Field and Wave Electromagnetics

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Chapter 3 Static Electric Field

Field: A spatial distribution of a scalar or vector quantity which may or may not be a function of time.

1. Fundamental postulates of electrostatic in free space
2. Coulomb's law
3. Gauss's law
4. Conductors in static electric field
5. Dielectrics in static electric field
6. Electric flux density
7. Boundary conditions for electrostatic fields
8. Capacitance and capacitors
9. Electrostatic energy and forces

3.2 Fundamental postulates of electrostatic in free space

- **Electric field intensity:**

$$\vec{E} = \lim_{q \to 0} \frac{F}{q}$$ (V/m); Force per unit charge

- **The two fundamental postulates:**

$$\nabla \cdot \vec{E} = \frac{r}{e_0}$$ (ρ: Volume density, ε_0: permittivity of free space)

$$\nabla \times \vec{E} = 0$$ → Static E fields are irrotational!

The two equations hold at every spatial point and are referred as the differential form of the postulates.

- Take volume integral and apply divergence theorem

$$\nabla \cdot \vec{E} = \frac{r}{e_0} \Rightarrow \int_V \nabla \cdot \vec{E}\, dv = \frac{1}{e_0} \int_V r\, dv \Rightarrow \oint_S \vec{E} \cdot d\vec{s} = \frac{Q}{e_0}$$

Gauss's Law: the total outward of the electric intensity over any closed surface is equal to the total charge enclosed by the surface.

- Take line integral and apply Stokes's theorem

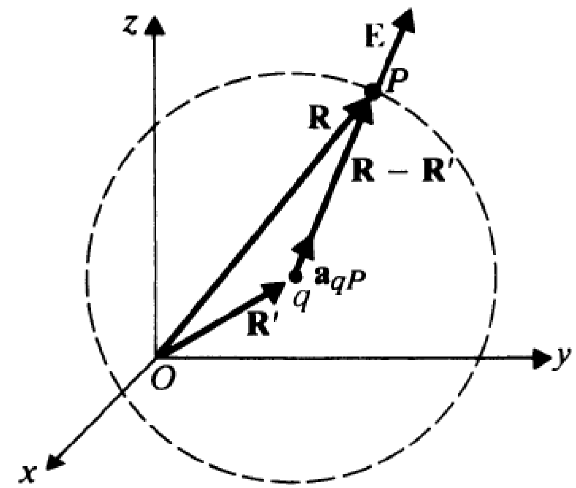
$$\nabla \times \vec{E} = 0 \Rightarrow \int_S \nabla \times \vec{E}\, d\vec{s} = 0 \Rightarrow \oint_C \vec{E} \cdot d\vec{l} = 0$$

- Equivalent to the Kirchhoff's voltage law;
- The line integral between two points is irrelavent to the path;
- Static electric field is irrotational (conservative)

These two postulates represent laws of nature.

3.3 Coulomb's Law

A point source of charge *q* at ***R'*** and find the electric field $\boldsymbol{E_p}$ at ***R*** (at point p):



$$\oint_S \vec{E}_p \cdot d\vec{s} = \oint_S \left(E_p \frac{\vec{R}-\vec{R}'}{\left|\vec{R}-\vec{R}'\right|}\right) \cdot \frac{\vec{R}-\vec{R}'}{\left|\vec{R}-\vec{R}'\right|} ds = \frac{q}{\varepsilon_0}$$

or $$E_p \oint_S ds = E_p \left(4\pi \left|\vec{R}-\vec{R}'\right|^2\right) = \frac{q}{\varepsilon_0}$$

$$\boxed{\vec{E}_p = \frac{q(\vec{R}-\vec{R}')}{4\pi\varepsilon_0 \left|\vec{R}-\vec{R}'\right|^3}}$$

Coulomb's Law

$$\boxed{\vec{F}_{12} = q_2 \vec{E}_{12} = \frac{q_1 q_2 (\vec{R}-\vec{R}')}{4\pi\varepsilon_0 \left|\vec{R}-\vec{R}'\right|^3}}$$

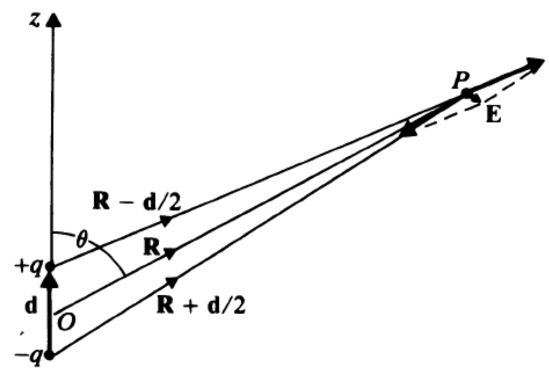
q_1 at ***R'*** and q_2 at ***R***

3.3.1 Electric field due to a system of discrete charges

- **Total field at a point *E*:** vector sum of the fields from all the individual charges

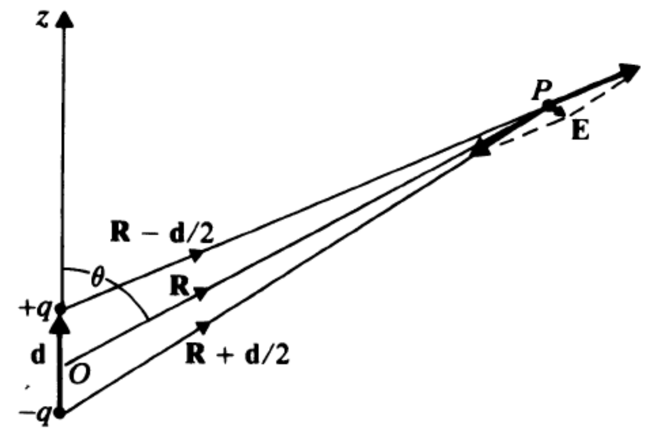
$$\boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{n}\frac{q_k(\vec{R}-\vec{R}')}{\left|\vec{R}-\vec{R}'\right|^3}}$$

- **Electric dipole:** a pair of equal and opposite charges $+q$ and $-q$ separated by a small distance d



$$\boxed{\vec{E} = \frac{q}{4\pi\varepsilon_0}\left\{\frac{\vec{R}-\frac{\vec{d}}{2}}{\left|\vec{R}-\frac{\vec{d}}{2}\right|^3} - \frac{\vec{R}+\frac{\vec{d}}{2}}{\left|\vec{R}+\frac{\vec{d}}{2}\right|^3}\right\}} \xrightarrow{d \ll R} \boxed{\vec{E} \cong \frac{q}{4\pi\varepsilon_0 R^3}\left[3\frac{\vec{R}\cdot\vec{d}}{R^2}\vec{R} - \vec{d}\right]}$$

◆ **Electric dipole moment, *p*:** $\boxed{\vec{p} = q\vec{d}}$



$$\boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0 R^3}\left[3\frac{\vec{R}\cdot\vec{p}}{R^2}\vec{R} - \vec{p}\right]}$$

If *p* is along *z* axis, then $\boxed{\vec{p} = \vec{a}_z p = p(\vec{a}_R\cos\theta + \vec{a}_\theta\sin\theta)}$

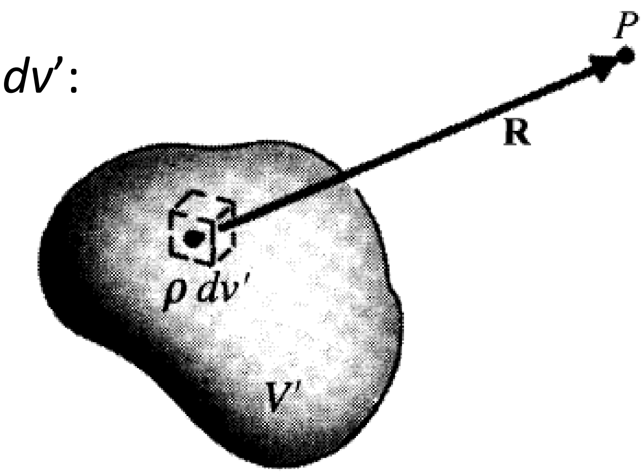
◆ Electric field ***E*** of a electric dipole in spherical coordinate

$$\boxed{\vec{E} = \frac{p}{4\pi\varepsilon_0 R^3}(\vec{a}_R 2\cos\theta + \vec{a}_\theta\sin\theta)}$$

Cubic dependence on *R*

3.3.2 Electric field due to a continuous distribution of charge

◆ Electric field ***E*** of a differential volume charge *ρdv'*:



$$\boxed{\vec{E} = \vec{a}_R \frac{\rho\, dv}{4\pi\epsilon_0 R^2}} \qquad (\vec{a}_R = \vec{R}/R)$$

◆ Total electric field ***E*** :

$$\boxed{(1) \quad \vec{E} = \frac{1}{4\pi\epsilon_0} \int_{V'} \vec{a}_R \frac{\rho}{R^2} dv}$$

(ρ: volume charge density)

$$\boxed{(2) \quad \vec{E} = \frac{1}{4\pi\epsilon_0} \int_{S'} \vec{a}_R \frac{\rho_s}{R^2} ds'}$$

(ρ_s: surface charge density)

$$\boxed{(3) \quad \vec{E} = \frac{1}{4\pi\epsilon_0} \int_{L'} \vec{a}_R \frac{\rho_l}{R^2} dl'}$$

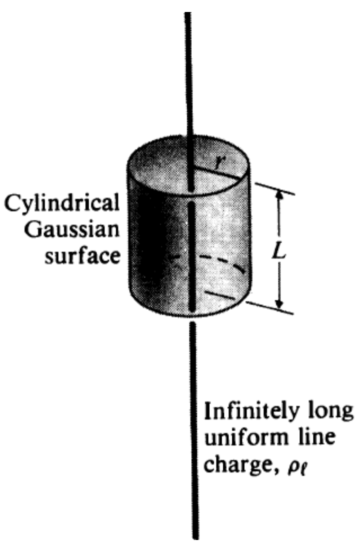
(ρ_l: line charge density)

3.4 Gauss's Law and Applications

$$\boxed{\oint_S \vec{E}\cdot d\vec{s} = \frac{Q}{\epsilon_0}}$$

Suitable for symmetry condition with proper surface

Example 1: Find ***E*** for an infinite uniform line charge



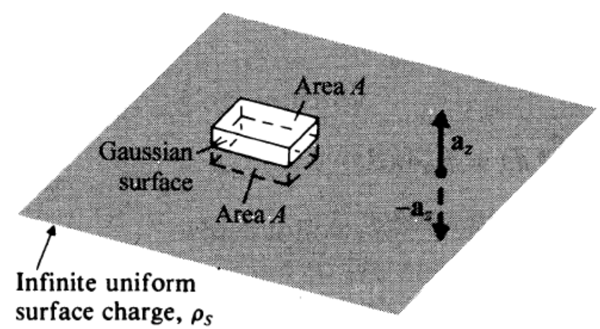
$$\oint_S \vec{E}\cdot d\vec{s} = \int_0^L \int_0^{2\pi} E_r r \, d\phi \, dz$$

$$= 2\pi r L E_r$$

$$\Rightarrow \; 2\pi r L E_r = \frac{\rho_l L}{\epsilon_0}$$

$$\Rightarrow \; \vec{E} = \vec{a}_r E_r = \vec{a}_r \frac{\rho_l}{2\pi\epsilon_0 r}$$

Example 2: Find ***E*** for an infinite uniform surface charge



$$\vec{E}\cdot d\vec{s} = (\vec{a}_z E_z)\cdot(\vec{a}_z ds) = E_z ds$$

$$\Rightarrow \; \vec{E} = \vec{a}_z E_r = \vec{a}_z \frac{\rho_s}{2\epsilon_0}, \quad (z>0)$$

$$\vec{E} = -\vec{a}_z E_r = -\vec{a}_z \frac{\rho_s}{2\epsilon_0}, \quad (z<0)$$

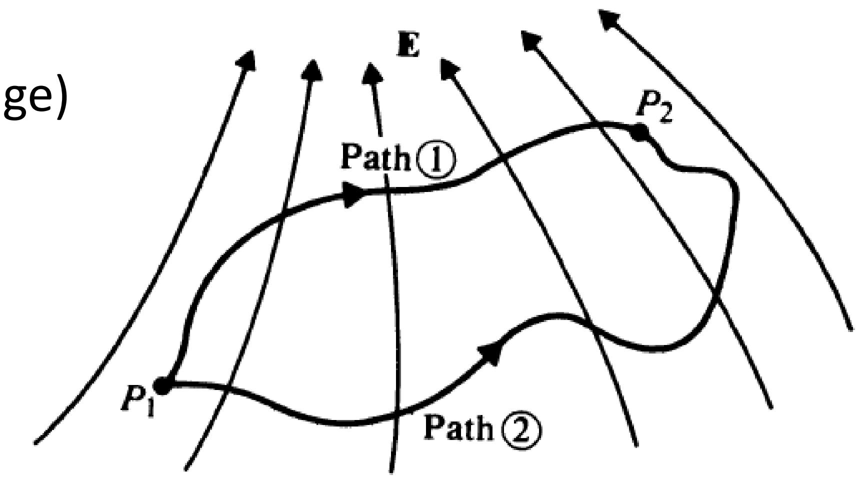
3.5 Electric Potential

◆ A scalar electric potential V:

$$\boxed{\vec{E} = -\nabla V} \quad \Leftarrow \begin{cases} \nabla \times \vec{E} = 0 \\ \nabla \times (\nabla A) = 0 \end{cases}$$

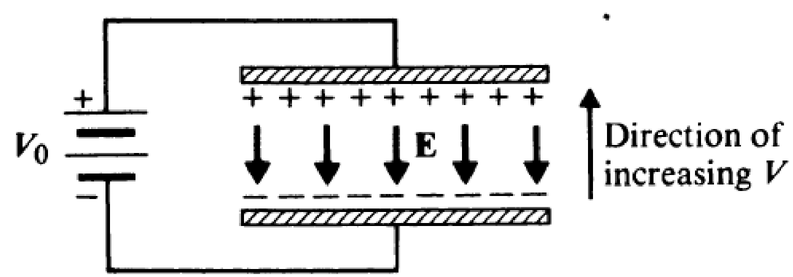
◆ Potential difference (electrostatic voltage) between points P_2 and P_1:

$$\boxed{V_2 - V_1 = -\int_{P_1}^{P_2} \vec{E} \cdot d\vec{l}}$$



In most cases, the zero-potential point is taken at infinity.

◆ Field lines are perpendicular to equipotential lines (surface) everywhere.



3.5.1 Electric Potential due to a charge distribution

◆ Electric potential of a point at a distance ***R*** from a point charge *q*:

$$\boxed{V = \frac{q}{4\pi\varepsilon_0 R}} \Leftarrow \boxed{V = -\int_{\infty}^{R} \left(\vec{a}_R \frac{q}{4\pi\varepsilon_0 R^2}\right) \cdot \left(\vec{a}_R dR\right)}$$

◆ Electric potential at ***R*** from a charge distribution:

(a) From n charges

$$\boxed{V = \frac{1}{4\pi\varepsilon_0} \sum_{k=1}^{n} \frac{q_k}{\left|\vec{R} - \vec{R}_k'\right|}}$$

(b) From charges of volume density ρ

$$\boxed{V = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\vec{R}')}{\left|\vec{R} - \vec{R}'\right|} dv}$$

$\Leftarrow$ Easier to determine E by taking the negative gradient, i.e.,

$$\boxed{\vec{E} = -\nabla V}$$

3.6 Conductors in Static Electric Field

◆ **Conductor, semiconductor, insulator (dielectric)**

◆ Inside a conductor, under static conductions we have

$$\boxed{r = 0, \quad \vec{E} = 0}$$

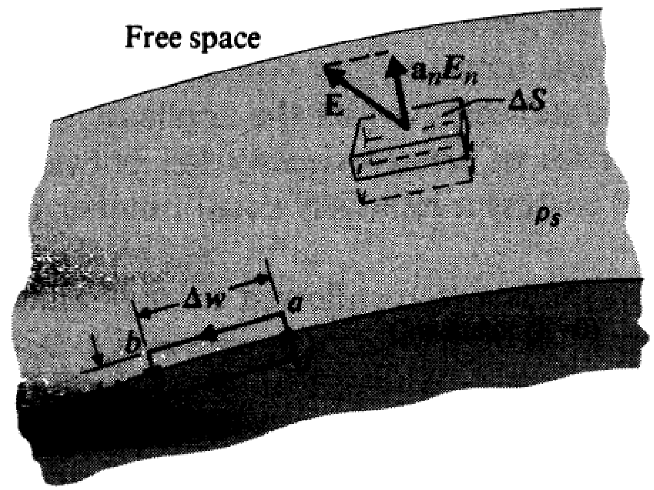
◆ **Boundary conditions** at a conductor/free-space interface:

Tangent component:

$$\boxed{\oint_{abcda} \vec{E} \cdot d\vec{l} = E_t \mathrm{D} w = 0 \quad \Rightarrow \quad E_t = 0}$$

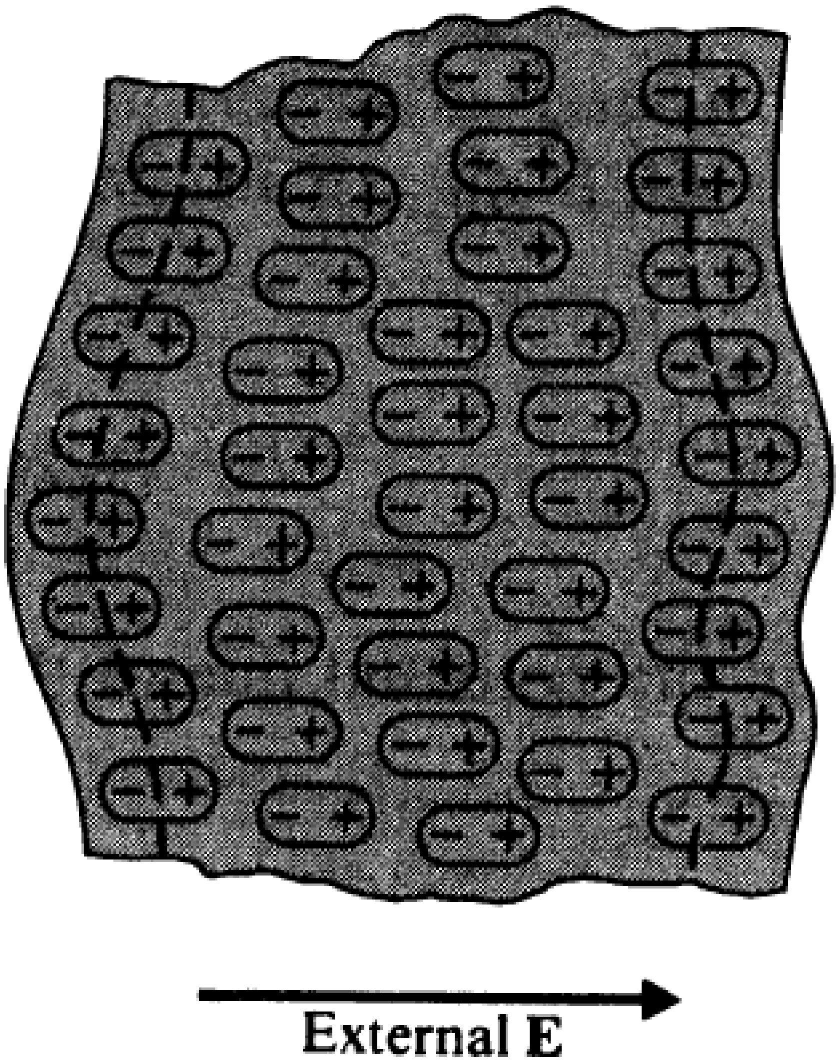
Normal component:

$$\boxed{\oint_S \vec{E} \cdot d\vec{s} = E_n \mathrm{D} S = \frac{r_S \mathrm{D} S}{e_0} \quad \Rightarrow \quad E_n = \frac{r_S}{e_0}}$$



For a neutral conductor placed in a static field, the induced electric field will cancel the external field both inside the conductor and tangent to its surface.

3.7 Dielectrics in Static Electric Field



Induced electric dipoles

- **Polar molecule**
- **electret**

◆ **Polarization vectors *P***, as

$$\mathbf{P} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \mathbf{p}_k}{\Delta v} \qquad (\mathrm{C/m^2}),$$

P is the volume density of electric dipole moment. The dipole moment *d**p*** of an element volume *dv'* is *d**p***=***P****dv'*, which produces a potential:

$$dV = \frac{\mathbf{P} \cdot \mathbf{a}_R}{4\pi\epsilon_0 R^2}\, dv'.$$

The potential *V* due to the dielectric:

$$V = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\mathbf{P} \cdot \mathbf{a}_R}{R^2}\, dv'$$

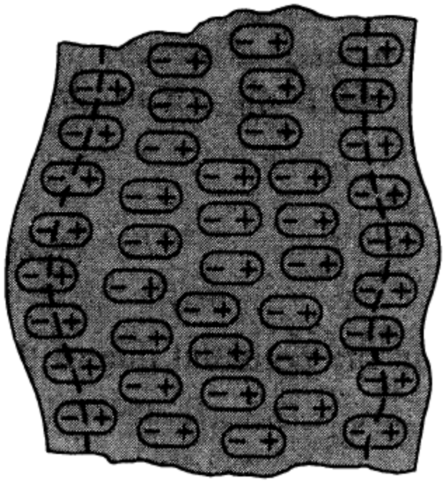
(*R* is the distance from the volume *dv*' to the fixed field point)

$$V = \frac{1}{4\pi\epsilon_0}\int_{V'} \frac{\mathbf{P}\cdot\mathbf{a}_R}{R^2}\,dv$$

$$\boldsymbol{\nabla}'\left(\frac{1}{R}\right) = \frac{\mathbf{a}_R}{R^2}$$

$$V = \frac{1}{4\pi\epsilon_0}\int_{V'} \mathbf{P}\cdot\boldsymbol{\nabla}'\left(\frac{1}{R}\right) dv$$

$$\boldsymbol{\nabla}'\cdot(f\mathbf{A}) = f\boldsymbol{\nabla}'\cdot\mathbf{A} + \mathbf{A}\cdot\boldsymbol{\nabla}' f$$



$$V = \frac{1}{4\pi\epsilon_0}\left[\int_{V'} \boldsymbol{\nabla}'\cdot\left(\frac{\mathbf{P}}{R}\right) dv' - \int_{V'} \frac{\boldsymbol{\nabla}'\cdot\mathbf{P}}{R}\,dv'\right]$$

Divergence theorem

$$V = \frac{1}{4\pi\epsilon_0}\oint_{S'} \frac{\mathbf{P}\cdot\mathbf{a}_n'}{R}\,ds' + \frac{1}{4\pi\epsilon_0}\int_{V'} \frac{(-\boldsymbol{\nabla}'\cdot\mathbf{P})}{R}\,dv$$

Boundary charge density:

$$\rho_{ps} = \mathbf{P}\cdot\mathbf{a}_n$$

Polarization charge density:

$$\rho_p = -\boldsymbol{\nabla}\cdot\mathbf{P}$$

$$V = \frac{1}{4\pi\epsilon_0}\oint_{S'} \frac{\rho_{ps}}{R}\,ds' + \frac{1}{4\pi\epsilon_0}\int_{V'} \frac{\rho_p}{R}\,dv'$$

◆ Total net charges Q flowing out of a surface S boundary a volume V:

$$Q = -\oint_S \mathbf{P} \cdot \mathbf{a}_n \, ds$$
$$= \int_V (-\nabla \cdot \mathbf{P}) \, dv = \int_V \rho_p \, dv,$$

◆ For a originally neutral dielectric body:

$$\text{Total charge} = \oint_S \rho_{ps} \, ds + \int_V \rho_p \, dv$$
$$= \oint_S \mathbf{P} \cdot \mathbf{a}_n \, ds - \int_V \nabla \cdot \mathbf{P} \, dv = 0,$$

3.8 Electric Flux Density and Dielectric Constant

- Gauss's equation in the dielectric:

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}(\rho + \rho_p) \quad \text{or} \quad \nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho.$$

- **Electric flux density or electric displacement, *D*:**

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \qquad (\mathrm{C/m^2}).$$

Gauss's equation

$$\nabla \cdot \mathbf{D} = \rho \qquad (\mathrm{C/m^3}) \qquad \text{(Differential form)}$$

Free charges

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q \qquad (\mathrm{C}). \qquad \text{(Integral form)}$$

$$\boxed{\oint_S \mathbf{D} \cdot d\mathbf{s} = Q}$$

The total outward flux of the electric displacement over any closed surface is equal the total free charge enclosed in the surface.

◆ Polarization for linear and isotropic system:

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$$

χ_e: Electric susceptibility

Then,

$$\mathbf{D} = \epsilon_0(1 + \chi_e)\mathbf{E}$$
$$= \epsilon_0 \epsilon_r \mathbf{E} = \epsilon \mathbf{E} \qquad (\mathrm{C/m^2}),$$

Where ***relative permittivity or dielectric constant*** is defined by

$$\epsilon_r = 1 + \chi_e = \frac{\epsilon}{\epsilon_0}$$

Dielectric constant (Complex in general)

Anisotropic medium

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}.$$

Biaxial medium

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}.$$

Uniaxial anisotropic medium

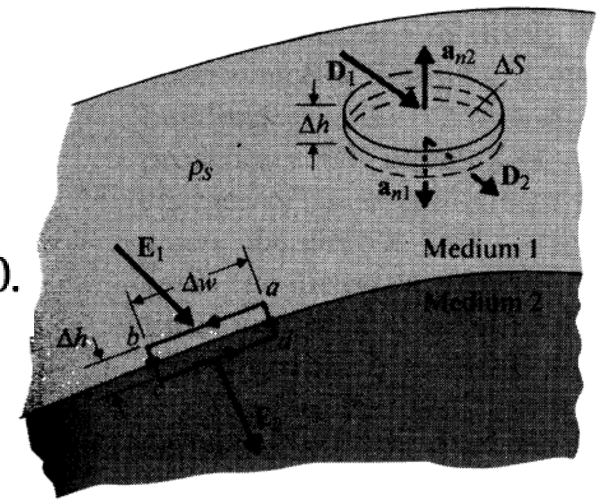
$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}.$$

Isotropic medium

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_1 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}.$$

3.9 Boundary Conditions for Electrostatic Fields

At the interface of two general media, we write the line integral of ***E*** around a closed path by



(1)

$$\oint_{abcda} \mathbf{E} \cdot d\ell = \mathbf{E}_1 \cdot \Delta\mathbf{w} + \mathbf{E}_2 \cdot (-\Delta\mathbf{w}) = E_{1t}\Delta w - E_{2t}\Delta w = 0.$$

Therefore, $\boxed{E_{1t} = E_{2t}}$

- *The tangential component of an E field is continuous.*

(2)

$$\begin{aligned}\oint_S \mathbf{D} \cdot d\mathbf{s} &= (\mathbf{D}_1 \cdot \mathbf{a}_{n2} + \mathbf{D}_2 \cdot \mathbf{a}_{n1})\Delta S \\ &= \mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2)\Delta S \\ &= \rho_s \Delta S,\end{aligned}$$

$\longrightarrow \boxed{D_{1n} - D_{2n} = \rho_s}$

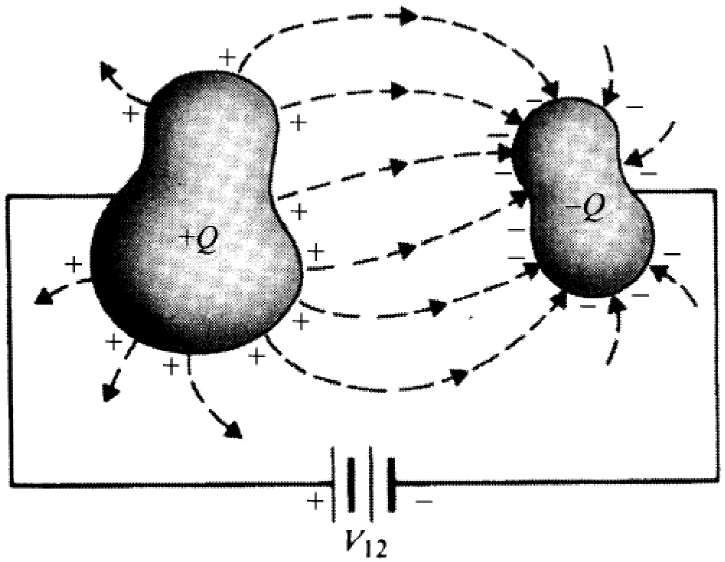
- *The normal component of D field is discontinuous and the discontinuity is equal to the surface charges.*

3.10 Capacitance and Capacitors

◆ Capacitance C:

$$C = \frac{Q}{V_{12}} \qquad \text{(F)}.$$

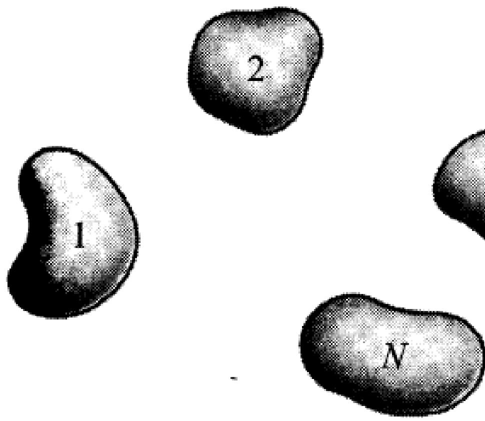
C is dependent on the geometry of the conductors and on the permittivity the surrounding medium.

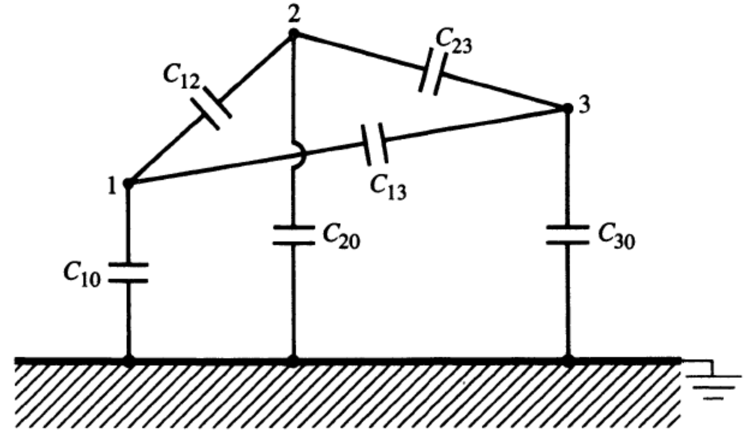


How to find C:

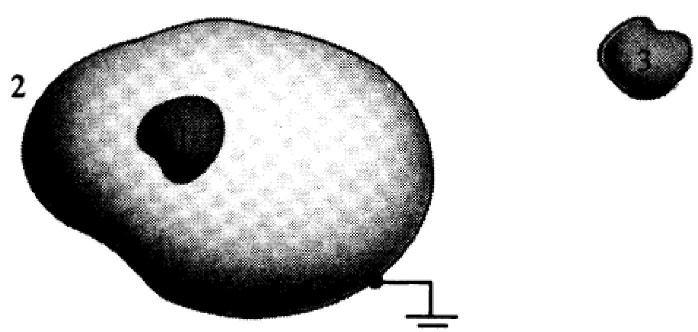
1. Choose an appropriate coordinate system for the given geometry.
2. Assume charges $+Q$ and $-Q$ on the conductors.
3. Find $\mathbf{E}$ from Q by Eq. (3–122), Gauss's law, or other relations.
4. Find V_{12} by evaluating
$$V_{12} = -\int_2^1 \mathbf{E} \cdot d\boldsymbol{\ell}$$
from the conductor carrying $-Q$ to the other carrying $+Q$.
5. Find C by taking the ratio Q/V_{12}.

- Capacitance in multiconductor systems





- Electrostatic Shielding



3.11 Electrostatic Energy and Forces

- Electric potential at a point in an electric field is the work required to bring a unit charge from infinity to that point.

$$V = \frac{Q}{4\pi\epsilon_0 R}$$

- The work W_2 required to move a charge Q_2 in the field of charge Q_1:

$$\boxed{W_2 = Q_2 V_2 = Q_2 \frac{Q_1}{4\pi\epsilon_0 R_{12}}.}$$

Or equally,

$$\boxed{W_2 = Q_1 \frac{Q_2}{4\pi\epsilon_0 R_{12}} = Q_1 V_1}$$

$$\boxed{W_2 = \tfrac{1}{2}(Q_1 V_1 + Q_2 V_2).}$$

(R_{12} is the moved distance)

◆ Bring another charge Q_3:

- Additional work ΔW required:

$$\Delta W = Q_3 V_3 = Q_3\left(\frac{Q_1}{4\pi\epsilon_0 R_{13}} + \frac{Q_2}{4\pi\epsilon_0 R_{23}}\right)$$

- Total work ΔW required:

$$\begin{aligned}
W_3 = W_2 + \Delta W &= \frac{1}{4\pi\epsilon_0}\left(\frac{Q_1 Q_2}{R_{12}} + \frac{Q_1 Q_3}{R_{13}} + \frac{Q_2 Q_3}{R_{23}}\right) \\
&= \frac{1}{2}\left[Q_1\left(\frac{Q_2}{4\pi\epsilon_0 R_{12}} + \frac{Q_3}{4\pi\epsilon_0 R_{13}}\right) + Q_2\left(\frac{Q_1}{4\pi\epsilon_0 R_{12}} + \frac{Q_3}{4\pi\epsilon_0 R_{23}}\right)\right. \\
&\quad \left. + Q_3\left(\frac{Q_1}{4\pi\epsilon_0 R_{13}} + \frac{Q_2}{4\pi\epsilon_0 R_{23}}\right)\right] \\
&= \tfrac{1}{2}(Q_1 V_1 + Q_2 V_2 + Q_3 V_3).
\end{aligned}$$

◆ General expression of potential energy W_e for n discrete charges:

$$\boxed{W_e = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k \qquad \text{(J)}}$$

Potential at Q_k: $$V_k = \frac{1}{4\pi\epsilon_0}\sum_{\substack{j=1 \\ (j\neq k)}}^{N} \frac{Q_j}{R_{jk}}$$

Two remarks:

- W_e can be negative;
- W_e represents the interaction energy (mutual energy), not including the energy to assemble the charges themselves.

For continuous charges ρdv:

$$\boxed{W_e = \frac{1}{2}\int_{V'} \rho V \, dv \qquad \text{(J).}}$$

◆ Energy unit: Joule (J) and electron voltage (eV)

$$1 \quad (\text{eV}) = (1.60 \times 10^{-19}) \times 1 = 1.60 \times 10^{-19}$$

3.11.1 Electrostatic Energy In Terms of Field Quantities

◆ W_e in terms of ***E*** and/or ***D***:

$$\nabla \cdot D = r$$

$$W_e = \frac{1}{2}\int_{V'} (\nabla \cdot \mathbf{D}) V \, dv \quad \Leftarrow \quad \boxed{W_e = \frac{1}{2}\int_{V'} \rho V \, dv}$$

Using $\nabla \cdot (V\mathbf{D}) = V\nabla \cdot \mathbf{D} + \mathbf{D} \cdot \nabla V$, we have

$$W_e = \frac{1}{2}\int_{V'} \nabla \cdot (V\mathbf{D}) \, dv - \frac{1}{2}\int_{V'} \mathbf{D} \cdot \nabla V \, dv$$

$$= \frac{1}{2}\oint_{S'} V\mathbf{D} \cdot \mathbf{a}_n \, ds + \frac{1}{2}\int_{V'} \mathbf{D} \cdot \mathbf{E} \, dv,$$

As we let R → □□

$$\oint_{S'} V\vec{D} \cdot \vec{a}_n \, ds \to 0$$

$$\boxed{W_e = \frac{1}{2}\int_{V'} \mathbf{D} \cdot \mathbf{E} \, dv}$$

Use $\boldsymbol{D} = \varepsilon\boldsymbol{E}$, we have

$$\boxed{W_e = \frac{1}{2}\int_{V'} \epsilon E^2\, dv \qquad \text{(J)}}$$

and

$$\boxed{W_e = \frac{1}{2}\int_{V'} \frac{D^2}{\epsilon}\, dv \qquad \text{(J).}}$$

◆ **Electrostatic energy density w_e**

$$\boxed{W_e = \int_{V'} w_e\, dv.}$$

$$w_e = \frac{1}{2}\mathbf{D}\cdot\mathbf{E} \qquad (\mathrm{J/m^3})$$

$$w_e = \frac{1}{2}\epsilon E^2 \qquad (\mathrm{J/m^3})$$

$$w_e = \frac{D^2}{2\epsilon} \qquad (\mathrm{J/m^3}).$$

3.11.2 Electrostatic Forces

◆ Limit of Coulomb's law in solving the force of complex charge systems

$$\vec{F}_{12} = q_2\vec{E}_{12} = \frac{q_1 q_2 (\vec{R} - \vec{R}')}{4\pi\varepsilon_0 \left|\vec{R} - \vec{R}'\right|^3}$$

◆ Calculate **electrostatic forces *F_Q*** from the energy:

Method: Principle of virtual displacement

The mechanic work *dW* done by displacing the one charged body by a virtual differential distance:

$$\boxed{dW = \mathbf{F}_Q \cdot d\ell,} \xrightarrow[\text{system}]{\text{Isolated}} \boxed{dW = -dW_e = \mathbf{F}_Q \cdot d\ell.}$$

Gradient of a scalar (2-28, p43)

$$dV = (\nabla V) \cdot d\vec{l}$$

$$\boxed{dW_e = (\nabla W_e) \cdot d\ell.}$$

Electrostatic Force:

$$\boxed{\mathbf{F}_Q = -\nabla W_e \qquad \text{(N).}}$$