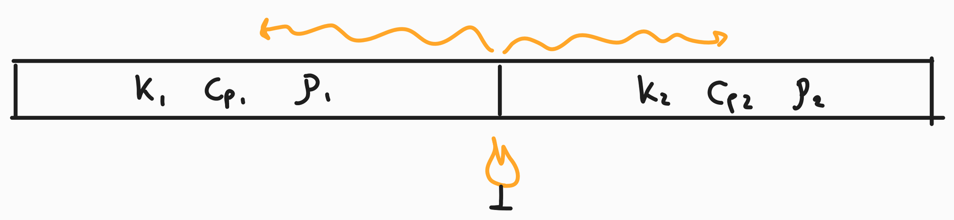
1D Heat transfer in composite material



Let's construct our model:

$$C_p \rho \frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2} + x \qquad (k, C_p, \rho) = \begin{cases} (k_1, C_{p_1}, \rho_1) & x < L/2 \\ (k_2, C_{p_2}, \rho_2) & x > \frac{L}{2} \end{cases}$$

$$u(x,0) = 25$$

$$\boxed{\frac{\partial u}{\partial x}(x,0) = 0 \qquad \frac{\partial u}{\partial x}(L,0) = 0}$$

why using Dirichlet is wrong?

$$u\left(\frac{L}{2}, 0\right) = T_{source} = 200$$

Find analytical solution for material 1:

$$u(x,t) = v(x) + w(x,t) \qquad \alpha = \frac{k}{C_p \rho}$$

$v(x)$: $\alpha \frac{\partial^2 v}{\partial x^2} + \frac{x}{C_p \rho} = 0$

$$k \frac{\partial^2 v}{\partial x^2} + x = 0 \longrightarrow v(x) = \frac{-x^3}{6k} + C_1 x + C_2$$

$$\hat{L} = \frac{L}{2} \qquad \frac{\partial v(0,t)}{\partial x} = 0 \qquad v(\hat{L}, t) = 200$$

$$\begin{cases} 0 = 0 + C_1 \rightarrow C_1 = 0 \\ 200 = \frac{-\hat{L}^3}{6k} + C_2 \rightarrow C_2 = 200 + \frac{\hat{L}^3}{6k} \end{cases}$$

$$v(x) = \frac{-x^3}{6k} + 200 + \frac{\hat{L}^3}{6k}$$

$w(x,t)$: $w = X(x) T(t)$ $\qquad \frac{\partial w(0,t)}{\partial t} = 0 \qquad w(\hat{L}, t) = 0$

$$\frac{T'}{\alpha T \cdot T} = \frac{X''}{\alpha X \cdot X} = -\lambda$$

$$T = T_0 e^{-\alpha \lambda t}$$

$$X = A\cos(\sqrt{\lambda} x) + B \sin(\sqrt{\lambda} x)$$

$$\begin{cases} X'(0) = 0 + B\sqrt{\lambda} = 0 \longrightarrow B = 0 \\ X(\hat{L}) = A \cos \sqrt{\lambda} \hat{L} = 0 \longrightarrow \lambda = \left(\frac{(2n+1)\pi}{2\hat{L}}\right)^2 \end{cases}$$

$$25 - v(x) \xrightarrow{t=0} w(x,t) = \sum_{n=0}^{\infty} C_n \cos\left(\frac{(2n+1)\pi x}{2\hat{L}}\right) e^{-\alpha \left(\frac{(2n+1)\pi}{\hat{L}}\right)^2 t}$$

$$C_n = \frac{2}{\hat{L}} \int_0^{\hat{L}} (25 - v(x)) \cos\left(\frac{\pi x}{\hat{L}}\right) \cos\left(\frac{(2n+1)\pi}{\hat{L}} x\right) dx$$

Perform same steps for material 2

Numerical solution: (explicit)

$$C_p \rho \frac{u_i^{n+1} - u_i^n}{\Delta t} = k \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} + x_i$$

$$u_i^{n+1} = \alpha \left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) \frac{\Delta t}{\Delta x^2} + \frac{x_i \Delta t}{C_p \rho} + u_i^n$$

von Neumann stability analysis

$$N = D + \underbrace{\varepsilon}_{\text{plug to } u}$$

$$\frac{u_i^{n+1} - u_{i-1}^n}{\Delta t} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

$$\varepsilon_i^{n+1} = \frac{\Delta t}{\Delta x^2}\left(\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n\right) + \varepsilon_i^n$$

apply Fourier expansion:

$$\varepsilon(x,t) = \sum_{m=1}^{+\infty} b_m(t) e^{ik_m x} \quad \Rightarrow \langle b_m, e^{ik_m x} \rangle = 0$$

$$\varepsilon_m(x,t) = e^{\alpha t} e^{ik_m x}$$

since $\langle e^{\alpha t}, e^{ik_m x} \rangle = \int e^{\alpha t} e^{ik_m x} dx =$

$$= \frac{e^{\alpha t}}{ik_m} e^{ik_m x} - \frac{-ik_m e^{\alpha t} e^{ik_m x}}{k_m^2} = \frac{-i}{k_m} \langle e^{\alpha t}, e^{ik_m x} \rangle$$

$$\Rightarrow \langle e^{\alpha t}, e^{ik_m x} \rangle \left(1 + \frac{i}{k_m}\right) = 0$$

$$\Rightarrow \langle e^{\alpha t}, e^{ik_m x} \rangle = 0$$

$$e^{\alpha(t+\Delta t)} e^{ik_m x} - e^{\alpha t} e^{ik_m x} = \frac{\Delta t}{\Delta x^2}\left(e^{\alpha t} e^{ik_m(x+\Delta x)} - 2e^{\alpha t} e^{ik_m x} + e^{\alpha t} e^{ik_m(x-\Delta x)}\right)$$

divide by $e^{\alpha t} e^{ik_m x}$

$$e^{\alpha \Delta t} = 1 + \frac{\Delta t}{\Delta x^2}\left(\underbrace{e^{ik_m \Delta x}} - 2 + \underbrace{e^{-ik_m \Delta x}}\right) \qquad \cos\beta = \frac{e^{i\beta} + e^{-i\beta}}{2}$$

$$e^{\alpha \Delta t} = 1 - 2 \cdot \frac{\Delta t}{\Delta x^2}\left(\cos k_m x - 1\right) \qquad \sin^2\frac{\beta}{2} = \frac{1 - \cos\beta}{2}$$

$$\Rightarrow \left| e^{\alpha \Delta t} \right| = \left| 1 - 4 \frac{\Delta t}{\Delta x^2} \sin^2 \frac{k_m x}{2} \right| \leq 1$$

$$\left| 1 - 4 \frac{\Delta t}{\Delta x^2} \right| \leq 1$$

$$\Rightarrow 0 < \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$$

$$\Delta t \leq \frac{\Delta x^2}{2 \cdot \alpha_{max}}$$