Report: Midsem Exam Coding Assignment (AE675A)

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Q1. 1-D bar problem with end load P_0, P_L and distributed force f(x), also a spring acting on its boundary k(x) using polynomial basis function .

1.1 problem description:

Given,
length of bar L=1,
Area times young's modulus Ea. =1
Load at x=1, Pl=10
Load at x=0, P_0= 5
Value of distributed load,

Case a: f(x)=10
Case b: f(x)=10x
Case c: f(x)=$10x^2$

Value of stiffness of spring,
k = [0,10]

Order of polynomials we have to iterate for
N=2, N=4, N=6, N=8
Basis function: here U is approximated by polynomial (n-1) order. Basis function $\{x^{(n-1)}\}$.

Thus we have 6 cases to code for

1. k=10	f(x)=10	Exact solution possible
2. k=10	f(x)=10x	Exact solution possible
3. k=10	f(x)=$10x^2$	Exact solution possible
4. k=0	f(x)=10	Exact solution not possible
5. k=0	f(x)=10x	Exact solution not possible
6. k=0	f(x)=$10x^2$	Exact solution not possible

Case 1. (k=10, f=10)

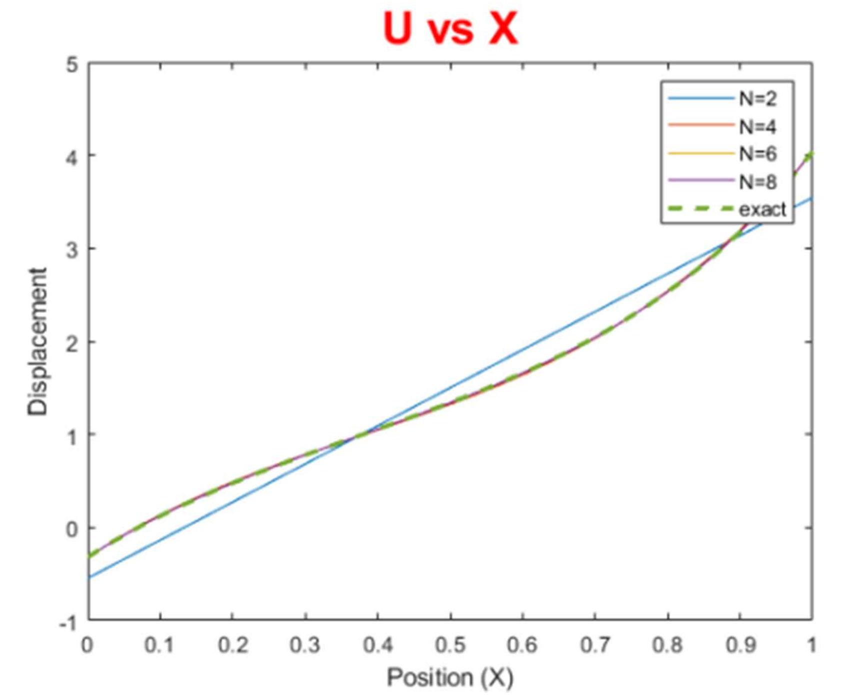


Figure 1

Force Plot

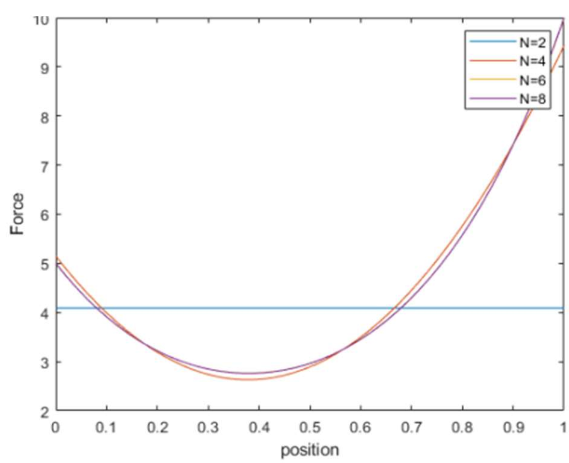


Fig 2

Error Plot

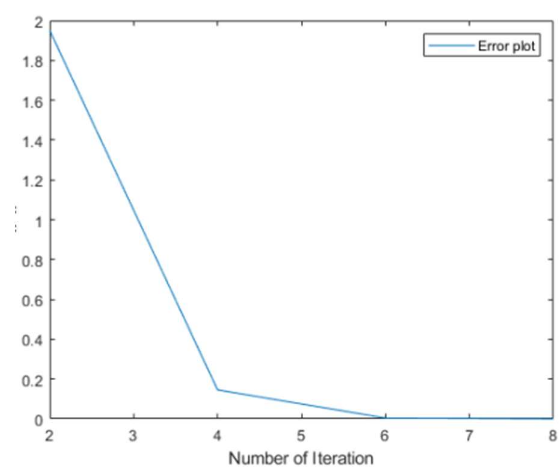


Fig 3

Case 2. (k=10, f=10)

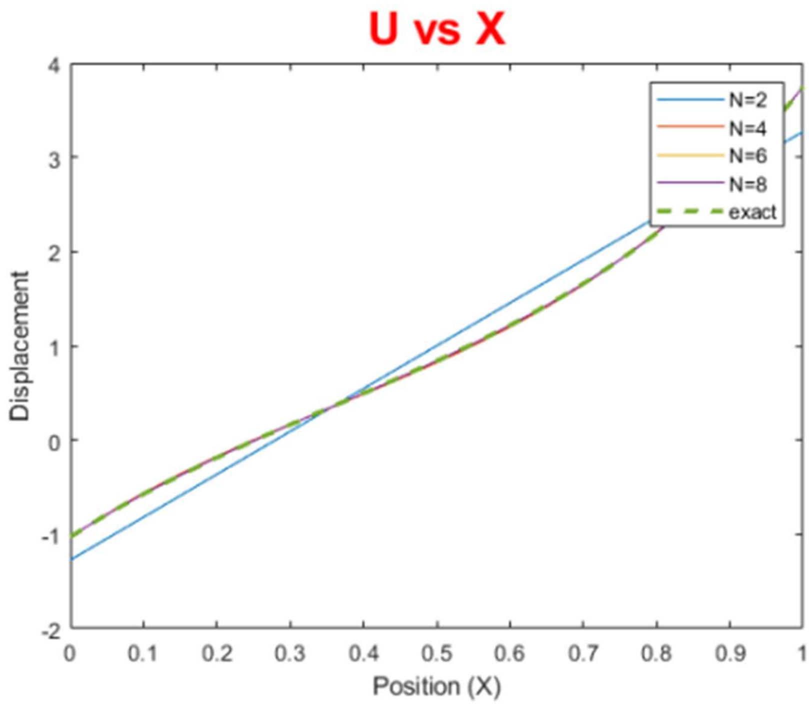


Fig 4

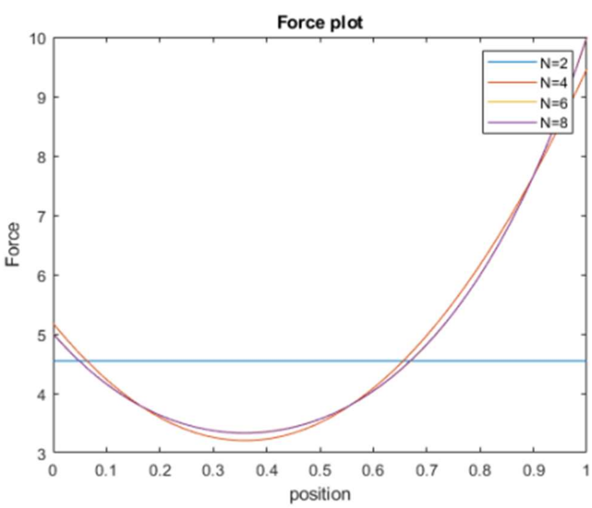


Fig 5

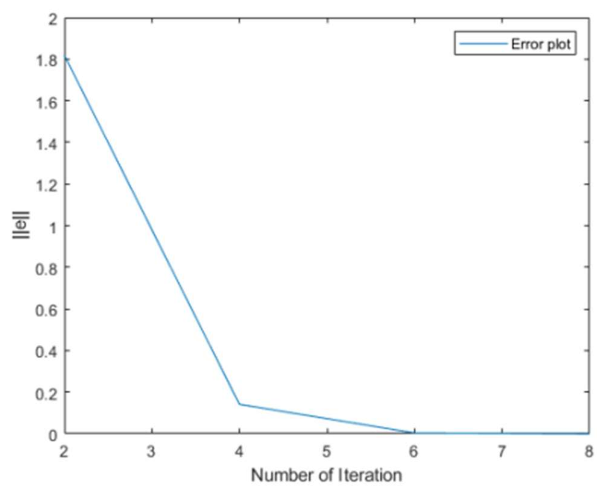


Fig 6

Case 3. (k=10, $f=10x^2$)

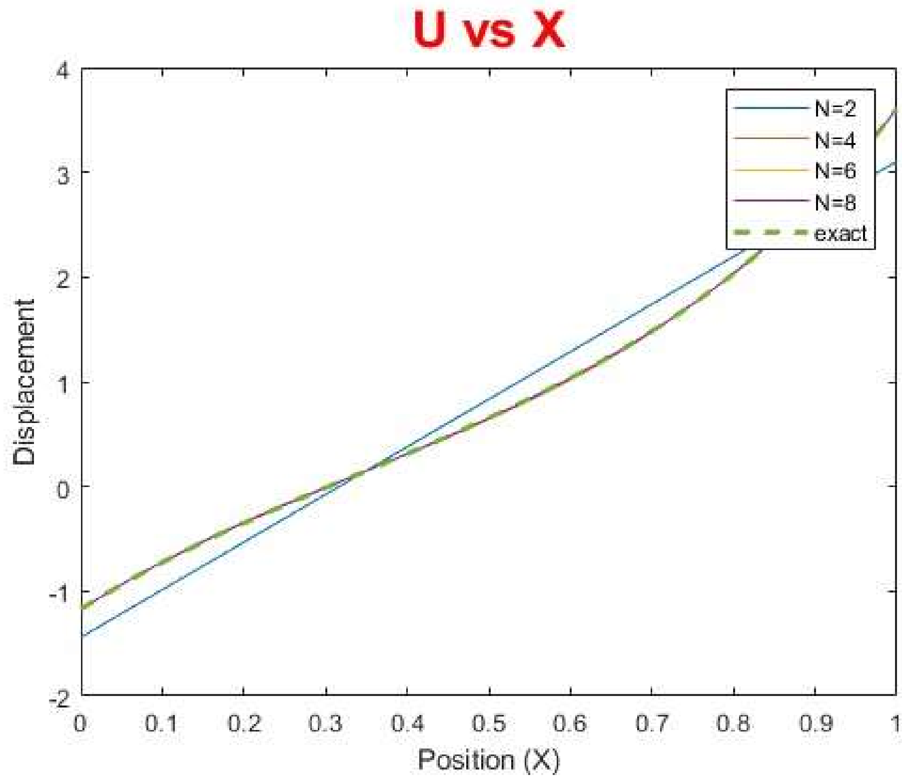


Figure 8

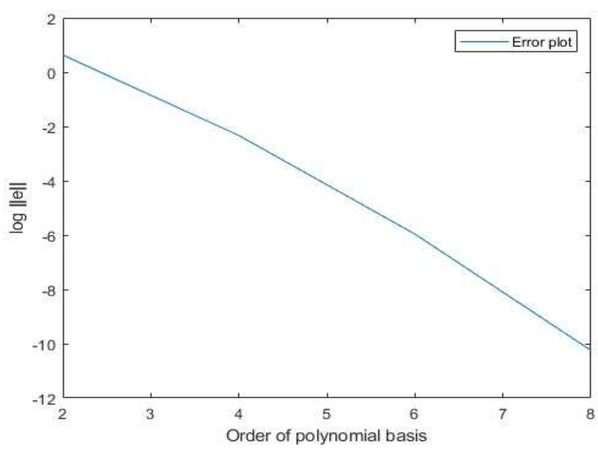


Figure 9

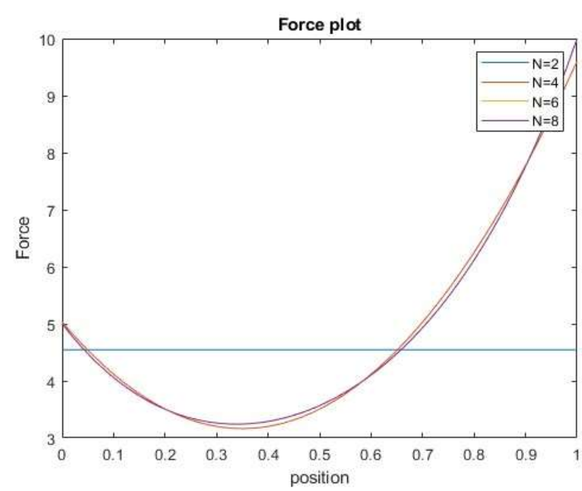


figure 10

1.2 Observation for all cases k=10

- As can be seen plot of N=2 is not giving better results due to lowest degree of polynomial approximation.
  Whereas for higher degree of polynomial basis function gives better results.

- Also, from plots it can be seen that for Displacement it converges at same plots for N=4 onwards but for force plot it converges after N=6 and onwards.
  This is due to reduction of order of polynomial on differentiation.

- The mod error vs. N figure also demonstrates that the error is highest for N=2 and minimum for N=8, i.e., the error drops monotonically with increasing N.

- So, it can be said that here we are using p- version of FEM as in the p-version, the order of the polynomial used to approximate the solution is increased, allowing for a more accurate solution to be obtained with fewer elements.

- The error in the p-version of the finite element method (FEM) can be calculated using the following formula:

  $$B\,(e, e) \leq Ch^2p$$

  Where C is constant depending on mesh size and equation and, h is maximum element size in mesh, p is order of the polynomial used in the approximation.

- Also, the value of |er| calculated using the $\sqrt{B(er, er)}$ equals the value calculated using $\sqrt{B(u, u) - B(un, un)}$ where 'un' is the approximate solution of FEM with n terms.

Cases 4,5,6

Here only posting results for $f(x)=10x^2$ and $k=0$

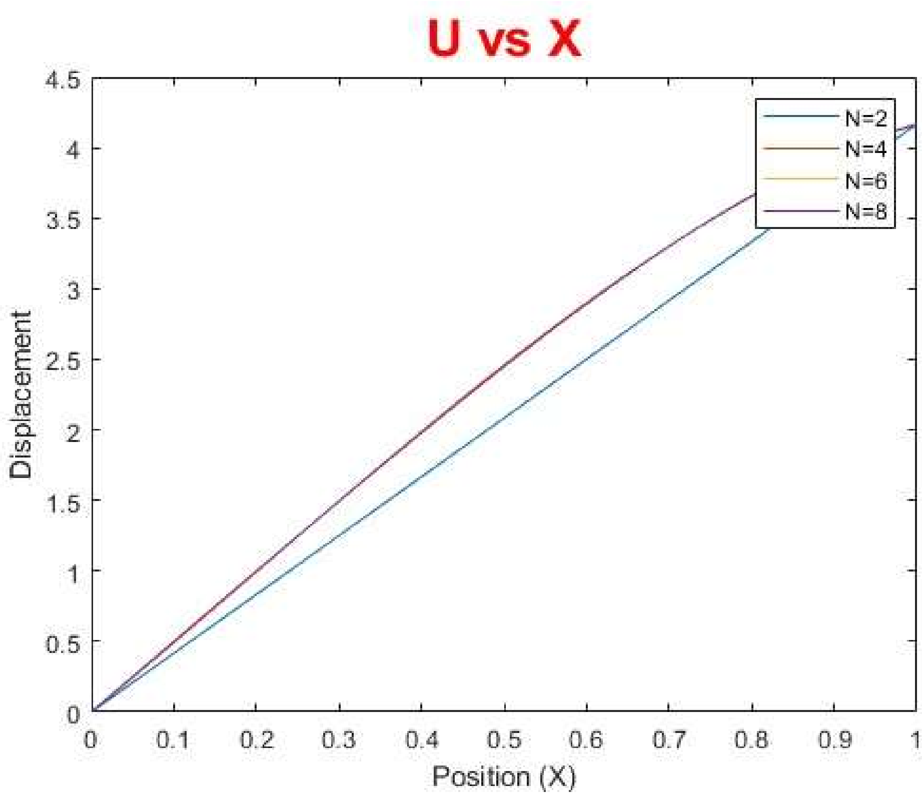


Figure 11

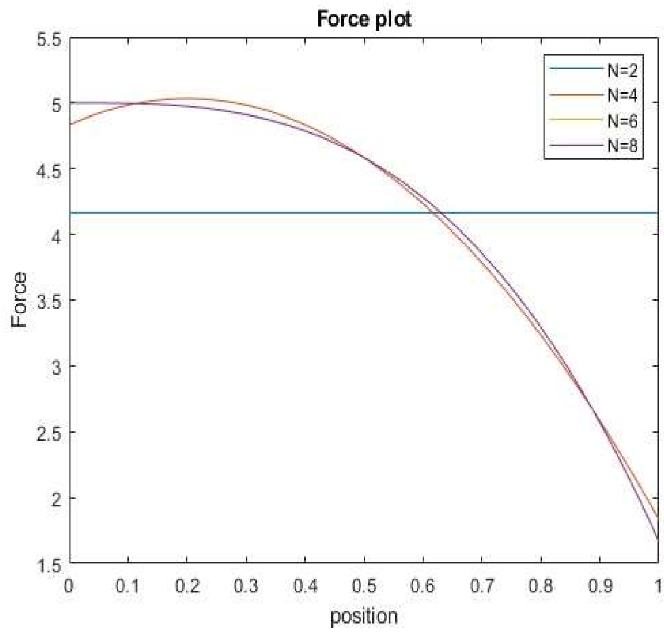


Figure 12

- Exact solution plot is abscent Due to unavailability of unique exact solution Ode for k(x)= 0 case
- Also FEM approx solution is not possible, as in case of k(x)=0 and without any constraint will get an non invertible stiffness matrix i.e (with zero first row and coloumn).
- If k(x) is zero so it has a rigid translation in one direction that's why the stiffness matrix becomes 1 rank deficient. To avoid such scenario, we must add a constraint to stop the translation motion.
- To avoid rank deficiency, we tamper the [K] and {F} matrices by forcing alpha1(first coefficient) equal to zero.
- To force alpha1= 0, we make K (1,1) =1 and take value of P_L such that F (1) =0.
- We calculate P_L by force balance equation on Bar.

1.3 Observation for all cases k(x)=0

- ❖ As to remove rigid body mode we have forced U = 0, It can also be seen from our plot of U vs position fig (11).

- ❖ Also, from plots it can be seen that for Displacement it converges at same plots for N=4 onwards but for force plot it converges after N=6 and onwards. This is due to reduction of order of polynomial on differentiation.

Q2.1-D An axial bar problem with end load two fixed side and distributed force f(x), also a spring acting on its boundary k(x). Using hierarchy basis function (Lagrange Polynomials).

2.1 problem description:

Given,
length of bar L=1,
Area times young's modulus Ea. =1
Load at x=1, Pl=10
Load at x=0, P_0= 5
Value of distributed load,

Case a: f(x)=10
Case b: f(x)=10x
Case c: $f(x)=10x^2$

Value of stiffness of spring,

k = [0,10]

Order of polynomials we have to iterate for

N=2, N=4, N=6, N=8

Basis function: here U is approximated by Using hierarchy basis function (Lagrange Polynomials).

Thus, we have 6 cases to code for

1. k=10	f(x)=10	Exact solution possible
2. k=10	f(x)=10x	Exact solution possible
3. k=10	$f(x)=10x^2$	Exact solution possible
4. k=0	f(x)=10	Exact solution not possible
5. k=0	f(x)=10x	Exact solution not possible
6. k=0	$f(x)=10x^2$	Exact solution not possible

For k =0 Given two Dirichlet boundary conditions, U=0 at x=0 and U=0 at x=L, the singular solution will be present. The boundary condition will cause the stiffness matrix([K]) to be invertible.

Hierarchy basis function has to be employed.

To apply the boundary condition the alpha 1 and alpha n should be forced to be 1 as at x=0 only phi-1 is 1 and rest all are zero and at x=L phi-n is 1 and rest are all zero

Case 1,2,3:

for $k(x)=10$, $f(x)=10x^2$

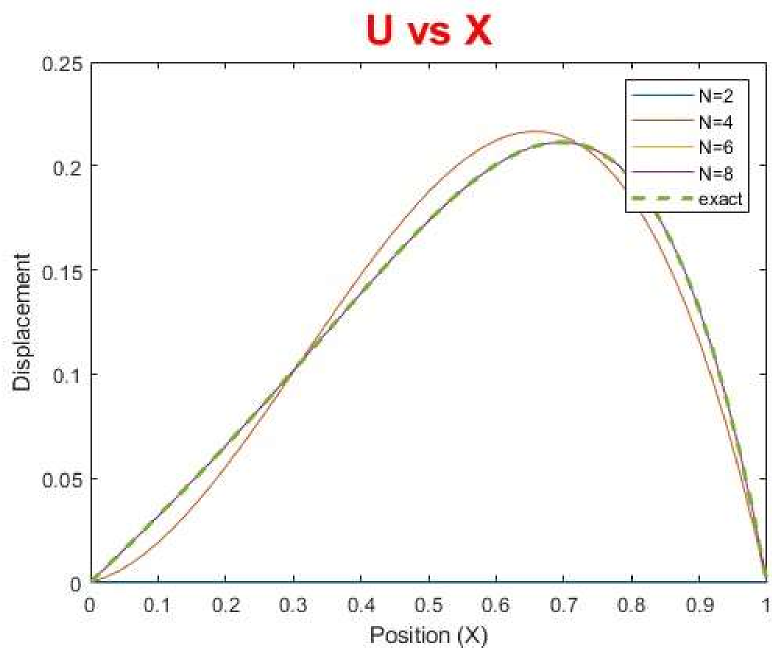
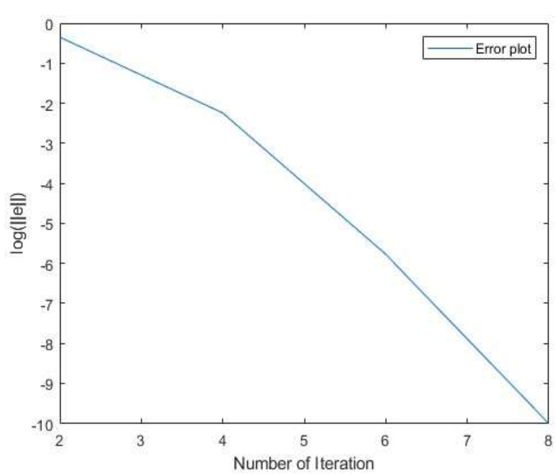


Figure 13



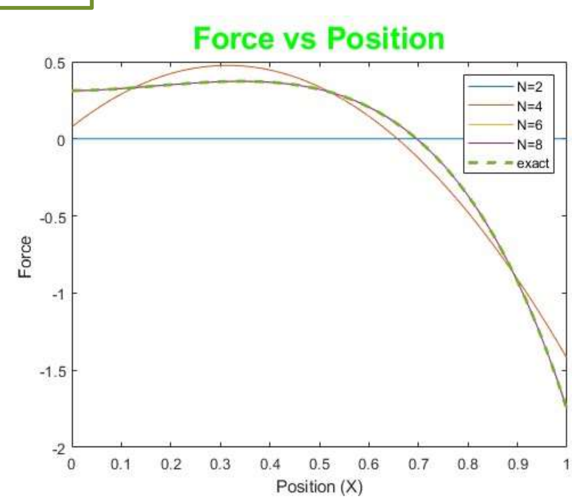


Fig 14 Fig 15

2.2 Observation for Lagrange basis function and k=10

- Form figure 13 we can see that essential boundary condition (displacement boundary condition) is always satisfied for any N.
- here for figure 13, N=2 order basis function is giving worst results as it consists of lowest order Lagrange polynomials (i.e., Linear functions). Whereas N=4, N=6, N=8 giving good results comparatively.
- Also, from plots it can be seen that for Displacement it converges at N=4 onwards but for force plot it converges after N=6 and onwards.
  This is due to reduction of order of basis function on differentiation.
- Here |err|= $\sqrt{B(er, er)}$ or $\sqrt{B(u,u) - B(un, un)}$
- As we are using p version of FEM so it can be seen as we increase our order of polynomial, 'log |e|' is decreasing exponentially.

Case 4,5,6

For $k(x)=0$, $f(x)=10x^2$

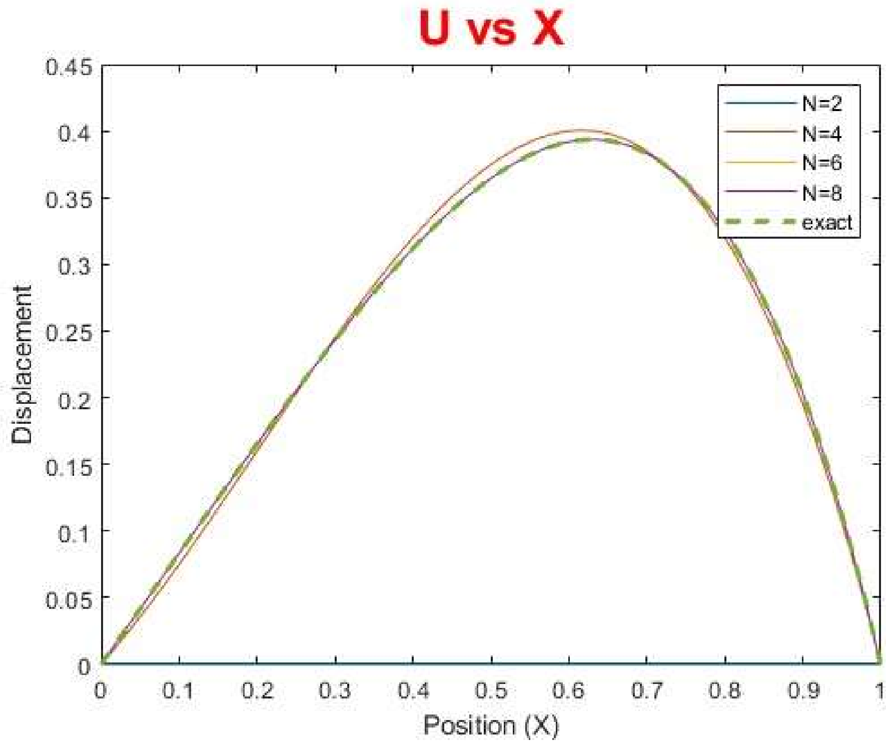


Fig 16

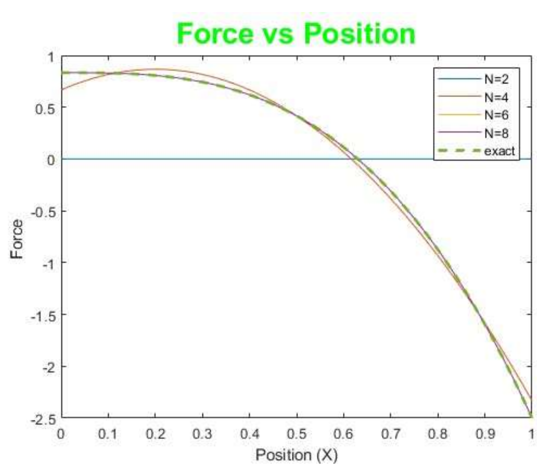


Fig 17

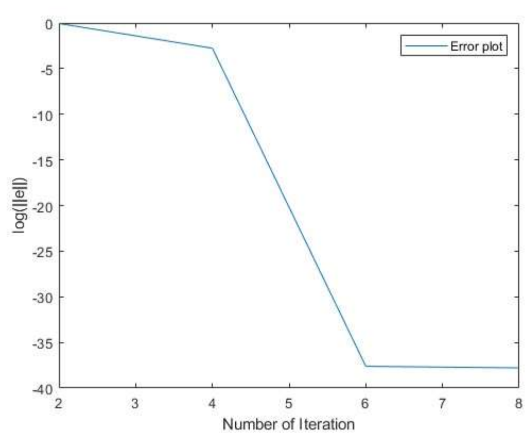


Fig 18

Observation for case 6

- Here also form figure 16 we can see that essential boundary condition (displacement boundary condition) is always satisfied for any N.

- And N=2 order basis function is giving worst results as it consists of lowest order Lagrange polynomials (i.e., Linear functions). Whereas N=4, N=6, N=8 giving good results comparatively.

- Also, from plots it can be seen that for Displacement it converges at N=4 onwards but for force plot it converges after N=6 and onwards.
  This is due to reduction of order of Basis function on differentiation.

- Here error plot is only due to rounding off error in Gauss quad method. It can also be seen from values of log |err| in figure 18.

Q3. Repeat Q1 and Q2 using hat basis function.

3.1 Problem description

For comparison that how much effect the choice of basis has on the approximate solution, only two cases (one from each prob) will be sufficient.

1. Problem 1 k=10, f(x)=$10x^2$, N= 2, 4, 8
2. Problem 2 k=10, f(x)=$10x^2$, N= 2, 4, 8

Hat basis functions consists of only linear functions. And over the whole domain there are N+1 basis functions, here N= number of elements

Case 1: for problem 1 with end loads

Here k(x)=10, f(x)=$10x^2$

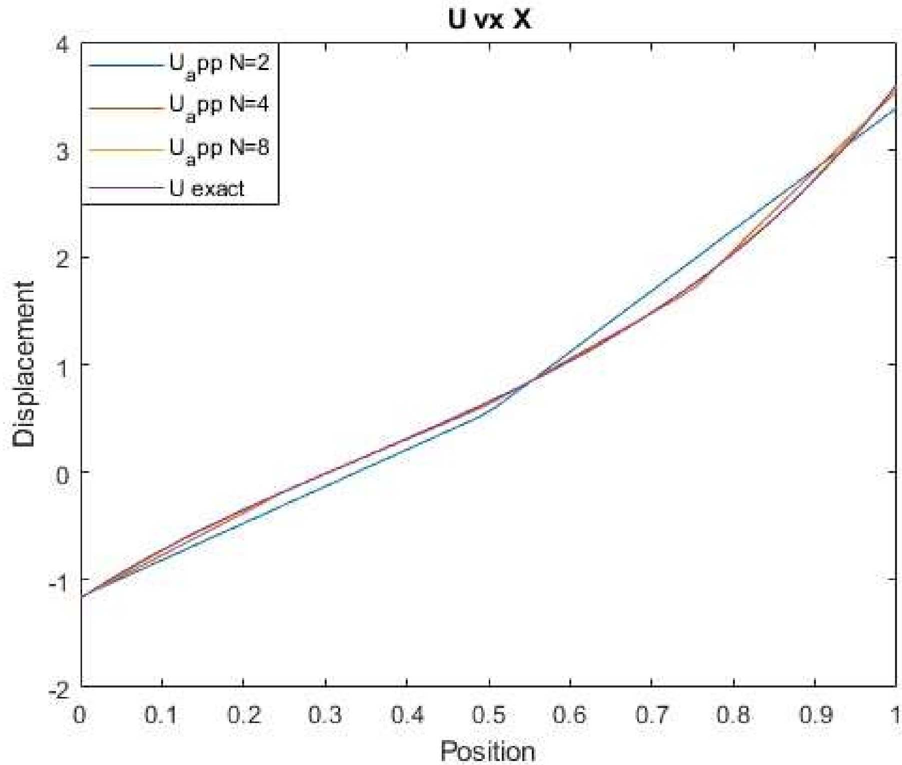


Fig 19

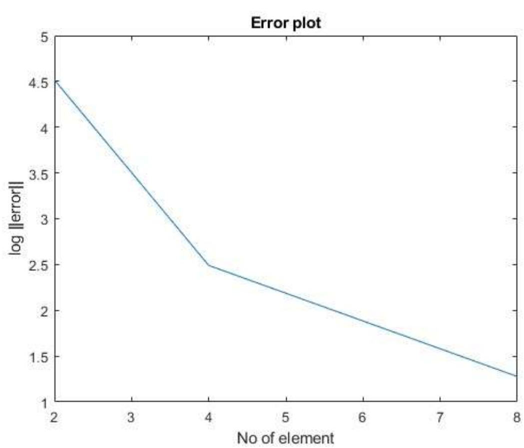


Fig 20

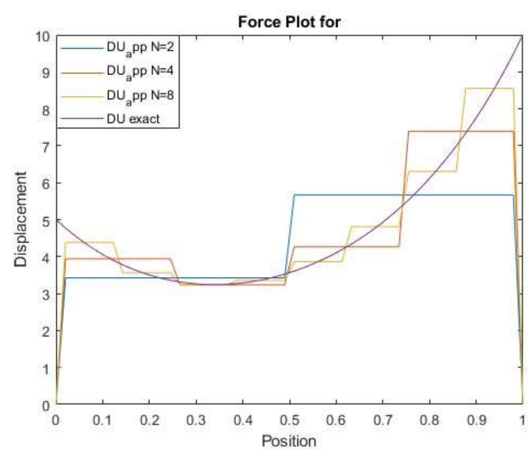


Fig 21

3.2 Observation of case 1

- ❖ With the help of hat basis function, we can take care of local point loads and avoid jumps in plot.
- ❖ Here using lower value of p=1 we are able to approximate a non-linear curve.
- ❖ The curve for force vs displacement is discontinuous at the node and constant over an element as seen in the figure where with polynomial basis we get a smooth curve for force.

- As here we are putting p= const and only increasing h hence rate of convergence is slow.

- This is H version of FEM

Case 2: for problem 2 with fixed support

Here k(x)=10, f(x)=10x^2

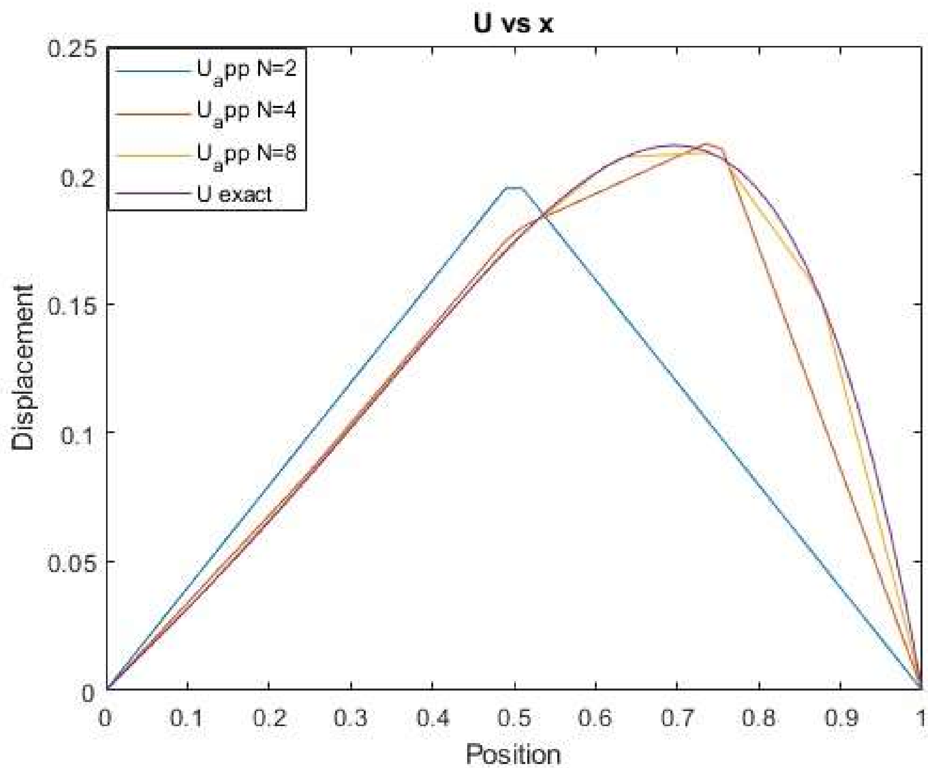
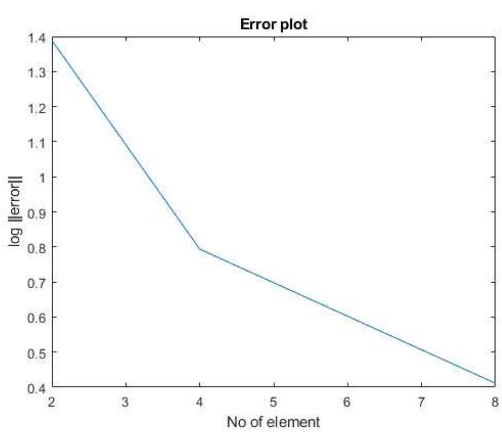


Fig 22



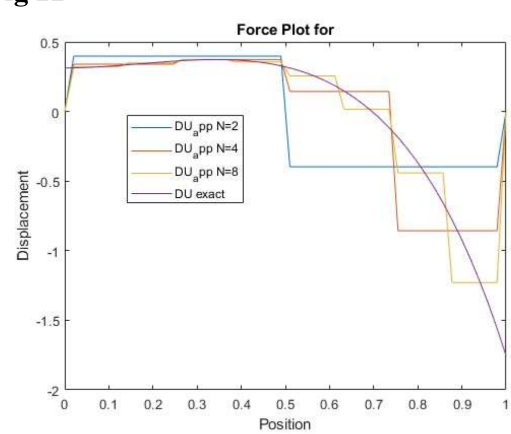


Fig 23 Fig 24

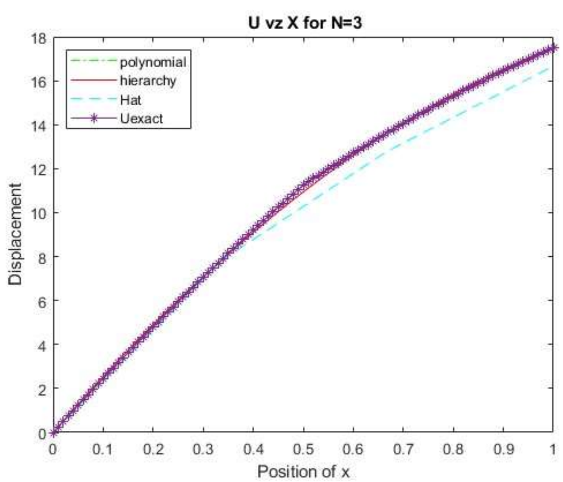
On comparing with problem 2 following observation can be made

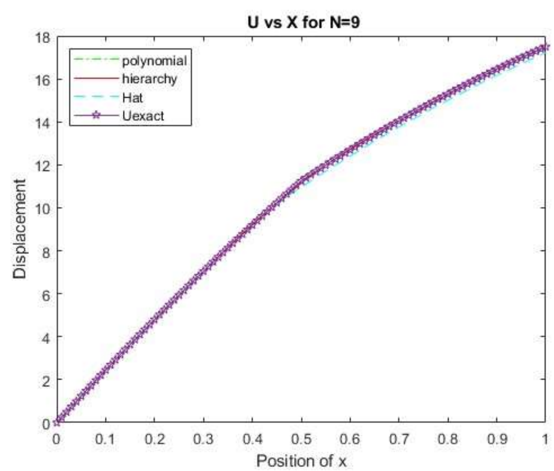
- N=2 gives a pretty decent result as compared with the Hierarchy basis
- And rate of convergence is slower due to H version of FEM

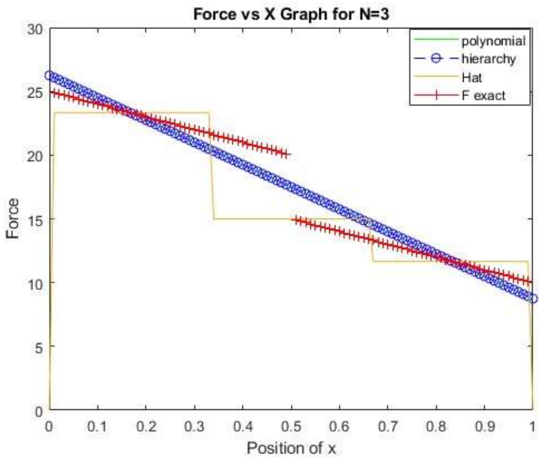
Q4. An axial bar with fixed at L = 0 and at L = 1 P_L load is applied. f(x) body force is acting on the bar and there is a point load is acting on the bar at L = 0.5, $P_{1/2}$ = 5. Approximating the Ans for [3,5,9] no of element

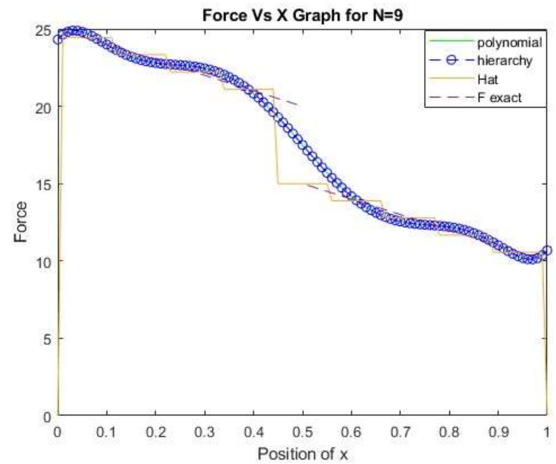
4.1 problem description

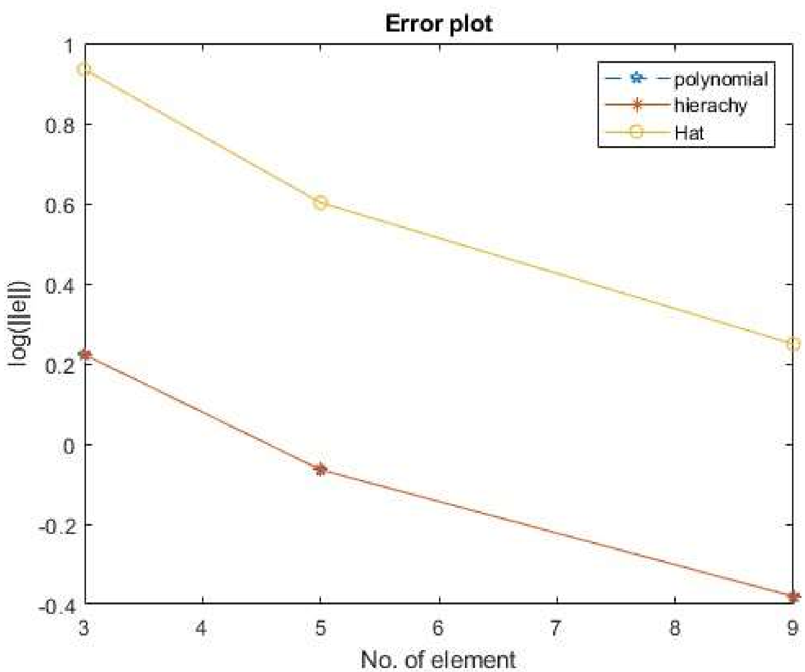
Pl=10, $P_{0.5L}$=5 and U(x=0) =0 also f(x)= $10x^2$
Here we are considering odd no of terms











Observation for ques 4

- for lower N hierarchy and polynomial basis function giving better approximation than the hat basis function, but as no of element increases hat basis function also start giving better approximation.

- ❖ The force vs. x plots show that neither the polynomial nor the hierarchy basis can accurately approximate the discontinuity in the solution (caused by the point load) because both contain continuous functions.

- ❖ Hat basis functions are piecewise linear and have continuous first derivatives, which makes them smoother than discontinuous piecewise polynomial basis functions such as the Lagrange basis functions. Hence this is leading to more accurate approximations and faster convergence rates.

- ❖ The hat basis functions still give constant force over the element.

- ❖ The rate of convergence for hat is higher than polynomial and hierarchy for this specific problem as it is able to capture the discontinuity due to its piecewise nature.

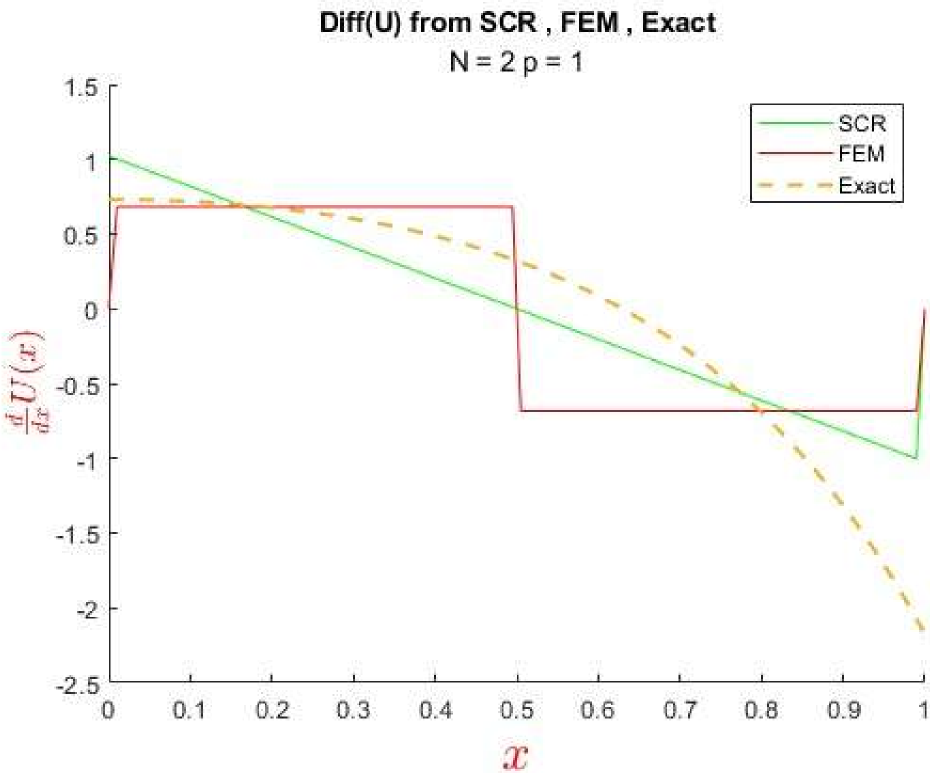
5 Postprocessing (SCR) and Use of gauss quad integration method

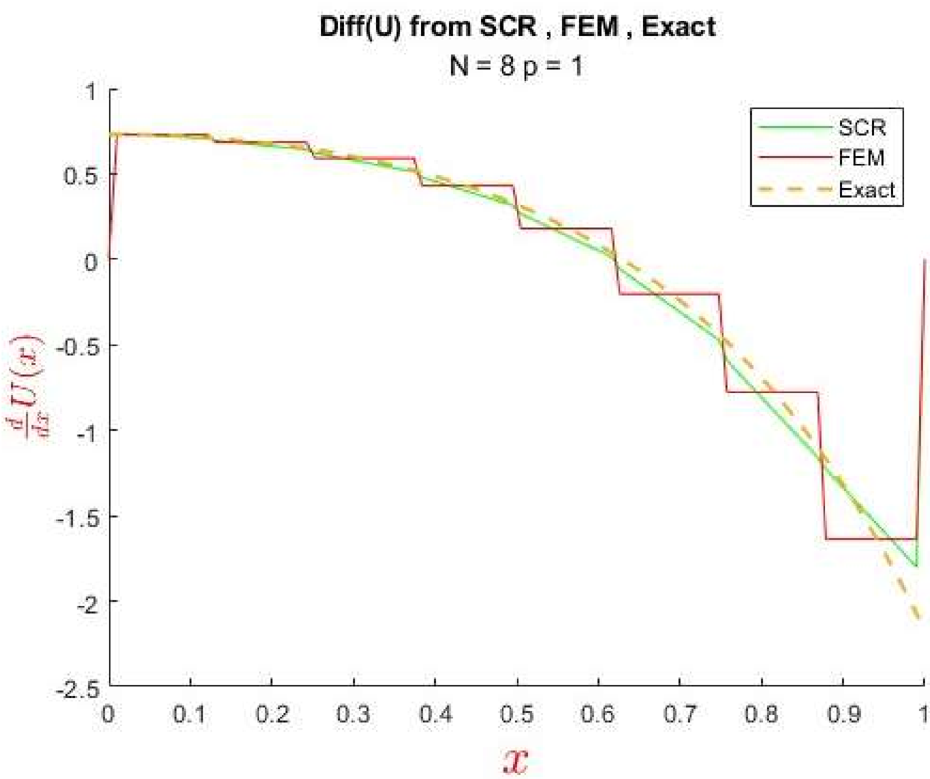
5.1 Problem statement:

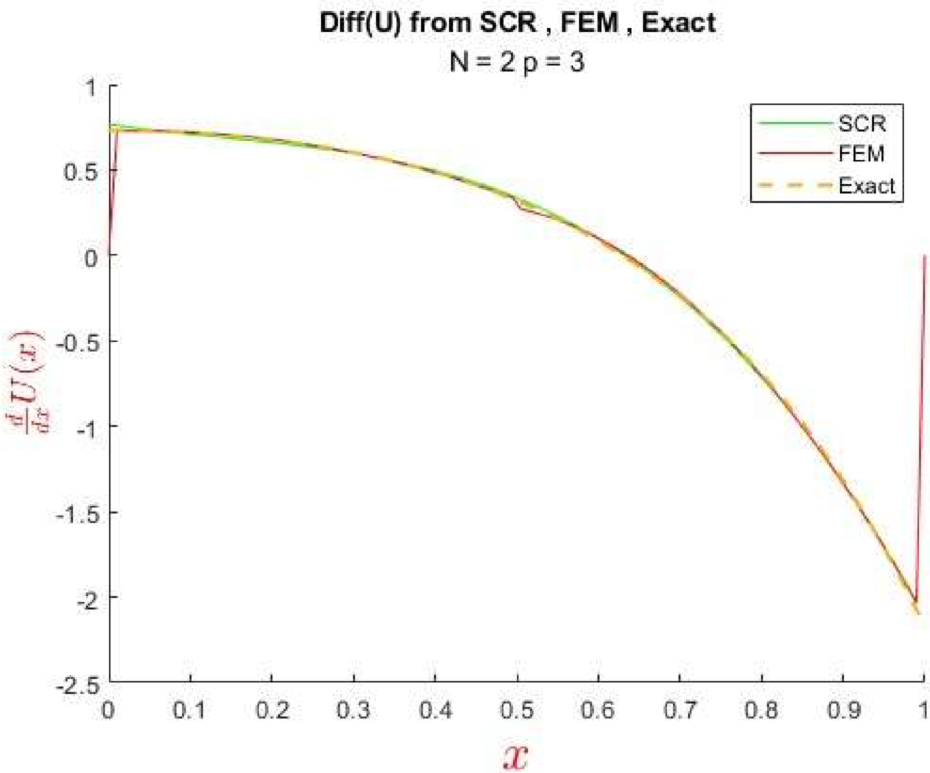
Here in this we will use the problem statement of question 3 and we compare the exact solution obtained by solving the governing differential equation with the approximation of the finite element solution achieved by using gauss quadrature and super convergent patch recovery theory.

5.2 Related plots:

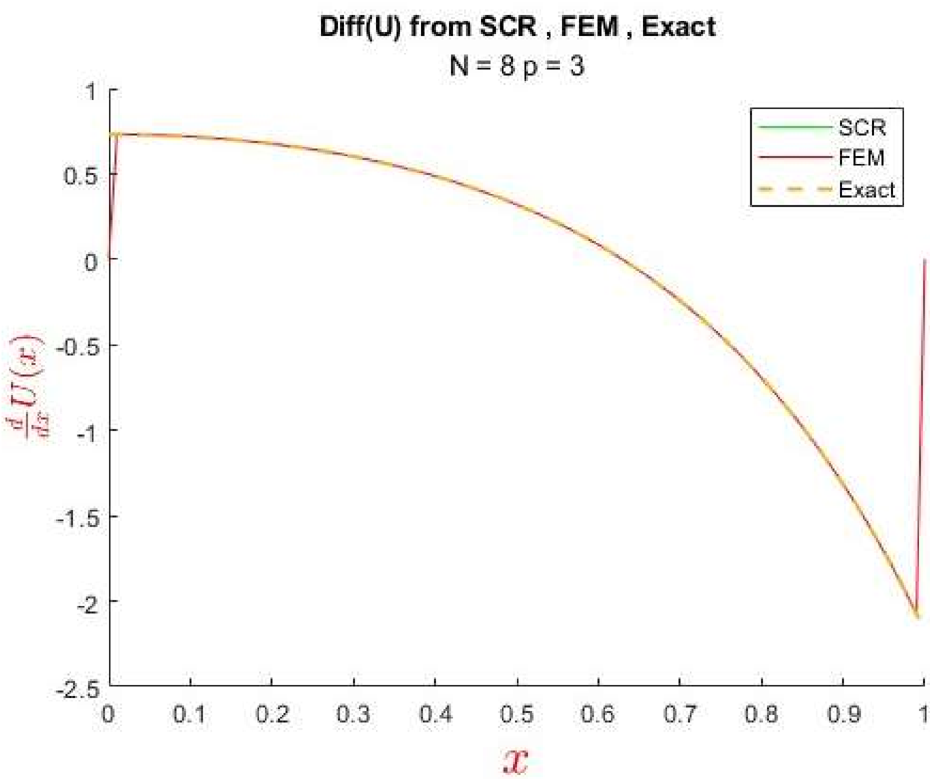
Here we have plot for order p=1,3
And for N=2,8
The code uses Gauss quadrature to numerically evaluate the integrals involved in the calculation of K and F.


Diff(U) from SCR , FEM , Exact
N = 2 p = 1
SCR
FEM
Exact
$\frac{d}{dx}U(x)$
x
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1


Diff(U) from SCR , FEM , Exact
N = 8 p = 1
SCR
FEM
Exact
$\frac{d}{dx}U(x)$
x
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1



Diff(U) from SCR , FEM , Exact
N = 2 p = 3
SCR
FEM
Exact
$\frac{d}{dx}U(x)$
x
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1



Diff(U) from SCR , FEM , Exact
N = 8 p = 3
SCR
FEM
Exact
$\frac{d}{dx}U(x)$
x
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

5.3 Conclusion:

- As can be seen Strains from FE solution are good at the center of each element but they are having a bad representation at nodes.

- In case of SPR we do least square fit to improve and refine overall results.

- **T**he SPR approach, however, gives a considerably closer approximation to the actual solution.