Chapter 5 Steady Electric Currents

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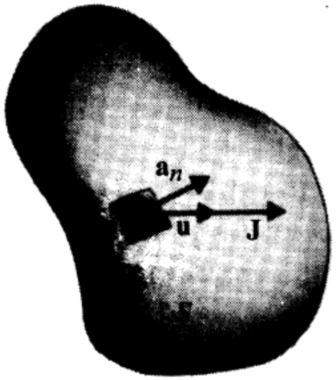
§ 5-1 Introduction

Types of electric currents:

- **Conduction current**: Drift motion of conduction electrons and/or holes in conductors and semiconductors; very low for the average drift velocity of the electrons (1/1000 m/s);
- **Electrolytic (电解) current**: Migration of positive and negative ions;
- **Convection current**: Motion of electrons and/or ions (positively or negatively charged particles) in vacuum or rarefied gas; not governed by Ohm's law.

§ 5-2 Current density and Ohm's law

Consider a steady motion of one kind of charges q over a differential surface Δs with velocity $\boldsymbol{u}$ and the total charges ΔQ passing this surface in a time interval Δt is



$$\Delta Q = Nq\mathbf{u}\cdot\mathbf{a}_n\,\Delta s\,\Delta t \qquad \text{(C)}.$$

Total current over the differential surface Δs:

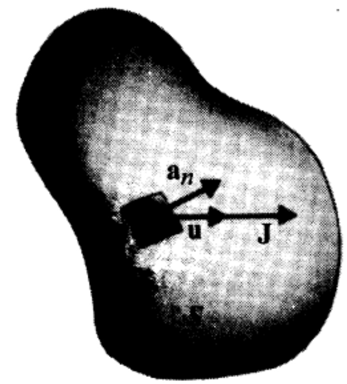
$$\Delta I = \frac{\Delta Q}{\Delta t} = Nq\mathbf{u}\cdot\mathbf{a}_n\,\Delta s = Nq\mathbf{u}\cdot\Delta\mathbf{s} \qquad \text{(A)}.$$

Current density: $\mathbf{J} = Nq\mathbf{u} \qquad (\mathrm{A/m^2}),$ $\Longrightarrow$ $\Delta I = \mathbf{J}\cdot\Delta\mathbf{s}.$

N: number of charge carriers per unit volume

The total current I flowing through an arbitrary surface S:

$$\boxed{I = \int_S \mathbf{J} \cdot d\mathbf{s} \qquad \text{(A).}}$$



Noting that the charge density $\rho = Nq$, we have

$$\boxed{\mathbf{J} = \rho\mathbf{u} \qquad (\text{A/m}^2),} \quad \Leftarrow \quad \mathbf{J} = Nq\mathbf{u}$$

Which is the relation between the ***convection current density*** and the ***velocity*** of the charge carrier

In the case of conduction currents there may be more than one kind of charge carriers (electrons, holes, and ions) drifting with different velocities. Equation (5–3) should be generalized to read

$$\mathbf{J} = \sum_i N_i q_i \mathbf{u}_i \qquad (\mathrm{A/m^2}).$$

- Drift motion of charge carriers under electric field
- Atoms remain neutral (ρ=0)

For most of metallic conductors in which the average electron drift is proportional to electric field, we write

$$\mathbf{u} = -\mu_e \mathbf{E} \qquad (\mathrm{m/s}),$$

Electron mobility μ_e:

Cu: $3.2 \times 10^{-3}\ (\mathrm{m^2/V \cdot s})$

Al: $1.4 \times 10^{-4}\ (\mathrm{m^2/V \cdot s})$

Then we have,

$$\mathbf{u} = -\mu_e \mathbf{E}$$

$$\mathbf{J} = -\rho_e \mu_e \mathbf{E}; \qquad \mathbf{J} = \rho \mathbf{u}$$

Where ρ_e=-Ne is the charge density of the drifting electrons. We rewrite it by

$$\boxed{\mathbf{J} = \sigma \mathbf{E} \qquad (\mathrm{A/m^2}),}$$

Conductivity, σ = -$\rho_e \mu_e$; a macroscopic parameter of a medium;
Unit: amper per volt-meter (A/Vm) or siemens per meter (S/m)

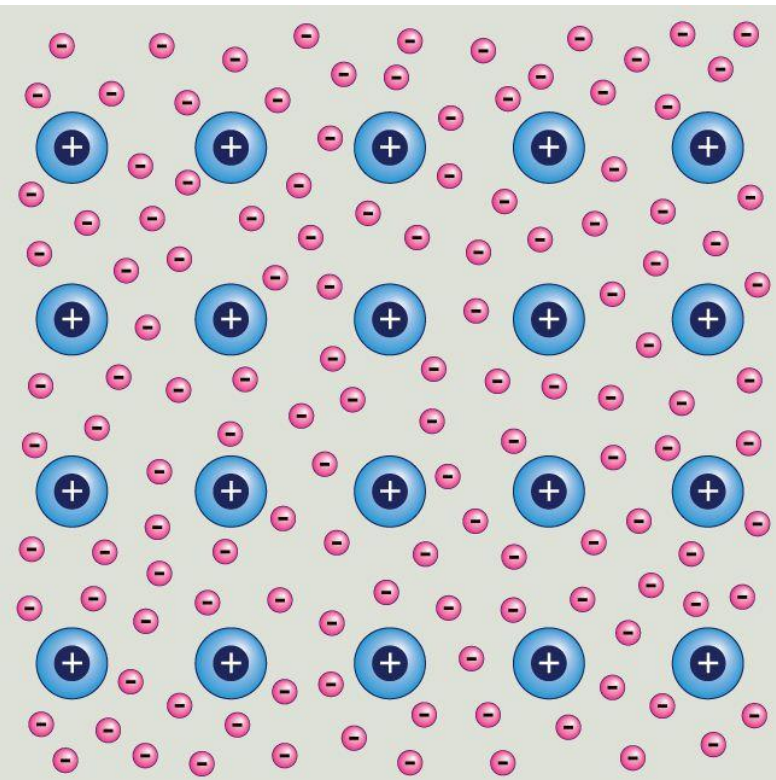
For semiconductors, conductivity depends on the concentration and mobility of both electrons and holes:

$$\sigma = -\rho_e \mu_e + \rho_h \mu_h, \qquad \text{Graphene: } 1.5\ (\mathrm{m^2/V\ s}) \qquad (5\text{–}22)$$

where the subscript h denotes hole. In general, $\mu_e \neq \mu_h$. For germanium, typical values are $\mu_e = 0.38$, $\mu_h = 0.18$; for silicon, $\mu_e = 0.12$, $\mu_h = 0.03$ $(\mathrm{m^2/V \cdot s})$.

Drude model: conductivity for metals
(Microscopic Ohm's law)

Metal and Free electron gas



Nuclei fixed; electron are free moving

Harmonic oscillation of electrons under the excitation of electric fields

$$m_e \vec{a}_e = \vec{F}_{E\_local} + \vec{F}_{damping}$$

$$m_e \frac{d\vec{u}}{dt} = -e\vec{E}_L e^{-i\omega t} - m_e \gamma \vec{u}$$

Collision frequency

$$\Rightarrow \quad \vec{u} = \frac{e}{i m_e \omega - m_e \gamma} \vec{E}_L e^{-i\omega t}$$

$$\Rightarrow \quad \vec{J} = -Ne\vec{u} = \left(\frac{-Ne^2}{i m_e \omega + m_e \gamma}\right) \vec{E}_L e^{-i\omega t}$$

$$\boxed{\Rightarrow \quad \sigma(\omega) = \frac{Ne^2}{-i m_e \omega + m_e \gamma} = \frac{Ne^2}{m_e \gamma} \frac{1}{(1 - i\omega/\gamma)}}$$

Conductivity (static)

$$\boxed{s(0) = \frac{Ne^2}{m_e g}}$$

Material	Conductivity, σ(S/m)	Material	Conductivity, σ(S/m)
Silver	6.17×10^7	Fresh water	10^{-3}
Copper	5.80×10^7	Distilled water	2×10^{-4}
Gold	4.10×10^7	Dry soil	10^{-5}
Aluminum	3.54×10^7	Transformer oil	10^{-11}
Brass	1.57×10^7	Glass	10^{-12}
Bronze	10^7	Porcelain	2×10^{-13}
Iron	10^7	Rubber	10^{-15}
Seawater	4	Fused quartz	10^{-17}

Resistivity: the reciprocal of conductivity (Ω•m)

Count Alessandro Giuseppe Antonio Anastasio Volta, 1745--1827, Italy

André-Marie Ampère, 1775--1836, France

Georg Simon Ohm, 1789—1854, German

Resistance and conductance

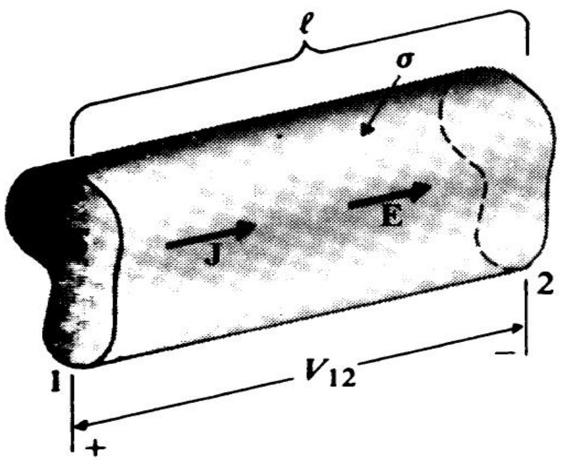


Fig. 5-3 Homogeneous conductor with a constant cross section

Voltage between terminals 1 and 2:

$$V_{12} = E\ell$$

The total current:

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} = JS$$

Point or microscopic form of Ohm's law

$$\boxed{\mathbf{J} = \sigma \mathbf{E}}$$

$$\frac{I}{S} = \sigma \frac{V_{12}}{\ell}$$

$$V_{12} = \left(\frac{\ell}{\sigma S}\right) I = RI,$$

Macroscopic Ohm's law

Resistance

$$\boxed{R = \frac{\ell}{\sigma S} \qquad (\Omega).}$$

Resistance and conductance

The ***conductance***, G, or the reciprocal of resistance, is useful in combining resistances in parallel. The unit for conductance is (Ω^{-1}), or siemens (S).

$$G = \frac{1}{R} = \sigma \frac{S}{\ell} \qquad \text{(S)}. \tag{5–28}$$

From circuit theory we know the following:

a) When resistances R_1 and R_2 are connected in series (same current), the total resistance R is

$$\boxed{R_{sr} = R_1 + R_2.} \tag{5–29}$$

b) When resistances R_1 and R_2 are connected in parallel (same voltage), we have

$$\boxed{\frac{1}{R_{||}} = \frac{1}{R_1} + \frac{1}{R_2}} \tag{5–30a}$$

§ 5-3 Electromotive force and Kirchhoff's voltage law

Ohmic material under static electric field

$$\oint_C \frac{1}{\sigma} \mathbf{J} \cdot d\boldsymbol{\ell} = 0.$$

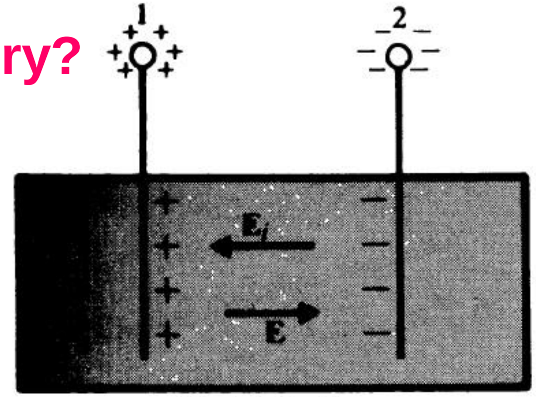
Static electric field is **conservative**

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = 0.$$

A steady current cannot be maintained in the same direction in a closed circuit by an electrostatic field. ***(D. K. Cheng, p. 206)***

(steady: motion and constant velocity; static: no motion)

Electric fields inside an electric battery?



Electric battery

Impressed(外加) electric field, E_i

- non-conservative electric field caused by chemical action;

- **Electromotive force (emf)** is the integral from the negative to the positive electrode **inside the battery**.

$$\boxed{\mathscr{V} = \int_2^1 \mathbf{E}_i \cdot d\ell = -\int_2^1 \mathbf{E} \cdot d\ell.}$$

The conservative electrostatic field satisfies

$$\oint_C \mathbf{E} \cdot d\ell = \underbrace{\int_1^2 \mathbf{E} \cdot d\ell}_{\text{Outside the source}} + \underbrace{\int_2^1 \mathbf{E} \cdot d\ell}_{\text{Inside the source}} = 0.$$

emf: $$\mathscr{V} = \underbrace{\int_1^2 \mathbf{E} \cdot d\ell}_{\text{Outside the source}} = V_{12} = V_1 - V_2.$$

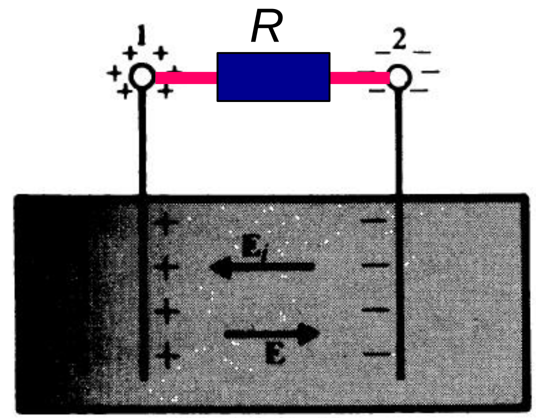
Two different electric fields:

(1) **Conservative fields** from the charges, both inside and outside the battery;

(2) **Nonconservative fields-impressed fields**, from chemical reaction, only inside the battery

Connect the battery electrodes by resistance *R*

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{E}_i),$$



Electric battery

Emf: (integral over the closed loop)

$$\mathcal{V} = \oint_C (\mathbf{E} + \mathbf{E}_i) \cdot d\boldsymbol{\ell} = \oint_C \frac{1}{\sigma} \mathbf{J} \cdot d\boldsymbol{\ell}.$$

$$\boxed{R = \frac{\ell}{\sigma S}}$$

$$\boxed{J = I/S}$$

- For one source of electromagnetic force $\mathcal{V} = RI.$
  (if the internal resistance is negligible),
- For more than one source of electromagnetic force and more than one resistor,

Kirchhoff's voltage law

$$\boxed{\sum_j \mathcal{V}_j = \sum_k R_k I_k \qquad \text{(V)}.}$$

Around a closed path in an electric circuit, the algebraic sum of the emf's (voltage rises) is equal to the algebraic sum of the voltage drops across the resistances.

§ 5-4 Equation of continuity and Kirchhoff's current law

From the ***principle of conservative of charge***, the current leaving one region is the total outward flux of the current density through the enclosing surface *S*,

$$I = \oint_S \mathbf{J} \cdot d\mathbf{s} = -\frac{dQ}{dt} = -\frac{d}{dt}\int_V \rho \, dv.$$

Apply divergence theorem, we obtain for a stationary volume

$$\int_V \nabla \cdot \mathbf{J} \, dv = -\int_V \frac{\partial \rho}{\partial t} \, dv.$$

(ρ is a time-space function)

The above equation holds regardless of V, and the integrands must be equal.

Equation of continuity

$$\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \qquad (\mathrm{A/m^3}).}$$

The charge conservation law can also be given as follows

The electric charge density $\rho = \rho(\vec{x}(t), t)$

$$0 = \frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \frac{d\vec{x}}{dt} \cdot \frac{\partial\rho}{\partial\vec{x}} = \frac{\partial\rho}{\partial t} + \vec{v} \cdot \frac{\partial\rho}{\partial\vec{x}} = \frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\vec{v})$$

$$= \frac{\partial\rho}{\partial t} + \nabla \cdot \vec{J}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \frac{\partial}{\partial\vec{x}}$$

Note: For such a function $\rho = \rho(\vec{x}(t), t)$, $\frac{d}{dt} \neq \frac{\partial}{\partial t}$.

$\frac{d}{dt}$ is the total derivative, while $\frac{\partial}{\partial t}$ is the partial derivative.

For steady currents, $\partial\rho/\partial t = 0.$

$$\nabla \cdot \mathbf{J} = 0. \quad \Longleftarrow \quad \nabla \cdot \mathbf{J} = -\frac{\partial\rho}{\partial t}$$

Steady electric currents are divergenceless or solenoidal. Over any enclosed surface, we have

$$\oint_S \mathbf{J} \cdot d\mathbf{s} = 0,$$

Which can be written as

Kirchhollf's current law

$$\boxed{\sum_j I_j = 0 \qquad \text{(A).}}$$

The algebraic sum of all currents flowing out of a junction in an electric circuit is zero.

Relaxation time of charges in a metal?

Combining the Ohm's law with the continuity equation and assuming σ, we have

$$\sigma \nabla \cdot \mathbf{E} = -\frac{\partial \rho}{\partial t}. \quad \longleftarrow \quad \nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

In a simple medium, $\nabla \cdot \mathbf{E} = \rho/\epsilon$,

$$\frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon}\rho = 0.$$

The solution is

$$\boxed{\rho = \rho_0 e^{-(\sigma/\epsilon)t} \qquad (\mathrm{C/m^3}),}$$

Relaxation time, τ

$$\tau = \frac{\epsilon}{\sigma} \qquad (\mathrm{s}).$$

Eg: Copper
$\tau = 1.52 \times 10^{-19}\mathrm{s}$

The charge density at a given location will decrease with time exponentially.

§ 5-5 Power dissipation and Joule's law

Power p provided by an electric field $\boldsymbol{E}$ in moving a charge q:

$$p = \lim_{\Delta t \to 0} \frac{\Delta w}{\Delta t} = q\mathbf{E} \cdot \mathbf{u},$$

Total power P delivered to all charge carriers in a volume dv:

$$dP = \sum_i p_i = \mathbf{E} \cdot \Big(\underbrace{\sum_i N_i q_i \mathbf{u}_i}_{= \boldsymbol{J} \text{ (current density)}} \Big) \, dv,$$

$$dP = \mathbf{E} \cdot \mathbf{J} \, dv$$

or

$$\frac{dP}{dv} = \underbrace{\mathbf{E} \cdot \mathbf{J}}_{= \text{power density}} \qquad (\mathrm{W/m^3}).$$

Total power converted into heat in a volume V

Joule's law

$$\boxed{P = \int_V \mathbf{E} \cdot \mathbf{J} \, dv}$$

In a conductor of a constant cross section, *dv=ds•dl*, with *dl* measured in the current direction

$$\boxed{P = \int_V \mathbf{E} \cdot \mathbf{J} \, dv} \Rightarrow \boxed{\begin{aligned} &P = \int_L E \, d\ell \int_S J \, ds = VI, \\ &P = I^2 R \qquad \text{(W)}. \end{aligned}}$$

$p = \vec{J} \cdot \vec{E} = \frac{J^2}{\sigma} = \sigma E^2$ **is the microscopic form of Joule's law**

$P = IU = I^2 R = \frac{U^2}{R}$ **is the macroscopic form of Joule's law**

§ 5-6 Boundary conditions for current density

In the absence of non-conservative energy source, we shall have

Governing Equations for Steady Current Density	
Differential Form	Integral Form
$\nabla \cdot \mathbf{J} = 0$	$\oint_S \mathbf{J} \cdot d\mathbf{s} = 0$
$\nabla \times \left(\dfrac{\mathbf{J}}{\sigma}\right) = 0$	$\oint_C \dfrac{1}{\sigma} \mathbf{J} \cdot d\boldsymbol{\ell} = 0$

We can obtain the boundary conditions for J (as in Fig. 3-23 and in Sec. 3-9):

(1) Normal component of boundary current, J_n

$$\boxed{J_{1n} = J_{2n}} \Longleftarrow \nabla \cdot \mathbf{J} = 0$$

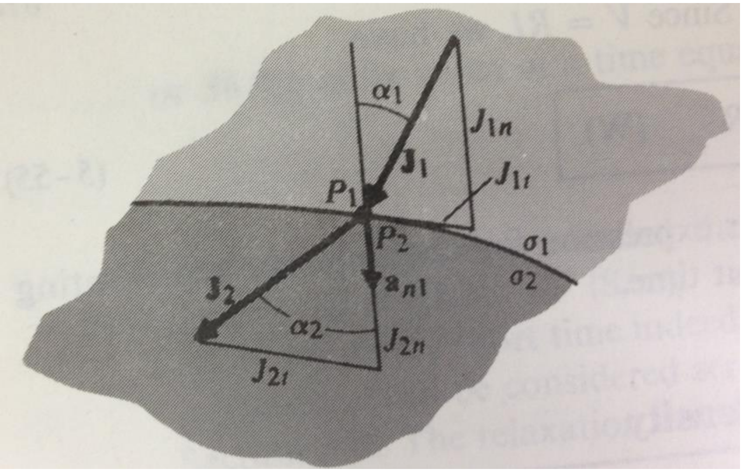
The normal components of J at two sides of an interface is continuous.

(2) Tangent component of boundary current, J_t

$$\boxed{\frac{J_{1t}}{J_{2t}} = \frac{\sigma_1}{\sigma_2}} \Longleftarrow \nabla \times (\mathbf{J}/\sigma) = 0$$

The ratio of the tangential components of J at two sides of an interface is equal to the ratio of the electric conductivities.

Q5-3 Two conducting media with conductivities σ_1 and σ_2 are separated by an interface. The steady current density in medium 1 at point P_1 has a magnitude J_1 and makes an angle α_1 with the normal. Determine the magnitude and direction of the current density at point P_2 in medium 2.



$$J_1 \cos\alpha_1 = J_2 \cos\alpha_2$$

$$\sigma_2 J_1 \sin\alpha_1 = \sigma_1 J_2 \sin\alpha_2$$

$$\frac{\tan\alpha_2}{\tan\alpha_1} = \frac{\sigma_2}{\sigma_1}$$

If medium 1 is a much better conductor than medium 2 $\sigma_1 >> \sigma_2$ or $\dfrac{\sigma_2}{\sigma_1} \to 0$

α_2 →zero, and J_2 emerges almost perpendicularly to the interface (normal to the surface of the good conductor). The magnitude of J_2 is

$$J_2 = \sqrt{J^2{}_{2t} + J^2{}_{2n}} = \sqrt{(J_2 \sin\alpha_2)^2 + (J_2 \cos\alpha_2)^2}$$

$$= [(\frac{\sigma_2}{\sigma_1} J_1 \sin\alpha_1)^2 + (J_1 \cos\alpha_1)^2]^{1/2} = J_1[(\frac{\sigma_2}{\sigma_1}\sin\alpha_1)^2 + \cos^2\alpha_1]^{1/2}$$

Boundary between two lossy dielectrics

When a steady current flows across the boundary between two different lossy dielectrics (dielectrics with permittivities ϵ_1 and ϵ_2 and finite conductivities σ_1 and σ_2),

$$E_{2t} = E_{1t};$$

$$J_{1n} = J_{2n} \rightarrow \sigma_1 E_{1n} = \sigma_2 E_{2n}$$

$$D_{1n} - D_{2n} = \rho_s \rightarrow \epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s,$$

$$\rho_s = \left(\epsilon_1 \frac{\sigma_2}{\sigma_1} - \epsilon_2\right) E_{2n} = \left(\epsilon_1 - \epsilon_2 \frac{\sigma_1}{\sigma_2}\right) E_{1n}.$$