# 第五讲

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$L[y] = f(x)$, $y = \sum C_i y_i + y^*$ ($f$), $y^* = x^k P(x) e^{\alpha x}$

(二): $f(x) = P_m(x) e^{\alpha x} \cos\beta x$ 或 $Q_l(x) e^{\alpha x} \sin\beta x$

或 $P_m(x) e^{\alpha x} \cos\beta x + Q_l(x) e^{\alpha x} \sin\beta x$

$y^* = ?$ (Euler 公式, $e^{ix} = \cos x + i \sin x$)

Taylor

$f(x) = Re\left(P_m(x) e^{(\alpha+i\beta)x}\right)$

或 $Im\left(Q_l(x) e^{(\alpha+i\beta)x}\right)$

$\tilde{f} \leftrightarrow \boxed{L[\tilde{y}^*] = \tilde{f}}$

$y^* = Re(\tilde{y}^*)$

$\tilde{y}^* = y^* + i y_1^*$

$\tilde{f} = L[\tilde{y}^*] = L[y^*] + i L[y_1^*]$

$\parallel$

$f + i f_1$

$\tilde{y}^* = x^k R_m(x) e^{(\alpha+i\beta)x}$，其中 $k$ 代表 $\alpha+i\beta$ 是特征方程根的重数

例: $y'' - y = e^{x} \overset{\cos}{\cancel{\sin 2}}\, x$

解: 齐次方程: $y'' - y = 0$ 的通解为:

$y = C_1 e^{x} + C_2 e^{-x} \longleftarrow \boxed{\begin{array}{c}(\lambda^2-1)=0 \\ \lambda = \pm 1\end{array}}$

$e^{x} \sin 2x = Re\left(\square e^{(1+2i)x}\right)$

由于 $1+2i$ 不是特征根，$\tilde{y}^* = Ae^{(1+2i)x}$

$$A(1+2i)^2 e^{(1+2i)x} - A(1+2i)e^{(1+2i)x} = e^{(1+2i)x}$$

$$\Rightarrow A = \frac{1}{(1+2i)^2-1} = -\frac{1}{8}(1+i)$$

$$\therefore y^* = Re(\tilde{y}^*)$$

$$= Re\left(-\frac{1}{8}(1+i)e^{(1+2i)x}\right)$$

$$= Re\left(-\frac{1}{8}(1+i)(\cos 2x + i\sin 2x)e^x\right)$$

$$= -\frac{1}{8}e^x(\cos 2x - \sin 2x)$$

提示：$f = P_m(x)e^{\alpha x}\cos\beta x$

$\updownarrow$

$y^* = R(x)e^{\alpha x}\cos\beta x + Q(x)e^{\alpha x}\sin\beta x$ （✓）

一般情形：

$$y'' + py' + qy = f(x) = P_m(x)e^{\alpha x}\cos\beta + Q_l(x)e^{\alpha x}\sin\beta x$$

则：$y^* = x^k\left(R_h(x)e^{\alpha x}\cos\beta x + S_h(x)e^{\alpha x}\sin\beta x\right)$

$$k = \begin{cases} 0, & \text{当 } \alpha \pm i\beta \text{ 不是特征根} \\ 1, & \text{当 } \alpha \pm i\beta \text{ 是单重特征根} \end{cases}$$

$h = \max\{m, l\}$ （$2h+2$）个待定系数

例：$y'' - 2y' + 5y = xe^x\cos 2x + e^x\sin 2x$

召　齐次方程的通解为　　$\lambda^2 - 2\lambda + 5 = 0$

解：齐次方程的通解为：

$\lambda^2-2\lambda+5=0$

$\lambda=1\pm 2i$

$$y=C_1e^x\cos 2x+C_2e^x\sin 2x$$

由于 $1\pm 2i$ 是特征根，

则：令 $y^*(x)=x\left[(Ax+B)e^x\cos 2x+(Cx+D)e^x\sin 2x\right]$ 代入方程

$(y^*)'=[\quad\quad]+x[A$

$(y^*)''=\cdots$

$\vdots$

例：求 $y''+4y=2\cos 2x$，满足 $y|_{x=0}=0$，$y'|_{x=0}=2$ 的解。

解：先求齐次方程：$y=C_1\cos 2x+C_2\sin 2x$

然后：$y^*=x(A\cos 2x+B\sin 2x)$

$(y^*)'=A\cos 2x+B\sin 2x+x(-2A\sin 2x+2B\cos 2x)$

$(y^*)''=2(-2A\sin 2x+2B\cos 2x)+x(-4A\cos 2x-4B\sin 2x)$

$\therefore (y^*)''+4(y^*)=(-4A\sin 2x+4B\cos 2x)\cancel{-4x(A\cos 2x+B\sin 2x)}$
$\cancel{+4x(A\cos 2x+B\sin 2x)}$

$\|$

$2\cos 2x$

$\Rightarrow$：$A=0$，$B=\frac{1}{2}$.

$\therefore\ y^* = \frac{1}{2}x\sin 2x.$

$\therefore\ y = C_1\cos 2x + C_2\sin 2x + \frac{1}{2}x\sin 2x$

再根据 $y(0)=0,\ y'(0)=2$

$\Rightarrow: C_1=0,\quad C_2=1$

$\therefore\ y=\sin 2x + \frac{1}{2}x\sin 2x$

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总结：常系数线性常微分方程求解基本步骤.

1°：齐次方程（特征方程）：基本解组.

2°：特解 $(y^*)$：（待定系数）

推到高阶方程.

例：$f(x)$ 连续，且满足：$f(x)=e^{-x}+\frac{1}{2}\int_0^x (x-t)^2 f(t)\,dt$

求 $f(x)$.

解：令 $y=f(x)$，$y' = -e^{-x} + \left[\cancel{\frac{1}{2}\times 1\,(x-x)^2 f(x)} = 0 + \frac{1}{2}\int_0^x 2(x-t)f(t)\,dt\right]$

$= -e^{-x} + \int_0^x (x-t)f(t)\,dt$

$y'' = e^{-x} + \int_0^x f(t)\,dt$

$\boxed{y''' = -e^{-x} + y}$

$y(0)=f(0)=1,\ y'(0)=-1,\ y''(0)=1$

1°：（齐次方程）$\lambda^3-1=0,\ \Rightarrow: \lambda_1=1,\ \lambda_{2,3}=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

$$y = C_1 e^{x} + C_2 e^{\frac{1}{2}x}\cos\frac{\sqrt{3}}{2}x + C_3 e^{\frac{1}{2}x}\sin\frac{\sqrt{3}}{2}x$$

2°：特解 $(y^*)$：

$$y^* = Ae^{-x}.$$

$$-Ae^{-x} = e^{-x} + Ae^{-x} \Rightarrow A = \frac{1}{2}.$$

$$\therefore y = C_1 e^{x} + C_2 e^{\frac{1}{2}x}\cos\frac{\sqrt{3}}{2}x + C_3 e^{\frac{1}{2}x}\sin\frac{\sqrt{3}}{2}x + \frac{1}{2}e^{-x}$$

再根 $y(0)=1,\ y'(0)=-1,\ y''(0)=1$

得到：

$$C_1 = -\frac{1}{6}$$
$$C_2 = \frac{2}{3}$$
$$C_3 = 0.$$

$\therefore y = \cdots$

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一般线性常微分方程的解法

- （可转化为常系数的微分方程）.（Euler 方程）

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_{n-1}x\frac{dy}{dx} + a_n y = f(x)$$

以二阶方程为例：

$$a_0 x^2 y'' + a_1 x y' + a_2 y = f(x)$$

$$f(x) \to \tilde{f}(t)$$

令 $e^t = x\ (x>0)$，$t = \ln x$，或 $-e^t = x\ (x<0)$

$$y(x) = \tilde{y}(t)$$

$$y'(x) = \frac{d\tilde{y}}{dt}\frac{dt}{dx} = \frac{d\tilde{y}}{dt}\times\frac{1}{x}.$$

$$y''(x) = -\frac{1}{x^2}\frac{d\tilde{y}}{dt} + \left(\frac{d^2\tilde{y}}{dt^2}\times\frac{dt}{dx}\right)\times\frac{1}{x}$$

$$= \frac{1}{x^2}\left(\frac{d^2\tilde{y}}{dt^2} - \frac{d\tilde{y}}{dt}\right)$$

得到：$a_0 x^2 \times \frac{1}{x^2}\left(\frac{d^2\tilde{y}}{dt^2} - \frac{d\tilde{y}}{dt}\right) + a_1 x \times \frac{1}{x}\frac{d\tilde{y}}{dt} + a_2\tilde{y}$

$$\boxed{= a_0\left(\frac{d^2\tilde{y}}{dt^2} - \frac{d\tilde{y}}{dt}\right) + a_1\frac{d\tilde{y}}{dt} + a_2\tilde{y} = \tilde{f}(t)}$$

常系数…

$x^n y^{(n)} = \underline{D(D-1)(D-n+1)}\tilde{y}$，其中 $D = \frac{d}{dt}$

$D^2 = \frac{d^2}{dt^2}$

$D^3 = \frac{d^3}{dt^3}$

⋮

例：$\underline{x^2\frac{d^2y}{dx^2}} + x\frac{dy}{dx} = 6\ln x - \frac{1}{x}$.

解：令 $x = e^t$，$\tilde{y}(t) = y(x)$.

则原方程可化为：

$$\underline{D(D-1)\tilde{y} + D\tilde{y}} = 6t - e^{-t}$$

$$\| $$

$$D^2\tilde{y} = \frac{d^2\tilde{y}}{dt^2} = 6t - e^{-t}.$$

$y'' = f(x)$

$\boxed{y'' = f(x, y')}$

$y'' = f(y, y')$

积分两次可得：

$$\tilde{y} = c_1 + c_2 t + t^3 - e^{-t}$$

$$y = C_1 + C_2\ln x + (\ln x)^3 - \frac{1}{x}$$

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一般二阶齐次线性微分方程．$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$

一般二阶齐次线性微分方程. $\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=0$

$y'+p(x)y=f(x)$ (通解)

思想 (降阶)

设 $y_1(x)$ 是原方程的解.

$P_{115}$: 26, 29, 35, 49, 50