OLG: Steady-state Optimality and Competitive Equilibrium

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Outline

1. **Overview**

2. **Competitive equilibrium**
   - Log-utility example

3. **Long-run Optimality**
   - Long-run feasibility
   - Golden rule
   - Long-run feasibility

Overview

This lecture:

- Restate the recursive competitive equilibrium (RCE) characterization
- Welfare properties of RCE in the OLG model (in a steady state)
- (Steady state) Competitive equilibrium of OLG model may/may not be Pareto optimal

Definition

Given k_0, a RCE is a price system $\{w_t(k_t), r_t(k_t)\}_{t=0}^{\infty}$ and allocation $\{k_{t+1}(k_t), c_t^t(k_t), c_{t+1}^t(k_t)\}_{t=0}^{\infty}$ that satisfies, for each $t \in \mathbb{N}$:

1. Consumer's lifetime utility maximization:

$$\beta \frac{U_c(c_{t+1}^t)}{U_c(c_t^t)} = \frac{1}{1+r_{t+1}}, \text{ and, } c_t^t + \frac{c_{t+1}^t}{(1+r_{t+1})} = w_t \cdot 1.$$

2. Firm's profit maximization:

$$f'(k_t) = r_t + \delta, \text{ and, } f(k_t) - k_t f'(k_t) = w_t.$$

3. Market clearing in the credit/capital market:

$$(1+n)k_{t+1} = (w_t \cdot 1 - c_t^t).$$

Recursive competitive equilibrium ...

Young-age budget constraint:

$$\begin{aligned} c_t^t &= w_t - s_t \\ &= [f(k_t) - k_t f'(k_t)] - s_t \equiv w(k_t) - s_t \end{aligned}$$

and old-age budget constraint:

$$\begin{aligned} c_{t+1}^t &= (1 + r_{t+1})s_t \\ &= [f_k(k_{t+1}) + 1 - \delta]s_t \equiv R(k_{t+1}) \cdot s_t \end{aligned}$$

Recursive competitive equilibrium (cont'd) ...

Let $R_{t+1} := R(k_{t+1})$. From Euler equation, denote for all $t \in \mathbb{N}$:

$$E(s_t, w_t, R_{t+1}) \equiv -U_c(w_t - s_t) + \beta R_{t+1} U_c(R_{t+1} s_t) = 0,$$

In words, we have:

- a necessary sequence of FOC's (Euler equations) characterizing the optimal savings trajectory $\{s_t\}_{t=0}^{\infty}$ (of all generations);
- Given (i.e. taken as parametric by consumer) market terms of trades (w_t, R_{t+1}), this Euler equation implicitly defines the solution as some function $s : \mathbb{R}^2_{++} \mapsto \mathbb{R}_+$ such that $s_t = s(w_t, R_{t+1})$.

Recall assumptions on primitive U:

- U is continuous on $\mathbb{R}_+$
- For all $c > 0$, $U_c(c) > 0$, and, $U_{cc}(c) < 0$ exist
- $\lim_{c \searrow 0} U_c(c) = +\infty$

Then the function $(w, R) \mapsto s(w, R)$, such that

$$s_t = s(w_t, R_{t+1}),$$

is well-defined and $s_w(w, R)$, and $s_R(w, R)$ exist for every $(w, R) \in \mathbb{R}^2_{++}$.

Definition (IES)

Given per-period utility function U, the intertemporal elasticity of substitution, evaluated at a point c is

$$\sigma(c) = -\frac{U_c(c)}{U_{cc}(c) \cdot c}$$

Remark: Note similarity to Arrow-Pratt measure of relative risk aversion? How?

Recursive competitive equilibrium (cont'd) ...

From Euler equation (dropping t subscripts),

$$E(s, w, R) \equiv -U_c(w - s) + \beta R U_c(Rs) = 0,$$

We can use the implicit function theorem to obtain:

$$E_s ds + E_w dw + E_R dR = 0,$$

where:

- $E_s := \partial E(s, w, R)/\partial s = U_{cc}(w - s) + \beta R^2 U_{cc}(Rs) < 0$
- $E_w := \partial E(s, w, R)/\partial w = -U_{cc}(w - s) > 0$
- $E_R := \partial E(s, w, R)/\partial R = \beta U_c(Rs) \left[1 - \frac{1}{\sigma(Rs)}\right] \lesseqqgtr 0$

Hold R constant (i.e. $dR = 0$), we have

$$s_w(w, R) = -\frac{E_w}{E_s} = \left[1 + \frac{\beta R^2 U_{cc}(Rs)}{U_{cc}(w - s)}\right]^{-1} \in (0, 1);$$

i.e. the *marginal propensity to save* out of w (equiv. lifetime income) is

- endogenous, and depends (in general) on aggregate state (relative prices) (w, R),
- is bounded in the set $(0, 1)$. Why? Because (c_t^t, c_{t+1}^t) are normal goods!

Hold w constant (i.e. $dw = 0$), we have

$$\begin{aligned} s_R(w, R) &= -\frac{E_R}{E_s} \\ &= -\frac{\beta U_c(Rs)[1 - 1/\sigma(Rs)]}{U_{cc}(w - s) + \beta R^2 U_{cc}(Rs)} \lesseqqgtr 0, \text{ if } \sigma(Rs) \lesseqqgtr 0. \end{aligned}$$

i.e. effect of the rate of return on capital on saving:

- is ambiguous ...
- depends on $\sigma(Rs) \lesseqqgtr 0$, and therefore on, specification of U.

Given an optimal savings rule (equiv. consumption demand functions), $s(w(k_t), R(k_{t+1}))$, a RCE sequence of allocations $\{s_t, c_t^t, c_{t+1}^t, k_{t+1}\}_{t\in\mathbb{N}}$ satisfies for all $t \in \mathbb{N}$:

- $s_t = s(w(k_t), R(k_{t+1}))$,
- $(1+n)k_{t+1} = s_t$,
- $c_t^t = w(k_t) - s_t$, and
- $c_{t+1}^t = R(k_{t+1})s_t$,

for $k_0 > 0$ given.

Specific Example

Exercise

1. *Derive, and therefore, show that* $s(w, R)$ *does not depend on* R *in the case of* $U(c) = \ln(c)$.
2. *Explain why this is the case. Hint: You have learned this in consumer theory from intermediate microeconomics.*
3. *Depict this in the* (c_t^t, c_t^{t+1})*-space using the geometric devices of indifference and budget sets.*

Optimality: steady states

Focus: long-run steady state.

We'll study this in three successive components:

- Long-run feasibility
- Long-run maximal consumption: the Golden Rule
- Optimal long-run: Diamond's "Golden Age"

Long-run feasibility I

Consider a long run (steady state), where per worker capital is k.

Definition (Long-run feasibility)

A steady-state $k \geq 0$ is feasible if net production at k is non-negative:

$$\phi(k) := f(k) - (\delta + n)k \geq 0.$$

Notes:

- $f(k)$: gross output at a steady state k
- $(\delta + n)k$: claims on gross output at k

Long-run feasibility II

Recall assumption:

- f continuous on $\mathbb{R}_+$
- $f_k(k) > 0, f_{kk}(k) < 0$ for all $k \in \mathbb{R}_+$
- f satisfies Inada conditions ... (What are they?!)

Long-run feasibility III

Since $f_k(k) > 0, f_{kk}(k) < 0$ for all $k \geq 0$, then:

- $\phi_k(k) = f_k(k) - (\delta + n) \lesseqqgtr 0$,
- $\phi_{kk}(k) = f_{kk}(k) < 0$;

so that $\phi(k)$ is strictly concave.

Also note that:

- $\phi(0) = f(0) \geq 0$,
- $\lim_{k \searrow 0} \phi_k(k) = \lim_{k \searrow 0} f_k(k) - (\delta + n)$, and
- $\lim_{k \nearrow \infty} \phi_k(k) = \lim_{k \nearrow \infty} f_k(k) - (\delta + n)$.

Long-run feasibility IV

Long-run feasible sets: If ...

F1. $\phi_k(k) > 0$, for all $k \geq 0$, any $k \in \mathbb{R}_+$ is long-run feasible.

F2. $\phi_k(k) < 0$, for all $k \geq 0$, and,

- **(a)** if $f(0) > 0$, then $[0, \hat{k}]$ is long-run feasible, for some $\hat{k} \in (0, \infty)$.
- **(b)** if $f(0) = 0$, then only $k = 0$ is long-run feasible.

F3. $\phi(k)$ non-monotonic. ...

... And $\exists \overline{k} \in (0, \infty)$ s.t. $f(\tilde{k}) - (\delta + n)\tilde{k} = 0$, then any $k \in (0, \overline{k})$, is long-run feasible.

Long-run feasibility V

Exercise (Long-run-feasible sets of k)

Given assumptions about $f_k > 0$, $f_{kk} < 0$, and $f(0) \geq 0$, illustrate (in two respective diagrams) the graphs of:

1. $k \mapsto f(k)$ *and* $k \mapsto (\delta + n)k$, *and therefore,*
2. $k \mapsto \phi(k)$;

and show the corresponding long-run feasible sets, if F1, F2, or F3 were to hold.

Long-run feasibility VI

Exercise (Long-run-feasible sets of k (cont'd))

The golden rule I

Consider cases:

F1. Iff $\lim_{k \nearrow \infty} f_k(k) \geq (\delta + n) \Rightarrow \lim_{k \nearrow \infty} \phi_k(k) > 0$, then ϕ is strictly increasing on $\mathbb{R}_+$.

F2. If $\phi_k(k) < 0$, then ϕ is strictly decreasing. Not interesting — largest net production is at $k = 0$: $\phi(0) \geq 0$.

F3. If $\lim_{k \nearrow \infty} f_k(k) < (\delta + n) < f_k(0)$ then ϕ is non-monotonic:

- There exists a unique $k_{GR} \in (0, \infty)$ such that $\phi_k(k_{GR}) = 0$: i.e. net production is maximized, and
- ϕ is increasing on $(0, k_{GR})$ and decreasing on (k_{GR}, ∞).

The golden rule II

Proposition (Golden rule)

Assume production function f such that $\lim_{k \nearrow \infty} f_k(k) < (\delta + n) < f_k(0)$.

Then there exists a unique $k_{GR} \in (0, \infty)$ such that $\phi_k(k_{GR}) = f_k(k_{GR}) - (\delta + n) = 0$: i.e. net production is maximized.

The golden rule III

Exercise

Illustrate the last proposition using appropriate diagrams.

The golden rule IV

Exercise

Show that the regularity condition $\lim_{k\nearrow\infty} f_k(k) < (\delta + n) < f_k(0)$ *does not apply to the Cobb-Douglas family of functions* $f(\cdot;\alpha)$, $\alpha \in (0,1)$.

Remark: However, in Cobb-Douglas $f(k;\alpha) = k^\alpha$ case with $\alpha \in (0,1)$, $k_{GR} \in (0,\infty)$ still exists.

The golden rule V

Remarks: In any steady state k,

... given regularity conditions on U and f,

... we know from the RCE conditions, (s_t, c_t^t, c_{t+1}^t) must converge to a well-defined limit (s, c^y, c^o):

- savings function, $s = s(w(k), R(k))$,
- consumption (young), $c^y = w(k) - s$, and
- consumption (old), $c^o = R(k)s$.

The golden rule VI

Therefore, the golden-rule proposition implies that there is a steady state golden-rule consumption level for each young and old agent, (c^y_{GR}, c^o_{GR}).

... The Solow-Swan golden-rule, per-se, says nothing about Pareto optimality in the long run! Why?

... What of *steady state optimality* in this model? Relation to the golden rule in this model?

Golden Age: optimal steady state I

Optimal steady state: “The Golden Age” (Diamond, 1965)

- Suppose we have the condition: $\lim_{k \nearrow \infty} f_k(k) < (\delta + n) < f_k(0)$. This is guaranteed by the Inada conditions on f.
- On a steady state path, $k_t = k$, $c_t^t = c^y$ and $c_{t+1}^t = c^o$ for all t.
- The resource constraint is then: $f(k) = (\delta + n)k + c^y + (1+n)^{-1}c^o$.

Golden Age: optimal steady state II

- A Pareto allocation of consumption across periods of life along the steady state trajectory solves:

$$\max_{(k,c^y,c^o)\in\mathbb{R}^3_+}\left\{U(c^y)+\beta U(c^o) : f(k) = (\delta+n)k+c^y+(1+n)^{-1}c^o\right\}$$

- This is still an intertemporal allocation problem, albeit stationary.

Golden Age: optimal steady state III

Characterization of Pareto-optimal steady state

1. The maximum feasible net production is attained when:

$$\phi_k(k) := f_k(k) - (\delta + n) = 0 \Rightarrow k = k_{GR}.$$

(i.e. this is just the same condition characterizing the golden-rule per-worker capital stock, at steady state!)

2. Given assumption on f such that case F3 prevails, we then know $k_{GR} \in (0, \infty)$.

Golden Age: optimal steady state IV

3. Also, the maximum of $U(c) + \beta U(c^o)$ s.t. $\phi(k) = c^y + (1+n)^{-1} c^o$ is characterized by:

$$\phi(k_{GR}) = c^y_{GR} + \frac{c^o_{GR}}{1+n},$$

and,

$$U_c(c^y_{GR}) = \beta(1+n) U_c(c^o_{GR}).$$

Golden Age: optimal steady state V

Proposition (Optimal steady state)

Given assumptions above, a unique Pareto-optimal steady state exists: k_{GR} *satisfying*

$$f_k(k_{GR}) - (\delta + n) = 0; \qquad \text{(Golden rule)}$$

and c^y_{GR} *and* c^o_{GR}*, respectively, satisfy*

$$\phi(k_{GR}) = c^y_{GR} + \frac{c^o_{GR}}{1+n}, \qquad \text{(Resource constraint)}$$

and,

$$U_c(c^y_{GR}) = \beta(1+n)U_c(c^o_{GR}). \qquad \text{(Euler equation)}$$

Optimal vs. CE arbitrage I

If we decentralized previous Pareto planning problem ...

- Given relative price (btw. young-vs-old consumption) R_{t+1}, each consumer's optimal decisions (c_t^t, c_{t+1}^t) satisfy

$$U_c(c_t^t) = \beta R_{t+1} U_c(c_{t+1}^t). \qquad \text{(Euler eqn: at CE)}$$

Optimal vs. CE arbitrage II

- Optimal arbitrage: If $R_{t+1} = (1+n)$ for all t, i.e. Samuelson's (1958) "biological return" equals market terms of trade btw (c_t^t, c_{t+1}^t), so there is a triple (c^y, c^o, k) such that

$$U_c(c^y) = \beta(1+n)U_c(c^o),$$

and the actual value of lifetime expenditure on consumption (for each agent) is

$$\begin{aligned} c^y + \frac{c^o}{1+n} &= w(k) = f(k) - f_k(k)k \\ &= f(k) - (\delta + n)k. \end{aligned}$$

Optimal vs. CE arbitrage III

- But ... at $R = 1 + n$, market clearing at steady state requires

$$(1+n)k = s[w(k), 1+n] = w(k) - c^y.$$

- If we impose the optimal allocation, setting $k = k_{GR}$, then $c^y = c^y_{GR}$ and $c^o = c^o_{GR}$, in general,

$$(1+n)k_{GR} \neq s[w(k_{GR}), 1+n].$$

Optimal steady-state path, in general, not equivalent to the competitive equilibrium steady-state path.

Optimal vs. CE arbitrage IV

Proposition (Optimal allocation and life-cycle no-arbitrage)

The optimal steady state path $(k_{GR}, c^y_{GR}, c^o_{GR})$ *satisfies:*

- *the decentralized no-arbitrage condition of each consumer where the return on saving is* $R = f_k(k_{GR}) + (1-\delta) = 1+n$*; and*
- *her life-cycle income is* $w(k_{GR}) = f(k_{GR}) - f_k(k_{GR})k_{GR}$.

But her choice of saving is generally not equal to the level of Pareto-optimal invest: $s[w(k_{GR}), 1+n] \neq (1+n)k_{GR}$.

Optimal vs. CE arbitrage V

To prove this, all we need is a counter-example.

Example ($\delta = 1$)

Let $U(c) = \ln(c)$ and $f(k) = k^{\alpha}$. Then

- $k_{GR} = [\alpha/(1+n)]^{1/(1-\alpha)}$
- $\phi(k_{GR}) = w(k_{GR}) = (1-\alpha)k_{GR}^{\alpha}$
- $c_{GR}^{y} = (1+\beta)^{-1}\phi(k_{GR})$
- $c_{GR}^{o} = (1+\beta)^{-1}[(1+n)\beta]\phi(k_{GR})$
- $s[w(k_{GR}), 1+n] = \beta(1+\beta)^{-1}\phi(k_{GR})$.

Show that at a steady state $k = k_{GR}$ it is possible that it is not consistent with a RCE.

Optimal vs. CE arbitrage VI

Example (cont'd)

Observe that:

$$s[w(k_{GR}), 1+n] \lesseqqgtr (1+n)k_{GR},$$

if and only if:

$$\frac{\beta}{(1+\beta)}(1-\alpha)k_{GR}^{\alpha} \lesseqqgtr \alpha k_{GR}^{\alpha} \Leftrightarrow \frac{\beta}{1+\beta} \lesseqqgtr \frac{\alpha}{1-\alpha}.$$

Given α, if β too large (agent's too patient), then savings exceeds golden rule capital stock. Only in special case where $\beta/(1+\beta) = \alpha/(1-\alpha)$, do the two equal.

Optimal vs. CE arbitrage VII

What is the reasoning behind RCE allocation not necessarily being an optimal one?

- FWT states that a competitive equilibrium is also Pareto optimal, as long as there exist complete markets, agents are price-takers and preferences are locally non-satiated.
- This steady state analysis showed a breakdown of what is known as the First Welfare Theorem (FWT).

Optimal vs. CE arbitrage VIII

- The problem here is that in a CE each generation's old agents do not care about the next generation's young.
- The former eats up the total dividend from and the remainder of their capital stock.
- Competitive agents do not internalize the need of moving resources intertemporally across infinitely far generations.
- They only move private resources across time (through savings) insofar as it maximizes their own lifetime utilities.

Optimal vs. CE arbitrage IX

- A planner in an optimal steady state cares about every generation and maximizes the net production subject to that being feasible; and
- Planner allocates consumption intertemporally for each generation according to the biological rate of exchange.
- Pareto planner internalizes the effect of shifting resources across infinite sequences of generations; and
- planner's optimal allocation is feasible w.r.t. resource constraint that holds over all $t \in \mathbb{N}$.

Over/under accumulation of capital I

At a steady state $\bar{k}$ of an RCE:

- If $f_k(\bar{k}) > \delta + n$, then $\bar{k} < k_{GR}$ (under-accumulation).
- If $f_k(\bar{k}) < \delta + n$, then $\bar{k} > k_{GR}$ (over-accumulation).

Over/under accumulation of capital II

Note in both cases, for a given $\bar{k}$,

- the maximum life-cycle utility satisfies: $U_c(c^y) + \beta(1+n)U_c(c^o)$, given net production fixed at $\phi(\bar{k})$,

… but …

- the life-cycle utility at the competitive steady state satisfies: $U_c(\bar{c}^y) = \beta[f_k(\bar{k}) + 1 - \delta]U_c(\bar{c}^o)$.

Over/under accumulation of capital III

Implications:

- Competitive equilibrium over- or under-accumulation of $\bar{k}$ on a steady state path is not Pareto optimal.
- E.g. if $\bar{k} > k_{GR}$ (over-accumulation):
  - possible to increase total consumption by reducing k to yield total resources per period $\phi(k)$ forever.
  - If k reduces discretionarily to k_{GR} at some period, total consumption will be $\phi(k) + (k - k_{GR})(1+n) > \phi(k)$. Total consumption in that period rises.
  - For continuation periods, the surplus is now $\phi(k_{GR})$ forever. But by definition of golden rule, $\phi(k_{GR}) > \phi(k)$. So total consumption forever is maximized.
  - Therefore total consumption for every generation can be increased at all dates by moving k towards k_{GR}.