

Notes

- **Assignment**

- ◆ 2.10
- ◆ 2.11
- ◆ 2.22 (b) (e)
- ◆ 2.25
- ◆ 2.28 (a) (c) (e) (g)

- **Tutorial questions this week (Week 4)**

- ◆ Basic Problems with Answers 2.20
- ◆ Basic Problems 2.29
- ◆ Advanced Problems 2.40, 2.43, 2.47

That is ...

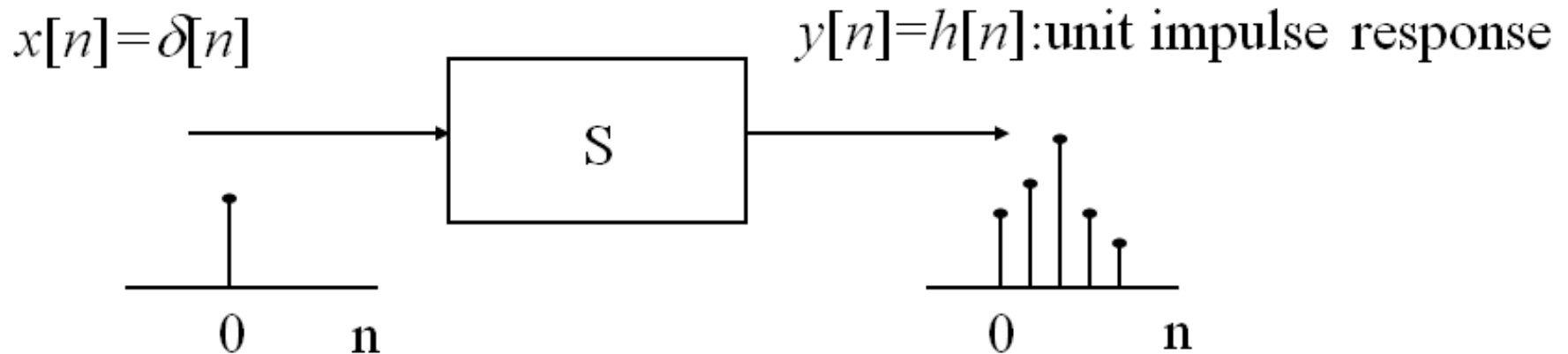
$$x[n] = \dots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \dots$$

⇓

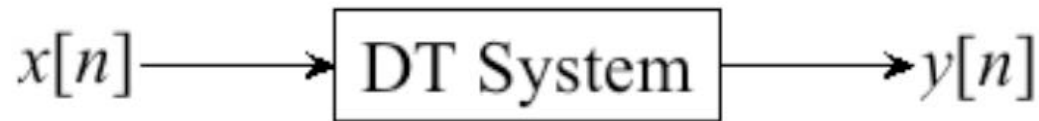
$$x[n] = \sum_{k=-\infty}^{+\infty} \underbrace{x[k]}_{\text{Coefficients}} \underbrace{\delta[n-k]}_{\text{Basic Signals}}$$

Important to note
the “-” sign

Unit Impulse Response



Response of DT LTI Systems



- Now suppose the system is **LTI**, and define the *unit impulse response* $h[n]$:

$$\delta[n] \longrightarrow h[n]$$



From **T**ime-**I**nvariance:

$$\delta[n - k] \longrightarrow h[n - k]$$

From **L**inearity:

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k] \delta[n - k] \longrightarrow y[n] = \underbrace{\sum_{k=-\infty}^{+\infty} x[k] h[n - k]}_{\text{convolution sum}} = x[n] * h[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Contribution to the output signal at time n

input signal

flipped version of $h[k]$ located at $k = n$

Convolution operation procedure:

$$h[k] \xrightarrow{\text{Flip}} h[-k] \xrightarrow{\text{Slide}} h[n-k] \xrightarrow{\text{Multiply}} x[k]h[n-k]$$

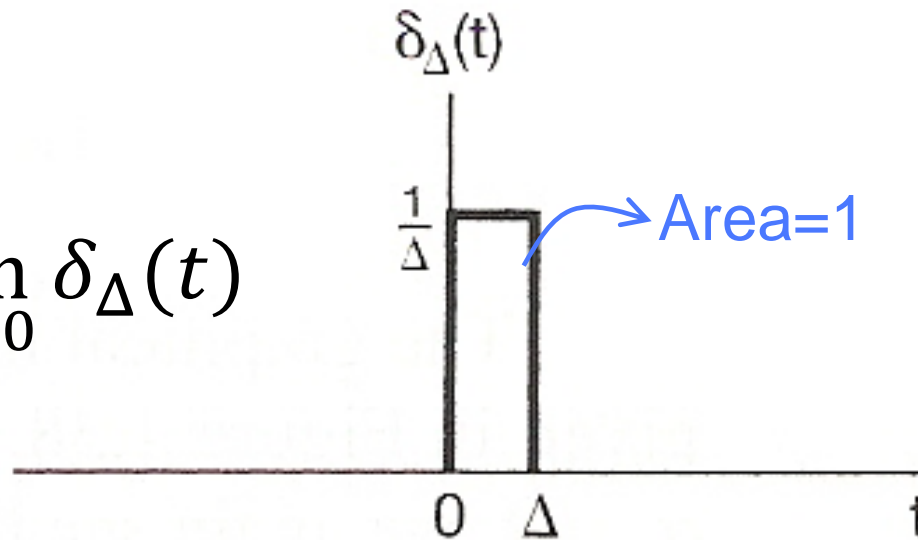
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$$\xrightarrow{\text{Sum}} \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Construction of the Unit-impulse function $\delta(t)$

One of the simplest way — rectangular pulse, taking the limit $\Delta \rightarrow 0$.

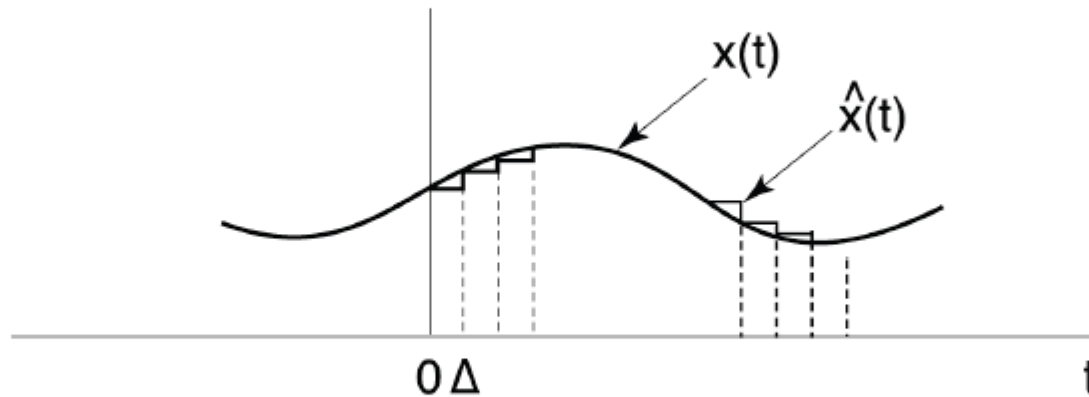
$$\delta(t) = \lim_{\Delta \rightarrow 0} \delta_{\Delta}(t)$$



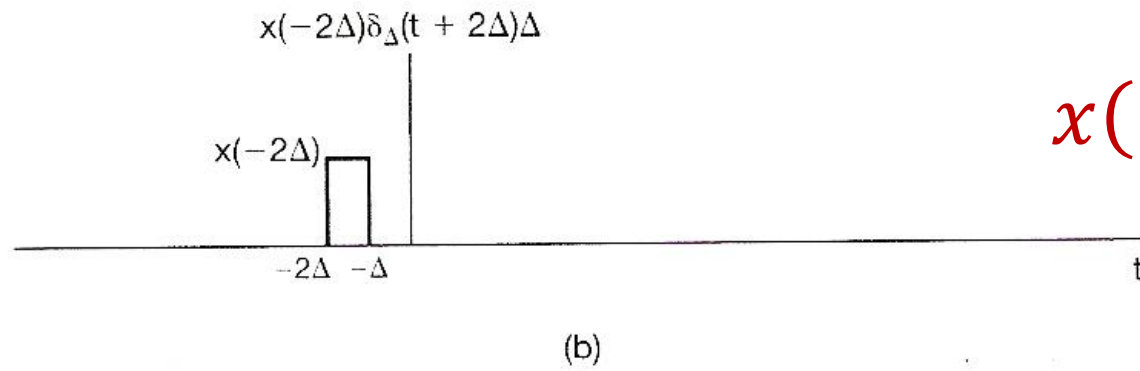
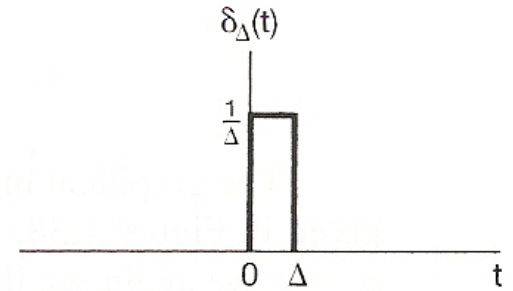
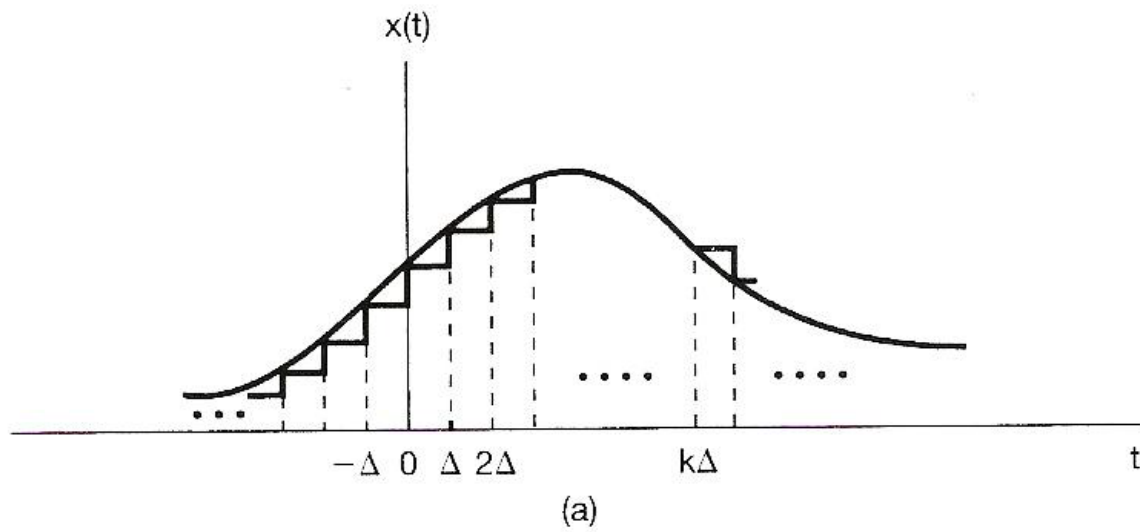
But this is by no means the only way. One can construct a $\delta(t)$ function out of many other functions, *Eg.* Gaussian pulses, triangular pulses, sinc functions, *etc.*, as long as the pulses are short enough — much shorter than the characteristic time scale of the system.

Representation of CT Signals

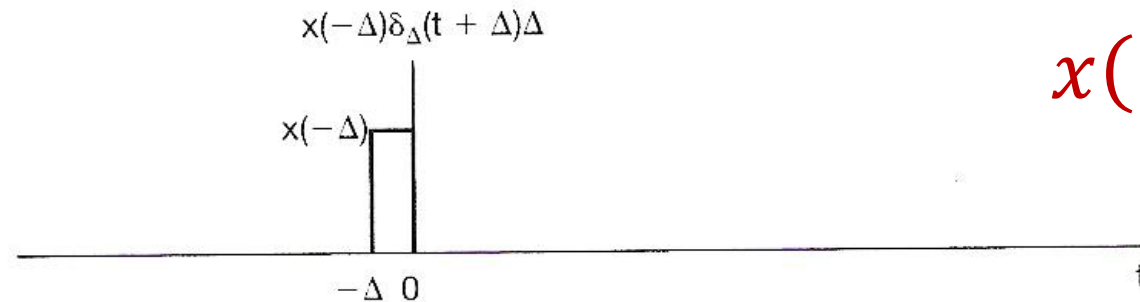
- Approximate any input $x(t)$ as a sum of shifted, scaled pulses (in fact, that is how we do integration)



$$x(t) = \lim_{\Delta \rightarrow 0} \hat{x}(t)$$

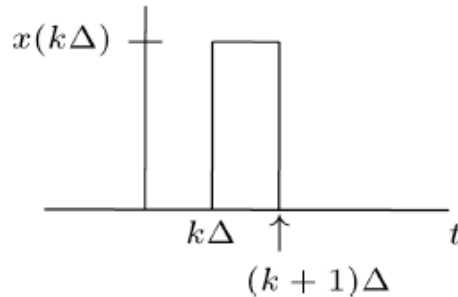


$$x(-2\Delta)\delta_{\Delta}(t + 2\Delta)\Delta$$



$$x(-\Delta)\delta_{\Delta}(t + \Delta)\Delta$$

Representation of CT Signals (cont.)



$$= x(k\Delta)\delta_{\Delta}(t - k\Delta)\Delta$$

\Downarrow

$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_{\Delta}(t - k\Delta)\Delta$$

\downarrow limit as $\Delta \rightarrow 0$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau$$

Sifting
property
of the unit
impulse

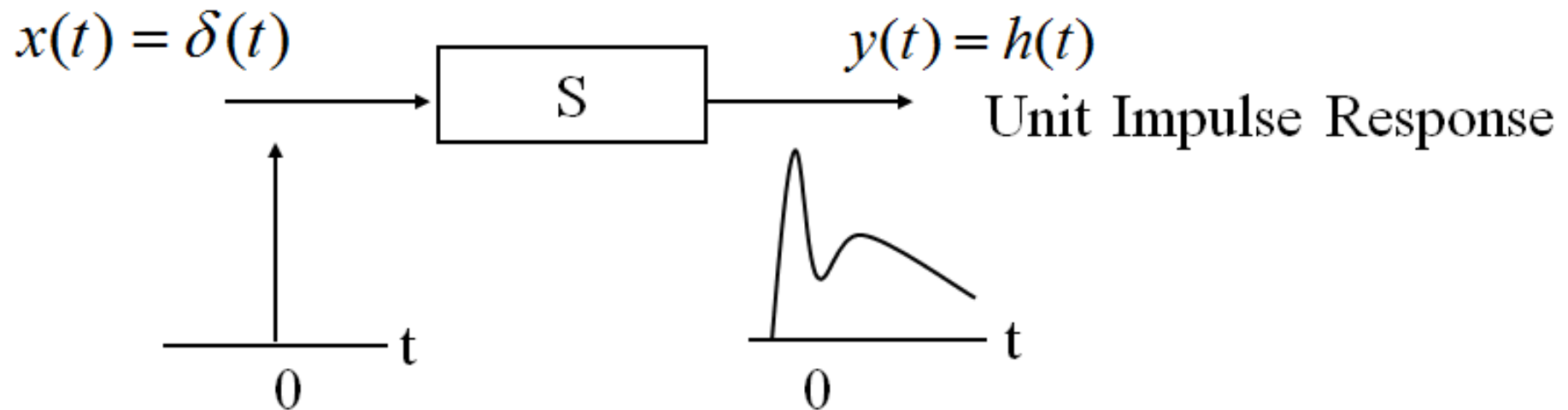
How to calculate the output of CT LTI systems?

Input: $x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] \rightarrow x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau$

LTI: $\delta[n] \rightarrow h[n] \rightarrow$

Output: $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \rightarrow$

Unit Impulse Response



Response of a CT LTI System



- Now suppose the system is **LTI**, and define the *unit impulse response* $h(t)$:

$$\delta(t) \longrightarrow h(t)$$



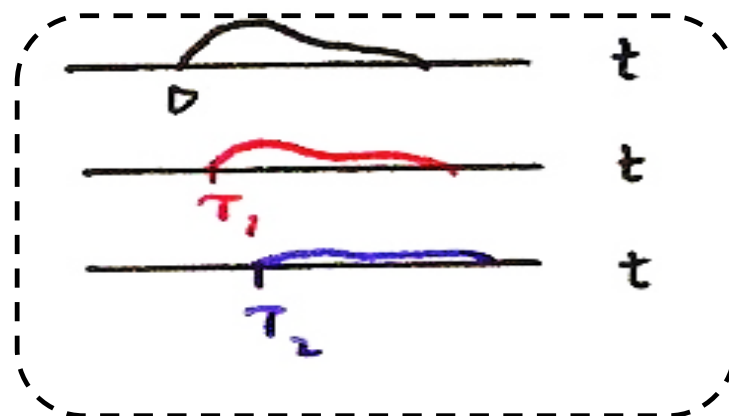
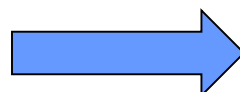
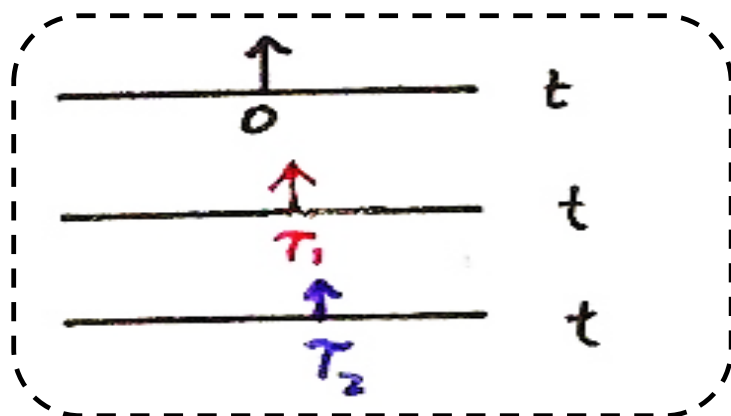
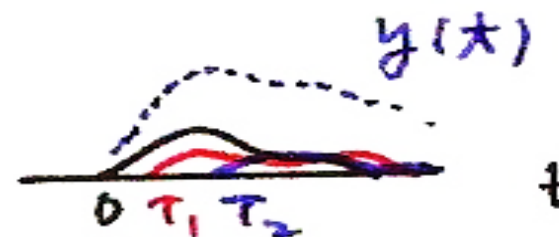
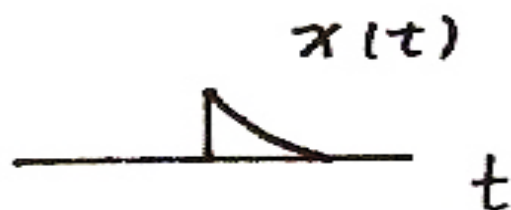
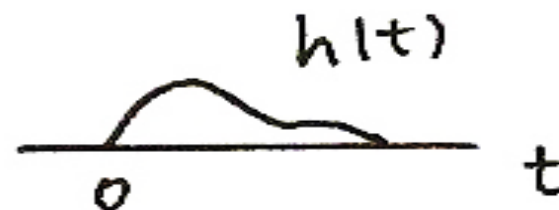
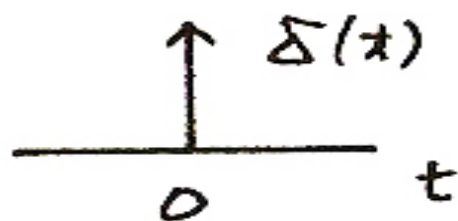
From **T**ime-**I**nvariance:

$$\delta(t - \tau) \longrightarrow h(t - \tau)$$

From **L**inearity:

$$x(t) = \int_{-\infty}^{+\infty} x(\tau) \delta(t - \tau) d\tau \longrightarrow y(t) = \underbrace{\int_{-\infty}^{+\infty} x(\tau) h(t - \tau) d\tau}_{\text{Convolution Integration}} = x(t) * h(t)$$

Response of a CT LTI System



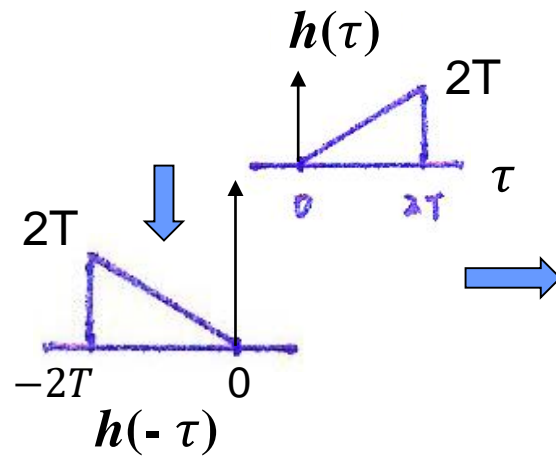
Summary

Input: $x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] \rightarrow x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau$

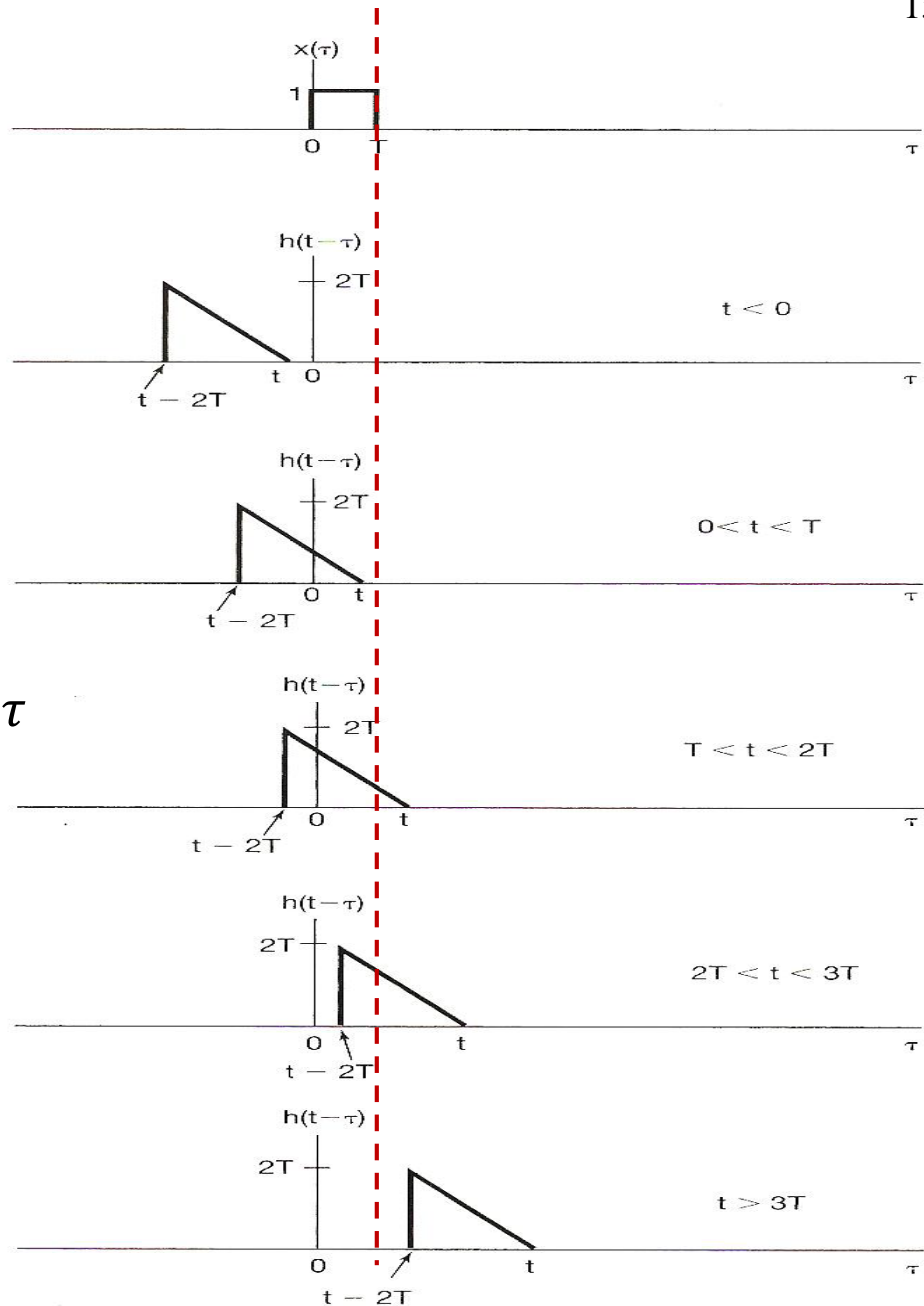
LTI: $\delta[n] \rightarrow h[n] \rightarrow \delta(t) \rightarrow h(t)$

Output: $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \rightarrow y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$
 $= x[n] * h[n] \rightarrow = x(t) * h(t)$

$$\begin{array}{ccccccc}
 h(\tau) & \xrightarrow{\text{Flip}} & h(-\tau) & \xrightarrow{\text{Slide}} & h(t-\tau) & \xrightarrow{\text{Multiply}} & \\
 & & & & & & \\
 x(\tau)h(t-\tau) & \xrightarrow{\text{Integrate}} & \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau & & & &
 \end{array}$$



$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t - \tau)d\tau$$



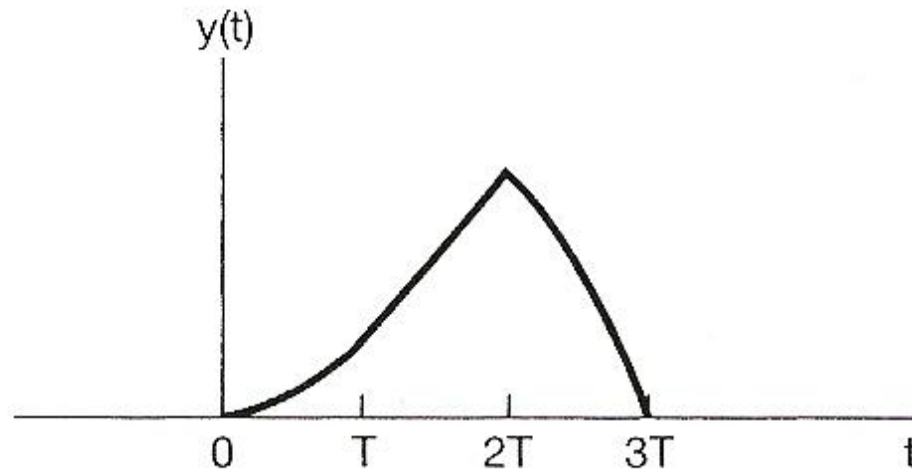


Figure 2.21 Signal $y(t) = x(t) * h(t)$ for Example 2.7.

→ Flip, slide, multiply, and integrate

One Important Convolution

$$x(t) * \delta(t - a)$$

$$= \int_{-\infty}^{+\infty} x(\tau) \delta(t - \tau - a) d\tau$$

$$= \int_{-\infty}^{+\infty} x(t - a) \delta(t - \tau - a) d\tau$$

$$= x(t - a) \int_{-\infty}^{+\infty} \delta(t - \tau - a) d\tau$$

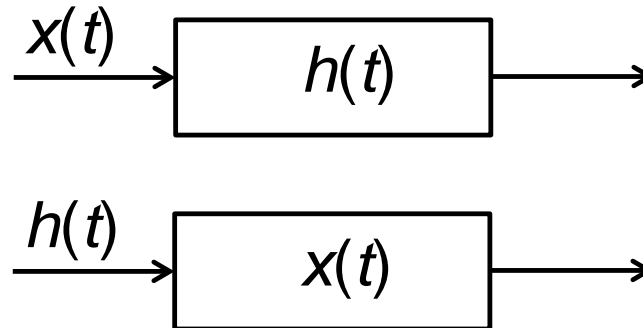
$$= x(t - a)$$

Similarly, $x[n] * \delta[n - k] = x[n - k]$

Property: Commutative

$$x(t) * h(t) = h(t) * x(t)$$

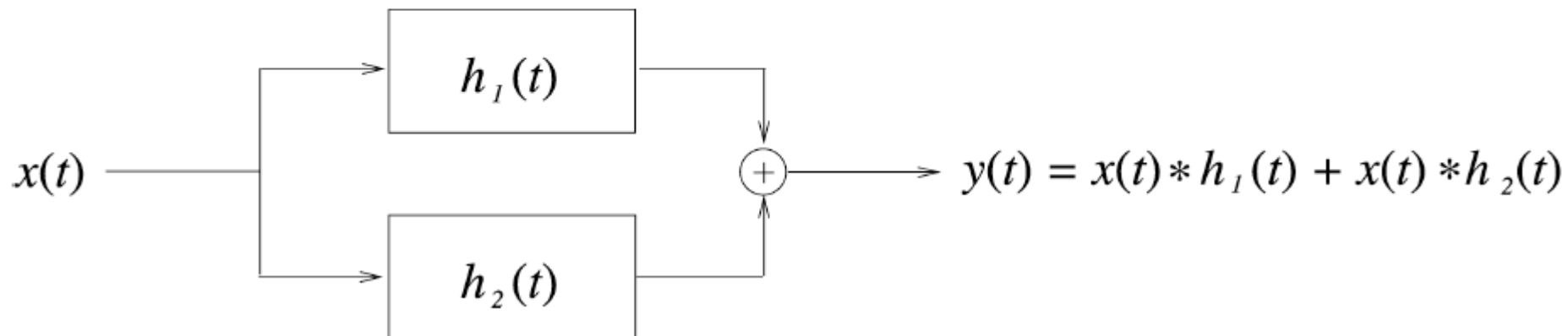
- The role of input signal and unit impulse response is **interchangeable**, giving the same output signal



Property: Distributive



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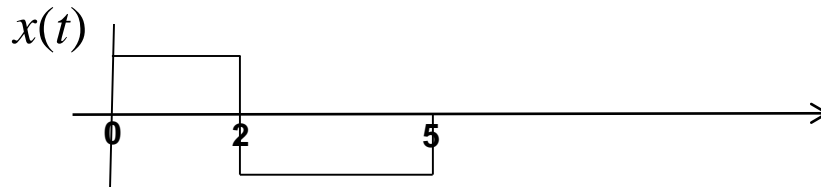
Property: Distributive (Cont.)

$$[x_1(t) + x_2(t)] * h(t) = x_1(t) * h(t) + x_2(t) * h(t)$$

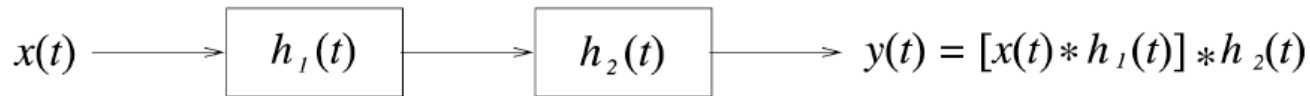
Problem 2.22 (b)

$$x(t) = u(t) - 2u(t - 2) + u(t - 5)$$

$$h(t) = e^{2t} u(1 - t)$$



Properties: Associative

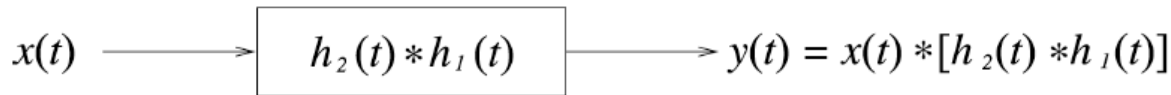


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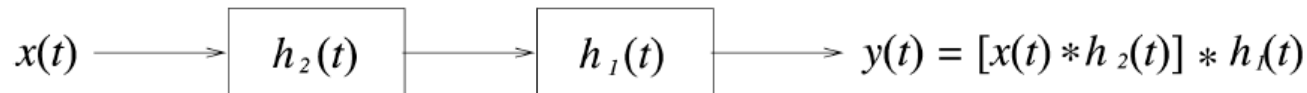


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← Commutativity



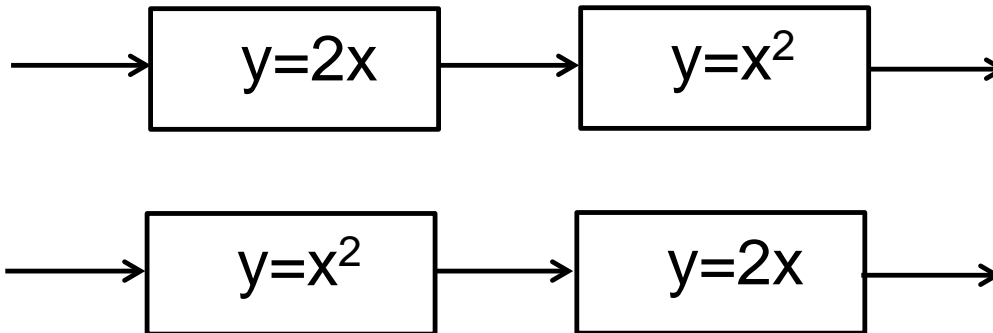
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- Cascade of two systems gives an unit impulse response which is the **convolution of the unit impulse responses** of the two individual systems
- The behavior of a cascade of two systems is **independent** of the order in which the two systems are cascaded

Properties: Associative (Cont.)

- The order in which non-linear systems are cascaded cannot be changed.
- e.g.



Property: Memory/Memoryless

- A linear, time-invariant, causal system is memoryless only

if $h[n] = K\delta[n]$ $h(t) = K\delta(t)$

$y[n] = Kx[n]$ $y(t) = Kx(t)$

if $K=1$ further, they are identity systems

$$y[n] = x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = x[n] * \delta[n]$$

$$y(t) = x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t) * \delta(t)$$

Property: Invertibility

2.50. Consider the cascade of two systems shown in Figure P2.50. The first system, A , is known to be LTI. The second system, B , is known to be the inverse of system A . Let $y_1(t)$ denote the response of system A to $x_1(t)$, and let $y_2(t)$ denote the response of system A to $x_2(t)$.



Figure P2.50

- (a) What is the response of system B to the input $ay_1(t) + by_2(t)$, where a and b are constants?
- (b) What is the response of system B to the input $y_1(t - \tau)$?

Property: Causality

Causality: CT LTI system is causal $\Leftrightarrow h(t) = 0$, at $t < 0$

- This is because that the input unit impulse function $\delta(t)=0$ at $t < 0$

As a result:

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t - \tau)d\tau = \int_{-\infty}^t x(\tau)h(t - \tau)d\tau$$

$t - \tau \geq 0$, or $\tau \leq t$

$y(t)$ only depends on $x(\tau < t)$.

Property: Stability

BIBO Stability: CT LTI system is stable $\Leftrightarrow \int_{-\infty}^{+\infty} |h(\tau)| d\tau < \infty$

→ Sufficient condition:

For $|x(t)| \leq x_{\max} < \infty$,

$$|y(t)| = \left| \int_{-\infty}^{+\infty} x(\tau) h(t - \tau) d\tau \right| \leq x_{\max} \left| \int_{-\infty}^{+\infty} h(t - \tau) d\tau \right| < \infty.$$

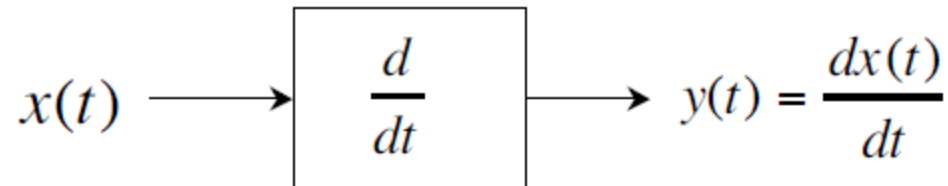
→ Necessary condition:

Suppose $\int_{-\infty}^{+\infty} |h(\tau)| d\tau = \infty$

Let $x(t) = h^*(-t)/|h^*(-t)|$, then $|x(t)| \equiv 1$ bounded

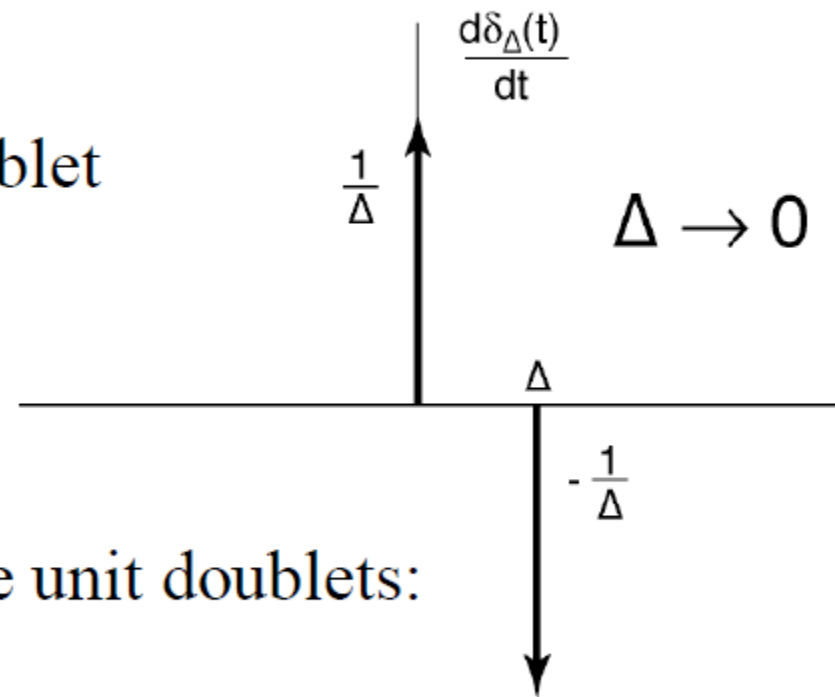
$$\text{But } y(0) = \int_{-\infty}^{+\infty} x(\tau) h(-\tau) d\tau = \int_{-\infty}^{+\infty} \frac{h^*(-\tau) h(-\tau)}{|h(-\tau)|} d\tau = \int_{-\infty}^{+\infty} |h(-\tau)| d\tau = \infty$$

Differentiator



Impulse response = unit doublet

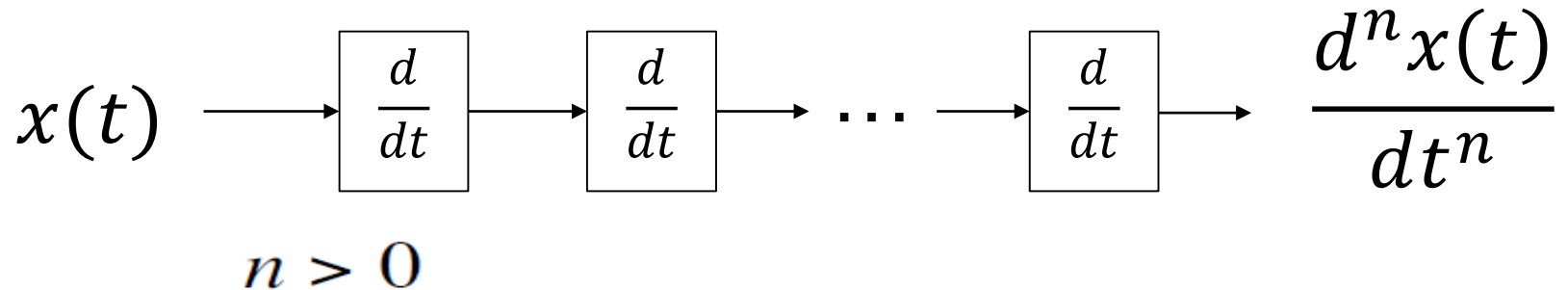
$$u_1(t) = \frac{d\delta(t)}{dt}$$



The operational definitions of the unit doublets:

$$x(t) * u_1(t) = \frac{dx(t)}{dt}$$

Triplets and beyond!



$$u_n(t) = \underbrace{u_1(t) * \dots * u_1(t)}_{n \text{ times}}$$

Operational definitions:

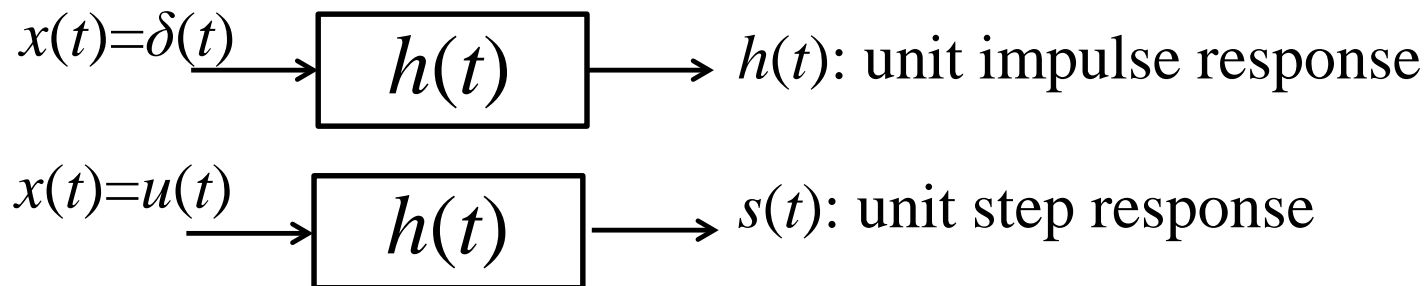
$$x(t) * u_n(t) = \frac{d^n x(t)}{dt^n} \quad (n > 0)$$

Unit Step Response

unit step function \rightarrow unit step response

Step response

$$s(t) = u(t) * h(t) = h(t) * u(t) = \int_{-\infty}^t h(\tau) d\tau$$



The **relation** between unit step function and unit impulse function

$$h(t) = \frac{ds(t)}{dt} = s'(t)$$

Integrators

$$x(t) \longrightarrow \boxed{\int} \longrightarrow y(t) = \int_{-\infty}^t x(\tau) d\tau$$

Impulse response:

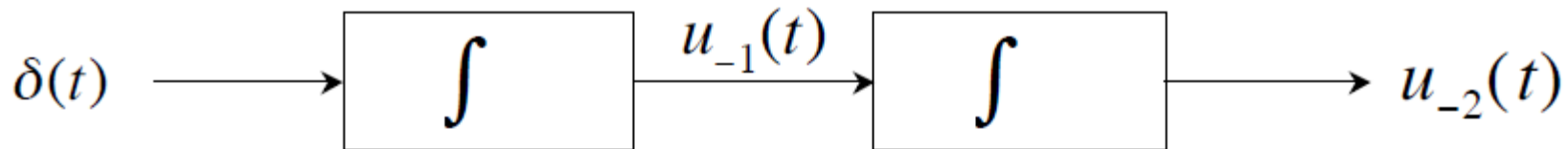
$$u_{-1}(t) \equiv u(t)$$

Operational definition: $x(t) * u_{-1}(t) = \int_{-\infty}^t x(\tau) d\tau$

Cascade of n integrators:

$$u_{-n}(t) = \underbrace{u_{-1}(t) * \cdots * u_{-1}(t)}_{n \text{ times}} \quad (n > 0)$$

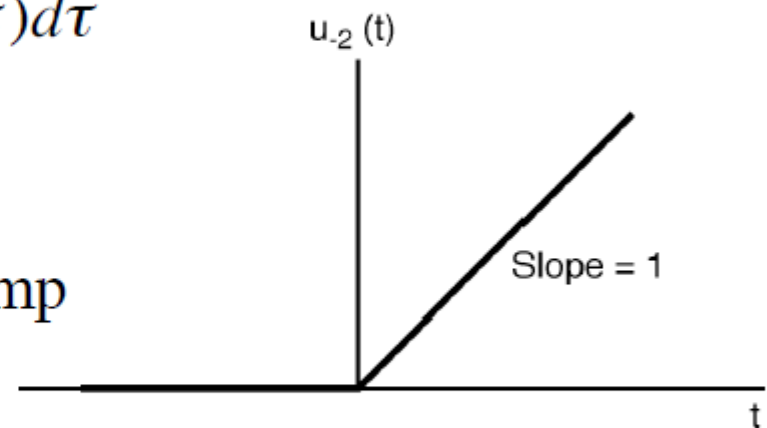
Integrators (Cont.)



$$u_{-2}(t) = \int_{-\infty}^t u_{-1}(\tau) d\tau = \int_{-\infty}^t u(\tau) d\tau$$

$$= u(t) \int_0^t d\tau$$

$$= t \cdot u(t) \quad \text{the unit ramp}$$



More generally, for $n > 0$

$$u_{-n}(t) = \frac{t^{(n-1)}}{(n-1)!} u(t)$$

Notation

Define

$$u_0(t) = \delta(t)$$

Then

$$u_n(t) * u_m(t) = u_{n+m}(t)$$

n and m can be \pm

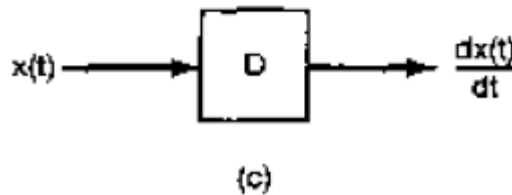
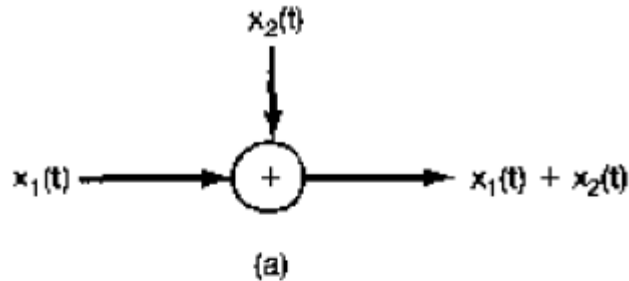
E.g.

$$u_1(t) * u_{-1}(t) = u_0(t)$$

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$$\left(\frac{d}{dt} u(t) \right) = \delta(t)$$

Block diagram representation - CT

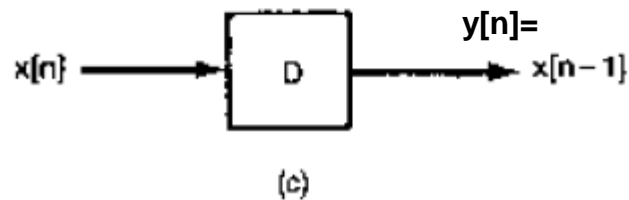
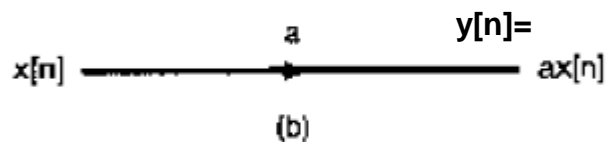
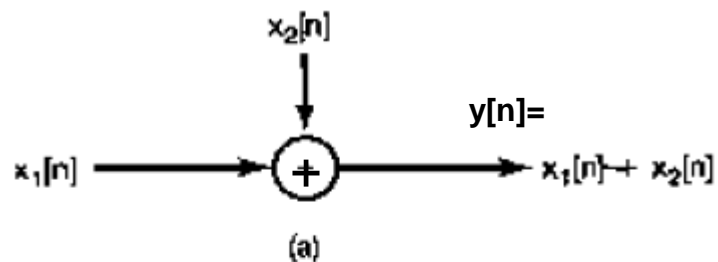


Problem 2.39

$$y(t) = -\frac{1}{2} \frac{dy(t)}{dt} + 4x(t)$$

$$\frac{dy(t)}{dt} + 3y(t) = x(t)$$

Block diagram representation - DT



Problem 2.38

$$y[n] = \frac{1}{3} y[n-1] + \frac{1}{2} x[n]$$

$$y[n] = \frac{1}{3} y[n-1] + x[n-1]$$

From block diagram to difference equation

