oblig_1_Trym_Erik_Nielsen

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1 Obligatory assignment - MEK4250 - *Trym Erik Nielsen*

1.1 Required imports

```
In [10]: from dolfin import *
         import numpy as np
         import pandas as pd
         from matplotlib import pyplot as plt
         from IPython.display import display
         %matplotlib inline
```

1.2 Boundary class definitions

```
In [11]: class Right(SubDomain):
             def inside(self, x, on_boundary, eps=1.e-14):
                 return x[0] > 1 - eps


         class Left(SubDomain):
             def inside(self, x, on_boundary, eps=1.e-14):
                 return x[0] < eps
```

1.3 Problem specific solver functions

```
In [18]: def solver_prob_1(N=8, deg=1, k=1):
             mesh = UnitSquareMesh(N, N)
             V = FunctionSpace(mesh, 'P', deg)

             #numerical and analytical expressions
             u_e = Expression('sin(pi*k*x[0])*cos(pi*k*x[1])', degree=deg, k=k)
             u_num = Function(V)

             #Boundary conditions
             bcs = [DirichletBC(V, Constant(0.0), Left()),
                    DirichletBC(V, Constant(0.0), Right())]

             u = TrialFunction(V)
```

```python
    v = TestFunction(V)

    f = Expression('2*pi*pi*k*k*sin(pi*k*x[0])*cos(pi*k*x[1])',degree=deg, k=k)
    a = dot(grad(u), grad(v)) * dx
    L = f * v * dx

    u = Function(V)
    solve(a == L, u, bcs)

    return mesh, u, V


def solver_prob_2(N=8, mu=1, deg=1, galerkin=False):

    mesh = UnitSquareMesh(N, N)
    V = FunctionSpace(mesh, 'P', deg)
    u_num = Function(V)

    bcs = [
        DirichletBC(V, Constant(0.0), Left()),
        DirichletBC(V, Constant(1.0), Right()),
    ]

    u = TrialFunction(V)
    v = TestFunction(V)

    if galerkin:
        beta = 0.5 * mesh.hmin()
        v   = v + beta * v.dx(0)

    f = Constant(0.0)
    g = Constant(0.0)

    a = mu * inner(grad(u), grad(v)) * dx + u.dx(0) * v * dx
    L = f * v * dx + g * v * ds

    solve(a==L, u_num, bcs)

    return u_num, V, mesh
```

1.4 Problem functions

We can now define problem functions that call our problem specific solvers. We use pandas dataframes to display the errornorms for different frequencies and mesh refinements using the FEniCS built in errornorm function. Since errornorm does not support the L_∞ errornorm, we project our expression for the exact solution onto our subspace and take the maximum of the absolute value of the difference with the numerical solution.

We also define the error fitting function that creates a linear best-fit between our error norms

L_2 and H_1

```
In [27]: def error_fit(L_2, H_1):
             N = L_2.index.values
             param = L_2.columns.values
             best_fit_coeffs = pd.DataFrame(index=param, columns=['alpha', 'C_alpha', 'beta', 'C
             h_log = [np.log(1.0 / n) for n in N]
             L_2_log = L_2.applymap(np.log)
             H_1_log = H_1.applymap(np.log)

             plt.plot(h_log, L_2_log, 'r', h_log, H_1_log, 'b')
             plt.title('LogLog plot of error norm VS cell size')
             plt.xlabel('log(h)')
             plt.ylabel('log(L_2) / Log(H_1)')
             plt.show()

             for p in param:
                 L_2_fit = np.polyfit(h_log, L_2_log[p], deg=1)
                 H_1_fit = np.polyfit(h_log, H_1_log[p], deg=1)

                 L_2_fit[1] = np.exp(L_2_fit[1])
                 H_1_fit[1] = np.exp(H_1_fit[1])

                 best_fit_coeffs.loc[p] = list(L_2_fit) + list(H_1_fit)


             return best_fit_coeffs

         def prob_1(deg=1):

             N = [8, 16, 32, 64]
             freqs = [1, 2, 4,8]
             L_2_error = pd.DataFrame(index=N, columns=freqs)
             L_inf_error = pd.DataFrame(index=N, columns=freqs)
             H_1_error = pd.DataFrame(index=N, columns=freqs)


             for n in N:
                 for freq in freqs:
                         mesh,u_num,V = solver_prob_1(N=n, deg=deg, k=freq)
                         u_ex = Expression('sin(k*pi*x[0])*cos(k*pi*x[1])', k=freq, degree=deg)

                         L_2 = errornorm(u_ex, u_num, 'l2', degree_rise=1)
                         H_1 = errornorm(u_ex, u_num, 'H1', degree_rise=1)

                         u_ex_values = interpolate(u_ex, V)
                         u_ex_vector = u_ex_values.vector()
                         u_num_vector = u_num.vector()
```

```
                L_inf = max(np.abs(u_ex_vector - u_num_vector))

                L_2_error.at[n, freq] = L_2
                L_inf_error.at[n, freq] = L_inf
                H_1_error.at[n, freq] = H_1


    print('----- problem 1 - part b -------' + '\n')
    print('L2 error norm' + '\n')
    display(L_2_error)
    print('L_inf error norm' + '\n')
    display(L_inf_error)
    print('H1 error norm' + '\n')
    display(H_1_error)
    print('\n\n\n')
    print('----- problem 1 - part c -------' + '\n')
    print('Best fit parameters')
    display(error_fit(L_2_error, H_1_error))

    #plot highest accuracy mesh and u
    if deg > 1:
        p = plot(u_num)
        p.set_cmap('plasma')
        p.set_clim(0.0, 1.0)
        plt.title('u solution')
        plt.colorbar(p)
        plt.show()


def prob_2(deg=1, galerkin=False):
    mu_values = [0.1, 0.3, 1.0]
    N = [8, 16, 32, 64]

    L_2_error = pd.DataFrame(index=N, columns=mu_values)
    L_inf_error = pd.DataFrame(index=N, columns=mu_values)
    H_1_error = pd.DataFrame(index=N, columns=mu_values)

    for mu in mu_values:
        for n in N:
            u_num, V, omega = solver_prob_2(N=n, mu=mu, deg=deg, galerkin=galerkin)
            u_ex = Expression('(exp(1 / mu * x[0]) - 1) / (exp(1 / mu) - 1)', mu=mu, de

            L_2 = errornorm(u_ex, u_num, 'L2', degree_rise=3)
            H_1 = errornorm(u_ex, u_num, 'H1', degree_rise=3)
            u_ex_values = interpolate(u_ex, V)
            u_ex_vector = u_ex_values.vector()
            u_num_vector = u_num.vector()
            L_inf = max(np.abs(u_ex_vector - u_num_vector))
```

```
            L_2_error.at[n, mu] = L_2
            L_inf_error.at[n, mu] = L_inf
            H_1_error.at[n, mu] = H_1

    print('----- problem 2 - part b -------' + '\n')
    print('L2 error norm' + '\n')
    display(L_2_error)
    print('L_inf error norm' + '\n')
    display(L_inf_error)
    print('H1 error norm' + '\n')
    display(H_1_error)
    print('\n\n\n')
    print('----- problem 2 - part c -------' + '\n')
    print('Best fit parameters')
    display(error_fit(L_2_error, H_1_error))

    if deg >= 1:
        p = plot(u_num)
        p.set_cmap('plasma')
        p.set_clim(0.0, 1.0)
        plt.title('u solution')
        plt.colorbar(p)
        plt.show()
```

1.4.1 H^P norm of u

The H^P norm of the function u can be expressed as

$$|| \, u \, ||_P = \left(\sum_{|\alpha| \leq p} \int_{\Omega} (\frac{\partial^{|\alpha|} u}{\partial x^{\alpha}})^2 \, dx \right)^{1/2}$$

with $\alpha = (i, j)$ in the two dimensional Cartesian space. Using our assumed u, and the fact that $f = -\Delta u = 2\pi^2 k^2 u$, we find

$$|| \, u \, ||_P = \frac{1}{2} \Big(\sum_{|\alpha| \leq p} (k\pi)^{2|\alpha|} \Big)^{1/2}$$

Since $\int_0^1 \sin(\pi k x) = \int_0^1 \cos(\pi k y) = \frac{1}{2}$

1.4.2 Problem 1, Lagrange elements order 1

In [28]: prob_1()

```
----- problem 1 - part b -------


L2 error norm
```

	1	2	4	8
8	0.0327134	0.126466	0.303989	0.421637
16	0.00845879	0.0360234	0.124987	0.298277
32	0.00213295	0.009314	0.0358724	0.124795
64	0.000534392	0.00234834	0.00929057	0.0359663

L_inf error norm

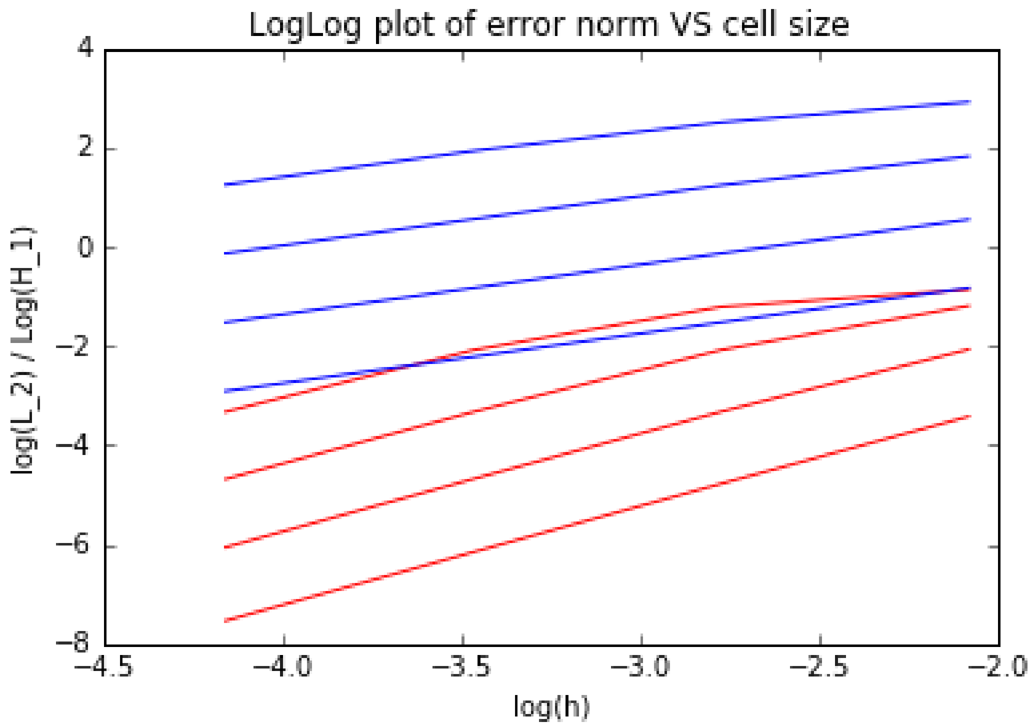
	1	2	4	8
8	0.0374752	0.173276	0.482934	9.79717e-16
16	0.00978353	0.0473498	0.171247	0.482693
32	0.0024616	0.0120996	0.0467505	0.171114
64	0.000616387	0.00304138	0.0119439	0.0467122

H1 error norm

	1	2	4	8
8	0.436629	1.73058	6.17599	18.48
16	0.218173	0.872448	3.44912	12.2594
32	0.109056	0.436258	1.74356	6.88985
64	0.0545238	0.218102	0.872358	3.48637

----- problem 1 - part c -------

Best fit parameters



	alpha	C_alpha	beta	C_beta
1	1.97951	2.02411	1.00047	3.49595
2	1.92043	7.09742	0.996444	13.7769
4	1.68972	11.6033	0.945522	45.5484
8	1.19109	6.3301	0.804985	105.779

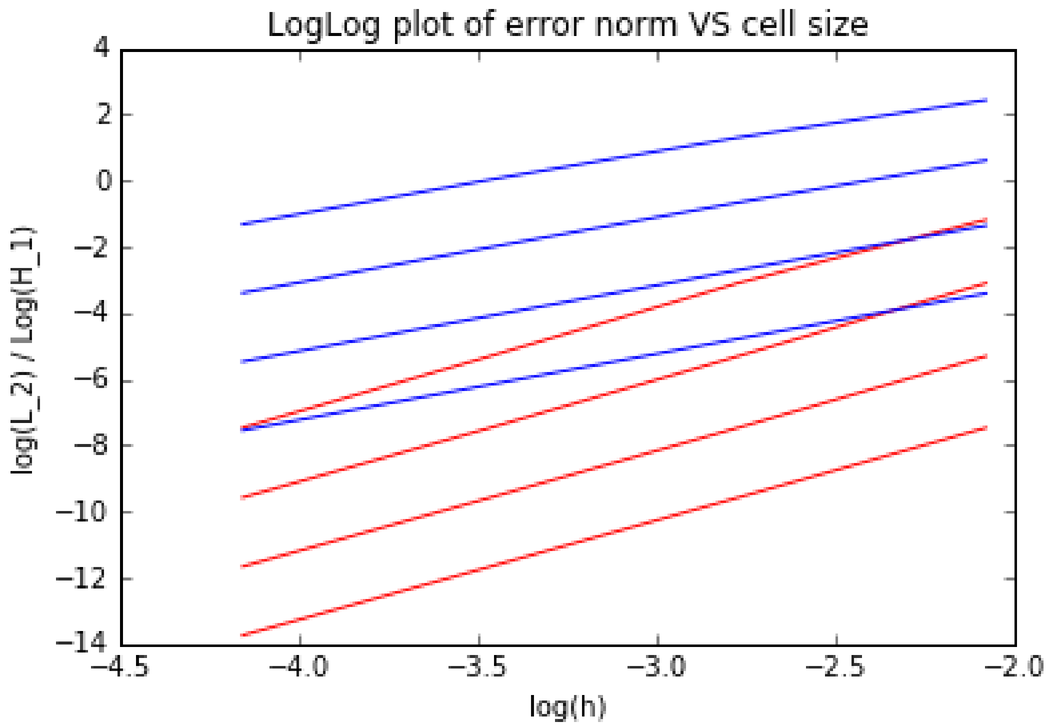
1.4.3 Problem 1, Lagrange elements order 2

```
In [23]: prob_1(2)
```

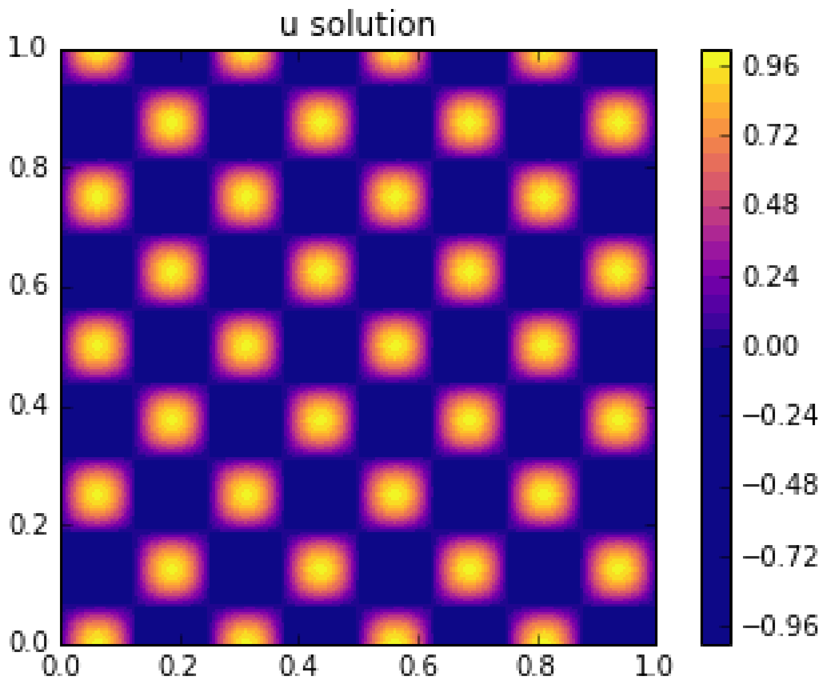
```
Calling FFC just-in-time (JIT) compiler, this may take some time.
----- problem 1 - part b -------

L2 error norm
```

	1	2	4	8
8	0.000568795	0.00495767	0.0450458	0.305222
16	6.93305e-05	0.000570275	0.00496087	0.0452697
32	8.61128e-06	6.94028e-05	0.000570722	0.00498072
64	1.07511e-06	8.61702e-06	6.9447e-05	0.000571757

```
L_inf error norm


              1            2            4            8
8   0.000735545   0.00665548    0.0639175    0.494596
16  8.80406e-05  0.000741005   0.00664197   0.0638915
32  1.08331e-05  9.01769e-05  0.000740577  0.00664111
64   1.3658e-06  1.15708e-05  8.93791e-05  0.00074055


H1 error norm


              1           2          4          8
8     0.0331409    0.256453    1.83608    11.3549
16   0.00838664    0.066496   0.514056    3.67091
32   0.00210537   0.0168044   0.133221    1.02936
64  0.000527159  0.00421483  0.0336408   0.266677




----- problem 1 - part c -------

Best fit parameters
```



	alpha	C_alpha	beta	C_beta
1	3.0151	0.298514	1.99167	2.0907
2	3.05434	2.78314	1.97657	15.7771
4	3.11435	28.5518	1.92589	103.625
8	3.03648	182.835	1.80706	515.869



1.4.4 Problem 2, without SUPG stablization, Lagrange elements order 2

```
In [29]: prob_2(1)
```

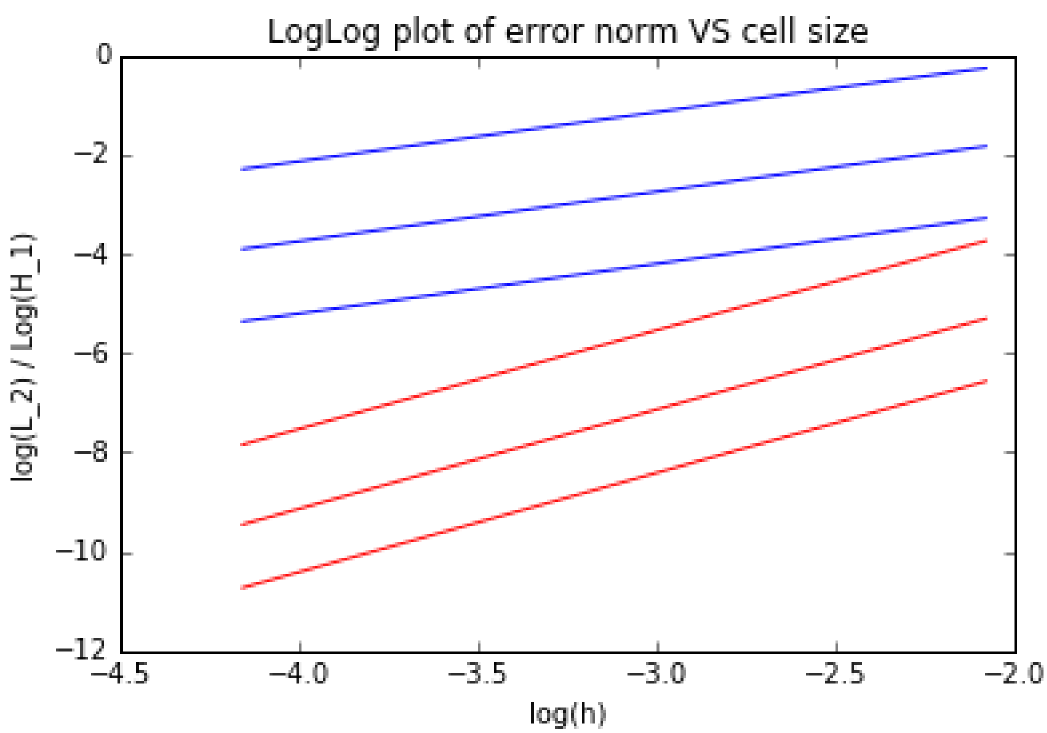
```
----- problem 2 - part b -------

L2 error norm



            0.1          0.3          1.0
8     0.0237473   0.00496957   0.00140249
16   0.00617686   0.00125284  0.000350758
32   0.00156133  0.000313907  8.76984e-05
64  0.000391471  7.85213e-05  2.19252e-05


L_inf error norm



           0.1          0.3          1.0
8     0.102246   0.00956397  0.000594796
16   0.0236434   0.00238083  0.000150234
32  0.00600694  0.000599617  3.76462e-05
```

```
64  0.00150666  0.000149844  9.41695e-06
```

```
H1 error norm
```

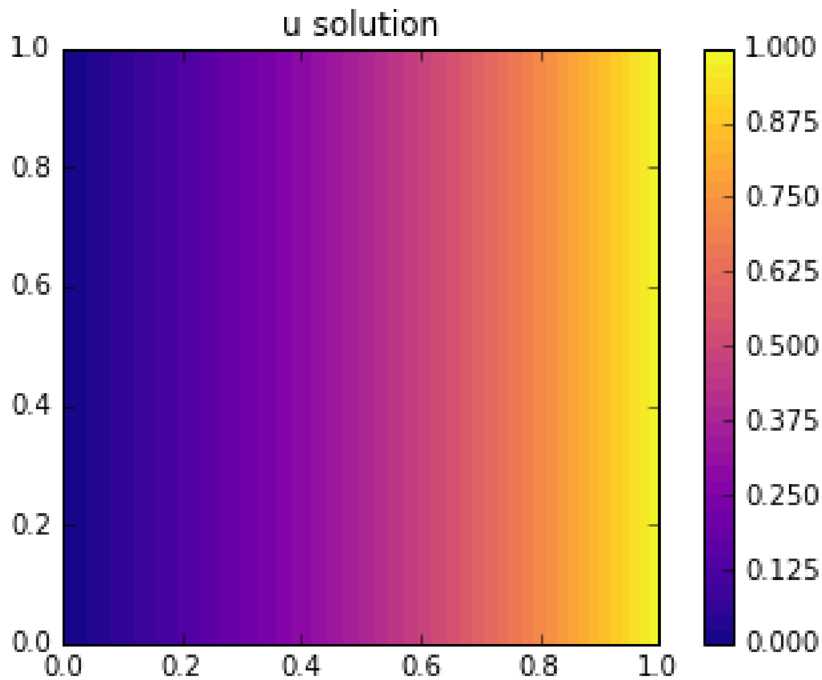
	0.1	0.3	1.0
8	0.769237	0.159755	0.0375224
16	0.398389	0.0803121	0.0187656
32	0.201077	0.0402122	0.00938338
64	0.100781	0.0201132	0.00469176

```
----- problem 2 - part c -------

Best fit parameters
```




```
    alpha    C_alpha      beta     C_beta
```

0.1	1.97522	1.45832	0.978303	5.93637
0.3	1.99485	0.31535	0.996692	1.27108
1.0	1.99976	0.0897242	0.999856	0.300108



1.4.5 Problem 2, with SUPG stabalization, Lagrange elements order 1

```
In [31]: prob_2(1, galerkin=True)
```

```
----- problem 2 - part b -------

L2 error norm
```

	0.1	0.3	1.0
8	0.116845	0.0508915	0.00817331
16	0.0633218	0.0271282	0.00396968
32	0.03313	0.0140583	0.00195639
64	0.0169821	0.00716368	0.000971209

```
L_inf error norm
```

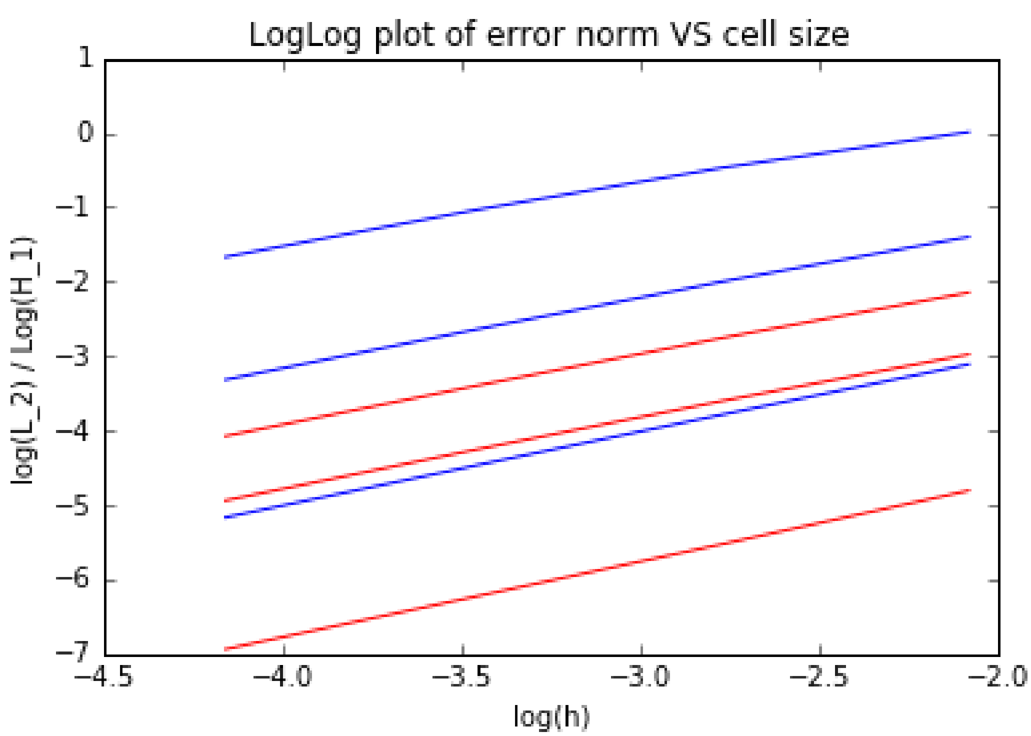
	0.1	0.3	1.0
8	0.230657	0.073797	0.010013
16	0.134348	0.0403927	0.00518187
32	0.0727037	0.0212317	0.00262968
64	0.0384774	0.0108983	0.00132507

H1 error norm

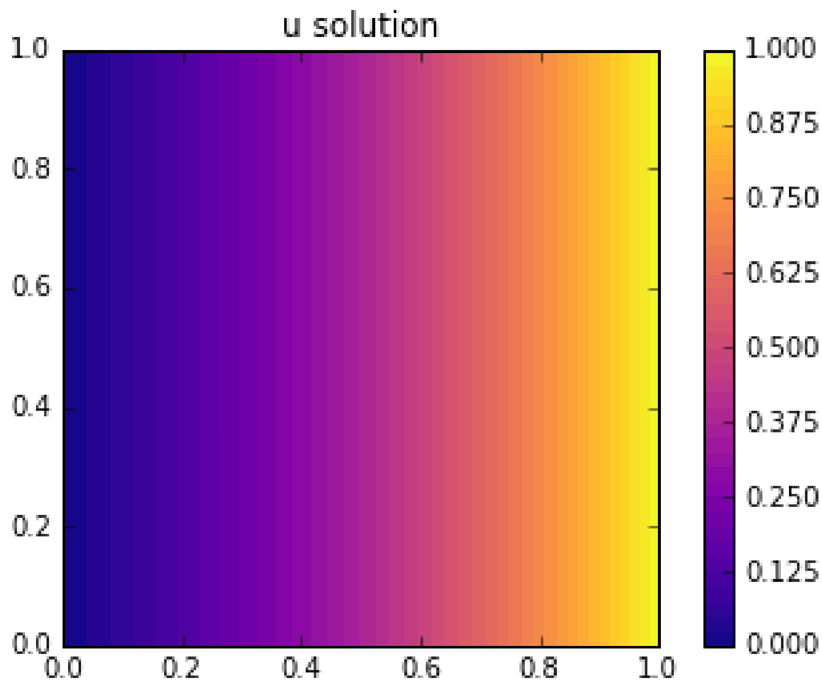
	0.1	0.3	1.0
8	1.00841	0.246454	0.044519
16	0.622238	0.134677	0.0225774
32	0.351801	0.0706637	0.0113699
64	0.188202	0.0362331	0.00570544

----- problem 2 - part c -------

Best fit parameters



	alpha	C_alpha	beta	C_beta
0.1	0.928208	0.81701	0.808787	5.62651
0.3	0.943433	0.366261	0.922828	1.70776
1.0	1.024	0.0683329	0.988169	0.348477



1.5 Problem 2 - discussion

Analytical Solution

Our boundary value problem reads:

$$-\mu\Delta + u_x = 0 \; in \; \Omega$$

with Dirichlet BC's;

$$u = 0 \; for \; x = 0$$

$$u = 1 \; for \; x = 1$$

and Neumann BC's:

$$\frac{\partial u}{\partial n} = 0 \; for \; y = 0 \; and \; y = 1$$

We can solve this analytically by the method of seperation of variables, we let $u(x,y) = g(x)h(y)$ and plugging into our original equation and dividing by μ gives us the two equations

$$g''(x) - \frac{1}{\mu} - C_0 g(x) = 0\,,$$
$$h''(y) + C_0 h(y) = 0\,,$$

Solving for h and using the above Neumann b.c, we obtain:

$$h(y) = C_1 \cos(n\pi y),\ for\, n = 0, 1, 2, ..$$

Choosing $n = 0$ we have $C_0 = 0$, and our expression for $h(y)$ becomes

$$h(y) = C_1$$

with $n = 0$, our expression for $g(x)$ can be expressed as

$$g''(x) - \frac{1}{\mu} g'(x) = 0$$

which gives the expression for $g(x)$

$$g(x) = C_2 e^{\frac{1}{\mu}x} + C_3$$

using the dirichlet conditions, we can solve for C_2 and C_3 we obtain the analytical expression for u as

$$u(x,y) = g(x)h(y) = \frac{e^{\frac{1}{\mu}x} - 1}{e^{\frac{1}{\mu}} - 1}$$

Numerical error

Problem 2 includes a convective u_x term, and we can therefore reasonably expect a upwinding scheme to stablize the solution of our system. Both the L_1 and H^1 tend to zero linearly with the mesh size for both Lagrange elements of degree and 2. However, the H^1 norm tends to zero faster than the L_1 norm for lagrange elements of order 1. We can also see from the C_α and C_β terms for upwinding leads to more accurate results for small μ where the u_x term dominates and low values of N