$$\begin{aligned}\tilde{h}[n] &= h[n], \quad \text{for } 0 \le n \le N-1 \\ \tilde{h}[n+mN] &= \tilde{h}[n], \quad \text{for all integer } m,\ 0 \le n \le N-1\end{aligned}$$

The periodic convolution between $\tilde{h}[n]$ and $\tilde{x}[n]$ is

$$\tilde{y}[n] = \sum_{k=0}^{N-1} \tilde{h}[k]\tilde{x}[n-k]$$

Use the relationship between $h[n], x[n]$ and $\tilde{h}[n], \tilde{x}[n]$ to prove that $\tilde{y}[n] = y[n], \quad M-1 \le n \le N-1$. That is, the periodic convolution is equal to the linear convolution at $L = N - M + 1$ values of n.

$$\tilde{y}[n] = \sum_{k=0}^{N-1} h[k]\tilde{x}[n-k] \qquad (1)$$

Now since $\tilde{x}[n] = x[n], \quad$ for $0 \le n \le N-1$, we know that

$$\tilde{x}[n-k] = x[n-k], \quad \text{for } 0 \le n-k \le N-1$$

In (1), the sum over k varies from 0 to $M-1$, and so the condition $0 \le n-k \le N-1$ is always satisfied provided $M-1 \le n \le N-1$. Substituting $x[n-k] = \tilde{x}[n-k]$, $M-1 \le n \le N-1$ into (1) yields

$$\begin{aligned}\tilde{y}[n] &= \sum_{k=0}^{M-1} h[k]x[n-k] \qquad M-1 \le n \le N-1 \\ &= y[n]\end{aligned}$$

(b) Show that we may obtain values of $y[n]$ other than those on the interval $M-1 \le n \le N-1$ by shifting $x[n]$ prior to defining $\tilde{x}[n]$. That is, if

$$\begin{aligned}\tilde{x}_p[n] &= x[n+pL], \quad 0 \le n \le N-1 \\ \tilde{x}_p[n+mN] &= \tilde{x}_p[n], \quad \text{for all integer } m,\ 0 \le n \le N-1\end{aligned}$$

and

$$\tilde{y}_p[n] = \tilde{h}[n] \circledast \tilde{x}_p[n]$$

then show

$$\tilde{y}_p[n] = y[n+pL], \quad M-1 \le n \le N-1$$

This implies that the last L values in one period of $\tilde{y}_p[n]$ correspond to $y[n]$ for $M-1+pL \le n \le N-1+pL$. Each time we increment p the N point periodic convolution gives us L new values of the linear convolution. This result is the basis for the so-called *overlap and save method* for evaluating a linear convolution with the DTFS.

Overlap and Save Method of Implementing Convolution

1. Compute the N DTFS coefficients $H[k]: \ h[n] \xleftrightarrow{DTFS;\, 2\pi/N} H[k]$
2. Set $p = 0$ and $L = N - M + 1$
3. Define $\tilde{x}_p[n] = x[n - (M-1) + pL], \quad 0 \leq n \leq N-1$
4. Compute the N DTFS coefficients $\tilde{X}_p[k]: \ \tilde{x}_p[n] \xleftrightarrow{DTFS;\, 2\pi/N} \tilde{X}_p[k]$
5. Compute the product $\tilde{Y}_p[k] = NH[k]\tilde{X}_p[k]$
6. Compute the time signal $\tilde{y}_p[n]$ from the DTFS coefficients, $\tilde{Y}_p[k]: \ \tilde{y}_p[n] \xleftrightarrow{DTFS;\, 2\pi/N} \tilde{Y}_p[k]$.
7. Save the L output points: $y[n + pL] = \tilde{y}_p[n + M - 1], \quad 0 \leq n \leq L - 1$
8. Set $p = p + 1$ and return to step 3.

Solutions to Computer Experiments

4.55. Repeat Example 4.7 using zero-padding and the MATLAB commands fft and fftshift to sample and plot $Y(e^{j\Omega})$ at 512 points on $-\pi < \Omega \leq \pi$ for each case.

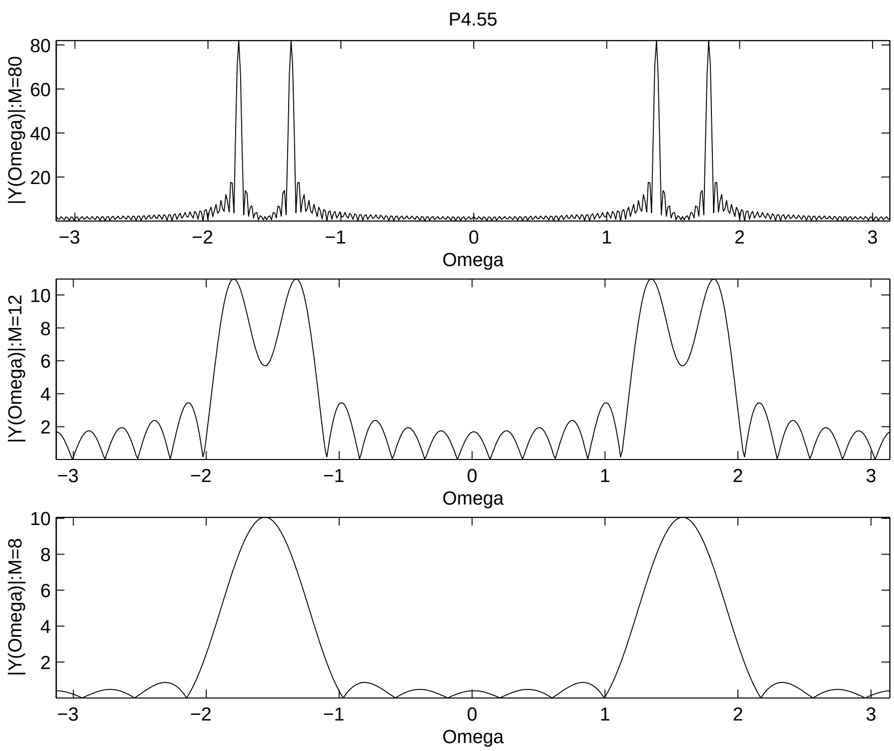


Figure P4.55. Plot of $Y(e^{j\Omega})$

4.56. The rectangular window is defined as

$$w_r[n] = \begin{cases} 1, & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases}$$

We may truncate a signal to the interval $0 \leq n \leq M$ by multiplying the signal with $w[n]$. In the frequency domain we convolve the DTFT of the signal with

$$W_r(e^{j\Omega}) = e^{-j\frac{M}{2}\Omega}\frac{\sin\left(\frac{\Omega(M+1)}{2}\right)}{\sin\left(\frac{\Omega}{2}\right)}$$

The effect of this convolution is to smear detail and introduce ripples in the vicinity of discontinuities. The smearing is proportional to the mainlobe width, while the ripple is proportional to the size of the sidelobes. A variety of alternative windows are used in practice to reduce sidelobe height in return for increased main lobe width. In this problem we evaluate the effect windowing time-domain signals on their DTFT. The role of windowing in filter design is explored in Chapter 8. The Hanning window is defined as

$$w_h[n] = \begin{cases} 0.5 - 0.5\cos(\frac{2\pi n}{M}), & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases}$$

(a) Assume $M = 50$ and use the MATLAB command fft to evaluate magnitude spectrum of the rectangular window in dB at intervals of $\frac{\pi}{50}$, $\frac{\pi}{100}$, and $\frac{\pi}{200}$.
(b) Assume $M = 50$ and use the MATLAB command fft to evaluate the magnitude spectrum of the Hanning window in dB at intervals of $\frac{\pi}{50}$, $\frac{\pi}{100}$, and $\frac{\pi}{200}$.
(c) Use the results from (a) and (b) to evaluate the mainlobe width and peak sidelobe height in dB for each window.

Using an interval of $\frac{\pi}{200}$, the mainlobe width and peak sidelobe for each window can be estimated from the figure, or finding the local minima and local nulls in the vicinity of the mainlobe.

	Ω:rad	(dB)
	Mainlobe width	Sidelobe height
Rectangular	0.25	-13.48
Hanning	0.5	-31.48

Note: sidelobe height is relative to the mainlobe.
The Hanning window has lower sidelobes, but at the cost of a wider mainlobe when compared to the rectangular window

(d) Let $y_r[n] = x[n]w_r[n]$ and $y_h[n] = x[n]w_h[n]$ where $x[n] = \cos(\frac{26\pi}{100}n) + \cos(\frac{29\pi}{100}n)$ and $M = 50$. Use the the MATLAB command fft to evaluate $|Y_r(e^{j\Omega})|$ in dB and $|Y_h(e^{j\Omega})|$ in dB at intervals of $\frac{\pi}{200}$. Does the window choice affect whether you can identify the presence of two sinusoids? Why?

Yes, since the two sinusoids are very close to one another in frequency, $\left(\frac{26\pi}{100} \text{ and } \frac{29\pi}{100}\right)$.
Since the Hanning window has a wider mainlobe, its capability to resolve these two sinusoid is inferior to the rectangular window. Notice from the plot that the existence of two sinusoids are visible for the rectangular window, but not for the Hanning.

(e) Let $y_r[n] = x[n]w_r[n]$ and $y_h[n] = x[n]w_h[n]$ where $x[n] = \cos(\frac{26\pi}{100}n) + 0.02\cos(\frac{51\pi}{100}n)$ and $M = 50$. Use the the MATLAB command fft to evaluate $|Y_r(e^{j\Omega})|$ in dB and $|Y_h(e^{j\Omega})|$ in dB at intervals of $\frac{\pi}{200}$. Does the window choice affect whether you can identify the presence of two sinusoids? Why?

Yes, here the two sinusoid frequencies are significantly different from one another. The separation of $\frac{\pi}{4}$ from each other is significantly larger than the mainlobe width of either window. Hence, resolution is not a problem for the Hanning window. Since the sidelobe magnitude is greater than 0.02 of the mainlobe in the rectangular window, the sinusoid at $\frac{51\pi}{100}$ is not distinguishable. In contrast, the sidelobes of the Hanning window are much lower, which allows the two sinusoids to be resolved.

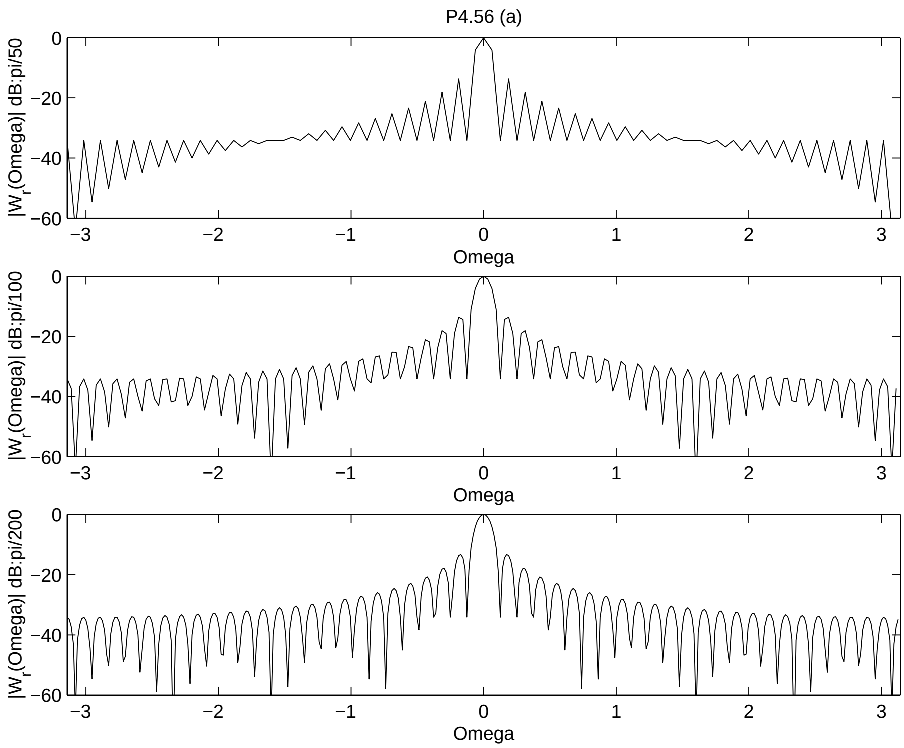


Figure P4.56. Magnitude spectrum of the rectangular window

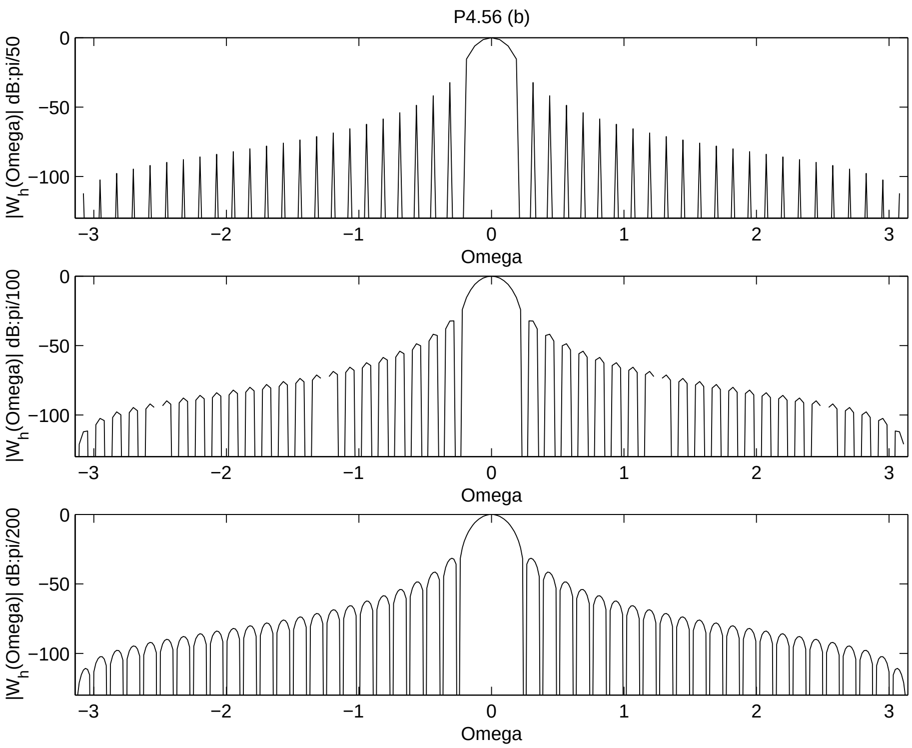


Figure P4.56. Magnitude spectrum of the Hanning window

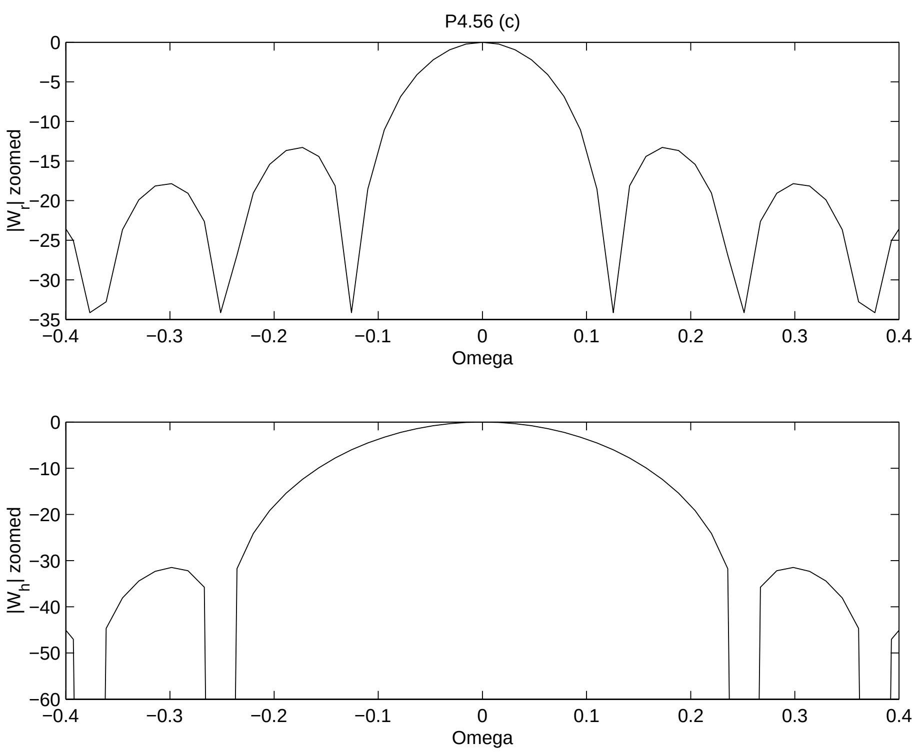


Figure P4.56. Zoomed in plots of $W_r(e^{j\Omega})$ and $W_h(e^{j\Omega})$

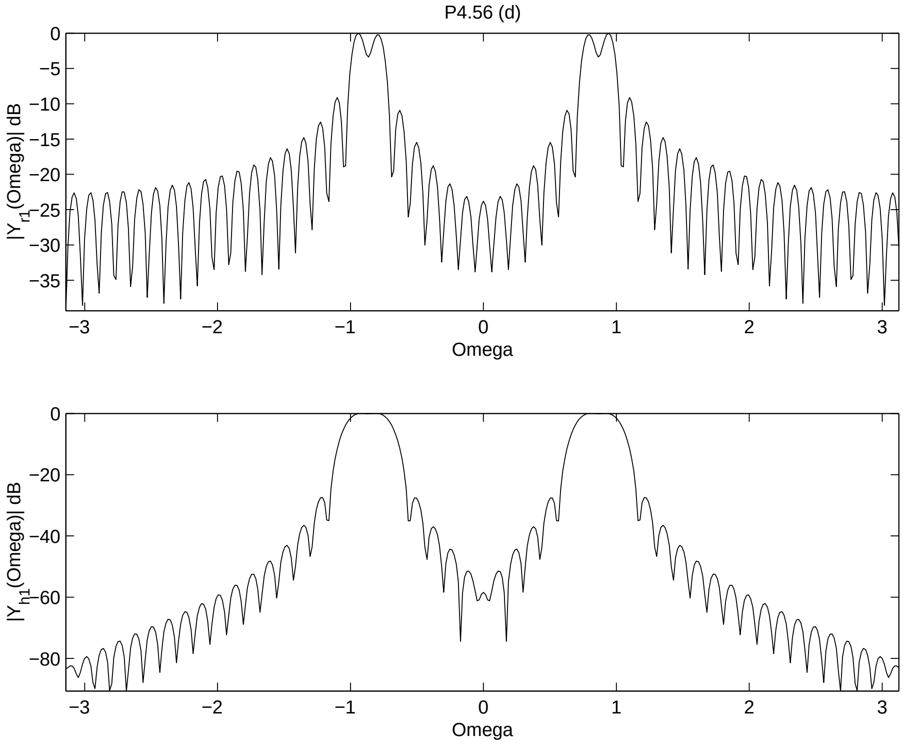


Figure P4.56. (d) Plots of $|Y_r(e^{j\Omega})|$ and $|Y_h(e^{j\Omega})|$

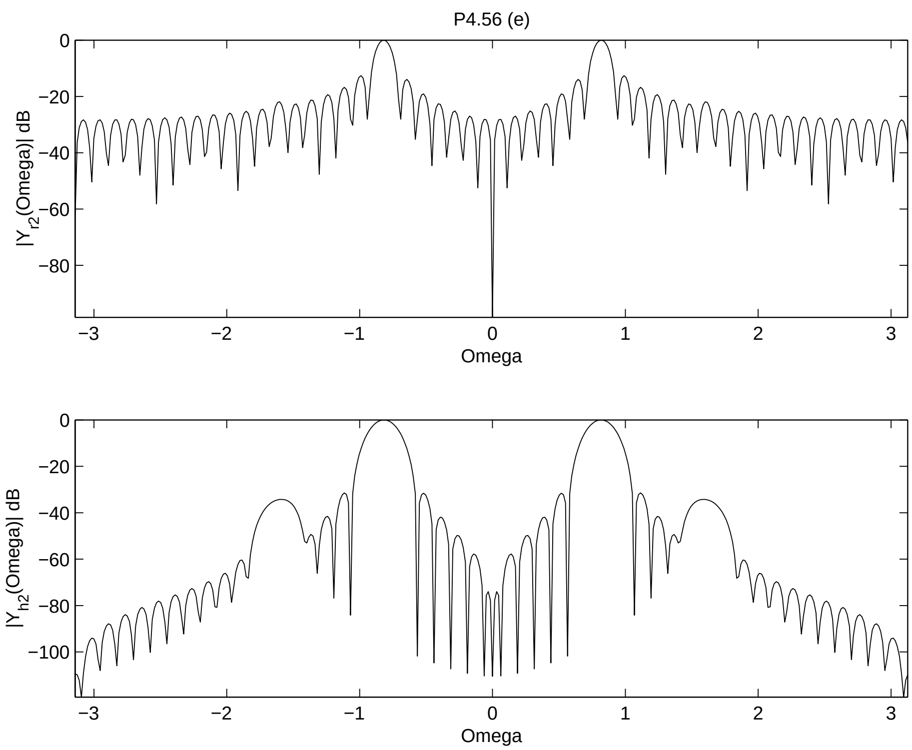


Figure P4.56. (e) Plots of $|Y_r(e^{j\Omega})|$ and $|Y_h(e^{j\Omega})|$

4.57. Let a discrete-time signal $x[n]$ be defined as

$$x[n] = \begin{cases} e^{-\frac{(0.1n)^2}{2}}, & |n| \leq 50 \\ 0, & \text{otherwise} \end{cases}$$

Use the MATLAB commands fft and fftshift to numerically evaluate and plot the DTFT of $x[n]$ and the following subsampled signals at 500 values of Ω on the interval $-\pi < \Omega \leq \pi$:
(a) $y[n] = x[2n]$
(b) $z[n] = x[4n]$

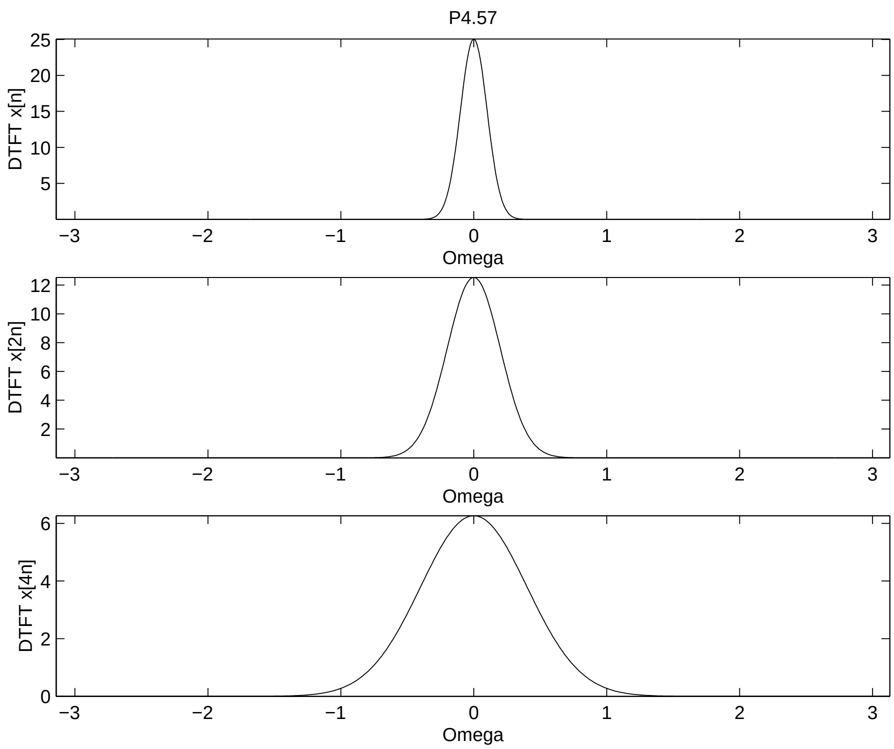


Figure P4.57. Plot of the DTFT of x[n]

4.58. Repeat Problem 4.57 assuming

$$x[n] = \begin{cases} \cos(\frac{\pi}{2}n)e^{-\frac{(0.1n)^2}{2}}, & |n| \leq 50 \\ 0, & \text{otherwise} \end{cases}$$

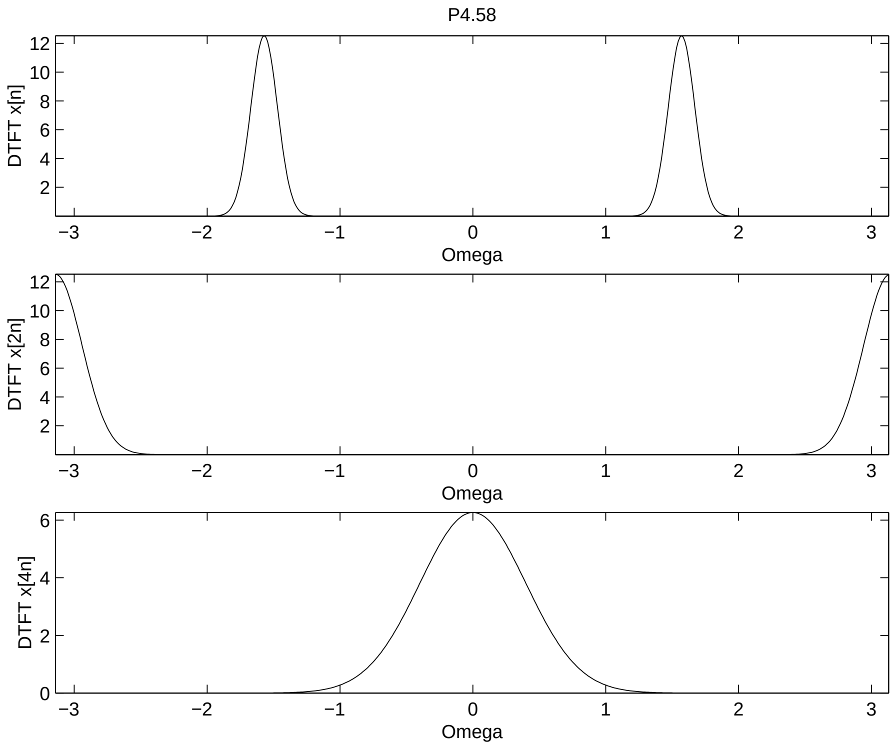


Figure P4.58. Plot of the DTFT of x[n]

4.59. A signal $x(t)$ is defined as

$$x(t) = \cos(\frac{3\pi}{2}t)e^{-\frac{t^2}{2}}$$

(a) Evaluate the FT $X(j\omega)$ and show that $|X(j\omega)| \approx 0$ for $|\omega| > 3\pi$.

$$\begin{aligned}
X(j\omega) &= \pi\left(\delta(\omega - \frac{3\pi}{2}) + \delta(\omega + \frac{3\pi}{2})\right) * \frac{e^{-\frac{\omega^2}{2}}}{\sqrt{2\pi}} \\
&= \sqrt{2\pi^3}\left(e^{\frac{-(\omega-\frac{3\pi}{2})^2}{2}} + e^{\frac{-(\omega+\frac{3\pi}{2})^2}{2}}\right) \\
\text{for } |\omega| > 3\pi & \\
|X(j\omega)| &= \sqrt{2\pi^3}\left|e^{\frac{-(\omega-\frac{3\pi}{2})^2}{2}} + e^{\frac{-(\omega+\frac{3\pi}{2})^2}{2}}\right| \\
|X(j\omega)| &\leq \sqrt{2\pi^3}\left|e^{\frac{-\left(\frac{3\pi}{2}\right)^2}{2}} + e^{\frac{-\left(\frac{9\pi}{2}\right)^2}{2}}\right| \\
&\approx 1.2 \times 10^{-4}
\end{aligned}$$

Which is small enough to be approximated as zero.

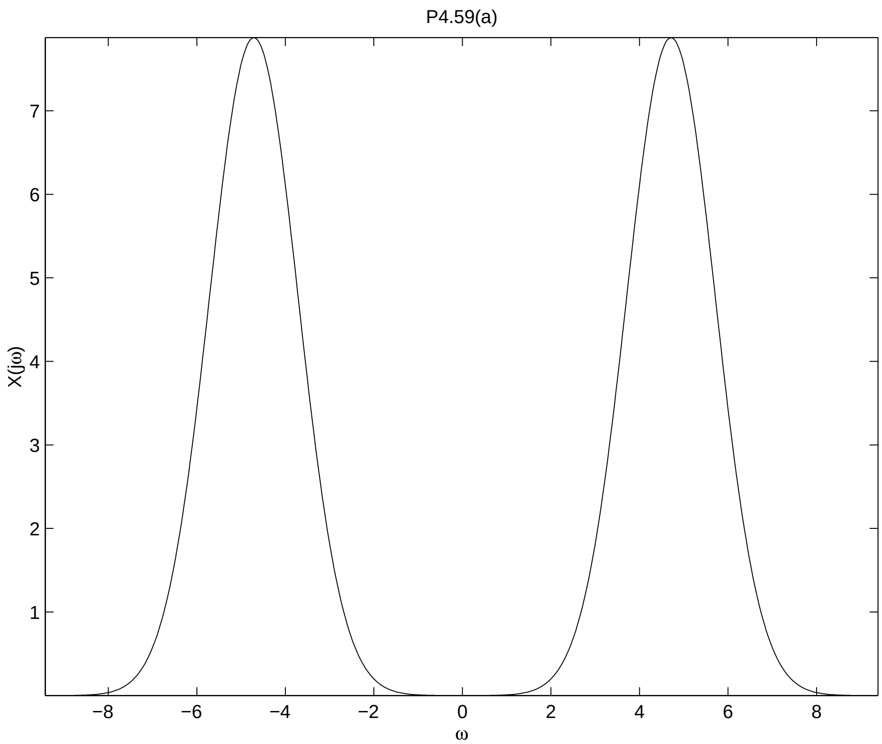


Figure P4.59. The Fourier Transform of $x(t)$

In parts (b)-(d), we compare $X(j\omega)$ to the FT of the sampled signal, $x[n] = x(nT_s)$, for several sampling intervals. Let $x[n] \xleftrightarrow{FT} X_\delta(j\omega)$ be the FT of the sampled version of $x(t)$. Use MATLAB to numerically determine $X_\delta(j\omega)$ by evaluating

$$X_\delta(j\omega) = \sum_{n=-25}^{25} x[n]e^{-j\omega T_s}$$

at 500 values of ω on the interval $-3\pi < \omega < 3\pi$. In each case, compare $X(j\omega)$ and $X_\delta(j\omega)$ and explain any differences.

Notice that $x[n]$ is symmetric with respect to n, so

$$\begin{aligned}
X_\delta(j\omega) &= x[0] + 2\sum_{n=1}^{25} x[n]\cos(\omega n T_s) \\
&= 1 + 2\sum_{n=1}^{25} x[n]\cos(\omega n T_s)
\end{aligned}$$

Aside from the magnitude difference of $\frac{1}{T_s}$, for each T_s, $X(j\omega)$ and $X_\delta(j\omega)$ are different within $[-3\pi, 3\pi]$ only when the sampling period is large enough to cause noticeable aliasing. It is obvious that the worst aliasing occurs when $T_s = \frac{1}{2}$.

(b) $T_s = \frac{1}{3}$

(c) $T_s = \frac{2}{5}$

(d) $T_s = \frac{1}{2}$

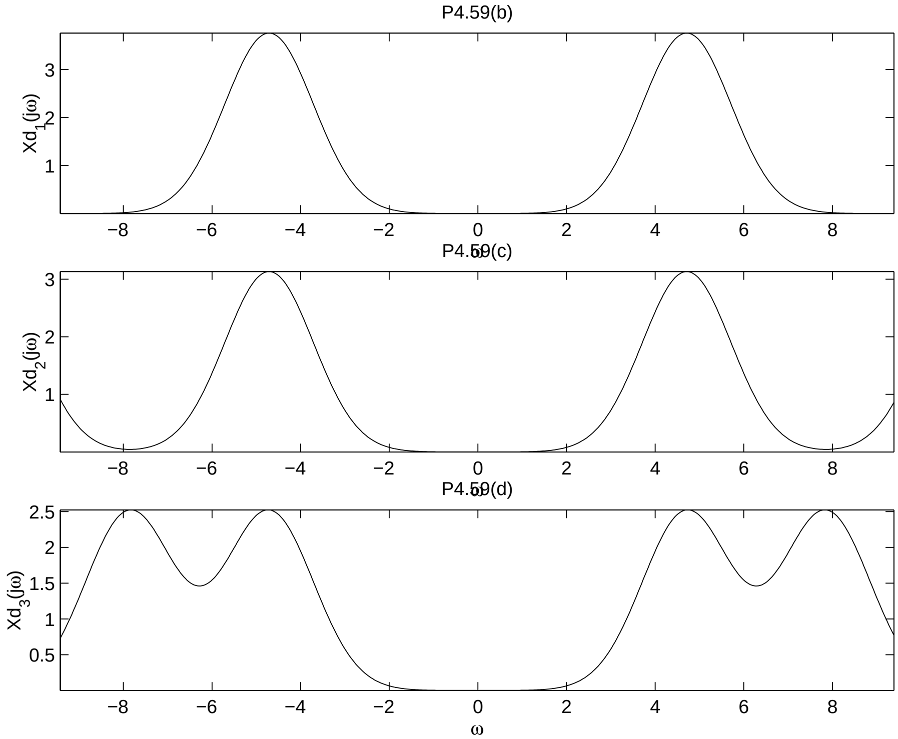


Figure P4.59. Comparison of the FT to the sampled FT

4.60. Use the MATLAB command fft to repeat Example 4.14.

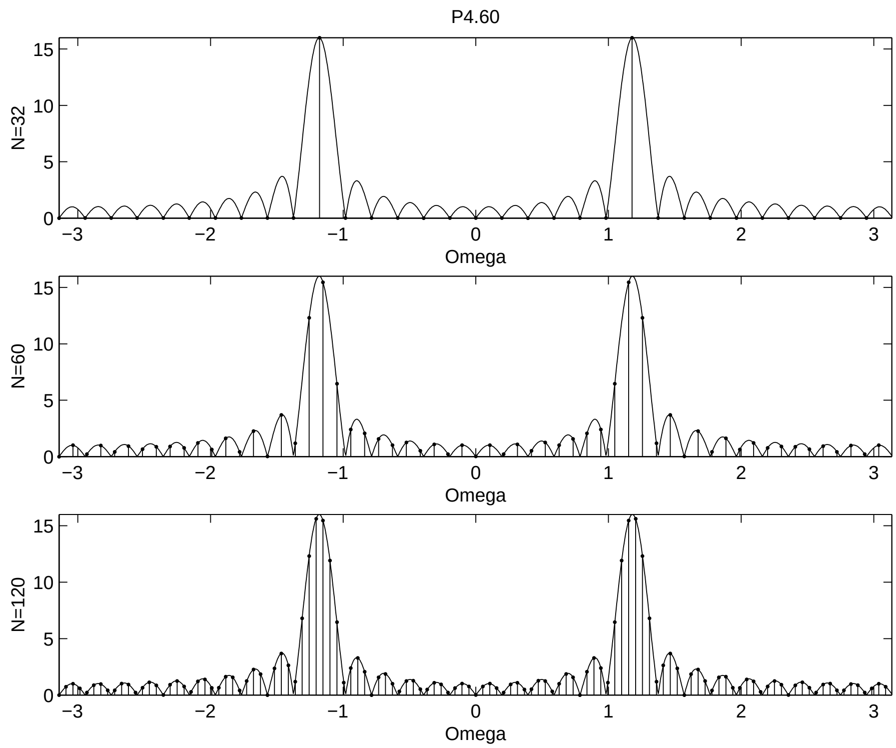
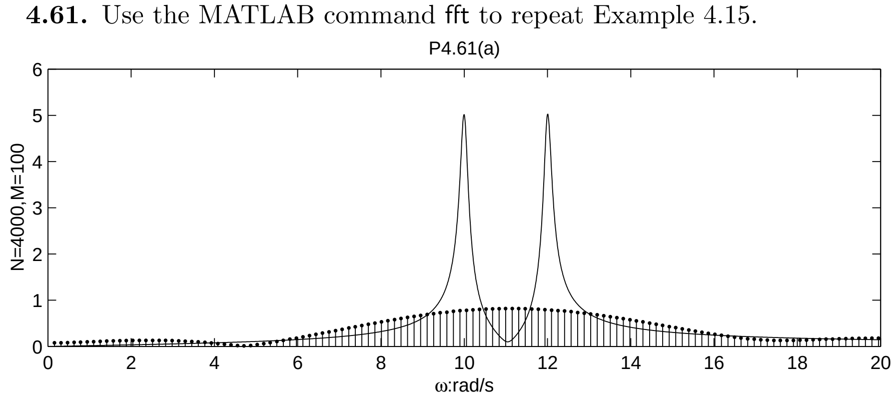


Figure P4.60. Plot of $|X(e^{j\Omega})|$ and $N|X[k]|$

4.61. Use the MATLAB command fft to repeat Example 4.15.



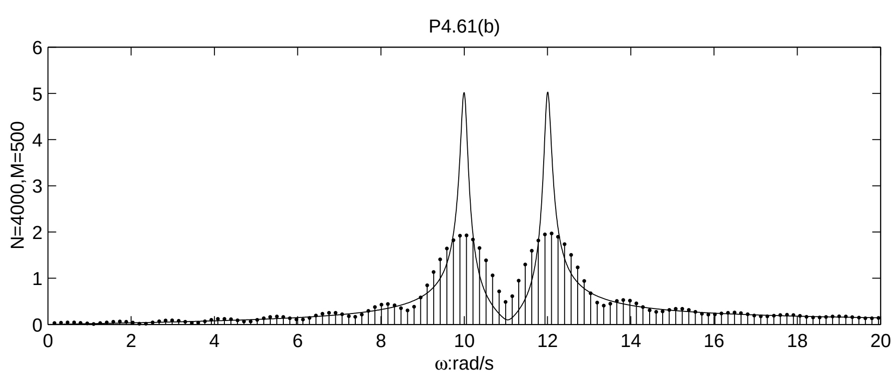


Figure P4.61. DTFT approximation to the FT, graphs of $|X(e^{j\Omega})|$ and $NT_s|Y[k]|$

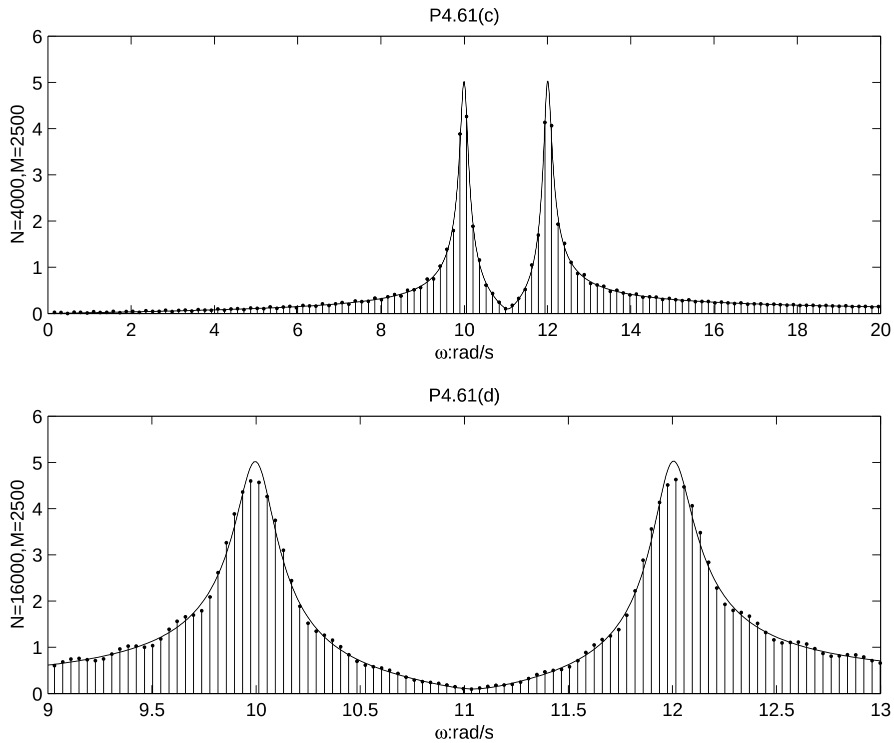


Figure P4.61. DTFT approximation to the FT, graphs of $|X(e^{j\Omega})|$ and $NT_s|Y[k]|$

4.62. Use the MATLAB command fft to repeat Example 4.16. Also depict the DTFS approximation and the underlying DTFT for $M = 2001$ and M=2005.

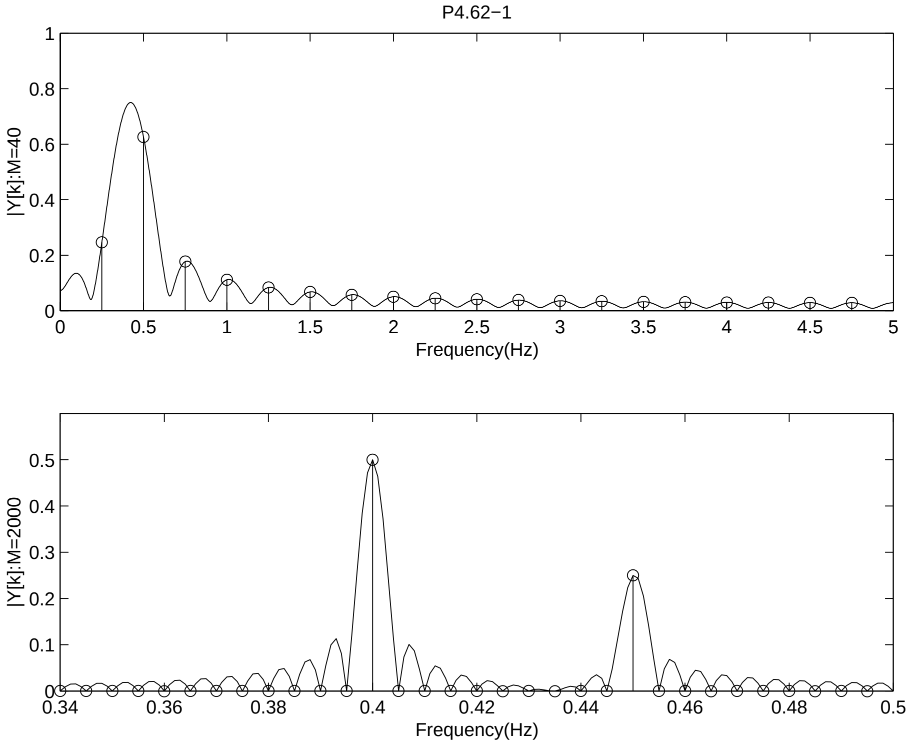


Figure P4.62. DTFS Approximation

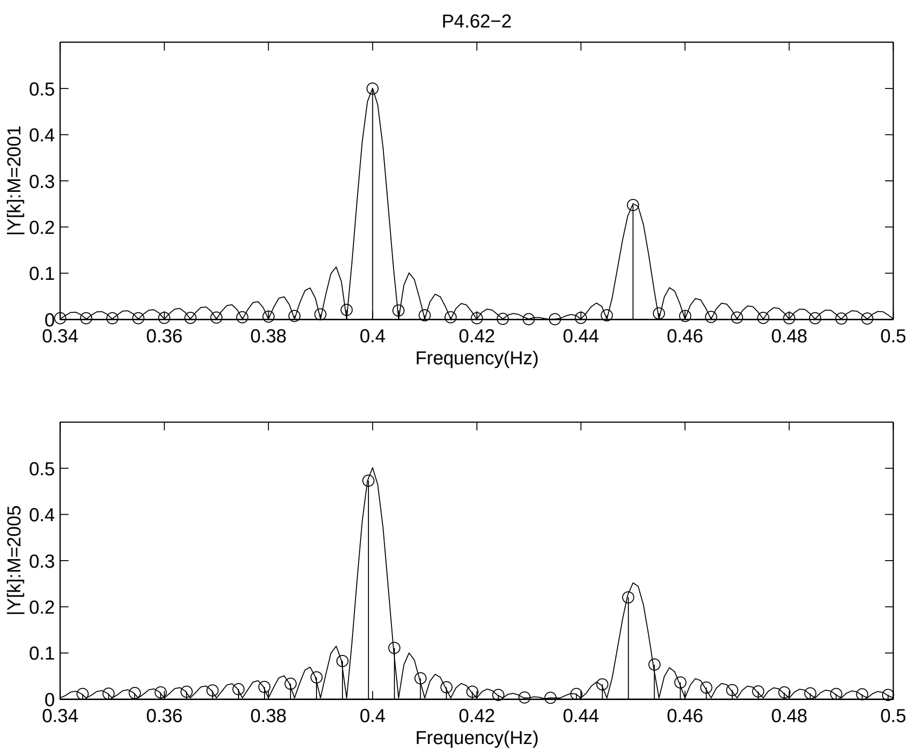


Figure P4.62. DTFS Approximation

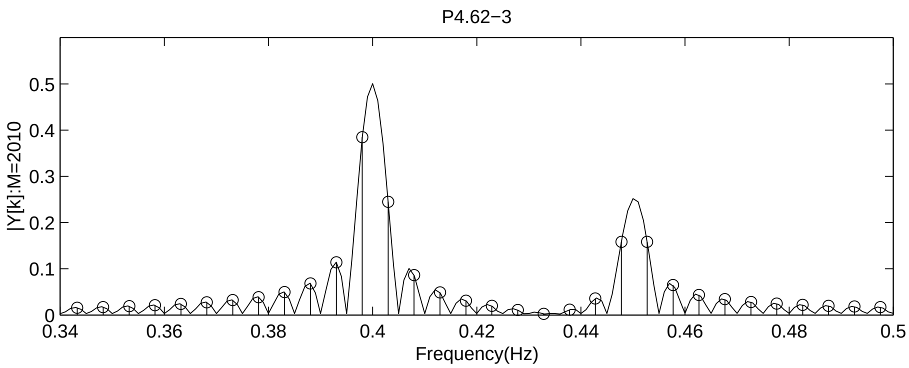


Figure P4.62. DTFS Approximation

4.63. Consider the sum of sinusoids

$$x(t) = \cos(2\pi t) + 2\cos(2\pi(0.8)t) + \frac{1}{2}\cos(2\pi(1.1)t)$$

Assume the frequency band of interest is $-5\pi < \omega < 5\pi$.
(a) Determine the sampling interval T_s so that the DTFS approximation to the FT of $x(t)$ spans the desired frequency band.

$$\begin{aligned}
\omega_a &= 5\pi \\
T_s &< \frac{2\pi}{3\omega_a} = 0.133 \\
\text{choose:} & \\
T_s &= 0.1
\end{aligned}$$

(b) Determine the minimum number of samples M_o so that the DTFS approximation consists of discrete-valued impulses located at the frequency corresponding to each sinusoid.

For a given M_o, the frequency interval for sampling the DTFS is $\frac{2\pi}{M_o}$

$$\begin{aligned}
\frac{2\pi}{M_o}k_1 &= 2\pi T_s \\
M_o &= \frac{k_1}{T_s} \\
\frac{2\pi}{M_o}k_2 &= 2\pi(0.8)T_s \\
M_o &= \frac{k_2}{0.8T_s} \\
\frac{2\pi}{M_o}k_3 &= 2\pi(1.1)T_s \\
M_o &= \frac{k_3}{1.1T_s}
\end{aligned}$$

where k_1, k_2, k_3 are integers.

By choosing $T_s = 0.1$, the minimum $M_o = 100$ with $k_1 = 10,\ k_2 = 8,\ k_3 = 11$.

(c) Use MATLAB to plot $\frac{1}{M}|Y_\delta(j\omega)|$ and $|Y[k]|$ for the value of T_s chosen in parts (a) and $M = M_o$.
(d) Repeat part (c) using $M = M_o + 5$ and $M = M_o + 8$.

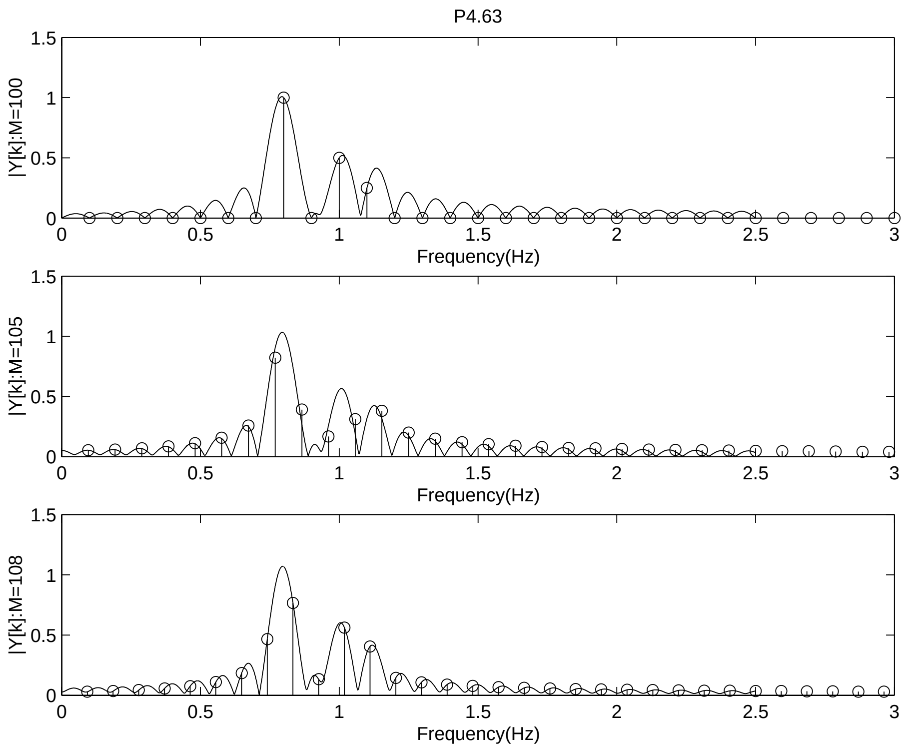


Figure P4.63. Plots of $(1/M)|Y_\delta(j\omega)|$ and $|Y[k]|$ for the appropriate values of T_s and M

4.64. We desire to use the DTFS to approximate the FT of a continuous time signal $x(t)$ on the band $-\omega_a < \omega < \omega_a$ with resolution ω_r and a maximum sampling interval in frequency of $\Delta\omega$. Find the sampling interval T_s, number of samples M, and DTFS length N. You may assume that the signal is effectively bandlimited to a frequency ω_m for which $|X(j\omega_a)| \geq 10|X(j\omega)|,\ \omega > \omega_m$. Plot the FT and the DTFS approximation for each of the following cases using the MATLAB command fft.

We use $T_s < \frac{2\pi}{\omega_m + \omega_a}$; $M \geq \frac{\omega_s}{\omega_r}$; $N \geq \frac{\omega_s}{\Delta\omega}$ to find the required $T_s,\ M,\ N$ for part (a) and (b).

(a) $x(t) = \begin{cases} 1, & |t| < 1 \\ 0, & \text{otherwise} \end{cases}$, $\omega_a = \frac{3\pi}{2}$, $\omega_r = \frac{3\pi}{4}$, and $\Delta\omega = \frac{\pi}{8}$

$$X(j\omega) = \frac{2\sin(\omega)}{\omega}$$

We want :

$$\left|\frac{\sin(\omega)}{\omega}\right| \leq \frac{1}{15\pi} \text{ for } |\omega| \geq \omega_m$$

which gives

$$
\begin{aligned}
\omega_m &= 15\pi \\
\text{Hence:} & \\
T_s &< \frac{2\pi}{15\pi + \frac{3\pi}{2}} \approx 0.12 \\
\text{choose} & \quad T_s = 0.1 \\
M &\geq 26.67 \\
\text{choose} & \quad M = 28 \\
N &\geq 160 \\
\text{choose} & \quad N = 160
\end{aligned}
$$

(b) $x(t) = \frac{1}{2\pi}e^{-\frac{t^2}{2}}$, $\omega_a = 3$, $\omega_r = \frac{1}{2}$, and $\Delta\omega = \frac{1}{8}$

$$
\begin{aligned}
X(j\omega) &= \frac{1}{\sqrt{2\pi}}e^{-\frac{\omega^2}{2}} \\
\text{We want :} & \\
e^{-\frac{\omega^2}{2}} &\leq \frac{1}{10}e^{-\frac{9}{2}} \\
\text{which gives} & \\
\omega_m &= 3.69 \\
\text{Hence:} & \\
T_s &< 0.939 \\
\text{choose} & \quad T_s = 0.9 \\
M &\geq 3.49 \\
\text{choose} & \quad M = 4 \\
N &\geq 55.85 \\
\text{choose} & \quad N = 56
\end{aligned}
$$

(c) $x(t) = \cos(20\pi t) + \cos(21\pi t)$, $\omega_a = 40\pi$, $\omega_r = \frac{\pi}{3}$, and $\Delta\omega = \frac{\pi}{10}$

$$
\begin{aligned}
\omega_s &= 2\omega_a = 80\pi \\
X(j\omega) &= \pi\left(\delta(\omega - 20\pi) + \delta(\omega + 20\pi)\right) \\
\text{Hence:} & \\
T_s &< \frac{2\pi}{3\omega_a} = 0.0167 \\
\text{choose} & \quad T_s = 0.01 \\
M &\geq 600 \\
\text{choose} & \quad M = 600 \\
\text{choose} & \quad N = M = 600
\end{aligned}
$$

(d) Repeat case (c) using $\omega_r = \frac{\pi}{10}$.
Hint: Be sure to sample the pulses in (a) and (b) symmetrically about $t = 0$.

$$\begin{aligned} \text{choose} \quad & T_s = 0.01 \\ M \quad \geq \quad & 2000 \\ \text{choose} \quad & M = 2000 \\ \text{choose} \quad & N = M = 2000 \end{aligned}$$

For (c) and (d), one needs to consider $X(j\omega) * W_\delta(j\omega)$, where

$$W_\delta(j\omega) \quad = \quad e^{-j\omega T_s\left(\frac{M-1}{2}\right)} \left(\frac{\sin(M\frac{\omega T_s}{2})}{\sin(\frac{\omega T_s}{2})} \right)$$

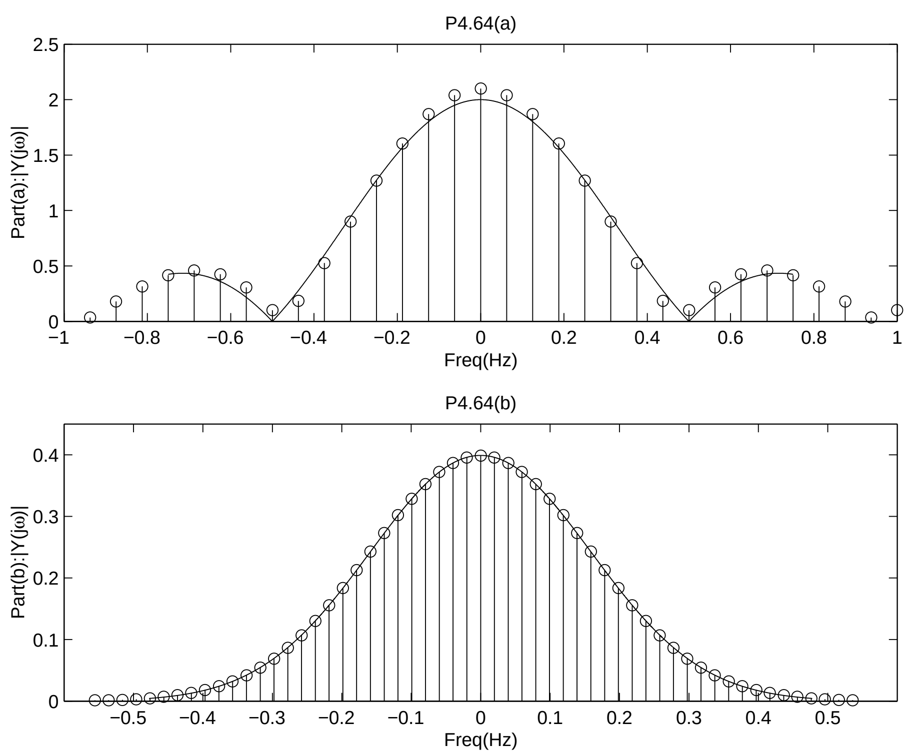


Figure P4.64. FT and DTFS approximation

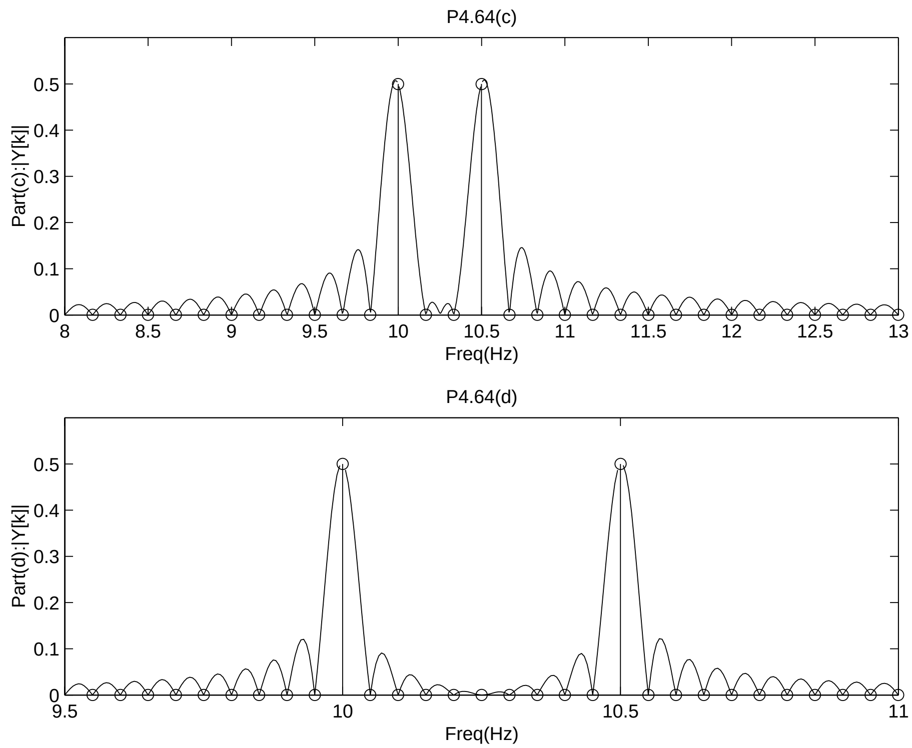


Figure P4.64. FT and DTFS approximation

4.65. The overlap and save method for linear filtering is discussed in Problem 4.54. Write a MATLAB m-file that implements the overlap and save method using fft to evaluate the convolution $y[n] = h[n] * x[n]$ on $0 \leq n < L$ for the following signals.
(a) $h[n] = \frac{1}{5}(u[n] - u[n-5])$, $x[n] = \cos(\frac{\pi}{6}n)$, $L = 30$

$$
\begin{aligned}
M &= 5 \\
L &= 30 \\
N &= 34
\end{aligned}
$$

(b) $h[n] = \frac{1}{5}(u[n] - u[n-5])$, $x[n] = (\frac{1}{2})^n u[n]$, $L = 20$

$$
\begin{aligned}
M &= 5 \\
L &= 20 \\
N &= 24
\end{aligned}
$$

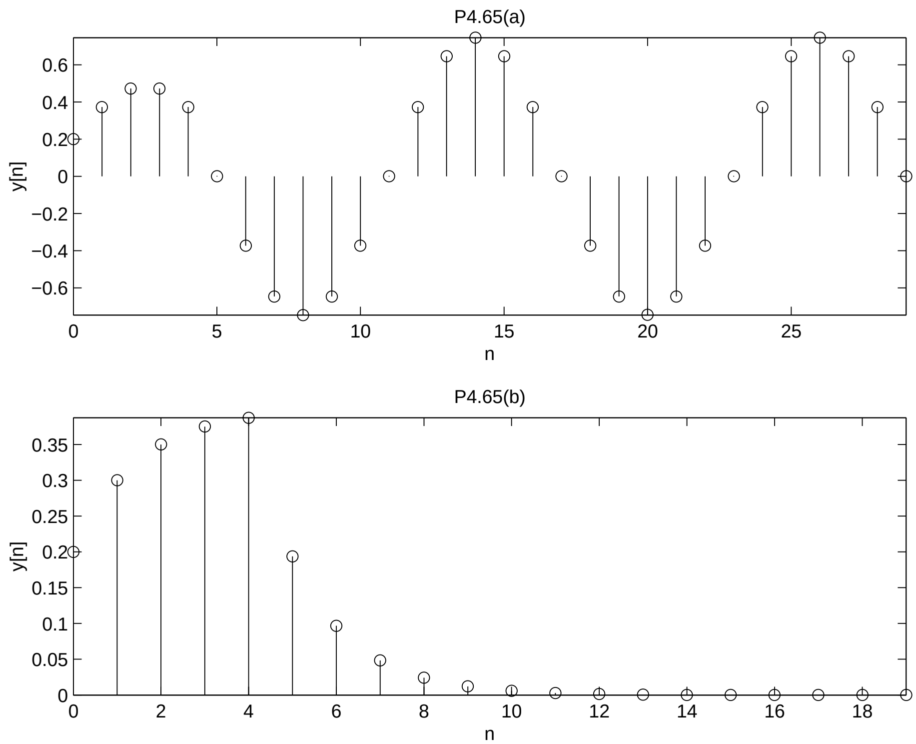


Figure P4.65. Overlap-and-save method

4.66. Plot the ratio of the number of multiplies in the direct method for computing the DTFS coefficients to that of the FFT approach when $N = 2^p$ for $p = 2, 3, 4, \ldots 16$.

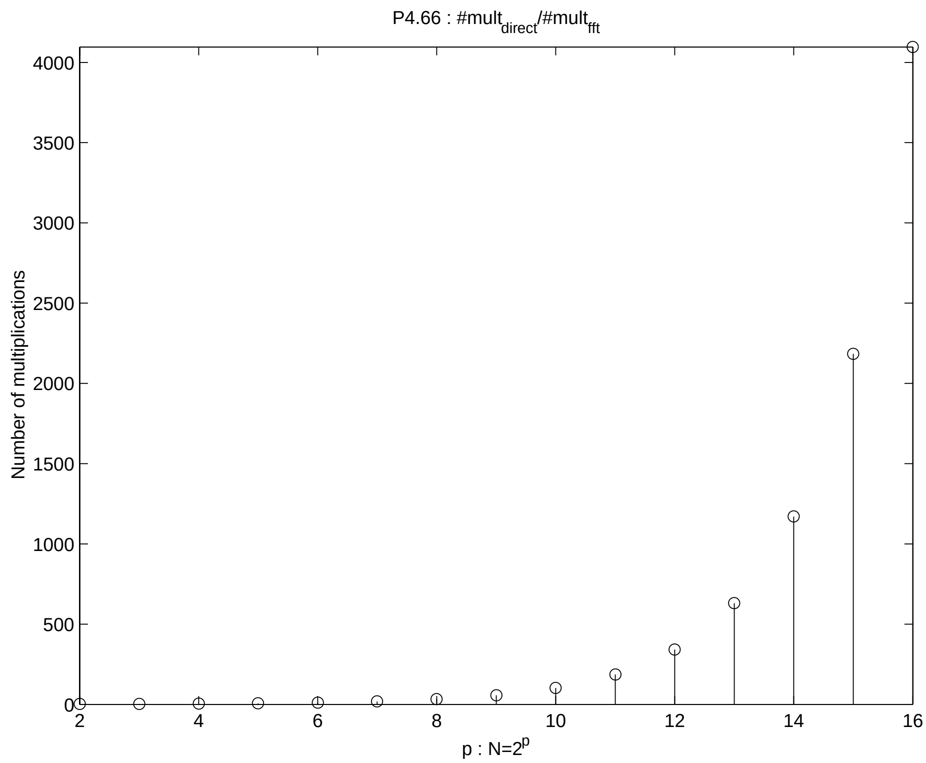


Figure P4.66. (a)

4.67. In this experiment we investigate evaluation of the time-bandwidth product with the DTFS. Let $x(t) \xleftrightarrow{FT} X(j\omega)$.

(a) Use the Riemann sum approximation to an integral

$$\int_a^b f(u)du \approx \sum_{m=m_a}^{m_b} f(m\Delta u)\Delta u$$

to show that

$$\begin{aligned} T_d &= \left[\frac{\int_{-\infty}^{\infty} t^2|x(t)|^2 dt}{\int_{-\infty}^{\infty} |x(t)|^2 dt}\right]^{\frac{1}{2}} \\ &\approx T_s \left[\frac{\sum_{n=-M}^{M} n^2|x[n]|^2}{\sum_{n=-M}^{M} |x[n]|^2}\right]^{\frac{1}{2}} \end{aligned}$$

provided $x[n] = x(nT_s)$ represents the samples of $x(t)$ and $x(nT_s) \approx 0$ for $|n| > M$.

By setting $t = nT_s$, $dt \approx \Delta t = T_s$, so

$$\begin{aligned} \int_{-\infty}^{\infty} t^2|x(t)|^2 dt &\approx \sum_{n=-M}^{M} (nT_s)^2|x(nT_s)|^2 T_s \\ &= T_s^3 \sum_{n=-M}^{M} n^2|x[n]|^2 \end{aligned}$$

similarly

$$\int_{-\infty}^{\infty} |x(t)|^2 dt \approx T_s \sum_{n=-M}^{M} |x[n]|^2$$

Therefore

$$T_d \approx T_s \left[\frac{\sum_{n=-M}^{M} n^2 |x[n]|^2}{\sum_{n=-M}^{M} |x[n]|^2} \right]^{\frac{1}{2}}$$

(b) Use the DTFS approximation to the FT and the Riemann sum approximation to an integral to show that

$$\begin{aligned} B_w &= \left[\frac{\int_{-\infty}^{\infty} \omega^2 |X(j\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega} \right]^{\frac{1}{2}} \\ &\approx \frac{\omega_s}{2M+1} \left[\frac{\sum_{k=-M}^{M} |k|^2 |X[k]|^2}{\sum_{k=-M}^{M} |X[k]|^2} \right]^{\frac{1}{2}} \end{aligned}$$

where $x[n] \xleftrightarrow{DTFS;\, \frac{2\pi}{2M+1}} X[k]$, $\omega_s = \frac{2\pi}{T_s}$ is the sampling frequency, and $X(jk\frac{\omega_s}{2M+1}) \approx 0$ for $|k| > M$.

Using the $(2M+1)$-point DTFS approximation, we have:

$$\omega_k = k \frac{\omega_s}{2M+1}$$

hence

$$\begin{aligned} d\omega &\approx \Delta\omega = \frac{\omega_s}{2M+1} \\ X[k] &= \frac{1}{(2M+1)T_s} X(jk\frac{\omega_s}{2M+1}) \\ \int_{-\infty}^{\infty} \omega^2 |X(j\omega)|^2 d\omega &= \left(\frac{\omega_s}{2M+1} \right)^3 \sum_{k=-M}^{M} k^2 |X[k]|^2 \\ \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega &= \left(\frac{\omega_s}{2M+1} \right) \sum_{k=-M}^{M} |X[k]|^2 \end{aligned}$$

therefore

$$B_w \approx \frac{\omega_s}{2M+1} \left[\frac{\sum_{k=-M}^{M} |k|^2 |X[k]|^2}{\sum_{k=-M}^{M} |X[k]|^2} \right]^{\frac{1}{2}}$$

(c) Use the result from (a) and (b) and Eq. (3.65) to show that the time-bandwidth product computed using the DTFS approximation satisfies

$$\left[\frac{\sum_{n=-M}^{M} n^2 |x[n]|^2}{\sum_{n=-M}^{M} |x[n]|^2} \right]^{\frac{1}{2}} \left[\frac{\sum_{k=-M}^{M} |k|^2 |X[k]|^2}{\sum_{k=-M}^{M} |X[k]|^2} \right]^{\frac{1}{2}} \geq \frac{2M+1}{4\pi}$$

$$\text{Note } T_d B_w \approx \frac{T_s \omega_s}{2M+1} \left[\frac{\sum_{n=-M}^{M} n^2 |x[n]|^2}{\sum_{n=-M}^{M} |x[n]|^2} \right]^{\frac{1}{2}} \left[\frac{\sum_{k=-M}^{M} |k|^2 |X[k]|^2}{\sum_{k=-M}^{M} |X[k]|^2} \right]^{\frac{1}{2}}$$

Since $T_d B_w \geq \frac{1}{2}$, and $\frac{T_s \omega_s}{2M+1} = \frac{2\pi}{2M+1}$, we have

$$\left[\frac{\sum_{n=-M}^{M} n^2 |x[n]|^2}{\sum_{n=-M}^{M} |x[n]|^2}\right]^{\frac{1}{2}} \left[\frac{\sum_{k=-M}^{M} |k|^2 |X[k]|^2}{\sum_{k=-M}^{M} |X[k]|^2}\right]^{\frac{1}{2}} \quad \geq \quad \frac{2M+1}{4\pi}$$

(d) Repeat Computer Experiment 3.115 to demonstrate that the bound in (c) is satisfied and that Gaussian pulses satisfy the bound with equality.

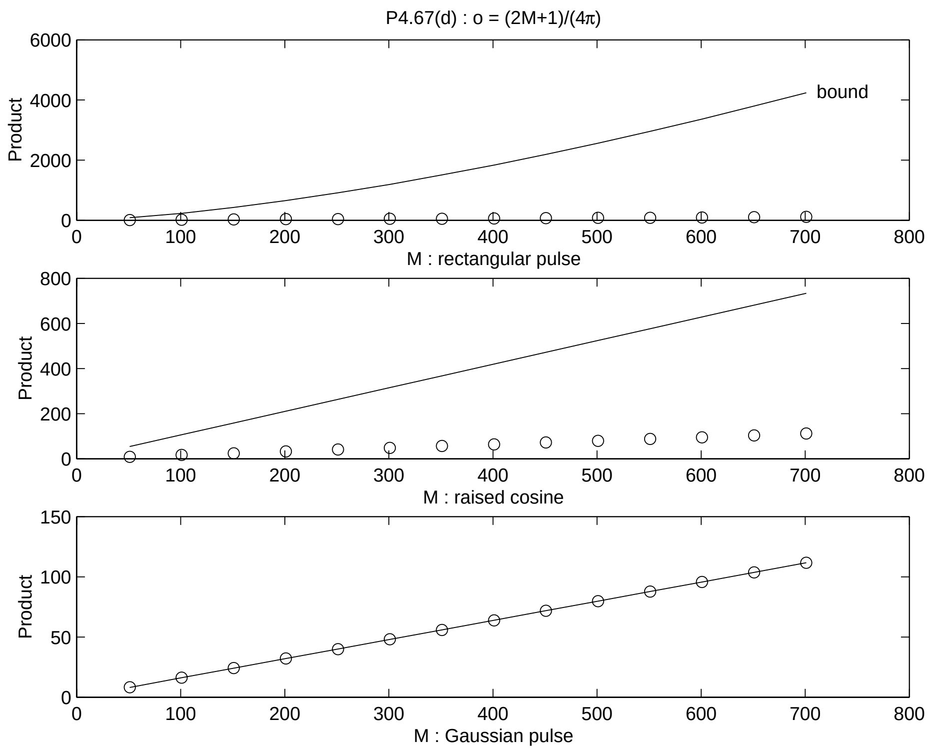


Figure P4.67. Plot of Computer Experiment 3.115