2.3 Calculating limits using limit laws

limit laws:

if $\lim_{x\to a} f(x)$ & $\lim_{x\to a} g(x)$ where $\neq \pm\infty$ then

$$\left.\begin{aligned}
&\lim_{x\to a}(f+g) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \\
&\lim_{x\to a}(f-g) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)
\end{aligned}\right\}$$

individually have to exist.

$$\lim_{x\to a} cg(x) = c\lim_{x\to a} g(x)$$

$$\lim_{x\to a} fg = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

$$\lim_{x\to a} \frac{f}{g} = \lim_{x\to a} f(x) \div \lim_{x\to a} g(x) \quad \text{where } \lim_{x\to a} g(x) \neq 0$$

Ex1:

$\lim_{x\to 1} f(x) = 2 \quad \lim_{x\to 1} g(x) = 5$

note: denominator shouldn't be equal to 0

$$\lim_{x\to 1} \frac{f}{2f-g} = \lim_{x\to 1} f(x) \div \left[2\lim_{x\to 1} f(x) - \lim_{x\to 1} g(x)\right]$$

$$\Rightarrow 2 \div [4-5] = \boxed{-2}$$

Further limit laws:

if $\lim_{x\to a} f(x)$ exists and $\neq \pm\infty$ then

$$\lim_{x\to a} f(x)^n = \left[\lim_{x\to a} f(x)\right]^n \quad \& \quad \lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$$

where n is positive

$$\lim_{x\to a} c = c$$

$$\lim_{x\to a} x^n = \left(\lim_{x\to a} x\right)^n = a^n$$

Ex2: $\lim_{x\to 6}\sqrt[3]{1\div(2x^2-8)}$

$$\left[\lim_{x\to 6} 1 \div \lim_{x\to 6}(2x^2-8)\right]^{1/3} \Leftrightarrow \left[1\div\left(\lim_{x-6} 2x^2 - \lim_{x-6} 8\right)\right]^{1/3}$$

$\Rightarrow \left(1\div(64-8)\right)^{1/3}$ and note $64-8 \neq 0$

Ex3: $\lim_{x-5}(x^3-2x^2+x+1)$

Be smart and don't use limit laws, just plug in the value $f(5)=5^3-2(5)^2+5+1 = 125-50+5-1$ 79

Direct Substitution Property: DSP

If f is a polynomial then

$$\lim_{x\to a} f(x) = f(a)$$

limit doesn't care what a is

Theorem

If $f(x)=g(x)$ whenever $x\neq a$ and if $\lim_{x\to a} g(x)$ exists, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$$

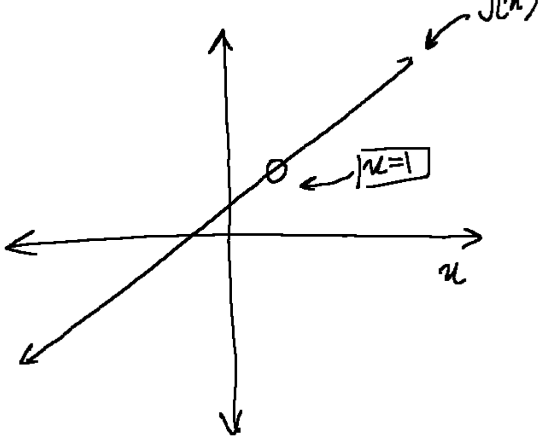
Ex1: $\lim_{x\to 1}\frac{x^2-1}{x-1}$

LL not allowed as "$\frac{0}{0}$"

$f(x) = \frac{x^2-1}{x-1}$ $\forall$ D: $\{x\in\mathbb{R} \mid x\neq 1\}$

if $x\neq 1$, then

$g(x) = x+1$ $\forall$ D: $\{x\in\mathbb{R}\}$



$\lim_{x\to 1} g(x) = x+1 = 2$

Strategy $\forall$ $\lim_{x\to} f(x)$: First simplify $f(x)$ assuming $x\neq a$. Then take the limit

Ex6: $\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t(t+1)}\right)$

"is not allowed due to $\infty-\infty$"

$$\Rightarrow \frac{t+1-1}{t(t+1)} = \boxed{\frac{1}{t+1}}$$

Using limit laws

$$\lim_{t\to 0}\frac{1}{t+1} = \frac{1}{+1} = +1$$

Ex5: $\lim_{x\to -3}\frac{x^3-8x+3}{x+3}$

"is ! allowed due to $\frac{0}{0}$"

Since -3 is factor of $f(x)$, we use polynomial division

$$(x^3-8x+3)\div(x+3) = x^2-3x+1$$

$$\lim_{x\to -3} x^2-3x+1 = h(-3)$$

$$9+9+1 = \boxed{19}$$

Then $\lim_{x\to -3}\frac{x^3-8x+3}{x+3} = 19$

Ex6: $\lim_{x\to 2}\frac{x^2-5x+6}{x^4-16}$

"is ! allowed due to $\frac{0}{0}$"

then $\rightarrow \frac{\cancel{(x-2)}(x-3)}{\cancel{(x-2)}(x+2)(x^2+4)} = \frac{(x-3)}{(x+2)(x^2+4)}$

$$\lim_{x\to 2}\frac{x-3}{(x+2)(x^2+4)} = \frac{2-3}{4(8)} = \boxed{\frac{-1}{32}}$$

Ex7: $\lim_{x\to 9^+}\frac{\sqrt{x}-3}{x-9}$

"is ! allowed due to $\frac{0}{0}$" conjugate

then $\lim_{x\to 9^+}\frac{(\sqrt{x}-3)}{(x-9)}\times\frac{(\sqrt{x}+3)}{(\sqrt{x}+3)} = \frac{\cancel{(x-9)}}{\cancel{(x-9)}(\sqrt{x}+3)}$

$$\lim_{x\to 9^+}\frac{1}{\sqrt{x}+3} = \frac{1}{\sqrt{9}+3} = \boxed{\frac{1}{6}}$$

Rule of thumb #1

If you see equal factors, combine all factors to simplify from in-determinate

Rule of thumb #2

If you need to take $\lim_{x\to a}\frac{P}{Q}$ (rational) & $P(a)=Q(a)=0$, then use polynomial division by $x-a$ on P & Q so yea.

Rule of thumb #3

If you see, difference of two square roots, then use the conjugate to try & simplify by mul & div.

Ex 8: $\lim_{x\to 0}\left[\frac{1}{x(x+1)^{1/2}}-\frac{1}{x}\right]$

then $\rightarrow \frac{1-\sqrt{x+1}}{x\sqrt{x+1}}\frac{(1+\sqrt{x+1})}{(1+\sqrt{x+1})}$

$\Rightarrow \frac{1-(x+1)}{x\sqrt{x+1}(1+\sqrt{x+1})} \Rightarrow \frac{-1}{\sqrt{x+1}+(x+1)}$

$\lim_{x\to 0}\frac{-1}{\sqrt{x+1}+x+1} = f(0)$

because denominator $\neq 0$

$\lim_{x\to 0}\frac{-1}{2} = \boxed{\frac{-1}{2}}$

"L ! allowed due to $\infty-\infty$"

Theorem

If $f(x)=g(x)$ whenever x is near a, but $x\neq a$ and if $\lim_{x\to a} g(x)$ exists then, $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$

Ex 9: $\lim_{x\to 1}\frac{x^2-3x+2}{|x-2|-1}$

If x is close 1 & $x\neq 1$, then $x-2>0$

so $|x-2| = -(x-2) = -x+2-1$

then $\lim_{x\to 1}\frac{x^2-3x+2}{-x+1}$ # Use the section ideas.

One sided limits:

All previous statements are also true for the one sided function limits $\lim_{x\to a^\pm} f(x)$ if we replace "$x\neq a$" with $x>a$ or $x<a$

Ex 10: $\lim_{x\to 0.5^-}\frac{2x-1}{|2x^3-x^2|}$

If x is close to 0.5 and $x<0.5$, then

$2x^3-x^2 = 2x^2(x-0.5) < 0$

so $|2x^3-x^2| = -(2x^3-x^2)$

$\lim_{x\to 0.5^-}\frac{2x-1}{-(2x^3-x^2)}$

Theorem:

If $f(x) \le g(x)$ whenever x is near a, $x \ne a$ and if both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$, then

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$$

(The same is true for one sided limits)

Ex9:

$\lim_{x\to 0^+} \frac{\sin x}{x}$ exists then, which is $\lim_{x\to 0^+} \frac{\sin x}{x} \le 1$

Try to show that:

$\frac{\sin x}{x} \le 1$ if $x > 0$

$\sin(x) \le x$ ← geometrically true

Then by Th.

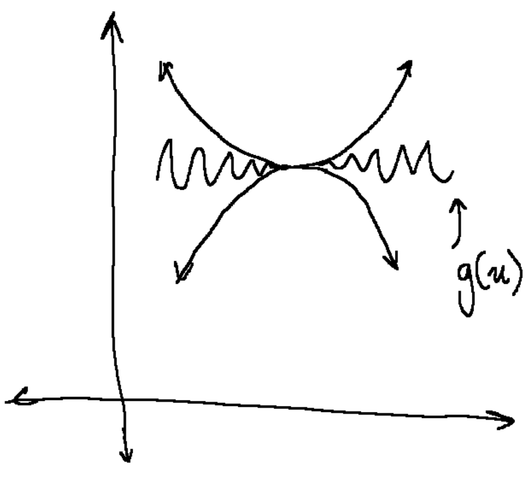
$\lim_{x\to 0^+} \frac{\sin x}{x} \le \lim_{x\to 0^+} 1 = 1$

Squeeze theorem:

if $f(x) \le g(x) \le h(x)$ whenever x is near a, but $x \ne a$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L \text{ then}$$

$\lim_{x\to a} g(x) = L$ [also true for one sided limits]



Ex10: $\lim_{x\to 0^+} x \sin\left(\frac{\pi}{x}\right)$

graph goes crazy

if $x > 0$

$f(x) = -x \le x \sin\left(\frac{\pi}{x}\right) \le x = h(x)$

$-1 \le \sin\left(\frac{\pi}{x}\right) \le 1$

then $\lim_{x\to 0^+} f(x) = 0 = \lim_{x\to 0^+} h(x)$

then $\lim_{x\to 0^+} g(x) = 0$