CS170– Spring 2022— Homework 7

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Linear Program

A linear program is an optimization problem that seeks the optimal assignment for a linear objective over linear constraints. Let $x \in \mathbb{R}^n$ be the set of variables and $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$. The canonical form of a linear program is:

$$\begin{gathered} \text{minimize } c^T x \\ \text{subject to } Ax \geq b \\ x \geq 0 \end{gathered}$$

Any linear program can be written in canonical form.

Dual

The dual of canonial LP is:

$$\begin{gathered} \text{maximize } b^T y \\ \text{subject to } A^T y \leq c \\ y \geq 0 \end{gathered}$$

Weak duality: The objective value of any feasible dual $\leq$ objective value of any feasible primal.
Strong duality: The optimal objective values of these two are equal.
Both are solvable in polynomial time by the Ellipsoid or Interior Point Method.

2 Modeling: Tricks of the Trade

Create a new variable z_i that will equal $|y_i(a + bx_i)|$ in the optimal solution.
Since the smallest value of z that satisfies $z \geq x$, $z \geq -x$ is $z = |x|$.
Now, consider the following linear programming problem:

$$\begin{gathered} \min \textstyle\sum_{i=1}^{n} z_i \\ \text{subject to } \begin{cases} z_i \geq y_i - (a + bx_i) & \text{for } 1 \leq i \leq n \\ z_i \geq (a + bx_i) - y_i & \text{for } 1 \leq i \leq n \end{cases} \end{gathered}$$

If for some solution we have that $z_i > |y_i(a + bx_i)|$, then by setting $z_i = |y_i(a + bx_i)|$ we will get a solution with a smaller value of the objective function, therefore the initial solution was not optimal. Hence, the constraints requires that the optimal solution will set $z_i = |y_i(a + bx_i)|$, so the new problem is indeed equivalent to the original problem.. And now it is a linear programming problem.

3 Jeweler

(a) Let $x =$ number of necklaces,
$y =$ number of engagement rings.
Our goal is:

$$\text{maximize } 60x + 30y$$
$$\text{subject to: } \begin{cases} 4x + y \leq 80 \\ 2x + 3y \leq 90 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

And the feasible region:

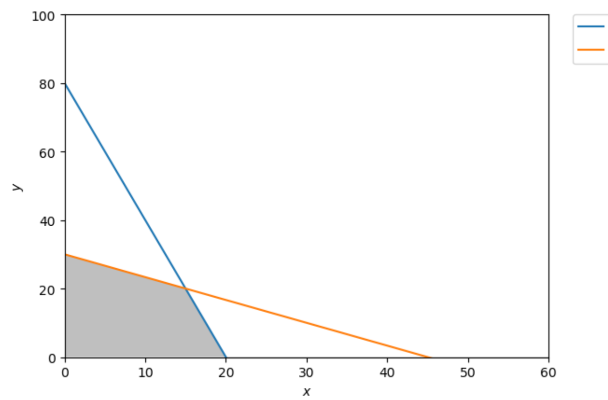

Figure 1: Feasible Region

The vertices are $(20, 0)$, $(15, 20)$, $(0, 30)$, and the objective is maximized at $(15, 20)$, where $60x + 30y = 1500$.

(b) $(0, 30)$: $C \leq 20$
$(15, 20)$: $20 \leq C \leq 120$
$(20, 0)$: $C \geq 120$

4 Standard Form LP

(a) $\min -\sum_i c_i x_i$

(b) $x_1 + s_1 = b_1$
$s_1 \geq 0$

(c) $x_2 - s_2 = b_2$
$s_2 \geq 0$

(d) break into $x_3 \leq b_1$ and $x_3 \geq b_2$, use constraints above.

(e) $x_1 + x_2 + x_3 + s_3 = b_3$
$s_3 \geq 0$

(f) $\min t$, $t \geq y_1$, $t \geq y_2$

(g) Replace x_4 by $x^+ - x^-$ and $x^+ \geq 0, x^- \geq 0$