# 4.9 Antiderivatives:

Ex1: $F'(x)=x^2$; $F(x)=\frac{x^3}{3}+C$ Ex2: $F'(x)=e^{3x}$; $F(x)=\frac{e^{3x}}{3}+C$

## Antiderivative & theorem:

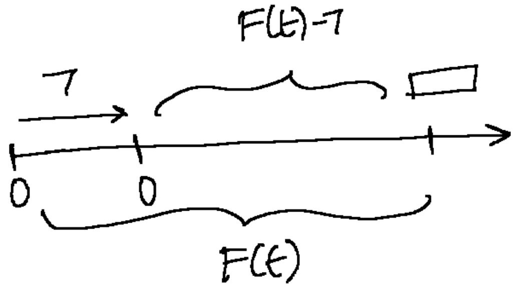
A function $F$ is called antiderivative of $f$ if $F'(x)=f(x)$ and if $F$ is an antiderivative of $f$, then all antiderivatives of $f$ are of the form: $F(x)+C$

Ex3:

Velocity at time $t$: $t^2=f(t)=F'(t)$

Position at $t$: $F(t)=\frac{t^3}{3}-7$

where $F(t)$ is antiderivative




## Functions w/ derivatives:

$0\to 0+C$; $1\to x+C$; $e^x\to e^x+C$; $\sin x\to -\cos x+C$; $\cos x\to \sin x+C$

$x^b\to \frac{x^{b+1}}{b+1}+C$; $1/x^2\to -1/x+C$; $1/x\to \ln|x|+C$; $1/(1+x^2)\to \tan^{-1}(x)+C$

Ex4: $F(x)=\frac{2x}{1+x^2}$; $F(0)=3$

$f(x)=\ln(1+x^2)+C$; $\ln(1)=0$; $C=3$

$f(x)=\ln(1+x^2)+3$

Ex5: $h(0)=2$; $h'(0)=8$; $h''(t)=-9$

$h'(t)=-9t+C$; $h'(t)=-9t+8$

$h(t)=\frac{-9t^2}{2}+8t+C$; $h(t)=\frac{-9t^2}{2}+8t+2$

Ex6: $f'''(x)=1$; $f''(0)=1$; $f'(0)=1$; $f(0)=1$

$f''(t)=x+1$; $f'(t)=\frac{x^2}{2}+x+1$

$f(t)=\frac{x^3}{6}+\frac{x^2}{2}+x+1$

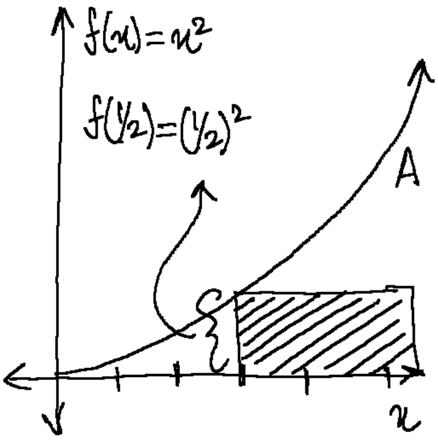
Ex7: $f(x)=e^{-x^2}$, Gaussian

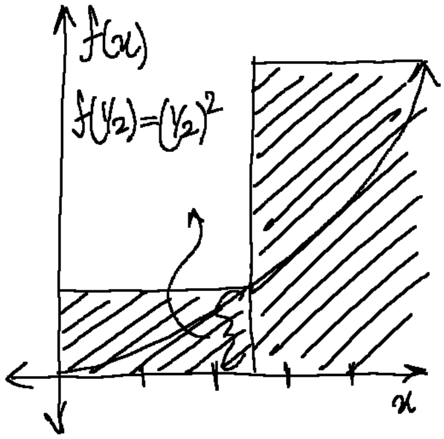
$F(x)=$

exists, but cannot be expressed w/ known facts

## 5.1 Areas and distances:

**Ex 1:** Find the area of A below the graph of $f(x)=x^2$ between 0,1




$A \geq \left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) = 0.125$

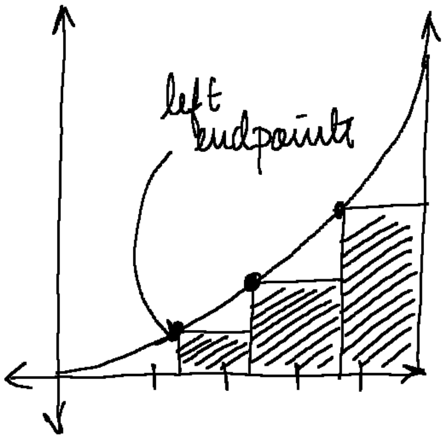
$A \leq \left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) + \left(\frac{2}{2}\right)^2\left(\frac{1}{2}\right) = 0.625$

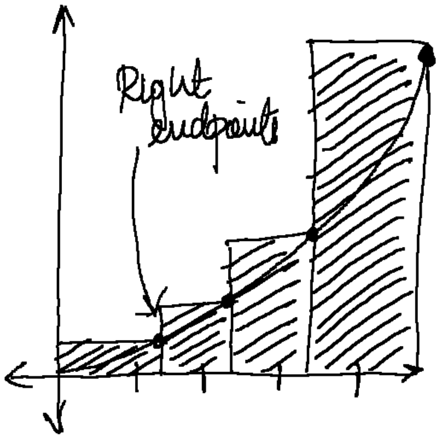
$0.125 \leq A \leq 0.625$

$A \geq \left(\frac{1}{4}\right)^2\left(\frac{1}{4}\right) + \left(\frac{2}{4}\right)^2\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)$
$\approx 0.218\ldots$

$A \leq \left(\frac{1}{4}\right)^2\left(\frac{1}{4}\right) + \left(\frac{2}{4}\right)^2\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right) + \left(\frac{4}{4}\right)^2\left(\frac{1}{4}\right) \approx 0.468\ldots$

$0.218 \leq A \leq 0.468$




Increase the number of rectangles w/ smaller widths then

10 rect: $0.285 = L_{10} \leq A \leq R_{10} = 0.385$

100 rect: $0.328 = L_{100} \leq A \leq R_{100} = 0.338$

## General Form of $L_n$ and $R_n$:

width of each rect: $\Delta x = 1/n$

Left Bound

height of rect 1: $f(0)$
rect 2: $f(1/n)$
⋮
rect n: $f\left(\frac{n-1}{n}\right)$

Right Bound

height of rect 1: $f(1/n)$
rect 2: $f(2/n)$
⋮
rect n: $f(n/n)$

$L_n = f(0)\left(\frac{1}{n}\right) + f\left(\frac{1}{n}\right)\left(\frac{1}{n}\right) \cdots f\left(\frac{n-1}{n}\right)\left(\frac{1}{n}\right)$ ; $R_n = f\left(\frac{1}{n}\right)\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right)\left(\frac{1}{n}\right) + \cdots + f\left(\frac{n}{n}\right)\left(\frac{1}{n}\right)$

$L_n \leq A \leq R_n$

## Area & distance Continued:

$R_n: \left(\frac{1}{n}\right)^2\left(\frac{1}{n}\right) + \left(\frac{2}{n}\right)^2\left(\frac{1}{n}\right) + \cdots + \left(\frac{n}{n}\right)^2\left(\frac{1}{n}\right)$

$\Rightarrow \frac{1^2+2^2\cdots n^2}{n^3} \Rightarrow \frac{n(n+1)(2n+1)}{6n^3}$

$1^2+2^2+3^2\cdots n^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

$L_n: (0)^2\left(\frac{1}{n}\right) + \left(\frac{1}{n}\right)^2\left(\frac{1}{n}\right) + \cdots + \left(\frac{n-1}{n}\right)^2\left(\frac{1}{n}\right)$

$\Rightarrow \frac{1^2+2^2+3^2+\cdots(n-1)^2}{n^3} \Rightarrow \frac{n(n-1)(2n-1)}{6n^3}$

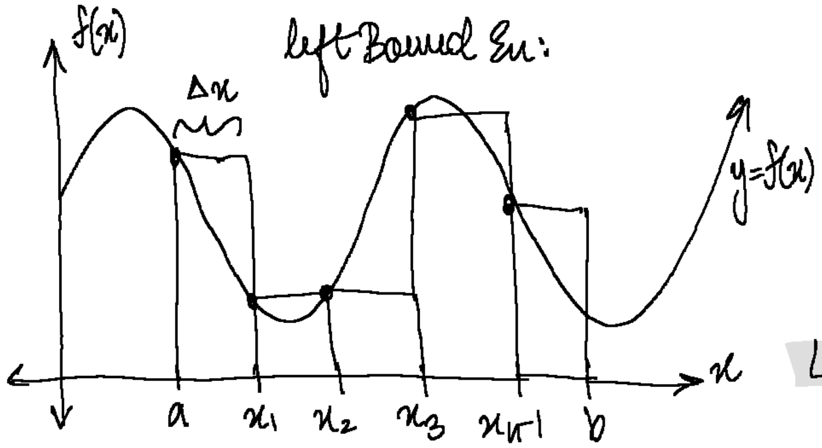
$1^2+2^2+3^2\cdots(n-1)^2 = \sum_{i=1}^{n-1} i^2 = \frac{n(n-1)(2n-1)}{6}$

## Summary:

$\frac{(n-1)(2n-1)}{6n^2} \le A \le \frac{(n+1)(2n+1)}{6n^2} \therefore A \le \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} = \boxed{1/3}$ & $\lim_{n\to\infty} \frac{(n-1)(2n-1)}{6n^2} = \boxed{\frac{1}{3}}$

Therefore the area of $f(x) = x^2 = \boxed{1/3}$

## Summary: Area under Graph of a continuous function




Width Rect: $\frac{b-a}{n}$

$x_0 = a$

$x_1 = a \cdot \Delta x$

$x_2 = a \cdot 2\Delta x$

$x_n = a \cdot n\Delta x = b$

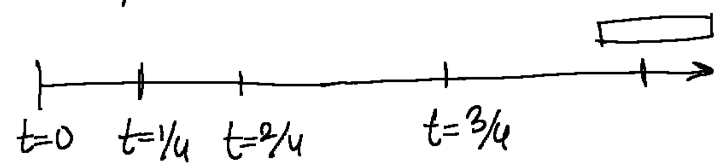
$\Big\} \; x_i = a_i \Delta x$

$L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x; \quad R_n = \sum_{i=1}^{n} f(x_i)\Delta x$

**Definition:** The area of the region under the graph of $f$ between $x = a$ and $x = b$ is defined to be

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n = \sum_{i=0 \text{ or } 1}^{n-1 \text{ or } n} f(x_i)\Delta x$$

**Ex2:** The distance Problem:



Ex2: distance Problem:

velocity at $t = f(t) = t^2$

pos at $t$: $t(0)=0$

pos at $t$: $t(1) = ?$

min total distance & max:

$$\left(\frac{0}{4}\right)^2\left(\frac{1}{4}\right)+\left(\frac{1}{4}\right)^2\left(\frac{1}{4}\right)+\left(\frac{2}{4}\right)^2\left(\frac{1}{4}\right)+\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)=L_4 \approx 0.218\cdots$$

$$\left(\frac{1}{4}\right)^2\left(\frac{1}{4}\right)+\left(\frac{2}{4}\right)^2\left(\frac{1}{4}\right)+\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)+\left(\frac{4}{4}\right)^2\left(\frac{1}{4}\right)=R_4 \approx 0.468\cdots$$

$L_n \leq \{\text{total distance}\} \leq R_n$; $\frac{1}{3} \leq \{\text{total distance}\} \leq \frac{1}{3}$

if $f(t)$ is the instantaneous velocity of an object at time $t$, then the distance that the object travels between time a and b is the same as the area bound b/w a and b.