Assignments (Week 3)
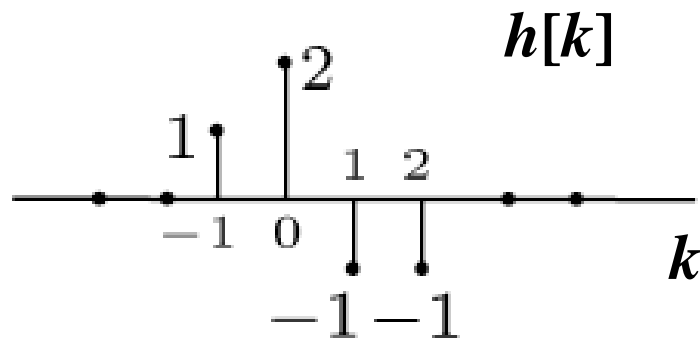
- **2.4**
- **2.6**
- **2.19**
- **2.21 (c) (d)**

Tutorial Problems (Week 4)

- **Basic Problems with Answers 2.3, 2.7, 2.13**
- **Basic Problems 2.24, 2.26**

Ch 1 Review (1)

- **Time-shift and flip**

$h[k]$

k

What is the plot for $h[n\text{-}k]$??
n is a constant

$$h[k] \xrightarrow{\text{Flip}} h[-k] \xrightarrow{\text{Slide}} h[n-k]$$

$h[n-k]$

$n-2$

k

$n+1$

Ch 1 Review (2)
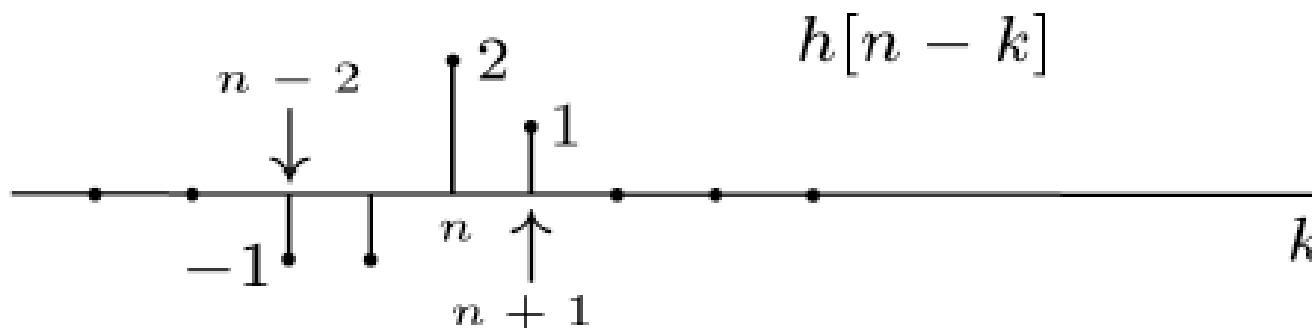
- **Unit impulse function (unit sample function)**

Discrete-time

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$
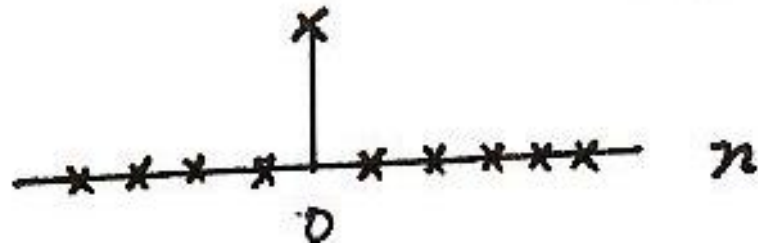
- We can use unit impulse function to represent any other different signals, or it is a building function (or basic signal) --- it will be explained soon.

System properties:

1. **With memory or memoryless**

   $y[n]=f (x[n])$

2. **Invertible**

   for a system $x{\rightarrow}y$, if $x_1{\neq}x_2$, then $y_1{\neq}y_2$

3. **Causal**

   … up to that time n …

4. **Stable**

   either prove the system is stable, or find a specific counterexample

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5. **Time-invariant**

$$\text{If} \qquad x[n] \rightarrow y[n]$$

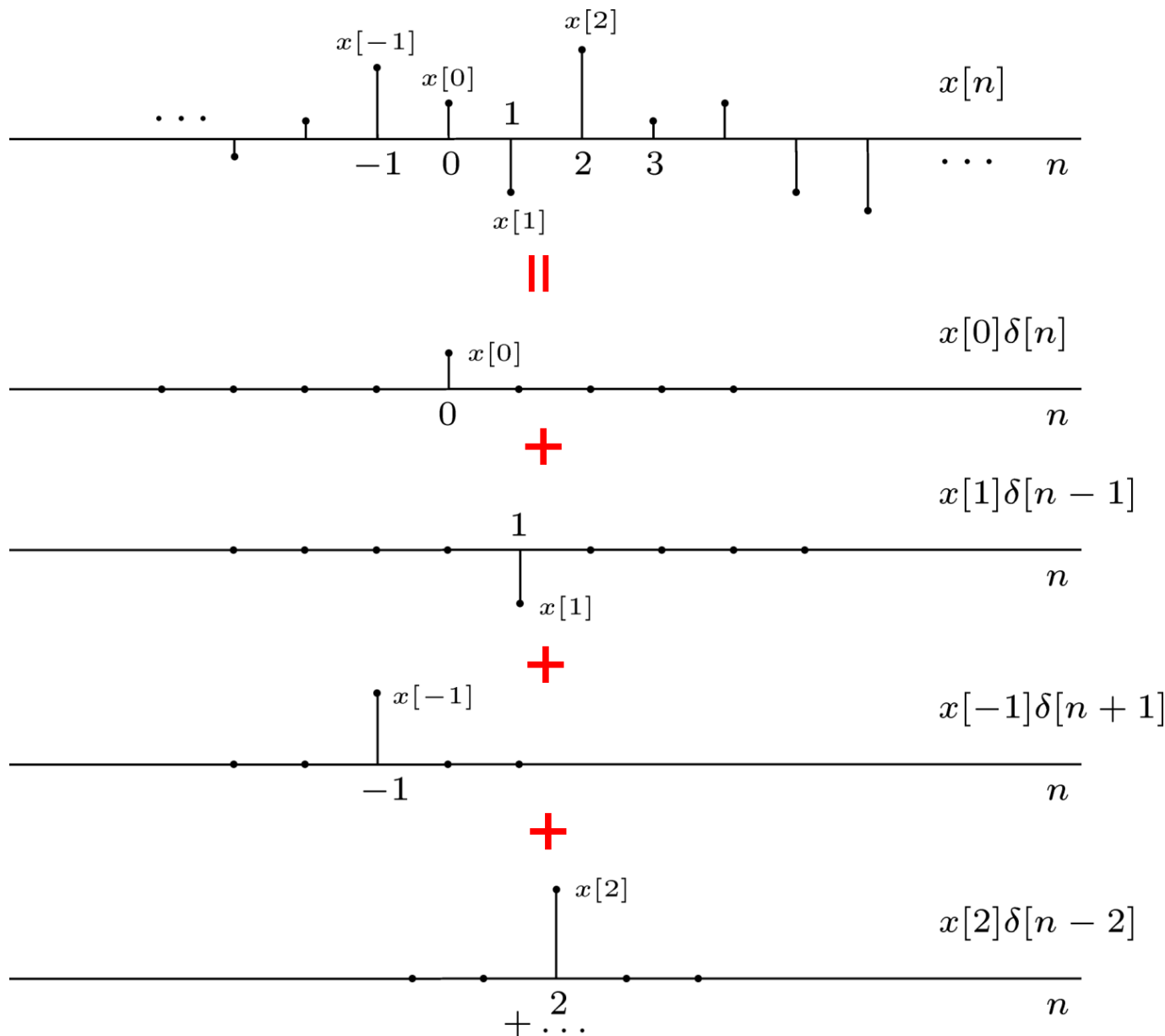$$\text{then} \qquad x[n - n_0] \rightarrow y[n - n_0] .$$

6. **Linear**

A (CT) system is linear if it has the superposition property:

$$\text{If} \qquad x_1(t) \rightarrow y_1(t) \quad \text{and} \quad x_2(t) \rightarrow y_2(t)$$

$$\text{then} \qquad ax_1(t) + bx_2(t) \quad \rightarrow \quad ay_1(t) + by_2(t)$$

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Chapter 2: Linear Time-invariant (LTI) Systems

Representation of DT Signals Using Unit Samples

$x[-1]$

$x[0]$

$x[2]$

$x[n]$

$\cdots$

1

$-1 \quad 0 \quad 2 \quad 3$

$\cdots \quad n$

$x[1]$

$=$

$x[0]\delta[n]$

$x[0]$

$0 \qquad n$

$+$

$x[1]\delta[n-1]$

1

n

$x[1]$

$+$

$x[-1]$

$x[-1]\delta[n+1]$

$-1 \qquad n$

$+$

$x[2]$

$x[2]\delta[n-2]$

$+\cdots \quad 2 \qquad n$

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That is ...

$$x[n] = ... + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + ...$$

$\Downarrow$

Important to note the "-" sign

$$x[n] = \sum_{k=-\infty}^{+\infty} \underbrace{x[k]}_{\text{Coefficients}} \underbrace{\delta[n-k]}_{\text{Basic Signals}}$$

Arbitrary DT signal can be written as a linear combination of impulse functions with different time shifting, i.e., linear combination of signals $\{\delta[n-k] | k = \cdots, -2, -1, 0, 1, 2, ...\}$.
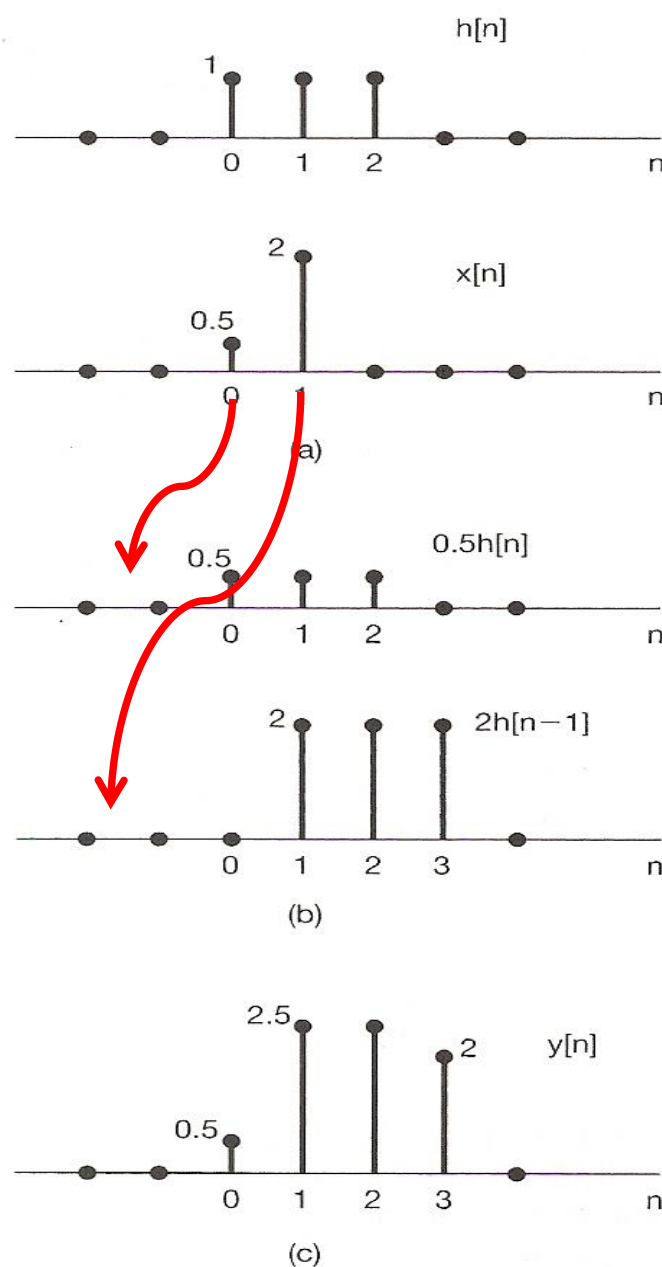
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Unit Impulse Response (Unit Sample Response)

● Define the output for an unit impulse input as the unit impulse response

$x[n]=\delta[n]$ → S → $y[n]=h[n]$:unit impulse response
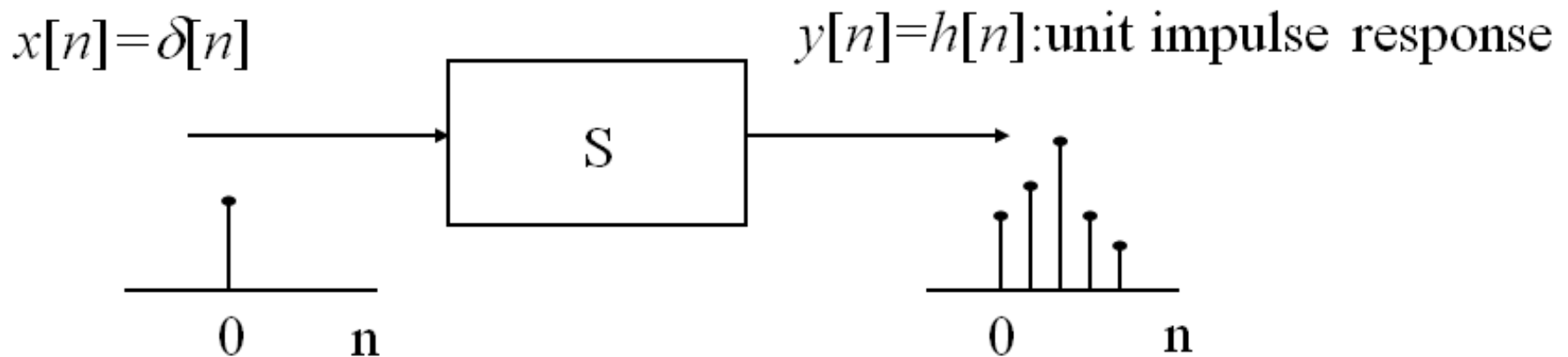
Example: $y[n]=x[n]+2x[n\text{-}1]+4x[n\text{-}2]$
What is unit impulse response?

Example

10

$$x[n] = 0.5\delta[n] + 2\delta[n-1]$$

By linearity:
$$0.5\delta[n] \Rightarrow 0.5h[n]$$
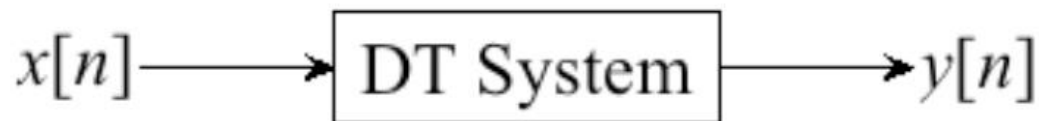
By LTI:
$$2\delta[n-1] \Rightarrow 2h[n-1]$$

y[n]=0.5h[n]+2h[n-1]

Figure 2.3 (a) The impulse response $h[n]$ of an LTI system and an input $x[n]$ to the system; (b) the responses or "echoes," $0.5h[n]$ and $2h[n-1]$, to the nonzero values of the input, namely, $x[0] = 0.5$ and $x[1] = 2$; (c) the overall response $y[n]$, which is the sum of the echos in (b).

Response of DT LTI Systems

$$x[n] \longrightarrow \boxed{\text{DT System}} \longrightarrow y[n]$$

- Now suppose the system is **LTI**, and define the *unit impulse response* $h[n]$:

$$\delta[n] \longrightarrow h[n]$$

$$\Downarrow$$

From **T**ime-**I**nvariance:

$$\delta[n-k] \longrightarrow h[n-k]$$

From **L**inearity:

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k] \longrightarrow y[n] = \underbrace{\sum_{k=-\infty}^{+\infty} x[k]\,h[n-k]}_{convolution\ sum} = x[n] * h[n]$$

The output for an arbitrary input signal is the superposition of a series of "shifted, scaled unit impulse response"
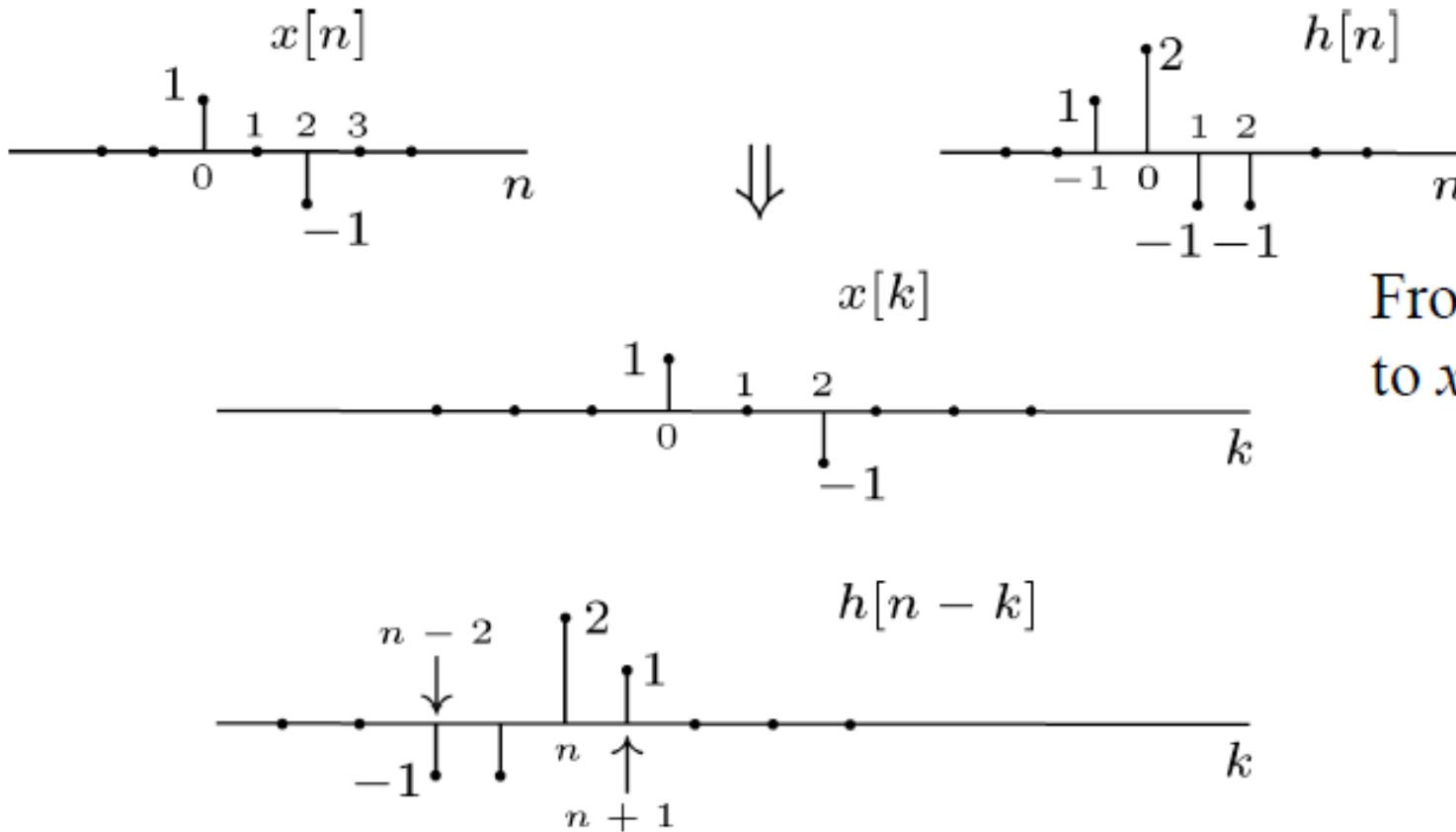
Hence a Very Important Property of LTI Systems:

The output of *any* DT LTI System is a convolution of the input signal with the unit-sample response, *i.e.*

$$\text{Any DT LTI} \quad \longleftrightarrow \quad y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$$

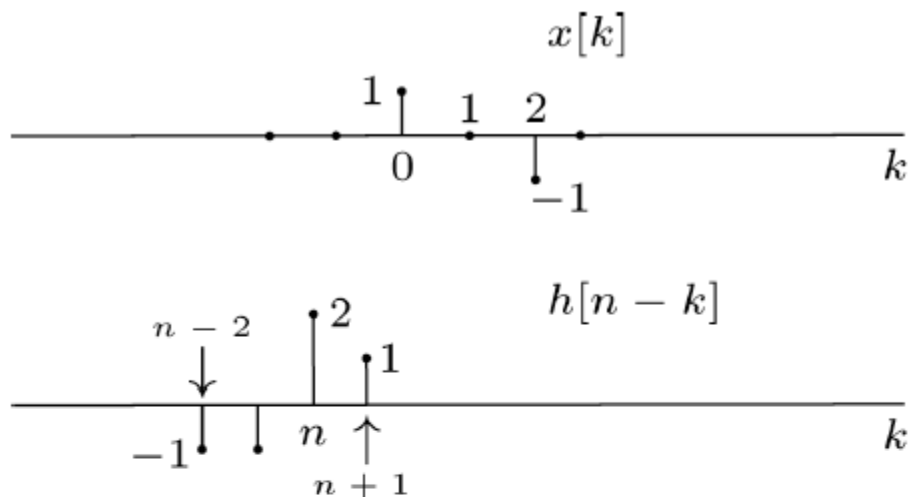As a result, any DT LTI Systems are *completely characterized* by its unit sample response

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Example: Flip h[n]

$x[n]$

$h[n]$

$x[k]$

From $x[n]$ and $h[n]$ to $x[k]$ and $h[n-k]$

$h[n-k]$

Calculating Successive Values: Shift, Multiply, Sum

$$x[k]$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$h[n-k]$$

$$y[n] = 0 \qquad \text{for } n <$$
$$y[-1] =$$
$$y[0] =$$
$$y[1] =$$
$$y[2] =$$
$$y[3] =$$
$$y[4] =$$
$$y[n] = 0 \qquad \text{for } n >$$

Signals and Systems

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Convolution operation procedure:

$$h[k] \xrightarrow{\text{Flip}} h[-k] \xrightarrow{\text{Slide}} h[n-k] \xrightarrow{\text{Multiply}} x[k]h[n-k]$$

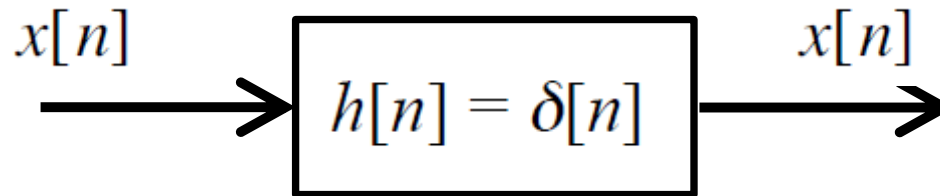$$\xrightarrow{\text{Sum}} \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

F-S-M-S for every fixed n

Observations:

- Convolution of two finite signals leads to another finite signal
- What's the relation on their non-zero duration?

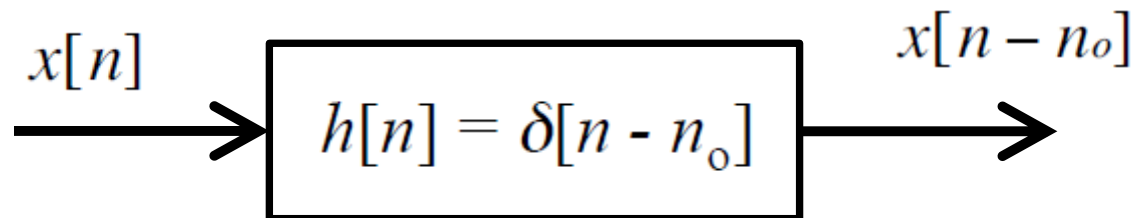南方科技大学 SUSTC

Examples of Convolution and DT LTI Systems

Ex. #1: $\quad h[n] = \delta[n]$

$$y[n] = x[n] * \delta[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$$= x[n] \quad - \quad \text{An Identity system}$$

$x[n]$ $\longrightarrow$ $\boxed{h[n] = \delta[n]}$ $\longrightarrow$ $x[n]$

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Ex. #2: $\quad h[n] = \delta[n - n_{\mathrm{o}}]$

$$y[n] = x[n] * \delta[n - n_o] = \sum_{k=-\infty}^{\infty} x[k]\delta[n - n_o - k]$$

$$= x[n - n_o] \quad — \quad \text{A Shift}$$

$x[n]$ $\boxed{h[n] = \delta[n - n_{\mathrm{o}}]}$ $x[n - n_o]$
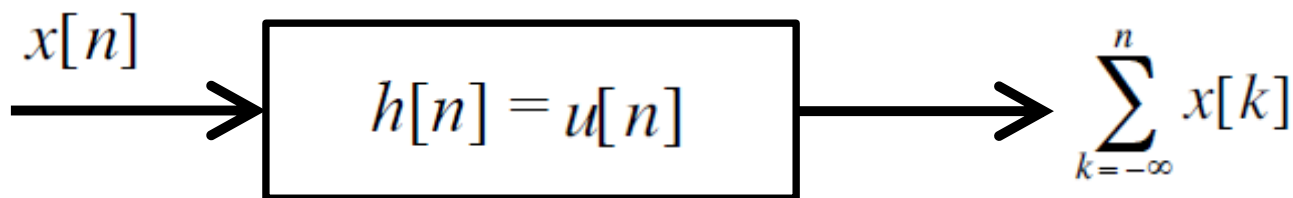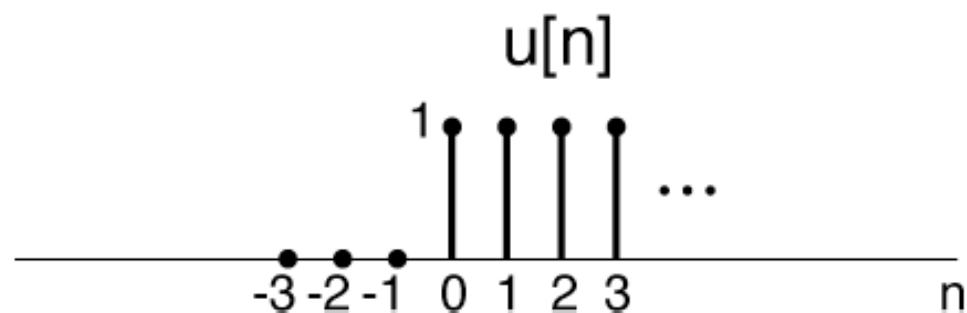
Signals and Systems

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Ex. #3 $y[n] = \sum\limits_{k=-\infty}^{n} x[k]$ – An accumulator

Unit Sample response

$$h[n] = \sum_{k=-\infty}^{n} \delta[k] = u[n]$$

u[n]

$$\Downarrow$$

$$x[n] * u[n] = \sum_{k=-\infty}^{n} x[k]$$

$$x[n] \quad \boxed{h[n] = u[n]} \quad \sum_{k=-\infty}^{n} x[k]$$

Signals and Systems

Ex. #4 (Example 2.3)

$0<\alpha<1$

$x[n] = \alpha^n u[n]$

(a)

$h[n] = u[n]$

(b)
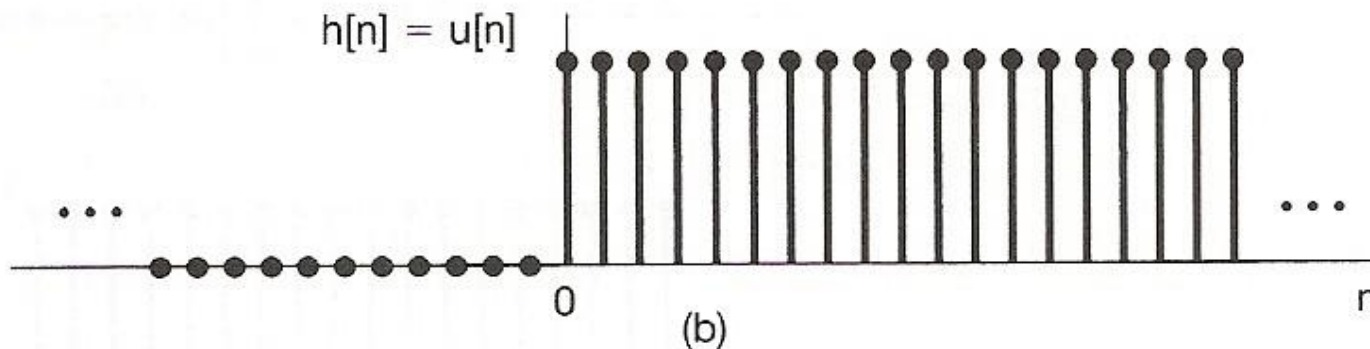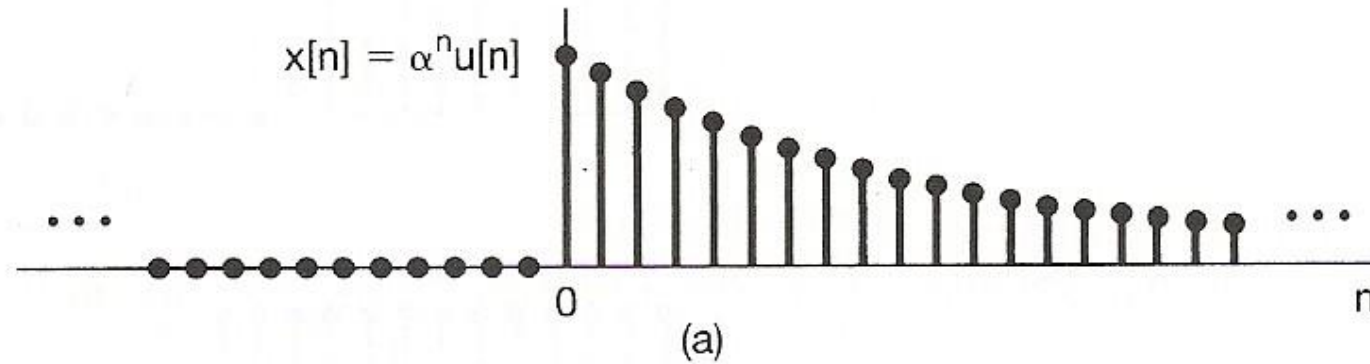
Figure 2.5 The signals $x[n]$ and $h[n]$ in Example 2.3.

$x[k] = \alpha^k u[k]$

$h[-k]$

$n= 0$

$y[0]$

$h[-1-k]$

$n= ?$

$y[-1]$

$h[1-k]$

$n= ?$

$y[1]$

$h[n-k]$

$n>0$

$h[n-k]$

$n<0$
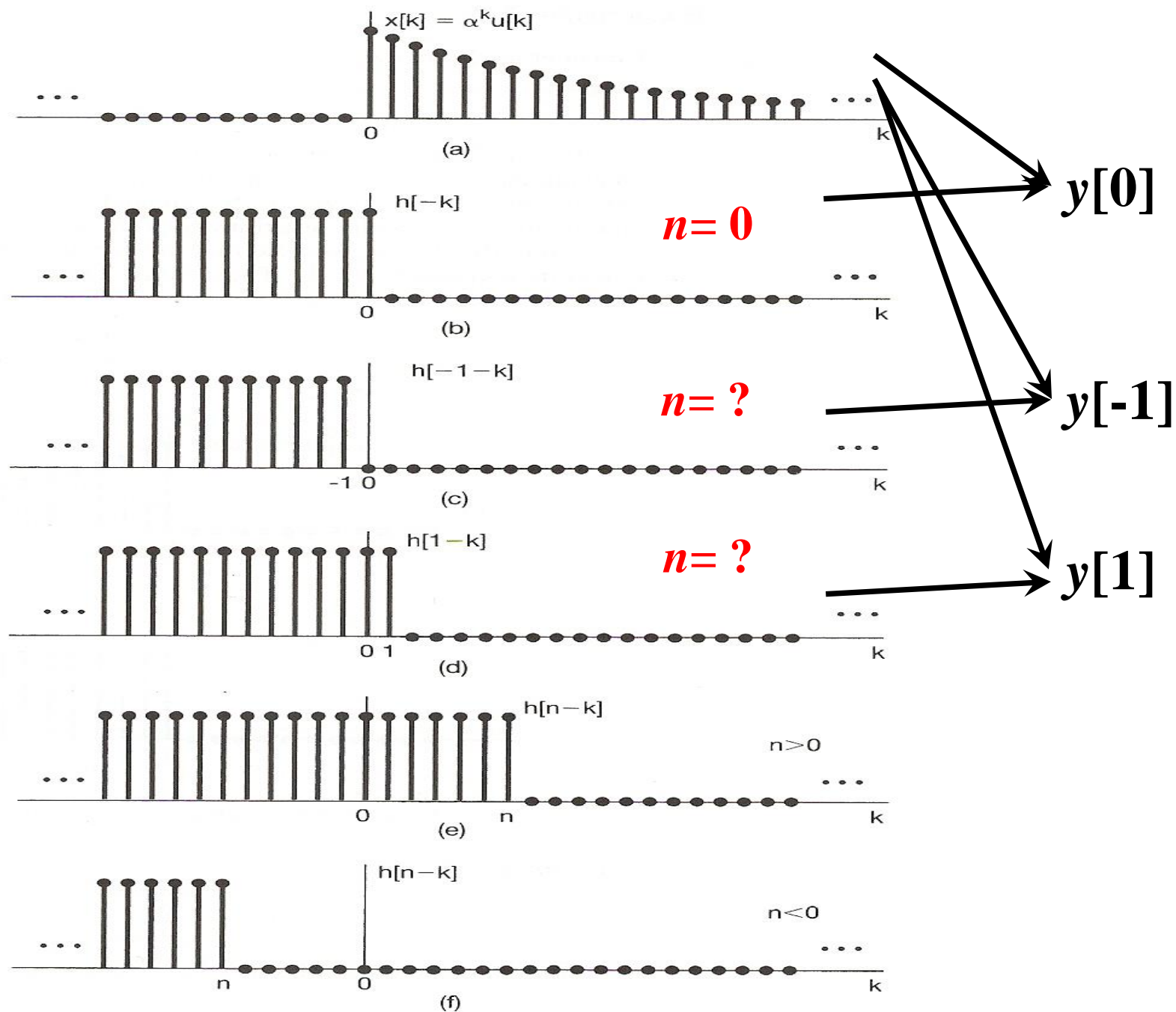
Figure 2.6 Graphical interpretation of the calculation of the convolution sum for Example 2.3.

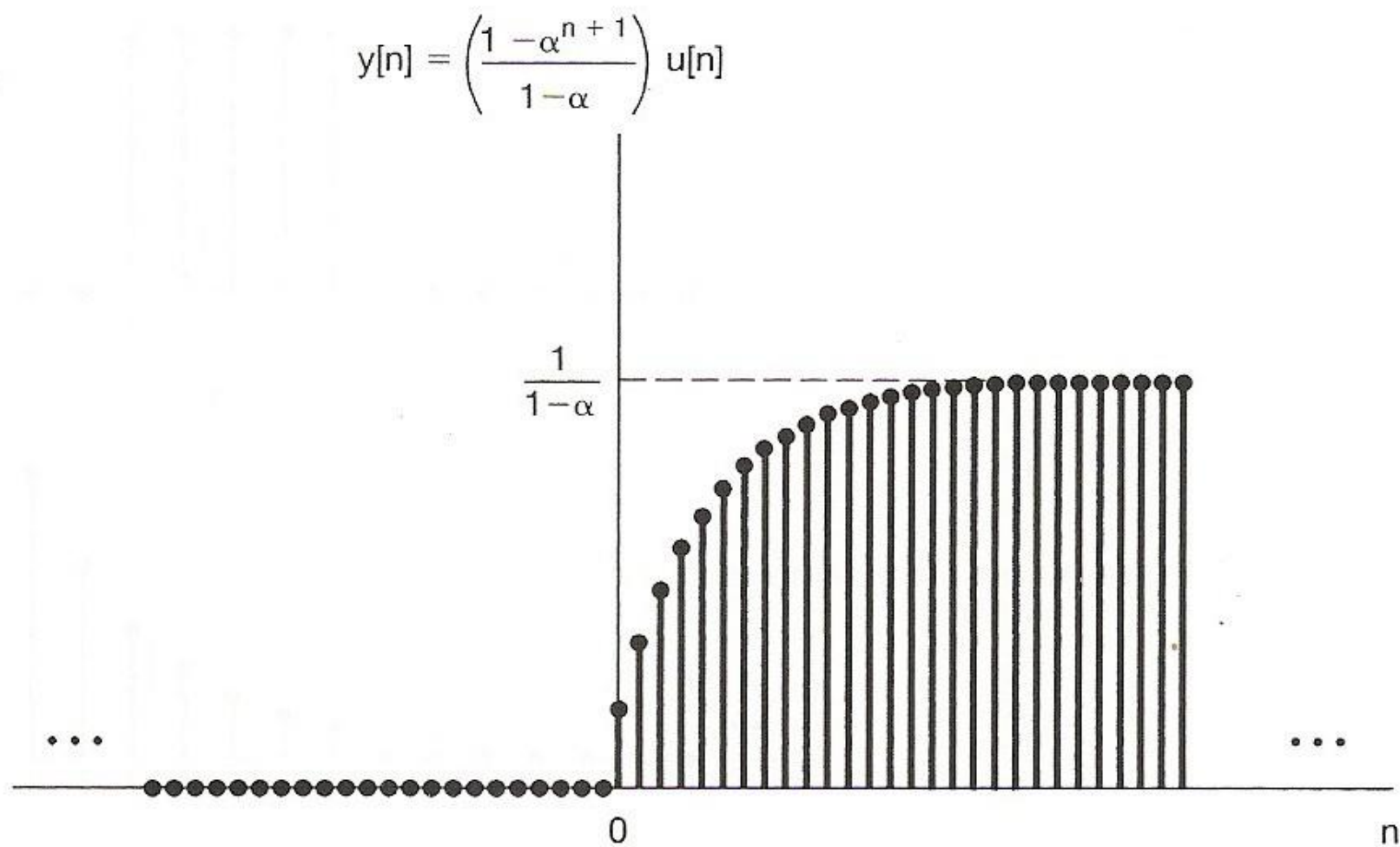$$y[n] = \left(\frac{1-\alpha^{n+1}}{1-\alpha}\right) u[n]$$

Figure 2.7 Output for Example 2.3.

Characteristics of an LTI system are completely determined by its impulse response.

● *What if the system is nonlinear?*

Consider a discrete-time system with unit impulse response

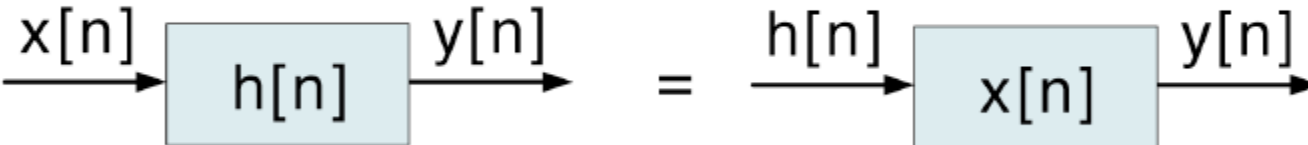$$h[n] = \begin{cases} 1 & n = 0,1 \\ 0 & \text{otherwise} \end{cases}$$

If the system is LTI, the input/output relationship is

$$y[n] = x[n] + x[n-1].$$

On the other hand, there are *many* nonlinear systems with the same response to the input $\delta[n]$.

$$y[n] = (x[n] + x[n-1])^2,$$
$$y[n] = \max(x[n], x[n-1]).$$

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The Commutative Property of Convolution

$$y[n] = x[n] * h[n] \qquad = \qquad h[n] * x[n]$$

$$\xrightarrow{x[n]} \boxed{h[n]} \xrightarrow{y[n]} \quad = \quad \xrightarrow{h[n]} \boxed{x[n]} \xrightarrow{y[n]}$$

The Distributive Property of Convolution

$$x[n] * \{h_1[n] + h_2[n]\} = x[n] * h_1[n] + x[n] * h_2[n]$$

Interpretation
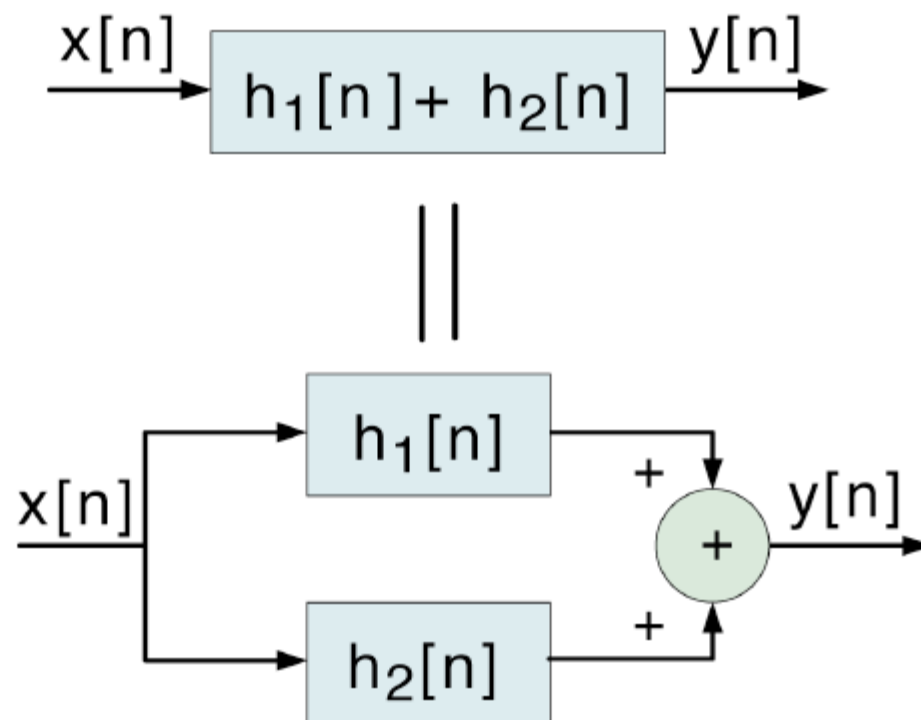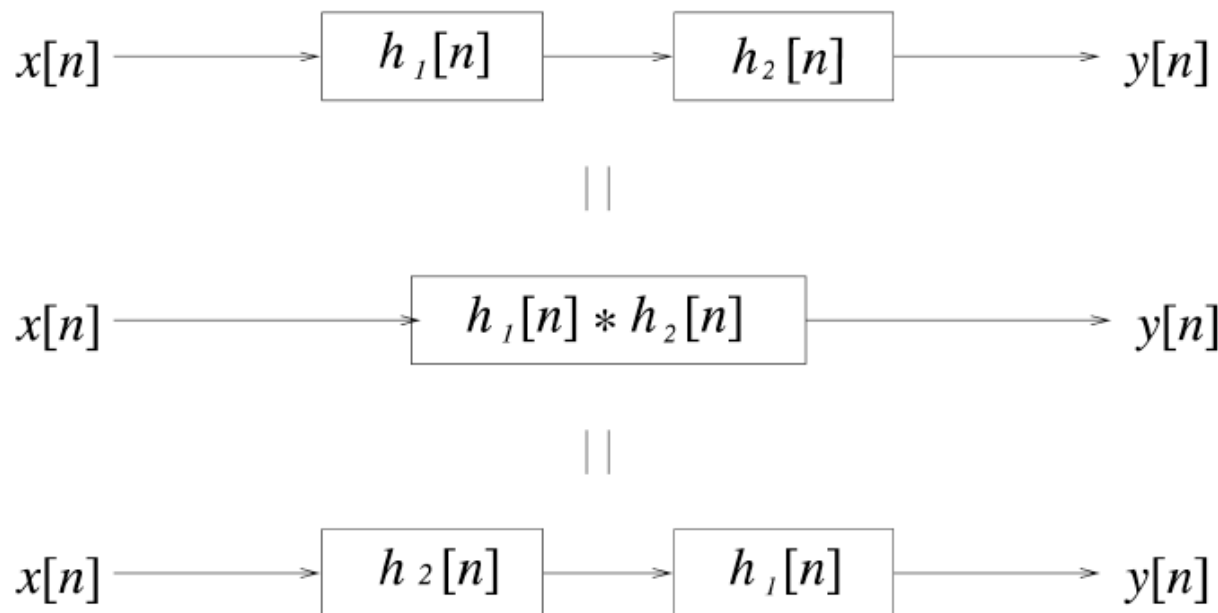
The Associative Property of Convolution

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$$

(Commutativity) $||$

$$x[n] * (h_2[n] * h_1[n]) = (x[n] * h_2[n]) * h_1[n]$$

Implication (Very special to LTI Systems)

$x[n]$ ⟶ [$h_1[n]$] ⟶ [$h_2[n]$] ⟶ $y[n]$

$||$

$x[n]$ ⟶ [$h_1[n] * h_2[n]$] ⟶ $y[n]$

$||$

$x[n]$ ⟶ [$h_2[n]$] ⟶ [$h_1[n]$] ⟶ $y[n]$

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Properties of Convolution

Combining the Commutative property,

$$y[n] = x[n] * h[n] \qquad = \qquad h[n] * x[n]$$

Distributive property,

$$x[n] * \{h_1[n] + h_2[n]\} = x[n] * h_1[n] + x[n] * h_2[n]$$

and Associative property,

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$$

symbolically, we can treat "$*$" as a "$\times$". Easy, piece of cake!

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Some Useful Properties of LTI Systems

1) Causality $\Leftrightarrow$ $h[n] = 0$ for all $n < 0$

2) Stability $\Leftrightarrow$ $\sum_{k=-\infty}^{+\infty} |h[k]| < \infty$

BIBO — Bounded Input $\Rightarrow$ Bounded Output

➡ Sufficient condition: For $|x[n]| \leq x_{max} < \infty$.

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} x[k]h[n-k] \right| \leq x_{max} \left| \sum_{k=-\infty}^{\infty} h[n-k] \right| < \infty.$$

➡ Necessary condition: If $\sum_{k=-\infty}^{\infty} |h[k]| = \infty$

Let $x[n] = h*[-n]/|h[-n]|$, then $|x[n]| \equiv 1$ bounded

But $y[0] = \sum_{k=-\infty}^{\infty} x[k]h[-k] = \sum_{k=-\infty}^{\infty} h*[-k]h[-k]/|h[-k]| = \sum_{k=-\infty}^{\infty} |h[-k]| = \infty$

- Memoryless / with Memory

    - A linear, time-invariant, causal system is memoryless only

    if $\quad h[n] = K\delta[n] \qquad h(t) = K\delta(t)$

    $\qquad y[n] = Kx[n] \qquad y(t) = Kx(t)$

    if k=1 further, they are identity systems

    $$y[n] = x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = x[n] * \delta[n]$$

    $$y(t) = x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t) * \delta(t)$$

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Summary

Understand the following new concepts:

1. **Use unit impulse function to represent any function**

2. **Unit impulse response h[n]**

   - ◆ **Given the system input/output equation, how to decide the unit impulse response?**

3. **Convolution, its properties, and calculation steps (FSMS)**

   - ◆ **Understand the meaning of index 'k' and index 'n'**

4. **Decide LTI system property by using unit impulse response h[n]**

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