Lab 8 for Math 1A (Fall 2023)

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1 Numerical Integration using spreadsheets

In this exercise we want to approximate a definite integral through numerical integration. To illustrate this, let's focus on the integral

$$\int_1^2 \frac{1}{x}\, dx \tag{1.1}$$

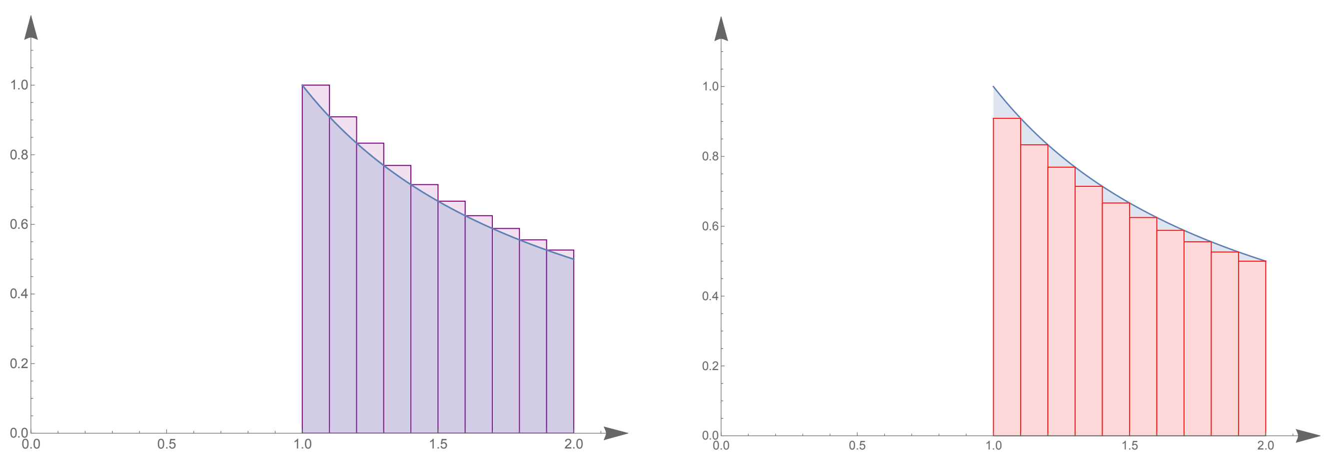
Our aim is to approximate this integral by computing the left and right endpoint approximations, denoted as L_n and R_n, respectively.

Numerical integration becomes imperative when an integral cannot be solved algebraically or when the function is only given via data points. The integral in this example, however, can be solved algebraically using the Fundamental Theorem of Calculus (see Chapter 5.3):

$$\int_1^2 \frac{1}{x}\, dx = \Big[\ln|x| \Big]_1^2 = \ln 2 - \ln 1 = \ln 2 = 0.6931\ldots$$

We can use this fact later to test the precision of our numerical approximation.

Let us now review the computation of the left and right endpoint approximations L_n, R_n. Consider the figures below, which correspond to L_{10} and R_{10}.



In each figure there are 11 (!!) sample points:

$$x_0 = 1, \quad x_1 = 1.1, \quad \ldots, \quad x_{10} = 2$$

spaced $\Delta x = \frac{2-1}{10} = 0.1$ far apart. The left endpoint approximation L_{10} is the sum of the areas of 10 rectangles, whose widths are Δx and whose heights are $f(x_0), f(x_1), \ldots, f(x_9)$; here $f(x) = \frac{1}{x}$. By contrast, the rectangles in the right endpoint approximation R_{10} have the heights $f(x_1), f(x_2), \ldots, f(x_{10})$. In terms of formulas, this means:

$$L_{10} = \sum_{i=0}^{9} f(x_i)\Delta x, \qquad R_{10} = \sum_{i=1}^{10} f(x_i)\Delta x. \tag{1.2}$$

So the sole distinction between both formulas lies in the range of i. To address potential misconceptions, it is essential to recongize that the 10 rectangles in the left figure are all distinct from the 10 rectangles in the right figure. However, rectangles 2 through 10 in the left figure align with rectangles 1 through 9 from the right figure after a leftward shift by Δx. Consequently, all but one term in both summations in (1.2) are the same.

We will now compute these sums using spreadsheets. We will denote the endpoints $1, 2$ by a, b. This will allow us to change the values of the endpoints easily.

1. Open a new spreadsheet and enter:
   - `a` in cell `A1`.
   - `b` in cell `B1`.
   - `n` in cell `C1`.
   - `Delta x` in cell `D1`.
   - `1` in cell `A2`.
   - `2` in cell `B2`.
   - `10` in cell `C2`.
2. Compute the width of the rectangles via the formula $\Delta x = \frac{b-a}{n}$. So enter `=(B2-A2)/C2` in cell `D2`.
3. Enter:
   - `i` in cell `A4`.
   - `x_i` in cell `B4`.
   - `f(x_i)` in cell `C4`.
   - `f(x_i) * Delta x` in cell `D4`.
4. Fill the cells `A5-A15` with the numbers $0, 1, \ldots, 10$. To do this, enter `0` in cell `A5` and `=A5+1` in cell `A6` and drag down the blue dot. Note again that these are 11 numbers, because we are starting with 0.

5. Compute $x_i = a + i \cdot \Delta x$. To do this, enter `=A$2+A5*D$2` in cell `B5` and drag down the blue dot (don't forget the dollar signs!). Verify that the first number is 1 and the last number is 2.

6. Compute $f(x_i) = \frac{1}{x_i}$. So enter `=1/B5` in cell `C5` and drag down the blue dot. Cells `C5-C15` show all 11 possible heights of the rectangles.

7. Compute the areas of the rectangles via the formula $f(x_i)\Delta x$. So enter `=C5*D$2` in cell `D5` and drag down the blue dot.

8. Remember that L_{10} is equal to the sum of the first 10 areas and R_{10} is equal to the sum of the last 10 areas. Enter `L_n` in cell `E4` and `=SUM(D5:D14)` in cell `E5`. Next, enter `R_n` in cell `F4` and `=SUM(D6:D15)` in cell `F5`. Note that `D5:D14` means the range of the cells `D5-D14` and `D6:D15` denotes the range of the cells `D6-D15`.

	A	B	C	D	E	F
1	a	b	n	Delta x		
2	1	2	10	0.1		
3						
4	i	x_i	f(x_i)	f(x_i) * Delta x	L_n	R_n
5	0	1	1	0.1	0.7187714032	0.6687714032
6	1	1.1	0.9090909091	0.09090909091		
7	2	1.2	0.8333333333	0.08333333333		
8	3	1.3	0.7692307692	0.07692307692		
9	4	1.4	0.7142857143	0.07142857143		
10	5	1.5	0.6666666667	0.06666666667		
11	6	1.6	0.625	0.0625		
12	7	1.7	0.5882352941	0.05882352941		
13	8	1.8	0.5555555556	0.05555555556		
14	9	1.9	0.5263157895	0.05263157895		
15	10	2	0.5	0.05		

Cells `E5`, `F5` display the values for L_{10} and R_{10}. Note that these numbers are approximations for $\ln 2 = 0.6931\ldots$. In contrast to the example from class we have $R_{10} < L_{10}$, because the function f is decreasing.

> The error in our approximation for (1.1) is ≈ 0.03. A general rule of thumb is that L_n and R_n approximate the definite integral of any differentiable function up to an error of $\approx \frac{1}{n}$. So if you increase n by a factor of 10, then the error decreases by a factor of 10. In other words, if you choose 10 times as many sample points, then the precision increases by an additional digit.

Can you explain geometrically why this rule of thumb is true? Think about the error in terms of the area under the graph that is not covered by rectangles, or as the area covered by rectangles that lies above the graph.

Modify your spreadsheet to compute L_{100} and R_{100} and verify that the precision of the approximation improves by about a factor of 10. Note that you will need to change more than just the value in cell `C2`.

The textbook also introduces the *midpoint approximation* in which the height of each rectangle is equal to the value $f(x_i^*)$ at the midpoint $x_i^* = \frac{x_i + x_{i+1}}{2}$ of each interval $[x_i, x_{i+1}]$. The midpoint approximation is more precise than the left or right endpoint approximation, because the error is $\approx \frac{1}{n^2}$. In other words, increasing the number of sample points by a factor of 10 results into an improvement of the precision by *two* additional digits. Try to modify your spreadsheet to compute the midpoint approximation of (1.1) and verify that the approximation is of higher quality.

Lastly, let us apply numerical integration to approximate π. To do this, recall that the graph of the function

$$f(x) = \sqrt{1 - x^2}$$

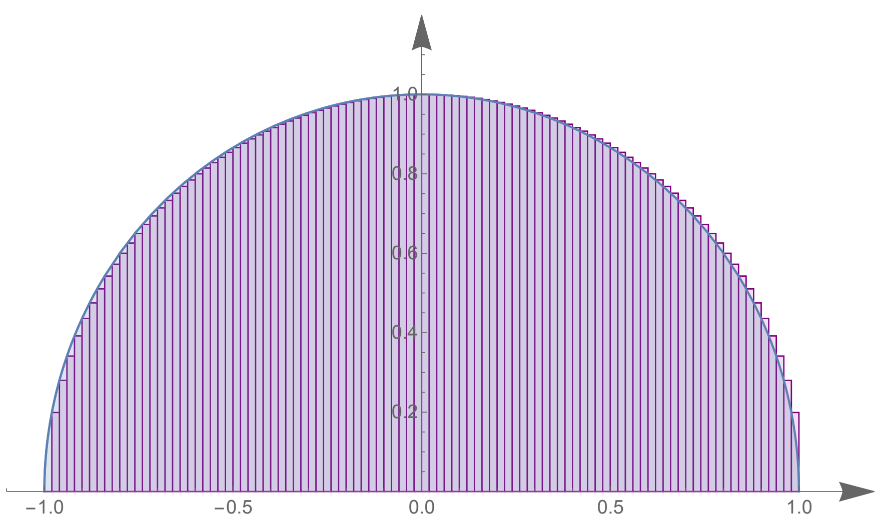
is a semicircle with radius 1. So its area must be

$$\int_{-1}^{1} f(x)dx = \frac{\pi}{2}. \tag{1.3}$$

Consequently,

$$\pi = 2\int_{-1}^{1} f(x)dx$$

So if we compute L_n and R_n for the integral (1.3), then $2L_n$ and $2R_n$ are approximations for π.



Exercise 1 Modify your spreadsheet to compute L_{100} for the integral (1.3). Report $2L_{100}$ to Gradescope (this number should be close to π). You may round to 5 digits after the decimal point.

You may notice that $L_{100} = R_{100}$. Explain why this is the case.

2 Understanding the Fundamental Theorem of Calculus via Spreadsheets

The goal of this exercise is to gain a better understanding as to why the Fundamental Theorem of Calculus is true. Although we won't derive a precise proof of this theorem, we will analyze its underlying mechanisms within a related setting. The Fundamental Theorem of Calculus consists of two parts, which we will analyze separately in the following subsections.

The integral of the derivative

Let us first analyze the following part of the Fundamental Theorem of Calculus:

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

We will choose as an example $f(x) = x^2$ and $a = 1$, $b = 2$. Our strategy will be as follows:

1. First we evaluate f at a sequence of sample points $x_0, x_1, \ldots$, spaced Δx apart from one another.

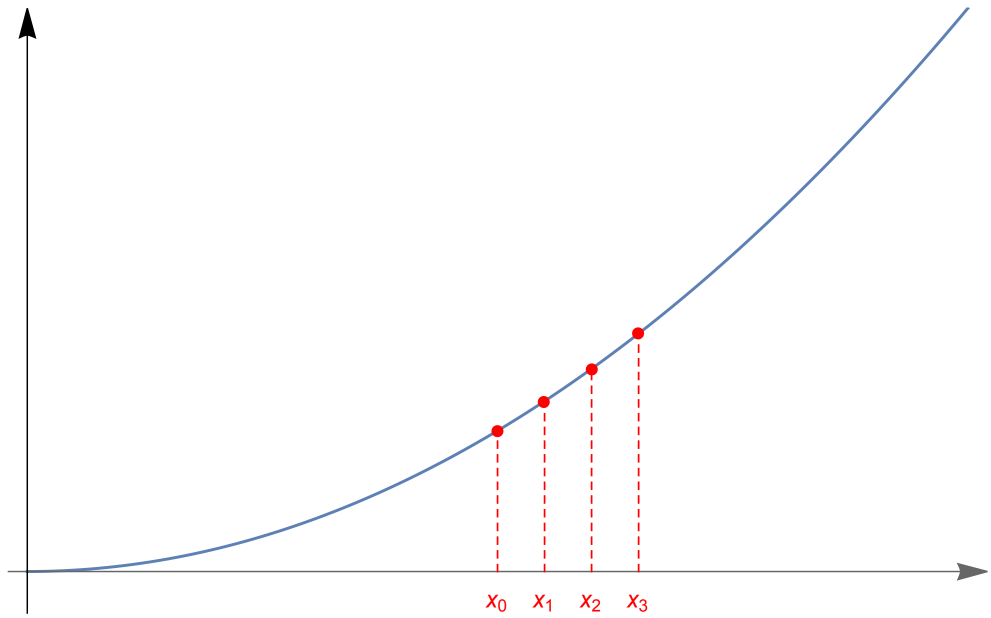
2. Then we approximate $f'(x_i)$ at these sample points using the formula

$$f'(x_i) \approx \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x} = \frac{f(x_{i+1}) - f(x_i)}{\Delta x}.$$

   Note that the term in the middle is just the fraction occurring in the definition of the derivative, where we use $h = \Delta x$. So this approximation for $f'(x_i)$ becomes better and better the smaller we choose Δx. Also note that $x_{i+1} = x_i + \Delta x$, because consecutive sample points are spaced Δx apart from one another.

   We should also clarify that we could have computed directly that $f'(x) = 2x$. However, we want our spreadsheet to work for any function given f via its data points $f(x_0), f(x_1), \ldots$. In addition, the main objective – understanding why the Fundamental Theorem of Calculus is true – can be achieved more easily if we work with the numerical approximation for $f'(x_i)$.

3. Lastly, we approximate $\int_1^2 f'(x)dx$ from the values of $f'(x_i)$ using the left endpoint approximation.



1. Open a new spreadsheet and enter:
   - `Delta x` in cell `A1`.
   - `i` in cell `A3`.
   - `x_i` in cell `B3`.
   - `f(x_i)` in cell `C3`.
   - `f’(x_i)` in cell `D3`.
   - `f’(x_i) * Delta x` in cell `E3`.
2. Choose $\Delta x = 0.1$. (If you want, you can choose a smaller value later to obtain better approximations.) Enter `0.1` in cell `B1`.
3. Fill the cells `A4-A14` with the numbers from 0 to 10. So enter `0` in cell `A4` and `=A4+1` in cell `A5` and drag down the little blue dot.
4. The sample points should be spaced Δx apart from one another and start at $x_0 = 1$. To achieve this, you can enter `=1+A4*B$1` in cell `B4` and drag down the blue dot. Alternatively, you could enter `1` in cell `B4` and `=B4+B$1` in cell `B5` and drag down the blue dot.
5. Evaluate f at the sample points x_i. So enter `=B4^2` in cell `C4` and drag down the blue dot.

6. Next, we want to compute $f'(x_i) \approx \frac{f(x_{i+1})-f(x_i)}{\Delta x}$. In other words, we need to take the difference between two consecutive numbers in column `C` and divide it by Δx. Enter `=(C5-C4)/B$1` in cell `D4` and drag down the blue dot. Note that you have to stop in row `13`, because the difference requires the value of the succeeding value $f(x_{i+1})$, which is not computed for $i = 10$. This will not cause any issues as we will not need this value in the next step.

7. Lastly, we want to compute the left endpoint approximation

$$\int_1^2 f'(x)\,dx \approx f'(x_0)\Delta x + f'(x_1)\Delta x + \ldots + f'(x_9)\Delta x = L_{10}.$$

   We will first compute each individual summand $f'(x_i)\Delta x$ by entering `=D4*B$1` in cell `E4` and dragging down the blue dot to row `13`. These are the areas of the 10 rectangles whose combined area approximates the desired integral.

   Then we compute the sum of these areas. Enter `sum` in cell `E14` and `=SUM(E4:E14)` in cell `E15`.

	A	B	C	D	E
1	Delta x	0.1			
2					
3	i	x_i	f(x_i)	f'(x_i)	f'(x_i) * Delta x
4	0	1	1	2.1	0.21
5	1	1.1	1.21	2.3	0.23
6	2	1.2	1.44	2.5	0.25
7	3	1.3	1.69	2.7	0.27
8	4	1.4	1.96	2.9	0.29
9	5	1.5	2.25	3.1	0.31
10	6	1.6	2.56	3.3	0.33
11	7	1.7	2.89	3.5	0.35
12	8	1.8	3.24	3.7	0.37
13	9	1.9	3.61	3.9	0.39
14	10	2	4		sum
15					3

You should obtain the value 3. So, in other words,

$$\int_1^2 f'(x)\,dx \approx 3 = f(2) - f(1),$$

which is the difference between the values in cells `B14` and `B4`. This is precisely the statement of the Fundamental Theorem of Calculus. In fact, this identity is always true.

8. Enter arbitrary numbers in cells `B4-B14`. The value in cell `E15` will always be equal to the difference between the values in cells `C14` and `C4`.

? Explain why this is true. You can explain this fact only based on the formulas that you've entered into your spreadsheet. You don't need to use concepts like derivatives or integrals. Try to figure it out yourself before you continue reading.

Explanation: We should first realize that the values in column `E` are just the differences between consecutive values in column `C`. Indeed, the values in column `E` are Δx times the values in column `D`. The values in this column are equal to differences between consecutive values in column `C` divided by Δx. So the multiplication by Δx just undoes the division by Δx. To summarize:

- the value in cell `E4` is equal to the increase from cell `C4` to cell `C5`,
- the value in cell `E5` is equal to the increase from cell `C5` to cell `C6`,
- and so on.

If we add up these successive increases, then we obtain the total increase (i.e., the difference) from cell `C4` to cell `C14`. In other words, the sum in cell `C15` is equal to

$$\begin{aligned} &f'(x_0)\Delta x + f'(x_1)\Delta x + \ldots + f'(x_9)\Delta x \\ &\qquad = \frac{f(x_1)-f(x_0)}{\Delta x}\Delta x + \frac{f(x_2)-f(x_1)}{\Delta x}\Delta x + \ldots \frac{f(x_{10})-f(x_9)}{\Delta x}\Delta x \\ &\qquad = \big(f(x_1)-f(x_0)\big) + \big(f(x_2)-f(x_1)\big) + \ldots + \big(f(x_{10})-f(x_9)\big) \\ &\qquad = -f(x_0) + f(x_{10}) \end{aligned}$$

The derivative of the integral

Next, let us analyze the second part of the Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

We will again choose $f(t) = t^2$ and $a = 1$ for simplicity. Similar to the last subsection we will employ the following strategy:

1. We choose the same sample points x_i from before and evaluate f at the sample points.
2. Then we compute $F(x_i) = \int_1^{x_i}$ via the left endpoint approximation L_i. So we approximate this integral using a different number of rectangles in each step. In doing so, we can ensure that the width of every rectangle is always Δx.

3. Then we approximate the derivative $F'(x_i)$ again via

$$F'(x_i) \approx \frac{F(x_i + \Delta x) - F(x_i)}{\Delta x} = \frac{F(x_{i+1}) - F(x_i)}{\Delta x}.$$

9. Duplicate the spreadsheet from the previous part. You may have to undo Step 8.

10. Delete the content of columns `D-F` and enter:

    - `f(x_i) * Delta x` in cell `D3`.
    - `F(x_i)` in cell `E3`.
    - `F'(x_i)` in cell `F3`.

11. We will compute the left endpoint approximation

$$F(x_i) = \int_0^{x_i} f(t)\,dt \approx f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{i-1})\Delta x = L_i.$$

    To do this, we first compute each summand. Enter `=C4*B$1` in cell `D4` and drag down the blue dot.

Now we need to add up these terms. For example, the value in cell `E6` should be the sum of the values in cells `D4`, `D5`. The value in cell `E7` should be the sum of the values in cell `D4`, `D5`, `D6`, and so on. So we can either compute these sums individually, or we can realize the each sum contains one more term, which can be added on to the previous sum. So for example, the value in cell `E7` should be the value in cell `E6` plus the additional value in cell `D6`.

12. Enter `0` in cell `E4`. The value in this cell is essentially a sum with no summands, so its value should be zero. Enter `=E4+D4` in cell `E5`, then select this cell and drag down the blue dot. Notice that this ensures that:

    - The value in cell `E5` is the value in cell `D4`.
    - The value in cell `E6` is the sum of the values in cells `D4`, `D5`.
    - The value in cell `E7` is the sum of the values in cells `D4`, `D5`, `D6`.
    - and so on

    Verify that this is indeed the case.

13. Lastly, we approximate the derivative $F'(x_i)$ via

$$F'(x_i) = \frac{F(x_{i+1}) - F(x_i)}{\Delta x}.$$

Enter `=(E5-E4)/B$1` in cell `F4` and drag down the blue dot. Notice again that you need to leave out row `14`.

	A	B	C	D	E	F
1	Delta x	0.1				
2						
3	i	x_i	f(x_i)	f(x_i) * Delta x	F(x_i)	F'(x_i)
4	0	1	1	0.1	0	1
5	1	1.1	1.21	0.121	0.1	1.21
6	2	1.2	1.44	0.144	0.221	1.44
7	3	1.3	1.69	0.169	0.365	1.69
8	4	1.4	1.96	0.196	0.534	1.96
9	5	1.5	2.25	0.225	0.73	2.25
10	6	1.6	2.56	0.256	0.955	2.56
11	7	1.7	2.89	0.289	1.211	2.89
12	8	1.8	3.24	0.324	1.5	3.24
13	9	1.9	[redacted]		1.824	[redacted]
14	10	2	4	0.4	2.185	

The values in column `F` should be exactly the same as the values in column `C`. In other words

$$F'(x_i) = f(x_i),$$

which is the statement of the Fundamental Theorem of Calculus. As in the previous subsection, this fact is still true if we enter arbitrary numbers in column `C`.

? Explain why this is the case. As before, just refer to the formulas that you've entered in the spreadsheet. You don't need to use terms like "derivative" or "integral". Your explanation should be similar to the previous explanation. Try to find it yourself before reading on!

Explanation: In order to compute the values in column `F`, we need to take differences between consecutive values in column `E` and then divide these differences by Δx. But as we discussed above, the values in column `E` differ exactly by the values in column `D`. Since the values in column `D` are just the values in column `C` times Δx, we obtain these values after from column `C` after division by Δx. In formulas:

$$\begin{aligned} F'(x_i) &\approx \frac{F(x_{i+1}) - F(x_i)}{\Delta x} \\ &= \frac{\big(f(x_0)\Delta x + \ldots + f(x_{i-1})\Delta x + F(x_i)\Delta x\big) - \big(f(x_0)\Delta x + \ldots + f(x_{i-1})\Delta x\big)}{\Delta x} \\ &= \frac{f(x_i)\Delta x}{\Delta x} \\ &= f(x_i) \end{aligned}$$

Summary

In rough terms, the Fundamental Theorem of Calculus can be viewed as an infinitesimal version of the statements that:

- The sums of successive increments equals the total increment.
- The difference between two successive cumulative sums is equal to the extra term.

Exercise 2 To show that you have completed this exercise, consider the spreadsheet that you obtained in Step 13. Report the number from cell `F13` to Gradescope (this number is redacted in the picture above).

3 The Monte Carlo Method

Here is another way of thinking about the average of a function on an interval. Let us consider the example of the function $f(x) = x^2$ over the interval $[0, 2]$. Imagine x as a random number chosen from the interval $[0, 2]$, for example, take twice the decimal part of the time it took you to get to campus today (in seconds). It's important that x is chosen with uniform probability. This means, for instance, that the probability that $x \in [0, 1]$ is $\frac{1}{2} = 50\%$. Now $f(x) = x^2$ yields a random number between 0 and 4, but its randomness doesn't obey a uniform probability distribution. For example, $f(x)$ is more likely less than 2 than greater than 2. To realize this, note that $f(x) \leq 2$ is true if $x \leq \sqrt{2}$, which happens with a probability of $\frac{\sqrt{2}}{2} \approx 71\%$.

Now, consider repeating this experiment numerous times. On average, we would expect that x is about 1 (the midpoint of $[0, 2]$). However, since $f(x)$ tends to be more likely to be less than 2 than greater than 2, the average of all values $f(x)$ is expected to be less than 1. This expected average is the same as the average of the function $f(x)$ on $[0, 2]$:

$$\frac{1}{2}\int_0^2 f(x)\,dx = \frac{1}{2}\int_0^2 x^2\,dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 = \frac{1}{2}\frac{8}{3} = \frac{4}{3} = 1.333\ldots$$

In probability theory – a branch of mathematics – we regard x as a "random variable" and deduce that the "expected value" of $f(x)$ equals the integral above.

Our observation has an important application. Suppose that you want to compute the integral $\int_a^b f(x)\,dx$ of a continuous function over an interval $[a, b]$. To do this, you can just sample a large number of random numbers x from the interval $[a, b]$ and compute the average over all resulting values $f(x)$. This average is then close to the average of $f(x)$ over $[a, b]$

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

So in order to obtain $\int_a^b f(x)\,dx$, we just need to multiply this average by $b - a$. This numerical method is called the *Monte Carlo Method,* referring to the Monte Carlo Casino in Monaco, a place emblematic for randomness.

Let us test the Monte Carlo Method by computing the average of the function $f(x) = x^2$ on the interval $[0, 2]$. Admittedly, a numerical simulation is not necessary here, because we already know the answer. However, applying this method in this familiar scenario enables us to compare our outcome to the established solution.

1. Open a new spreadsheet and enter:
   - `x` in cell `A1`.
   - `f(x)` in cell `B1`.
2. We want to generate a random number between 0 and 2 in cell `A2`. Recall that `RAND()` outputs a random number between 0 and 1. So we just need to multiply `RAND()` with 2. Enter `=2*RAND()` into cell `A2`. Recall that the random number updates if you input text into another cell and press enter.
3. Compute $f(x)$ by entering `=A2^2` in cell `B2`.
4. Repeat this experiment 10 times. To do this, select cells `A2-B2` and drag down the blue dot until you get to row `11`.
5. Enter `average` into cells `A12` and `B12`.
6. Compute the average of all numbers in row `A` by entering `=AVERAGE(A2:A11)` in cell `A13`.
7. Compute the average of all numbers in row `B` by entering `=AVERAGE(B2:B11)` in cell `B13`.

	A	B
1	x	f(x)
2	1.957043256	3.830018306
3	0.638944648	0.4082502632
4	1.769320784	3.130496035
5	1.094795309	1.198576769
6	0.2430717542	0.05908387769
7	0.9270329532	0.8593900963
8	1.689844082	2.855573023
9	0.7138634392	0.5096010099
10	0.4251991401	0.1807943088
11	0.6429801592	0.4134234852
12	average	average
13	1.010209552	1.344520717

Note that the average in cell `A13` (i.e., the average over all values of x) is about 1, which is in line what we discussed. The average of all values in cell `B13` (i.e., the average over all values of $f(x)$) is about $1.33\ldots$, which is the average of the function. Of course, these numbers are the result of randomness. For example, you could be extremely unlucky and all values of x could be less than 1. In this case, all values of $f(x)$, including their average would also be less than 1, resulting in a bad approximation. But such an unfortunate event would only happen with a probability of $(\frac{1}{2})^{10} \approx 0.1\%$. As you increase the number of sample points, the averages of all x and $f(x)$ values become closer to 1 and $1.333\ldots$, respectively, and the probability for outliers goes to zero. By the way, you can always update the choice of the random number by inputing text into a free cell and pressing enter.

? Modify your spreadsheet to increase the number of samples (i.e., add more rows) to 100 or 1000. Check whether the averages become closer to 1 and $1.333\ldots$, respectively.

It can be shown that the Monte Carlo Method converges at a rate of about $\frac{1}{\sqrt{n}}$. This means that in order to gain one additional digit of precision (i.e, decrease the error by a factor of 10), you need to increase the number of samples by a factor of 100. This is not as good as the left/right endpoint approximation or the midpoint approximation, which converged at rates $\frac{1}{n}$ and $\frac{1}{n^2}$, respectively. However, the Monte Carlo Method has advantages in certain situations, for example:

- The fact that the sample points are chosen randomly makes the Monte Carlo Method more robust to functions with periodic patterns. For example, suppose you want to compute the integral $\int_0^{2\pi} f(x)dx$ of the function $f(x) = \cos(2520x)$. This integral equals 0, however the left endpoint approximations $L_1, L_2, \ldots, L_{10}$ all return 2π, because the sample points always happen to be located at the local maxima. By contrast, this is not an issue for the Monte Carlo Method, because its sample points do not form a periodic pattern.

  In addition, the precision of the left/right/midpoint approximation normally deteriorates the more "complicated" the function f is. The precision of the Monte Carlo approximation doesn't.

- In many practical applications it is necessary to compute higher dimensional integrals. In other words, one needs to evaluate iterated integrals of the form

  $$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_d}^{b_d} f(x_1, x_2, \ldots, x_d)\, dx_1\, dx_2 \ldots dx_d.$$

  If the dimension d is large, then the left/right endpoint approximation converges very slowly, at a rate of $\frac{1}{\sqrt[d]{n}}$. So for example, suppose that the dimension (i.e., the number of parameters) is 10, which is not that uncommon. In order to improve the precision of the approximation by one additional digit, you would need to increase the number of samples by a factor of $10^{10} = 10,000,000,000$, which is often impossible. For the Monte Carlo Method, you just need to increase the number of steps by a factor of 100, no matter how high the dimension is.

Exercise 3 The number

$$\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx$$

is an important number in statistics (and many experimental sciences). It expresses the probability of a measurement falling within one standard deviation of the actual value, assuming a Gaussian error distribution. Approximate this integral using the Monte Carlo Method and using at least 100 sample points (i.e., rows). Remember that you have to adjust the formulas in columns `A` and `B`. Note that Google Spreadsheets incorrectly interprets `-A2^2` to mean "the square of `-A2`". So you need to set additional parentheses as follows to compute the negative of the square: `-(A2^2)`. Don't forget to multiply your result by the length of the interval, because the question is asking for the value of the integral and not for the average of the function!

Report your approximation of the above integral to Gradescope and upload a screenshot (select File, Download, PDF and Export to save your work as a PDF and upload this PDF). You may round to 2 digits after the decimal point. It may happen that your approximation is slightly off in which case we will look at your screenshot and assign points manually.