Class 24

5.4 Indefinite Integrals and the Net Change Theorem
5.5 The Substitution Rule

5.4 Indefinite Integrals and the Net Change Theorem

Definition (Indefinite Integral)

If F is the antiderivative of f, then we write

$$\int f(x)\,dx = F(x) + C$$

Ex 1

$$\int_2^3 x^2\,dx = \left.\frac{x^3}{3}\right]_2^3 = \frac{3^3}{3} - \frac{2^3}{3}$$

<u>definite</u>

$$\int x^2\,dx = \frac{x^3}{3} + C$$

<u>indefinite</u>

Rules

$$\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int (f(x) - g(x))\,dx = \int f(x)\,dx - \int g(x)\,dx$$

$$\int cf(x)\,dx = c\int f(x)$$

$$\int k\,dx = kx + C$$

$$\int x^b \, dx = \frac{x^{b+1}}{b+1} + C \qquad b \neq -1$$

$$\int \frac{1}{x} \, dx = \ln|x| + C$$

$$\int e^x \, dx = e^x + C$$

$$\int \sin(x) \, dx = -\cos(x) + C$$

$$\int \cos(x) \, dx = \sin(x) + C$$

Ex 2

$$\int ((-x + \cancel{2}7\sin(x)) - 3) \, dx = \int (-x + 5\sin x) \, dx - \int 3 \, dx$$

$$= \int \underbrace{(-x)}_{(-1)x} dx + \int 7 \sin x \, dx - \int 3 \, dx$$

$$= (-1) \int x \, dx + 7 \int \sin x \, dx - \int 3 \, dx$$

$$= -\left(\frac{x^2}{2} + C_1\right) + 7\left(-\cos x + C_2\right) - \left(3x + C_3\right)$$

$$= -\frac{x^2}{2} - \cos x - 3x + C$$

Ex 3 $\int (x+1)(x+2)\,dx = \int (x^2+3x+2)\,dx = \frac{x^3}{3} + \frac{3}{2}x^2 + 2x + C$

Net Change Theorem

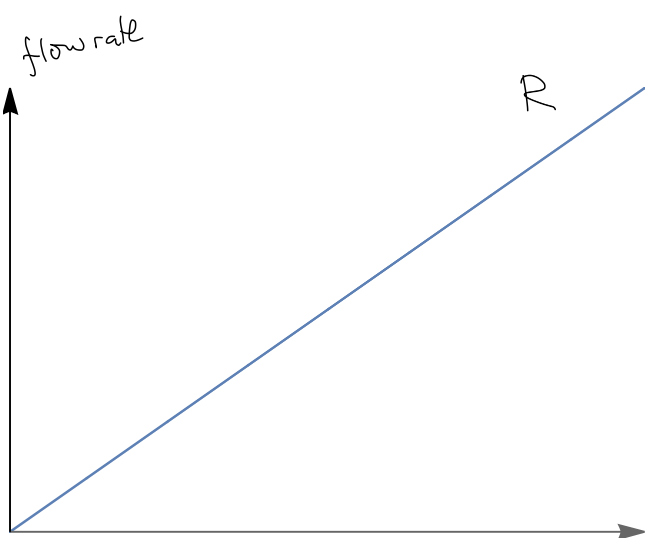
$$\int_a^b F'(x)\,dx = \int_a^b \underbrace{\frac{dF}{\cancel{dx}}\,\cancel{dx}}_{dF} = F(b) - F(a) = \Delta F$$

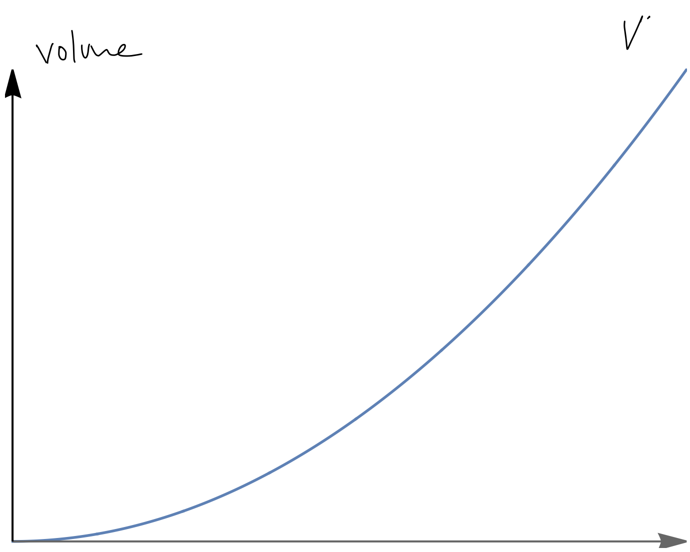
Ex 4 **Filling a pool**

Volume of Pool (gal): 1000

Flow Rate $\left(\frac{\text{gal}}{\text{s}}\right)$: $t = R(t)$

After how many seconds is the pool full?





$V(t)$ = volume (gal) of water at time t

$R(t)$ = flow rate $\left(\frac{\text{gal}}{\text{s}}\right) = V'(t)$

Net Change Theorem

$$\int_0^T \underbrace{\frac{dV}{dt}}_{R} \, dt = V(T) - \underbrace{V(0)}_{=0} = V(T)$$

$$V(T) = \int_0^T \frac{dV}{dt} \, dt = \int_0^T \underbrace{R(t)}_{=t} dt = \int_0^T t \, dt$$

$$= \left.\frac{t^2}{2}\right]_0^T = \frac{T^2}{2}$$

When is the pool full?

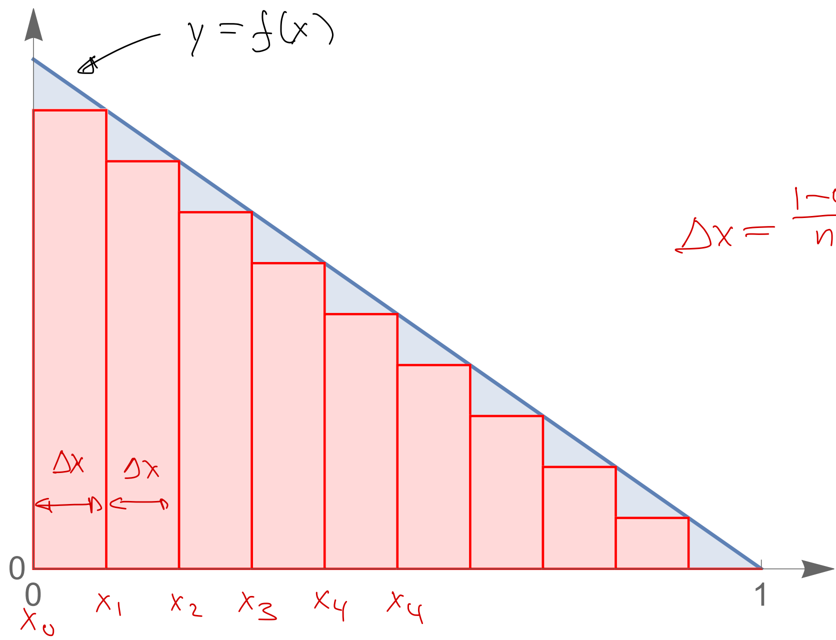
$$V(T) = 1,000$$

$$\frac{T^2}{2} = 1000$$

$$T = \sqrt{2000}$$

5.5 The Substitution Rule

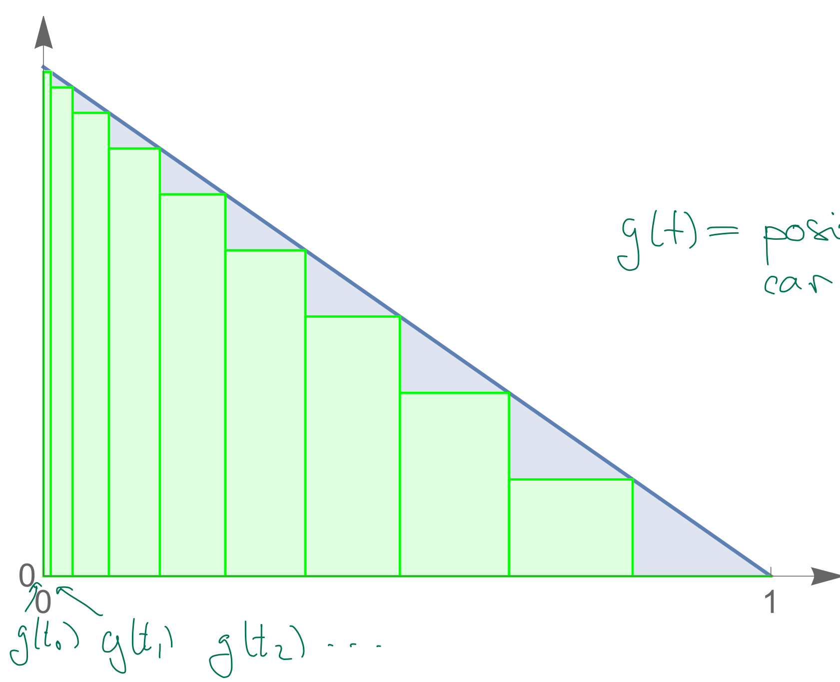
Motivating Intuition for Substitution Rule



$\Delta x = \frac{1-0}{n} = \frac{1}{n}$

constant speed

$$\int_0^1 f(x)\,dx \approx R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$
$$= \left(f(x_1) + \cdots + f(x_n)\right)\frac{1}{n} = \frac{f(x_1) + \cdots + f(x_n)}{n}$$



$g(t) =$ position of car

varying speed

$$\int_0^1 f(x)\,dx \approx f(g(t_1)) \cdot \underbrace{(g(t_1)-g(t_0))}_{\approx g'(t_1)\,\Delta t}$$

$$+ f(g(t_2)) \cdot \underbrace{(g(t_2)-g(t_1))}_{\approx g'(t_2)\,\Delta t} + \cdots$$

$$\approx f(g(t_1))\, g'(t_1)\,\Delta t$$

$$+ f(g(t_2))\, g'(t_2)\,\Delta t$$

$$+ \cdots$$

Summary

$$\int_0^1 f(x)\,dx = \int_0^T f(g(t))\, \underbrace{g'(t)}_{\text{"density" or "weight"}}\, dt$$

A more theoretic approach

Chain Rule: If F is the antiderivative of f:

$$\frac{d}{dx}(F(g(x))) = F'(g(x))\, g'(x) = f(g(x))\, g'(x)$$

Substitution Rule, Indefinite Integral

If F is the antiderivative of f:

$$\int f(g(x))\, g'(x)\, dx = F(g(x)) + C \;\text{"}{=}\text{"}\; \underbrace{\int f(x)\, dx}_{F(x)}$$

but plug in $g(x)$ later"

Ex 1 $\frac{d}{dx}\sqrt{\sin x} = \frac{1}{2\sqrt{\sin x}} \cos x$

$$\int \frac{1}{2\sqrt{\sin x}} \cos x \, dx = \sqrt{\sin x} + C$$

Ex 2 $\int \underbrace{2x}_{g'(x)}\, e^{x^2}\, dx = \int g'(x)\, f(g(x))\, dx$

$g(x)$

$\boxed{f(x) = e^x}$

$= F(g(x)) + C = e^{x^2} + C$

Find antiderivative of f

$\rightsquigarrow F(x) = e^x$

Ex 1 (revisited)

$$\int \frac{1}{2\sqrt{\sin x}} \cos x \, dx = \int \frac{1}{2\sqrt{u}} \frac{du}{dx} \, dx$$

$g(x) = \sin x = u$

$g'(x) = \cos x = \frac{du}{dx}$

$$= \int \frac{1}{2\sqrt{u}} \, du$$

substitution Rule

$$= \sqrt{u} + C$$

substitute back

$$= \sqrt{\sin x} + C$$

Substitution Rule, Indefinite Integral (reworded)

If $u = g(x)$, then

$$\int f(\underbrace{g(x)}_{u}) \underbrace{g'(x)}_{\frac{du}{dx}} \, dx = \int f(u) \, du$$

Ex 2 (revisited)

$$\int 2x \, e^{x^2} \, dx = \int e^u \, du = e^u + C$$

$$= e^{x^2} + C$$

$u = x^2$

$\frac{du}{dx} = 2x$

$(du = 2x \, dx)$

Ex 3 $\int \sin(2x)\ dx = \frac{1}{2}\int \sin(2x)\cdot 2\ dx$

$u = 2x$

$\frac{du}{dx} = 2$

$$= \frac{1}{2}\int \sin(u)\ du$$

$$= \frac{1}{2}\left(-\cos u + C\right)$$

$$= -\frac{1}{2}\cos(2x) + C$$

Ex 4 $\int \frac{\ln x}{x}\ dx = \int \ln x \cdot \frac{1}{x}\ dx = \int u\ dx$

$$\boxed{\begin{aligned} u &= \ln x \\ \frac{du}{dx} &= \frac{1}{x} \end{aligned}}$$

$$= \frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C$$

Check: $\frac{d}{dx}\left(\frac{(\ln x)^2}{2} + C\right) = \cdots = \frac{\ln x}{x}$

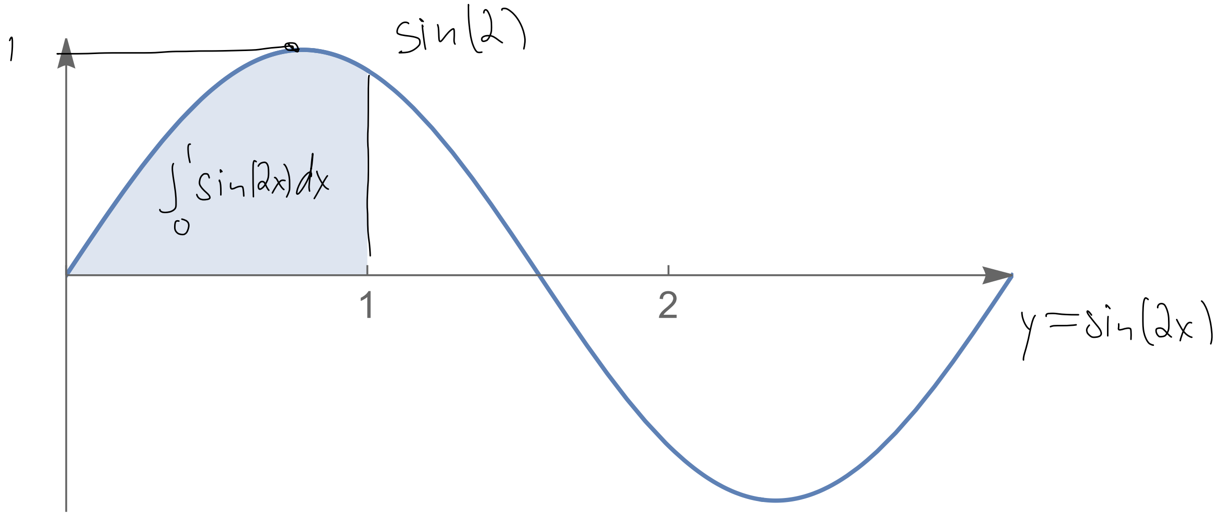
Ex 5 $\int_0^1 \sin(2x)\,dx \underset{FTC}{=} \frac{1}{2}\left(-\cos(2x)\right)\Big]_0^1$

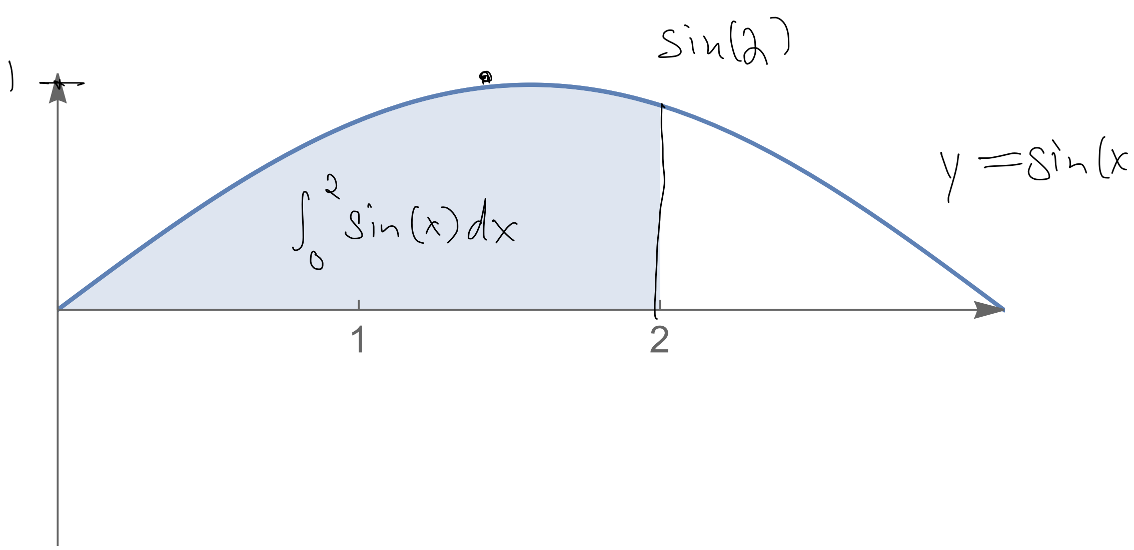
$$= -\frac{1}{2}\cos(2) + \frac{1}{2}\cos(0)$$

$$= \frac{1}{2}\int_0^2 \sin(x)\,dx$$

Summary:

$$\int_0^1 \sin(2x)\,dx = \frac{1}{2}\int_0^2 \sin(u)\,du$$





General computation:

$$\int_a^b F'(g(x))\,g'(x)\,dx = F(g(x))\Big]_a^b = F(g(b)) - F(g(a))$$

$$= F(u)\Big]_{g(a)}^{g(b)} = \int_{g(a)}^{g(b)} F'(u)\,du$$

Substitution Rule, ~~In~~definite Integral (reworded)

If $u = g(x)$, then

$$\int_a^b f(\underbrace{g(x)}_{u})\,\underbrace{g'(x)}_{\frac{du}{dx}}\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Ex 6

$$\int_e^{e^2} \frac{1}{x \ln x}\,dx = \int_{x=e}^{x=e^2} \frac{1}{\ln x}\,\frac{1}{x}\,dx$$

$$u = \ln x$$

$$= \int_{u=\ln e}^{u=\ln(e^2)} \frac{1}{u}\,\cancel{\frac{du}{dx}\,dx}\ du$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$du = \frac{1}{x}\,dx$$

$$= \int_1^2 \frac{1}{u}\,du$$

$$= \ln|u|\Big]_1^2 = \ln 2 - \ln 1$$

$$= \ln 2$$

Integrals of even and odd functions

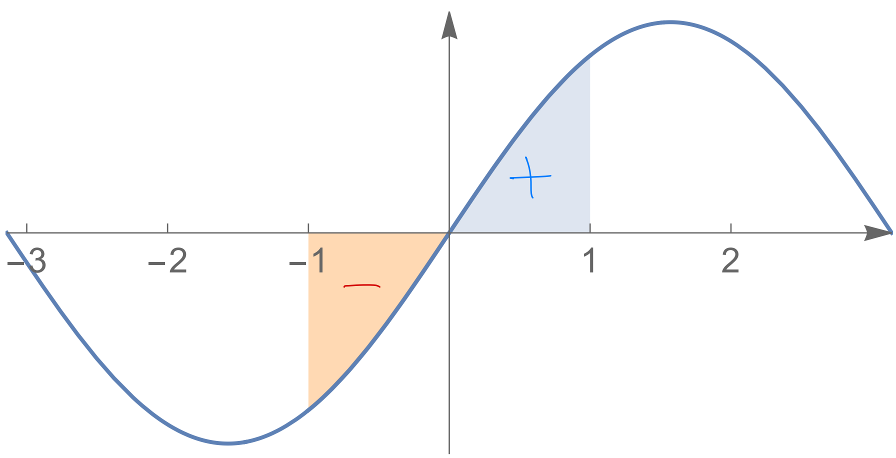
Ex 7 $\displaystyle\int_{-1}^{1} \sin x \; dx = \underbrace{\int_{-1}^{0} \sin x \, dx}_{} + \int_{0}^{1} \sin x \, dx = 0$

$u = -x$

$du = -dx$

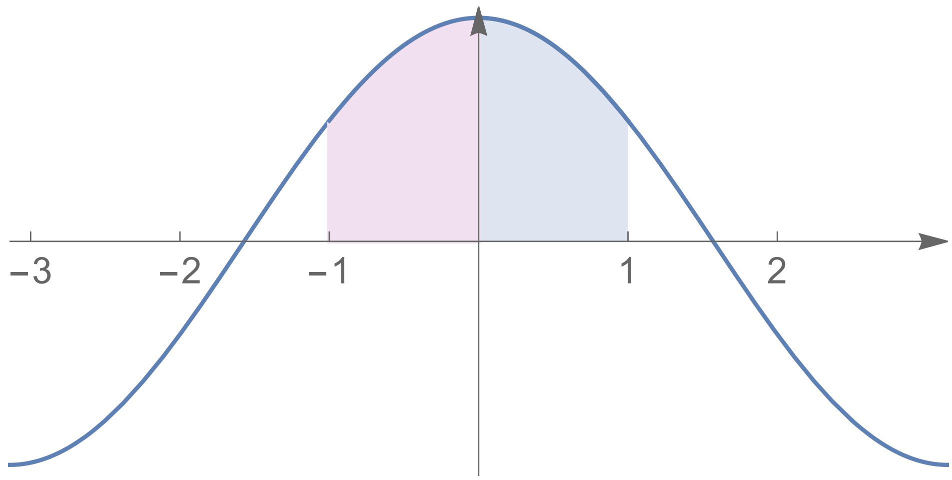
$$= \int_{1}^{0} \sin(-u)(-du) = \int_{1}^{0} (-\sin(u))(-du)$$

$$= \int_{1}^{0} \sin(u)\,du = -\int_{0}^{1} \sin(u)\,du = -\int_{0}^{1} \sin(x)\,dx$$



Ex 7 $\displaystyle\int_{-1}^{1} \cos x \; dx = 2\int_{0}^{1} \cos x \; dx$

exercise: verify using substitution.



Theorem

If f is <u>odd</u>, then

$$\int_{-a}^{a} f(x)\,dx = 0$$

If f is <u>even</u>, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$