北京大学 23/24 学年第 1 学期

高数 B 期中试题答案

原文链接：数竞之捷。由 Arthals 校对。如有问题请查阅原文。

1.

$$\lim_{n\to\infty}(1+\frac{1}{ne})^n \overset{t=ne}{=} \lim_{t\to+\infty}(1+\frac{1}{t})^{t\cdot\frac{1}{e}} = \mathrm{e}^{\frac{1}{e}}. \tag{1}$$

2. 对于任意 $x>0$，存在 $n\in\mathbb{N}_+$，使得 $n\leq x<n+1$，此时有 $[x]=n$。

   因此，有:

$$n\sin\frac{1}{[x]} \leq x\sin\frac{1}{[x]} < (n+1)\sin\frac{1}{n}. \tag{2}$$

   从而:

$$\lim_{n\to\infty} n\sin t \leq \lim_{x\to+\infty} x\sin\frac{1}{[x]} \leq \lim_{n\to\infty}(n+1)\sin\frac{1}{n}. \tag{3}$$

   令 $\frac{1}{n}=t$，则有:

$$\lim_{t\to0^+}\frac{\sin t}{t} \leq \lim_{x\to+\infty} x\sin\frac{1}{[x]} \leq \lim_{t\to0^+}(t+1)\cdot\frac{\sin t}{t}. \tag{4}$$

   即，

$$1 \leq \lim_{x\to+\infty} x\sin\frac{1}{[x]} \leq 1. \tag{5}$$

   由夹逼定理可得:

$$\lim_{x\to+\infty} x\sin\frac{1}{[x]} = 1. \tag{6}$$

3. 由变上限积分函数的性质和链式法则得

$$f'(x) = \sqrt{1+e^{\ln x}}(\ln x)' = \frac{\sqrt{1+x}}{x}. \tag{7}$$

4. 有理式分拆得

$$P = \frac{4x^2+4x-11}{(2x-1)(2x+3)(2x-5)} = \frac{a}{2x-1}+\frac{b}{2x+3}+\frac{c}{2x-5}, \tag{8}$$

   将右侧通分，则分子满足

$$4x^2+4x-11 = a(2x+3)(2x-5)+b(2x-1)(2x-5)+c(2x-1)(2x+3). \tag{9}$$

   令

$$x=\frac{1}{2}, 得a=\frac{1}{2}; 令x=-\frac{3}{2}, 得b=-\frac{1}{4}; 令x=\frac{1}{2}, 得c=\frac{3}{4}. \tag{10}$$

于是，

$$P=\frac{1}{2}\cdot\frac{1}{2x-1}-\frac{1}{4}\cdot\frac{1}{2x+3}+\frac{3}{4}\cdot\frac{1}{2x-5}. \tag{11}$$

从而，

$$\begin{aligned}&\int\frac{4x^2+4x-11}{(2x-1)(2x+3)(x-5)}\,\mathrm{d}x\\&=\frac{1}{2}\int\frac{\mathrm{d}x}{2x-1}-4\int\frac{\mathrm{d}x}{2x+3}+\frac{3}{4}\int\frac{\mathrm{d}x}{2x-5}\\&=\frac{1}{8}\ln\left|\frac{(2x-1)^2(2x-5)^3}{2x+3}\right|+C.\end{aligned} \tag{12}$$

方法二：待定系数法

$$\begin{aligned}&4x^2+4x-11\\&=(4a+4b+4c)x^2-(4a+12b-4c)x+(-15a+5b-3c).\end{aligned} \tag{13}$$

由待定系数法，

$$\begin{cases}4a+4b+4c=4\\-4a-12b+4c=4\\-15a+5b-3c=-11\end{cases} \tag{14}$$

解得

$$\begin{cases}a=\frac{1}{2}\\b=-\frac{1}{4}\\c=\frac{3}{4}\end{cases} \tag{15}$$

余下同上.

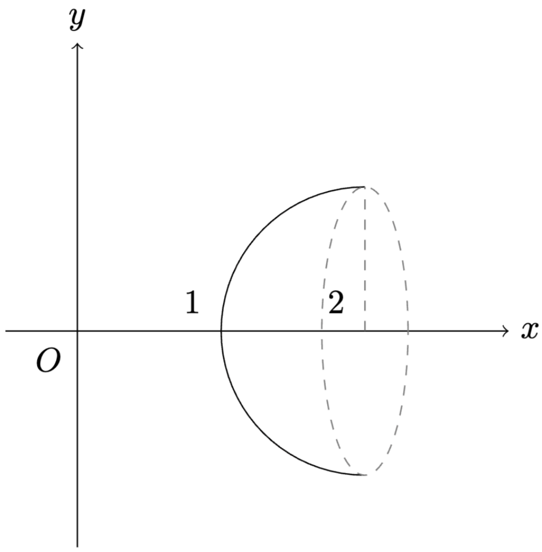
$$y'=\frac{1}{2}(\sqrt{x^2-1}+\frac{x^2}{\sqrt{x^2-1}}-\frac{1+\frac{x}{\sqrt{x^2-1}}}{x+\sqrt{x^2-1}})=\sqrt{x^2-1}\,, \tag{16}$$

5. 所求弧长

$$L=\int_1^2\sqrt{1+y'^2}\,\mathrm{d}x=\int_1^2 x\,\mathrm{d}x=\frac{1}{2}x^2\Big|_1^2=\frac{3}{2}. \tag{17}$$

6.

$$y=\frac{\ln x}{\sqrt{2\pi}}\quad(1\leq x\leq 2) \tag{18}$$



$$
\begin{aligned}
V =&\pi \int_1^2 y^2 \, \mathrm{d}x = \frac{1}{2} \int_1^2 \ln^2 x \, \mathrm{d}x = \frac{1}{2} x \ln^2 x \Big|_1^2 - \frac{1}{2} \int_1^2 x \, \mathrm{d}(\ln^2 x) \\
=&(\ln 2)^2 - \int_1^2 \ln x \, \mathrm{d}x = (\ln 2)^2 - x \ln x |_1^2 + \int_1^2 x \, \mathrm{d}(\ln x) \\
=&(\ln 2)^2 - 2\ln 2 + 1 = (\ln 2 - 1)^2.
\end{aligned} \tag{19}
$$

7. 一方面，$a_1 > b_1 > 0$，故

$$a_2 = \frac{a_1 + b_1}{2} > \sqrt{a_1 b_1} = b_2 > 0 \tag{20}$$

于是归纳地，$a_n > b_n > 0$.

另一方面，

$$a_{n+1} = \frac{a_n + b_n}{2} < \frac{a_n + a_n}{2} = a_n \tag{21}$$

故 $\{a_n\}$ 单调递减.

综上，序列 $\{a_n\}$ 单调有界，故

$$\lim_{n \to \infty} a_n \tag{22}$$

存在有限.

8. (a) 证明：

令 $g(x) = \frac{4 \sin x}{3 + \sin^2 x}, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

则 $g'(x) = \frac{4 \cos x \left(3 - \sin^2 x\right)}{\left(3 + \sin^2 x\right)^2} > 0$, 故 $g(x)$ 在 $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ 上单调递增, 从而 $g\left(-\frac{\pi}{2}\right) < g(x) < g\left(\frac{\pi}{2}\right)$

, 即 $-1 < \frac{4 \sin x}{3 + \sin^2 x} < 1$;

(b)

$$
\begin{aligned}
f'(x) =& \frac{1}{\sqrt{1-g^2(x)}} \cdot g'(x) \\
=& \frac{1}{\sqrt{1-\left(\frac{4\sin x}{3+\sin^2 x}\right)^2}} \cdot \frac{4\cos x\left(3-\sin^2 x\right)}{\left(3+\sin^2 x\right)^2} \\
=& \frac{4\cos x\left(3-\sin^2 x\right)}{\left(3+\sin^2 x\right)\sqrt{9-10\sin^2 x+\sin^4 x}} \\
=& \frac{4\cos x\left(3-\sin^2 x\right)}{\sqrt{1-\sin^2 x}\sqrt{9-\sin^2 x}\left(3+\sin^2 x\right)} \\
=& \frac{4\left(3-\sin^2 x\right)}{\sqrt{9-\sin^2 x}\left(3+\sin^2 x\right)}
\end{aligned} \tag{23}
$$

(c) 证明:

当 $x \in [0, \frac{\pi}{2}]$, 令 $f(x) = \theta$, $[0, \frac{\pi}{2}]$, 则 $\sin\theta = \frac{4\sin x}{3+\sin^2 x}$,

$$
\mathrm{d}\theta = f'(x)\mathrm{d}x = \frac{4(3-\sin x)}{\sqrt{9-\sin^2 x}(3+\sin^2 x)}\mathrm{d}x, \tag{24}
$$

$$
\sin^2\theta = \frac{16\sin^2 x}{(3+\sin^2 x)^2}, \cos^2\theta = \frac{9-10\sin^2 x+\sin^4 x}{(3+\sin^2 x)^2}, \tag{25}
$$

故

$$
\sqrt{4\cos^2\theta+\sin^2\theta} = \sqrt{\frac{4(3-\sin^2 x)^2}{(3+\sin^2 x)^2}} = \frac{2(3-\sin^2 x)}{3+\sin^2 x}, \tag{26}
$$

从而

$$
\begin{aligned}
&\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{4\cos^2\theta+\sin^2\theta}} \\
=& \int_0^{\frac{\pi}{2}} \frac{4(3-\sin^2 x)}{\sqrt{9-\sin^2 x}(3+\sin^2 x)} \frac{3+\sin^2 x}{2(3-\sin^2 x)}\mathrm{d}x \\
=& \int_0^{\frac{\pi}{2}} \frac{2\mathrm{d}x}{\sqrt{9-\sin^2 x}} = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{\sqrt{\frac{9}{4}\cos^2 x+2\sin^2 x}},
\end{aligned} \tag{27}
$$

于是,

$$
\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{\sqrt{4\cos^2 x+\sin^2 x}} = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{\sqrt{\frac{9}{4}\cos^2 x+2\sin^2 x}}. \tag{28}
$$

9. 设 $h(x) = f(x) - g(x), x \in [0,1]$，依题意 $h(x)$ 在 $[0,1]$ 上连续。

因为 $\cos f(1) = \cos g(1)$，$\sin f(1) = \sin g(1)$，所以 $\sin(h(1)) = 0$，从而 $h(1) = 2m\pi, m \in \mathbb{Z}$ ①。

又

$$
\forall x \in [0,1], [\cos f(x)+\cos g(x)]^2 + [\sin f(x)+\sin g(x)]^2 = 2[1+\cos h(x)] \neq 0, \tag{29}
$$

所以 $\forall x \in [0,1], h(x) \neq (2n+1)\pi, n \in \mathbb{Z}$ ②。

在 ① 中，假设 $m > 0$，则 $h(0) = 0, h(1) = 2m\pi$，由介值定理知：

$$\exists \xi_1 \in [0,1], h(\xi_1) = (2m-1)\pi, \tag{30}$$

这与 ② 矛盾（此时 $n = m-1$）。

假设 $m < 0$，同理 $\exists \xi_2 \in [0,1], h(\xi_2) = (2m+1)\pi$ 与 ② 矛盾（此时 $n = m$），所以 $m = 0$。即 $f(1) = g(1)$。