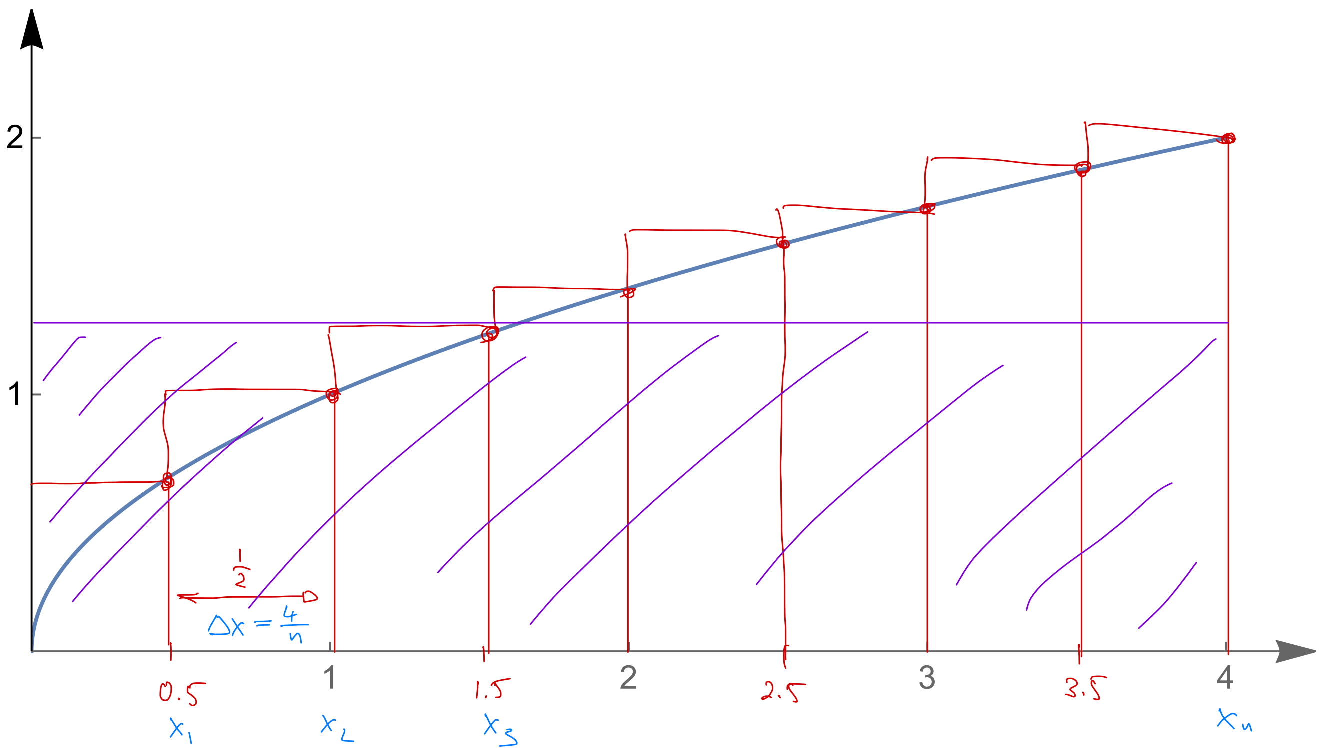
Class 25

6.5 Average of a Function
6.1 Areas Between Curves

6.5 Average of a Function

Ex 1 Find the "average" of $f(x) = \sqrt{x}$ over $[0,4]$.



$$\text{average} \approx \frac{f(0.5) + f(1) + f(1.5) + \cdots + f(4)}{8}$$

$$= \frac{1}{\cancel{8}\,4}\left(\frac{1}{2} f(0.5) + \frac{1}{2} f(1) + \cdots + \frac{1}{2} f(4)\right)$$

$$= \frac{1}{4} R_8$$

General Computation:

$$\text{average} \approx \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}$$

$$= \frac{1}{4}\left(\frac{4}{n} f(x_1) + \cdots + \frac{4}{n} f(x_n)\right)$$

$$\underbrace{\quad}_{\Delta x} \qquad \underbrace{\quad}_{\Delta x}$$

$$= \frac{1}{4} R_4$$

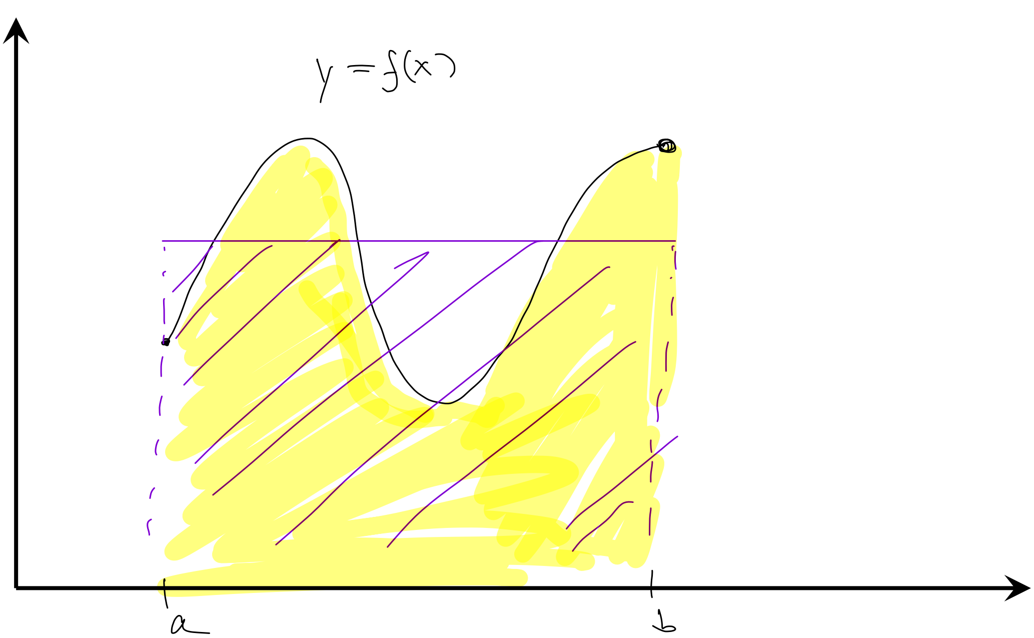
$$\text{average} = \frac{1}{4}\int_0^4 \underbrace{f(x)}_{\sqrt{x}}\,dx = \frac{1}{4}\int_0^4 \sqrt{x}\,dx$$

$$= \frac{1}{4}\cdot\frac{2}{3}x^{3/2}\Big]_0^4 = \frac{1}{4}\cdot\frac{2}{3}\cdot 4^{3/2} = \frac{4}{3} = 1.33\ldots$$

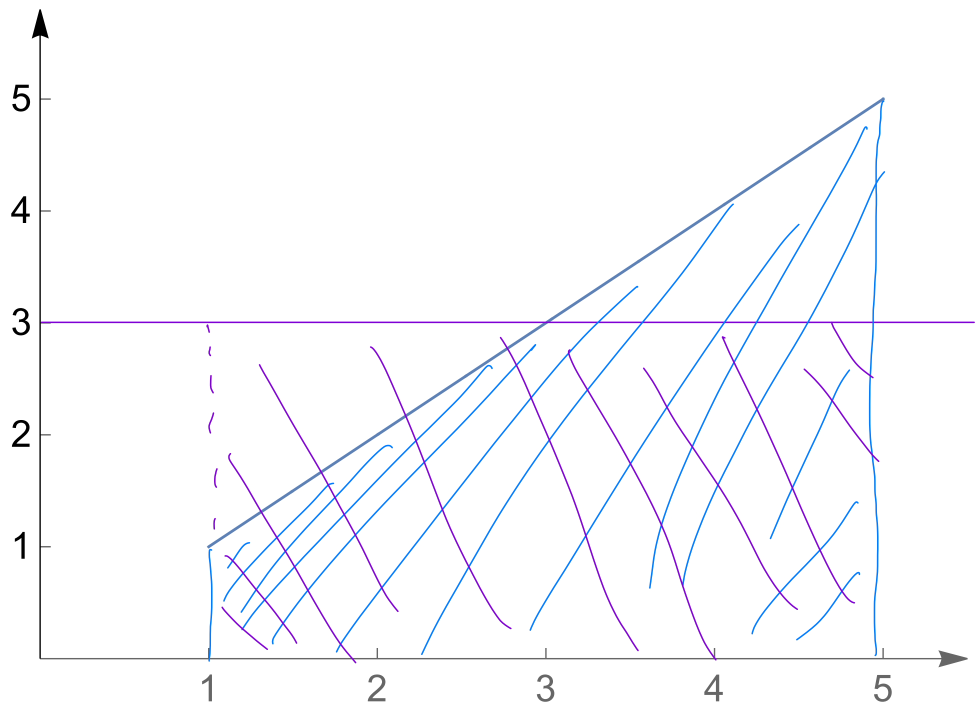
Definition

The average of a function f over an interval $[a,b]$ is defined to be

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$



Ex 2 Find the average of $f(x) = x$ over $[1,5]$.



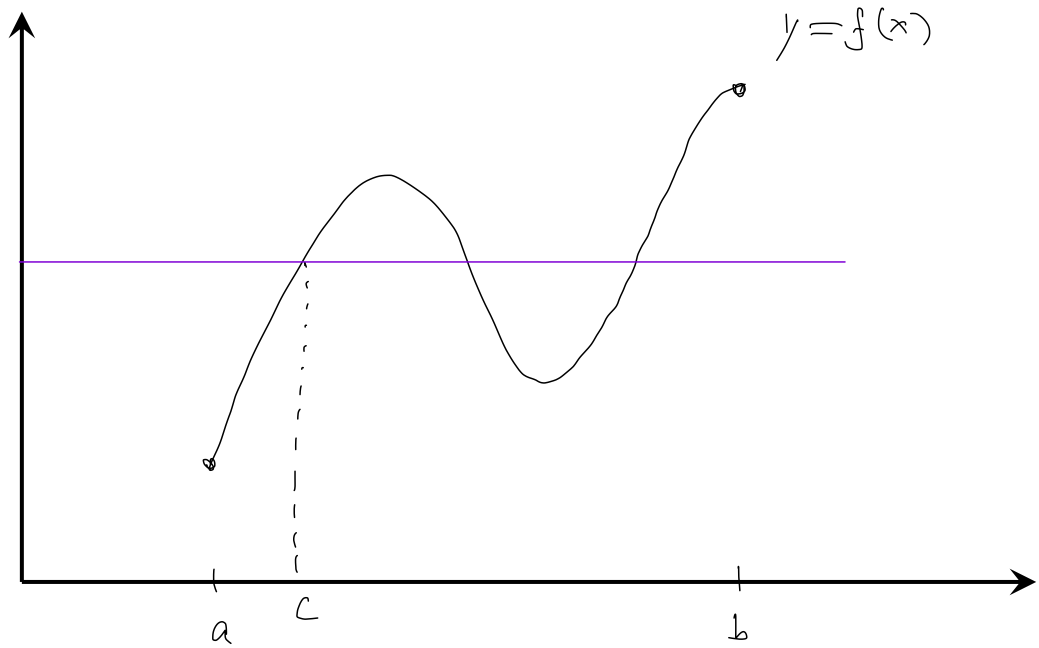
$$\text{average} = \frac{1}{5-1}\int_1^5 x\,dx = \frac{1}{4}\left.\frac{1}{2}x^2\right]_1^5$$

$$= 3$$

Mean Value Theorem for Integrals

If f is continuous on $[a, b]$, then there is a number $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

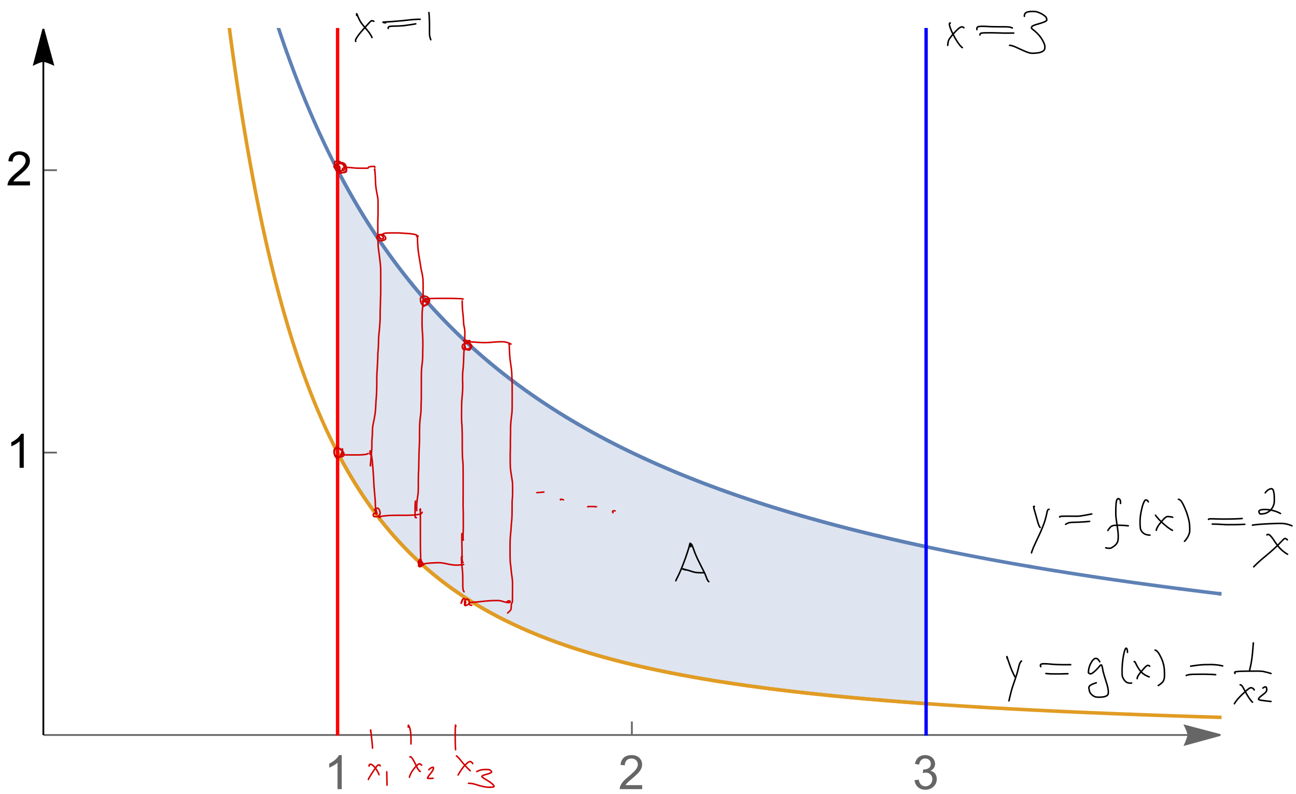


6.1 Areas Between Curves

Ex 1 Find the area between the curves

$$y = f(x) = \frac{2}{x} \qquad y = g(x) = \frac{1}{x^2}$$

$$x = 1 \qquad x = 3$$



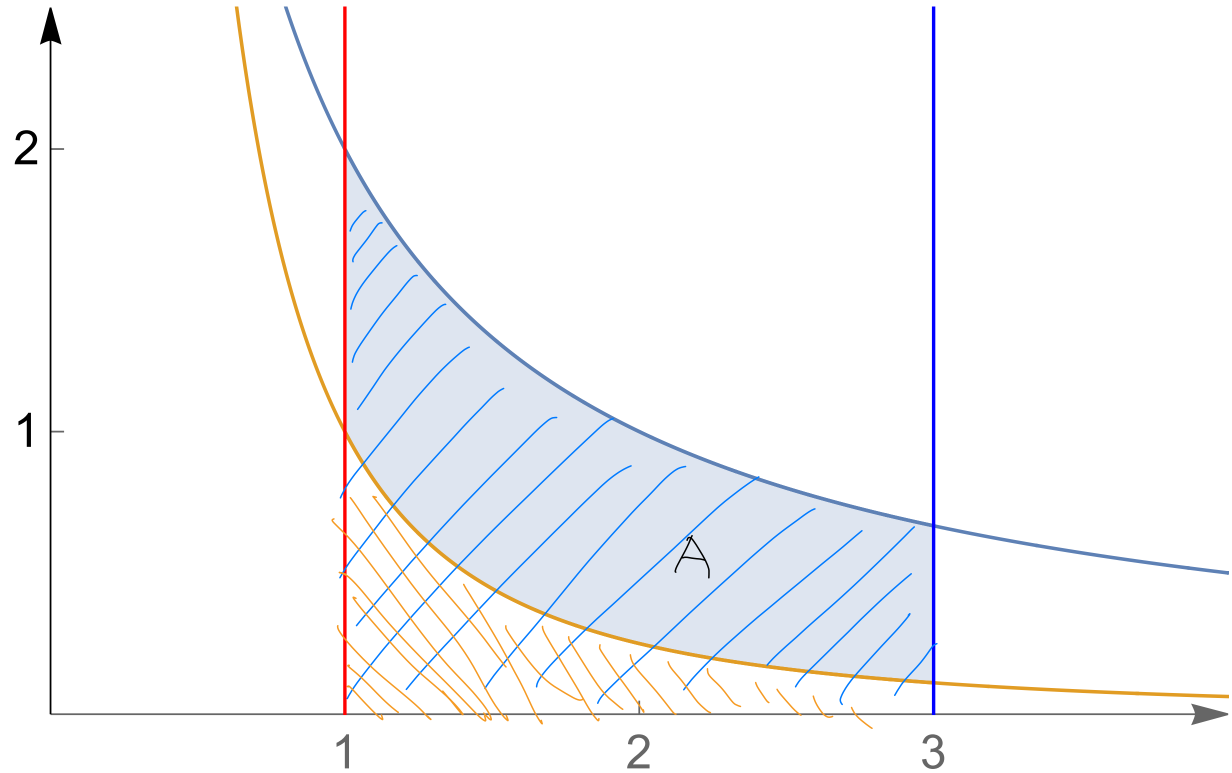
$L_n = (f(x_0) - g(x_0))\Delta x + (f(x_1) - g(x_1))\Delta x + \cdots$

$=$ left endpoint approximation for $f(x) - g(x)$

$$A = \lim_{n\to\infty} L_n = \int_1^3 (f(x) - g(x))\,dx = \int_1^3 \left(\frac{2}{x} - \frac{1}{x^2}\right)dx$$

$$= \int_1^3 \frac{2}{x}\,dx - \int_1^3 \frac{1}{x^2}\,dx = 2\ln|x|\Big]_1^3 - \left(\left(-\frac{1}{x}\right)\Big]_1^3\right)$$

$$= 2\ln 3 - 2\ln 1 - \left(-\frac{1}{3} + \frac{1}{1}\right) = \cdots = 2\ln 3 - \frac{2}{3}$$



Alternate approach:

$$A = \int_1^3 f(x)\,dx - \int_1^3 g(x)\,dx = \int_1^3 \left(f(x) - g(x)\right) dx$$

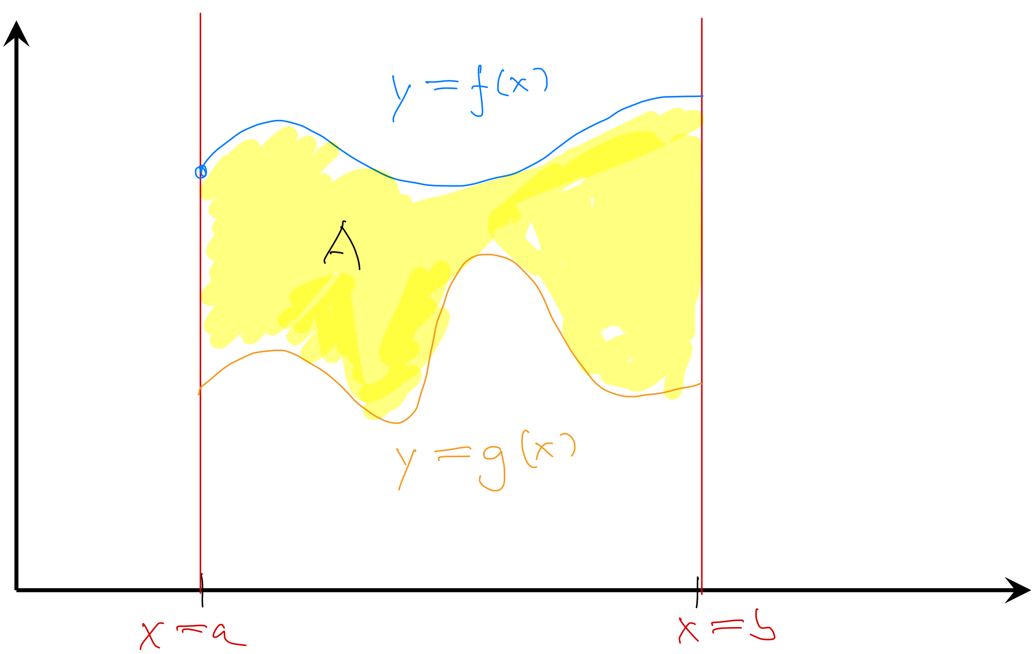
The area of the region bounded by the curves

$y = f(x)$ $\quad y = g(x)$ $\quad x = a$ $\quad x = b$

equals

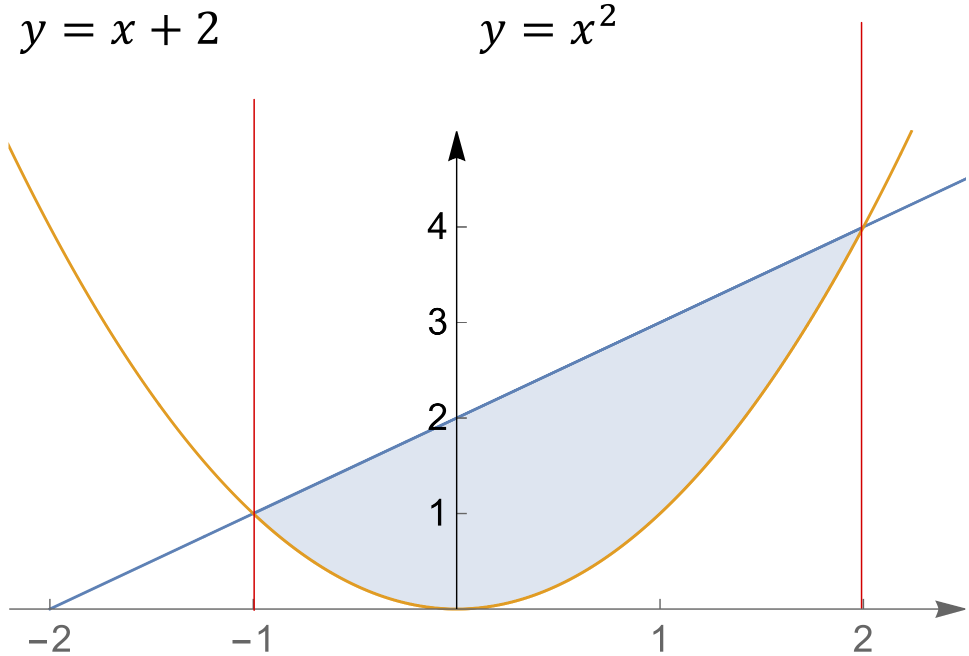
$$\int_a^b \left(f(x) - g(x)\right) dx$$

assuming that $f(x) \geq g(x)$



Ex 2 Find the area between the curves

$y = x + 2$ $\qquad$ $y = x^2$



Find the intersection points

$$x+2 = y = x^2$$

$$x+2 = x^2$$

$$x^2 - x - 2 = 0$$

$$x_{1/2} = -1, 2$$

Add the vertical lines $x=-1$ and $x=2$.

$$A = \int_{-1}^{2} \left((x+2) - x^2 \right) dx$$

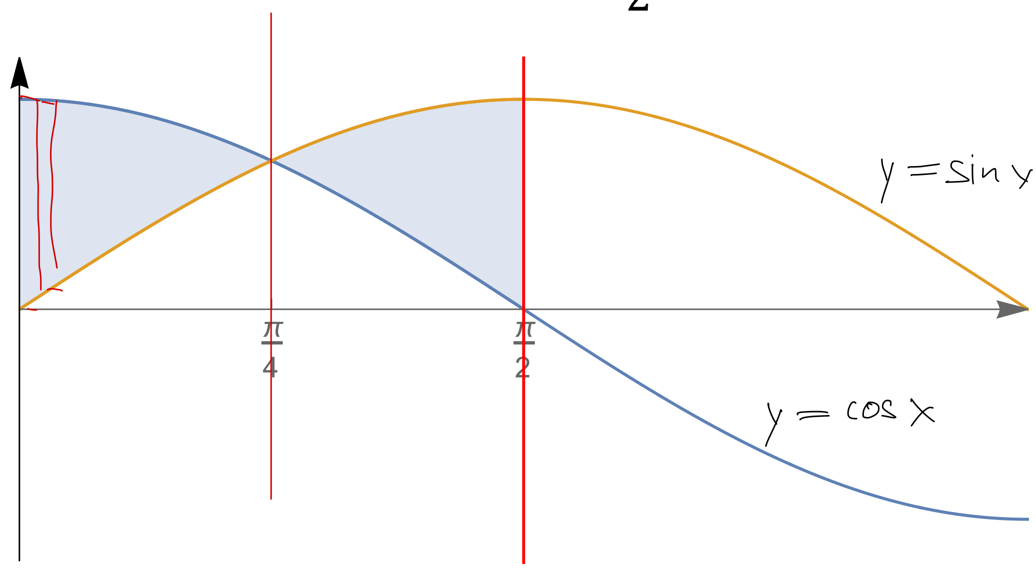
$$= \left(\frac{1}{2}x^2 + 2x - \frac{1}{3}x^3 \right) \Big]_{-1}^{2}$$

$$= \left(\frac{1}{2}4 + 2\cdot 2 - \frac{1}{3}8 \right) - \left(\frac{1}{2} - 2 - \frac{1}{3} \right) = \cdots$$

Ex 3 Find the area between the curves

$y = \sin x \qquad\qquad y = \cos x$

$x = 0 \qquad\qquad x = \dfrac{\pi}{2}$



$$A = \int_0^{\pi/2} |\cos x - \sin x|\, dx = \int_0^{\pi/4} \underbrace{|\cos x - \sin x|}_{\cos x - \sin x} dx + \int_{\pi/4}^{\pi/2} \underbrace{|\cos x - \sin x|}_{\substack{-(\cos x - \sin x) \\ = \sin x - \cos x}} dx$$

$$= \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx$$

$$= (\sin x + \cos x)\Big]_0^{\pi/4} + (-\cos x - \sin x)\Big]_{\pi/4}^{\pi/2}$$

$$= \Big(\underbrace{\sin\left(\tfrac{\pi}{4}\right)}_{\frac{1}{\sqrt{2}}} + \underbrace{\cos\left(\tfrac{\pi}{4}\right)}_{\frac{1}{\sqrt{2}}}\Big) - \big(\underbrace{\sin 0}_{=0} + \underbrace{\cos 0}_{=1}\big) + \cdots$$

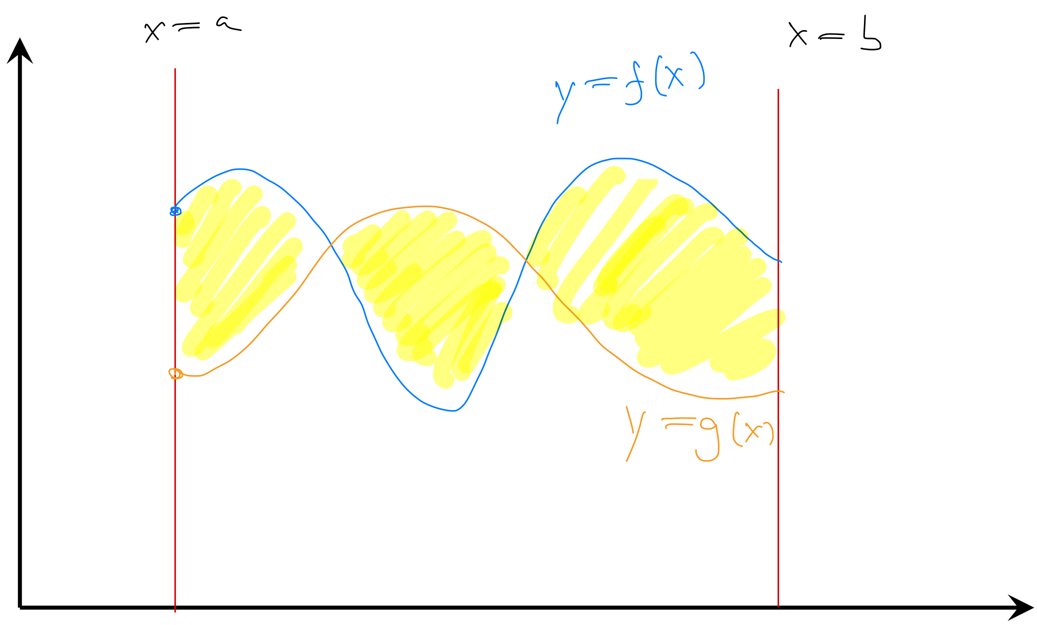
$$= \cdots = 2\sqrt{2} - 2$$

The area of the region bounded by the curves

$$y = f(x) \qquad y = g(x) \qquad x = a \qquad x = b$$

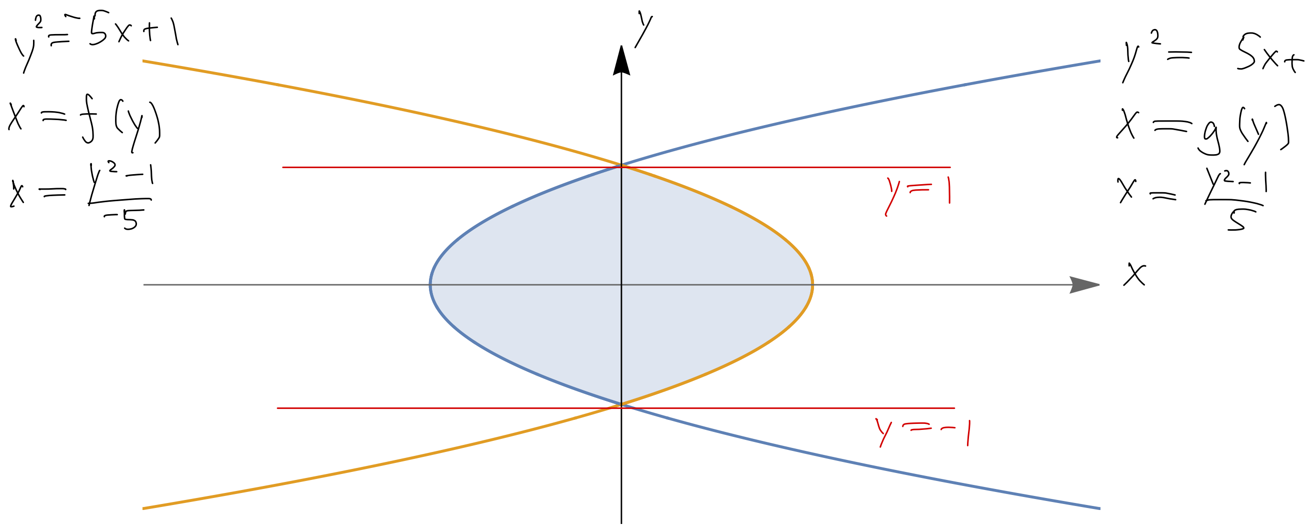
equals

$$\int_a^b |f(x) - g(x)|\, dx$$



Ex 4 Find the area between the curves

$$y^2 = 5x + 1 \qquad y^2 = -5x + 1$$



Switch the roles of x, y

$$y^2 = -5x + 1 \qquad\qquad y^2 = 5x + 1$$

$$y^2 - 1 = -5x \qquad\qquad y^2 - 1 = 5x$$

$$\frac{y^2-1}{-5} = x \qquad\qquad \frac{y^2-1}{5} = x$$

$$x = f(y) = \frac{y^2-1}{-5} \qquad\qquad x = g(y) = \frac{y^2-1}{5}$$

Find intersection points

$$\frac{y^2-1}{-5} = x = \frac{y^2-1}{5}$$

$$-(y^2-1) = y^2-1$$

$$0 = 2y^2 - 2$$

$$y^2 = 1$$

$$y = \pm 1$$

$$A = \int_{-1}^{1} (f(y) - g(y))\, dy$$

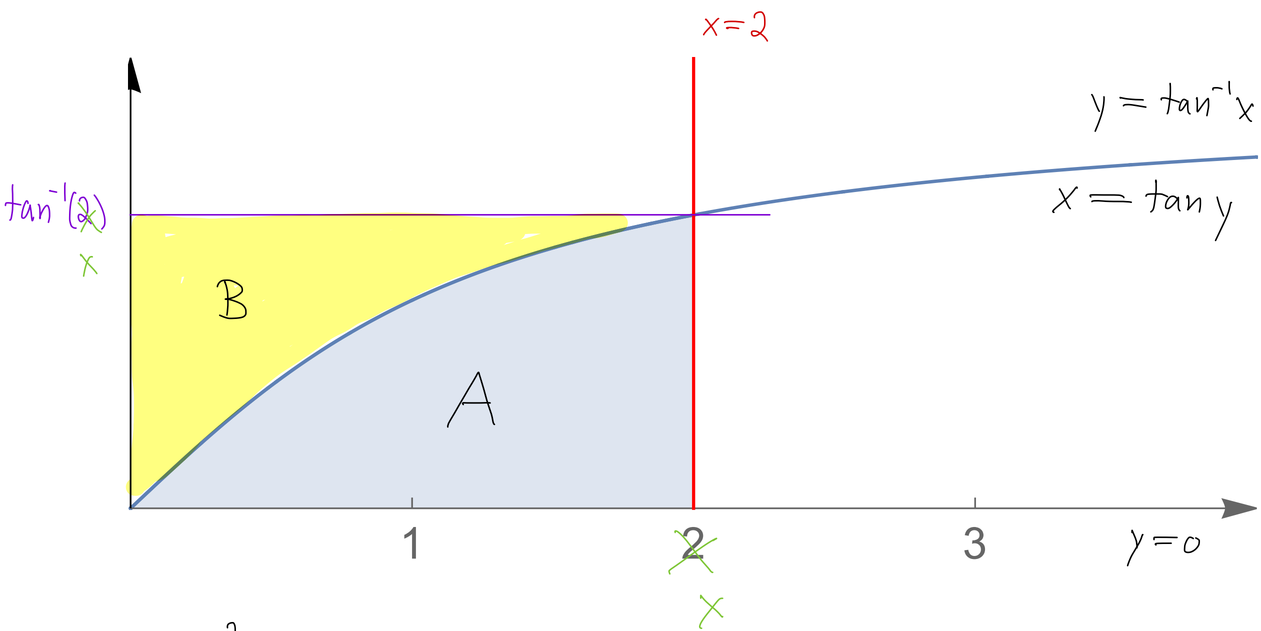
$$= \int_{-1}^{1} \left(\frac{y^2-1}{-5} - \frac{y^2-1}{5} \right) dy = -\frac{2}{5} \int_{-1}^{1} (y^2-1)\, dy$$

$$= -\frac{2}{5} \left(\frac{1}{3} y^3 - y \right) \Big]_{-1}^{1} = -\frac{2}{5} \left(\left(\frac{1}{3} - 1\right) - \left(-\frac{1}{3} + 1\right) \right)$$

$$= 4/5$$

Ex 5 Find the area between the curves

$$y = \tan^{-1} x \qquad \cancel{y}\,x = 2 \qquad \cancel{x}\,y = 0$$



$$A = \int_0^2 \tan^{-1} x \, dx$$

$$B = \int_0^{\tan^{-1}(2)} \tan y \, dy = -\ln|\cos y| \Big]_0^{\tan^{-1}(2)}$$

$$= -\ln|\cos(\tan^{-1}(2))| + \ln|\underbrace{\cos(0)}_{=1}|$$

$$\tan y = \frac{\sin y}{\cos y} \rightsquigarrow -\ln|\cos y|$$

$$= -\ln|\cos(\tan^{-1}(2))|$$

$$A = \underbrace{2\tan^{-1}(2)}_{\text{area of rectangle}} - \Big(\underbrace{-\ln|\cos(\tan^{-1}(2))|}_{B}\Big)$$

$$= 2\tan^{-1}(2) + \ln|\cos(\tan^{-1}(2))|$$

Conclusion

$$\int_0^{2} \tan^{-1}(\underset{\overset{\downarrow}{u}}{x})\, d\underset{\overset{\downarrow}{u}}{x} = 2\tan^{-1}(2) + \ln\left|\cos\left(\tan^{-1}(2)\right)\right|$$

More generally

$$\int_0^{x} \tan^{-1}(u)\, du = x\tan^{-1}(x) + \ln\left|\cos\left(\tan^{-1}(x)\right)\right|$$

$$\tan^{-1}(x) \underset{FTC}{=} \frac{d}{dx}\int_0^{x} \tan^{-1}(u)\, du = \frac{d}{dx}\left(x\tan^{-1}(x) + \ln\left|\cos\left(\tan^{-1}(x)\right)\right|\right)$$

The antiderivative of $\tan^{-1}(x)$ is

$$x\tan^{-1}(x) + \ln\left|\cos\left(\tan^{-1}(x)\right)\right| + C$$