4.4 Indeterminate form & l'hospital Rule:

Ex0: $\lim_{x\to 1} \frac{x+1}{x+3} = \frac{1}{2}$ **Ex1:** $\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty$ **Ex2:** $\lim_{x\to 0} \frac{\sin x}{x} = 1$ "$\frac{0}{0}$" indeterminate

Ex3: $\lim_{x\to 0} \frac{\ln(1+2x)}{x} = \frac{2}{1+2x} = 2$ "$\frac{0}{0}$" **Ex4:** $\lim_{x\to 0} \frac{\ln(1+2x)}{\sin x} = \frac{2/1+2x}{\cos x} = 2$ "$\frac{0}{0}$"

Alternatively:

linear approximation of num at 0:

$f(x) \approx 2\cdot(x-0)+0 = 2x$ if $x \approx 0$

linear approximation of den at 0:

$g(x) \approx 1\cdot(x-0)+0 = x$ if $x \approx 0$

$\lim_{x\to 0} \frac{\ln(1+2x)}{\sin x} = \frac{2x}{x} = 2$

$\forall f = \ln(1+2x)$ & $g = \sin x$

General Case:

Ex4: Suppose that f and g are differentiable at a and

$f(a) = g(a) = 0 \quad g'(a) \neq 0$

Solved ind

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{\frac{f(x)-f(0)}{x-0}}{\frac{g(x)-g(0)}{x-0}} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} \text{ \& } g'(a) \neq 0 \quad \text{"}\frac{0}{0}\text{"}$$

l'hospital theorem: Such a g

suppose that:

1. f and g are differentiable near a, but possible not at a
2. $g'(x) \neq 0$ if x close to a, but not possible $x = a$
3. $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \pm\infty$ & $\lim_{x\to a} g(x) = \pm\infty$

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ $\forall$ "$\frac{0}{0}$" & "$\frac{\infty}{\infty}$"

works $\forall$ any indeterminate form

Ex5: $\lim_{x\to 2} \frac{x^2-5x+16}{x^4-16} = \text{"}\frac{0}{0}\text{"}$

then $\lim_{x\to 2} \frac{2x-5}{4x^3} = \frac{-1}{32}$

Ex6: $\lim_{x\to\infty} \frac{2x^3-x+1}{5x^3+x+7} = \frac{2}{5} = \text{"}\frac{\infty}{\infty}\text{"}$

then $\lim_{x\to\infty} \frac{2\cdot 3x^2-1}{5\cdot 3x^2-2x} = \frac{2\cdot 3\cdot 2x}{5\cdot 3\cdot 2x-2} = \boxed{\frac{2}{5}}$

Ex6: $\lim_{x\to 2} \frac{e^x 1-x}{x^2} = \text{"}\frac{0}{0}\text{"}$

then $\lim_{x\to 2} \frac{e^x 1}{2x} = \frac{e^x}{2} = \boxed{\frac{1}{2}}$

Ex8: $\lim_{x\to\infty} \frac{e^x}{x^3} = \lim_{x\to\infty} = \frac{e^x}{3x^2} = \frac{e^x}{6x} = \frac{e^x}{6}$

then $\lim_{x\to\infty} \frac{e^x}{6} = \boxed{\infty}$

Indeterminate Products:

Ex9: $\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \text{"}\frac{0}{0}\text{"}$

then $\lim_{x\to 0^+} \frac{1}{x} \times \frac{-x^2}{1} = -x = \boxed{0}$

Ex10: $\lim_{x\to -\infty} x^3 e^x = \lim_{x\to -\infty} \frac{x^3}{1/e^x} = \text{"}\frac{0}{\infty}\text{"}$

then $\lim_{x\to -\infty} \frac{x^3}{e^{-x}} = \frac{3x^2}{-e^{-x}} = \frac{6x}{e^{-x}} = \frac{6}{-e^{-x}} = \boxed{0}$

Indeterminate differences:

Ex11: $\lim_{x\to 0^+} \left(\frac{1}{e^x-1} - \frac{1}{x}\right) = \frac{x-e^x+1}{x(e^x-1)} = \text{"}\frac{0}{0}\text{"}$

$\lim_{x\to 0^+} \frac{1-e^x}{xe^x+e^x-1} = \frac{-e^x}{xe^x+2e^x} = \boxed{\frac{-1}{2}}$

Indeterminate Powers:

Ex12: $\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^x = \frac{\ln\left(1+\frac{a}{x}\right)}{1/x} = \text{"}\frac{\infty}{\infty}\text{"}$

then $\lim_{x\to\infty} \frac{\ln(1+a/x)}{1/x} = \frac{a}{1+\frac{a}{x}} = \boxed{a} = \boxed{e^a}$

Ex13: $\lim_{n\to\infty} \left(\frac{3^{1/n}+5^{1/n}}{2}\right)^n =$ indeterminate at form "1^∞"

$e^{\frac{\ln(3)+\ln(5)}{2}}$ simplify this

$e^y = \lim_{n\to\infty} \frac{\ln\left(\frac{3^{1/n}+5^{1/n}}{2}\right)}{1/n} = \frac{-n^2}{\frac{3^{1/n}+5^{1/n}}{2}} \frac{d}{dn}\left(\frac{3^{1/n}+5^{1/n}}{2}\right) = \boxed{\sqrt{15}}$

skipped many

4.5 Summary of Curve Sketching

Information:

1. Domain and Range : inferred later on usually.
2. x/y intercepts : $f(0)$ and zeros
3. Symmetry : even/odd functions
4. Horizontal and vertical asymptotes
   w/ other asymptotic behavior
5. Critical Numbers
6. Intervals where functions increase & decrease
7. Inflection points
8. Intervals where functions concave up & down.
9. local and absolute extreme

Sometimes we may not determine all information directly. Some properties could be inferred from other properties.