Dynamics and Exogenous Economic Growth: the Swan-Solow Model

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Revision and Some More Detailed Analyses

Outline

Solow-Swan growth and transitional dynamics

- We begin by recalling our first example in Macroeconomics on dynamics.
- The questions in Macro we asked were:
  - What explains economic growth?
  - Why do some countries catch up in average incomes and others don't?
  - Theoretically, how long would it take for countries to catch up to each other, all else equal?
  - What explain lack of convergence in living standards?
- It turns out the simple Solow-Swan model can answer most of these questions. But some remain open.

Solow-Swan growth and transitional dynamics

- This lecture: We focus on the mechanics in the model.
- First tutorial problem set: you apply this model to answer and revisit some of the economic questions raised.

Solow-Swan growth and transitional dynamics

Recall undegraduate textbook model:

- Solow (1956, QJE) and Swan (1956, Econ Record).
- No theory of consumption choice, only fixed consumption rule.
- Dynamics arising from transition law for capital accumulation.
- Homogeneous of degree one production function.
- Competitive factor and product markets.
- Deterministic evolution of the state of the economy.
- natural time: $t \in \mathbb{N} := \{0, 1, ...\}$.

Solow-Swan recap

Consumer's *ad-hoc* consumption decision function $C : \mathbb{R}_+ \to \mathbb{R}_+$:

$$C_t = C(Y_t) = (1-s)Y_t, \qquad s \in (0,1).$$

Firm's production function $F : \mathbb{R}_+^2 \to \mathbb{R}$:

$$Y_t = F(K_t, N_t).$$

New investment flow: I_t. Law of motion for **state** of *this* economy (capital stock):

$$K_{t+1} = (1-\delta)K_t + I_t.$$

Exogenous population/labor force growth:

$$N_{t+1} = (1+n)N_t, \qquad n > -1, N_0 \text{ given.}$$

Solow-Swan recap

Assumption. F on $\mathbb{R}^2_+$ is homogenous of degree one function.

Assumption. $F \in C^2(\mathbb{R}^2_+)$. That is, F is a member of the family of twice continuously differentiable functions at all $(K, N) \in \mathbb{R}^2_+$.

Assumption. F is strictly concave, $F_K > 0, F_N > 0$, and $F_{ii} < 0$ for $i = K$ or N.

Solow-Swan recap

Definition. A **competitive equilibrium** in this economy is a sequence of allocations $\{C_t, K_{t+1}\}_{t\in\mathbb{N}}$ satisfying:

1. $C_t = C(Y_t) = (1-s)Y_t$,
2. Market clearing: $Y_t = F(K_t, N_t) = C_t + I_t$, and
3. Aggregate capital transition law: $K_{t+1} = (1-\delta)K_t + I_t$.

given initial conditions $(K_0, N_0) \in \mathbb{R}^2_+$.

Note: This is an unusual "competitive equilibrium" in the sense that there are no explicit prices. We'll come back to it later when we look at optimal growth models.

Solow-Swan recap

- We want to work with a transformed model that has a fixed steady-state.
- Exogenous trend n makes a long-run or steady-state outcome in terms of K_t or Y_t non-existent.
- So we can define variables relative to trending variable:

$$k_t := \frac{K_t}{N_t}, \qquad y_t := \frac{Y_t}{N_t}, \qquad c_t := \frac{C_t}{N_t}, \qquad i_t := \frac{I_t}{N_t}.$$

- Notational convention: let $x' := x_{t+1}$ and $x := x_t$.

Solow-Swan recap

Exercise. Show that at $(K, N) \in \mathbb{R}^2_+$,

1. $f'(k) = F_K$.
2. $F_N = f(k) - kf'(k)$.
3. $F_{KK} = f''(k)/N < 0$.
4. $F_{NN} = k^2 f''(k)/N < 0$.

Solow-Swan recap

Exercise.

1. Show that the "competitive equilibrium" can then be characterized in terms of the evolution of per-worker capital:

$$k' = \frac{sf(k)}{1+n} + \frac{(1-\delta)k}{1+n} := g(k).$$

2. Define a deterministic steady state as one where $k' = k = k^*$. Write down the relationship that solves for k^*. Also sketch this relationship in a diagram.
3. Sketch the paths of k, c, y and K, C, Y if $k_0 > k^*$.
4. Now assume $F(K, N) = AK^{\alpha}N^{1-\alpha}$, $\alpha = 1/3$, $n = 0.09$, $\delta = 0.1$, and pick any $A > 0$. Write a little program to generate the numerical outcomes for 2-3 above. Generate a finite path of length $T = 50$ (years).

Solow-Swan recap

We are now big boys and girls. We want more precise analysis of what we just drew using intuition in the phase diagram...

Existence and Uniqueness of Steady State
Definition. (Lyapunov Stability) We say that k^* is a *stable* fixed point of the map g if for an $\epsilon > 0$ there exists some $\delta \in (0, \epsilon)$ such that

$$\|k_s - k^*\| < \delta \Rightarrow \|k_t - k^*\| < \epsilon$$

for all $t \geq s$.

Note: Let state space be $X := \mathbb{R}_+$. Since $g : X \to X$, we can define the norm $\| \cdot \|$ as the usual metric $| \cdot |$. E.g. $\|b - a\| := |b - a|$ for $a, b \in \mathbb{R}_+$.

Definition. (Asymptotic Stability) The state k^* is *asymptotically stable* if it is stable and there is a $\delta > 0$ such that if $\|k_s - k^*\| < \delta$ for any $s \in \mathbb{N}$, then $\|k_t - k^*\| \to 0$ as $t \to \infty$.

Remark. The (open ball) neighborhood $B_\delta(k^*) := \{k_t \in X : \|k_t - k^*\| < \delta\}$ is the region of asymptotic stability of k^*; a.k.a. the *basin of attraction*.

Examples:

1. Periodic orbits of planets around the sun – stable, but not asymptotically stable.
2. Solow model – stable and asymptotically stable.

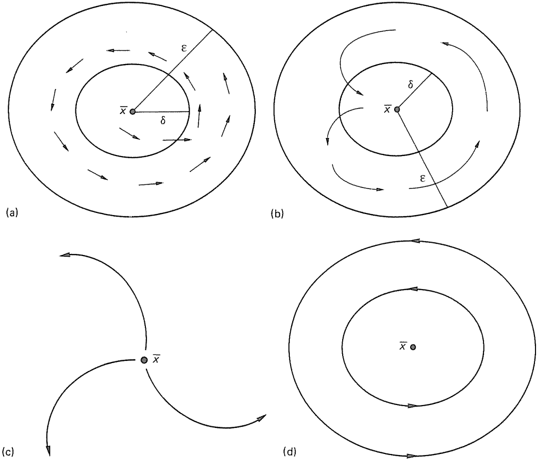

Figure 1.3 (a) $\overline{x}$ is stable; (b) $\overline{x}$ is asymptotically stable; (c) $\overline{x}$ is unstable; (d) $\overline{x}$ is stable but not asymptotically stable.

Source: Azariadis, C. (1993). *Intertemporal Macroeconomics*. Blackwell Publishers.

OK, back in the saddle again. Let's see how we can "make" the Solow-Swan model exhibit the stability properties.

Note that the equilibrium law of motion in the Solow-Swan model fits into the class of dynamical systems with an *increasing law of motion*.

Condition. (Condition PI) Let $X = \mathbb{R}_+$ and $g : X \to X$ be a function that is continuous and nondecreasing on X, and satisfies the following condition:

There is a unique $k^* > 0$ such that

1. $g(k) > k$, for all $k \in (0, k^*)$; and
2. $g(k) < k$, for all $k \in (k^*, \infty)$.

Let $\tau(k_0) := \{g^t(k_0)\}_{t=1}^{\infty}$, i.e. the trajectory from k_0.

Theorem 1. Let $X = \mathbb{R}_+$ and $g : X \to X$ be a function that is continuous and nondecreasing on X, and satisfies (PI).

Then for any $k_0 > 0$, the trajectory $\tau(k_0)$ converges to k^*. If $k < k^*$, $\tau(k_0)$ is a nondecreasing sequence. Otherwise, if $k > k^*$, $\tau(k_0)$ is a nonincreasing sequence.

Proof.

Case 1. Suppose $k_0 \in (0, k^*)$:

- Then $k_1 = g(k_0) > k_0$ since g satisfies (PI).
- Also, by (PI) of g, $g(k_0) \leq g(k^*) = k^*$.
- Note that $k_t = g^t(k_0) := g(g^{t-1}(k_0))$, for $t = 1, 2, ...$ So by induction,

  $$0 < k_0 < g^t(k_0) \leq k^*.$$

  So $\{k_t\}$ is nondecreasing and bounded above by k^*.
- So there is a $\tilde{k}$ and $\delta > 0$ such that if $\|k_s - \tilde{k}\| < \delta$ for any $s \in \mathbb{N}$, then $\|k_t - \tilde{k}\| \to 0$ as $t \to \infty$. Or, $\Rightarrow \lim_{t\to\infty} k_t = \tilde{k}$ exists.
- Since g continuous on X, $\tilde{k} = g(\tilde{k}) > 0$.
- By (PI) of g, $k^* = g(k^*)$ is unique. So then $\tilde{k} = k^*$.

Proof (continued)

Case 2. Suppose $k_0 = k^*$. Then $k_t = k^* = g(k^*)$ for all $t \in \mathbb{N}$.

Case 3. Suppose $k_0 \in (k^*, \infty)$. Exercise.

$\blacksquare$

A sufficient condition for Condition PI to hold.

Lemma 1. (Uzawa-Inada condition) Let state space $X = \mathbb{R}_+$, and, the self-map $g : X \to X$ be a function that is continuous and nondecreasing on X, and satisfies the Uzawa-Inada condition:

1. $G(k) := \frac{g(k)}{k}$, is decreasing in $k > 0$;
2. for some $\underline{k} > 0$, $G(\underline{k}) > 1$; and
3. for some $\overline{k} > \underline{k} > 0$, $G(\overline{k}) < 1$.

Then the condition (PI) holds.

Proof.

- Let $I = (\underline{k}, \overline{k})$.
- Since g is continuous on X, by the intermediate value theorem, $\exists k^* \in I$, s.t.

$$G(k^*) = 1 \Leftrightarrow g(k^*) = k^* > 0.$$

- Since we assumed $G(k)$ is decreasing,

$$G(k) \begin{cases} > 1 & \text{for } k < k^* \\ < 1 & \text{for } k > k^* \end{cases} \quad \Rightarrow \quad g(k) \begin{cases} > k & \text{for } k \in (0, k^*) \\ < k & \text{for } k \in (k^*, \infty) \end{cases}.$$

- That is, condition (PI) holds.

$\blacksquare$

Theorem 2. Let $X : \mathbb{R}_+$ and $g : X \to X$ be a function that is continuous on X, twice continuously differentiable at $k > 0$, and satisfying

A1 $\lim_{k \searrow 0} g'(k) = 1 + c_1, c_1 > 0$,

A2 $\lim_{k \nearrow \infty} g'(k) = 1 - c_2, c_2 > 0$, and

A3 $g'(k) > 0, g''(k) < 0$ at $k > 0$.

Then the Uzawa-Inada condition holds.

* Conditions 1 and 2 place upper and lower bounds on first derivatives. Condition 3 assumes strict concavity.

Proof.

- Consider an interval $I = (\underline{k}, \overline{k})$, s.t. $\overline{k} > \underline{k} > 0$.
- By the mean value theorem, $\exists k \in I$ such that

$$g'(k) = \frac{g(\overline{k}) - g(\underline{k})}{\overline{k} - \underline{k}}.$$

- Since $g'(k) > 0$, then $g(\overline{k}) > g(\underline{k})$.
- Since $\overline{k} > \underline{k} > 0$, there is a $t \in (0,1)$ s.t. $\underline{k} = t\overline{k} + (1-t)0$.
- Since $g''(k) < 0$ (strict concavity), then

$$\begin{aligned} g(\underline{k}) &= g(t\overline{k} + (1-t)0) \\ &> tg(\overline{k}) + (1-t)g(0) \\ &\geq tg(\overline{k}). \end{aligned}$$

Proof (continued).

- Since $\underline{k} = t\overline{k}$, we have

$$\frac{g(\underline{k})}{\underline{k}} > \frac{tg(\overline{k})}{t\overline{k}} = \frac{g(\overline{k})}{\overline{k}},$$

  so $G(k) := \frac{g(k)}{k}$ is decreasing in $k > 0$ (UI, #1).

- By A1-A3, $\exists \underline{k} > 0$ s.t. $g'(k) > 1$ for all $k \in (0, \underline{k}]$.
- By m.v.t. again, $\exists k \in (0, \underline{k})$ s.t.

$$g(\underline{k}) = g(0) + g'(k)\underline{k} \geq g'(k)\underline{k},$$

  or

$$G(\underline{k}) := \frac{g(\underline{k})}{\underline{k}} \geq g'(\underline{k}) > 1.$$

  i.e. (UI, #2).

Proof (continued).

- Now to proof that (UI #3) holds, there are two possible cases, depending on whether g is bounded function or not.
- Case 1. If g is bounded above, viz. $\exists N > 0$ s.t. $g(k) \leq N \Rightarrow g(k)/k \leq N/k$.
- So $\exists \overline{k} \in X$ sufficiently large s.t.

$$\frac{g(\overline{k})}{\overline{k}} \leq \frac{N}{\overline{k}} < 1.$$

. So we have (UI #3). Or, ...

- Case 2. If g not bounded above, we can find a sequence of points $\{k_t\}$ such that $g(k_t) \to \infty$ as $t \to \infty$.
- Then, using L'Hôpital's rule,

$$\lim_{t\to\infty} \frac{g(k_t)}{k_t} = \lim_{t\to\infty} g'(k_t).$$

- So we can find a $\overline{k}$ sufficiently large s.t. $G(\overline{k}) := \frac{g(k_t)}{k_t} < 1$. So we have (UI #3).

∎

Taking Stock

What have we done so far? Let's recount our step.

We want to verify that our model, summarized by a self-map $g : X \to X$, has the nice properties of unique steady state and global or asymptotic stability.

Step 1. Verify Theorem 2 on model. Check that g satisfies the Uzawa-Inada (UI) property.

Step 2. If (UI) condition holds, then (PI) holds (Lemma 1).

Step 3. If (PI) holds, then exists unique steady state *and* steady state is asymptotically stable. (Theorem 1).

Solow-Swan: Existence and Uniqueness of Steady State

And now the application to Solow-Swan!

Proposition. Assume $\lim_{k \to 0} f'(k) = \infty$ and $\lim_{k \to \infty} f'(k) = 0$. There exists a unique $k^* > 0$ such that

$$k^* = g(k^*).$$

If $k < k^*$, $\tau(k_0)$ is nondecreasing and converges to k^*. If $k > k^*$, $\tau(k_0)$ is nonincreasing and converges to k^*.

Proof. Exercise!

Summary

What have we done?

- Recap on undergrad Solow-Swan model.
- First encounter of a dynamical economic model.
- Basis of much modern macro-models.
- Study assumptions of model.
- Informal analysis of model dynamics and stability (undergrad stuff!).
- Formal analysis of model dynamics and stability.

Next, we build on this framework, as did Cass and Koopmans (1965), and subsequent modern business cycle models using decision theory and microfoundations.

Appendix

Theorem. (Bolzano's Intermediate Value Theorem) Let I be an interval and let $f : I \to \mathbb{R}$ be continuous on I. If $a, b \in I$ and if $d \in \mathbb{R}$ satisfies $f(a) < d < f(b)$, then there exists a point $c \in I$ such that $f(c) = d$.

Theorem. (Mean Value Theorem) Let f be a continuous function on $I := [a, b]$, and f has a derivative in the open interval (a, b). Then there exists at least one point $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a).$$