Lab 4 for Math 1A (Fall 2023)

Richard Bamler

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1 Analyzing indeterminate limits

An indeterminate limit is a limit that cannot be broken down using the limit laws. Colloquially, we refer to these limits as limits of type "$\frac{\infty}{\infty}$", "$\frac{0}{0}$", "$\infty - \infty$", etc. To compute such limits, it generally necessary to simplify the expression behind the limit sign first. The goal of this exercise is less the arithmetic involved in this process. We rather want to illustrate such limits using actual numbers. We also want to get a feeling for why such limits sometimes still produce a reasonable value, even though their components don't. In the process, we also want to see more concretely why some of the arithmetic tricks that we've learned in class work.

Limits of type "$\frac{\infty}{\infty}$"

We first consider a limit of the type "$\frac{\infty}{\infty}$". As an example, we want to compute the limit

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{g(x)}{h(x)} = \lim_{x\to\infty} \frac{2x^2+3x+4}{5x^2+6x+7}.$$

1. Open a new spreadsheet.
2. Enter:
   - `x` in cell `A1`.
   - `g(x)` in cell `B1`.
   - `h(x)` in cell `C1`.
   - `f(x)=g(x)/h(x)` in cell `D1`.
3. Choose $x = 10$, so enter `10` in cell `A2`.
4. Enter `=2*A2^2+3*A2+4` in cell `B2` and `=5*A2^2+6*A2+7` in cell `C2`.
5. Compute $f(x) = \frac{g(x)}{h(x)}$ by entering `=B2/C2` in cell `D2`.

B2 | fx =2*A2^2+3*A2+4

	A	B	C	D
1	x	g(x)	h(x)	f(x)=g(x)/h(x)
2	10	234	567	0.4126984127

Enter larger and larger values for x in cell `A2`. Try the values $100, 1000, 10000, \ldots$ and observe what happens to the values in cells `B2-D2`. Can you observe a pattern? Can you explain it?

Let us now review the arithmetic computation of the limit above:

$$\lim_{x\to\infty} \frac{2x^2+3x+4}{5x^2+6x+7} = \lim_{x\to\infty} \frac{\frac{2x^2+3x+4}{x^2}}{\frac{5x^2+6x+7}{x^2}} = \lim_{x\to\infty} \frac{2+\frac{3}{x}+\frac{4}{x^2}}{5+\frac{6}{x}+\frac{7}{x^2}} = \frac{2}{5} = 0.4.$$

We want to understand this computation using the numbers in our example. The first step in the computation is to divide the numerator and denominator by x^2.

6. Enter `=B2/A2^2` in cell `B3` and `=C2/A2^2` in cell `C3`.
7. Select cell `D2` and drag down the blue dot to fill cell `D3`, which computes the quotient of the values in cells `B3` and `C3`. Note that you should get the same number as in cell `D2`.

D3 | fx =B3/C3

	A	B	C	D
1	x	g(x)	h(x)	f(x)=g(x)/h(x)
2	10	234	567	0.4126984127
3		2.34	5.67	0.4126984127

Enter different numbers, such as $10, 100, 1000, 10000, \ldots$ in cell `A2` and observe what happens. Relate this to the arithmetic computation above.

Limits of the form "$\frac{0}{0}$"

Next, we will investigate a limit of the form "$\frac{0}{0}$". We will consider the following example:

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{g(x)}{h(x)} = \lim_{x\to 1} \frac{x^2-1}{x^2-x}.$$

8. Delete the content from cells `B3-D3`.
9. Replace the content of cell `B2` with `=A2^2-1` and the content of cell `C2` with `=A2^2-A2`.

? Enter numbers in cell `A2` that are closer and closer to 1, such as $1.1, 1.01, 1.001$, and observe the pattern.

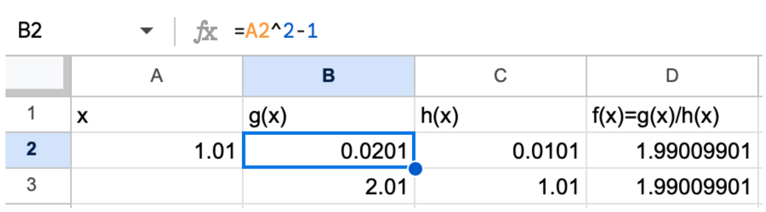
Again, we consider the arithmetic computation of this limit:

$$\lim_{x\to 1} \frac{x^2-1}{x^2-x} = \lim_{x\to 1} \frac{(x+1)(x-1)}{x(x-1)} = \lim_{x\to 1} \frac{x+1}{x} = 2.$$

We want to implement this idea as before, so we will divide numerator and denominator by $x-1$.

10. Enter `=B2/(A2-1)` in cell `B3` and `=C2/(A2-1)` in cell `C3`.
11. Select cell `D2` and drag down the blue dot to fill cell `D3`. Note again that you should get the same number as in cell `D2`.

? Again, choose different numbers in cell `A2`, such as $1.1, 1.01, 1.001$, and observe the pattern. Relate it to the arithmetic computation above.



Limits of the form "$\infty - \infty$"

Lastly, we want to consider a limit of the form "$\infty - \infty$":

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \big(g(x) - h(x)\big) = \lim_{x\to\infty} \left(\sqrt{x^2+2x+2} - \sqrt{x^2+1}\right).$$

12. Delete the content from cells `B3-D3`.
13. Replace the content of cell `D1` with `f(x)=g(x)-h(x)`.
14. Replace the content of cell `B2` with `=SQRT(A2^2+2*A2+2)` and the content of cell `C2`

with `=SQRT(A2^2+1)`.

15. Replace the content of cell `D2` with `=B2-C2`.

D2 ▾ | fx =B2-C2

	A	B	C	D
1	x	g(x)	h(x)	f(x)=g(x)-h(x)
2	10	11.04536102	10.04987562	0.9954853961

Choose $x = 2, 3, 4, \ldots$. Do you notice that the numbers $g(x), h(x)$ are just offset by one, meaning that $g(x) = f(x+1)$? Can you verify this using a computation? Also, do you notice that $g(x)$ and $f(x)$ are "not too far away" from x for large x? Can you make this more concrete? What do you observe as you choose x very large? What does this have to do with the limit?

Compute the limit above arithmetically and verify that the value in cell `D2` indeed approaches the correct limit.

You may want to implement the arithmetic computation of the limit in your spreadsheet. However, it seems less clear how to do this. Maybe you can find a way.

Exercise 1 Enter `20` in cell `A2` and report the value (rounded to the first 5 digits after the decimal point) of cell `D2` to Gradescope.

2 The Bisection Method

In this exercise we want to implement the bisection method to approximate a root of a continuous function. As an example, we will consider the function

$$f(x) = x^2 - 2$$

and our goal will be to approximate its second root $\sqrt{2} = 1.41\ldots$. Recall that the method works as follows:

1. We choose an interval $[a_1, b_1]$ on which f is continuous. In our example, we choose $[a_1, b_1] = [1, 2]$. Then $f(1) = -1 < 0$ and $f(2) = 1 > 0$. So by the Intermediate Value Theorem there has to be a root $c \in [1, 2]$ with $f(c) = 0$.

2. Denote the midpoint of the interval $[1, 2]$ by $m_1 = 1.5$. Bisect the interval $[a_1, b_1] = [1, 2]$ into the intervals $[a_1, m_1] = [1, 1.5]$ and $[m_1, b_1] = [1.5, 2]$. Since there is a root in $[1, 2]$, we

conclude that there must be a root in at least one of the smaller intervals $[1, 1.5]$ and $[1.5, 2]$. To find out which of these smaller intervals contains a root, we compute $f(m_1) = 1.25 > 0$ and remember that $f(a_1) < 0$ and $f(b_1) > 0$. So by the Intermediate Value Theorem, there must be a root in the first interval $[a_1, m_1]$.

3. Set $[a_2, b_2] = [a_1, m_1] = [1, 1.5]$ and let $m_2 = 1.25$ be its midpoint. Bisect the interval into smaller intervals $[1, 1.25]$ and $[1.25, 1.5]$. We remember that again $f(a_2) < 0$ and $f(b_2) > 0$, but this time we compute that $f(m_2) < 0$. So the root must lie in the second smaller interval $[1.25, 1.5]$.

4. Set $[a_3, b_3] = [1.25, 1.5]$ and continue the process.

1. Open a new spreadsheet.
2. Enter:
   - `n` in cell `A1`.
   - `a_n` in cell `B1`.
   - `m_n` in cell `C1`.
   - `b_n` in cell `D1`.
   - `f(a_n)` in cell `E1`.
   - `f(m_n)` in cell `F1`.
   - `f(b_n)` in cell `G1`.
3. Fill the cells `A2-A11` (or further) with the numbers 1, 2, . . . : Enter `1` in cell `A2`, `=A2+1` in `A3` and drag down the little blue dot.
4. We want to start our algorithm with the interval $[a_1, b_1] = [1, 2]$. So enter `1` in cell `B2` and `2` in cell `D2`.
5. Compute the midpoint $m_1 = \frac{a_1+b_1}{2}$ by entering `=(B2+D2)/2` in cell `C2`.
6. We now compute $f(a_1)$, $f(m_1)$ and $f(b_n)$, where $f(x) = x^2 - 2$. To do this, enter `=B2^2-2` in cell `E2` and drag the little blue dot to the right to fill the cells `E2-G2`. Verify that the content of the cells `F2` and `G2` is `=C2^2-2` and `=D2^2-2`, respectively.

Next, we need to determine the values of a_2 and b_2. To make this decision, we rely on the Intermediate Value Theorem to ascertain whether there exists a root of the function f within the interval $[a_1, m_1]$ (or, in other words, $f(x) = 0$ for some $x \in [a_1, m_1]$). If the theorem guarantees a root within this interval, we set $[a_2, b_2]$ equal to $[a_1, m_1]$; otherwise, we set it as $[m_1, b_1]$.

The Intermediate Value Theorem guarantees a root of f within $[a_1, m_1]$ if 0 is between or equal to $f(a_1)$ and $f(m_1)$. This is the case if:

- $f(a_1) \geq 0$ and $f(m_1) \leq 0$, or
- $f(a_1) \leq 0$ and $f(m_1) \geq 0$.

Now, here's a clever trick we can employ. Instead of checking both of the aforementioned cases separately, we can simplify our evaluation to a single condition:

$$f(a_1) \cdot f(m_1) \leq 0.$$

This is because the product of two numbers can only be non-negative if one of the numbers is non-negative, while the other is non-positive.

7. Enter `=IF(E2*F2<=0,B2,C2)` in cell `B3` and `=IF(E2*F2<=0,C2,D2)` in cell `D3`

The command `=IF(E2*F2<=0,B2,C2)` returns `B2` if the condition `E2*F2<=0`, which means "`E2` $\cdot$ `F2` ≤ 0", is met. Otherwise, it returns `C2`. So the structure is: `=IF(` 'condition'`,`'return value if condition is met'`,`'return value if condition is not met'`)`.

8. Compute the midpoint of the interval $[a_2, b_2]$, i.e., the average of the numbers in cells `B3` and `D3`, by selecting the cell `C2` and dragging down the blue dot to fill cell `C3`.
9. Compute the values $f(a_2), f(m_2)$ and $f(b_2)$ by selecting the cells `E2-G2` and dragging down the blue dot to fill the cells `E3-G3`.
10. Repeat the algorithm by selecting the cells `B3-G3` and dragging town the blue dot to fill the rows `4-11` (or further).

B3 | `=IF(E2*F2<=0,B2,C2)`

	A	B	C	D	E	F	G
1	n	a_n	m_n	b_n	f(a_n)	f(m_n)	f(b_n)
2	1	1	1.5	2	-1	0.25	2
3	2	1	1.25	1.5	-1	-0.4375	0.25
4	3	1.25	1.375	1.5	-0.4375	-0.109375	0.25
5	4	1.375	1.4375	1.5	-0.109375	0.06640625	0.25
6	5	1.375	1.40625	1.4375	-0.109375	-0.0224609375	0.06640625
7	6	1.40625	1.421875	1.4375	-0.0224609375	0.02172851563	0.06640625
8	7	1.40625	1.4140625	1.421875	-0.0224609375	-0.00042724609:	0.02172851563
9	8	1.4140625	1.41796875	1.421875	-0.00042724609:	0.01063537598	0.02172851563
10	9	1.4140625	1.416015625	1.41796875	-0.00042724609:	0.005100250244	0.01063537598
11	10	1.4140625	1.415039063	1.416015625	-0.00042724609:	0.002335548401	0.005100250244

Look at the numbers in columns `B-D`. Can you carry out the first few steps in your head (or using a calculator)? Explain why these columns contain an approximation of $\sqrt{2}$ with an error that halves in every step: $1, \frac{1}{2}, \frac{1}{4}, \ldots$

What happens if you adjust the endpoints of the interval $[a_1, b_1]$, i.e., the values in cells `B2` and `D2`. Experiment with different values. What happens if you choose an initial interval that does not contain a root of f? How do you need to choose the interval $[a_1, b_1]$ so that you capture the other root, $-\sqrt{2}$, of f?

It is a very deep mathematical theorem that the root of the function $f(x) = x^5 - x - 1$ cannot be computed using the basic arithmetic operations and n-th roots. Modify the content of columns `F-G` (and possibly the starting values for $[a_1, b_1]$) to compute an approximation for this root.

Exercise 2 Modify the spreadsheet to approximate $\arctan(0.5) = \tan^{-1}(0.5)$. Proceed as follows: Modify the content in columns `E-G` in a way that amounts to changing the function $f(x) = x^2 - 2$ to $f(x) = \tan(x) - c$, where c is a constant that you need to choose correctly (choosing this constant is part of the exercise). Modify the content in cells `B2` and `D2` so that the starting interval is $[a_1, b_1] = [0, 1]$. What is a_5 (this is the number in row `6`)? Enter this value (the first 4 digits after the decimal point) in Gradescope.
Be aware that $\tan(x)$ is not continuous on the interval $[1, 2]$, so the Intermediate Value Theorem does not apply to this interval.

3 Illustrating the precise definition of the limit

In this exercise, we want to (once again) illustrate the definition of the limit. For practical reasons we will consider the definition of the limit at infinity:

$$\lim_{x \to \infty} f(x) = L,$$

which is:

For any $\varepsilon > 0$ there is a number N such that if $x > N$, then $|f(x) - L| < \varepsilon$.

Let us rephrase this sentence as:

For any $\varepsilon > 0$ the bound $|f(x) - L| < \varepsilon$ is true, as long as x is chosen larger than some number N.

A more colloquial way would be to to say that $|f(x) - L| < \varepsilon$ is true *eventually,* as x is chosen larger and larger. This will be the idea of our "simulation"; we want to *see* that the condition is true *eventually* no matter how small we choose ε. As an example, we will analyze the limit:

$$\lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$$

1. Open a new spreadsheet.
2. Enter:
   - `x` in cell `A1`.
   - `f(x)` in cell `B1`.
   - `|f(x)-L|` in cell `C1`.
   - `|f(x)-L|<epsilon?` in cell `D1`.
   - `L` in cell `E1`.
   - `epsilon` in cell `F1`.
3. Choose:
   - $x = 1$ by entering `1` in cell `A2`.
   - $f(x) = \arctan(x)$ by entering `=ATAN(A2)` in cell `B2`.
   - $L = \frac{\pi}{2}$ by entering `=PI()/2` in cell `E2`.
   - $\varepsilon = 0.1$ by entering `0.1` in cell `F2`.
4. Compute $|f(x) - L|$ by entering `=ABS(B2-E$2)` in cell `C2`. Note the dollar sign, whose purpose will be clear later.
5. It remains to answer the question in cell `D1`. To do this, enter `=IF(C2<F$2,"YES","NO")` in cell `D2`. Note the dollar sign, whose purpose will be clear later. So if the number in cell `C2` is less than the number in the cell `F2`, then the answer in cell `D2` is `YES`, and otherwise `NO`.

? Enter bigger and bigger numbers in cell `A2` and/or cell `F2` and see how the numbers in the other cells and the answer in cell `D2` change.

We now want to illustrate more clearly that the answer to the question in cell `D1` is true *eventually.* To do this we will evaluate $\arctan(x)$ for $x = 1, 2, 3, \ldots$. Of course, the limit $\lim_{x\to\infty}$ does not require that x is an integer, but we can't consider *all* numbers in our simulation.

6. Enter `=A2+1` in cell `A3` and drag down the blue dot to fill column `A` with the numbers $x = 1, 2, \ldots$ for as far as you want.

7. Select the cells `B2-D2` and drag down the blue dot to the same row.

`D2` `=IF(C2<F$2,"YES","NO")`

	A	B	C	D	E	F
1	x	f(x)	|f(x)-L|	|f(x)-L|<epsilon?	L	epsilon
2	1	0.7853981634	0.7853981634	NO	1.570796327	0.1
3	2	1.107148718	0.463647609	NO		
4	3	1.249045772	0.3217505544	NO		
5	4	1.325817664	0.2449786631	NO		
6	5	1.373400767	0.1973955598	NO		
7	6	1.405647649	0.1651486774	NO		
8	7	1.428899272	0.1418970546	NO		
9	8	1.446441332	0.1243549945	NO		
10	9	1.460139106	0.1106572212	NO		
11	10	1.471127674	0.09966865249	YES		
12	11	1.48013644	0.0906598872	YES		
13	12	1.487655095	0.08314123189	YES		
14	13	1.494024436	0.07677189127	YES		
15	14	1.499488862	0.07130746479	YES		
16	15	1.504228163	0.06656816378	YES		

? Note in column `D` all answers are YES after a certain point. Adjust the number for ε in cell `F2` (for example $\varepsilon = 0.01$) and see what happens to the answers in row `D`.

? Choose a different value for L in cell `E2`. Is it true that the answers in column `D` are YES after a certain point *no matter how small the value in cell* `F2`?

? Remember the exercise ‘Slow vs fast convergence’ from the previous lab. What does ‘slow’ and ‘fast’ mean in the context of the dependence of column `D` on the value in cell `F2`? Does the point after which all answers are YES appear sooner or later for the ‘slow’ convergence?

? In our example there is a definite point up to which all answers are NO; after this point all answers are YES. However, this does not need to be the case. For example, the answers could be of the form NO, YES NO, YES, YES, YES NO, NO, NO YES, YES, . . . The important statement in the definition above is that the answers are eventually all YES.

Exercise 3 Modify the spreadsheet above by replacing the function $f(x) = \arctan(x)$ with the function $f(x) = \frac{\sin x}{x}$ and choose $L = 0$. In other words, simulate the limit $\lim_{x\to\infty} \frac{\sin x}{x} = 0$. Choose a value for ε in cell `F2` in such a way that the answers are not of the form NO, ..., NO, YES, ... (as discussed in the question above). In other words, choose ε so that the answer switches from YES back to NO at some point. The answers can start with YES or NO and there can be many switches (YES to NO as well as NO to YES) in between. Once you have achieved such a pattern, select File, Download, PDF and Export to save your work as a PDF. Upload this PDF to Gradescope.