2.5 Continuity:

Definition:

A function f is called continuous at a number "a" if

$$\lim_{x\to a} f(x) = f(a)$$

This entails: $\lim_{x\to a} f(x)$ exists & $f(a)$ is defined & LHS=RHS

Ex1: $f(x) = \dfrac{x}{x^3-25x}$ continuous?

$D: \{x\in\mathbb{R} \mid x\neq 0, \pm 5\}$; if $a\in D$, then $\lim_{x\to a} f(x) = \dfrac{a}{a^3-25a} = f(a)$

$\therefore$ Continuous in $\{x\in\mathbb{R} \mid x\neq 0,\pm 5\}$

Note: $\lim_{x\to 0} f(x) = -\frac{1}{25}$, but func is not defined there! continuous

Extend Definition

A function f is called continuous from the left or right at a number a if $\lim_{x\to a^\pm} = f(a)$

Ex2: $f(x) = \begin{cases} 0 \text{ if } x<0 \\ 1 \text{ if } x\geq 0 \end{cases}$

$D: \{x\in\mathbb{R} \mid x\neq 0\}$

if $a>0$: $\lim_{x\to a} f(x) = \lim_{x\to a} 1 = 1$

then continuous

if $a<0$: $\lim_{x\to a} f(a) = \lim_{x\to a} 0 = 0$

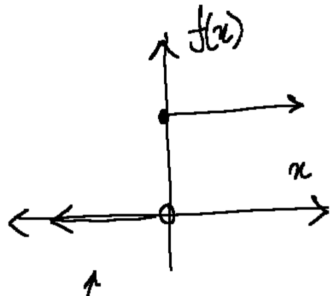
then continuous

if $a=0$:

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 0 = 0$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 1 = 1 = f(0)$

right continuity only!



Ex3: Where is the function Continuous?

Ex3:
$$f(x) = \begin{cases} x^2 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

if $a \neq 0$

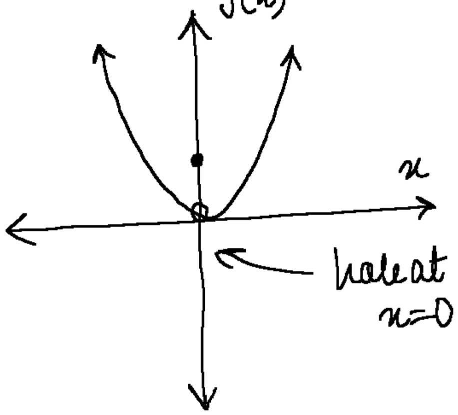
$$\lim_{x \to a} f(x) = \lim_{x \to a} x^2 = a^2 = f(a)$$

Continuous at $a \neq 0$

if $a = 0$

$$\lim_{x \to a^\pm} f(x) = \lim_{x \to 0^\pm} x^2 = 0 \neq f(0)$$

! cont at left or right



Definition

A function f is called continuous on a interval if it is

→ Continuous at every number in the interior of the interval and:

→ left/right continuous at every endpoint.

Idea: Continuous if you can "draw" w/o lifting your pen.

Ex4: Determine c such that function is continuous on the interval $x \in \mathbb{R}$.

$$f(x) = \begin{cases} x^2 & x < 3 \\ x + c & x \geq 3 \end{cases}$$

if $a < 3$:

$$\lim_{x \to a} f(x) = \lim_{x \to a} x^2 = a^2 = f(a)$$

Continuous

if $a > 3$:

$$\lim_{x \to a} f(x) = \lim_{x \to a} x + c = a + c = f(a)$$

Continuous

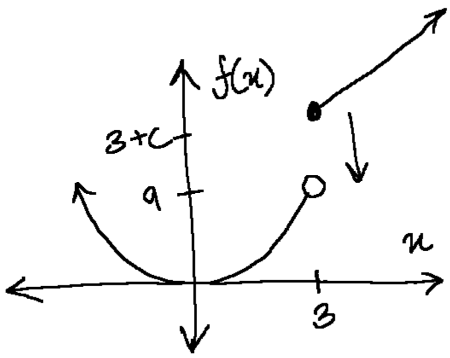
if $a = 3$:

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} x^2 = 9$$

$$\lim_{x \to 3^+} 3 + c \ \ \& \ \ f(3) = 3 + c$$

$\boxed{c = 6}$

+ Continuity



Ex5: Verify $f(x)=\sqrt{\frac{1}{x}-2}$ is cont over $(0,\frac{1}{2}]$

$0 \le x \le 1/2$: D: $x \in (0, 1/2]$

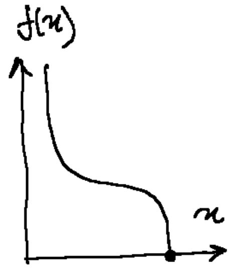
check:

① continuous at $(0, 1/2)$

② left continuity at $1/2$

① $\lim_{x \to a} \sqrt{\frac{1}{x}-2} = \sqrt{\frac{1}{a}-2} = f(a)$
continuous

② $\lim_{x \to 1/2^-} \sqrt{\frac{1}{x}-2} = \sqrt{0} = f(1/2)$
left continuous



Theorem

if f and g are continuous at a, then so are:

$f+g$; $f-g$; $f \times g$; $f \div g$ as $g \neq 0$

Polynomials $\longrightarrow$ cont at $x \in (-\infty, \infty)$

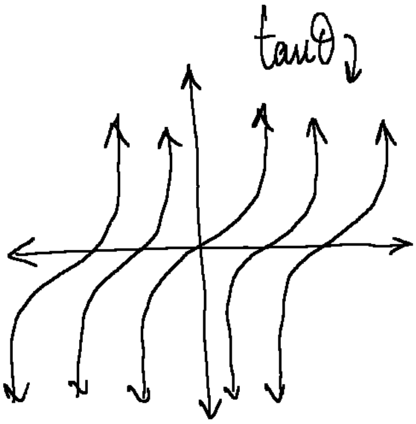
Rational functions $\longrightarrow$ if $Q(a) \neq 0$

$f(x) = x^{1/n} \longrightarrow$ $[0,\infty)$ if even & $\mathbb{R}$ at odd

$f(x) = \log_b x \longrightarrow (0,\infty)$

$f(x) = \sin x, \cos x \longrightarrow (-\infty, \infty)$

$f(x) = \arcsin, \arccos, \arctan \longrightarrow$ domain



Ex6: Where is $f(x) = \tan\theta$ continuous?

$f(x) = \tan\theta = \frac{\sin\theta}{\cos\theta}$ then $\cos(a) \neq 0$ → Visualize graph!

$\cos(a) \neq 0$ if $a = (k + 1/2)\pi$; $\{x \in \mathbb{R} \mid x \neq (k+1/2)\pi\}$

Ex7: Verify continuity?

$$f(x) = \begin{cases} x\sin\left(\frac{\pi}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

if $a \neq 0$

$\lim_{x \to a} f(x) = a\sin\frac{\pi}{a} = f(a)$
continuous

if $a = 0$; Sq th.

$\lim_{x \to 0^{\pm}} f(x) = 0$; $\lim_{x \to 0} f(x) = 0 = f(0)$
Continuous

Visualize the graph!

Theorem:

If f is continuous at b & $\lim_{x\to a} g(x) = b$ then

$$\lim_{x\to a} f(g(x)) = \lim_{x\to a} g(f(x))$$

The same is true $\forall$ one sided limits.

Ex7: $\lim_{x\to 2} \tan^{-1}\left(\frac{x^2-3x+2}{x-2}\right)$ where $a=2$, $f=\tan^{-1}$, $g=\theta$

check if arctan is continuous at 1 which is true!

$$b = \lim_{x\to 2} \frac{x^2-3x+2}{x-2} = \frac{\cancel{(x-2)}(x-1)}{\cancel{x-2}} = 1$$

then $\lim_{x\to 2} \tan^{-1}(1) = \boxed{\frac{\pi}{4}}$

Ex8: $\lim_{x\to 1^-} \ln(1-x) - \ln(1-\sqrt{x}) \Rightarrow \lim_{x\to 1^-}\left(\frac{1-x}{1-\sqrt{x}}\right)$

check if $\ln(2)$ is continuous at 2 which is true

$$b = \lim_{x\to 1^-} \frac{(1+\sqrt{x})\cancel{(1-\sqrt{x})}}{\cancel{(1-\sqrt{x})}} = 1+\sqrt{x} = 2$$

then $\lim_{x\to 2} \ln(x) = \ln(2)$

Theorem:

If: (also true $\forall$ one sided functions):

1. f is continuous from left of b &
2. $\lim_{x\to a} g(x) = b$ and
3. $g(x) < b$ sufficiently close to a, but $x \neq a$ then

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$

Similar statements are true $\forall$ one-sided limits and for continuity from the right.

Ex 9:

$$f(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}$$

$\lim_{x \to \pi/2} f(\sin x - 1) = ?$

$b = \lim_{x \to \frac{\pi}{2}} \sin x - 1 = \sin(\frac{\pi}{2}) - 1 = 0$

Theorem:

If g is continuous at a and f is continuous at $g(a)$ then $f \circ g$ is continuous at a.

Ex 10: show that $h(x) = \ln(\sin x + 2)$ is cont at 0

$f(x) = \ln(x)$ continuous at $g(0) = 2$

$g(x) = \sin x + 2$ continuous at $0 = 2$

then $h(x) = \ln(\sin x + 2)$ is continuous at zero

Intermediate Value th:

suppose f is continuous on the closed interval $[a,b]$ then let N be a # strictly b/w $f(a)$ & $f(b)$ where $f(a) \ne f(b)$, then there exists a # $c \in [a,b]$ $f(c) = N$.

Ex 11: $f(x) = x^3 + x - 1$

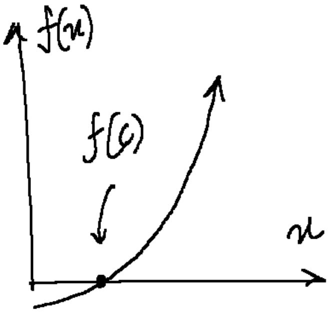
$a = 0$ $f(a) = -1$

$b = 2$ $f(b) = 9$

$N = 0$

f is continuous $[0,2]$

IVT tells $c \in [0,2]$ where $f(c) = 0$



Bisection Method

f root at $[0,2]$

mid $= 1$

$f(1) = 1 > 0$ then

f root at $[0,1]$

⋮

close approx.