《概率论与数理统计》

课后题详解

概率论与数理统计课后题详解（意见征集稿）

编写人员：（根据编写章节顺序排名）计试81朱吉羽、能动B71王培宇、能动C71吴家豪、能动D71张翼霄、计算机钱71徐子介、能动B71杨松、材料72杨卓凡、新能源71曹瑞曦

勘误及排版：能动 B71 杨松

加水印：电气 713 余希裴

感谢学业辅导中心各位工作人员与志愿者的努力工作，使本资料可以按时完工。由于编者们的能力与精力限制，难免有错误之处。如果同学们在本资料中发现错误，请联系仲英学业辅导中心：**XJTUzyxuefu@163.com**，我们将在修订时予以更正。

从第5周开始，**每晚19:30-21:30**，学辅志愿者在东21舍118学辅办公室值班，当面为学弟学妹们答疑。

同时，我们也有线上答疑平台——学粉群。**17 级学粉群：656224943，**697672133；**18 级学粉群：**646636875，928740856。以及微信公众号。

期中考试与期末考试前，我们还会举办考前讲座。学辅还有新生专业交流会，转专业交流会，英语考试讲座等活动，消息会在学粉群和公众号上公布，欢迎同学们参与。

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仲英书院学业辅导中心

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学辅公众号

1. (1) X的分布律:

$x < -1, F(x) = P\{x \le x\} = 0;$

$-1 \le x < 1, F(x) = P\{x \le x\} = \frac{1}{3} = P\{x = -1\};$

$1 \le x < 3, F(x) = P\{x \le x\} = P\{x = -1\} + P\{x = 1\} = \frac{5}{6};$

$x \ge 3, F(x) = P\{x \le x\} = 1.$

X	-1	1	3
P	$\dfrac{1}{3}$	$\dfrac{1}{2}$	$\dfrac{1}{6}$

X的分布函数:

$$F(x) = \begin{cases} 0 & x < -1 \\ \dfrac{1}{3} & -1 \le x < 1 \\ \dfrac{5}{6} & 1 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

(2) $P\{x \le 0\} = F(0) = \frac{1}{3}$ $P\{-1 < x \le 2\} = F(2) - F(-1) = \frac{1}{2}$

$P\{-1 \le x \le 2\} = F(2) - F_-(-1) = P\{-1 < x \le 2\} + \{x = -1\} = \frac{5}{6}$

补充: $P\{a < x \le b\} = F(b) - F(a)$ $P\{a \le x \le b\} = F(b) - F_-(a)$

$P\{a \le x < b\} = F_-(b) - F_-(a)$ $P\{a < x < b\} = F_-(b) - F(a)$
$P\{x = a\} = F(a) - F_-(a)$

2. $x < 0, F(x) = 0; 0 \le x < R, F(x) = P\{x \le x\} = \frac{\pi x^2}{\pi R^2} = \frac{x^2}{R^2};$

$x > R, F(x) = P\{x \le x\} = 1.$

$$综上: \ F(x) = \begin{cases} 0, \ x < 0 \\ \frac{x^2}{R^2}, \ 0 \le x < R \\ 1, \ x > R \end{cases}$$

3. (1) $F_1(+\infty) + F_2(+\infty) = 2$ ∴不是

(2) $a_1 F_1(+\infty) + a_2 F_2(+\infty) = a_1 + a_2 = 1$ 且非降, 右连续 ∴是

(3) $F_1(+\infty)F_2(+\infty) = 1 \ F_1(-\infty)F_2(-\infty) = 0$ 且单增, 右连续 ∴是

(4)不能保证$\int_{-\infty}^{+\infty} f_1(x)f_2(x)\,dx = 1$ 如$f_1(x) = \begin{cases} e^{-x}, x > 0 \\ 0, x \le 0 \end{cases}$

$$f_2(x) = \begin{cases} e^{-2x}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

此时 $\int_{-\infty}^{+\infty} f_1(x)f_2(x)\,dx = \frac{2}{3} \ne 1$ ∴不是

4. (1) $F(+\infty) = A + B \times \frac{\pi}{2} = 1$ $F(+\infty) = A - B \times \frac{\pi}{2} = 0$ ∴ $A = \frac{1}{2}$ $B = \frac{1}{\pi}$

(2) $P\{-1 < x \le 1\} = F(1) - F(-1) = \frac{1}{2}$

5. $P\{x = -1\} = F(-1) - F_-(-1) = 0.125$ $P\{x = 0\} = F(0) - F_-(0) = 0.5$
$P\{x = 0.5\} = F(0.5) - F_-(0.5) = 0.25$ $P\{x = 1\} = F(1) - F_-(1) = 0.125$

∴ X 的分布律为:

X	-1	0	0.5	1
P	0.125	0.5	0.25	0.125

6. $P\{x = 0\} = \frac{8}{10} = \frac{4}{5}$ $P\{x = 1\} = \frac{2}{10} \times \frac{8}{9} = \frac{8}{45}$ $P\{x = 2\} = \frac{2}{10} \times \frac{1}{9} \times \frac{8}{8} = \frac{1}{45}$

∴ X 的分布律为:

X	0	1	2
P	$\dfrac{4}{5}$	$\dfrac{8}{45}$	$\dfrac{1}{45}$

X 的分布函数:

$$F(x) = \begin{cases} 0, & x < 0 \\ \dfrac{4}{5}, & 0 \le x < 1 \\ \dfrac{44}{45}, & 1 \le x < 2 \\ 1, & x \ge 2 \end{cases}$$

7. (1) $P\{x = k\} = (1 - 0.8)^{k-1} \times 0.8$ $X \sim G(p)$ 几何分布

(2) $P\{x = k\} = C_{k-1}^{r-1} 0.2^{k-r} \times 0.8^r$, $k = r, r+1, r+2 \cdots$ $X \sim NB(p)$ 负二项分布

8. (1) $X \sim P(4)$ $p_1 = P\{x \ge 6\} - P\{x \ge 7\} = P\{x = 6\}$ 查表得 $p_1 = 0.1042$

(2) $p_2 = P\{x \ge 5\} - P\{x \ge 11\}$ 查表得 $p_2 = 0.3683$

9. $X \sim B(500, 0.005)$ 近似 $X \sim P(2.5)$ $p = 1 - P\{x \ge 6\}$ 查表得 $p = 0.9580$

10. (1) $P\{x = k\} = \begin{cases} 0.7 \times 0.3^{\frac{k-1}{2}} \times 0.2^{\frac{k-1}{2}} = 0.7 \times 0.6^{\frac{k-1}{2}}, & k = 1,3,5 \cdots \\ 0.8 \times 0.3 \times 0.3^{\frac{k-2}{2}} \times 0.2^{\frac{k-2}{2}} = 0.24 \times 0.6^{\frac{k-1}{2}}, & k = 2,4,6 \cdots \end{cases}$

(2) $P\{x = k\} = 0.3^{k-1} \times 0.2^{k-1} \times (0.7 + 0.3 \times 0.8) = 0.94 \times 0.6^{k-1}$, $k = 1,2,3 \cdots$

(3) $P\{x=0\}=0.7 \quad P\{x=k\}=0.282 \times 0.06^{k-1}, \ k=1,2,3 \cdots$

11. (1) $\int_{-\infty}^{+\infty} Ae^{-|x|}\,dx = -Ae^{-x}\big|_0^{+\infty} + Ae^x\big|_{-\infty}^0 = A + A = 1 \quad \therefore A = \frac{1}{2}$

(2) $x < 0$ 时，$F(x) = \int_{-\infty}^x \frac{1}{2}e^x\,dx = \frac{1}{2}e^x$

$x \geq 0$ 时，$F(x) = \int_{-\infty}^0 \frac{1}{2}e^x\,dx + \int_0^x \frac{1}{2}e^x\,dx = 1 - \frac{1}{2}e^x$

$$\text{综上：} \ F(x) = \begin{cases} \frac{1}{2}e^x, & x < 0 \\ 1 - \frac{1}{2}e^x, & x \geq 0 \end{cases}$$

(3) $P\{-1 < x < 2\} = F(2) - F(-1) = 0.7484$

12. (1) $\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-1}^1 \frac{A}{\sqrt{1-x^2}}\,dx = A\arcsin x\big|_{-1}^1 = A\pi = 1 \quad \therefore A = \frac{1}{\pi}$

(2) $x < -1, F(x) = 0$

$-1 \leq x < 1, F(x) = \int_{-1}^x \frac{1}{\pi\sqrt{1-x^2}}\,dx = \frac{1}{2} + \frac{1}{\pi}\arcsin x$

$x \geq 1, F(x) = 1$

$$\text{综上：} \ F(x) = \begin{cases} 0, & x < -1 \\ \frac{1}{2} + \frac{1}{\pi}\arcsin x, & -1 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

(3) $P\left\{-\frac{1}{2} \leq x \leq \frac{1}{2}\right\} = F\left(\frac{1}{2}\right) - F\left(-\frac{1}{2}\right) = \frac{1}{3}$

13. $x < 1, F(x) = 0, f(x) = 0$

$1 \leq x < e, F(x) = \int_1^x \ln x\,dx = \ln x \ f(x) = \frac{1}{x}$

$x \geq e, F(x) = 1 \ f(x) = 0$

$$\text{综上：} \ f(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{x}, & 1 \leq x < e \\ 0, & x \geq e \end{cases}$$

14. (1) $P\{x > -1\} = 1 - \Phi\left(\frac{-1+2}{3}\right) = 1 - 0.6293 = 0.3707$

(2) $P\{-5 \leq x \leq 3\} = \Phi\left(\frac{5}{3}\right) - \Phi(-1) = \Phi\left(\frac{5}{3}\right) - 1 + \Phi(1) = 0.7938$

(3) $P\{0 < x < 5\} = \Phi\left(\frac{7}{3}\right) - \Phi\left(\frac{2}{3}\right) = 0.2415$

(4) $P\{|x| > 1\} = P\left\{Y = \frac{X+2}{3} < \frac{1}{3} \text{或} Y > 1\right\} = \Phi\left(\frac{1}{3}\right) + 1 - \Phi(1) = 0.7880$

(5) $P\{|x+2| < 4\} = P\left\{|Y| < \frac{4}{3}\right\} = \Phi\left(\frac{4}{3}\right) - \Phi\left(-\frac{4}{3}\right) = 2\Phi\left(\frac{4}{3}\right) - 1 = 0.8164$

(6) $P\{|x-a| < a\} = 0.01 = P\left\{\frac{2}{3} \leq Y < \frac{2a+2}{3}\right\} = \Phi\left(\frac{2a+2}{3}\right) - \Phi\left(-\frac{2}{3}\right) = 0.01$

$\Phi\left(\frac{2a+2}{3}\right) = 0.7586 \quad$ 查表得 $\frac{2a+2}{3} \approx 0.70 \ a \approx 0.05$

15. $X \sim N(10.05, 0.06^2)$

$P\{|x - 10.05| < 0.12\} = P\left\{\dfrac{|x - 10.05|}{0.06} < 2\right\} = 2\Phi(2) - 1 = 0.9544$

不合格概率$p = 1 - 0.9544 = 0.0456$

16.

$P\{120 < x \leq 200\} = P\left\{-\dfrac{40}{\sigma} < \dfrac{x - 160}{\sigma} \leq \dfrac{40}{\sigma}\right\} = 2\Phi\left(\dfrac{40}{\sigma}\right) - 1 \geq 0.80 \ \ \Phi\left(\dfrac{40}{\sigma}\right) \geq 0.90$

查表得$\dfrac{40}{\sigma} \geq 1.28, \sigma \leq 31.25$

17. $P\{|x| \leq 30\} = P\left\{-\dfrac{5}{4} < \dfrac{x - 20}{40} \leq \dfrac{1}{4}\right\} = \Phi\left(\dfrac{1}{4}\right) - \Phi\left(-\dfrac{5}{4}\right) = 0.4931$

$\therefore p = 1 - (1 - 0.4931)^3 = 0.8698$

18. $X \sim exp\left(\dfrac{1}{5}\right) \ P\{X > 10\} = 1 - \int_0^{10} \dfrac{1}{5} e^{-\frac{1}{5}x}\, dx = 1 - (1 - e^{-2}) = e^{-2}$

$Y \sim B(5, e^{-2}) \quad P\{Y = k\} = C_k^5 e^{-2k}(1 - e^{-2})^{5-k}, \ k = 0,1 \cdots 5$

$P\{Y \geq 1\} = 1 - P\{Y = 0\} = 0.5167$

19. $X \sim U[0,5] \ P\{\Delta \geq 0\} = P\{x \leq 1.5\} = \int_0^{1.5} \dfrac{1}{5}\, dx = 0.3$

20. $X \sim exp(0.001)$ 设T为仪器寿命，x_i表示第i个元器件寿命 $(i = 1,2,3)$

$P\{1000 < T < 1500\} = P\{T > 1000\} - P\{T > 1500\}$
$= P\{x_1, x_2, x_3 > 1000\} - P\{x_1, x_2, x_3 > 1500\}$
$= (P\{x_1 > 1000\})^3 - (P\{x_2 > 1500\})^3 = e^{-3} - e^{-4.5}$

21. $\because P\{-1 < X \leq 1, -1 < Y \leq 1\}$
$= F(1,1) - F(-1,1) - F(1,-1) + F(-1,-1)$
$\qquad\qquad = -10 < 0$

$\therefore F(x, y)$不是分布函数。

21 题图

22. $(1) P\{X \leq a\} = F(a, +\infty)$
$(2) P\{Y > b\} = 1 - F(+\infty, b)$
$(3) P\{X > a, Y > b\} = 1 - F(a, +\infty) - F(+\infty, b) + F(a, b)$
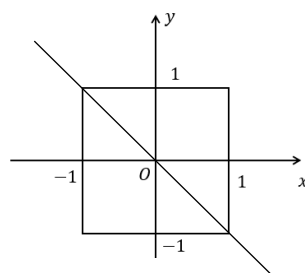$(4) P\{a < X \leq b, Y > c\} = F(b, c) - F(a, c)$

23. $(1) F(-\infty, -\infty) = A\left(B + \dfrac{\pi}{2}\right)\left(C + \dfrac{\pi}{2}\right) = 1$

$F(-\infty, -\infty) = A\left(B - \dfrac{\pi}{2}\right)\left(C - \dfrac{\pi}{2}\right) = 0$

$\therefore B = C = \dfrac{\pi}{2}, A = \dfrac{1}{\pi^2}$

$(2) P\{0 < X \leq 2, 0 < Y \leq 3\} = F(2,3) - F(2,0) - F(0,3) + F(0,0) = \dfrac{1}{16}$

(3) $P\{X > 2, Y > 3\} = 1 - F(+\infty, 3) - F(2, +\infty) + F(2,3) = \frac{1}{16}$

(4) $F_X = F(x, +\infty) = \frac{1}{\pi}\left(\frac{\pi}{2} + arctan\frac{x}{2}\right), F_Y = F(y, +\infty) = \frac{1}{\pi}\left(\frac{\pi}{2} + arctan\frac{y}{3}\right)$

24.

$Y\backslash X$	0	1	2	3	$P_{\cdot j}$
1	0	3/8	3/8	0	3/4
3	1/8	0	0	1/8	1/4
$P_{i\cdot}$	1/8	3/8	3/8	1/8	1

25. (1)

$$f(x,y) = \begin{cases} \dfrac{1}{2} & ,(x,y) \in G \\ 0 & ,其他 \end{cases}$$

$$G = \{(x,y)|y \geq x, x \geq 1, y \leq 3\} = \{(x,y)|1 \leq x \leq y \leq 3\}$$

(2)

$$f_X(x) = \begin{cases} \displaystyle\int_x^3 \frac{1}{2}dy = \frac{1}{2}(3-x), 1 \leq x \leq 3 \\ 0 \qquad ,其他 \end{cases}$$

$$f_Y(y) = \begin{cases} \displaystyle\int_1^y \frac{1}{2}dx = \frac{1}{2}(y-1), 1 \leq y \leq 3 \\ 0 \qquad ,其他 \end{cases}$$

(3)

$$P\{Y - X \leq 1\} = \iint\limits_{\substack{y-x\leq 1 \\ 1\leq x\leq y\leq 3}} f(x,y)dxdy = 1 - \iint\limits_{\substack{y-x>1 \\ 1\leq x\leq y\leq 3}} \frac{1}{2}dxdy = 1 - \frac{1}{4} = \frac{3}{4}$$

26. (1)

$$P\{2X \leq Y\} = \iint\limits_{2x\leq y} f(x,y)dxdy = \int_\varphi^{\varphi+\pi} d\varphi \int_0^{+\infty} \frac{1}{4\pi^2} e^{-\frac{\rho^2}{4\pi}}\rho d\rho = \frac{1}{2}$$

令$X = x - 1, Y = y - 2$ 则$dxdy = dXdY$ ，用极坐标有$\rho^2 = X^2 + Y^2$

(2)

$$P\{(X,Y) \in G\} = \int_0^{2\pi} d\varphi \int_{\sqrt{\pi}}^{2\sqrt{\pi}} \frac{1}{4\pi^2} e^{-\frac{\rho^2}{4\pi}}\rho d\rho = e^{-\frac{1}{4}} - e^{-1}$$

27. (1)

$$\int_{-1}^1 dx \int_{x^2}^1 Cx^2 ydy = \frac{4C}{21} = 1 \ , \quad C = \frac{21}{4}$$

(2)

$$P\{|X| \le Y\} = \int_0^1 dx \int_x^1 \frac{21}{4} x^2 y dy + \int_{-1}^0 dx \int_{-x}^1 \frac{21}{4} x^2 y dy = \frac{7}{10}$$

(3)

$$f_X(x) = \begin{cases} \int_{x^2}^1 \frac{21}{4} x^2 y dy = \frac{21}{8}(x^2 - x^6) & , \quad -1 \le x \le 1 \\ 0 & , \quad \text{其他} \end{cases}$$

$$f_Y(y) = \begin{cases} \int_{-\sqrt{y}}^{\sqrt{y}} \frac{21}{4} x^2 y dx = \frac{7}{2} y^{\frac{5}{2}} & , \quad 0 \le y \le 1 \\ 0 & , \quad \text{其他} \end{cases}$$

28. (1)

$$f_X(x) = \begin{cases} \int_x^{+\infty} e^{-y} dy = e^{-x} & , \quad x > 0 \\ 0 & , \quad \text{其他} \end{cases}$$

$$f_Y(y) = \begin{cases} \int_o^y e^{-y} dx = y e^{-y} & , \quad y > 0 \\ 0 & , \quad \text{其他} \end{cases}$$

(2)

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{f(x,y)}{f_Y(y)} = \dfrac{1}{y}, & 0 < x < y \\ 0 & , \quad \text{其他} \end{cases}$$

$$f_{Y|X}(y|x) = \begin{cases} \dfrac{f(x,y)}{f_X(x)} = e^{-y}, & 0 < x < y \\ 0 & , \quad \text{其他} \end{cases}$$

(3)

$$P\{X > 2 | Y < 4\} = 1 - P\{X \le 2 | Y < 4\} = 1 - \frac{\int_0^2 dx \int_x^4 e^{-y} dy}{\int_0^4 y e^{-y} dy} = \frac{e^{-2} - 3e^{-4}}{1 - 5e^{-4}}$$

29.

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{f(x,y)}{f_Y(y)} = \dfrac{24y(1 - \frac{1}{2} - y)}{(\int_0^x 24y(1-x-y)dy)|_{x=\frac{1}{2}}} = 24y - 48y^2, & 0 < y < \frac{1}{2} \\ 0 & , \quad \text{其他} \end{cases}$$

$$P\left\{Y > \frac{1}{4} \middle| X = \frac{1}{2}\right\} = \frac{\int_{\frac{1}{4}}^{\frac{1}{2}} 24y\left(1 - \frac{1}{2} - y\right)dy}{\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2} = \int_{\frac{1}{4}}^{\frac{1}{2}}(24y - 48y^2)dy$$

30. (1)$P\{X = n, Y = k\} = C_n^k (\frac{1}{2})^n \frac{\lambda^n e^{-\lambda}}{n!} = \frac{1}{k!(n-k)!}\left(\frac{\lambda}{2}\right)^n e^{-\lambda}(k = 0,1,\dots; n = 0,1\dots)$

(2)$Y \sim P\left(\frac{\lambda}{2}\right)$ $P\{Y = k\} = \frac{1}{k!}\left(\frac{\lambda}{2}\right)^k e^{-\frac{\lambda}{2}}, k = 0,1,\dots$

(3)当$k = 0,1,\dots$ 时 $P\{X = n | Y = k\} = \frac{1}{(n-k)!}\left(\frac{\lambda}{2}\right)^{n-k} e^{-\frac{\lambda}{2}} (n = k, k+1, \dots)$

31. (1)
$$f(x,y) = \begin{cases} f_{Y|X}(y|x) \cdot f_X(x) = xe^{-xy}, & 0 \le x \le 1, y > 0 \\ 0 &, \quad 其他 \end{cases}$$

(2)
$$f_Y(y) = \begin{cases} \int_o^y xe^{-xy}dx = \frac{1}{y^2}[1 - (1+y)e^{-y}] &, \quad y > 0 \\ 0 &, \quad 其他 \end{cases}$$

(3)
$$f_{X|Y}(x|y) = \begin{cases} \frac{f(x,y)}{f_Y(y)} = \frac{xy^2 e^{(1-x)y}}{e^y - (1+y)}, & 0 < x \le 1 \\ 0 &, \quad 其他 \end{cases}$$

32.
$$P\{X = 1, Y = 0\} = \frac{1}{9} = P\{X = 1\} \cdot P\{Y = 0\} = \frac{1}{3} \cdot \left(\frac{1}{9} + a\right), \therefore a = \frac{2}{9}$$
$$P\{X = 1, Y = 1\} = \frac{1}{18} = P\{X = 1\} \cdot P\{Y = 1\} = \frac{1}{3} \cdot \left(\frac{1}{18} + b\right), \therefore b = \frac{1}{9}$$

33. (1)
$$f_X(x) = \int_0^2 \left(x^2 + \frac{xy}{3}\right)dy = 2x^2 + \frac{2}{3}x \ , \ f_Y(y) = \int_0^1 \left(x^2 + \frac{xy}{3}\right)dx = \frac{1}{3} + \frac{y}{6}$$

$$f(x,y) \ne f_X(x) \cdot f_Y(y) , \ 故X,Y不独立$$

(2)
$$f_X(x) = \int_0^1 6x^2 y dy = 3x^2 \ , \ f_Y(y) = \int_0^1 6x^2 y dx = 2y$$

$$f(x,y) = f_X(x) \cdot f_Y(y), \ 故X,Y独立$$

(3)

$$f_X(x) = \int_{-x}^{x} \frac{3}{2}xdy = 3x^2 \ , \ \ f_Y(y) = \int_0^1 \frac{3}{2}xdx = \frac{3}{4}$$

$f(x,y) \neq f_X(x) \cdot f_Y(y),$故$X,Y$不独立

(4)

$$f_X(x) = \int_0^{+\infty} \frac{1}{2}e^{-y}dy = \frac{1}{2} \ , \ \ f_Y(y) = \int_0^2 \frac{1}{2}e^{-y}dx = e^{-y}$$

$f(x,y) = f_X(x) \cdot f_Y(y),$ 故X,Y独立

34. 当$-1 \leq x,y,z \leq 1$ 时,

$$f_X(x) = \int_{-1}^1 dy \int_{-1}^1 \frac{1}{8}(1-xyz)dz = \frac{1}{2} , f_Y(y) = \frac{1}{2} , f_Z(z) = \frac{1}{2}$$

$$\therefore f(x,y) = f_X(x) \cdot f_Y(y) , f(x,z) = f_X(x) \cdot f_Z(z) , f(y,z) = f_Y(y) \cdot f_Z(z)$$
$$\Rightarrow F(x,y) = F_X(x) \cdot F_Y(y) , F(x,z) = F_X(x) \cdot F_Z(z) , F(y,z) = F_Y(y) \cdot F_Z(z)$$

但$f(x,y,z) \neq f_X(x) \cdot f_Y(y) \cdot f_Z(z)$ $\therefore X,Y,Z$两两独立，但不相互独立

35. 列表:

X	-2	-1	0	1	2	3
$2X+1$	-3	-1	1	3	5	7
$\|1-2X\|$	5	3	1	1	3	5
$1-X^2$	-3	0	1	0	-3	-8
$cos\frac{\pi X}{4}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$
P_k	$\frac{1}{15}$	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{3}{10}$	$\frac{1}{30}$

得:

(1)Y_1的分布律为:

Y_1	-3	-1	1	3	5	7
P	$\frac{1}{15}$	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{3}{10}$	$\frac{1}{30}$

(2)Y_2的分布律为:

Y_2	1	3	5
P	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{10}$

(3)Y_3的分布律为：

Y_3	-8	-3	0	1
P	$\dfrac{1}{30}$	$\dfrac{11}{30}$	$\dfrac{13}{30}$	$\dfrac{1}{6}$

(4)Y_4的分布律为：

Y_4	$-\dfrac{\sqrt{2}}{2}$	0	$\dfrac{\sqrt{2}}{2}$	1
P	$\dfrac{1}{30}$	$\dfrac{11}{30}$	$\dfrac{13}{30}$	$\dfrac{1}{6}$

36. 设国徽向上的次数为X，则$X \sim B(5, \frac{1}{2})$，$P\{X = k\} = C_5^k (\frac{1}{2})^5, k = 0,1,2,3,4,5$

令$Y = X(5 - X)$,则$Y = 0,4$ 或 6

$$P\{Y = 0\} = P\{X = 0 \text{ 或 } 5\} = P\{X = 0\} + P\{X = 5\} = \frac{1}{16}$$

$$P\{Y = 4\} = P\{X = 1 \text{ 或 } 4\} = P\{X = 1\} + P\{X = 4\} = \frac{5}{16}$$

$$P\{Y = 6\} = P\{X = 2 \text{ 或 } 3\} = P\{X = 2\} + P\{X = 3\} = \frac{5}{8}$$

则所求分布律为：

Y	0	4	6
P	$\dfrac{1}{16}$	$\dfrac{5}{16}$	$\dfrac{5}{8}$

37. X的概率密度函数为$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, x > 0 \\ 0, \quad x \leq 0 \end{cases}$

(1)$Y_1 = g_1(X) = X^3, X = g_1^{-1}(Y_1) = \sqrt[3]{Y_1}$，则$Y_1$的概率密度函数为：

$$f_{Y_1}(y) = f_X(\sqrt[3]{y}) \cdot \left| (\sqrt[3]{y})' \right| = \begin{cases} \dfrac{\lambda}{3} e^{-\lambda \sqrt[3]{y}} \cdot y^{-\frac{2}{3}}, y > 0 \\ 0 \qquad , y \leq 0 \end{cases}$$

(2)$Y_2 = g_2(X) = e^{-\lambda x} > 0, X = g_2^{-1}(Y_2) = -\frac{1}{\lambda} \ln Y_2$，则$Y_2$的概率密度函数为：

$$f_{Y_2}(y) = \begin{cases} f_X\left(-\dfrac{1}{\lambda} \ln y\right) \cdot \left| \left(-\dfrac{1}{\lambda} \ln y\right)' \right|, y > 0 \\ 0 \qquad , \text{其他} \end{cases} = \begin{cases} 1, 0 < y < 1 \\ 0, \text{其他} \end{cases}$$

38. 出现正面的次数$Y \sim B\left(5, \frac{1}{2}\right), P\{Y = k\} = C_5^k (\frac{1}{2})^5, \; k = 0,1,2,3,4,5$

则有$X = Y + (-1) \cdot (5 - Y) = 2Y - 5$

$$P\{X = 2k - 5\} = P\{2Y - 5 = 2k - 5\} = P\{Y = k\} = C_5^k (\frac{1}{2})^5, k = 0,1,2,3,4,5$$

即为X的分布律。

39. Y的分布函数为

$$F_Y(y) = P\{Y \le y\} = \begin{cases} P\{-y \le X - \mu \le y\}, y > 0 \\ 0 \quad\quad, y \le 0 \end{cases} = \begin{cases} \Phi\left(\frac{y}{\sigma}\right) - \Phi\left(-\frac{y}{\sigma}\right), y > 0 \\ 0 \quad\quad, y \le 0 \end{cases}$$

$$= \begin{cases} 2\Phi\left(\frac{y}{\sigma}\right) - 1, y > 0 \\ 0 \quad\quad, y \le 0 \end{cases}, \text{ 求导可得}Y\text{的概率密度函数为：}$$

$$f_Y(y) = F_Y'(y) = \begin{cases} \frac{2}{\sigma}\varphi\left(\frac{y}{\sigma}\right), y > 0 \\ 0 \quad, y \le 0 \end{cases} = \begin{cases} \frac{2}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}}, y > 0 \\ 0 \quad, y \le 0 \end{cases}$$

40. 列表：

(X,Y)	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$X + Y$	0	1	1	2
$2X$	0	0	2	2
XY	0	0	0	1
X^2	0	0	1	1
p	$(\frac{1}{2})^2$	$(\frac{1}{2})^2$	$(\frac{1}{2})^2$	$(\frac{1}{2})^2$

得：

(1)$X + Y$的分布律为：

$X + Y$	0	1	2
p	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

(2)$2X$的分布律为：

$2X$	0	2
p	$\frac{1}{2}$	$\frac{1}{2}$

(3)XY的分布律为：

XY	0	1
p	$\frac{3}{4}$	$\frac{1}{4}$

(4)X^2的分布律为:

X^2	0	1
p	$\dfrac{1}{2}$	$\dfrac{1}{2}$

41. X的概率密度函数为$f_X(x) = \begin{cases} \dfrac{1}{2} &, -1 < x < 1 \\ 0 &, 其他 \end{cases}$, Y与X同分布

由于X和Y相互独立，则X和Y的联合概率密度函数为

$$f(x,y) = f_X(x) \cdot f_Y(y) = \begin{cases} \dfrac{1}{4} &, -1 < x, y < 1 \\ 0 &, 其他 \end{cases}$$
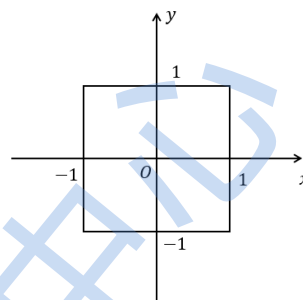
则(X,Y)在$-1 < x, y < 1$区域上服从二维正态分布

Z的分布函数为 $F_Z(z) = P\{Z \le z\} = P\{XY \le z\} = \iint_{xy \le z} f(x,y)dxdy$

①$z \ge 1$, $F_Z(z) = 1$

②$z \le -1$, $F_Z(z) = 0$

③$z = 0$, $F_Z(z) = \dfrac{1}{2}$

④$0 < z < 1$, $F_Z(z) = 1 - 2\int_z^1 dx \int_{\frac{z}{x}}^1 \dfrac{1}{4}dy = \dfrac{1}{2} + \dfrac{1}{2}z(1 - lnz)$

⑤$-1 < z < 0$, $F_Z(z) = 2\int_{-1}^z dx \int_{\frac{z}{x}}^1 \dfrac{1}{4}dy = \dfrac{1}{2} + \dfrac{1}{2}z(1 - ln|z|)$

综上所述，Z的分布函数为$F_Z(z) = \begin{cases} 0 &, z \le -1 \\ \dfrac{1}{2} + \dfrac{1}{2}z(1 - ln|z|) &, 0 < |z| < 1 \\ \dfrac{1}{2} &, z = 0 \\ 1 &, z \ge 1 \end{cases}$
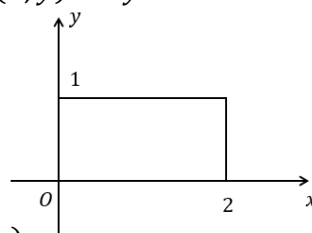
42. X和Y的联合概率密度函数为 $f(x,y) = \begin{cases} \dfrac{1}{2} &, (x,y) \in G \\ 0 &, 其他 \end{cases}$

S的分布函数为 $F(s) = P\{S \le s\} = P\{XY \le s\} = \iint_{xy \le s} f(x,y)dxdy$

①$s \ge 2$, $F(s) = 1$

②$s \le 0$, $F(s) = 0$

③$0 < s < 2$, $F(s) = 1 - \int_s^2 dx \int_{\frac{s}{x}}^1 \dfrac{1}{2}dy = \dfrac{1}{2}(s + sln2 - slns)$

故S的概率密度为$f(s) = F'(s) = \begin{cases} \frac{1}{2}(ln2 - lns) ,0 < s < 2 \\ \qquad 0 \qquad ,其他 \end{cases}$

43. X的概率密度函数为$f_X(x) = \begin{cases} \lambda_1 e^{-\lambda_1 x}, x > 0 \\ 0, \quad x \le 0 \end{cases}$

Y的概率密度函数为$f_Y(y) = \begin{cases} \lambda_2 e^{-\lambda_2 x}, y > 0 \\ 0, \quad y \le 0 \end{cases}$

由于X和Y相互独立，故可用卷积公式，$Z = X + Y$的概率密度函数为

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(x)f_Y(z-x)dx$$

$$= \begin{cases} \int_0^z \lambda_1 e^{-\lambda_1 x} \cdot \lambda_2 e^{-\lambda_2(z-x)} dx, z > 0 \\ \qquad\qquad 0, z \le 0 \end{cases} = \begin{cases} \int_0^z \lambda_1\lambda_2 e^{(\lambda_2-\lambda_1)x - \lambda_2 z} dx, z > 0 \\ \qquad\qquad 0, z \le 0 \end{cases}$$

①$\lambda_1 = \lambda_2$，$f_Z(z) = \begin{cases} \lambda_1^2 z e^{-\lambda_1 z} dx, z > 0 \\ \quad 0, z \le 0 \end{cases}$

②$\lambda_1 \neq \lambda_2$，$f_Z(z) = \begin{cases} \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left(e^{-\lambda_1 z} - e^{-\lambda_2 z}\right)dx , z > 0 \\ \qquad\quad 0 \qquad , z \le 0 \end{cases}$

44. 设两周和三周的需求量分别为Z_2和Z_3,第i周的需求量为X_i

则有 $Z_2 = X_1 + X_2$，$Z_3 = Z_2 + X_3$

X_1和X_2相互独立，可用卷积公式，则Z_2的概率密度函数为

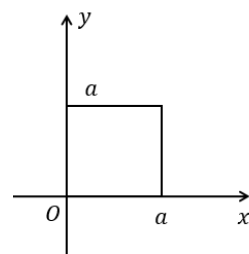$$f_{Z_2}(z_2) = \int_{-\infty}^{+\infty} f(x_1)f(z_2 - x_1)dx_1$$

$$= \begin{cases} \int_0^{z_2} f(x_1)f(z_2 - x_1)dx_1, z_2 > 0 \\ \qquad\quad 0 \qquad , z_2 \le 0 \end{cases} = \begin{cases} \frac{1}{6}z_2^3 e^{-z_2} \quad , z_2 > 0 \\ \qquad 0 \qquad , z_2 \le 0 \end{cases}$$

Z_2和X_3相互独立，可用卷积公式，则Z_3的概率密度函数为

$$f_{Z_3}(z_3) = \int_{-\infty}^{+\infty} f_{z_2}(z_2)f(z_3 - z_2)dz_2 = \begin{cases} \int_0^{z_3} f_{z_2}(z_2)f(z_3 - z_2)dz_2, z_3 > 0 \\ \qquad\qquad 0, z_3 \le 0 \end{cases}$$

$$= \begin{cases} \int_0^{z_3} \frac{1}{6}z_2^3 e^{-z_2} \cdot (z_3 - z_2)e^{-(z_3-z_2)}dz_2, z_3 > 0 \\ \qquad\qquad 0 \qquad\quad , z_3 \le 0 \end{cases} = \begin{cases} \frac{1}{120}z_3^5 e^{-z_3} \quad , z_3 > 0 \\ \qquad 0 \qquad , z_3 \le 0 \end{cases}$$

45. 由于X和Y相互独立，则X和Y的联合概率密度函数为

$$f(x,y) = f_X(x) \cdot f_Y(y) = \begin{cases} \dfrac{1}{a^2} & ,0 \le x,y \le a \\ 0 & ,其他 \end{cases}$$

则(X,Y)在$0 \le x,y \le a$区域上服从二维正态分布

(1)令$Z = X - Y$,Z的分布函数为

$$F(z) = P\{Z \le z\} = P\{X - Y \le z\} = \iint\limits_{x-z \le y} f(x,y)dxdy$$

①$z \ge a, F(z) = 1$

②$z \le -a, F(z) = 0$

③$-a < z \le 0, F(z) = \dfrac{1}{2}(a+z)^2 \cdot \dfrac{1}{a^2} = \dfrac{(z+a)^2}{2a^2}$

④$0 < z < a, F(z) = 1 - \dfrac{1}{2}(a-z)^2 \cdot \dfrac{1}{a^2} = 1 - \dfrac{(z-a)^2}{2a^2}$

所以Z的概率密度为$f(z) = F'(z) = \begin{cases} \dfrac{1}{a^2}(a - |z|) & ,-a \le z \le a \\ 0 & ,其他 \end{cases}$

(2)令$Z = |X - Y|$,Z的分布函数为

$$F(z) = P\{Z \le z\} = P\{|X - Y| \le z\} = \iint\limits_{|x-y| \le z} f(x,y)dxdy$$

①$z \ge a, F(z) = 1$

②$z \le 0, F(z) = 0$

③$0 < z < a, F(z) = 1 - 2 \cdot \dfrac{1}{2}(a-z)^2 \cdot \dfrac{1}{a^2} = 1 - \dfrac{(z-a)^2}{a^2}$

所以Z的概率密度为$f(z) = F'(z) = \begin{cases} \dfrac{2}{a^2}(a - z) & ,0 < z < a \\ 0 & ,其他 \end{cases}$

46. 由于X和Y独立性暂未知，故Z的概率密度为$f_Z(z) = \int_{-\infty}^{+\infty} f(x, z-x)dx$

$x \in [-1,1] \cap [z-1, z+1]$时，$f(x, z-x) \ne 0$

①$z < -2$或$z > 2$时，$f(x, z-x) = 0$，故$f_Z(z) = 0$

②$-2 \le z \le 0$时，$f_Z(z) = \int_{-1}^{z+1} f(x, z-x)dx = \dfrac{1}{24}(z^3 - 8)$

③$0 < z \leq 2$时，$f_Z(z) = \int_{z-1}^{1} f(x, z-x) dx = \frac{1}{24}(8 - z^3)$

综上所述，Z的概率密度为$f_Z(z) = \begin{cases} \frac{1}{24}(8 - |z|^3), & |z| \leq 2 \\ 0, & \text{其他} \end{cases}$

47. X和Y的联合概率密度函数为 $f(x, y) = \begin{cases} \frac{1}{ab}, & 0 \leq x \leq a, \ 0 \leq y \leq b \\ 0, & \text{其他} \end{cases}$

Z的概率密度为$f_Z(z) = \int_{-\infty}^{+\infty} |y| f(yz, y) dy$

$y \in [0, b] \cap [0, \frac{a}{z}]$时，$f(yz, y) \neq 0$

①$z < 0$，$f(yz, y) = 0$，故$f_Z(z) = 0$

②$z = 0$，$f_Z(z) = \int_{-\infty}^{+\infty} |y| f(0, y) dy = \int_0^b y \frac{1}{ab} dy = \frac{b}{2a}$

③$0 < z < \frac{a}{b}$，$f_Z(z) = \int_{-\infty}^{+\infty} |y| f(yz, y) dy = \int_0^b y \frac{1}{ab} dy = \frac{b}{2a}$

④$z \geq \frac{a}{b}$，$f_Z(z) = \int_{-\infty}^{+\infty} |y| f(yz, y) dy = \int_0^{\frac{a}{z}} y \frac{1}{ab} dy = \frac{a}{2bz^2}$

综上所述，Z的概率密度为$f_Z(z) = \begin{cases} 0, & z < 0 \\ \frac{b}{2a}, & 0 \leq z < \frac{a}{b} \\ \frac{a}{2bz^2}, & z \geq \frac{a}{b} \end{cases}$

48. X和Y相互独立,则Z的概率密度函数为$f_Z(z) = \int_{-\infty}^{+\infty} |y| f(yz) f(y) dy$

$y \in [100, +\infty] \cap [\frac{100}{z}, +\infty]$时，$f(yz)f(y) \neq 0$

①$z \leq 0$，$f(yz)f(y) = 0$，故$f_Z(z) = 0$

②$0 < z < 1$，$f_Z(z) = \int_{\frac{100}{z}}^{+\infty} |y| f(yz) f(y) dy = \int_{\frac{100}{z}}^{+\infty} \frac{10000}{y^3 z^2} dy = \frac{1}{2}$

③$z \geq 1$，$f_Z(z) = \int_{100}^{+\infty} |y| f(yz) f(y) dy = \int_{100}^{+\infty} \frac{10000}{y^3 z^2} dy = \frac{1}{2z^2}$

综上所述，Z的概率密度为$f_Z(z) = \begin{cases} 0, & z \leq 0 \\ \frac{1}{2}, & 0 < z < 1 \\ \frac{1}{2z^2}, & z \geq 1 \end{cases}$

49. X和Y同分布，概率密度函数为$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$

X和Y独立，则联合概率密度函数为$f(x, y) = f(x)f(y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$

令$R = \sqrt{X^2 + Y^2}$，则R分布函数

$$F_R(r) = P\{R \leq r\} = P\{X^2 + Y^2 \leq r^2\} = \iint\limits_{x^2+y^2 \leq r^2} f(x,y)dxdy$$

$$= \begin{cases} \int_0^{2\pi} d\theta \int_0^r \dfrac{1}{2\pi\sigma^2} e^{-\frac{\rho^2}{2\sigma^2}}\rho d\rho, r \geq 0 \\ 0, r < 0 \end{cases} = \begin{cases} \dfrac{1}{2}\left(1 - e^{-\frac{r^2}{\sigma^2}}\right), r \geq 0 \\ 0 \quad, \quad r < 0 \end{cases}$$

所以R的概率密度为$f_R(r) = F_R'(r) = \begin{cases} \dfrac{r}{\sigma^2} e^{-\frac{r^2}{\sigma^2}}, r \geq 0 \\ 0 \quad, r < 0 \end{cases}$

50. X的概率密度函数为$f_X(x) = \begin{cases} \dfrac{1}{5}, 0 \leq x \leq 5 \\ 0, \quad 其他 \end{cases}$

Y的概率密度函数为$f_Y(y) = \begin{cases} 5e^{-5y}, y > 0 \\ 0, \quad y \leq 0 \end{cases}$

(1)令$W = X + Y$，由于X和Y相互独立，故可用卷积公式，W的概率密度为

$$f_W(w) = \int_{-\infty}^{+\infty} f_X(x)f_Y(w-x)dx$$

①$w \leq 0$，此时$f_X(x)f_Y(w-x) = 0$，故$f_W(w) = 0$

②$0 < w < 5$, $f_W(w) = \int_0^w \dfrac{1}{5} \cdot 5e^{-5(w-x)}dx = \dfrac{1}{5}(1 - e^{-5w})$

③$w \geq 5$, $f_W(w) = \int_0^5 \dfrac{1}{5} \cdot 5e^{-5(w-x)}dx = \dfrac{1}{5}(e^{25-5w} - e^{-5w})$

综上所述，W的概率密度为$f_W(w) = \begin{cases} 0 \quad, w \leq 0 \\ \dfrac{1}{5}(1 - e^{-5w}), 0 < w < 5 \\ \dfrac{1}{5}(e^{25} - 1)e^{-5w}, w \geq 5 \end{cases}$

(2)$P\{Z = 1\} = P\{Y \geq X\}$

$$= \int_{-\infty}^{+\infty} dx \int_x^{+\infty} f(x,y)dy = \int_0^5 dx \int_x^{+\infty} \dfrac{1}{5} \cdot 5e^{-5y}dy = \dfrac{1}{25}(1 - e^{-25})$$

$P\{Z = 0\} = 1 - P\{Z = 1\} = \dfrac{1}{25}(24 + e^{-25})$，则$Z$的分布律为

Z	0	1
P	$\dfrac{24 + e^{-25}}{25}$	$\dfrac{1 - e^{-25}}{25}$

51. 由于$X_1, X_2, \cdots, X_n$独立同分布，其分布函数都为$F(x) = \begin{cases} 0 \quad, x < 0 \\ x, 0 \leq x < 1 \\ 1 \quad, x \geq 1 \end{cases}$

故Y_n的分布函数为$F_{Y_n}(y) = [F(y)]^n = \begin{cases} 0 \quad, y < 0 \\ y^n, 0 \leq y < 1 \\ 1 \quad, y \geq 1 \end{cases}$

$P\{Y_n \geq 0.99\} \geq 0.95$

$\rightarrow P\{Y_n < 0.99\} \leq 0.05$

$\rightarrow F_{Y_n}(0.99) \leq 0.05$

$\rightarrow 0.99^n \leq 0.05$

$\rightarrow n \geq log_{0.99}(0.05) = 298.07$，故$n \geq 299$且$n \in N^*$

52. 列表（因版面问题未能横着列表）

(X,Y)	$X+Y$	$max(X,Y)$	$min(X,Y)$	P
(0,0)	0	0	0	0
(0,1)	1	1	0	0.01
(0,2)	2	2	0	0.02
(1,0)	1	1	0	0.05
(1,1)	2	1	1	0.09
(1,2)	3	2	1	0.11
(2,0)	2	2	0	0.08
(2,1)	3	2	1	0.12
(2,2)	4	2	2	0.13
(3,0)	3	3	0	0.12
(3,1)	4	3	1	0.15
(3,2)	5	3	2	0.12

得:

$(1)Z = X + Y$的分布律为$(Z = 0$不能不写$)$

Z	0	1	2	3	4	5
P	0	0.06	0.19	0.35	0.28	0.12

$(2)U = max(X,Y)$的分布律为$(U = 0$不能不写$)$

U	0	1	2	3
p	0	0.15	0.46	0.39

$(3)V = min(X,Y)$的分布律为

V	0	1	2
P	0.28	0.47	0.25

53. 边缘概率密度:

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)dy = \begin{cases} \dfrac{1}{2}, |x| \leq 1 \\ 0, 其他 \end{cases}, \qquad f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)dx = \begin{cases} \dfrac{1}{2}, |y| \leq 1 \\ 0, 其他 \end{cases}$$

$f_X(x) \cdot f_Y(y) \neq f(x,y)$，故$X$与$Y$不独立

令 $U = X^2, V = Y^2$

U 的分布函数 $F_U(u) = P\{U \le u\} = P\{X^2 \le u\}$

$$= \begin{cases} 0 & ,u \le 0 \\ P\{-\sqrt{u} \le X \le \sqrt{u}\} & ,0 < u < 1 \\ 1 & ,u \ge 1 \end{cases} = \begin{cases} 0 & ,u \le 0 \\ \sqrt{u} & ,0 < u < 1 \\ 1 & ,u \ge 1 \end{cases}$$

U 的概率密度为 $f_U(u) = F_U'(u) = \begin{cases} \frac{1}{2\sqrt{u}} & ,0 < u < 1 \\ 0 & ,其他 \end{cases}$

同理可得，V 的概率密度为 $f_V(v) = \begin{cases} \frac{1}{2\sqrt{v}} & ,0 < v < 1 \\ 0 & ,其他 \end{cases}$

(U,V) 的联合分布函数 $F(u,v) = P\{U \le u, V \le v\} = P\{X^2 \le u, Y^2 \le v\}$

$$= \begin{cases} P\{-\sqrt{u} \le X \le \sqrt{u}, -\sqrt{v} \le Y \le \sqrt{v}\} & ,0 < u,v \le 1 \\ 1 & , \quad u,v > 1 \\ 0 & , \quad 其他 \end{cases}$$

$$= \begin{cases} \int_{-\sqrt{u}}^{\sqrt{u}} dx \int_{-\sqrt{v}}^{\sqrt{v}} \frac{1+xy}{4} dy & ,0 < u,v \le 1 \\ 1 & , \quad u,v > 1 \\ 0 & , \quad 其他 \end{cases} = \begin{cases} \sqrt{uv} & ,0 < u,v \le 1 \\ 1 & , \quad u,v > 1 \\ 0 & , \quad 其他 \end{cases}$$

则 (U,V) 的联合概率密度 $f(u,v) = \frac{\partial^2 F(u,v)}{\partial u \partial v} = \begin{cases} \frac{1}{4\sqrt{uv}} & , \ 0 \le u,v \le 1 \\ 0 & ,其他 \end{cases}$

由于 $f_U(u) \cdot f_V(v) = f(u,v)$，故 U 与 V 独立,即 X^2 与 Y^2 独立

注：本题说明,若随机变量 X 与 Y 不独立，则 $f(X)$ 与 $g(Y)$ 不一定不独立

54. (泊松分布的可加性)

$P\{X = k\} = \frac{\lambda_1^k}{k!} e^{-\lambda_1}, k = 0,1,2 \cdots$, $P\{Y = k\} = \frac{\lambda_2^k}{k!} e^{-\lambda_2}, k = 0,1,2 \cdots$

则 $Z = X + Y$ 的分布律为

$$P\{Z = n\} = \sum_{k=0}^{n} P\{X = k, Y = n-k\} = \sum_{k=0}^{n} P\{X = k\} \cdot P\{Y = n-k\}$$

$$= \sum_{k=0}^{n} \frac{\lambda_1^k}{k!} e^{-\lambda_1} \cdot \frac{\lambda_2^{n-k}}{(n-k)!} e^{-\lambda_2} = e^{-(\lambda_1+\lambda_2)} \sum_{k=0}^{n} \frac{1}{n!} \cdot \frac{n!}{k! \cdot (n-k)!} \lambda_1^k \cdot \lambda_2^{n-k}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{k=0}^{n} C_n^k \cdot \lambda_1^k \cdot \lambda_2^{n-k} = \frac{e^{-(\lambda_1+\lambda_2)}}{n!} (\lambda_1 + \lambda_2)^n = \frac{(\lambda_1 + \lambda_2)^n}{n!} e^{-(\lambda_1+\lambda_2)}$$

即$Z \sim P(\lambda_1 + \lambda_2)$

55. 各随机变量有相同的概率密度和分布函数，分别设为$f(x)$和$F(x)$

则$P\{X_n > max\{X_1, X_2, \cdots, X_{n-1}\}\} = P\{X_n > X_1, X_n > X_2, \cdots, X_n > X_{n-1}\}$

$= P\{X_n > X_1\}P\{X_n > X_2\} \cdots P\{X_n > X_{n-1}\}$

$= \sum P\{X_n = x_n\} \cdot P\{x_n > X_1\}P\{x_n > X_2\} \cdots P\{x_n > X_{n-1}\}$

$$= \int_{-\infty}^{+\infty} f(x_n)d(x_n) \cdot \int_{-\infty}^{x_n} f(x_1)d(x_1) \cdot \int_{-\infty}^{x_n} f(x_2)d(x_2) \cdots \int_{-\infty}^{x_n} f(x_{n-1})d(x_{n-1})$$

$$= \int_{-\infty}^{+\infty} f(x_n)d(x_n) \cdot F(x_n) \cdot F(x_n) \cdots F(x_n) = \int_{-\infty}^{+\infty} [F(x_n)]^{n-1} f(x_n)d(x_n)$$

$$= \int_{x_n=-\infty}^{x_n=+\infty} [F(x_n)]^{n-1} d(F(x_n)) = \frac{1}{n} [F(x_n)]^n |_{x_n=-\infty}^{x_n=+\infty} = \frac{1}{n}(1^n - 0^n) = \frac{1}{n}$$

第三章

1. X的可能取值为 0,1,2,3 $P\{X=0\}=\frac{C_{12}^5 C_3^0}{C_{15}^5}=\frac{24}{91}$, $P\{X=1\}=\frac{C_{12}^4 C_3^1}{C_{15}^5}=\frac{45}{91}$,

   $P\{X=2\}=\frac{C_{12}^3 C_3^2}{C_{15}^5}=\frac{20}{91}$, $P\{X=3\}=\frac{C_{12}^2 C_3^3}{C_{15}^5}=\frac{2}{91}$

   $\therefore E(X)=0\times\frac{24}{91}+1\times\frac{45}{91}+2\times\frac{20}{91}+3\times\frac{2}{91}=1$

2. $E(X)=\int_{-\infty}^{+\infty}\frac{1}{2}xe^{-|x|}dx=0$

3. $E(X)=\int_0^{+\infty}\frac{x^2}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}dx=\int_0^{+\infty}\sqrt{2}\sigma\left(\frac{x^2}{2\sigma^2}\right)^{\frac{1}{2}}e^{-\frac{x^2}{2\sigma^2}}d(\frac{x^2}{2\sigma^2})=\sqrt{2}\sigma\times\Gamma\left(\frac{3}{2}\right)=\sqrt{\frac{\pi}{2}}\sigma$

4. $E(X)=\int_a^{+\infty}xd(1-\frac{a^3}{x^3})=\int_a^{+\infty}\frac{3a^3}{x^3}dx=\frac{3}{2}a$

5. $E(X)=\int_{-\infty}^{+\infty}\frac{x}{\pi(1+x^2)}dx=\frac{1}{2\pi}\ln(1+x^2)\mid_{-\infty}^{+\infty}$,

   因反常积分$\int_{-\infty}^{+\infty}\frac{x}{1+x^2}dx\sim\int_{-\infty}^{+\infty}\frac{1}{x}dx$不绝对收敛，故柯西分布的$E(X)$不存在

6. $E(X)=E(sinX)=\int_0^{2\pi}\sin x\frac{1}{2\pi-0}dx=0$

7. $(1)E(|X-E(X)|)=E(|X-\mu|)$

   $=-\int_{-\infty}^{\mu}\frac{x-\mu}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx+\int_{\mu}^{+\infty}\frac{x-\mu}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx$

   $=2\int_0^{+\infty}\frac{\sigma}{\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}d\left(\frac{(x-\mu)^2}{2\sigma^2}\right)=2\frac{\sigma}{\sqrt{2\pi}}\Gamma(1)=\sqrt{\frac{2}{\pi}}\sigma$

   $(2)E(Y)=\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x^2}{2\sigma^2}}dx=\frac{1}{\sqrt{2\pi}\sigma}\sqrt{2}\sigma\Gamma(1)=1$

8. $E(X)=\frac{1}{2}m\int_0^{+\infty}\frac{4x^4}{a^3\sqrt{\pi}}e^{-\frac{x^2}{a^2}}dx=\frac{2m}{a^3\sqrt{\pi}}\int_0^{+\infty}\frac{a^5}{2}\left(\frac{x^2}{a^2}\right)^{\frac{3}{2}}e^{-\frac{x^2}{a^2}}d(\frac{x^2}{a^2})$

   $=\frac{2m}{a^3\sqrt{\pi}}\frac{a^5}{2}\Gamma\left(\frac{5}{2}\right)=\frac{3}{4}ma^2$

9. 设净利润为Y元，$P\{Y=-200\}=P\{X\le1\}=\int_0^1\frac{1}{4}e^{-\frac{x}{4}}dx=1-e^{-\frac{1}{4}}$,

   $P\{Y=100\}=1-P\{Y=-200\}=e^{-\frac{1}{4}}$, $E(Y)=300e^{-\frac{1}{4}}-200$

10. (1)由$E(X) = -(a + 0.1) + (0.2 + c) = 0$,

$E(Y) = (a + 0.2) + 0.4 + 3(b + c) = 2, a + b + c + 0.4 = 1$,

解得$a = 0.2, b = 0.3, c = 0.1$

(2)$Z = (X - Y)^2$, Z的分布律为 $\begin{array}{cccccc} Z & 0 & 1 & 4 & 9 & 16 \\ P & 0.1 & 0.2 & 0.3 & 0.4 & 0 \end{array}$ 故$E(Z) = 5$

(3)$Z = X^2Y$, Z 的分布律为 $\begin{array}{ccccc} Z & 0 & 1 & 2 & 3 \\ p & 0.4 & 0.3 & 0.2 & 0.1 \end{array}$ 故$E(Z) = 1$

11. $E(X) = \int_0^1 dy \int_0^2 \frac{1}{3}(x + y)x\,dx = \frac{11}{9}$,

$$E(Y) = \int_0^1 dy \int_0^2 \frac{1}{3}(x + y)y\,dx = \frac{5}{9}, E(XY) = \int_0^1 dy \int_0^2 \frac{1}{3}xy(x + y)dx = \frac{2}{3},$$

$$E(X^2 + Y^2) = \int_0^1 dy \int_0^2 \frac{1}{3}(x^2 + y^2)(x + y)dx = \frac{13}{6}$$

12. (X, Y)服从 $0 \le x \le a, 0 \le y \le a$上的均匀分布,

$$E(|X - Y|) = \int_0^a dx \int_0^x \frac{1}{a^2}(x - y)dy + \int_0^a dx \int_x^a \frac{1}{a^2}(y - x)dy = \frac{a}{3}$$

13. (X, Y)服从$x^2 + y^2 \le R^2$上的均匀分布, $R = \sqrt{X^2 + Y^2}$,

$$E(R) = \int_0^{2\pi} d\varphi \int_0^R \frac{\rho}{\pi R^2}\rho d\rho = \frac{2}{3}R$$

14. 设随机变量$X_i = \begin{cases} 0, 不配对 \\ 1, 配对 \end{cases}$, $X = \sum_{i=1}^n X_i, E(X_i) = \frac{1}{n}, E(X) = E(\sum_{i=1}^n X_i) = 1$

15. (1)$X \sim U(0, 2), Y \sim exp(2), E(X + Y) = E(X) + E(Y) = 1 + \frac{1}{2} = \frac{3}{2}$,

$$E(X^2 - 2Y + 1) = E(X^2) - 2E(Y) + 1 = \frac{1}{12}(12 - 0)^2 + 1 - 2 \times \frac{1}{2} + 1 = \frac{4}{3}$$

(2)由X与Y相互独立, $E(XY) = E(X) \cdot E(Y) = \frac{1}{2}$

16. $P\left(X > \frac{\pi}{3}\right) = \int_{\frac{\pi}{3}}^{\pi} \frac{1}{2}\cos\frac{x}{2}dx = \frac{1}{2}$, $Y \sim B\left(4, \frac{1}{2}\right), E(Y^2) = D(Y) + E^2(Y) = 5$

17. (1)X, Y的联合分布律为
$\begin{array}{c|cc} Y\backslash X & -1 & 1 \\ \hline -1 & \frac{1}{4} & \frac{1}{2} \\ 1 & 0 & \frac{1}{4} \end{array}$

(2)$E(X) = \frac{1}{2}, E(Y) = -\frac{1}{2}, E(X^2) = 1, E(Y^2) = 1, D(X) = \frac{3}{4}, D(Y) = \frac{3}{4}$,

$$E(XY) = 0, Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{4},$$

$$故 D(X+Y) = DX + DY + 2Cov(X,Y) = 2$$

18. (1)$D(X) = E(X^2) - E(X)^2 = \frac{572}{1001}$

(2)$D(X) = \int_{-\infty}^{+\infty} \frac{1}{2} x^2 e^{-|x|} dx - 0 = 2$

(3)$E(X^2) = \int_0^{+\infty} \frac{x^3}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx = 2\sigma^2, D(X) = \left(2 - \frac{\pi}{2}\right)\sigma^2$

(4)$E(X^2) = \int_a^{+\infty} \frac{3a^3}{x^2} dx = 3a^2, D(X) = \frac{3}{4}a^2$

19. $E(Y^2) = \int_0^{2\pi} \sin^2 x \frac{1}{2\pi} dx = \frac{1}{2}, D(Y) = \frac{1}{2} - 0 = \frac{1}{2}$

20. $f_X(x) = \begin{cases} 2x, 0 \le x \le 1 \\ 0, other \end{cases}, f_Y(y) = \begin{cases} 1 - |y|, |y| \le 1 \\ 0, other \end{cases},$

$$E(X^2) = \int_0^1 2x^3 dx = \frac{1}{2}, E(X) = \int_0^1 2x^2 dx = \frac{2}{3},$$

$$E(Y^2) = \int_0^1 y^2(1-y) dy + \int_{-1}^0 y^2(1+y) dy = \frac{1}{6},$$

$$E(Y) = \int_{-1}^1 y(1 - |y|) dy = 0, D(X) = \frac{1}{18}, D(Y) = \frac{1}{6}$$

21. $E(X-Y)^2 = D(X-Y) + E^2(X-Y) = DX + DY = 2\sigma^2$

22.
XY	0	1	2	-1	-2
P	0.6	0.2	0	0.2	0

$E(XY) = 0, E(X) = 0.4, E(Y) = 0,$

$Cov(X,Y) = 0, \rho_{XY} = 0$

23. $E(XY) = \int_0^1 dx \int_0^1 xy(2 - x - y) dy = \frac{1}{6},$

$$E(X) = \int_0^1 dx \int_0^1 x(2 - x - y) dy = \frac{5}{12},$$

$$E(Y) = \int_0^1 dx \int_0^1 y(2 - x - y) dy = \frac{5}{12},$$

$$E(X^2) = \int_0^1 dx \int_0^1 x^2(2 - x - y) dy = \frac{1}{4},$$

$$E(Y^2) = \int_0^1 dx \int_0^1 y^2(2-x-y)dy = \frac{1}{4},$$

$$Cov(X,Y) = \frac{1}{6} - \left(\frac{5}{12}\right)^2 = -\frac{1}{144}, D(X) = D(Y) = \frac{11}{144},$$

$$\rho_{XY} = \frac{Cov(X,Y)}{\sqrt{DX}\sqrt{DY}} = -\frac{1}{11},$$

$$D(2X - Y + 1) = 4D(X) + D(Y) - 4Cov(X,Y) = \frac{59}{144}$$

24. $E(XY) = \int_0^2 dy \int_0^2 xy\frac{1}{8}(x+y)dx = \frac{4}{3}, E(X) = E(Y) = \frac{7}{6},$

$$E^2(X) = E^2(Y) = \frac{5}{3}, D(X) = D(Y) = \frac{11}{36},$$

$$Cov(X,Y) = -\frac{1}{36}, \rho_{XY} = -\frac{1}{11}, D(X+Y) = \frac{5}{9}$$

25. $D(X)$与$D(Y)$均存在且大于零, $Cov(X,Y) = 0 \Leftrightarrow E(XY) = E(X) \cdot E(Y),$

$$\rho_{XY} = 0 \Leftrightarrow X与Y \text{ 不相关} \Leftrightarrow D(X+Y) = D(X) + D(Y)$$

26. $Cov(X_1 - X_2, X_1X_2) = E(X_1^2 X_2) - E(X_2^2 X_1),$

因X_1与X_2相互独立, 方差存在, 故X_1^2与X_2, X_2^2与X_1也相互独立

$\therefore Cov(X_1 - X_2, X_1X_2) = E(X_1^2)E(X_2) - E(X_2^2)E(X_1) = 0 \therefore Y_1$与$Y_2$不相关

27. $(1)E(W) = E(X) + E(Y) + E(Z) = 1$
$(2)D(W) = D(X+Y) + D(Z) + 2Cov(X+Y,Z)$

$= D(X+Y) + D(Z) + 2\big(Cov(X,Z) + Cov(Y,Z)\big),$

$\because \rho_{XY} = 0 \therefore D(X+Y) = DX + DY = 2, \because \rho_{XZ} = \dfrac{Cov(X,Z)}{\sqrt{D(X)}\sqrt{D(Z)}} = \dfrac{1}{2},$

$\rho_{YZ} = \dfrac{Cov(Y,Z)}{\sqrt{D(Y)}\sqrt{D(Z)}} = -\dfrac{1}{2}, \therefore D(Z) = 1, Cov(X,Z) = \dfrac{1}{2}, Cov(Y,Z) = -\dfrac{1}{2},$

$\therefore D(W) = 3$

28. XY的分布律为 $\begin{array}{c|ccc} XY & 0 & 1 & -1 \\ \hline P & 1 & 0 & 0 \end{array}$ $E(XY) = 0, EX = EY = 0,$

$\therefore Cov(X,Y) = 0, X$与$Y$不相关

$\because P\{X = -1, Y = -1\} = 0 \neq P\{X = -1\} \cdot P\{Y = -1\} = \dfrac{1}{16} \therefore X$与$Y$不独立

29. $\because D(X) = 1, Cov(X,Y) = -1, \rho_{XY} = -\frac{1}{2} \therefore D(Y) = 4$

$\therefore D(X - Y) = 1 + 4 - 2 \times (-1) = 7$

30. 若 $\rho_{XY} = 0$, 则 X 与 Y 不相关 $\therefore E(XY) = P(AB) = E(X) \cdot E(Y) = P(A)P(B)$,

即事件 A 与 B 相互独立，又由于下列事件中只要有一对相互独立，其他三对也相互独立

$$(A, B), (\overline{A}, B), (A, \overline{B}), (\overline{A}, \overline{B})$$

$$P\{X = 0, Y = 0\} = P(\overline{A} \cap \overline{B}) = P(\overline{A})P(\overline{B}) = P\{X = 0\}P\{Y = 0\},$$

$$P\{X = 0, Y = 1\} = P(\overline{A} \cap B) = P(\overline{A})P(B) = P\{X = 0\}P\{Y = 1\},$$

$$P\{X = 1, Y = 0\} = P(A \cap \overline{B}) = P(A)P(\overline{B}) = P\{X = 1\}P\{Y = 0\},$$

$$P\{X = 1, Y = 1\} = P(A \cap B) = P(A)P(B) = P\{X = 1\}P\{Y = 1\}$$

$\therefore$ 随机变量 X 与 Y 相互独立

31. (1) $\because Cov(XY, X) = E(X^2) \cdot E(Y) - E^2(X) \cdot E(Y) = (\sigma_1^2 + \mu_1^2)\mu_2 - \mu_1^2\mu_2 = \mu_2\sigma_1^2$, $D(XY) = E(X^2)E(Y^2) - (EX)^2(EY)^2 = \mu_1^2\sigma_2^2 + \mu_2^2\sigma_1^2 + \sigma_1^2\sigma_2^2$

$$\therefore \rho_{ZX} = \frac{\mu_2\sigma_1^2}{\sqrt{\mu_1^2\sigma_2^2 + \mu_2^2\sigma_1^2 + \sigma_1^2\sigma_2^2}\sigma_1} = \frac{\mu_2\sigma_1}{\sqrt{\mu_1^2\sigma_2^2 + \mu_2^2\sigma_1^2 + \sigma_1^2\sigma_2^2}}$$

(2) 由(1)知, 当 $\rho_{ZY} = 0$ 即 $\mu_1 = 0$,

Z 与 Y 不相关，但由于相关系数只是描述线性关系的量，故不可能有严格的线性关系

第四章

1. $P\{|x - \mu| < \varepsilon\sigma\} \geq 1 - \frac{D(x)}{(\varepsilon\sigma)^2} = 1 - \frac{1}{9} = \frac{8}{9}$ 故概率下界为$\frac{8}{9}$

2. $P\{E(X) - \varepsilon < X < E(X) + \varepsilon\} = P\{|X - E(X)| < \varepsilon\} \geq 1 - \left(\frac{0.3}{\varepsilon}\right)^2$

   令$1 - \left(\frac{0.3}{\varepsilon}\right)^2 \geq 0.9 \Rightarrow \varepsilon \geq 0.9487 \Rightarrow \varepsilon_{min} = 0.9487$

3. 设正面出现的次数为$X, X \sim B(1000, 0.5), E(X) = 500, D(X) = 250$

   $P\{400 < X < 600\} = P\{|X - 500| < 100\} \geq 1 - \frac{250}{100^2} = 0.975$

4. 设一天的生产量为$X, E(X) = 500, D(X) = 9^2$

   $P\{455 < X < 545\} = P\{|X - 500| < 45\} \geq 1 - (9/45)^2 = 24/25$

5. $E(X_i) = -2^n \cdot 2^{-(2n+1)} + 2^n \cdot 2^{-(2n+1)} = 0$

   由切比雪夫大数定律得，$\lim\limits_{n \to \infty} P\left\{\left|\frac{1}{n}\sum_{k-1}^{n} X_k - 0\right| < \varepsilon\right\} = 1$

6. 由独立同分布的大数定律，$E(X_i) = \frac{1+5}{2} = 3$ $\therefore Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{P} 3$

7. $X_i \sim exp(2) \ (i = 1, 2, \cdots, n), E(X_i) = 0.5, D(X_i) = 0.25, E(X_i^2) = 0.25 + 0.5^2 = 0.5$

   由独立同分布的大数定律，$Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{P} E(X_i^2) = 0.5$

8. 设X_i为第i只部件的使用寿命$(i = 1, 2, \cdots, 30)$ $E(X_i) = 10, D(X_i) = 100$

   $$P\left\{\sum_{i=1}^{30} X_i > 350\right\} = P\left\{\frac{\sum_{i=1}^{30} X_i - 30 \times 10}{\sqrt{30} \times 10} > \frac{350 - 300}{\sqrt{30} \times 10}\right\} \approx 1 - \Phi\left(\frac{5}{\sqrt{30}}\right)$$
   $$\approx 1 - 0.8186 = 0.1814$$

9. 设$X_i = \begin{cases} 0, 第i道题答错 \\ 1, 第i道题答对 \end{cases}$, $(i = 1, 2, \cdots, 100)$ $X_i \sim \left(1, \frac{1}{4}\right), E(X_i) = \frac{1}{4}, D(X_i) = \frac{3}{16}$

   $$P\left\{\sum_{i=1}^{100} X_i > 40\right\} = P\left\{\frac{\sum_{i=1}^{100} X_i - 100 \times \frac{1}{4}}{\sqrt{100} \times \sqrt{\frac{3}{16}}} > \frac{40 - 25}{\sqrt{\frac{100 \times 3}{16}}}\right\} \approx 1 - \Phi\left(\frac{6}{\sqrt{3}}\right) \approx 1 - 1$$
   $$= 0$$

10. 设X_i为第i袋大米的重量$(i = 1, 2, \cdots, 100)$ $E(X_i) = 10, D(X_i) = 0.1$

$$P\left\{\left|\sum_{i=1}^{100} X_i - 1000\right| < 10\right\}$$

$$= P\left\{-\frac{10}{\sqrt{100 \times 0.1}} < \frac{\sum_{i=1}^{100} X_i - 1000}{\sqrt{100 \times 0.1}} < \frac{10}{\sqrt{100 \times 0.1}}\right\}$$

$$\approx \Phi(\sqrt{10}) - \Phi(-\sqrt{10}) \approx 0.9986$$

11. 设终端使用的次数为 X，$X \sim B(100,0.02)$

   则由二项分布可得：$P(X \geq 1) = 1 - P(X = 0) = 1 - 0.98^{100} \approx 0.8674$

   而由中心极限定理得：$P(X \geq 1) = P\left\{\frac{X - 100 \times 0.02}{\sqrt{100 \times 0.02 \times 0.98}} \geq \frac{1 - 100 \times 0.02}{\sqrt{100 \times 0.02 \times 0.98}}\right\} \approx 1 -$

$$\Phi(-0.71)$$
$$= \Phi(0.71) = 0.7611$$

12. 设投掷次数为 n，$X_i = \begin{cases} 0, \text{出现反面} \\ 1, \text{出现正面} \end{cases}$ $(i = 1,2,\cdots,100)$，$E(X_i) = 0.5$，$D(X_i) =$

   0.25

   由切比雪夫不等式有：

$$P\left\{\left|\frac{1}{n}\sum_{i=1}^{n} X_i - 0.5\right| < 0.1\right\} = P\left\{\left|\sum_{i=1}^{n} X_i - 0.5n\right| < 0.1n\right\} \geq 1 - \frac{\frac{1}{4} \cdot n}{(0.1n)^2}$$

$$= 1 - \frac{25}{n}$$

   令 $1 - \frac{25}{n} \geq 0.9 \Rightarrow n \geq 250$

   由中心极限定理有：

$$P\left\{0.4 \leq \frac{1}{n}\sum_{i=1}^{n} X_i \leq 0.6\right\} = P\left\{\frac{0.4 - 0.5}{\sqrt{\frac{1}{4} \cdot \frac{1}{n}}} \leq \frac{\frac{1}{n}\sum_{i=1}^{n} X_i - 0.5}{\sqrt{\frac{1}{4} \cdot \frac{1}{n}}} < \frac{0.6 - 0.5}{\sqrt{\frac{1}{4} \cdot \frac{1}{n}}}\right\}$$

$$\approx \Phi(0.2\sqrt{n}) - \Phi(-0.2\sqrt{n}) = 2\Phi(0.2\sqrt{n}) - 1 \geq 0.9$$

   查表可得 $0.2\sqrt{n} \geq 1.65 \Rightarrow n \geq 68$

13. 设需要 n 个车位，且第 i 户有车辆数为 $X_i (i = 1,2,\cdots,2000)$ $E(X_i) = 0.8$ $D(X_i) =$

   0.36

$$P\left\{\sum_{i=1}^{2000} X_i \leq n\right\} = P\left\{\frac{\sum_{i=1}^{2000} X_i - 2000 \times 0.8}{\sqrt{2000 \times 0.36}} < \frac{n - 1600}{\sqrt{2000 \times 0.36}}\right\} \approx \Phi\left(\frac{n - 1600}{12\sqrt{5}}\right)$$

$$\geq 0.95$$

查表后可得$\frac{n-1600}{12\sqrt{5}} \geq 1.65 \Rightarrow n \geq 1644$

14. (1)设售出第i只蛋糕的价格为$X_i(i = 1,2,\cdots,300), E(X_i) = 4.2, D(X_i) = 0.31$

$$P\left\{\sum_{i=1}^{300} X_i \geq 1200\right\} = P\left\{\frac{\sum_{i=1}^{300} X_i - 300 \times 4.2}{\sqrt{300 \times 0.31}} \geq \frac{1200 - 300 \times 4.2}{\sqrt{300 \times 0.31}}\right\}$$

$$\approx 1 - \Phi(-6.22) \approx 1$$

(2)设$Y_i = \begin{cases} 0, \text{出售价格不为 4 元} \\ 1, \text{出售价格恰为 4 元} \end{cases}$ $(i = 1,2,\cdots,300), E(Y_i) = 0.5, D(Y_i) = 0.25$

$$P\left\{\sum_{i=1}^{300} Y_i > 100\right\} = P\left\{\frac{\sum_{i=1}^{300} Y_i - 300 \times 0.5}{\sqrt{300 \times 0.25}} > \frac{100 - 300 \times 0.5}{\sqrt{300 \times 0.25}}\right\}$$

$$\approx 1 - \Phi(-5.77) \approx 1$$

15. 设灯的使用盏数为$X, X \sim B(1000,0.7)$

$$P\{7800 \leq X \leq 8200\} = P\left\{\frac{7800 - 7000}{\sqrt{2100}} \leq \frac{X - 7000}{\sqrt{2100}} < \frac{8200 - 7000}{\sqrt{2100}}\right\}$$

$$\approx \Phi\left(\frac{1200}{\sqrt{2100}}\right) - \Phi\left(\frac{800}{\sqrt{2100}}\right) \approx \Phi(26.19) - \Phi(17.64) = 0$$

16. 设$X_i = \begin{cases} 0, \text{第}i\text{件产品为劣质产品} \\ 1, \text{第}i\text{件产品为优质产品} \end{cases}$, $(i = 1,2,\cdots,6000)$ $X_i \sim \left(1, \frac{1}{6}\right)$,

$$E(X_i) = \frac{1}{6}, D(X_i) = \frac{5}{36}$$

$$P\left\{\frac{1}{n}\left|\sum_{i=1}^{100} X_i - \frac{1}{6}\right| < 0.01\right\} = P\left\{\frac{-0.01}{\sqrt{\frac{5}{36n}}} \leq \frac{\frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{6}}{\sqrt{\frac{5}{36n}}} < \frac{0.01}{\sqrt{\frac{5}{36n}}}\right\}$$

$$\approx 2\Phi\left(\frac{0.01}{\sqrt{\frac{5}{36n}}}\right) - 1 = 2 \times 0.9808 - 1 = 0.9616$$

17. 设 正 品 数 为 $\eta_n, \eta_n \sim B(10000,0.8), X_i = \begin{cases} 0, \text{第}i\text{只显像管为次品} \\ 1, \text{第}i\text{只显像管为正品} \end{cases}$, $(i = 1,2,\cdots,n)$

设每月生产n只, $Y_n = \sum_{i=1}^{n} X_i$

$$P\{Y_n < 10000\} < 0.003 \Rightarrow P\left\{\frac{Y_n - n \cdot 0.8}{\sqrt{n \cdot 0.8 \cdot 0.2}} < \frac{10000 - n \cdot 0.8}{\sqrt{n \cdot 0.8 \cdot 0.2}}\right\}$$

$$\approx \Phi\left(\frac{10000 - 0.8n}{\sqrt{0.16n}}\right) < 0.003$$

查表得$\frac{10000-0.8n}{\sqrt{0.16n}} \le -2.75 \Rightarrow n \ge 12654.68$

故最少生产 12655 只显像管

18. 设$Z_n = \frac{1}{n}\sum_{i=1}^{n} X_i^2$

$\because X_1, X_2, \cdots, X_n$独立同分布，故$X_1^2, X_2^2, \cdots, X_n^2$也独立同分布，且$E(X_i^2)$与

$D(X_i^2)$均存在。

由中心极限定理，n充分大时，Z_n近似服从于正态分布

$E(X_i^2) = \alpha_2, D(X_i^2) = E(X_i^4) - \left(E(X_i^2)\right)^2 \quad \therefore Z_n \sim N(\alpha_2, \frac{\alpha_4 - \alpha_2^2}{n})$

19. $E(X) = \int_0^{+\infty} \frac{x^{n+1}e^{-x}}{n!}dx = \frac{\Gamma(n+2)}{n!} = n + 1$

$$D(X) = E(X^2) - (E(X))^2 = \int_0^{+\infty} \frac{X^{n+1}e^{-X}}{n!}dx - (n+1)^2$$

$$= \frac{\Gamma(n+3)}{n!} - (n+1)^2 = n + 1$$

由切比雪夫不等式得:

$$P\{0 < x < 2(n+1)\} = P\{|X - (n+1)| < n+1\} \ge 1 - \frac{n+1}{(n+1)^2} = \frac{n}{n+1}$$

3. $f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i) = \begin{cases} \left(\frac{1}{b-a}\right)^n & a \leq x_1 \leq \cdots \leq x_n \\ 0 & \text{其它} \end{cases}$

4. $P\{X_1 = x_1, \ldots, X_n = x_n\} = \prod_{i=1}^{n} P(X_i = x_i) = \prod_{i=1}^{n} \frac{\lambda^{x_i} \cdot e^{-\lambda}}{(x_i)!} = \frac{\lambda^{\sum_{i=1}^{n} x_i} \cdot e^{-n\lambda}}{\prod_{i=1}^{n}(x_i)!}$

5. $F_3(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{3} & -1 \leq x < 0 \\ \frac{2}{3} & 0 \leq x < 6 \\ 1 & x \geq 6 \end{cases}$

7. $E(\bar{X}) = E(X_i) = 0$

   $D(\bar{X}) = D\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} \cdot n \cdot D(X_i) = \frac{1}{3n}$

8. 根据题意可得: $\overline{X_1} \sim N(\mu, \frac{\sigma^2}{n})$ $\overline{X_2} \sim N(\mu, \frac{\sigma^2}{n})$

又因为上述两者相互独立, 有: $\overline{X_1} - \overline{X_2} \sim N(0, \frac{2\sigma^2}{n})$

因此有: $P\{|\overline{X_1} - \overline{X_2} - 0| > \sigma\} = 1 - P\{|\overline{X_1} - \overline{X_2}| \leq \sigma\} = 2\left[1 - \Phi\left(\frac{\sqrt{n}\sigma}{\sqrt{2}\sigma}\right)\right] \leq 0.01$

于是: $\sqrt{\frac{n}{2}} \geq 2.58$, n取14

9. (1)$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{1}{n}\sum_{i=1}^{n}(a + cY_i) = a + \frac{c}{n}\sum_{i=1}^{n} Y_i = a + c\bar{Y}$

$$S_Y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} Y_i^2 - n\bar{Y}^2\right]$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n}\left(\frac{X_i - a}{c}\right)^2 - n\left(\frac{\bar{X} - a}{c}\right)^2\right]$$

$$= \frac{1}{c^2} \cdot \frac{1}{n-1}\left[\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right] = \frac{1}{c^2}S_X^2$$

(2) $E(\bar{Y}) = \frac{1}{c}[E(\bar{X}) - a] = \frac{1}{c}(\mu - a)$

   $E(S_Y^2) = \frac{1}{c^2}E(S_X^2) = \frac{\sigma^2}{c^2}$

10. (1) $\sum_{i=1}^{n}(X_i - a)^2 = \sum_{i=1}^{n}(X_i - \overline{X_n} + \overline{X_n} - a)^2$
    $= \sum_{i=1}^{n}(X_i - \overline{X_n})^2 + 2\sum_{i=1}^{n}(X_i - \overline{X_n})(\overline{X_n} - a) + n(\overline{X_n} - a)^2$
    $= \sum_{i=1}^{n}(X_i - \overline{X_n})^2 + n(\overline{X_n} - a)^2$

其中: $2\sum_{i=1}^{n}(X_i - \overline{X_n})(\overline{X_n} - a) = \sum_{i=1}^{n}(X_i\overline{X_n} - aX_i - \overline{X_n}^2 + a\overline{X_n})$

$$= n\overline{X_n}^2 - a \cdot n\overline{X_n} - n\overline{X_n}^2 + a \cdot n\overline{X_n} = 0$$

(2) $\overline{X_{n+1}} = \dfrac{1}{n+1}\sum_{i=1}^{n+1} X_i = \dfrac{1}{n+1}\left(X_{n+1} + \sum_{i=1}^{n} X_i\right)$

$\qquad\qquad = \dfrac{1}{n+1}(X_{n+1} + n\overline{X_n} + \overline{X_n} - \overline{X_n})$

$\qquad\qquad = \overline{X_n} + \dfrac{1}{n+1}(X_{n+1} - \overline{X_n})$

$\qquad S_{n+1}^2 = \dfrac{1}{n}\sum_{i=1}^{n+1}(X_i - \overline{X_{n+1}})^2 = \dfrac{1}{n}[\sum_{i=1}^{n+1} X_i^2 - (n+1)\overline{X_{n+1}}^2]$

$\qquad S_n^2 = \dfrac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X_n})^2 = \dfrac{1}{n-1}[\sum_{i=1}^{n} X_i^2 - n\overline{X_n}^2]$

因此：$S_{n+1}^2 - \dfrac{n-1}{n}S_n^2 = \dfrac{1}{n}\left[\sum_{i=1}^{n+1} X_i^2 - (n+1)\overline{X_{n+1}}^2\right] - \dfrac{1}{n}[\sum_{i=1}^{n} X_i^2 - n\overline{X_n}^2]$

$\qquad\qquad = \dfrac{X_{n+1}^2}{n} - \dfrac{n+1}{n}\overline{X_{n+1}}^2 + \overline{X_n}^2 = \dfrac{X_{n+1}^2}{n} - \dfrac{n+1}{n}\left[\overline{X_n} + \dfrac{1}{n+1}(X_{n+1} - \overline{X_n})\right]^2 + \overline{X_n}^2$

$\qquad\qquad = \dfrac{1}{n+1}(X_{n+1} - \overline{X_n})^2$

11. 由于 $X \sim P(\lambda)$，且 $X_1, X_2, \ldots, X_n$ 相互独立，故有 $\sum_{i=1}^{n} X_i \sim P(n\lambda)$（泊松分布的

可加性）

所以 $P(\sum_{i=1}^{n} X_i = k) = P\left(\bar{X} = \dfrac{k}{n}\right) = \dfrac{(n\lambda)^k}{k!}e^{-n\lambda}(k = 0,1,\ldots)$

12. $X_\tau \sim \Gamma(\alpha,\beta)$ $(i = 1,2,\ldots)$ 由 $Gamma$ 分布第一参数可加性：$\sum_{i=1}^{n} X_i \sim \Gamma(n\alpha,\beta)$

下面以此来求 $\dfrac{1}{n}\sum_{i=1}^{n} X_i$ $(X > 0)$ 的分布规律：

$P\left(\dfrac{1}{n}\sum_{i=1}^{n} X_i \leq x\right) = P(\sum_{i=1}^{n} X_i \leq nx) = \int_0^{nx} \dfrac{\beta^{n\alpha}}{\Gamma(n\alpha)} t^{n\alpha-1} \cdot e^{-\beta t} dt$

对上式求导得到概率密度函数：

$f_{\bar{X}}(x) = \begin{cases} n\dfrac{\beta^{n\alpha}}{\Gamma(n\alpha)}(nx)^{n\alpha-1} \cdot e^{-n\beta x} = \dfrac{(n\beta)^{n\alpha}}{\Gamma(n\alpha)} x^{n\alpha-1} \cdot e^{-n\beta x} & (x > 0) \\ 0 & (x \leq 0) \end{cases} \sim \Gamma(n\alpha, n\beta)$

13. 由于 $X_1 - 2X_2 \sim N(0,20)$，故 $\dfrac{(X_1 - 2X_2)^2}{20} \sim \chi^2(1)$

同时 $3X_3 - 4X_4 \sim N(0,100)$，故 $\dfrac{(3X_3 - 4X_4)^2}{100} \sim \chi^2(1)$

所以 $Y = \dfrac{(X_1 - 2X_2)^2}{20} + \dfrac{(3X_3 - 4X_4)^2}{100} \sim \chi^2(2)$，于是 $a = \dfrac{1}{20}$，$b = \dfrac{1}{100}$

14. (1) $P\left\{\dfrac{S^2}{\sigma^2} \leq 2.041\right\} = P\left\{\dfrac{(16-1)S^2}{\sigma^2} \leq 15 \times 2.041\right\}$ 又因为 $\dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$

因此 $P\{\chi^2(15) \leq 15 \times 2.041\} = 0.99$

$(2) D(S^2) = (\frac{\sigma^2}{n-1})^2 \cdot D\left(\frac{(n-1)S^2}{\sigma^2}\right) = \frac{\sigma^4}{(n-1)^2} \cdot 2(n-1) = \frac{2\sigma^4}{15}$

15. $Y = \frac{1}{\sigma^2} \sum_{i=1}^{20} (x_i - \mu)^2 \sim \chi^2(20)$

16. $\sum_{i=1}^{m} X_i \sim N(0, m), \quad \sum_{i=m+1}^{n} X_i \sim N(0, n-m)$

故有: $\frac{1}{m}(\sum_{i=1}^{m} X_i)^2 \sim \chi^2(1), \quad \frac{1}{n-m}(\sum_{i=m+1}^{n} X_i)^2 \sim \chi^2(1)$

因此: $Y \sim \chi^2(2)$

17. $X \sim exp(\frac{1}{2})$ 故: $X \sim \chi^2(2)$ $Y = \sum_{i=1}^{n} X_i \sim \chi^2(2n)$ (根据χ^2分布的可加性)

18. $T = -2\sum_{i=1}^{n} lnF(X_i) \qquad Y_i = -2lnF(X_i)$

下证: $Y_i \sim \chi^2(2)$

当$y \leq 0$时, $F_Y(y) = P(Y \leq y) = 0$

当$y > 0$时, $F_Y(y) = P(Y \leq y) = P\{-2lnF(x) \leq y\} = P\left\{F(x) \geq e^{-\frac{1}{2}y}\right\}$

$$= P\left\{x \geq F^{-1}\left(e^{-\frac{1}{2}y}\right)\right\} = 1 - P\left\{x \leq F^{-1}\left(e^{-\frac{1}{2}y}\right)\right\}$$

$$= 1 - F\left(F^{-1}\left(e^{-\frac{1}{2}y}\right)\right) = 1 - e^{-\frac{1}{2}y}$$

也即: $Y \sim exp(\frac{1}{2})$, $Y_i \sim \chi^2(2)$ 因此$T \sim \chi^2(2n)$ (由于$T = \sum_{i=1}^{n} Y_i$且Y_i相互独立)

19. $U = X_{n+1} - \overline{X_n} \sim N\left(0, \sigma^2\left(1 + \frac{1}{n}\right)\right)$

$V = \frac{(n-1)S_n^2}{\sigma^2} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \overline{X_n})^2 = \chi^2(n-1)$

因此: $\frac{U/\sqrt{\frac{n+1}{n}}\sigma}{\sqrt{V/(n-1)}} = \frac{X_{n+1} - \overline{X_n}}{\sqrt{\frac{n+1}{n}}S_n} = \sqrt{\frac{n}{n+1}} \cdot \frac{X_{n+1} - \overline{X_n}}{S_n} \sim t(n-1)$ 即$c = \sqrt{\frac{n}{n+1}}$

20. $Y_1 = \frac{\sqrt{m}\sum_{i=1}^{n} X_i}{\sqrt{n}\sqrt{\sum_{i=n+1}^{n+m} X_i^2}} = \frac{\frac{1}{\sqrt{n}} \cdot \sum_{i=1}^{n} \frac{X_i}{\sigma}}{\sqrt{\frac{1}{m} \cdot \sum_{i=n+1}^{n+m} (\frac{X_i}{\sigma})^2}} \sim t(m)$

$(2) Y_2 = \frac{\sum_{i=1}^{n} X_i^2/n}{\sum_{i=n+1}^{n+m} X_i^2/m} \sim F(n, m)$

21. $P\{X > 1\} = P\left\{\frac{1}{X} < 1\right\}$ 由于$X \sim F(n, n)$, 故: $\frac{1}{X} \sim F(n, n)$

于是有: $P\{X > 1\} = \frac{1}{2}\{P(X > 1) + P(X \leq 1)\}$ 且$P\{X = 1\} = 0$

因此$P\{X > 1\} = 0.5$

22. 由于$X_1 + X_2 \sim N(0, 2\sigma^2)$　　$X_1 - X_2 \sim N(0, 2\sigma^2)$

故有：$\dfrac{(X_1+X_2)^2}{2\sigma^2} \sim \chi^2(1)$　　$\dfrac{(X_1-X_2)^2}{2\sigma^2} \sim \chi^2(1)$

因此：$Y = \dfrac{(X_1+X_2)^2}{(X_1-X_2)^2} = \dfrac{\frac{(X_1+X_2)^2}{2\sigma^2}}{\frac{(X_1-X_2)^2}{2\sigma^2}} \sim F(1,1)$

23. $Y = \dfrac{\sum_{i=1}^{10} X_i^2}{2\sum_{i=11}^{15} X_i^2} = \dfrac{\sum_{i=1}^{10}\left(\frac{X_i}{\sigma}\right)^2}{2\sum_{i=11}^{15}\left(\frac{X_i}{\sigma}\right)^2} = \dfrac{\sum_{i=1}^{10}\left(\frac{X_i}{\sigma}\right)^2/10}{2\sum_{i=11}^{15}\left(\frac{X_i}{\sigma}\right)^2/10} = \dfrac{\sum_{i=1}^{10}\left(\frac{X_i}{\sigma}\right)^2/10}{\sum_{i=11}^{15}\left(\frac{X_i}{\sigma}\right)^2/5} \sim F(10,5)$

24. 由于$X \sim t(n)$　令$X = \dfrac{U}{\sqrt{V/n}}$　故：$X^2 = \dfrac{U^2/1}{V/n}$　又因为：$U^2 \sim \chi^2(1)$　$V \sim \chi^2(n)$

因此：$X^2 \sim F(1, n)$

25. $F = \dfrac{\sum_{i=1}^{n_1}\left(\frac{X_i-\mu_1}{\sigma_1}\right)^2/n_1}{\sum_{i=1}^{n_2}\left(\frac{X_i-\mu_2}{\sigma_2}\right)^2/n_2} \sim F(n_1, n_2)$

26. $Y = 2\lambda \cdot \sum_{i=1}^{n} X_i$　　令$T_i = 2\lambda \cdot X_i$

下证：$T_i \sim \chi^2(2)$　$(i = 1, \ldots, n)$

$f_T(t) = \begin{cases} \left|\frac{1}{2\lambda}\right| \lambda \cdot e^{-\frac{\lambda t}{2\lambda}} = \frac{1}{2} e^{-\frac{1}{2}t} & t \geq 0 \\ 0 & \text{其它} \end{cases}$　因此：$T_i \sim exp(\frac{1}{2})$，也即$T_i \sim \chi^2(2)$

又因为$Y = \sum_{i=1}^{n} T_i$且T_i相互独立，所以：$Y \sim \chi^2(2n)$

1. (1) $\alpha_1 = E(X) = \frac{1}{\lambda} = \bar{A} = \bar{X}$. 矩估计量$\hat{\lambda} = \frac{1}{\bar{X}}$, 矩估计值$\hat{\lambda} = \frac{1}{\bar{x}}$.

   $L(\lambda) = \prod_{i=1}^{n} \lambda\, e^{-\lambda X_i} = \lambda^n e^{-\lambda \sum_{i=1}^{n} X_i},\ X_i > 0.\ \ln L(\lambda) = n \ln \lambda - \lambda \sum_{i=1}^{n} X_i.$

   令$\frac{d \ln L(\lambda)}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} X_i = 0 \Rightarrow \hat{\lambda} = \frac{1}{\bar{X}}$. 极大的矩估计值$\hat{\lambda} = \frac{1}{\bar{X}}$, 矩估计量$\hat{\lambda} = \frac{1}{\bar{x}}$.

(2) $\alpha_1 = E(X) = mp = \bar{X}$. 矩估计量$\hat{p} = \frac{\bar{X}}{m}$, 矩估计值$\hat{p} = \frac{\bar{x}}{m}$.

$L(p) = \prod_{k=1}^{n} C_m^{X_k}(1-p)^{m-X_k} p^{X_k} = \prod_{k=1}^{n} C_m^{X_k}(1-p)^{m\cdot n - \Sigma_{k=1}^{n} X_k}\cdot p^{\Sigma_{k=1}^{n} X_k}$

   $\ln L(p) = \ln c + (mn - \sum_{k=1}^{n} X_k)\cdot \ln(1-p) + \sum_{k=1}^{n} X_k \cdot \ln p$

   令$\frac{d \ln L(p)}{dp} = -\frac{mn - \sum_{k=1}^{n} X_k}{1-p} + \sum_{k=1}^{n} X_k \cdot \frac{1}{p} = 0 \Rightarrow \hat{p} = \frac{1}{mn} \sum_{k=1}^{n} X_k = \frac{\bar{X}}{m}$

(3) $\begin{cases} \alpha_1 = E(X) = \dfrac{\theta_1 + \theta_2}{2} = \bar{X} \\[2mm] \alpha_2 = E(X^2) = (\dfrac{\theta_1 + \theta_2}{2})^2 + \dfrac{1}{12}(\theta_2 - \theta_1)^2 = \dfrac{1}{12}\sum_{i=1}^{n} X_i{}^2 \end{cases}$

   解得$\hat{\theta}_1 = \bar{X} - \sqrt{3B_2} = \bar{X} - \sqrt{\frac{3(n-1)}{n}}S$

   $\hat{\theta}_2 = \bar{X} + \sqrt{3B_2} = \bar{X} + \sqrt{\frac{3(n-1)}{n}}S$.

   $L(\theta_1, \theta_2) = \begin{cases} \dfrac{1}{(\theta_2 - \theta_1)^n}, \theta_1 \le X_1, \dots, X_n \le \theta_2 \\[2mm] \qquad 0, 其他 \end{cases}$

当$\theta_1 = X_{(1)},\ \theta_2 = X_{(n)}$时, $L(\theta_1, \theta_2)$有最大值, $\hat{\theta}_1 = X_{(1)},\ \hat{\theta}_2 = X_{(n)}$.

(4) $\alpha_1 = E(X) = \int_0^1 \sqrt{\theta}\, X^{\sqrt{\theta}-1+1} dX = \frac{\sqrt{\theta}}{\sqrt{\theta}+1} = \bar{X}$. 即$\hat{\theta} = (\frac{\bar{X}}{1-\bar{X}})^2$.

   $L(\theta) = \prod_{i=1}^{n} f(X_i) = \theta^{\frac{n}{2}}\cdot(\prod_{i=1}^{n} X_i)^{\sqrt{\theta}-1}.\ \ln L(\theta) = \frac{n}{2}\ln\theta + (\sqrt{\theta}-1)\sum_{i=1}^{n} \ln X_i$

   令$\frac{d \ln L(\theta)}{d\theta} = \frac{n}{2\theta} + \frac{1}{2\sqrt{\theta}}\cdot \sum_{i=1}^{n} \ln X_i = 0$. 得$\hat{\theta} = \frac{n^2}{(\sum_{i=1}^{n} \ln X_i)^2}$

(5) $\alpha_1 = E(X) = \int_c^{+\infty} \theta\cdot c^{\theta}\cdot X^{-\theta}\, dX = \frac{c\theta}{\theta-1} = \bar{X}$. 即$\hat{\theta} = \frac{\bar{X}}{\bar{X}-c}$.

   $L(\theta) = \prod_{i=1}^{n} \theta \cdot c^{\theta} \cdot X_i{}^{-(\theta+1)} = \theta^n \cdot c^{n\theta} \cdot (\prod_{i=1}^{n} X_i)^{-(\theta+1)}$

   $\ln L(\theta) = n \ln \theta + n\theta \ln c - (\theta+1)\cdot \sum_{i=1}^{n} \ln X_i.$

   令$\frac{d \ln L(\theta)}{d\theta} = \frac{n}{\theta} + n \ln c - \sum_{i=1}^{n} \ln X_i = 0$

得 $\theta = \frac{n}{\sum_{i=1}^{n} ln\, X_i - n\, ln\, c}$ $\therefore \hat{\theta} = \frac{n}{\sum_{i=1}^{n} ln\, X_i - n\, ln\, c}$

2. $\alpha_1 = E(X) = \int_0^1 (\beta + 1) X^{\beta+1}\, dX = \frac{\beta+1}{\beta+2} = \bar{X}$. 即 $\hat{\beta} = \frac{1-2\bar{X}}{\bar{X}-1}$

$L(\beta) = \prod_{i=1}^{n}(\beta + 1) X_i{}^{\beta} = (\beta + 1)^n (\prod_{i=1}^{n} X_i)^{\beta}$

$$ln\, L(\beta) = n\, ln\, L(\beta + 1) + \beta \sum_{i=1}^{n} ln\, X_i$$

令 $\frac{d\, ln\, L(\beta)}{d\beta} = \frac{n}{\beta+1} + \sum_{i=1}^{n} ln\, X_i = 0$, 得 $\hat{\beta} = -(1 + \frac{n}{\sum_{i=1}^{n} \ln X_i})$

3. $(1)\alpha_1 = E(X) = 1 \times \theta^2 + 2 \times 2\theta(1-\theta) + 3 \times (1-\theta)^2 = 3 - 2\theta = \bar{X}$

$\hat{\theta} = \frac{3-\bar{X}}{2}$ 当观测值为 $(1,2,1)$ 时, $\hat{\theta} = \frac{5}{6}$

$(2)L(\theta) = \prod_{i=1}^{n} P(X = X_i) = (\theta^2)^2 \cdot 2\theta \cdot (1-\theta) = 2\theta^5 \cdot (1-\theta) = 2\theta^5 - 2\theta^6$

令 $\frac{d\, L(\theta)}{d\theta} = 0$, 得 $\hat{\theta} = \frac{5}{6}$.

4. (1) $\begin{cases} \alpha_1 = E(X) = \int_{\theta_1}^{+\infty} \frac{X}{\theta_2} \cdot e^{-\frac{X-\theta_1}{\theta_2}}\, dX = \theta_1 + \theta_2 = \bar{X} \\ \alpha_2 = E(X^2) = \int_{\theta_1}^{+\infty} \frac{X^2}{\theta_2} \cdot e^{-\frac{X-\theta_1}{\theta_2}}\, dX = \theta_1{}^2 + 2\theta_1\theta_2 + 2\theta_2{}^2 = A_2 \end{cases}$

解得 $\hat{\theta}_1 = \bar{X} - \sqrt{B_2}$, $\hat{\theta}_2 = \sqrt{B_2}$

$(2)L(\theta_1, \theta_2) = \prod_{i=1}^{n} \frac{1}{\theta_2} \cdot e^{-\frac{X_i-\theta_1}{\theta_2}} = \frac{1}{\theta_2{}^n} \cdot e^{-\frac{\sum_{i=1}^{n} ln\, X_i - n\theta_1}{\theta_2}} = e^{-\frac{n\theta_1}{\theta_2}} \cdot \frac{1}{\theta_2{}^n} \cdot e^{-\frac{\sum_{i=1}^{n} X_i}{\theta_2}} (X_i > \theta_1)$

$ln\, L(\theta_1, \theta_2) = -n ln\, \theta_2 + \frac{n\theta_1}{\theta_2} - \frac{1}{\theta_2} \cdot \sum_{i=1}^{n} X_i$

令 $\frac{\partial\, ln\, L(\theta_1, \theta_2)}{\partial \theta_1} = \frac{n}{\theta_2} = 0$, $\frac{\partial\, ln\, L(\theta_1, \theta_2)}{\partial \theta_2} = -\frac{n}{\theta_2} + \frac{\sum_{i=1}^{n} X_i - n\theta_1}{\theta_2{}^2} = 0 \Rightarrow \theta_1 + \theta_2 = \bar{X}$.

只能通过定义求

当 $\theta_1 = X_{(1)}$ 时, $\theta_2 = \bar{X} - X_{(1)}$, $L(\theta_1, \theta_2)$ 有最大值. 故 $\hat{\theta}_1 = X_{(1)}$, $\hat{\theta}_2 = \bar{X} - X_{(1)}$

5. $X \sim P(\lambda)$ $P\{X = 0\} = \frac{\lambda^0 e^{-\lambda}}{0!} = e^{-\lambda}$, 由例 6.1.10, λ 的极大的矩估计量为 $\bar{X}$,

又由性质 6.1.1, $e^{-\hat{\lambda}}$ 也是极大的矩估计. $\therefore \hat{P}\{X = 0\} = e^{-\hat{\lambda}} = e^{-\hat{X}}$

6. $L(\theta) = \prod_{i=1}^{n} \theta X_i{}^{\theta-1} = \theta^n \cdot (\prod_{i=1}^{n} X_i)^{\theta-1}$, $0 < X_1, ..., X_n < 1$

$ln\, L(\theta) = n\, ln\, \theta + (\theta - 1) \sum_{i=1}^{n} X_i$ 令 $\frac{d\, ln\, L(\theta)}{d\theta} = \frac{n}{\theta} + \sum_{i=1}^{n} ln\, X_i = 0$

得 $\hat{\theta} = -\frac{n}{\sum_{i=1}^{n} ln\, X_i}$. 由性质 6.1.1, $\hat{T} = e^{\frac{\sum_{i=1}^{n} ln\, X_i}{n}}$.

7.　(1)$E(c \sum_{i=1}^{n-1} (X_{i+1} - X_i)^2)$

　　$= c \cdot D(\sum_{i=1}^{n-1}(X_{i+1} - X_i)) + c \cdot E^2(\sum_{i=1}^{n-1}(X_{i+1} - X_i))$

　　$= c \cdot (n-1) \cdot 2\sigma^2 + 0 = \sigma^2 \Rightarrow c = \frac{1}{2(n-1)}$

(2)$E(\bar{X}^2 - cS^2) = D(\bar{X}) + E^2(\bar{X}) - c \cdot E(S^2) = \frac{\sigma^2}{n} + \mu^2 - c \cdot \sigma^2 = \mu^2$

　　$\Rightarrow c = \frac{1}{n}$

8.　(1)$E(T_1) = \frac{1}{6} \cdot 2\theta + \frac{1}{3} \cdot 2\theta = \theta$. $E(T_2) = \frac{1}{5} \cdot (\theta + 2\theta + 3\theta + 4\theta) = 2\theta \neq \theta$

　　$E(T_3) = \frac{1}{4} \cdot 4\theta = \theta$　$\therefore T_1, T_3$ 为 θ 的无偏估计量

(2)$D(T_1) = \frac{1}{36} \cdot 2\theta^2 + \frac{1}{9} \cdot 2\theta^2 = \frac{5}{18}\theta^2$, $D(T_3) = \frac{1}{16} \cdot 4\theta^2 = \frac{1}{4}\theta^2$

　　$D(T_1) > D(T_3)$　$\therefore T_3$ 较为有效

9.　(1)$\because E(\bar{X}) = p$, $D(\bar{X}) = \frac{p(1-p)}{n}$　$\therefore E(\bar{X}^2) = p^2 + \frac{p(1-p)}{n} = \frac{n-1}{n}p^2 + \frac{p}{n}$

　　$\therefore E(\frac{n}{n-1}\bar{X}^2 - \frac{\bar{X}}{n-1}) = \frac{n}{n-1} \cdot E(\bar{X}^2) - \frac{1}{n-1} \cdot E(\bar{X}) = p^2$(配凑)

　　即 $\hat{p}^2 = \frac{n}{n-1}(\bar{X}^2 - \frac{1}{n}\bar{X})$

(2)由(1)知, $E\left(\bar{X} - \left(\frac{n}{n-1}\bar{X}^2 - \frac{\bar{X}}{n-1}\right)\right) = E(\bar{X}) - E\left(\frac{n}{n-1}\bar{X}^2 - \frac{\bar{X}}{n-1}\right) = p(1-p)$

　　即 $\widehat{p(1-p)} = \bar{X} - (\frac{n}{n-1}\bar{X}^2 - \frac{\bar{X}}{n-1}) = \frac{n}{n-1}\bar{X}(1-\bar{X})$

10.　$\because E(\bar{X}) = \lambda$, $D(\bar{X}) = \frac{\lambda}{n}$, $E(S^2) = \lambda$

$\therefore$ 对于任意的常数 k, $E(k\bar{X} + (1-k)S^2) = k\lambda + (1-k) \cdot \lambda = \lambda$

　　即 $k\bar{X} + (1-k)S^2$ 是 λ 的无偏估计量

11.　$\because E(\hat{\theta}^2) = D(\hat{\theta}) + E^2(\hat{\theta}) = D(\hat{\theta}) + \theta^2 > \theta^2$　$\therefore \hat{\theta}^2$ 不是 θ^2 的无偏估计量

12.　$\because E(\hat{\theta}_1) = E(2\bar{X}) = 2 \cdot \frac{\theta}{2} = \theta$　$\therefore 2\bar{X}$ 是 θ 的无偏估计量

　　$D(2\bar{X}) = 4 \cdot \frac{\frac{1}{12}\theta^2}{n} = \frac{\theta^2}{3n}$

　　$X_{(n)}$ 即 $max\{X_1, ..., X_n\}$ 的分布函数 $F(x) = \begin{cases} (\frac{x}{\theta})^n, & 0 < x < \theta \\ 0, & 其他 \end{cases}$

$$\therefore f(x) = F'(x) = \begin{cases} n\dfrac{x^{n-1}}{\theta^n}, & 0 < x < \theta \\ 0, & \text{其他} \end{cases}$$

$$\therefore E(X_{(n)}) = \int_0^\theta n \cdot \frac{x^n}{\theta^n} dx = \frac{n}{n+1}\theta \quad \therefore E(\frac{n}{n+1}X_{(n)}) = \theta$$

$$\therefore \frac{n+1}{n}X_{(n)}\text{是}\theta\text{的无偏估计量}$$

$$D(\frac{n+1}{n}X_{(n)}) = (\frac{n+1}{n})^2 \cdot (\int_0^\theta n \cdot \frac{x^{n+1}}{\theta^n} dx - \left(\frac{n}{n+1}\right)^2 \cdot \theta^2) = \frac{1}{n(n+2)}\theta^2 < D(2\bar{X}) \ (n \geq 2)$$

$$\therefore n \geq 2\text{时}, \ \hat{\theta}_2\text{比}\hat{\theta}_1\text{更有效}$$

13. ① $2\bar{X}$: $\lim\limits_{n\to\infty} P\{|2\bar{X} - \theta| < \varepsilon\} = \lim\limits_{n\to\infty} P\{\left|\bar{X} - \frac{\theta}{2}\right| < \frac{\varepsilon}{2}\} \geq \lim\limits_{n\to\infty}(1 - \frac{\frac{\theta^2}{12} \cdot \frac{1}{n}}{(\frac{\varepsilon}{2})^2}) = 1$

又 $P(\left|\bar{X} - \frac{\theta}{2}\right| < \frac{\varepsilon}{2}) \leq 1$, 由夹逼准则知, $\lim\limits_{n\to\infty} P\{|2\bar{X} - \theta| < \varepsilon\} = 1$

$\therefore 2\bar{X}$为θ的相合估计量

$$\because E(2\bar{X} - \theta)^2 = D(2\bar{X} - \theta) + E^2(2X - \theta) = 4D(\bar{X}) + (2E(X) - \theta)^2 = \frac{\theta^2}{3n}$$

$$\therefore \lim\limits_{n\to\infty} E(2\bar{X} - \theta)^2 = 0 \quad \therefore 2\bar{X}\text{为}\theta\text{的均方相合估计量}$$

② $X_{(n)}$: $X_{(n)}$的概率密度为 $f(x) = \begin{cases} n\dfrac{x^{n-1}}{\theta^n}, & 0 \leq x \leq \theta \\ 0, & \text{其他} \end{cases}$

$$E(X_{(n)}) = \int_0^\theta \frac{nx^n}{\theta^n} dx = \frac{n}{n+1}\theta$$

$$D(X_{(n)}) = \int_0^\theta \frac{nx^{n+1}}{\theta^n} dx - \left(\frac{n}{n+1}\theta\right)^2 = \frac{n}{(n+2)(n+1)^2}\theta^2 \quad D(\frac{n}{n+1}X_{(n)}) = \frac{n^3}{(n+2)(n+1)^4}\theta^2$$

$$\lim\limits_{n\to\infty} P\{|X_{(n)} - \theta| < \varepsilon\} = \lim\limits_{n\to\infty} P\{\left|\frac{n}{n+1}X_{(n)} - \frac{n}{n+1}\theta\right| < \frac{n\varepsilon}{n+1}\}$$

$$\geq \lim\limits_{n\to\infty}\left(1 - \frac{n\theta^2}{(n+2)(n+1)^2\varepsilon^2}\right) = 1$$

$\therefore X_{(n)}$为θ的相合估计量

$$\because E(X_{(n)} - \theta)^2 = D(X_{(n)}) + (E(X_{(n)}) - \theta)^2 = \frac{2}{(n+2)(n+1)}\theta$$

$$\therefore \lim\limits_{n\to\infty} E(X_{(n)} - \theta)^2 = 0$$

$\therefore X_{(n)}$为θ的均方相合估计量

14. $\because \hat{\theta}$是θ的一个渐进无偏估计量 $\therefore \lim\limits_{n\to\infty}\left(E(\hat{\theta})-\theta\right)=0$

$\because$对$\forall \varepsilon>0$ $\lim\limits_{n\to\infty}P\{|\hat{\theta}-\theta|<\varepsilon\}=\lim\limits_{n\to\infty}P\{|\hat{\theta}-E(\hat{\theta})|<\varepsilon\}\geq\lim\limits_{n\to\infty}(1-\frac{D(\hat{\theta})}{\varepsilon^2})=1$

$\therefore \hat{\theta}$是θ的相合估计量

$\because \lim\limits_{n\to\infty}E(\hat{\theta}-\theta)^2=\lim\limits_{n\to\infty}((E(\hat{\theta})-\theta)^2+D(\hat{\theta}))=0$ $\therefore \hat{\theta}$是θ的均方相合估计量

15. 由已知, $X_i\sim B(1,p)$ $(i=1,\dots,100)$ $\bar{X}=p=0.6,\ \alpha=0.05$

由大样本的区间估计, 置信区间为:

$$(\bar{X}-u_{\frac{0.05}{2}}\sqrt{\frac{\bar{X}(1-\bar{X})}{100}},\bar{X}+u_{\frac{0.05}{2}}\sqrt{\frac{\bar{X}(1-\bar{X})}{100}})=(0.50,0.69)$$

16. $\bar{X}=15.0617,\ u_{\frac{0.05}{2}}=1.96.$

μ的置信度为0.95的置信区间$(\bar{X}\pm\frac{\sigma}{\sqrt{n}}u_{\frac{\alpha}{2}})=(14.88,15.20)$

17. $\bar{X}=10,\ S^2=0.16,\ \sigma^2$未知. μ的置信区间$(\bar{X}\pm\frac{S}{\sqrt{n}}t_{\frac{\alpha}{2}})=(9.7868,10.2132)$

18. μ的置信区间长度$L=\frac{2\sigma}{\sqrt{n}}u_{\frac{\alpha}{2}}=\frac{2\sigma\times1.96}{\sqrt{n}}\leq k$ 即$n\geq(\frac{3.926}{k})^2$

19. σ的置信度为0.9的置信区间$(\sqrt{\frac{(n-1)S^2}{X_{\frac{\alpha}{2}}^2(n-1)}},\sqrt{\frac{(n-1)S^2}{X_{1-\frac{\alpha}{2}}^2(n-1)}})=(0.15,0.31)$

$S=0.2,\ X_{\frac{0.1}{2}}^2(11)=19.675,\ X_{1-\frac{0.1}{2}}^2(11)=4.575$

20. $\bar{X}=12.075,\ S^2=\frac{1}{15}\times0.0366=0.00244,$

$X_{\frac{0.05}{2}}^2(15)=27.49,\ X_{1-\frac{0.05}{2}}^2(15)=6.26$

σ^2的置信区间为$(\frac{(n-1)S^2}{X_{\frac{\alpha}{2}}^2(n-1)},\frac{(n-1)S^2}{X_{1-\frac{\alpha}{2}}^2(n-1)})=(0.0013,0.0058)$

21. $\bar{X}=999.853,\ S^2=23.5030,$

$t_{\frac{0.05}{2}}(9)=2.2622$ $X_{\frac{0.05}{2}}^2(9)=19.023,\ X_{1-\frac{0.05}{2}}^2(9)=2.7$

μ的置信区间为$(\bar{X}\pm\frac{S}{\sqrt{n}}t_{\frac{0.05}{2}})=(996.3852,1003.3211)$

σ^2的置信区间为$(\frac{(n-1)S^2}{X_{\frac{0.05}{2}}^2(n-1)},\frac{(n-1)S^2}{X_{1-\frac{0.05}{2}}^2(n-1)})=(11.1195,78.3433)$

σ的置信区间为$\left(\sqrt{\frac{(n-1)S^2}{X_{\frac{\alpha}{2}}^2(n-1)}}, \sqrt{\frac{(n-1)S^2}{X_{1-\frac{\alpha}{2}}^2(n-1)}}\right) = (3.3346, 8.8512)$

22. $\bar{X} = 0.14125$, $S_1^2 = 8.25 \times 10^{-6}$,

$\bar{y} = 0.1392$, $S_2^2 = 5.2 \times 10^{-6}$, $t_{\frac{0.05}{2}}(7) = 2.3646$, $S_w = 0.002551$

$\mu_1 - \mu_2$的置信区间为$\left(\bar{X} - \bar{Y} \pm t_{\frac{\alpha}{2}}(n_1 + n_2 - 2) \cdot S_W \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right) = (-0.0019, 0.0060)$

23. $\bar{X} = 650$, $\bar{y} = 480$, $\sigma_1 = 120$, $\sigma_2 = 106$, $u_{\frac{0.05}{2}} = 1.96$, $n_1 = n_2 = 25$

$\mu_1 - \mu_2$的置信区间为$\left(\bar{X} - \bar{Y} \pm u_{\frac{\alpha}{2}}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right) = (107.24, 232.76)$

24. $\bar{X} = 533.11$, $\bar{y} = 561.8$, $S_1^2 = 63.99$, $S_2^2 = 236.84$,

$F_{\frac{0.05}{2}}(8,9) = 4.10$, $F_{1-\frac{0.05}{2}}(8,9) = \frac{1}{4.36}$

$\frac{\sigma_1^2}{\sigma_2^2}$的置信区间为$\left(\frac{S_1^2}{S_2^2 F_{\frac{\alpha}{2}}(n_1-1,n_2-1)}, \frac{S_1^2}{S_2^2 F_{1-\frac{\alpha}{2}}(n_1-1,n_2-1)}\right) = (0.066, 1.178)$

$\therefore \frac{\sigma_1}{\sigma_2}$的置信区间为$(0.26, 1.08)$

25. $S_1^2 = 0.5419$, $S_2^2 = 0.6065$, $F_{\frac{0.05}{2}}(9,9) = 4.03$, $F_{1-\frac{0.05}{2}}(9,9) = \frac{1}{4.03}$

$\frac{\sigma_1^2}{\sigma_2^2}$的置信区间为$\left(\frac{S_1^2}{S_2^2 F_{\frac{\alpha}{2}}(n_1-1,n_2-1)}, \frac{S_2^2}{S_2^2 F_{\frac{\alpha}{2}}(n_1-1,n_2-1)}\right) = (0.2217, 3.6008)$

26. $\bar{X} = 0.921$, $S^2 = 5.272 \times 10^{-4}$, $S = 0.02296$, $t_{0.05}(5) = 2.0150$

M的置信下限为$\bar{X} - \frac{S}{\sqrt{n}}t_{\alpha}(n-1) = 0.9021$

27. $t_{\alpha}(n_1 + n_2 - 2) = t_{0.05}(7) = 1.8946$

$\mu_1 - \mu_2$的置信下限为$\bar{X} - \bar{Y} - t_{\alpha}(n_1 + n_2 - 2) \cdot S_W \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} = -0.0012$

28. $\frac{\sigma_1^2}{\sigma_2^2}$的置信上限为$\frac{S_1^2}{S_2^2 F_{1-\alpha}(n_1-1,n_2-1)} = \frac{0.5419}{0.6065 \times \frac{1}{3.18}} = 284$

第七章

1. 某种零件的长度服从正态分布，方差$\sigma^2 = 1.21$，随机抽取6件，记录其长度（单位：mm）分别为

$$32.46, \quad 31.54, \quad 30.10, \quad 29.76, \quad 31.67, \quad 31.23$$

在显著性水平$\alpha = 0.05$下，能否认为这批零件的平均长度为$32.50mm$?

解　H_0:可以认为平均长度是$32.50mm$　H_1:不认为平均长度是$32.50mm$

长度$X \sim N(\mu, 1.21)$，μ未知，σ^2已知。

设$U = \frac{\sqrt{n} \; (\bar{X} - \mu_0)}{\sigma_0} = \frac{\sqrt{6} \; (32.127 - 32.5)}{1.1} = -3.058$，而拒绝域为$U > u_{\frac{\alpha}{2}} \, or \, U < -u_{\frac{\alpha}{2}}$

U位于拒绝域，因此拒绝H_0，即不能认为平均长度为$32.50mm$。

2. 某厂计划投资1万元的广告费以提高某种食品的销售量，厂方认为此项计划可以使每周销售量达到$225kg$.实行此计划一个月后，调查了16家商店,计算得平均每周的销售量为$209kg$,标准差为$42kg$,问在$\alpha = 0.05$下，可否认为此项计划达到了该厂的预期效果（设每周销售量服从正态分布）？

解　设H_0：$\mu_0 = 225$，H_1：$\mu_0 \neq 225$.

销售量$X \sim N(\mu, \sigma^2)$，σ^2未知，$\bar{X} = 209$.

设$t = \frac{\sqrt{n}(\bar{X} - \mu_0)}{s} = \frac{4(209 - 225)}{43} = -1.5238$,

而拒绝域为$t > t_{\frac{\alpha}{2}}(15) \, or \, t < -t_{\frac{\alpha}{2}}(15)$,

$-t_{\frac{\alpha}{2}}(15) < t < t_{\frac{\alpha}{2}}(15)$，因此接受$H_0$，即可认为达到了预期效果。

3. 正常人的脉搏平均每分钟72次，某医生测得10例四乙基铅中毒患者的脉搏数如下

$$54, \quad 67, \quad 68, \quad 78, \quad 70, \quad 66, \quad 67, \quad 65, \quad 69, \quad 70$$

已知人的脉搏次数服从正态分布，问在显著性水平$\alpha = 0.05$下，四乙基铅中毒

患者的脉搏数和正常人的脉搏有无显著差异?

解　设H_0: $\mu_0 = 72$, H_1: $\mu_0 \neq 72$.

脉搏数$X \sim N(72, \sigma^2)$, σ^2未知, $\bar{X} = 67.4$, $S^2 = 35.156$.

设$t = \frac{\sqrt{n}(\bar{X} - \mu_0)}{s} = \frac{\sqrt{10}(67.4 - 72)}{\sqrt{35.156}} = -2.4534$,

而拒绝域为$t > t_{\frac{\alpha}{2}}(9) \ or \ t < -t_{\frac{\alpha}{2}}(9)$,

$t < -t_{\frac{\alpha}{2}}(9)$, 因此拒绝$H_0$, 即可认为有显著差异。

4. 某纯净水生产厂用自动灌装机灌装纯净水,该自动灌装机正常灌装量

　　$X \sim N(18, 0.4^2)$,

现测量某厂9个灌装样品的灌装量（单位: L）如下:

18.0,　　17.6,　　17.3,　　18.2,　　18.1,　　18.5,　　17.9,　　18.1,　　18.3
在显著性水平$\alpha = 0.05$下，试问

(1)该天灌装是否正常?

(2)灌装量精度是否在标准范围内?

解(1)先检验均值　设H_0: $\mu = 18$, H_1: $\mu \neq 18$.

σ^2已知, $\bar{X} = 18$, $S^2 = 0.1325$。

$u = \frac{\sqrt{n}(\bar{X} - \mu_0)}{s} = 0$　而拒绝域为$u > u_{\frac{\alpha}{2}} \ or \ u < -u_{\frac{\alpha}{2}}$

u位于接受域, 因此接受H_0;

再检验方差　设H_0: $\sigma^2 = 0.4^2$, H_1: $\sigma^2 \neq 0.4^2$, μ已知.

　　　$\chi^2 = \frac{\sum_{i=1}^{n}(x_i - \mu)^2}{\sigma_0^2} = 6.625$, $\chi_{1-\frac{\alpha}{2}}^2(8) < \chi^2 < \chi_{\frac{\alpha}{2}}^2(8)$,

因此接受H_0; 因此可以认为该天灌装正常。

(2)设H_0: $\sigma^2 \leq 0.4^2$, H_1: $\sigma^2 > 0.4^2$.

设$\chi^2 = \frac{\sum_{i=1}^{n}(x_i - \mu)^2}{\sigma_0^2} = 6.625$, $\chi^2 < \chi_{\alpha(8)}^2 = 16.607$,

因此接受H_0, 即可认为精度在标准范围内。

5. 某地区100个登记死亡人的样本中，其平均值寿命为71.8年，标准差为8.9，假设人的寿命X服从正态分布$N(\mu, \sigma^2)$，μ, σ^2均未知.问是否有理由认为该地区的平均寿命不低于70岁$(\alpha = 0.05)$?

解 设H_0：$\mu \geq 70$，H_1：$\mu < 70$. σ^2未知，$\bar{X} = 71.8$，$S = 8.9$。

设$t = \frac{\sqrt{n}(\bar{X} - \mu_0)}{S} = \frac{10(71.8 - 70)}{8.9} = 2.022$，因$t > -t_\alpha(99)$（书上附录没有$t_\alpha(99)$，可近似将$t_\alpha(45)$看作$t_\alpha(99)$）因此接受$H_0$，即可认为该地区平均寿命不低于70岁。

6. 某厂的生产管理员认为该厂第一道工序加工完的产品送到第二道工序进行加工之前的平均等待时间超过$90min$.现对100件产品进行随机抽样结果显示平均等待时间为$96min$,样本标准差为$30min$,设平均等待时间服从正态分布.问抽样的结果是否支持该管理员的看法？（$\alpha = 0.05$）

解 设H_0：$\mu \geq 90$，H_1：$\mu < 90$.

平均等待时间$X \sim N(\mu, \sigma^2)$，σ^2未知，$\bar{X} = 96$，$S = 30$。

设$t = \frac{\sqrt{n}(\bar{X} - \mu_0)}{S} = \frac{10（96 - 90）}{30} = 2$，$t > -t_\alpha(99)$（道理同上）

因此接受H_0，即支持管理员的看法。

7. 某汽车配件厂在新工艺下对加工好的25个活塞直径进行测量,得样本方差$S^2 = 0.00066$.已知旧工艺生产的活塞直径的方差为 0.00040,假设活塞直径服从正态分布.问革新后活塞直径的方差是否大于旧工艺的方差（$\alpha = 0.05$）？

解 设H_0：$\sigma_1^2 \geq \sigma_2^2$，$H_1$：$\sigma_1^2 < \sigma_2^2$，活塞直径$X \sim N(\mu, \sigma^2)$

$S_1^2 = 0.00066, S_2^2 = 0.0004$，$\mu_1, \mu_2$均未知。

设$f = \frac{S_1^2}{S_2^2} = 1.65$，$f > F_{1-\alpha}(24,24) = \frac{1}{F_\alpha(24,24)} = 0.5051$，

因此接受H_0，即可以认为革新后活塞直径的方差大于旧工艺的方差。

8. 某种导线的电阻服从正态分布$N(\mu, 0.005^2)$,从一批导线中抽取9根,测得这9

   根导线的电阻的样本标准差为0.008,能否认为这批导线电阻的标准差仍为

   0.005（$\alpha = 0.05$）？

解 设H_0：$\sigma^2 = 0.005^2$，H_1：$\sigma^2 \neq 0.005^2$，μ未知，电阻$X \sim N(\mu, 0.005^2)$

   设$\chi^2 = \frac{(n-1)S^2}{\sigma_0^2} = 20.48$，因$\chi^2 > \chi_{\frac{\alpha}{2}}^2(8)$，

   因此拒绝H_0，即不可认为这批导线电阻的标准差仍为0.005。

9. 无线电厂生产某种高频管,其中一项指标服从正态分布$N(\mu, \sigma^2)$.从该厂生产

   的一批高频管中随机抽取8个，测得该项指标的数据为

   68,43,70,65,55,56,60,72.

(1)若已知$\mu = 60$，检验假设H_0：$\sigma^2 = 49$，H_1：$\sigma^2 \neq 49(\alpha = 0.05)$;

(2)若μ未知，H_0：$\sigma^2 \leq 49$，H_1：$\sigma^2 > 49$（$\alpha = 0.05$）.

解 (1)设H_0：$\sigma^2 = 49$，H_1：$\sigma^2 \neq 49$，μ已知，$\bar{X} = 61.125$，$S^2 = 93.2679$

设$\chi^2 = \frac{\sum_{i=1}^{n}(X_i - \mu)^2}{\sigma_0^2} = \frac{663}{49} = 13.53$ 由于$\chi_{1-\frac{\alpha}{2}}^2(8) < \chi^2 < \chi_{\frac{\alpha}{2}}^2(8)$，因此接受$H_0$;

(2)设H_0：$\sigma^2 \leq 49$，H_1：$\sigma^2 > 49$，μ未知，

设$\chi^2 = \frac{(n-1)S^2}{\sigma_0^2} = 13.324$ $\chi^2 < \chi_{\alpha}^2(7)$，因此接受$H_0$。

10. 设有两个来自不同正态总体$N(\mu, \sigma^2)$的样本,$m = 4, n = 5$,

$\bar{x} = 0.60, \bar{y} = 2.25, S_1^2 = 15.07, S_2^2 = 10.81$,在显著性水平$\alpha = 0.05$下,试检验两个

样本是否来自相同方差的正态总体.

解 H_0：$\sigma_1^2 = \sigma_2^2$，H_1：$\sigma_1^2 \neq \sigma_2^2$. μ_1, μ_2未知，$S_1^2 = 15.07, S_2^2 = 10.81$.

   设$f = \frac{S_1^2}{S_2^2} = 1.3941$，$F_{1-\frac{\alpha}{2}}(3,4) = \frac{1}{15.10} = 0.0662$

   $F_{1-\frac{\alpha}{2}}(3,4) < f < F_{\frac{\alpha}{2}}(3,4)$，因此接受$H_0$，即可认为两个样本来自相同方差

的正态总体。

11. 为了提高振动板的硬度，热处理车间选择两种淬火温度T_1及T_2进行试验，测

得振动板的硬度数据如下：

$$T_1: 85.6, 85.9, 85.7, 85.8, 85.7, 86.0, 85.5, 85.4;$$

$$T_2: 86.2, 85.7, 85.5, 85.7, 85.8, 86.3, 86.0, 85.8.$$

假设两种淬火温度下的振动板的硬度服从正态分布，检验：

(1)两种淬火温度下振动板硬度的方差是否与显著差异（$\alpha = 0.05$）？

(2)淬火温度对振动板的硬度是否有显著影响（$\alpha = 0.05$）？

解 （1）设H_0： $\sigma_1^2 = \sigma_2^2$，H_1： $\sigma_1^2 \neq \sigma_2^2$. μ_1, μ_2未知，

$$\bar{x} = 85.7, \bar{y} = 85.875, S_1^2 = 0.04, S_2^2 = 0.07357$$

设$f = \frac{S_1^2}{S_2^2} = 0.5437$，$F_{1-\frac{\alpha}{2}}(7,7) = \frac{1}{4.99} = 0.2004$

$F_{1-\frac{\alpha}{2}}(7,7) < f < F_{\frac{\alpha}{2}}(7,7)$，因此接受$H_0$，即可认为方差无显著差异；

（2）由（1）得$\sigma_1^2 = \sigma_2^2 = \sigma^2$，设$H_0: \mu_1 = \mu_2$，$H_1$： $\mu_1 \neq \mu_2$，

设$t = \frac{\bar{x} - \bar{y}}{s_w \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{85.7 - 85.785}{\sqrt{\frac{1}{2}S_1^2 + \frac{1}{2}S_2^2}\sqrt{\frac{1}{8} + \frac{1}{8}}} = -1.4688$，$t_{\frac{\alpha}{2}}(14) = 2.1448$

$-t_{\frac{\alpha}{2}}(14) < t < t_{\frac{\alpha}{2}}(14)$，因此接受$H_0$，即可认为硬度无显著影响。

12. 对某地7岁儿童作身高调查，结果如下：

性别	人数	平均身高	样本标准差
男	384	118.64	4.53
女	377	117.86	4.56

假设身高服从正态分布，由以上数据能否说明性别对7岁儿童的身高有显著影

响．（$\alpha = 0.05$）

解 先检验方差，设H_0： $\sigma_1^2 = \sigma_2^2$，H_1： $\sigma_1^2 \neq \sigma_2^2$.

$\bar{x} = 118.64, \bar{y} = 117.86, S_1^2 = 4.53^2, S_2^2 = 4.56^2$.

设$f = \frac{S_1^2}{S_2^2} = 0.8688$，$f < F_{\frac{\alpha}{2}}(383,376)$，因此接受$H_0$，即$\sigma_1^2 = \sigma_2^2 = \sigma^2$；

再检验均值，设$H_0: \mu_1 = \mu_2$，$H_1:\ \mu_1 \neq \mu_2$，$\sigma_1^2 = \sigma_2^2 = \sigma^2$，

设$t = \frac{\bar{x}-\bar{y}}{s_w\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} = \frac{118.64-117.86}{\sqrt{\frac{383S_1^2+376S_2^2}{759}}\sqrt{\frac{1}{384}+\frac{1}{377}}} = 2.2907$，$t > t_{\frac{\alpha}{2}}(759)$，因此接受$H_0$；

因此不可以认为性别对7岁儿童的身高有显著影响。

13. 某药厂为比较新旧两种方法提取某有效成分的效率，用新旧方法各做了10

   次实验，提取有效成分的比率如下表所示：

新方法	79.1	81	77.3	79.1	80	79.1	79.1	77.3	80.2	82.1
旧方法	78.1	72.4	76.2	74.3	77.4	78.4	76	75.5	76.7	77.3

假设这两种样本分布分别取自正态分布总体，且两样本相互独立，试问新方法

的提取率比旧方法的提取率是否有所提高（$\alpha = 0.01$）？

解 先检验方差，设$H_0:\ \sigma_1^2 = \sigma_2^2$，$H_1:\ \sigma_1^2 \neq \sigma_2^2$，

$\bar{x} = 79.43, \bar{y} = 76.23, S_1^2 = 2.2245, S_2^2 = 3.3245$

设$f = \frac{S_1^2}{S_2^2} = 0.6691$，$F_{1-\frac{\alpha}{2}(9,9)} = \frac{1}{6.54} = 0.1529$，$F_{1-\frac{\alpha}{2}}(9,9) < f < F_{\frac{\alpha}{2}}(9,9)$

因此接受H_0，即$\sigma_1^2 = \sigma_2^2 = \sigma^2$；

再检验均值，设$H_0: \mu_1 \leq \mu_2$，$H_1:\ \mu_1 > \mu_2$，$\sigma_1^2 = \sigma_2^2 = \sigma^2$，

设$t = \frac{\bar{x}-\bar{y}}{s_w\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} = \frac{79.43-76.23}{\sqrt{\frac{9S_1^2+9S_2^2}{18}}\sqrt{\frac{1}{10}+\frac{1}{10}}} = 4.2958$，$t_{\alpha}(18) = 2.5524$，$t > t_{\alpha}(18)$

因此拒绝H_0，即可认为新方法比旧方法提取率有所提高。

14. 在某校大一学生中随机抽取 10 人，让他们分别采用A和B两套数学试卷进

   行测试，成绩如下：

试卷A	78	63	72	89	91	49	68	76	85	55
试卷B	71	44	61	84	74	51	55	60	77	39

假设学生成绩服从正态分布，试检验两套数学试卷是否有显著差异

（$\alpha = 0.01$）.

解 先检验方差，设H_0: $\sigma_1^2 = \sigma_2^2$, H_1: $\sigma_1^2 \neq \sigma_2^2$,

$\bar{x} = 72.6, \bar{y} = 61.6, S_1^2 = 198.04, S_2^2 = 217.82$,

设$f = \frac{S_1^2}{S_2^2} = 0.9092$, $F_{1-\frac{\alpha}{2}}(9,9) = \frac{1}{6.54} = 0.1529$, $F_{1-\frac{\alpha}{2}}(9,9) < f < F_{\frac{\alpha}{2}}(9,9)$,

因此接受H_0, 即$\sigma_1^2 = \sigma_2^2 = \sigma^2$;

再检验均值，$H_0: \mu_1 = \mu_2$, H_1: $\mu_1 \neq \mu_2$.

设$t = \frac{\bar{x} - \bar{y}}{s_w \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{72.6 - 61.6}{\sqrt{\frac{9S_1^2 + 9S_2^2}{18}} \sqrt{\frac{1}{10} + \frac{1}{10}}} = 1.7058$, $t_{\frac{\alpha}{2}}(18) = 2.8784$,

$-t_{\frac{\alpha}{2}}(18) < t < t_{\frac{\alpha}{2}}(18)$, 因此接受$H_0$, 即可以认为两套数学试卷无显著差异。