LaTeX command declarations here.

In [10]:

```
from __future__ import division

# scientific
%matplotlib inline
from matplotlib import pyplot as plt;
import matplotlib as mpl;
import numpy as np;
import sklearn as skl;
import sklearn.datasets;
import sklearn.cluster;
import sklearn.mixture;

# ipython
import IPython;

# python
import os;
import random;

#####################################################

# image processing
import PIL;

# trim and scale images
def trim(im, percent=100):
    print("trim:", percent);
    bg = PIL.Image.new(im.mode, im.size, im.getpixel((0,0)))
    diff = PIL.ImageChops.difference(im, bg)
    diff = PIL.ImageChops.add(diff, diff, 2.0, -100)
    bbox = diff.getbbox()
    if bbox:
        x = im.crop(bbox)
        return x.resize(((x.size[0]*percent)//100,
                         (x.size[1]*percent)//100), PIL.Image.ANTIA
LIAS);


#####################################################

# daft (rendering PGMs)
import daft;

# set to FALSE to load PGMs from static images
RENDER_PGMS = True;

# decorator for pgm rendering
def pgm_render(pgm_func):
    def render_func(path, percent=100, render=None, *args, **kwarg
s):
        print("render_func:", percent);
        # render
        render = render if (render is not None) else RENDER_PGMS;
```

```
        if render:
            print("rendering");
            # render
            pgm = pgm_func(*args, **kwargs);
            pgm.render();
            pgm.figure.savefig(path, dpi=300);

            # trim
            img = trim(PIL.Image.open(path), percent);
            img.save(path, 'PNG');
        else:
            print("not rendering");

        # error
        if not os.path.isfile(path):
            raise Exception("Error:  Graphical model image %s not f
ound.\
                You may need to set RENDER_PGMS=True." % path);

        # display
        return IPython.display.Image(filename=path);

    return render_func;

######################################################
```

EECS 545: Machine Learning

Lecture 17: GMMs & Hidden Markov Models

- Instructor: **Jacob Abernethy**
- Date: March 21, 2016

Lecture Exposition: Benjamin Bray

Gaussian Mixture Models

> (Left over from last time!)

Gaussian Mixture Models: Specification

Recall from earlier the **Gaussian Mixture Model**:

$$\begin{aligned}
\theta &= (\pi, \mu, \mathbf{\Sigma}) && \text{model parameters} \\
z_n &\sim \text{Categorical}[\pi] && \text{cluster indicators} \\
x_n | z_n, \theta &\sim \mathcal{N}(\mu_{z_n}, \Sigma_{z_n}) && \text{base distribution}
\end{aligned}$$

GMM: Complete Log Likelihood

The complete data log-likelihood for a single datapoint (x_n, z_n) is

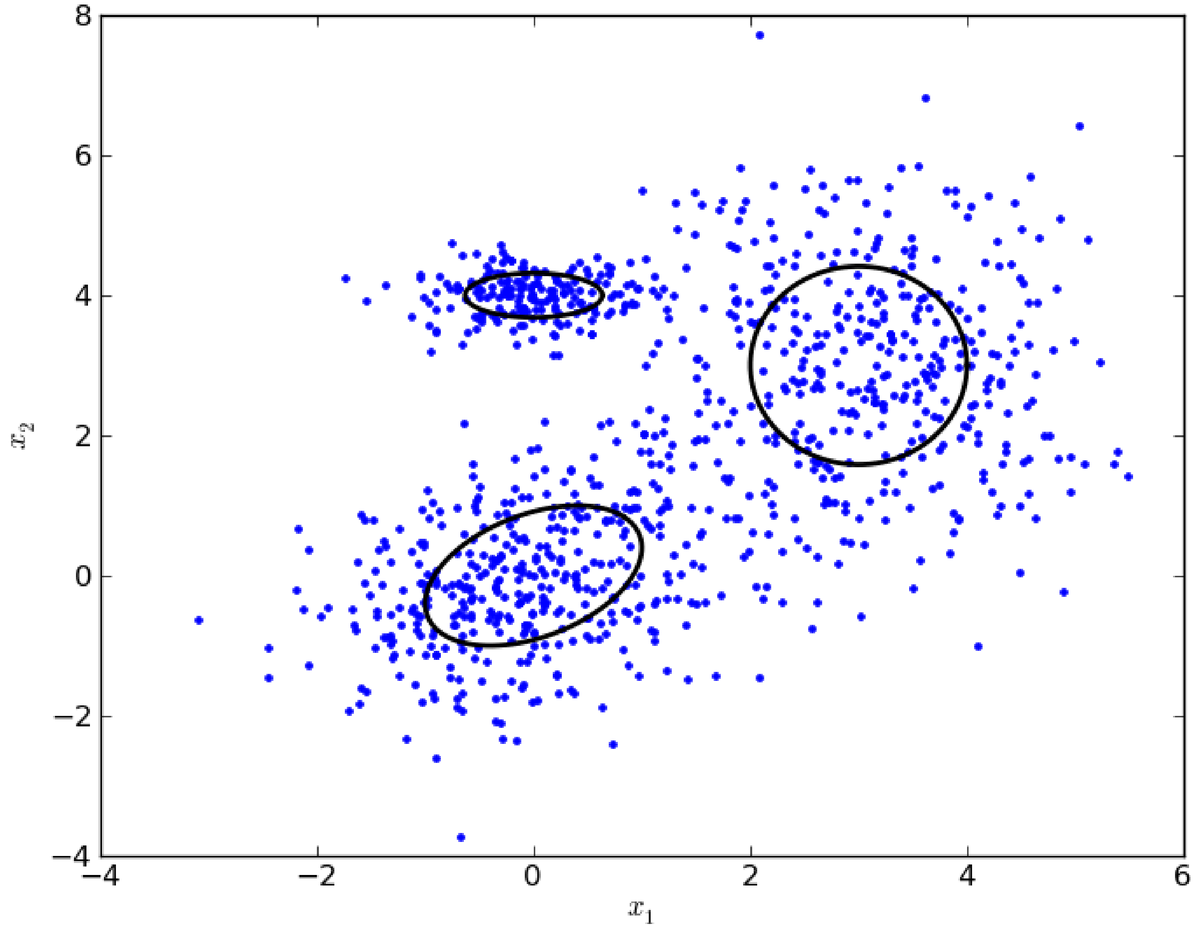
$$\begin{aligned}
\log p(x_n, z_n | \theta) &= \log \prod_{k=1}^{K} \pi_k \mathcal{N}(x_n \mid \mu_k, \Sigma_k)^{\mathbb{I}(z_n = k)} \\
&= \sum_{k=1}^{K} \mathbb{I}(z_n = k) \log \pi_k \mathcal{N}(x_n \mid \mu_k, \Sigma_k)
\end{aligned}$$

GMM: Hidden Posterior

The hidden posterior for a single point (x_n, z_n) can be found using Bayes' rule:

$$\begin{aligned}
p(z_n = k | x_n, \theta) &= \frac{P(z_n = k | \theta) p(x_n | z_n = k, \theta)}{p(x_N | \theta)} \\
&= \frac{\pi_k \mathcal{N}(x_n | \mu_k, \Sigma_k)}{\sum_{k'=1}^{K} \pi_{k'} \mathcal{N}(x_n | \mu_{k'}, \Sigma_{k'})}
\end{aligned}$$

GMM: Visual example



GMM: E-Step

The **expected complete likelihood** is $$ \begin{align} Eq[\log p(\X,Z|\theta)] &= \sum{n=1}^N \sum{k=1}^K E_q\big[\mathbb{I}(z_n = k) \big] \log \pi_k \mathcal{N}(x_n \mid \mu_k, \Sigma_k) \ &= \sum{n=1}^N \sum{k=1}^K r{nk} \log \pi_k

- \sum{n=1}^N \sum{k=1}^K

  ```
  r_{nk} \log \mathcal{N}(x_n \mid \mu_k, \Sigma_k)
  ```

  \end{align} $$

where $r_{nk} \equiv p(z_n = k \mid x_n, \theta_t)$ is the **responsibility** that cluster k takes for datapoint x_n after step t.

GMM: M-Step

During the M-Step, we optimize the lower bound with respect to $\theta = (\pi, \mu, \Sigma)$. Verify that the correct updates are

$$\pi_k = \frac{1}{N}\sum_{n=1}^{N} r_{nk} = \frac{r_k}{N} \qquad \mu_k = \frac{\sum_{n=1}^{N} r_{nk} x_n}{r_k}$$

$$\Sigma_k = \frac{\sum_{n=1}^{N} r_{nk} x_n x_n^T}{r_k} - \mu_k \mu_k^T$$

where $r_k = \sum_{n=1}^{N} r_{nk}$ is the *effective* sample size for cluster k.

Markov Models

> Uses material from **[MLAPP]**

Sequential Data

Some data has intrinsic sequential structure

- **Time Series:** Speech, EKGs, stock market, robot sensors, etc.
- **Spatial Data:** DNA, natural language, etc.

We could treat data points as iid samples

- e.g. "bag of words" assumption for spam classification
- But this is false! We are ignoring valuable constraints in the data.

Markov Models

A **Markov chain** is a series of random variables $X_1, \ldots, X_N$ satisfying the *Markov property*:

> The future is independent of the past, given the present.
>
> $$p(x_n | x_1, \ldots, x_{n-1}) = p(x_n | x_{n-1})$$

A chain is **stationary** if the transition probabilities do not change with time.

Example: Random Shakespeare (http://www.cs.princeton.edu/courses/archive/spr05/cos126/assignments/ma

```
DUKE SENIOR    Now, my co-mates and thus bolden'd, man, how now, monsie
ur Jaques,
    Unclaim'd of his absence, as the holly!
    Though in the slightest for the fashion of his absence, as the only
wear.

TOUCHSTONE    I care not for meed!
    This I must woo yours: your request than your father: the time,
    That ever love I broke
    my sword upon some kind of men
    Then, heigh-ho! sing, heigh-ho! sing, heigh-ho! sing, heigh-ho! unt
o the needless stream;
    'Poor deer,' quoth he,
    'Call me not so keen,
    Because thou the creeping hours of the sun,
    As man's feasts and women merely players:
    Thus we may rest ourselves and neglect the cottage, pasture?

[Exit]
```

Markov Models: PGM

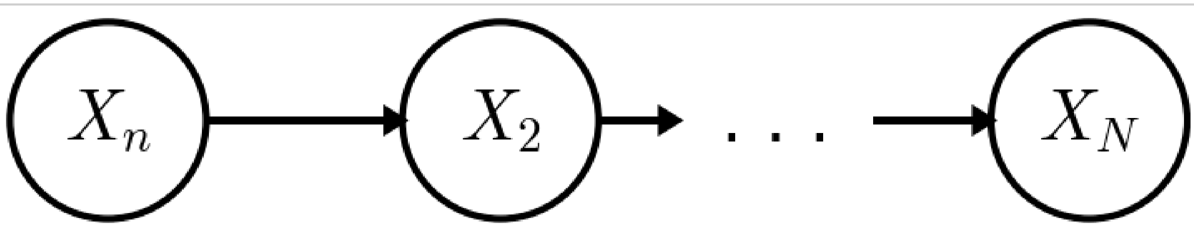
In [5]:

```
@pgm_render
def pgm_markov_chain():
    pgm = daft.PGM([6, 6], origin=[0, 0])

    # Nodes
    pgm.add_node(daft.Node("x1", r"$X_n$", 2, 2.5))
    pgm.add_node(daft.Node("x2", r"$X_2$", 3, 2.5))
    pgm.add_node(daft.Node("ellipsis", r" . . . ", 3.7, 2.5,
                           offset=(0, 0), plot_params={"ec" : "non
e"}))
    pgm.add_node(daft.Node("ellipsis_end", r"", 3.7, 2.5,
                           offset=(0, 0), plot_params={"ec" : "non
e"}))
    pgm.add_node(daft.Node("xN", r"$X_N$", 4.5, 2.5))

    # Add in the edges.
    pgm.add_edge("x1", "x2", head_length=0.08)
    pgm.add_edge("x2", "ellipsis", head_length=0.08)
    pgm.add_edge("ellipsis_end", "xN", head_length=0.08)

    return pgm;
```

In [6]:

```
%%capture
pgm_markov_chain("images/pgm/markov-chain.png")
```

Out[6]:



Markov Models: Joint Distribution

If a sequence has the Markov property, then the joint distribution factorizes according to

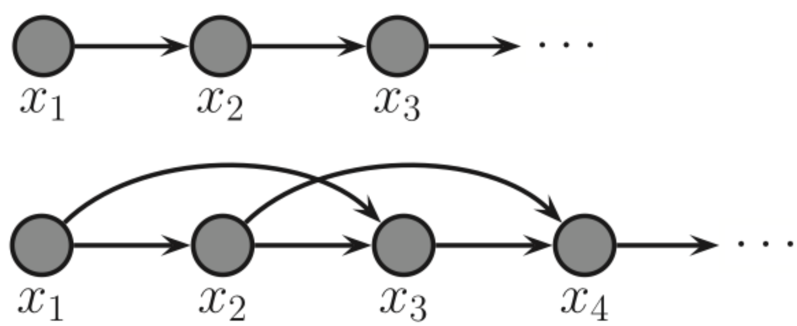
$$p(x_1, \dots, x_N) = p(x_1) \prod_{n=2}^{N} p(x_n | x_{n-1})$$

Example: Language Modeling

One important application is statistical **language models**.

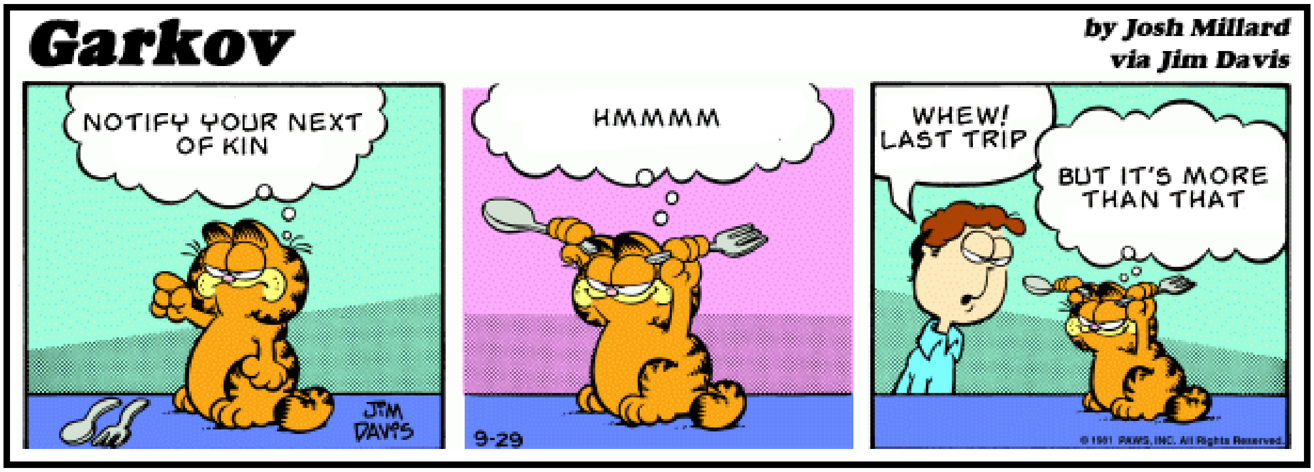
- Bag-of-words assumption is inadequate
- Assume each word X_t depends on the previous n words

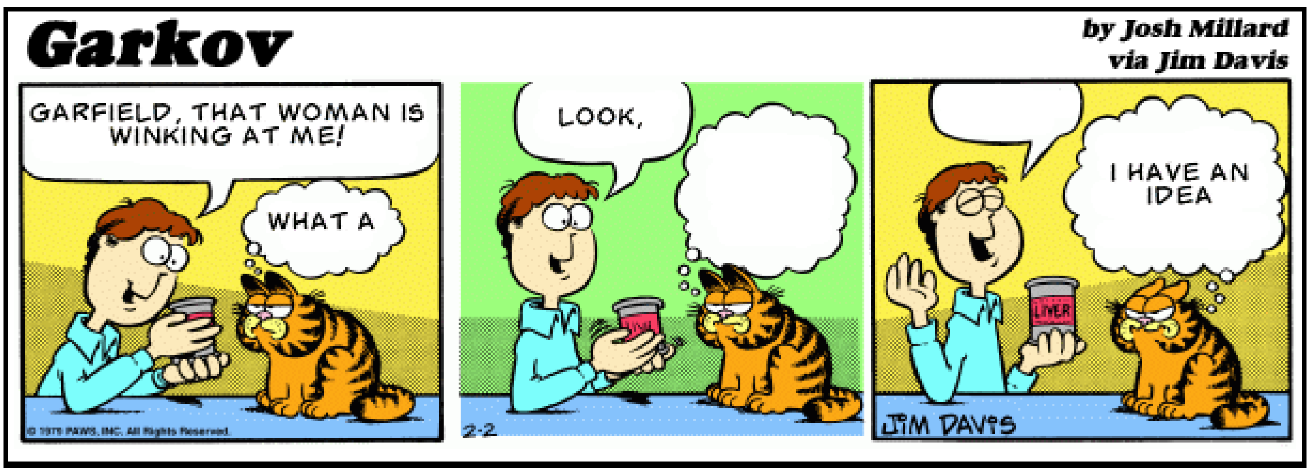
Can model longer-range dependencies

Zero-Order: $P(X_t)$ *First-Order:* $P(X_t \| X_{t-1})$ *Second-Order:* $P(X_t \| X_{t-1}, X_{t-2})$	

One we've learned an nth order Markov chain, we can use it to generate text!

Example: [Garkov Chain (http://joshmillard.com/garkov/)](http://joshmillard.com/garkov/)





Markov Models: Transition Matrix

Suppose $X_t \in \{1, \ldots, K\}$ is discrete. Then, a stationary chain with N states can be described by a **transition matrix**, $A \in \mathbb{R}^{N \times N}$ where

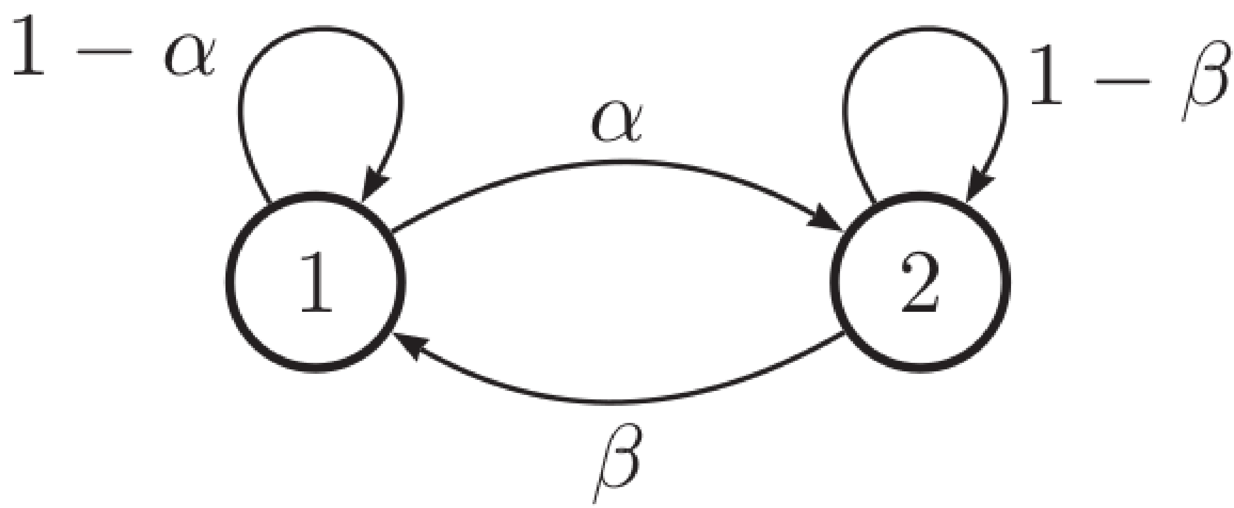
$$a_{ij} = p(X_t = j \mid X_{t-1} = i)$$

is the probability of transitioning from state i to j.

> Each row sums to one, $\sum_{j=1}^{K} A_{ij} = 1$, so A is a *row-stochastic matrix*.

Markov Models: Transition Diagram

Transitions between states can be represented as a graph:



> This is *not* a Bayesian network!

(Figure 17.4b from **[MLAPP]**)

Markov Models: State Vectors

Consider a row vector $x_t \in \mathbb{R}^{K\times 1}$ with entries $x_{tj} = p(X_t = j)$. Then,

$$\begin{aligned} p(X_t = j) &= \sum_{i=1}^{K} p(X_t = j \mid X_{t-1} = i) p(X_{t-1} = i) \\ &= \sum_{i=1}^{K} A_{ij} x_{t-1,i} \end{aligned}$$

Therefore, we conclude $x_t = x_{t-1}A$. Note $\sum_{j=1}^{K} x_{tj} = 1$.

Markov Models: Matrix Powers

Since $x_t = x_{t-1}A$, this suggests that in general,

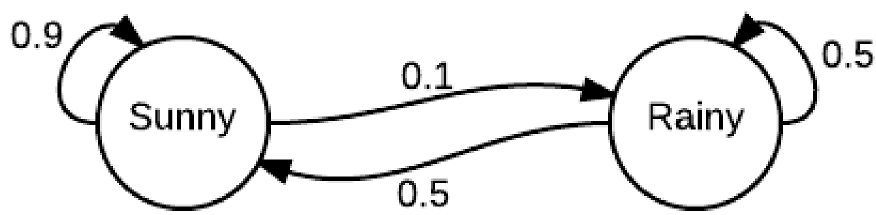
$$x_t = x_{t-1}A = x_{t-2}A^2 = \cdots x_0 A^t$$

If we know the initial state probabilities x_0, we can find the probabilities of landing in any state at time $t > 0$.

Example: Weather

Suppose the weather is either $R = \text{Rainy}$ or $S = \text{Sunny}$,

$$A = \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}$$



(Taken from [Wikipedia](https://en.wikipedia.org/wiki/Examples_of_Markov_chains#A_very_simple_weather_model))

Example: Weather

Suppose today is sunny, $x_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}$. We can predict tomorrow's weather,

$$x_1 = x_0 A = \begin{bmatrix} 0.9 & 0.1 \end{bmatrix}$$

The weather over the next several days will be

$$\begin{aligned} x_2 &= x_1 A = \begin{bmatrix} 0.86 & 0.14 \end{bmatrix} \\ x_3 &= x_2 A = \begin{bmatrix} 0.844 & 0.156 \end{bmatrix} \\ x_4 &= x_3 A = \begin{bmatrix} 0.8376 & 0.1624 \end{bmatrix} \end{aligned}$$

> **Question:** What happens to $x_0 A^n$ as $n \to \infty$?

Markov Chains: Stationary Distribution

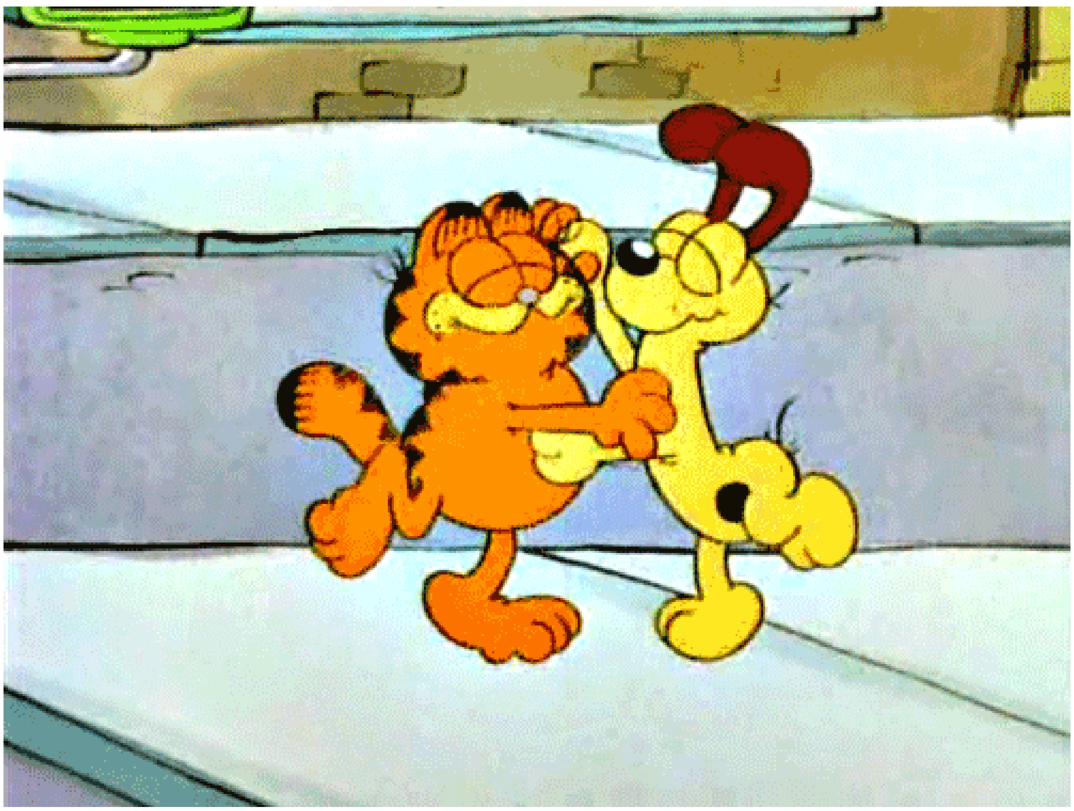
If we ever reach a stage x where

$$x = xA$$

then we have reached the **stationary distribution** of the chain.

- To find $x = v^T$, solve the eigenvalue problem $A^T v = v$
- Under certain conditions, the limiting distribution $\lim_{n \to \infty} x_0 A^n = x$
- Stationary distribution x does not depend on the starting state x_0

Break time!



Hidden Markov Models

> Uses material from **[MLAPP]**

Hidden Markov Models

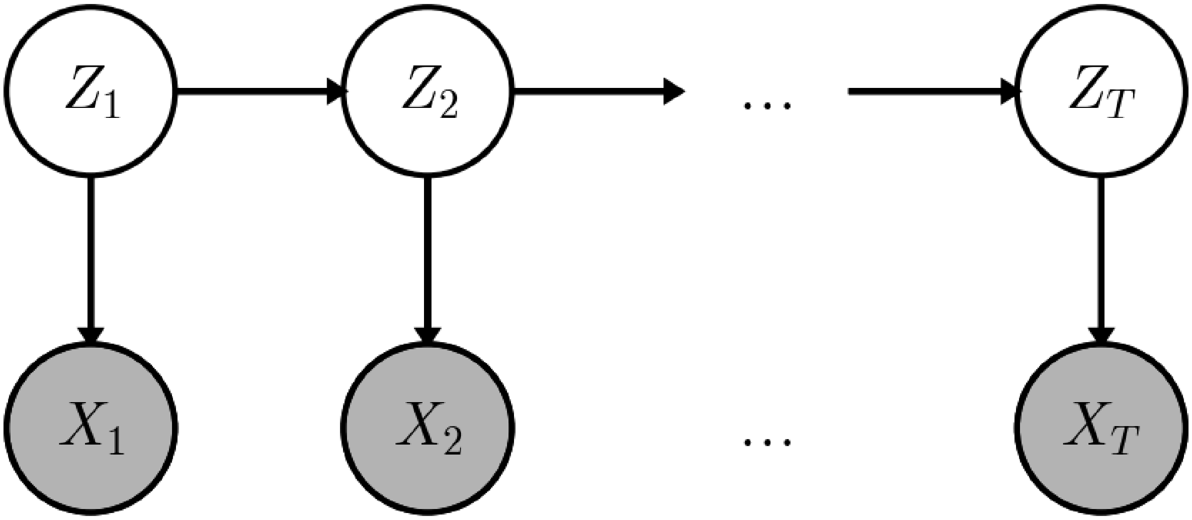
Noisy observations X_k generated from *discrete* hidden Markov chain Z_k.

$$P(\mathbf{X}, \mathbf{Z}) = P(Z_1)P(X_1 \mid Z_1) \prod_{k=2}^{T} P(Z_k \mid Z_{k-1})P(X_k \mid Z_k)$$

In [31]:
```
@pgm_render
def pgm_hmm():
    pgm = daft.PGM([7, 7], origin=[0, 0])

    # Nodes
    pgm.add_node(daft.Node("Z1", r"$Z_1$", 1, 3.5))
    pgm.add_node(daft.Node("Z2", r"$Z_2$", 2, 3.5))
    pgm.add_node(daft.Node("Z3", r"$\dots$", 3, 3.5, plot_params=
{'ec':'none'}))
    pgm.add_node(daft.Node("Z4", r"$Z_T$", 4, 3.5))

    pgm.add_node(daft.Node("x1", r"$X_1$", 1, 2.5, observed=True))
    pgm.add_node(daft.Node("x2", r"$X_2$", 2, 2.5, observed=True))
    pgm.add_node(daft.Node("x3", r"$\dots$", 3, 2.5, plot_params=
{'ec':'none'}))
    pgm.add_node(daft.Node("x4", r"$X_T$", 4, 2.5, observed=True))


    # Add in the edges.
    pgm.add_edge("Z1", "Z2", head_length=0.08)
    pgm.add_edge("Z2", "Z3", head_length=0.08)
    pgm.add_edge("Z3", "Z4", head_length=0.08)

    pgm.add_edge("Z1", "x1", head_length=0.08)
    pgm.add_edge("Z2", "x2", head_length=0.08)
    pgm.add_edge("Z4", "x4", head_length=0.08)

    return pgm;
```

In [32]:
```
%%capture
pgm_hmm("images/pgm/hmm.png")
```

Out[32]:



HMM: Parameters

For a Hidden Markov Model with N hidden states and M observed states, there are three *row-stochastic* parameters $\theta = (A, B, \pi)$,

- Transition matrix $A \in \mathbb{R}^{N \times N}$

$$A_{ij} = p(Z_t = j | Z_{t-1} = i)$$

- Emission matrix $B \in \mathbb{R}^{N \times M}$

$$B_{jk} = p(X_t = k | Z_t = j)$$

- Initial distribution $\pi \in \mathbb{R}^N$,

$$\pi_j = p(Z_1 = j)$$

HMM: Filtering Problem

Filtering means to compute the current *belief state* $p(z_t|x_1,\ldots,x_t,\theta)$.

$$p(z_t|x_1,\ldots,x_t) = \frac{p(x_1,\ldots,x_t,z_t)}{p(x_1,\ldots,x_t)}$$

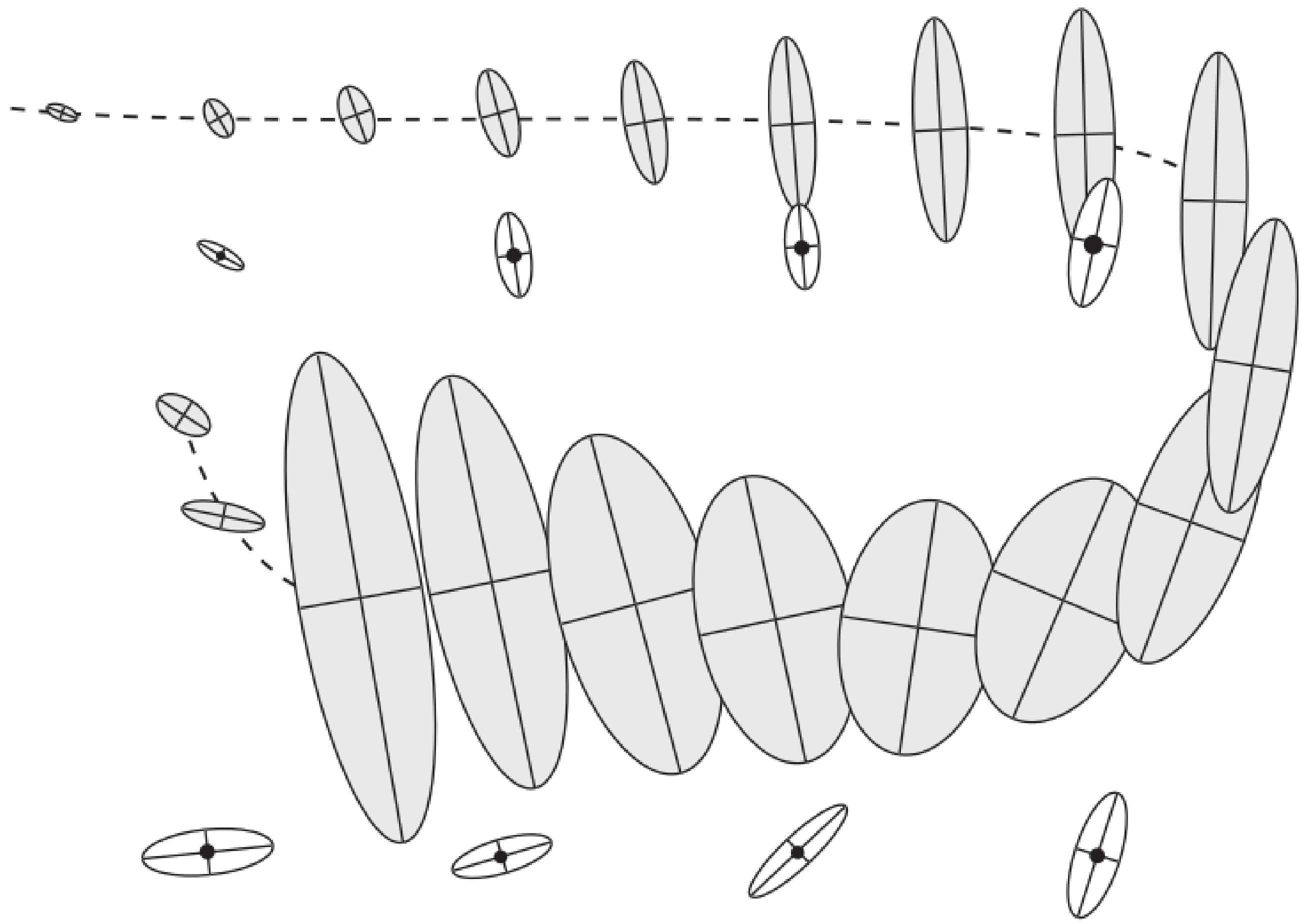
- Given observations $x_{1:t}$ so far, infer z_t.
- *Example*: Estimate robot position given previous sensor readings.

> Solved by the **forward algorithm**.

HMM: Filtering Problem

As an example of filtering, consider **robot localization**, in which we

- estimate the current location z_t
- given **noisy** sensor readings $x_1, \ldots, x_t$.



(Figure 18.3a from **[MLAPP]**)

HMM: Forward Algorithm

The **forward algorithm** computes $\alpha_t(z_t) \equiv p(x_1, \dots, x_t, z_t)$.

$$\begin{aligned}
\alpha_t(z_t) &= \sum_{z_{t-1}} p(x_1, \dots, x_t, z_{t-1}, z_t) \\
&= \sum_{z_{t-1}} p(x_1, \dots, x_{t-1}, z_{t-1}) p(z_t | z_{t-1}) p(x_t | z_t) \\
&= p(x_t | z_t) \sum_{z_{t-1}} \alpha_{t-1}(z_{t-1}) p(z_t | z_{t-1}) \\
&= B_{z_t, x_t} \sum_{z_{t-1}} \alpha_{t-1}(z_{t-1}) A_{z_{t-1}, z_t}
\end{aligned}$$

> Compute recursively starting from the *front* of the chain.

HMM: Smoothing Problem

Compute $p(z_t | \mathcal{X})$ offline, given all observations.

- Retroactively infer z_t. (Hindsight!)

We can break the chain into two parts, the *past* and *future*:

$$\begin{aligned}
p(z_t | \mathcal{X}) &= p(x_{1:t}, z_t, x_{t+1:T}) \frac{1}{p(\mathcal{X})} \\
&= p(x_{1:t}, z_t) p(x_{t+1:T} | x_{1:t}, z_t) \frac{1}{p(\mathcal{X})} \\
&= p(x_{1:t}, z_t) p(x_{t+1:T} | z_t) \frac{1}{p(\mathcal{X})}
\end{aligned}$$

HMM: Smoothing Problem

Overall, the **smoothing problem** is to compute

$$
\begin{aligned}
\gamma_t(j) \equiv p(z_t = j \mid \mathcal{X}) &= \frac{p(\mathcal{X} \mid z_t = j)p(z_t = j)}{p(\mathcal{X})} \\
&= \frac{p(\mathcal{X}_{1:t} \mid z_t = j)p(\mathcal{X}_{t+1:T}|z_t = j)p(z_t = j)}{p(\mathcal{X})} \\
&= \frac{\alpha_t(j)\beta_t(j)}{p(\mathcal{X})} = \frac{\alpha_t(j)\beta_t(j)}{\sum_k \alpha_t(k)\beta_t(k)}
\end{aligned}
$$

It is an easy exercise to check that, for any t, $p(\mathcal{X}) = \sum_k \alpha_t(k)\beta_t(x)$.

We solve this via the **forward-backward algorithm**, where

- $\alpha_t(z_t) \equiv p(x_1, \ldots, x_t, z_t)$ is found with the **forward algorithm**
- $\beta_t(z_t) \equiv p(x_{t+1}, \ldots, x_T|z_t)$ is found with the **backward algorithm**

HMM: Backward Algorithm

The **backward algorithm** computes $\beta_t(z_t) \equiv p(x_{t+1}, \ldots, x_T|z_t)$,

$$
\begin{aligned}
\beta(z_t) &= \sum_{z_{t+1}} p(x_{t+1}, \ldots, x_T, z_{t+1}|z_t) \\
&= \sum_{z_{t+1}} p(z_{t+1}|z_t)p(x_{t+1}, \ldots, x_T|z_{t+1}, z_t) \\
(\text{since } x_{t+1:T} \perp z_t \mid z_{t+1}) \quad &= \sum_{z_{t+1}} p(z_{t+1}|z_t)p(x_{t+1}, \ldots, x_T|z_{t+1}) \\
(\text{since } x_{t+2:T} \perp x_{t+1} \mid z_{t+1}) \quad &= \sum_{z_{t+1}} p(z_{t+1}|z_t)p(x_{t+1}|z_{t+1})p(x_{t+2}, \ldots, x_T|z_{t+1}) \\
&= \sum_{z_{t+1}} p(z_{t+1}|z_t)p(x_{t+1}|z_{t+1})\beta_{t+1}(z_{t+1}) \\
&= \sum_{z_{t+1}} A_{z_t, z_{t+1}} B_{z_{t+1}, x_{t+1}} \beta_{t+1}(z_{t+1})
\end{aligned}
$$

> Compute recursively starting from the *back* of the chain.

HMM: Decoding Problem

Decoding computes the most probable state sequence, given observations.

$$\mathbf{z}^* = \arg\max_{z_1,\ldots,z_T} p(z_1,\ldots,z_T|x_1,\ldots,x_T,\theta)$$

> The decoding problem is solved by the **Viterbi algorithm**, which uses dynamic programming. See **[PRML]** or **[MLAPP]** for more details.

Viterbi Algorithm

- Viterbi is another recursive procedure for computing the most likely sequence of hidden states.
- We define $V_t(z_t)$ to be the probability of the most likely sequence of states up to time t *that ended in state* z_t, given observed data
- These $V(z_t)$ values satisfies the recursion:

$$\begin{aligned} V_1(j) &= p(x_1 \mid z_1 = j) = B_{x_1,j} \\ V_{t+1}(j) &= p(x_{t+1} \mid z_{t+1} = j) \max_k \left\{ V_t(k) p(z_{t+1} = j \mid z_t = k) \right\} \\ &= B_{x_{t+1},j} \max_k \left\{ V_t(k) A_{j,k} \right\} \end{aligned}$$

- Can be computed very quickly!

HMM: Part-of-Speech Tagging

In English, some words can have multiple parts of speech. For instance,

- Business is going **well** (*Adverb*).
- All is **well** with us (*Adjective*).
- **Well**, who would have thought he could do it? (*Interjection*)
- The **well** was drilled fifty meters deep. (*Noun*)
- Tears **well** up in my eyes. (*Verb*)

(Example taken from [here](http://english.stackexchange.com/questions/46277/what-word-can-fulfill-the-most-parts-of-speech).)

HMM: Part-of-Speech Tagging

We can use a Hidden Markov Model to **disambiguate** the part of speech using context clues!

- Hidden states z_t are parts-of-speech
- Observed states x_t are words

Certain sequences of POS tags are unlikely. This allows us to infer the correct tags!

HMM: Learning Problem

It is usually necessary to **learn** the model parameters $\theta = (A, B, \pi)$ from data.

- Given observations $\mathcal{X} = \{x_1, \dots, x_T\}$
- Given model dimensions N and M
- Find parameters that best fit the data

> The learning problem is solved by the **Baum-Welch algorithm**, a special case of expectation maximization.

Recall: Expectation-Maximization

E-Step: Write down an expression for

$$Q(\theta_t, \theta) = E_q[\log p(\mathcal{X}, Z|\theta)] \qquad \text{where } q = q(\cdot|\theta_t)$$

M-Step: Maximize the auxiliary function,

$$\theta_{t+1} = \arg\max_{\theta} Q(\theta_t, \theta)$$

> Recall $q_t(Z) = p(Z|\mathcal{X}, \theta_t)$

HMM: Complete-Data Log-Likelihood

The joint likelihood of the hidden and observed states is

$$\begin{aligned}\log p(x_{1:T}, z_{1:T}|\theta) &= \log\left[p(z_1|\pi)\prod_{t=2}^{T} p(z_t|z_{t-1}, A)\prod_{t=1}^{T} p(x_t|z_t, B)\right] \\ &= \log p(z_1|\pi) + \sum_{t=2}^{T} \log p(z_t|z_{t-1}, A) \\ &\quad + \sum_{t=1}^{T} \log p(x_t|z_t, B)\end{aligned}$$

HMM: Complete-Data Log-Likelihood

Each term of the complete-data log-likelihood is:

$$\begin{aligned}\log p(z_1|\pi) &= \sum_{j=1}^{N} \mathbb{I}(z_t = j)\log \pi_j \\ \log p(z_t|z_{t-1}, A) &= \sum_{i=1}^{N}\sum_{j=1}^{N} \mathbb{I}(z_{t-1} = i)\mathbb{I}(z_t = j)\log A_{ij} \\ \log p(x_t|z_t, B) &= \sum_{j=1}^{N} \mathbb{I}(z_t = j)\log B_{j,x_t}\end{aligned}$$

HMM: Expected Complete Likelihood

The expected complete likelihood $Q(\theta_t, \theta)$ is

$$\begin{aligned}Q(\theta_t, \theta) &= E_q[\log p(\mathcal{X}, Z|\theta)] \\ &= E_q[\log p(z_1|\pi)] + E_q\left[\sum_{t=2}^{T} \log p(z_t|z_{t-1}, A)\right] \\ &\quad + E_q\left[\sum_{t=1}^{T} \log p(x_t|z_t, B)\right]\end{aligned}$$

HMM: Expected Complete Likelihood

Fixing $t > 1$, and taking expectations with respect to $q(Z) = p(Z|\mathcal{X}, \theta)$,

$$E_q[\log p(z_1|\pi)] = \sum_{j=1}^{N} q(z_1 = j) \log \pi_j$$

$$E_q[\log p(z_t|z_{t-1}, A)] = \sum_{i=1}^{N} \sum_{j=1}^{N} q(z_{t-1} = i, z_t = j) \log A_{ij}$$

$$E_q[\log p(x_t|z_t, B)] = \sum_{j=1}^{N} q(z_t = j) \log B_{j,x_t}$$

HMM: Baum-Welch

- The **E-Step** consists of computing the q terms from the previous slide, which can all be computed using the **forward-backward algorithm**!
- The **M-Step** consists of normalizing the expected transition and emission counts
  - Similar to MLE for complete data
  - Requires some careful calculations. See details on next slide.

You must compute the key quantities for $q()$ using the forward-backward algorithm. In the E step you can assume the parameters $\theta = (A, B, \pi)$, so we'll drop dependence on θ.

1. First note that $q(z_t = j) = p(z_t = j \mid \mathcal{X}, \theta) = \gamma_t(j)$ which we computed above.
2. Our last step is to compute $q(z_{t-1} = i, z_t = j) = p(z_{t-1} = i, z_t = j | \mathcal{X}, \theta)$. Again, dropping dependence on θ, we have

$$
\begin{aligned}
p(z_{t-1}, z_t | \mathcal{X}, \theta) &= \frac{p(\mathcal{X} \mid z_t, z_{t-1}) p(z_t, z_{t-1})}{p(\mathcal{X})} \\
&= \frac{p(\mathcal{X}_{1:t-1} \mid z_t, z_{t-1}) p(\mathcal{X}_{t:T} | z_t, z_{t-1}) p(z_t, z_{t-1})}{p(\mathcal{X})} \\
&= \frac{p(\mathcal{X}_{1:t-1} \mid z_{t-1}) p(\mathcal{X}_{t:T} | z_t) p(z_t, z_{t-1})}{p(\mathcal{X})} \\
&= \frac{p(\mathcal{X}_{1:t-1}, z_{t-1}) p(x_t \mid z_t) p(\mathcal{X}_{t+1:T} | z_t) p(z_t, z_{t-1})}{p(z_{t-1}) p(\mathcal{X})} \\
&= \frac{\alpha_{t-1}(z_{t-1}) \beta_t(z_t) p(z_t \mid z_{t-1}) p(x_t \mid z_t)}{p(\mathcal{X})} \\
&= \frac{\alpha_{t-1}(z_{t-1}) \beta_t(z_t) A_{z_{t-1}, z_t} B_{z_t, x_t}}{\sum_k \alpha_t(k) \beta_t(k)}
\end{aligned}
$$