2.6 limits at infinity: HA

Definition

we write $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to -\infty} f(x) = L$

if $|f(x) - L|$ can be made as small as we want, as long as x is chosen sufficiently large.

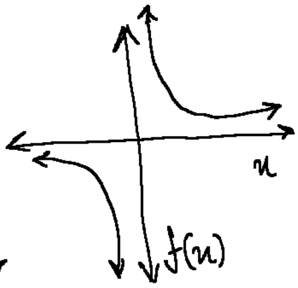
Note: Read the precise definitions in TB

Ex1: $f(x) = \frac{1}{x}$

$\lim_{x\to\infty} f(x) = 0$

$\lim_{x\to-\infty} f(x) = 0$

Graphically

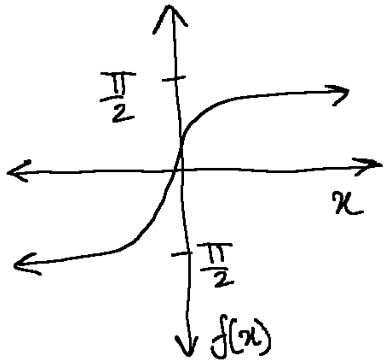


Ex2: $f(x) = \tan^{-1}(x)$

$\lim_{x\to\infty} f(x) = \frac{\pi}{2}$

$\lim_{x\to-\infty} f(x) = \frac{-\pi}{2}$

Graphically

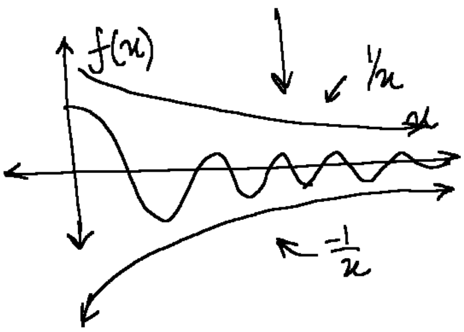


Definition

The line $y = L$ is called horizontal asymptote if

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L$$

Ex3: $\lim_{x\to\infty} \frac{\sin x}{x} = 0$



Squeeze theorem:

$-1 \le \frac{\sin x}{x} \le 1$

$\frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$

then $\lim_{x\to\infty} \frac{1}{x} = \boxed{0}$

$f(x) \le g(x) \le h(x)$

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} h(x) = 0$

then $\lim_{x\to\infty} g(x) = 0$

Squeeze th:

The squeeze th. is still true if we replace $\lim_{x\to a}$ w/ $\lim_{x\to\pm\infty}$

Limit laws:

The limit laws hold true even if the limit approaches infinity. As it just a huge number.

If f is continuous at b and $\lim_{x\to\infty} g(x) = b$ then

$$\lim_{x\to\infty} f\circ g(x) = f\left(\lim_{x\to\infty} g(x)\right) \text{ ; Similar to } \lim_{x\to-\infty} g(x)$$

Ex4:

if $\lim_{x\to\infty} g(x) \geq 0$ exists $\forall\, r > 0$, then

$$\lim_{x\to\infty} [g(x)]^r = \left[\lim_{x\to\infty} g(x)\right]^r ; \quad \lim_{x\to\infty} \sqrt[n]{g(x)} = \sqrt[n]{\lim_{x\to\infty} g(x)}$$

$$\text{and } \lim_{x\to\infty} \frac{1}{x^r} = \left[\lim_{x\to\infty} \frac{1}{x}\right]^r$$

Rule of th #3

Divide top & bottom w/ highest power of x

Ex5:

$$\lim_{x\to\infty} \sqrt{\frac{3}{x}+7}$$

$$\sqrt{3\lim_{x\to\infty}\frac{1}{x} + \lim_{x\to\infty} 7} = \sqrt{7}$$

then $\lim_{x\to\infty} \sqrt{\frac{3}{x}+7} = \sqrt{7}$

Ex6:

$$\lim_{x\to\infty} \frac{2x^3-x+1}{5x^3+x^2+7} \cdot \frac{1/x^3}{1/x^3}$$

$$\Rightarrow \frac{2-1/x^2+1/x^3}{5+1/x+7/x^3} = \boxed{\frac{2}{5}}$$

Then $\lim_{x\to\infty} f(x) = \boxed{\frac{2}{5}}$

Ex7:

$$\lim_{x\to\infty} x\left(\sqrt{x^2+1} - x^2\right)$$

$$\Rightarrow \lim_{x\to\infty} \frac{x(x^2+1-x^2)}{\sqrt{x^2+1}+x^2} \Rightarrow \text{Simplify it}$$

$$\Rightarrow \frac{x}{\sqrt{x^2+1}+x} \times \frac{1/x^2}{1/x^2} \Rightarrow \frac{1}{\sqrt{1+1/x^2}+1}$$

$$\lim_{x\to\infty} f(x) = \boxed{\frac{1}{2}}$$

Rule of th #3

Divide the number into a conjugate no matter what

Infinity limits at infinity:

Infinity limits at Infinity

$$\lim_{x \to \pm\infty} f(x) = \pm\infty$$

We write

$$\lim_{x \to \infty} f(x) = \infty$$

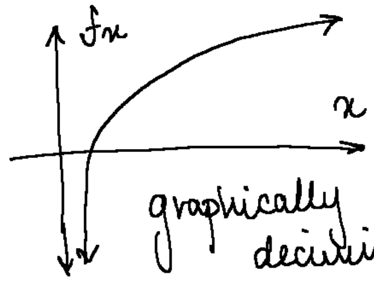
if $f(x)$ can be made as large as we want, as long as x is chosen sufficiently large.

Ex 7:

$$\lim_{x \to \pm\infty} x = \pm\infty \; ; \quad \lim_{x \to \pm\infty} x^2 = \infty \; ; \quad \lim_{x \to \pm\infty} x^3 = \pm\infty$$

Ex 8: $\lim_{x \to \infty} \log_2(x) = \infty$

$$\lim_{x \to \infty} \ln(x) = \infty$$



graphically deciding!

If n is a positive integer then,

$$\lim_{x \to \infty} x^n = \infty \qquad \lim_{x \to -\infty} x^n = \begin{cases} \infty & \text{even} \\ -\infty & \text{odd} \end{cases}$$

Computing w/ infinity

limit laws extend to determinate settings

if $\lim_{x \to a} f(x) = \infty$ & $\lim_{x \to a} g(x) \neq -\infty$, exists then

$$\lim_{x \to a} f(x) + g(x) = \infty \qquad \text{not } \infty - \infty$$

Indeterminate forms

if $\lim_{x \to a} f(x) = \infty$ & $\lim_{x \to a} g(x) \neq 0$, exists then

$$\lim_{x \to a} f(x) \cdot g(x) = \begin{cases} \infty & g(x) > 0 \\ -\infty & g(x) < 0 \end{cases} \qquad \text{not } 0 \cdot \infty$$

if $\lim_{x \to a} f(x) = \infty$ & $\lim_{x \to a} g(x) \neq \pm\infty$ exists then

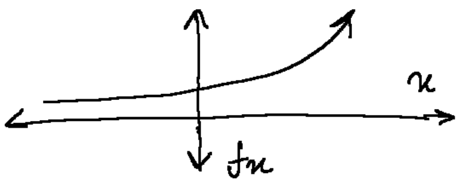
$$\lim_{x \to a} f(x) \div g(x) = \begin{cases} \infty & g(x) > 0 \text{ if } x \text{ is close to } a \\ -\infty & g(x) < 0 \text{ if } x \text{ is close to } a \end{cases}$$

$$\lim_{x \to a} g(x) \div f(x) = 0$$

$\lim_{x \to \infty} f(x)$ exists & $\lim_{x \to \infty} g(x)$ then $\lim_{x \to \infty} f \circ g = \lim_{x \to \infty} f(x)$

Ex9: $\lim_{x\to\infty} e^x = \infty$

$\lim_{x\to -\infty} e^x = 0$



Ex10: $\lim_{x\to\infty} (-7x^3+3x^2-1)$

$\Rightarrow \dfrac{-7x^3+3x^2-1 \cdot 1/x^3}{1/x^3} \to x^3(-7+3/x+1/x^3)$ then simplify

$\Rightarrow \lim_{x\to\infty} -7x^3 = \boxed{-\infty}$ as $\frac{3}{x}$ & $\frac{1}{x^3} \to \boxed{0}$

Definition

if $f(x) = a_n x^n + a_{n-1}x^{n-1} \cdots a_0$ is a polynomial of degree n and $a_n \neq 0$ then

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} (a_n x^n)$$

Ex11: $\lim_{x\to 0^+} e^{\frac{1}{x}} = \infty$

$\lim_{x\to 0^-} e^{1/x} = 0$

Ex12: $\lim_{x\to -\infty} \dfrac{1}{\ln(x^2+1)} = 0$

$\lim_{x\to -\infty} \tan\left(\frac{\pi}{2}+\frac{1}{x}\right) = \text{DNE}$

if $\frac{\pi}{2}^{\pm} = \mp\infty$