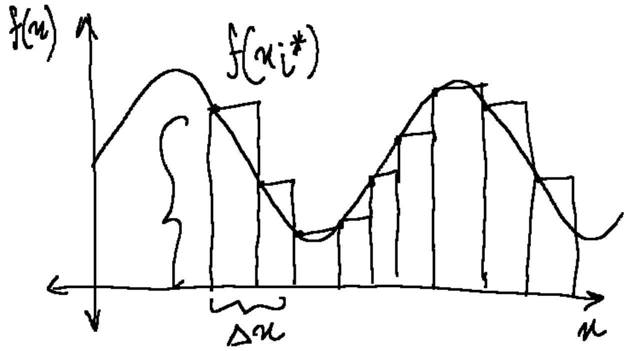
5.2 The definite integral:

Rieman Sum:



width of rectangle: $\Delta x = \frac{b-a}{n}$

subdivision points: $x_i = a + i\Delta x$

arbitrary sample $x_i^* \in [x_{i-1}, x_i]$

height of the rectangle: $i = f(x_i)$

Total area of all rectangles:

$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \ldots\ldots$

$\ldots f(x_n^*)\Delta x = \sum_{i=1}^{n} f(x_i^*)\Delta x$

$L_n: \sum_{i=1}^{n} f(x_{i-1})\Delta x$

$R_n: \sum_{i=1}^{n} f(x_i)\Delta x$

$M_n: \sum_{i=1}^{n} f(\frac{1}{2}(x_i + x_{i-1}))\Delta x$

Definition: The definite integral:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

if this limit exists then we call f and continuous then integratable

Theorem Continued:

if f is integratable then $= \int_a^b f(x)dx = \lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n = \lim_{n\to\infty} M_n$

Ex1: $\int_0^1 x^2 dx = \frac{x^3}{3}\Big]_0^1 = \frac{1}{3} = \lim_{n\to\infty} L_n$

Ex2: $\int_3^5 7dx = 7x\Big]_3^5 = 14 = \lim_{n\to\infty} 14 = 14$

Ex3: $\int_3^5 \sin(x)dx = -\cos(x)\Big]_3^5 = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} \sin(x_i)\Delta x$

$\Delta x = \frac{5-3}{n} = \frac{2}{n}$; $x_i = 3 + i\Delta x = 3 + \frac{2i}{n}$; $\lim_{n\to\infty} \sum_{i=1}^{n} \sin(3 + \frac{i2}{n}) \cdot \frac{2}{n})$

Anti-derivative

Ex4: $\int_{-2}^{1} x dx = \lim_{n\to\infty} L_n = \frac{x^2}{2}\Big]_{-2}^{1} = \frac{1}{2} - \frac{4}{2} = \boxed{\frac{-3}{2}}$

$\int_a^b f(x)dx$ counts area below the x-axis as negative area

Properties of the definite integral:

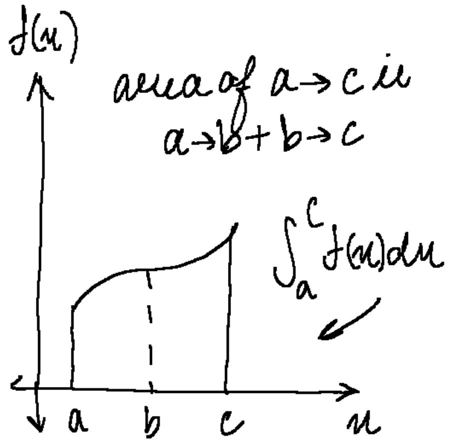
$\int_a^b c\,dx = c(b-a)$; $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$

$\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

$\int_a^b f(x)-g(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

$\int_a^a f(x)\,dx = 0$; $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

$\int_a^c f(x)\,dx = \int_a^b f(x)+dx + \int_b^c f(x)+dx$



Comparisons:

if $a \le b$ and if $\forall$ all $x \in [a,b]$ ....

... $f(x) \ge 0$, then $\int_a^b f(x)\,dx$

... $f(x) \le g(x)$, then $\int_a^b f(x)\,dx$

... $m \le f(x) \le M$, then $\int_a^b f(x)\,dx$

Ex4: $\int_0^1 5x^2+2\,dx = \frac{5x^3}{3}+2x\Big]_0^1$

$\frac{5(1)^3}{3}+2(1) - \frac{5(0)^2}{3} - 2(0) = \frac{5}{3}\cdot\frac{6}{3} = \boxed{\frac{11}{3}}$

Ex5: $\int_1^0 x^2\,dx = -\int_0^1 x^2\,dx = \frac{x^3}{3}\Big]_0^1$

then $A = \boxed{\frac{-1}{3}}$

5.3 The fundamental Theorem of Calculus:

Recall:

$$\begin{pmatrix}\text{signed area under} \\ \text{the graph of } f \\ \text{btw } x=a,\ x=b\end{pmatrix} = \int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \begin{pmatrix}\text{displacement of} \\ \text{an object w/ } \vec{v}\ f(t) \\ \text{btw } t=a,\ t=b\end{pmatrix}$$

velocity at $t = f(t) = F'(t)$; position at $= F(t)$

$$\int_a^b f(t)\,dt = (\text{displacement btw } t=a \ \&\ t=b) = F(b) - F(a)$$

Ex 1:

$$\int_1^2 e^{3t}\,dt = F(2) - F(1)$$

$F(t)$ = antiderivative of t

$f(t) = e^{3t}$; $F(t) = \frac{1}{3}e^{3t} + C$

position $C = 7$ then

$\frac{1}{3}e^{3\cdot 2} + 7 - \frac{1}{3}e^{3\cdot 1} - 7$ = displacement

position at given t.

Fundamental theorem of calculus Part 2:

if f is continuous on $[a,b]$ and F is the antiderivative of f then $\int_a^b f(x)\,dx = F(b) - F(a)$ or $\int_a^b \frac{dF}{dx}\,dx = \Delta F = F(b) - F(a)$

Ex 2: $\int_2^3 2xe^{x^2}\,dx = F(3) - F(2)$

$e^{3^2} - e^{2^2} = [e^{x^2}]_2^3$: $F(x) = e^{x^2}$

$u = x^2$; $du = 2x\,dx$

$F(x) = e^{x^2}\Big]_2^3 \Rightarrow e^{3^2} - e^{2^2}$ = Answer

Ex 3: $\int_1^2 \frac{1}{x}\,dx = F(2) - F(1)$

$1/x = \ln|x|$ then $\ln(x)\Big]_1^2$

$F(x) = \ln(x)$; $\ln(2) - \cancel{\ln(1)}$

then answer $= \ln(2)$

Ex 4: $g(x) = \int_1^x t^3\,dt = \frac{t^4}{4}\Big]_1^x$

$\Rightarrow \frac{x^4}{4} - \frac{1^4}{4} \Rightarrow \frac{x^4}{4} - 1/4$: but $g'(x) = x^3$

Fundamental Theorem of Calculus Part 1:

if f is continuous on $[a,b]$, then $g(x)=\int_a^x f(t)dt$ is continuous on $[a,b]$ and differentiable on (a,b) and we have $g'=f(x)$

Ex 5: $\frac{d}{dx}\int_5^x e^{-t^2}dt = e^{-x^2}$

because $g'(x)=f(x)$

$$\int_0^x e^{-t^2}dt = \int_0^5 e^{-t^2}dt + \int_5^x e^{-t^2}dt$$

Ex 6: $g(x)=\int_1^{x^3}\sqrt{1+t^2}\,dt = h(x^3)$

$h(x)=\int_1^x \sqrt{1+t^2}\,dt \; ; \; h'(x)=\sqrt{1+x^2}$

$g'(x)=\frac{d}{dx}(h(x^3)) = h'(x^3)3x^2$

Answer $\Rightarrow$ $3x^2\sqrt{1+x^6} = g'(x)$

Ex 7: $\frac{d}{dx}\int_x^{2x}\sin(t^2)dt$

$\Rightarrow \frac{d}{dx}\left[\int_x^0 \sin(t^2) + \int_0^{2x}\sin(t^2)\right]$

$-\sin(x^2) + 2\sin(4x^2) + C$

$g'(x)= 2\sin(4x^2)-\sin(x^2)$

Ex 8: $\frac{d}{dx}\int_x^2 \sin(t^2)\,dt = \frac{d}{dx} -\int_2^x \sin(t^2)dt$

$-\frac{d}{dx}\int_2^x \sin(t^2)dt \Rightarrow -\sin(x^2)+C$

$g'(x)=-\sin(x^2)$ w/ $f(x)$