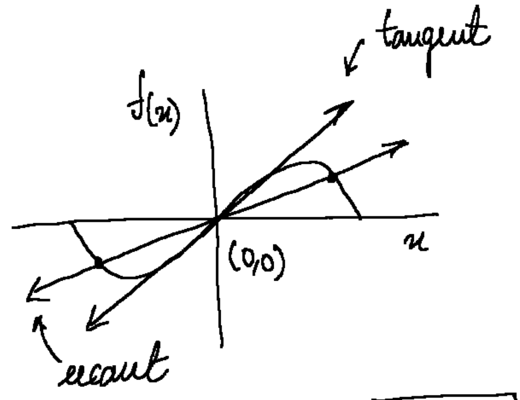
2.1 The tangent & velocity problems.

The tangent problem:

Ex1: $f(x) = \sin(x)$ ← intersects only once

Find the tangent line at line $(0,0)$



Idea: get the secants closer and closer to point giving an approximation of tangent. $\boxed{\text{slope} = 1}$

slope of secant thru $(0,0)$ and $(x, f(x))$

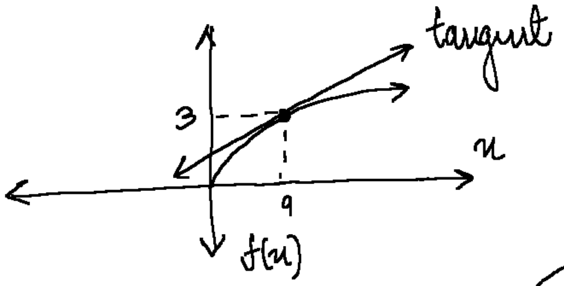
$$\Rightarrow \frac{f(x) - f(0)}{x - 0} = \frac{\sin x}{x} ; \quad g(x) = \frac{\sin x}{x}$$

x	2	1	0.01
$g(x)$	.45	.84	.9998

→ 1

Ex2:

$f(x) = \sqrt{x}$; tangent at $(9,3)$



function is universal
works for small & large vals

slope of secant thru $(0,0)$ & $(x, f(x))$

$$\Rightarrow \frac{f(x) - f(9)}{x - 9} = \frac{\sqrt{x} - 3}{x - 9}$$

x	10	9.1	9.001
$g(x)$	.162	.1662	.166662

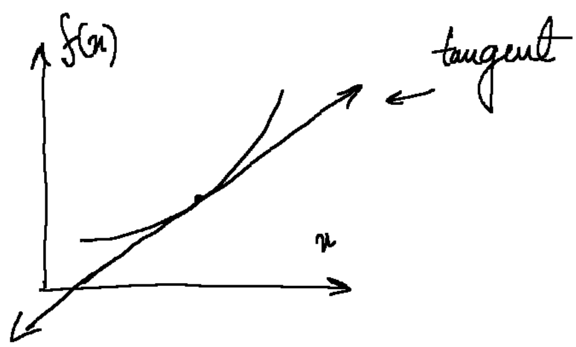
slope of tangent $\approx 0.166 \; \boxed{\frac{1}{6}}$

Approximation slope of tangent line $x = a$:

slope of secant line thru: $(a, f(a))$ & $(x, f(x))$

$$\frac{f(x) - f(a)}{x - a}$$

Precision depends on how close it is to a, but x != a



The velocity Problem:

average velocity: Δposition/Δtime

instantaneous velocity: tangent line idea.

Ex1:

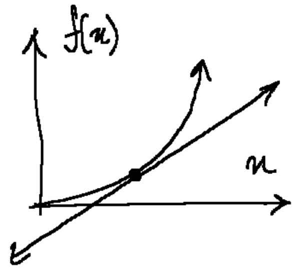
Dropping a ball:

$d(m) = f(t) = 5t^2$

Instantaneous $\vec{v}$ at time $t=1$

where time = seconds

Note: Works for smaller Value



$V_{Avg} = \frac{f(x)-f(a)}{x-a}$

(ie) $\frac{f(2)-f(1)}{2-1}$

t	1s [1,2]	0.1 [1,1.1]	0.01 [1,1.001]
V_{Avg}	15	10.5	10.05

→ 10

Approximation of instantaneous Velocity

Average velocity at time t w/ period

$$\frac{f(t+h)-f(t)}{h}$$

where the precision depends on how close h is 0, but != 0

Ex4:

Find vertical instantaneous $\vec{v}$ at 5400s

hard so just take average of two given points

2.2 The limit of a function:

Recall based on the previous ideology:

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \text{slope of tangent}$$

$$\lim_{h\to 0} \frac{f(t+h)-f(t)}{h} = \text{instant velocity}$$

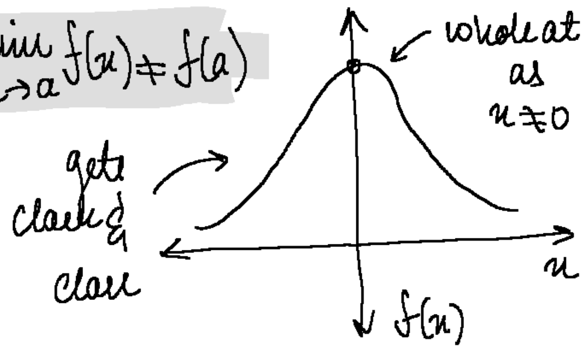
"A limit is a promise"

We can ensure that $f(x)$ is as $\lim_{x\to a} f(x) = L$ to L as we want, as long as x is chosen sufficiently close to a but never equal to a.

Ex 1:

$\lim_{x\to 2} 7 = 7$ $\lim_{x\to 3} x^2 = 9$ $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$\lim_{x\to a} f(x) \neq f(a)$



x	2	1.0	0.01
$\sin x$	.45	.84	.998

Visual representation of the graph of $\frac{\sin x}{x}$; $\boxed{x \neq 0}$

Ex 4:

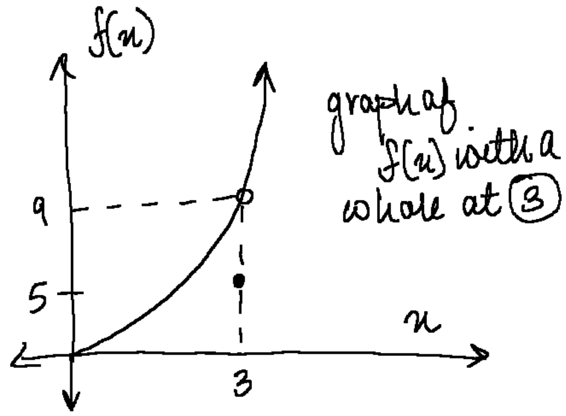
$$f(x) = \begin{cases} x^2 ; & x \neq 3 \\ 5 ; & x = 3 \end{cases}$$

$\lim_{x\to 2} f(x) = 4$

$\lim_{x\to a} f(a) \neq f(a)$

$\lim_{x\to 3} f(x) = 9$

as the points near it

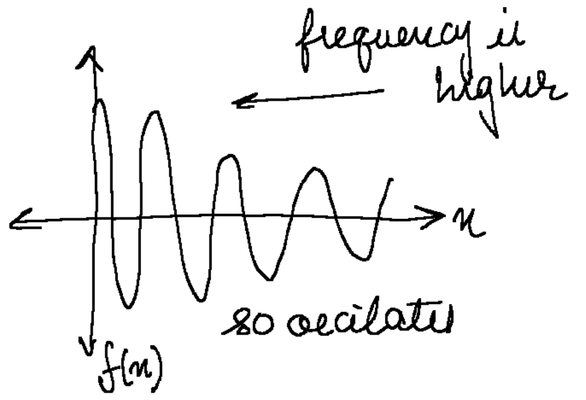


Ex5:

$f(x) = \sin\left(\frac{\pi}{x}\right)$ where $\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right) = ?$

x	1	1/2	1/3	1/100	0.08
$f(x)$	0	0	0	0	1

as $0 \neq 1$



thus $\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right) = \text{DNE}$

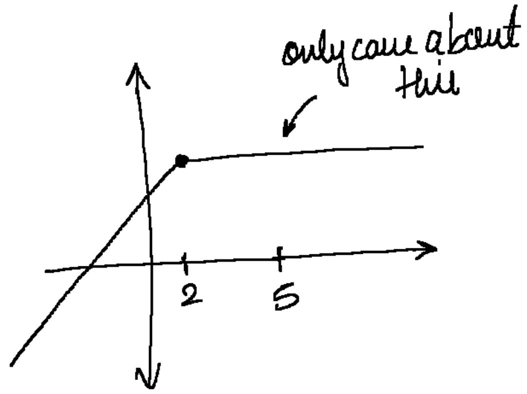
Ex6:

$f(x) = \overbrace{|x+3|}^{>0} - \overbrace{|x-1|}^{>0}$ where $\lim_{x \to 5} f(x) = ?$

x	5.1	5.01	5.001
$f(x)$	4	4	4.0

$\lim_{x \to 5} (x+3-x+1) = 4$

$\lim_{x \to 5} 4 = 4$



$\lim_{x \to a} f(x)$ only depends on value near $f(x)$ and x

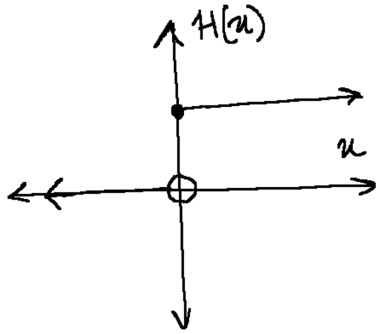
One sided limits:

Heavyside function: $H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$

$\lim_{x \to 0} H(x) = \text{DNE}$

$\lim_{x \to 0^-} H(x) = 0$

$\lim_{x \to 0^+} H(x) = 1$



$\lim_{x \to a^+} f(x) = L \iff$ We can ensure that $f(x)$ is as close to L as we want; as long as x is chosen sufficiently close to a, but $x > a$

Theorem:

$\lim_{x\to a} f(x) = L$ if and only if $\left.\begin{array}{l}\lim_{x\to a^+} f(x) = L \\ \lim_{x\to a^-} f(x) = L\end{array}\right\}$ and

Ex8: $f(x) = |x|$; $\lim_{x\to 0} f(x) = ?$

x	0.1	0.01	0.001
$f(x)$	0.1	0.01	0.001

and

x	-0.1	-0.01	-0.001
$f(x)$	0.1	0.01	0.001

right handed | left handed

because $\lim_{x\to 0^+} f(x) = (x) = 0$ & $\lim_{x\to 0^-} f(x) = (-x) = 0$ then $\lim_{x\to 0} f(x) = |x| = 0$

Ex9: $\lim_{x\to 2} \frac{x-2}{|x-2|} = ?$

$\lim_{x\to 2^+} \frac{x-2}{|x-2|} \Rightarrow \lim_{x\to 2^+} \frac{x-2}{x-2} \Rightarrow 1$ — if $x>2$ then $x-2 \geq 0$

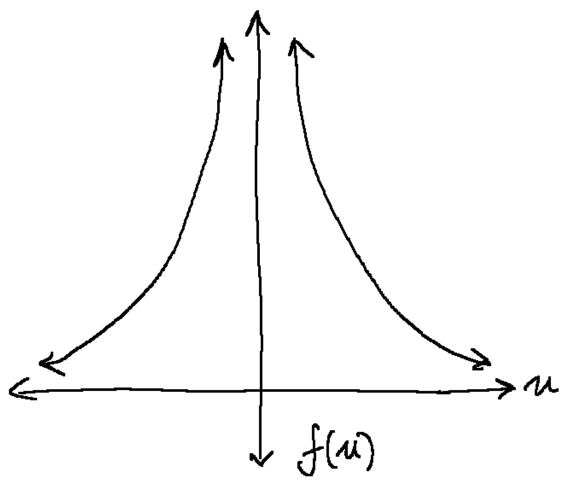
$\lim_{x\to 2^-} \frac{x-2}{|x-2|} \Rightarrow \lim_{x\to 2^-} \frac{x-2}{-(x-2)} \Rightarrow -1$ — if $x<2$ then $x-2<0$

based on the theorem $\lim_{x\to 2} \frac{x-2}{|x-2|} = DNE$

Infinite limit:

$f(x) = \frac{1}{x^2}$; $\lim_{x\to 0} f(x) = DNE$

x	0.1	0.01	0.001
$f(x)$	0.1	0.01	0.001



upgrade the limit

Infinite limits:

$\lim_{x\to a} f(x)=\infty \iff$ We can ensure that $f(x)$ is as large as we want if x is chosen sufficiently close to a, but $\neq a$.

similarily: $\lim_{x\to a^{\mp}} f(x) = \pm\infty$

Vertical Asymptotes:

$x=a$ is called a vertical asymptote of a graph $y=f(x)$ if at least one of the following is true:

$\lim_{x\to a^{\pm}} f(x)=\infty$ OR $\lim_{x\to a^{\pm}} f(x)=-\infty$

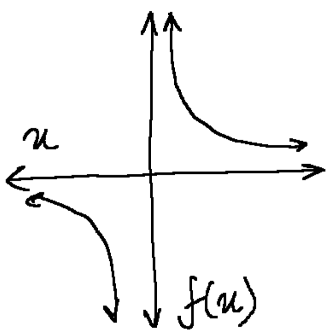
Example of possible VA

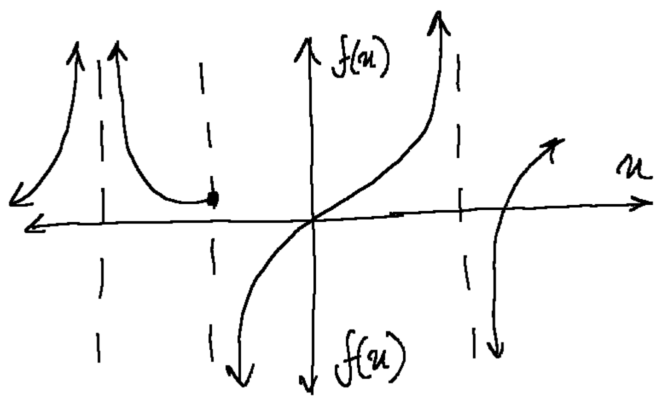
Ex: $f(x)=1/x$

$\lim_{x\to 0} f(x) = \text{DNE}$

$\lim_{x\to 0^-} f(x) = -\infty$

$\lim_{x\to 0^+} f(x) = +\infty$





Examples:

(a) $\lim_{x\to -1^+} \frac{x}{x+1} = \frac{-1}{0^+} = \boxed{-\infty}$

$\lim_{x\to -1^-} \frac{x}{x+1} = \frac{-1}{0^-} = \boxed{\infty}$

(b) $\lim_{x\to \pi/2^+} \tan(x) = \frac{\sin(\pi/2)}{\cos(\pi/2)} = \frac{1}{0^+} = \boxed{\infty}$

$\lim_{x\to \pi/2^-} \tan(x) = \frac{\sin(-\pi/2)}{\cos(-\pi/2)} = \frac{+1}{0^-} = \boxed{-\infty}$