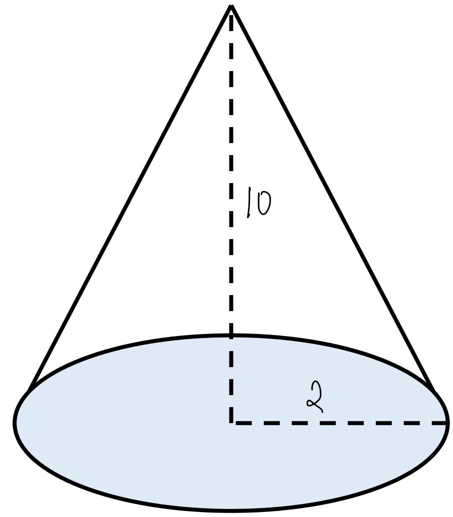
Class 26

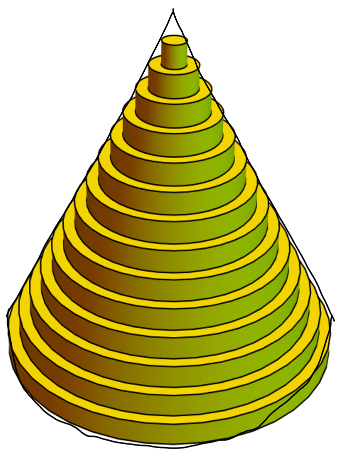
6.2 Volumes

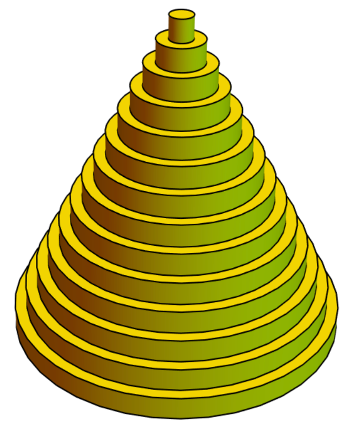
6.3 Volumes by Cylindrical Shells

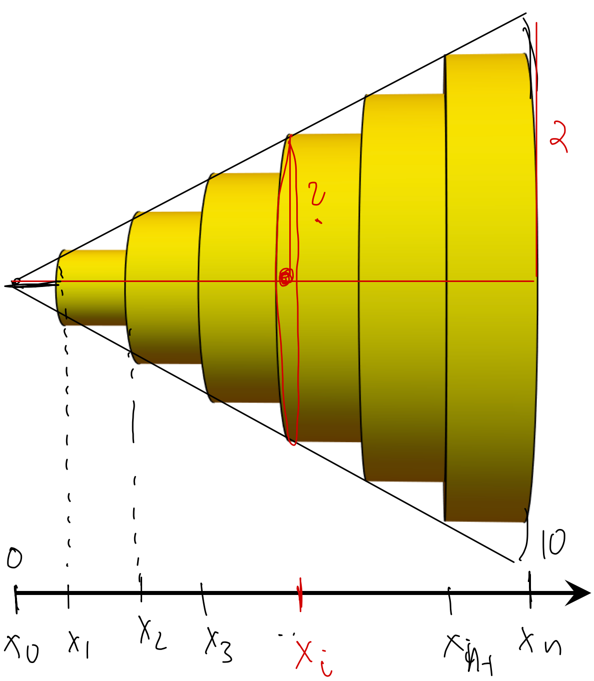
6.2 Volumes

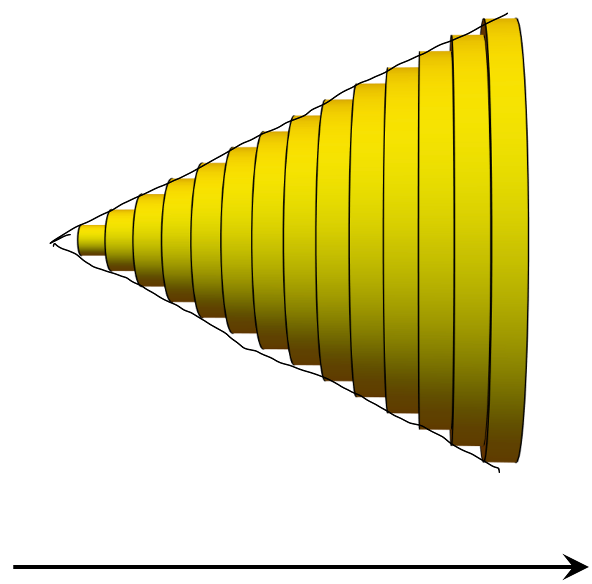
Ex 1 Find the volume of a cone of height 10 and base-radius 2.











$$\Delta x = \frac{10}{n}$$

$$\frac{\text{radius at } x_i}{x_i} = \frac{2}{10} \Rightarrow \text{radius at } x_i = \frac{2}{10} x_i$$

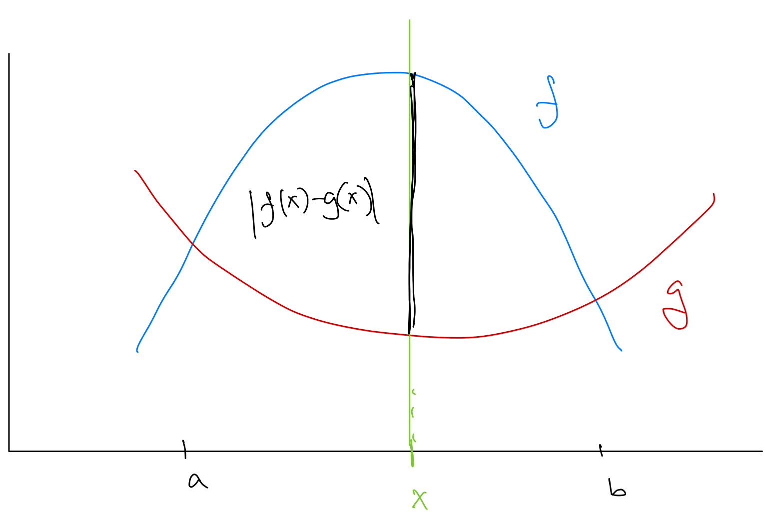
$$\text{volume of cone} \approx \sum_{i=0}^{n-1} \underbrace{\pi\left(\frac{2}{10}x_i\right)^2}_{\text{area of the circular base of disk } i} \cdot \underbrace{\Delta x}_{\frac{10}{n}}$$

$$= L_n \text{ approximation of } \int_0^{10} \pi\left(\frac{2}{10}x\right)^2 dx$$

$$\text{volume of cone} = \lim_{n\to\infty} L_n = \int_0^{10} \pi\left(\frac{2}{10}x\right)^2 dx$$

$$= \int_0^{10} \pi\left(\frac{1}{5}x\right)^2 dx = \frac{\pi}{25}\int_0^{10} x^2\,dx$$

$$= \frac{\pi}{25}\,\frac{1}{3}x^3\Big]_0^{10} = \frac{1000}{75}\pi$$

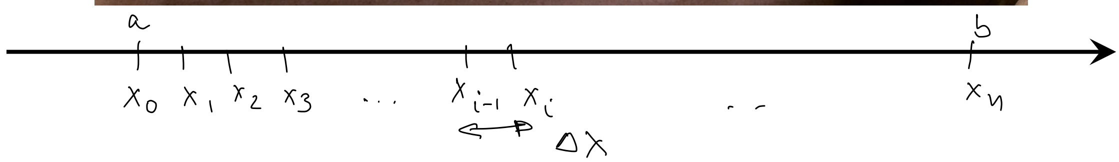


$$A = \int_a^b |f(x)-g(x)|\,dx$$

General formula

area of left side $A(x_{i-1})$
are of right side $A(x_i)$

i^{th} slice



$A(x)$ = area of cross-section at x

volume of i^{th} slice $\approx$ $A(x_{i-1})\,\Delta x$ $\quad$ OR $\quad$ $A(x_i)\,\Delta x$

volume of loaf $\approx$ $\underbrace{\sum_{i=1}^{n} A(x_{i-1})\Delta x}_{L_n}$ $\quad$ OR $\quad$ $\underbrace{\sum_{i=1}^{n} A(x_i)\Delta x}_{R_n}$

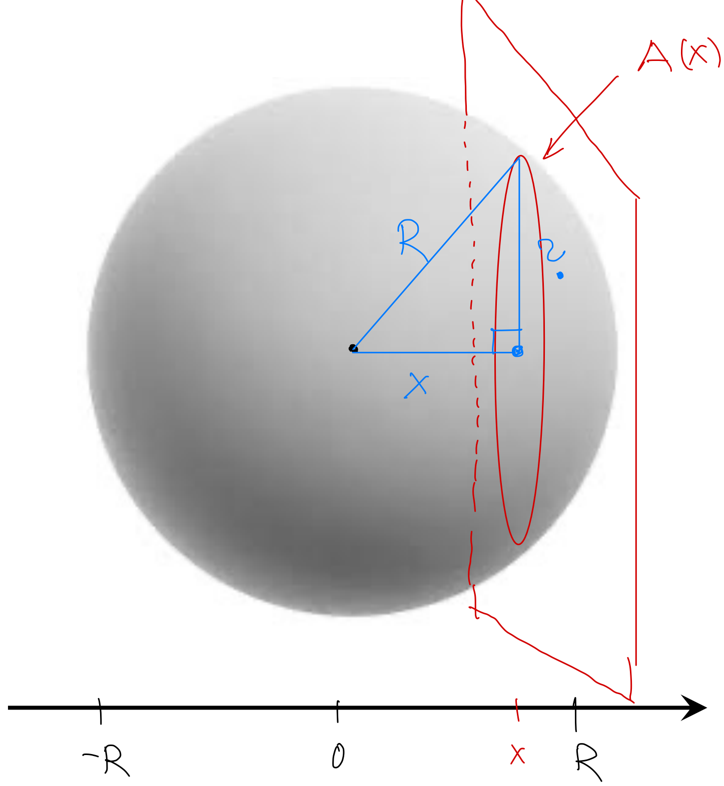
volume of loaf =

$$= \int_a^b A(x)\,dx$$

If $A(x)$ denotes the area of the cross-section of a solid at x, then

$$\text{volume} = \int_a^b A(x)\,dx$$

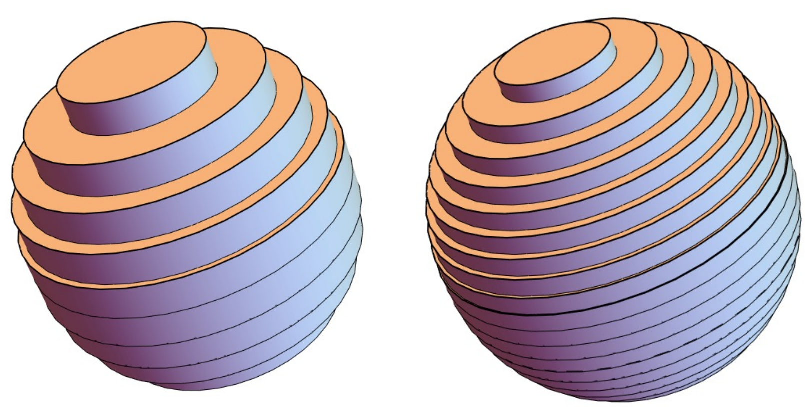
Ex 2 Find the volume of a sphere of radius R.



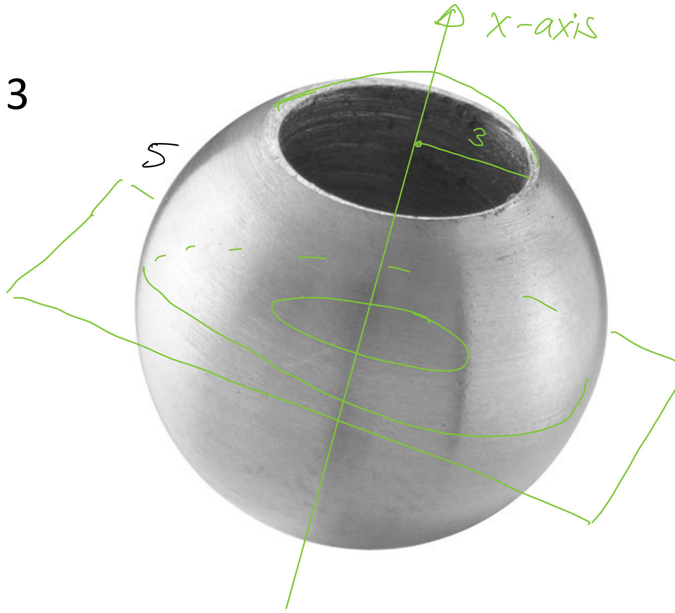
$$? = \sqrt{R^2 - x^2}$$

$$A(x) = \pi\left(\sqrt{R^2 - x^2}\right)^2$$
$$= \pi\left(R^2 - x^2\right)$$

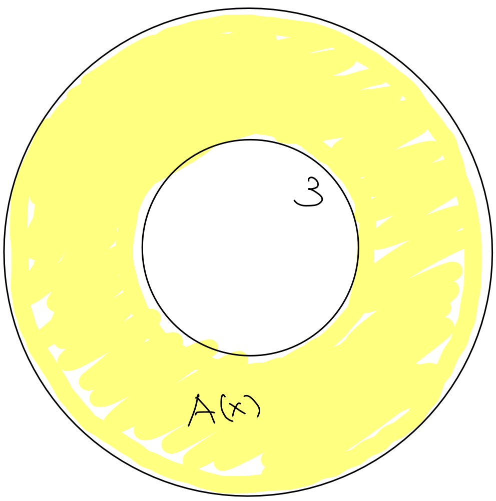
$$\text{volume} = \int_{-R}^{R} A(x)\,dx$$
$$= \int_{-R}^{R} \pi\left(R^2 - x^2\right)dx$$
$$= \left(\pi R^2 x - \tfrac{1}{3}x^3\right)\Big]_{-R}^{R}$$
$$= \underbrace{\left(\pi R^2 \cdot R - \tfrac{1}{3}R^3\right)}_{\pi R^3 - \frac{1}{3}R^3} - \underbrace{\left(\pi R^2(-R) - \tfrac{1}{3}(-R)^3\right)}_{-\pi R^3 + \frac{1}{3}R^3}$$
$$= \tfrac{2}{3}\pi R^3 + \tfrac{2}{3}\pi R^3 = \tfrac{4}{3}\pi R^3$$



Ex 3 A cylindrical hole of radius 3 is drilled into a sphere of radius 5.
What is the volume of the resulting object?



cross-section via plane at x



$\sqrt{5^2-x^2}$

boundary values:

$$A(x) = 0$$

$$\pi(16-x^2)=0$$

$$x = \pm 4$$

$$A(x) = \underbrace{\pi\left(\sqrt{5^2-x^2}\right)^2}_{\text{area of outer circle}} - \underbrace{\pi\, 3^2}_{\text{area of inner circle}}$$

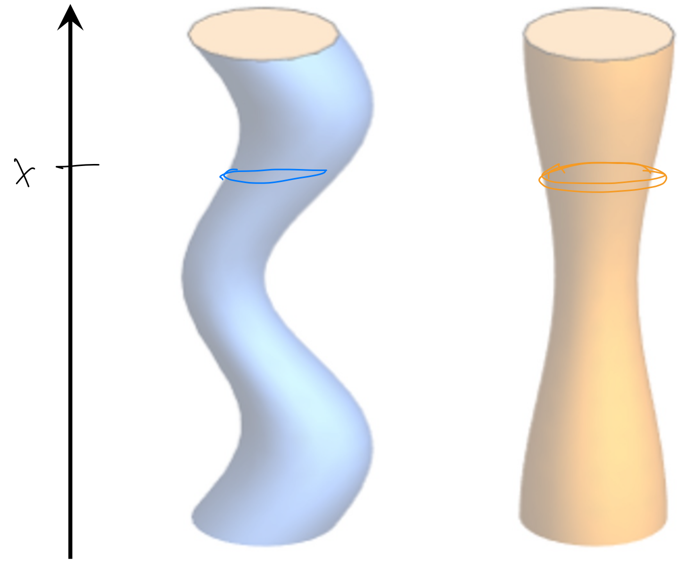
$$= \pi(25-x^2) - \pi \cdot 9 = \pi(16-x^2)$$

$$\text{volume} = \int_{-4}^{4} A(x)\,dx = \int_{-4}^{4} \pi(16-x^2)\,dx$$

$$= \left(16\pi x - \tfrac{1}{3}\pi x^3\right)\Big]_{-4}^{4}$$

$$= 2\cdot\left(16\pi\cdot 4 - \tfrac{1}{3}\pi\, 4^3\right) = \frac{256}{3}\pi$$

Cavalieri's Principle
If the cross-sections of two solids have equal areas at equal heights, then both objects have the same volume.



6.3 Volumes by Cylindrical Shells

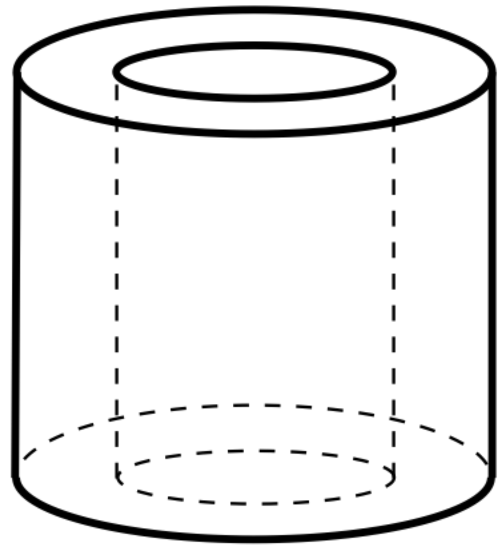
3D Cylindrical Shell

r = inner radius

w = width

h = height

$\bar{r}$ = average radius

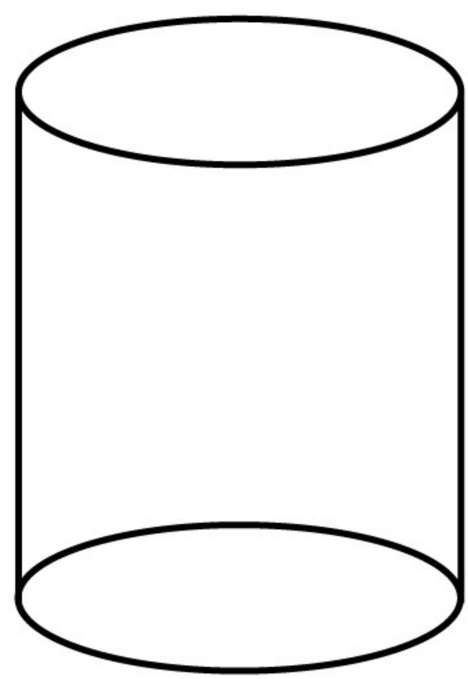


volume =

2D (uncapped) Cylinder

r = radius

h = height



area =

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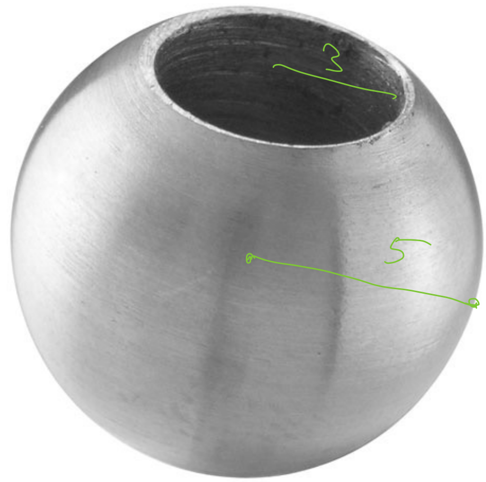
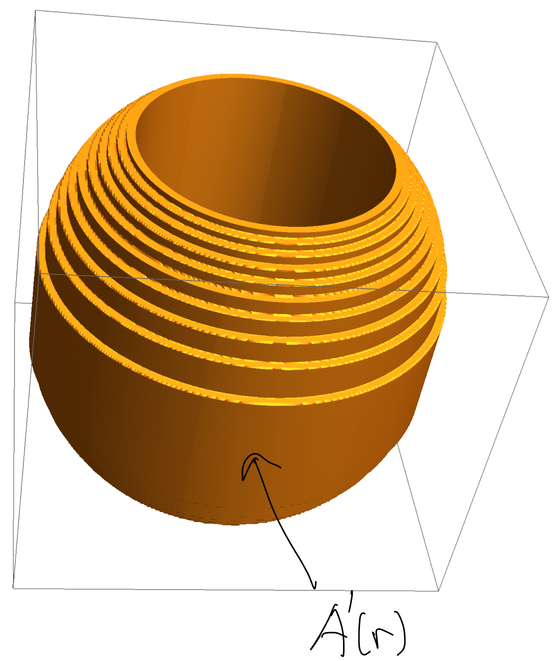


average radius of i^{th} shell $\approx$

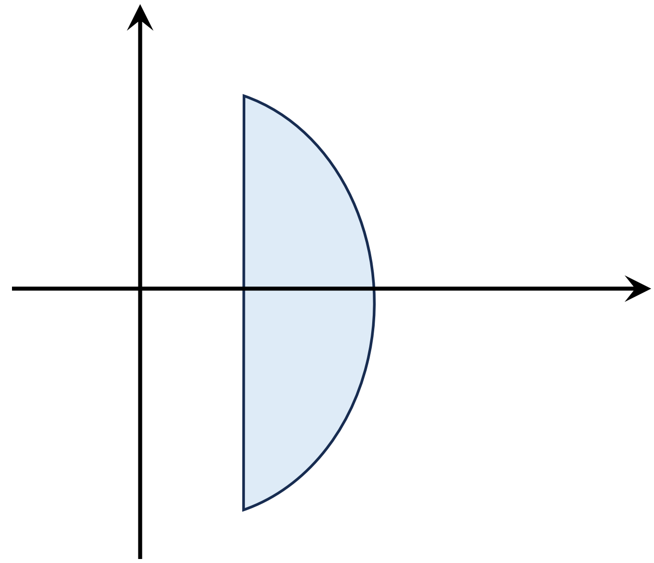
height of i^{th} shell $\approx$



total volume of shells $=$

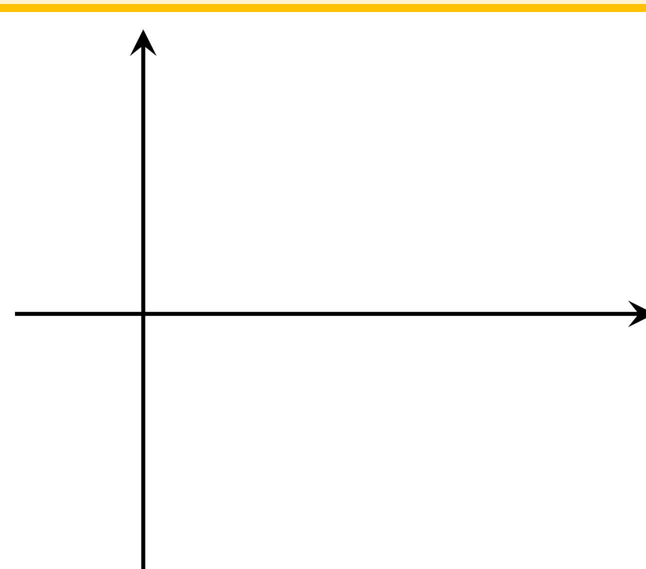
volume of solid $=$ $$\int_3^5 \underbrace{A'(r)}_{\text{surface area of cylindrical shell at radius } r}\, dr$$

area of cylindrical shell at radius r:



Suppose a solid of revolution is obtained by rotating a 2D region around the y-axis. If $h(a)$ is the height of the vertical line $x = a$ intersected with the this region, then the volume of the 3D object is:

volume $=$



Ex 2

Find the volume of a solid of revolution obtained from rotating the shape depicted below around the y-axis.