ishes rapidly as the distance between ω and ω_0 increases (see Figure 31.4). Hence this integral supports a rule of thumb stating that a $20N$ db/decade rate of gain decrease in the vicinity of frequency ω_0 implies that $\angle L(j\omega_0) \approx -90N^\circ$. Most transfer functions are sufficiently well behaved that the value of phase at a frequency is largely determined by that of the gain over a decade-wide interval centered at the frequency of interest [11].

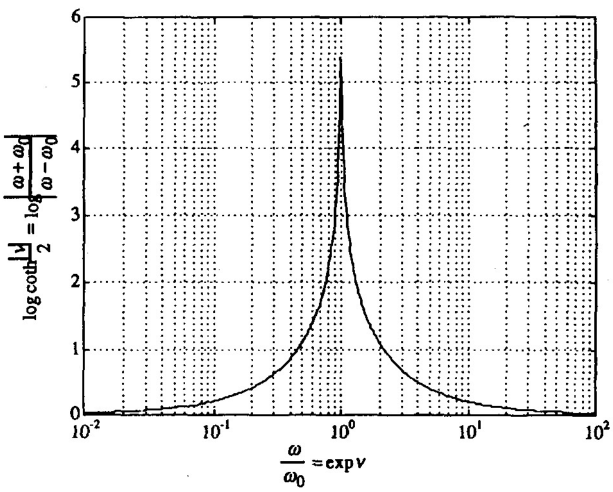

Figure 31.4 Weighting function in gain-phase integral.

The Bode gain-phase relation may be used to assess whether a design specification of the type shown in Figure 31.3 is achievable. Since a $20N$ db/decade rate of gain decrease in the vicinity of crossover implies that phase at crossover is roughly $-90N^\circ$, it follows that the rate of gain decrease cannot be much greater than 20 db/decade if the Nyquist stability criterion is to be satisfied and if an acceptable phase margin is to be maintained. One implication of this fact is that the frequency ω_L in Figure 31.3 cannot be too close to the frequency ω_H. Hence, the frequency range over which loop gain can be large to obtain sensitivity reduction is limited by the need to ensure stability robustness against uncertainty at higher frequencies, and to maintain reasonable feedback properties near crossover. As discussed in [2] and [3], relaxing the assumption that $L(s)$ has no right-half plane poles or zeros does not lessen the severity of this tradeoff. Indeed, the tradeoff only becomes more difficult to accomplish. If one is willing to accept a system that is only conditionally stable, Horowitz [2] claims that larger values of low-frequency gain may be obtained.

31.4.3 The Bode Sensitivity Integral

The purpose of this section is to present and discuss the constraint imposed by stability on the sensitivity function. This constraint was first developed in the context of feedback systems in [1]. This integral quantifies a tradeoff between sensitivity reduction and sensitivity increase which must be performed whenever the open-loop transfer function has at least two more poles than zeros.

The magnitude of the sensitivity function of a scalar feedback system can be obtained easily using a Nyquist plot of $L(j\omega)$. Indeed, since $S(j\omega) = 1/[1 + L(j\omega)]$, the magnitude of the sensitivity function is just the reciprocal of the distance from the Nyquist plot to the critical point. In particular, sensitivity is less than one at frequencies for which $L(j\omega)$ is outside the unit circle centered at the critical point. Sensitivity is greater than one at frequencies for which $L(j\omega)$ is inside this unit circle.

To motivate existence of the integral constraint, consider the open-loop transfer function $L(s) = \frac{2}{(s+1)^2}$. As shown in Figure 31.5, there exists a frequency range over which the Nyquist plot of $L(j\omega)$ penetrates the unit circle and sensitivity is thus greater than one. In practice, the open-loop transfer function will generally have at least two more poles than zeros [2]. If $L(s)$ is stable then, using the gain-phase relation (Equation 31.30), it is straightforward to show that $L(j\omega)$ will asymptotically have phase lag at least -180°. Hence, there will always exist a frequency range over which sensitivity is greater than one. This behavior may be quantified using a classical theorem due to Bode [1], which was extended in [4] to allow unstable poles in the loop transfer function.

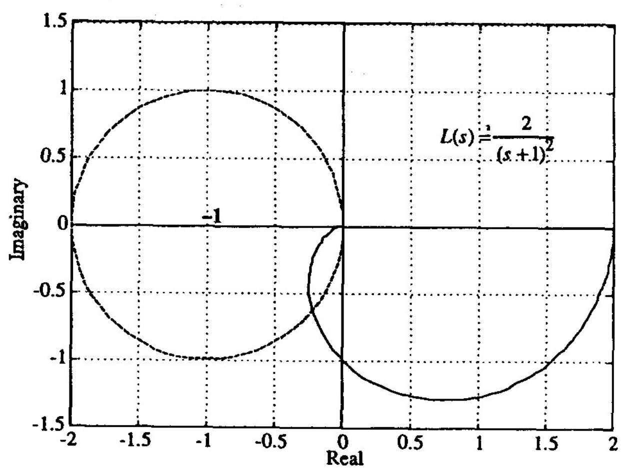

Figure 31.5 Effect of a two-pole rolloff upon the Nyquist plot.

THEOREM 31.2 *(Bode Sensitivity Integral): Suppose that the open-loop transfer function $L(s)$ is rational and has right-half plane poles $\{p_i : i = 1, \ldots, N_p\}$, with multiple poles included according to their multiplicity. If $L(s)$ has at least two more poles than zeros, and if the associated feedback system is stable, then the sensitivity function must satisfy*[2]

$$\int_0^\infty \log|S(j\omega)|d\omega = \pi \sum_{i=1}^{N_p} Re[p_i] \tag{31.32}$$

[2] Throughout this chapter, the function log(*) will be used to denote the natural logarithm.

This theorem shows that a tradeoff exists between sensitivity properties in different frequency ranges. Indeed, for stable open-loop systems, the area of sensitivity reduction must equal the area of sensitivity increase on a plot of the logarithm of the sensitivity versus linear frequency (see Figure 31.6). In this respect, the benefits and costs of feedback are balanced exactly.

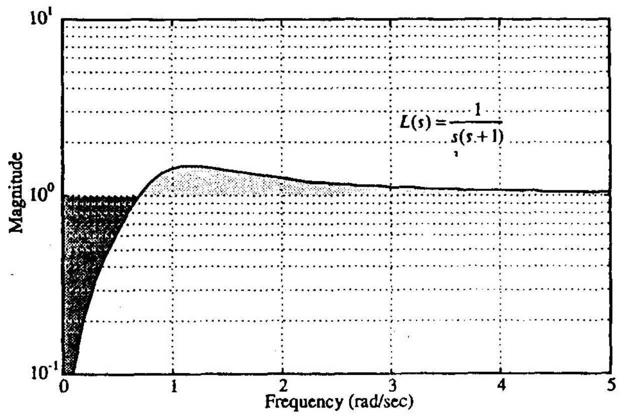

Figure 31.6 Areas of sensitivity reduction (dark gray) and sensitivity increase (light gray).

The extension of Bode's theorem to open-loop unstable systems shows that the area of sensitivity increase exceeds that of sensitivity reduction by an amount proportional to the distance from the unstable poles to the left-half plane. A little reflection reveals that this additional sensitivity increase is plausible for the following reason. When the system is open-loop unstable, then it is obviously necessary to use feedback to achieve closed-loop stability, as well as to obtain sensitivity reduction. One might expect that this additional benefit of feedback would be accompanied by a certain cost, and the integral (Equation 31.32) substantiates that hypothesis. Alternatively, we could interpret Equation 31.32 as implying that the area of sensitivity reduction must be less than that of sensitivity increase, thus indicating that a portion of the open-loop gain which could otherwise contribute to sensitivity reduction must instead be used to pull the unstable poles into the left-half plane.

By itself, the tradeoff quantified by Equation 31.32 does not impose a meaningful design limitation. Although it is true that requiring a large area of sensitivity reduction over a low-frequency interval implies that an equally large area of sensitivity increase must be present at higher frequencies, it does not follow that there must exist a peak in sensitivity which is bounded greater than one. It is possible to achieve an arbitrary large area of sensitivity increase by requiring $|S(j\omega)| = 1 + \delta, \forall\omega \in [\omega_1, \omega_2]$, where δ can be chosen arbitrarily small and the interval (ω_1, ω_2) is adjusted to be sufficiently large.

The analysis in the preceding paragraph ignores the effect of limitations upon system bandwidth that are always present in a practical design. For example, it is almost always necessary to decrease open-loop gain at high frequencies to maintain stability robustness against large modeling errors due to unmodeled dynamics. Small open-loop gain is also required to prevent sensor noise from appearing at the system output. Finally, requiring open-loop gain to be large at a frequency for which plant gain is small may lead to unacceptably large response of the plant input to noise and disturbances. Hence the natural bandwidth of the plant also imposes a limitation upon open-loop bandwidth.

One or more of the bandwidth constraints just cited is usually present in any practical design. It is reasonable, therefore, to assume that open-loop gain must satisfy a frequency-dependent bound of the form

$$|L(j\omega)| \le \varepsilon \left(\frac{\omega_c}{\omega}\right)^{1+k} \quad \forall\omega \ge \omega_c \tag{31.33}$$

where $\varepsilon < 1/2$ and $k > 0$. This bound imposes a constraint upon the rate at which loop gain rolls off, as well as the frequency at which rolloff commences and the level of gain at that frequency.

When a bandwidth constraint such as Equation 31.33 is imposed, it is obviously not possible to require the sensitivity function to exceed one over an arbitrarily large frequency interval. When Equation 31.33 is satisfied, there is an upper bound on the area of sensitivity increase which can be present at frequencies greater than ω_c. The corresponding limitation imposed by the sensitivity integral (Equation 31.32) and the rolloff constraint (Equation 31.33) is expressed by the following result [5].

COROLLARY 31.1 Suppose, in addition to the assumptions of Theorem 31.2, that $L(s)$ satisfies the bandwidth constraint (Equation 31.33). Then the tail of the sensitivity integral must satisfy

$$\left|\int_{\omega_c}^{\infty} \log|S(j\omega)|d\omega\right| \le \frac{3\varepsilon\omega_c}{2k} \tag{31.34}$$

The bound defined by Equation 31.34 implies that the sensitivity tradeoff imposed by the integral (Equation 31.32) must be accomplished primarily over a finite frequency interval. As a consequence, the amount by which $|S(j\omega)|$ must exceed one cannot be arbitrarily small. Suppose that the sensitivity function is required to satisfy the upper bound

$$|S(j\omega)| \le \alpha < 1 \quad \forall\omega \le \omega_l < \omega_c \tag{31.35}$$

If the bandwidth constraint (Equation 31.33) and the sensitivity bound (Equation 31.35) are both satisfied, then the integral constraint (Equation 31.32) may be manipulated to show [5] that

$$\sup_{\omega\in(\omega_l,\omega_c)} \log|S(j\omega)| \ge \frac{1}{\omega_c - \omega_l} \left\{\pi \sum_{i=1}^{N_p} \mathrm{Re}[p_i] + \omega_l \cdot \log\left(\frac{1}{\alpha}\right) - \frac{3\varepsilon\omega_c}{2k}\right\} \tag{31.36}$$

The bound given in Equation 31.36 shows that increasing the area of low-frequency sensitivity reduction by requiring α to be very small or ω_l to be very close to ω_c, will necessarily cause a