第一讲 复数预备知识

Sunday, September 16, 2018 7:37 PM

2018-秋-复变函数

扫一扫二维码，加入该群。

密码: 1234

复数: Cardano 研究三次方程求根 $40=(5+\sqrt{-15})(5-\sqrt{-15})$

笛卡尔 $\sqrt{-1}$ 称为虚数

$\updownarrow$

实

Euler: $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

$e^{i\pi} = -1$

Gauss, Cauchy, Abel

第一节: 复数的表示

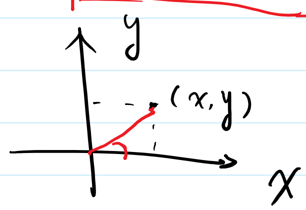
给出定义：一对有序的实数 (x, y) 构成一个复数

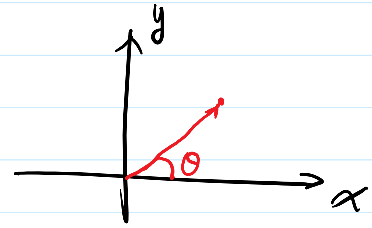
记号：$z = x + iy$　　实 $Re(z)$　　虚部 $Im(z)$

复数相等：$z = 0 \iff x = 0$ 且 $y = 0$.

表示：(代数形式)　$\boxed{z = x + iy}$ 或 (x, y)

(x, y)





指数形式.

令 $x = r\cos\theta$
　 $y = r\sin\theta$

$$z = x + iy = r\cos\theta + i r\sin\theta = r(\cos\theta + i\sin\theta)$$
$$= \boxed{re^{i\theta}}$$

$r = \sqrt{x^2 + y^2}$：代表向量的长度. 称为复数的模，$|z|$

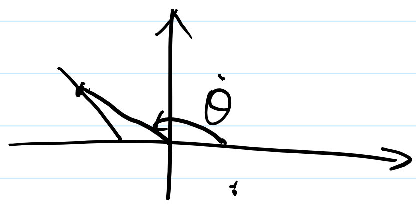
θ　：代表向量与x轴的夹角. 称为复数辐角. $Arg\, z$

辐角主值：落在 $(-\pi, \pi]$ 的θ 记作 $arg\, z$.

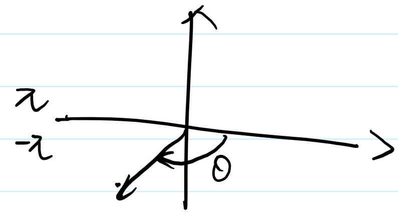
$$Arg\, z = arg\, z + 2k\pi$$

$$\mathrm{Arg}\, z = \underline{\arg z} + 2k\pi$$

$$\neq \operatorname{atan}\frac{y}{x} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$



复数形式转换.



1). $z = -\sqrt{12} - 2i$

解: $r = \sqrt{x^2+y^2} = \sqrt{12+4} = 4$

$$\arg z = \theta = \operatorname{atan}\frac{-2}{-\sqrt{12}} - \pi$$

$$= -\pi + \frac{\pi}{6} = -\frac{5}{6}\pi$$

$$z = 4e^{i(-\frac{5}{6}\pi)}$$

复数的运算.

四则运算, 加法/减法: $(x_1+iy_1) + (x_2+iy_2)$

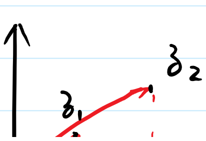
$$= (x_1+x_2) + i(y_1+y_2)$$

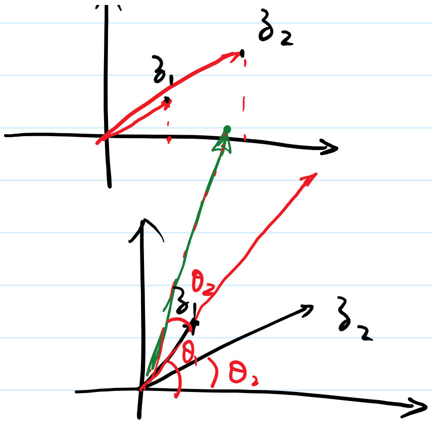
乘法 $(x_1+iy_1)(x_2+iy_2)$

$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

除法

$$z_1 z_2 = r_1 e^{i\theta_1} \times r_2 e^{i\theta_2}$$





$$= r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

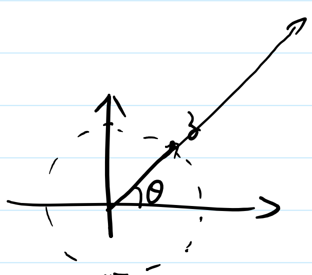
复数的运算满足交换律、结合律、分配律.

注意: $\mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2 = \mathrm{Arg}(z_1 \times z_2)$ (✓)

$\arg z_1 + \arg z_2 \boxed{\neq} \arg(z_1 z_2)$

乘幂. n次幂. $z^n = (re^{i\theta})^n$

$$= r^n e^{in\theta}$$



开方 $w^n = z$ 即: $w = \sqrt[n]{z}$

令 $w = \rho e^{i\varphi}$, $w^n = \rho^n e^{in\varphi} = re^{i\theta}$

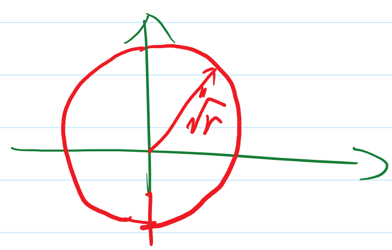
$\Rightarrow$: $\begin{cases} \rho^n = r \end{cases}$

$$\Rightarrow: \begin{cases} \rho^n = r \\ n\varphi = \theta + 2k\pi \end{cases}$$

$$\Rightarrow: \rho = \sqrt[n]{r}$$

$$\varphi_k = \frac{\theta}{n} + \frac{2k\pi}{n} \qquad k=0,1,\cdots,n-1$$

$\boxed{\omega_k} = \rho e^{i\varphi_k}$ 共有n个值



例：求 $z=1+i$ 的4次方根. $\qquad \sqrt[4]{1+i} = ?$

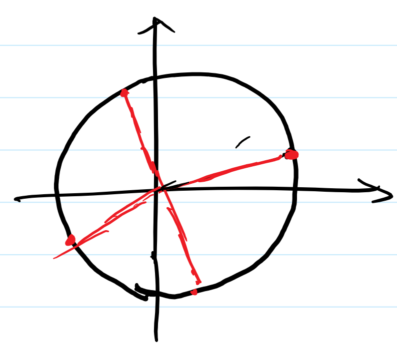
解：$z = 1+i = \sqrt{2}e^{i\frac{\pi}{4}}$

$$\omega_k = \sqrt[4]{z} = (\sqrt{2})^{\frac{1}{4}} e^{i\frac{\frac{\pi}{4}+2k\pi}{n}}, \quad k=0,1,2,3$$

$$= \sqrt[8]{2}\, e^{i\frac{\frac{\pi}{4}+2k\pi}{4}}, \quad k=0,1,2,3.$$

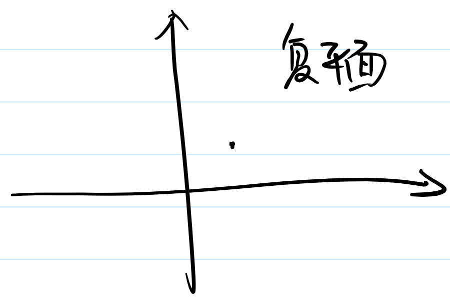
$$\omega_1 = \sqrt[8]{2}\left(\cos\frac{\pi}{16} + i\sin\frac{\pi}{16}\right)$$

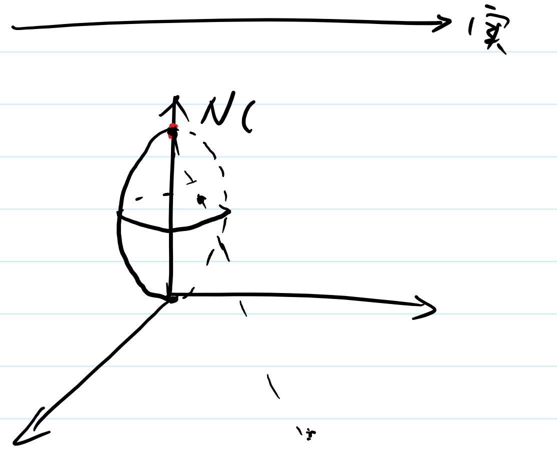
$$\omega_2 = \sqrt[8]{2}\left(\cos\frac{9}{16}\pi + i\sin\frac{9}{16}\pi\right)$$



复球面与无穷远点

(x,y)





引进北极点 $\infty^* \cup$ 复平面

称为 扩充复平面

$\infty^* + z = \infty^*$

$\infty^* \cdot z = z \cdot \infty^* = \infty^*$, $z \neq 0$

$\dfrac{\infty^*}{z} = \infty^*$, $\dfrac{z}{\infty^*} = 0$ $\quad z \neq \infty^*$

$\dfrac{z}{0} = \infty^*$, $\quad \forall z \neq 0$.

$|\infty^*| = \infty$, 没有辐角.

复平面上的点集

(1): 邻域(开集). $|z-z_0|<\delta$ 记作 $D(z_0, \delta)$

去心邻域.(开集): $0<|z-z_0|<\delta$.

"∞"的邻域？ $|z|>M$ M为很大的数

"∞"的去心邻域？ $M<|z|<\infty$

(2). 开集/内点. 集合 G, $z_0 \in G$.

内点：存在 z_0 的邻域，使得邻域落在 G 内.

$z_0 \in G$, $\exists \delta$, s.t. $\{z \mid |z-z_0|<\delta\} \subset G$.

$\forall z_0 \in G$, z_0 都是内点.

(3). 边界点.

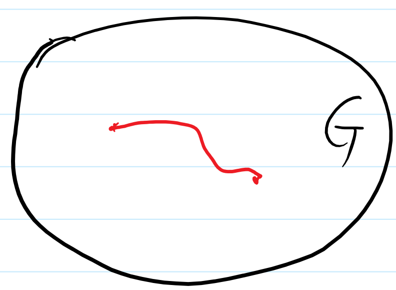
$z_0 \in G$, $\forall \delta>0$, $\exists z_1, z_2 \in D(z_0, \delta)$.

s.t. $\begin{cases} z_1 \in G \\ z_2 \notin G \end{cases}$

$z_0 \in \underline{\underline{\partial G}}$

(4). 区域.

$\begin{cases} 1.\ G\text{是开集} \\ 2.\ G\text{是连通} \end{cases}$

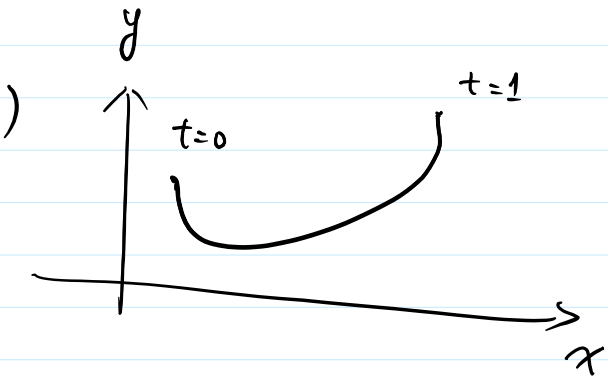


(5). 闭区域： $G \cup \partial G$

(6). 有界区域： $\forall z \in G$. $|z|<M$

(7). 简单曲线/光滑曲线

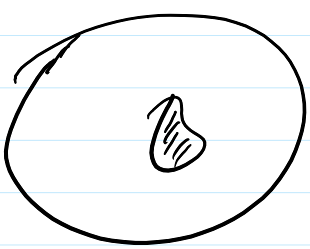
C: $z(t) = x(t) + i\,y(t)$



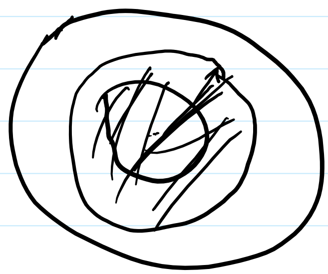
简单: 没有重合点

闭: 起点与终点重合.

单连通区域/多连通区域



单连通



二连通区域

作业: 习题一.

3(3), 4, 9, 10(1), 11, 13.

复平面

· 点 $\longleftrightarrow$ 复数

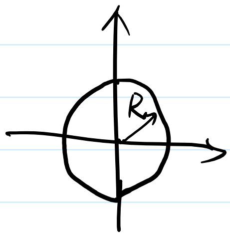
复平面

·点 ⟷ 复数

点集 ⟷ 复数集合

图形 ⟷ 复数方程/不等式

例:

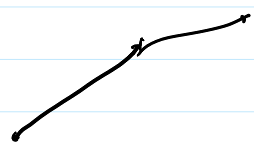


: $|z|=R$

以原点为中心，$\pm z_0$为焦点，长半轴为a的椭圆.

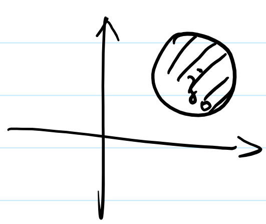
$$|z-z_0|+|z+z_0|=2a$$

z_1, z_2, z_3共线

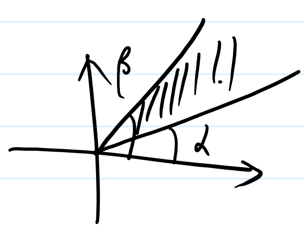


$$\frac{z_3-z_1}{z_2-z_1}=t,\quad t\text{为实数.}$$

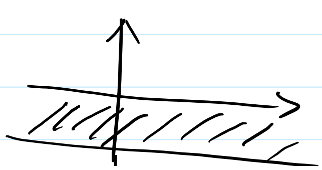
不等式:



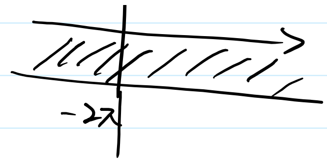
$|z-z_0|<R$ (开圆)



$\alpha < \arg z < \beta$



$-2\pi < \mathrm{Im}(z) < 0$



$-2\pi < \mathrm{Im}(z) < 0$

解析函数

复变函数

定义：设D是复平面上的点集，若对D中任意一点z，存在一个或多个w与之对应，则称w是定义在D上的复变函数。

D称为定义域，所有w的集合称为值域 G

记作：$w = f(z) = u + iv$

$= u(x,y) + iv(x,y)$

单值.

单射，满射，双射。 ← 既单又满（f^{-1}存在）

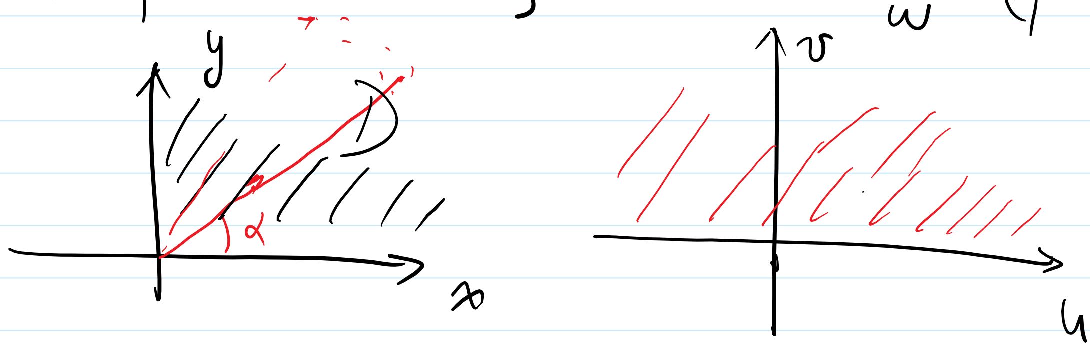
若 $z_1 \neq z_2 \in D$ 且 $f(z_1) \neq f(z_2)$

$\forall w \in G$，$\exists z \in D$，s.t. $f(z) = w$

例：$w = f(z) = z^2$.

例: $w = f(z) = z^2$.

$D = \{ z \mid \mathrm{Re}\, z > 0,\ \mathrm{Im}\, z > 0 \}$



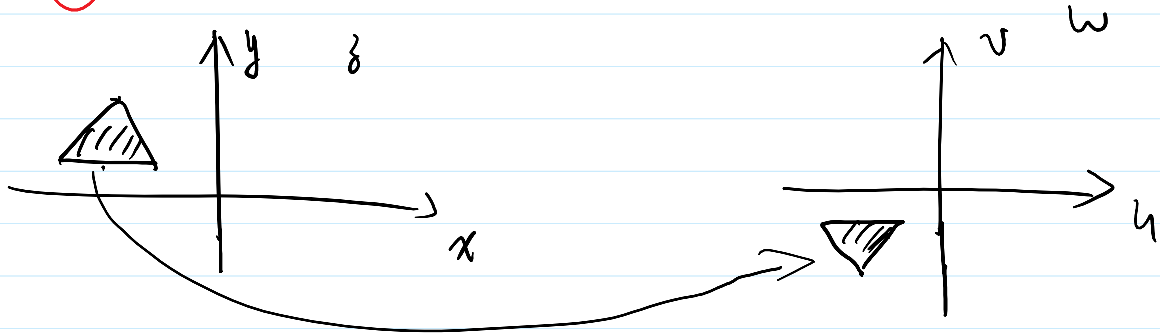
$G = \{ w \mid \mathrm{Re}(w) > 0 \}$

$f: D \to G$ $\boxed{\text{双射}}$

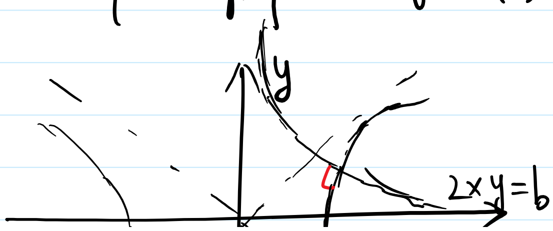
$f^{-1}: G \to D$

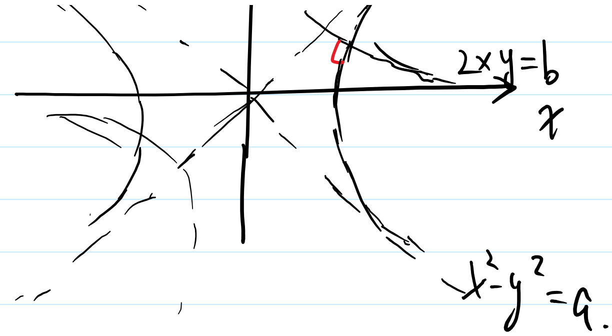
例2:

$f(z) = \bar{z} = x - iy$



例: $D: \{ (x,y) \mid x^2 - y^2 = a > 0 \} \cup \{ (x,y) \mid 2xy = b \}$.

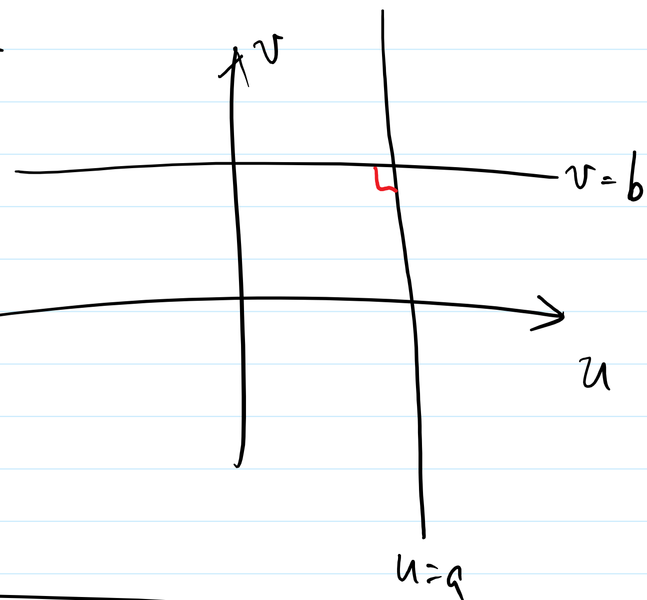




$$w = u + iv = z^2 = (x+iy)^2$$

$$= \underline{(x^2-y^2) + i\,2xy}$$

$$= a + bi$$



例：求下列曲线在映射下的象.

$x^2+y^2=8 \quad \underline{w=\frac{1}{z}}$

$$z = \frac{1}{w} = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$$

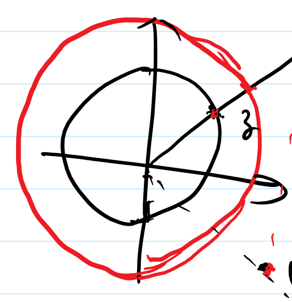
$\therefore \quad x = \frac{u}{u^2+v^2}$

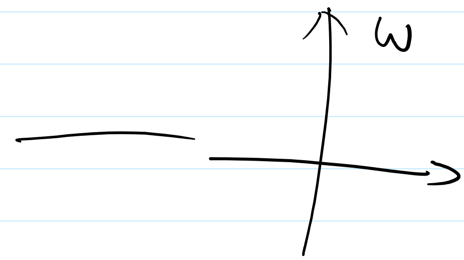
$\qquad y = -\frac{v}{u^2+v^2}$

由 $x^2+y^2 = \left(\frac{u}{u^2+v^2}\right)^2 + \left(-\frac{v}{u^2+v^2}\right)^2 = \frac{1}{u^2+v^2} = 8$

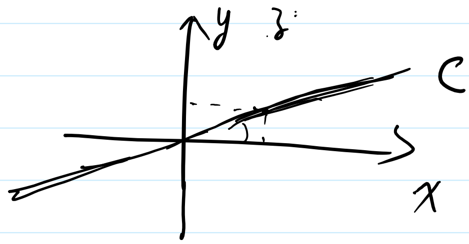
$\Rightarrow$: $u^2+v^2=\frac{1}{8}$

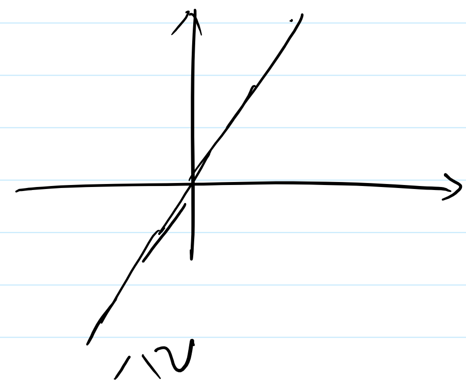
$\boxed{w=\frac{1}{z}}$





例: C: $z=(2+i)t$. $t\in R$ $\quad \boxed{f(z)=z^2} \rightarrow$



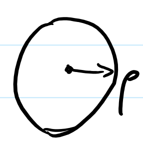


$$w=z^2=[(2+i)t]^2$$
$$=t^2(3+4i)$$
$$=u+iv.$$

$$\left.\begin{aligned}u&=3t^2\\ v&=4t^2.\end{aligned}\right\} \Rightarrow: \frac{u}{v}=\frac{3}{4}.\ 即: \boxed{v=\frac{4}{3}u}$$

极限/连续

$w=f(z)$, 在 z_0 处的去心邻域 $D(z_0,\rho)$ 内,



如果存在某个确定的复数 A, 使得:

$\forall\varepsilon>0$, 都存在 $\delta(\varepsilon)$, 且 $\delta(\varepsilon)<\rho$, $\forall z\in \underline{D(z_0,\delta(\varepsilon))} < D(z_0,\rho)$

$\forall \varepsilon>0$, 都存在 $\delta(\varepsilon)$，且 $\delta(\varepsilon)<\rho$, $\underline{D(z_0, \delta(\varepsilon))} < D(z_0, \rho)$

有： $|f(z)-A| < \varepsilon$.

则称 A 为 $f(z)$ 在 z_0 处的极限.

记作： $\lim\limits_{z \to z_0} f(z) = A$.

$\forall \varepsilon>0, \exists \delta(\varepsilon)>0$, s.t.: $\forall z \in D(z_0, \delta(\varepsilon))$, $|f(z)-A|<\varepsilon$.

定理：如果极限存在，一定是唯一的.

极限的运算：

$\lim\limits_{z\to z_0} f(z)=A$, $\lim\limits_{z\to z_0} g(z)=B$.

则： $\lim\limits_{z\to z_0} f(z)\pm g(z) = A \pm B$

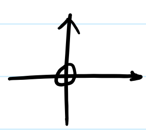
$\lim\limits_{z\to z_0} f(z)g(z) = AB$

$\lim\limits_{z\to z_0} \dfrac{f(z)}{g(z)} = \dfrac{A}{B}$ $(B\neq 0)$

若 $\lim\limits_{z\to A} g(z) = C$.

则： $\lim\limits_{z\to z_0} g(f(z)) = C$.

例： $f(z)=\dfrac{\mathrm{Re}(z)}{|z|}$ 在 $z\to 0$ 时极限.



函数的连续性.

- 若 $f(z)$ 在 z_0 极限存在. $\lim\limits_{z\to z_0} f(z) = A = f(z_0)$
- 若 $f(z)$ 在 D 内任一点连续，$f(z)$ 在 D 上连续.

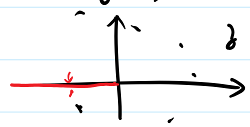
$f(z) = u+iv = \underline{u(x,y)} + i\,\underline{v(x,y)}$

- 四则运算以及复合 保持连续性.

$|f(z)|$ 也是连续

例： $f(z) = \arg z$, $z \in \mathbb{C}$

例: $f(z)=\arg z$, $z\in\mathbb{C}$



$(-\pi,\pi]$

$f(z)$在负实轴上不连续

例: 讨论 $f(z)=\dfrac{z\,\mathrm{Im}\,z^2}{|z|^2}$ $(z\neq 0)$ 的连续性.

令 $z=x+iy$,

$$f(z)=\frac{(x+iy)\times 2xy}{x^2+y^2}=\frac{2x^2y}{x^2+y^2}+i\frac{2xy^2}{x^2+y^2}$$

$z\to 0$, $z=re^{i\theta}$, $r\to 0$, $\theta\in(-\pi,\pi]$

$$f(z)=\frac{re^{i\theta}r^2\sin 2\theta}{r^2}$$

$$\boxed{\begin{array}{l} z=re^{i\theta} \\ z^2=r^2e^{2i\theta}\end{array}}$$

$$=re^{i\theta}\sin 2\theta\to 0$$

$$\lim_{z\to 0}f(z)=0$$

补充定义: $f(z)=0$ if $z=0$.

则: $f(z)=\begin{cases}\dfrac{z\,\mathrm{Im}\,z^2}{|z|^2} & z\neq 0\\ 0 & z=0\end{cases}$

---

3 解析函数.

作业:

P19: 14(3)(6)(8)

P47: 1, 5