OLG: Economic Policy (Part 2)

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Outline

1. Signpost
2. Lump-sum transfers
   - RCE with lump-sum transfers
3. Optimal Allocation
   - Modified steady-state optimum and golden rule
4. 2nd Welfare Theorem

Overview

- Previously, we considered long-run steady state optimum and competitive equilibria.
- Now, we consider dynamic equilibria, and, dynamic Pareto-optimal allocations.
- Two redistributive policy settings:
  1. Decentralization of Pareto allocation if lump sum taxes available: Second Welfare Theorem
  2. Lump-sum transfers and pensions; effect on capital accumulation:
     - Unfunded pensions: PAYG social security
     - Fully funded social security

RCE with lump-sum transfers I

Recall our intermediate goal ...

- Extend previous OLG model: now assume $\exists$ a transfer system in place:
  - Lump sum taxes on young: a_t
  - Lump sum taxes on old: z_t
- Use this extended vehicle to study various transfer (fiscal) policies.
- Consider for now, *per-period* balanced-budget policies.

RCE with lump-sum transfers II

Definition (RCE recap)

Given k_0 and a sequence of lump-sum transfers $\{a_t\}_{t\in\mathbb{N}}$, a RCE (with perfect foresight) and lump-sum transfers is a sequence of allocations $\{k_{t+1}\}_{t\in\mathbb{N}}$ and relative prices $\{R_{t+1}, w_t\}_{t\in\mathbb{N}}$ such that for all $t \in \mathbb{N}$,

1. $w_t = f(k_t) - f'(k_t)k_t \equiv w(k_t)$;
2. $(1+n)k_{t+1} = \tilde{s}\left(w_t - a_t, z^e_{t+1}, R^e_{t+1}\right) > 0$;
3. $R^e_{t+1} = R_{t+1} = f'(k_{t+1}) + 1 - \delta$; and
4. $z^e_{t+1} = (1+n)a_{t+1}$.

RCE with lump-sum transfers III

- Conditions 1 and 3: firm maximizes profit
- Condition 2: Capital market clears
- Condition 4: Transfer system's (or "government") budget constraint satisfied

Pareto-optimal Allocation I

- As a benchmark, we consider what a Pareto planner would do.
- We won't fully solve for the Pareto-optimal trajectory.
- We'll just characterize the necessary conditions for a path to be Pareto optimal.

Goal: We will use this part later on when we consider whether such a Pareto-optimal allocation can be decentralized through market allocations—i.e. through competitive equilibrium.

Pareto-optimal Allocation II

Suppose, more generally, we have a Pareto planner who:

- Discounts different generations' payoff by a factor $\gamma \in (0, 1)$
- Maximizes the total lifetime payoff of all generations
- Faces resource constraint

Pareto-optimal Allocation III

A little notational trick for convenience

Denote total resources at state k as

$$\tilde{f}(k) :=: f(k) + (1-\delta)k.$$

Pareto-optimal Allocation IV

The planner's problem is thus:

$$\max_{c_0^o, \{c_t^y, c_{t+1}^o, k_{t+1}\}_{t\geq 0}} \Bigg\{ \beta\gamma^{-1}U(c_0^o) + \sum_{t=0}^{\infty} \gamma^t \left[U(c_t^y) + \beta U(c_{t+1}^o)\right] :$$

$$c_t^o = (1+n)\left[\tilde{f}(k_t) - (1+n)k_{t+1} - c_t^y\right], \forall t \geq 0$$

$$k_0 \text{ given} \Bigg\}$$

Interpretation of $\{\gamma^t : t \in \mathbb{N}\}$: Importance a planner attaches to a date-t generation's lifetime welfare.

Pareto-optimal Allocation V

Initial (date-0) old can be thought of as being born at date $t = -1$, and their lifetime utility is $U(c^y_{-1}) + \beta U(c^o_0)$.

The planner attaches the Pareto weight $\gamma^t = \gamma^{-1}$ to them.

Since the planner's problem starts from date $t = 0$, then the date -1 young agent's utility $\gamma^{-1} U(c^y_{-1})$ is ignored in the planner's objective function.

Hence the first term in the objective function is $\beta\gamma^{-1} U(c^o_0)$, which is the Pareto-weighted date-0 welfare of old agents.

Pareto-optimal Allocation VI

Equivalently, the planner's problem can be written as:

$$\max_{\{c_t^y, k_{t+1}\}_{t\geq 0}} \left\{ \sum_{t=0}^{\infty} \gamma^t \left[U(c_t^y) + \beta\gamma^{-1} U(c_t^o) \right] : \right.$$

$$c_t^o = (1+n)\left[\tilde{f}(k_t) - (1+n)k_{t+1} - c_t^y\right], \forall t \geq 0$$

$$\left. k_0 \text{ given} \right\}$$

Pareto-optimal Allocation VII

If you're worried what this looks like ... try expanding out the objective function, i.e. the infinite sum ...

This space was brought to you by one less tree.

Pareto-optimal Allocation VIII

An interior optimal allocation satisfies the FONCs:

$$U'(c_t^y) = \beta\gamma^{-1}(1+n)U'(c_t^o),$$

and,

$$U'(c_t^o) = \frac{\tilde{f}'(k_{t+1})\gamma}{1+n}U'(c_{t+1}^o),$$

and,

$$\tilde{f}(k_t) - (1+n)k_{t+1} - c_t^y - \frac{c_t^o}{1+n} = 0.$$

for all $t \geq 0$.

Pareto-optimal Allocation IX

What do these necessary conditions say?

1. Intra-temporal Optimal allocation of (c_t^y, c_t^o) between current young and current old.
   - Equate *planner's* $MRS(c_t^y, c_t^o; \beta)$ to biological return $(1+n)$.
2. Inter-temporal Optimal allocation of (c_t^o, c_{t+1}^o) between current old and future old.
   - Equate *planner's* $MRS(c_t^o, c_{t+1}^o; \gamma)$ to population growth discounted return of capital, $\tilde{f}'(k_{t+1})/(1+n)$.
3. These two intra- and intertemporal trade-offs must also be feasible (resource constraint must hold), for all $t \in \mathbb{N}$.

Pareto-optimal Allocation X

Combining the intra- and inter-temporal optimal trade-offs:

$$U'(c_t^y) = \beta U'(c_{t+1}^o)\tilde{f}'(k_{t+1}).$$

Optimal planner's trade-off for each generation:

- within each generation's lifetime, the planner commands that each *agent's* $MRS(c_t^y, c_{t+1}^o; \beta)$ equals the marginal rate of transformation, $MRT(c_t^y, c_{t+1}^o) = \tilde{f}'(k_{t+1})$.
- identical to what individual agents would choose if they expected the gross return on saving, $R_{t+1}^e = \tilde{f}'(k_{t+1})$.

Pareto-optimal Allocation XI

Remarks:

- We characterized necessary conditions for a trajectory (or allocation path) to be Pareto optimal.
- These are necessary but not sufficient conditions.
- A sufficient condition also requires an infinite-horizon version of a boundary/terminal condition for pinning down the trajectory that satisfies the planner's FONC.
  - "Transversality condition": $\lim_{t\to+\infty}\gamma^t U'(c_t^y)\tilde{f}'(k_t)k_t = 0$.
  - Intuitively, in the limit of the indefinite future, the marginal utility value of capital income should go to zero.
  - Mathematically, the planner's optimal allocation is a solution to a second order difference equation in k_t. Requires two boundary conditions.
- We won't attempt to solve for the Pareto allocation here. It requires dynamic programming tools.

Modified steady-state optimum and golden rule I

- Earlier we consider the golden rule and its relation to Diamond's golden age.
- Now, if we consider a steady state consistent with our γ-planner ...
- ... we will derive a version of this called the modified golden rule, and its corresponding steady state optimum.

Modified steady-state optimum and golden rule II

Consider steady-state path such that $(c_t^y, c_t^o, k_{t+1}) = (c^y, c^o, k)$ for all $t \geq 0$.

- Then we have:

$$U'(c^y) = \beta \tilde{f}'(k) U'(c^o)$$

- and, the *modified golden rule*

$$\tilde{f}'(k) = \gamma^{-1}(1+n).$$

Modified steady-state optimum and golden rule III

- so together, the optimal arbitrage between young- and old-age consumption for each generation is described by:

$$U'(c^y) = \gamma^{-1}\beta(1+n)U'(c^o),$$

along the modified golden rule steady state trajectory.

Modified steady-state optimum and golden rule IV

$$U'(c^y) = \gamma^{-1}\beta(1+n)U'(c^o),$$

In words: At planner's steady-state solution ...

- planner commands that each generation's (steady-state) intertemporal $MRS(c_t^y, c_{t+1}^o) \equiv MRS(c^y, c^o)$ to equal the planner's discount factor, adjusted for populations growth, $\gamma/(1+n)$.
- This coincides with the best-response of a consumer when the gross return on capital is $(1+n)/\gamma$, ...

  ... i.e. when the per-worker capital stock is at the modified golden rule.

Second Welfare Theorem I

Now we are ready to study:

- competitive equilibrium, lump-sum transfers ...
- its relation to the γ-planner's optimal allocation ...
- a version of the Second Welfare Theorem of general equilibrium

Second Welfare Theorem II

Proposition

For any feasible allocation $\{c_t^y, c_t^o, k_{t+1}\}_{t \geq 0}$ *beginning from* $k_0 = \breve{k}_0$*, which satisfies for all* $t \geq 0$*:*

$$U'(c_t^y) = \beta U'(c_{t+1}^o)[f'(k_{t+1}) + 1 - \delta],$$

there exists a sequence of lump sum transfers $\{a_t\}_{t \geq 0}$ *such that this trajectory is a perfect-foresight recursive competitive equilibrium.*

Second Welfare Theorem III

Proof:

- Suppose for all $t \in \mathbb{N}$,

$$a_t = \frac{z_t}{1+n} = \frac{c_t^o - \tilde{f}'(k_t)(1+n)k_t}{1+n}.$$

Where does this conjectured lump-sum tax amount come from?

Second Welfare Theorem IV

- The transfer a_t from current young allows the *current old* the ability to consume:

$$\begin{aligned} c_t^o &= \tilde{f}'(k_t)s_{t-1} + (1+n)a_t \\ &= \tilde{f}'(k_t)(1+n)k_t + (1+n)a_t. \end{aligned}$$

- From resource constraint:

$$\begin{aligned} 0 &= \tilde{f}(k_t) - (1+n)k_{t+1} - c_t^y - \frac{c_t^o}{1+n} \\ \Rightarrow a_t &= w(k_t) - c_t^y - (1+n)k_{t+1}, \end{aligned}$$

where $w(k) = \tilde{f}(k) - \tilde{f}'(k)k = f(k) - f'(k)k$.

Second Welfare Theorem V

- Agents take $w(k_t)$, $\tilde{f}'(k_{t+1})$, a_t, and z_{t+1} as exogenous to their decisions.
- Under perfect-foresight equilibrium, beliefs are such that, $R^e_{t+1} = \tilde{f}'(k_{t+1})$ and $z^e_{t+1} = z_{t+1}$, at any date t, at given k_t.
- Given these forecasts, the optimal decisions of the time-t young agents $(\breve{c}^y_t, \breve{c}^o_{t+1}, \breve{s}_t)$ satisfy their FONCS:

$$\begin{aligned}
U'(\breve{c}^y_t) &= \beta U'(\breve{c}^o_{t+1})\tilde{f}'(k_{t+1}) \\
\breve{c}^y_t &= w(k_t) - a_t - \breve{s}_t \\
&= c_t + (1+n)k_{t+1} - \breve{s}_t \\
\breve{c}^o_{t+1} &= z_{t+1} + \tilde{f}'(k_{t+1})\tilde{s}_t \\
&= c^o_{t+1} - \tilde{f}'(k_{t+1})(1+n)k_{t+1} + \tilde{f}'(k_{t+1})\breve{s}_t.
\end{aligned}$$

Second Welfare Theorem VI

- In a perfect-foresight RCE, market clearing must also hold for all $t \geq 0$, so that $(1+n)k_{t+1} = \breve{s}_t$.
- Use this fact in the agents' FONCs.
- For the first old generation, we have $\breve{c}_0^o = \tilde{f}'(k_0)(1+n)k_0 + z_0 = c_0^o$ by definition of z_0.
- Therefore there is a RCE under a lump-sum transfer system, such that $\breve{c}_t^o = c_t^o$ and $\breve{c}_t^y = c_t^y$ for all dates $t \geq 0$.

$\blacksquare$

Second Welfare Theorem VII

The last proposition states that:

- There always exists transfers ...
- ... that allow for the decentralization of a feasible allocation ...
- ... and that these transfers satisfy the intertemporal arbitrage condition:

$$U'(c_t^y) = \beta U'(c_{t+1}^o)\tilde{f}'(k_{t+1}).$$

Second Welfare Theorem VIII

- Now ...
  1. All Pareto-optimal allocations (or trajectories), by construction, are feasible ...
  2. and they satisfy the intertemporal arbitrage condition.
- Therefore, we have the following theorem as a consequence ...

Theorem

For any Pareto-optimal trajectories $\{c_t^y, c_t^o, k_{t+1}\}_{t\geq 0}$*, there exists a sequence of lump sum transfers* $\{a_t\}_{t\geq 0}$ *such that this trajectory is a perfect-foresight recursive competitive equilibrium.*