Chapter 2

Optical Resonator and Gaussian Beam optics

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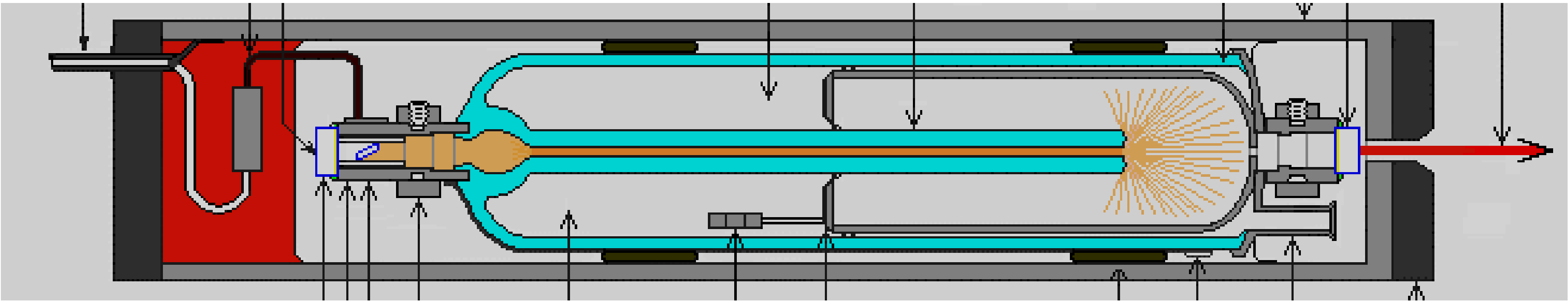
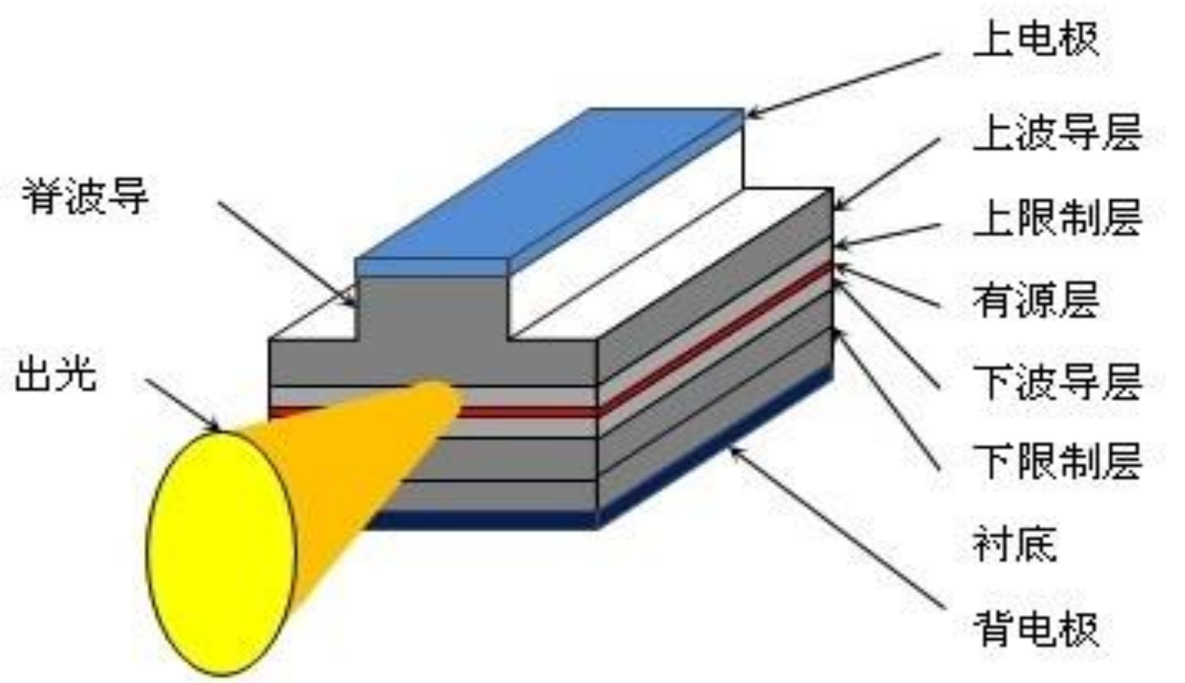
He-Ne laser



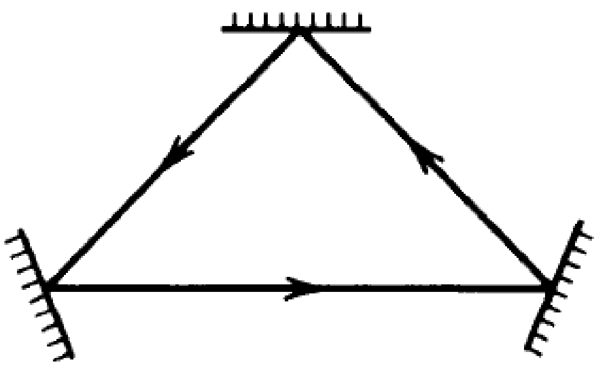

上电极
上波导层
脊波导
上限制层
有源层
出光
下波导层
下限制层
衬底
背电极

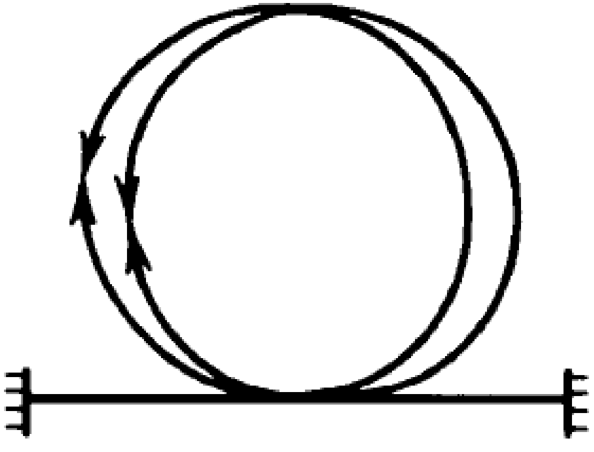
Why do we present the optical resonator?

- It is the fundamental optical phenomena
- It is also enhancer in many optical detection, and multiplication
- It is the cause that to create the Gaussian beam.
- It is the fundamental part of laser, keeps laser light to be narrow band wavelength (coherent), good directionality (high brightness) ,and beam form.

What is an optical resonator?

An optical resonator, the optical counterpart of an electronic resonant circuit, confines and stores light at certain resonance frequencies. It may be viewed as an optical transmission system incorporating feedback; light circulates or is repeatedly reflected within the system, without escaping.

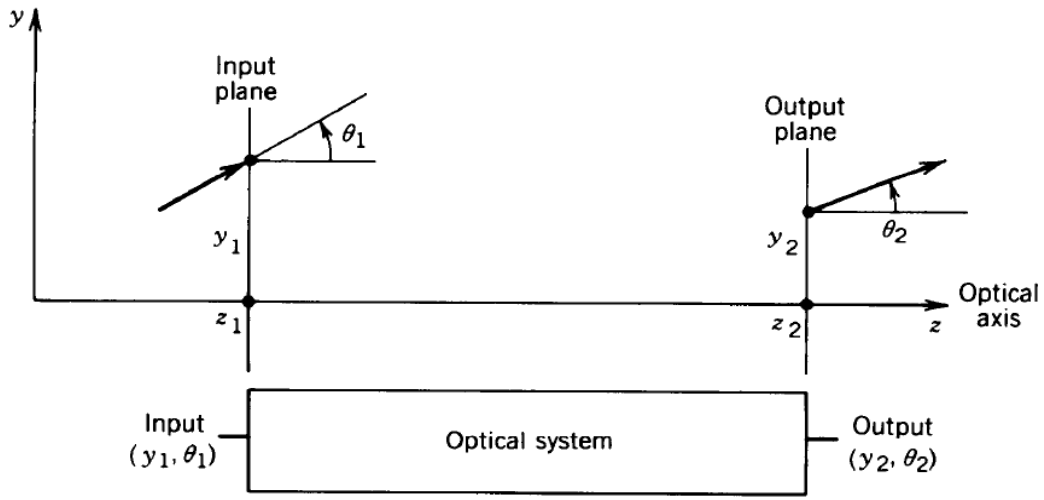




Contents

2.1 Brief review of Matrix optics

Light propagation in a optical system, can use a matrix M, whose elements are A, B, C, D, characterizes the optical system Completely (known as the ***ray-transfer matrix***.) to describe the rays transmission in the optical components.

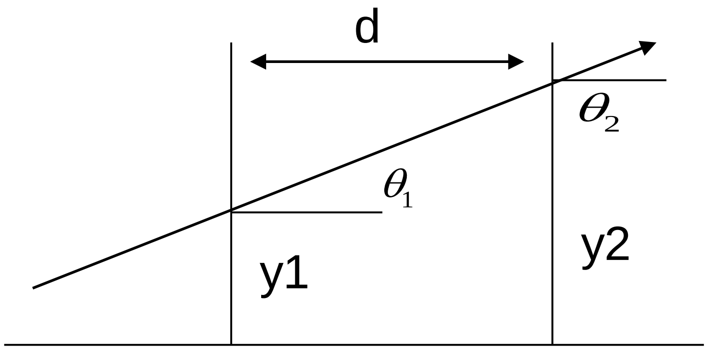


One can use two parameters:

y: the high

θ: the angle above z axis

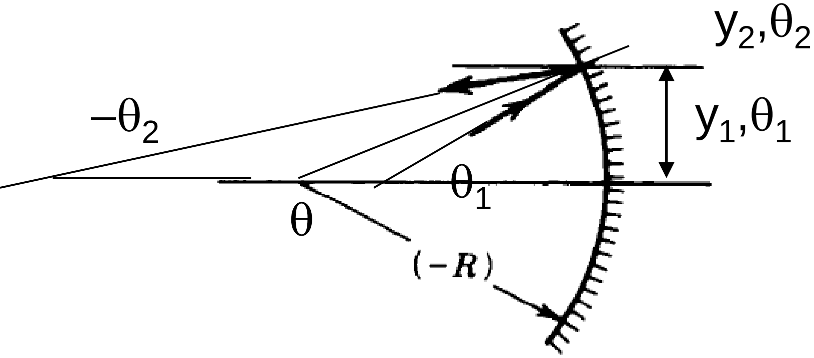
$$\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix}$$



$$y_2 = y_1 + d \cdot tg\theta_1$$
$$\theta_2 = \theta_1$$

For the paraxial rays $\qquad tg\theta \approx \theta$

$$\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix}$$



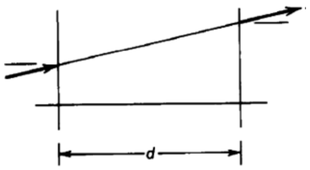
Concave, $R<0$; convex, $R>0$

$$y_2 = y_1$$
$$\theta_2 = \frac{2}{R} y_1 + \theta_1$$

$$\theta \approx \frac{y_1}{-R}$$

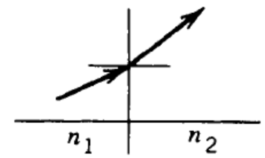
Along z upward angle is positive, and downward is negative

Free-Space Propagation



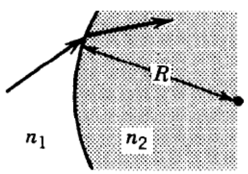
$$M=\begin{bmatrix}1 & d\\ 0 & 1\end{bmatrix}$$

Refraction at a Planar Boundary



$$M=\begin{bmatrix}1 & 0\\ 0 & \dfrac{n_1}{n_2}\end{bmatrix}$$

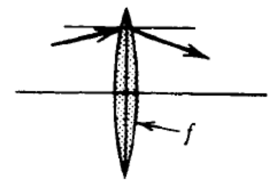
Refraction at a Spherical Boundary



Convex, $R>0$; concave, $R<0$

$$M=\begin{bmatrix}1 & 0\\ -\dfrac{(n_2-n_1)}{n_2 R} & \dfrac{n_1}{n_2}\end{bmatrix}$$

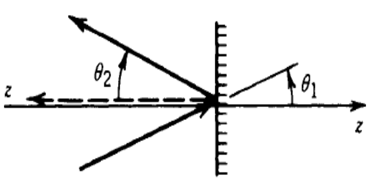
Transmission Through a Thin Lens



Convex, $f>0$; concave, $f<0$

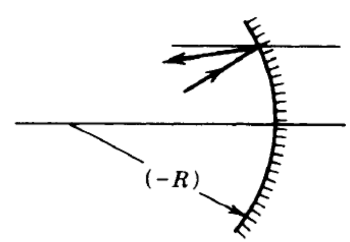
$$M=\begin{bmatrix}1 & 0\\ -\dfrac{1}{f} & 1\end{bmatrix}$$

Reflection from a Planar Mirror



$$M=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

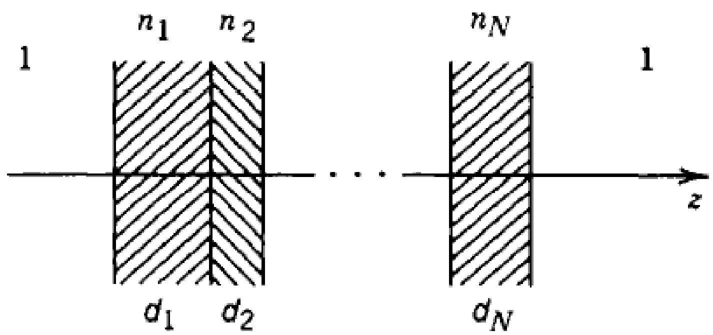
Reflection from a Spherical Mirror



Concave, $R<0$; convex, $R>0$

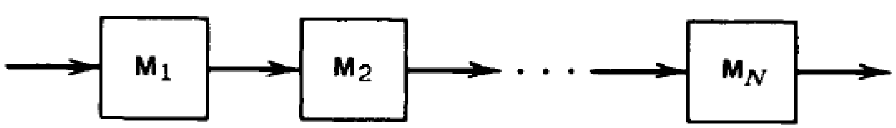
$$M=\begin{bmatrix}1 & 0\\ \dfrac{2}{R} & 1\end{bmatrix}$$

A Set of Parallel Transparent Plates.



$$M = \begin{bmatrix} 1 & \sum \frac{d_i}{n_i} \\ 0 & 1 \end{bmatrix}$$

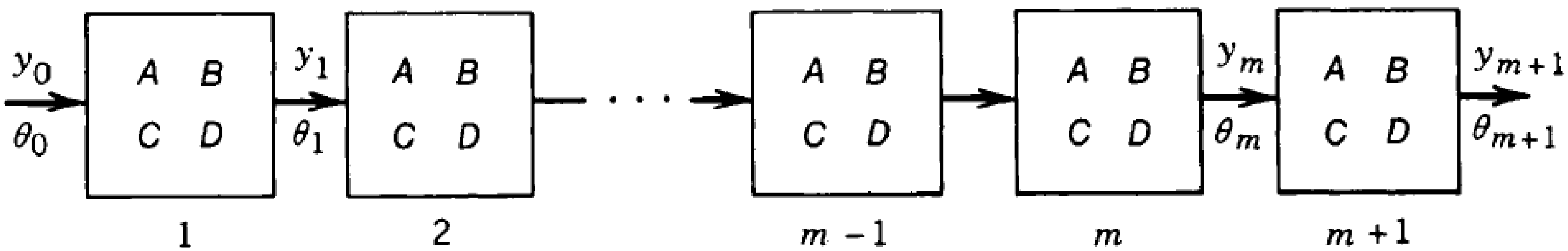
Matrices of Cascaded Optical Components



$$\boxed{M = M_N M_{N-1} \ldots M_1}$$

Periodic Optical Systems

The reflection of light between two parallel mirrors forming an optical resonator is a periodic optical system is a cascade of identical unit system.



A periodic system is composed of a cascade of identical unit systems (stages), each with a ray-transfer matrix (A, B, C, D). A ray enters the system with initial position y_0 and slope θ_0. To determine the position and slope (y_m,θ_m) of the ray at the exit of the m^{th} stage, we apply the ABCD matrix m times,

$$\begin{bmatrix} y_m \\ \theta_m \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^m \begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix} \Rightarrow \begin{aligned} y_{m+1} &= Ay_m + B\theta_m \\ \theta_{m+1} &= Cy_m + D\theta_m \end{aligned}$$

$$y_{m+1} = Ay_m + B\theta_m$$
$$\theta_{m+1} = Cy_m + D\theta_m$$

From these equation, we have

So that

$$\left.\begin{aligned}\theta_m &= \frac{y_{m+1} - Ay_m}{B} \\ \theta_{m+1} &= \frac{y_{m+2} - Ay_{m+1}}{B}\end{aligned}\right\}$$

And then:

$$y_{m+2} = 2by_{m+1} - F^2 y_m$$

linear differential equations,

where

$$b = {(A+D)}\big/{2}$$

and

$$F^2 = Ad - BC = \det[M]$$

If we assumed:

$$y_m = y_0 h^m$$

So that, we have

$$h^2 - 2bh + F^2 = 0 \quad \Rightarrow \quad h = b \pm i\sqrt{F^2 - b^2}$$

If we defined $\phi = \cos^{-1}\left(b/F\right)$

We have $b = F\cos\phi$ $\sqrt{F^2 - b^2} = F\sin\phi$

then $h = F(\cos\phi \pm i\sin\phi) = Fe^{\pm i\phi}$ $\Rightarrow$ $y_m = y_0 F^m e^{\pm im\phi}$

A general solution may be constructed from the two solutions with positive and negative signs by forming their linear combination. The sum of the two exponential functions can always be written as a harmonic (circular) function,

$$y_m = y_0 F^m \sin(m\phi + \phi_0) = y_{\max} F^m \sin(m\phi + \phi_0)$$

If F=1, then $$y_m = y_{\max}\sin(m\phi+\phi_0)$$

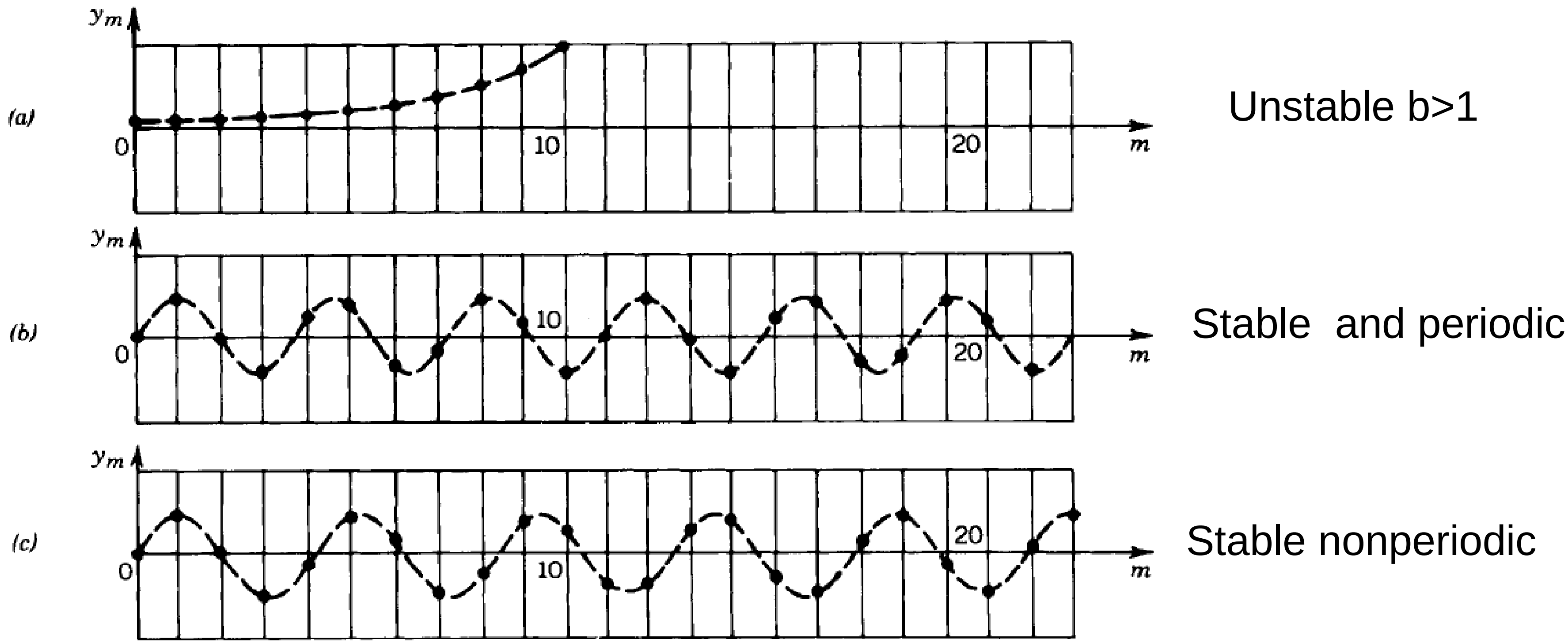
Condition for a Harmonic Trajectory: *if y_m be harmonic, the ϕ=cos^{-1}b must be real,* We have condition

$$|b| \le 1 \qquad \text{or} \qquad \left|\frac{A+D}{2}\right| \le 1$$

The bound $|b| \le 1$ therefore provides a condition of stability (boundedness) of the ray trajectory

If, instead, |b| > 1, ϕ is then imaginary and the solution is a hyperbolic function (cosh or sinh), which increases without bound. A harmonic solution ensures that y, is bounded for all m, with a maximum value of y_{max}. The bound |b|< 1 therefore provides a ***condition of stability* (boundedness) of the ray trajectory.**

Condition for a Periodic Trajectory



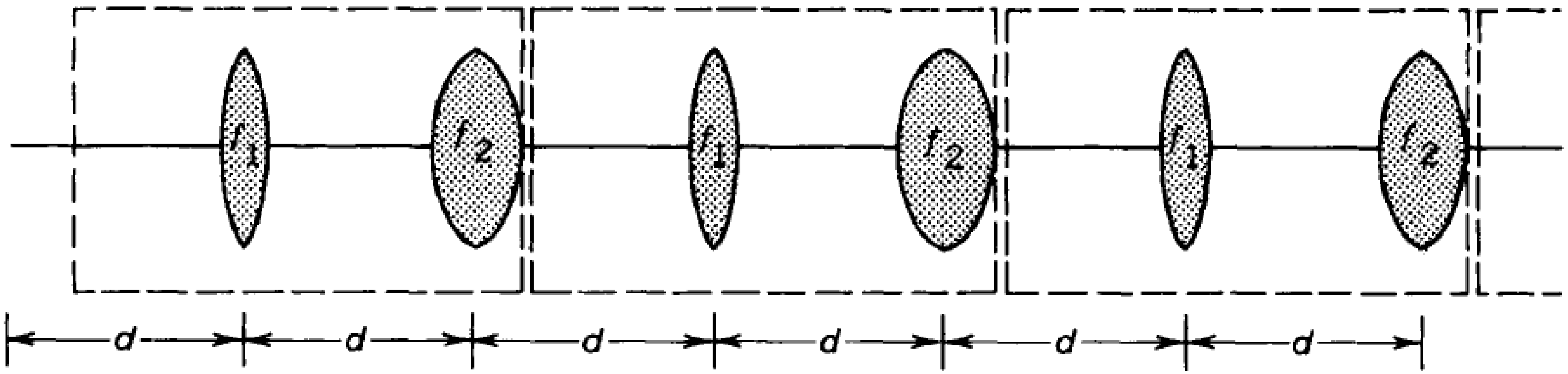
The harmonic function is periodic in m, if it is possible to find an integer s such that $y_{m+s} = y_m$, for all m. The smallest such integer is the period.

The necessary and sufficient condition for a periodic trajectory is:

$\boldsymbol{s\phi = 2\pi q}$, where q is an integer

EXERCISE

A Periodic Set of Pairs of Different Lenses. Examine the trajectories of paraxial rays through a periodic system composed of a set of lenses with alternating focal lengths f_1 and f_2 as shown in Fig. Show that the ray trajectory is bounded (stable) if



$$0 \le \left(1 - \frac{d}{2f_1}\right)\left(1 - \frac{d}{2f_2}\right) \le 1.$$

$$M = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{bmatrix}\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{bmatrix}\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\frac{d}{f_1} & 2d-\frac{d^2}{f_1} \\ \frac{d}{f_1 f_2}-\frac{1}{f_1}-\frac{1}{f_2} & -\frac{d}{f_2}+(1-\frac{d}{f_1})(1-\frac{d}{f_2}) \end{bmatrix}$$

Home work

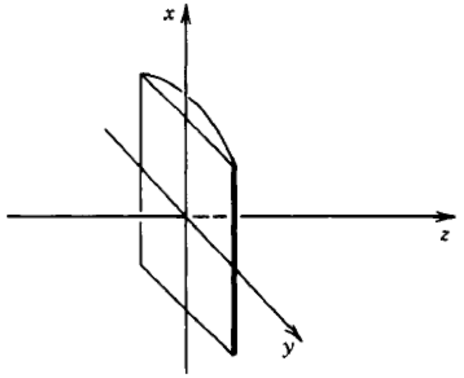
1. Ray-Transfer Matrix of a Lens System. Determine the ray-transfer matrix for an optical system made of a thin convex lens of focal length f and a thin concave lens of focal length -f separated by a distance f. Discuss the imaging properties of this composite lens.

Home works

2. 4 X 4 Ray-Transfer Matrix for Skewed Rays. Matrix methods may be generalized to describe skewed paraxial rays in circularly symmetric systems, and to astigmatic (non-circularly symmetric) systems. A ray crossing the plane z = 0 is generally characterized by four variables-the coordinates (x, y) of its position in the plane, and the angles (e,, ey) that its projections in the x-z and y-z planes make with the z axis. The emerging ray is also characterized by four variables linearly related to the initial four variables. The optical system may then be characterized completely, within the paraxial approximation, by a 4 X 4 matrix.

(a) Determine the 4 x 4 ray-transfer matrix of a distance d in free space.

(b) Determine the 4 X 4 ray-transfer matrix of a thin cylindrical lens with focal length f oriented in the y direction. The cylindrical lens has focal length f for rays in the y-z plane, and no focusing power for rays in the x-z plane.



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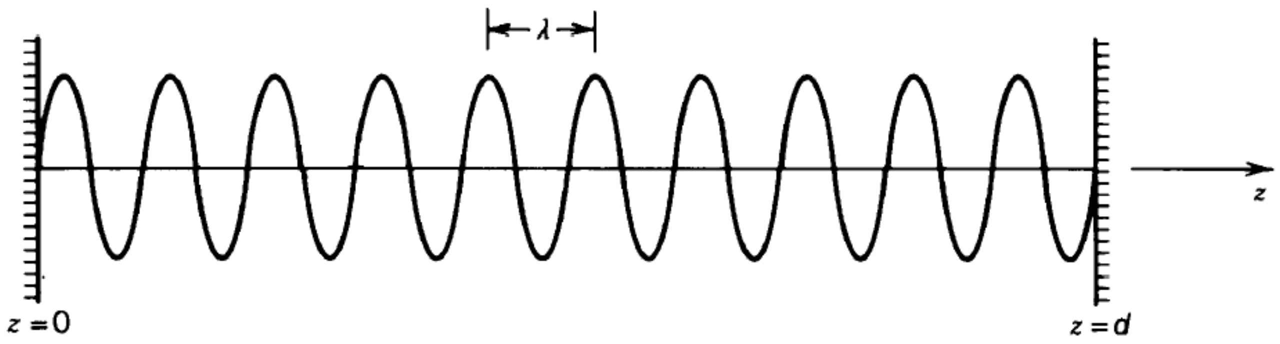
2.2 Planar Mirror Resonators

Charles Fabry (1867-1945),

Alfred Perot (1863-1925),

2.2 Planar Mirror Resonators



This simple one-dimensional resonator is known as a Fabry-Perot etalon.

A. Resonator Modes

Resonator Modes as Standing Waves

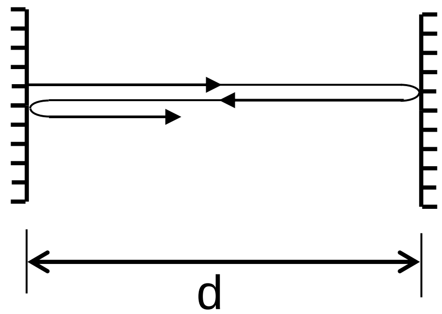
A monochromatic wave of frequency v has a wavefunction as

$$u(r,t) = \mathrm{Re}\left\{U(r)\exp(i2\pi vt)\right\}$$

Represents the transverse component of electric field.
The complex amplitude U(r) satisfies the Helmholtz equation; $\nabla^2 U + k^2 U = 0,$

Where *k =2πv/c* called wavenumber, *c* speed of light in the medium the modes of a resonator must be the solution of Helmholtz equation with the boundary conditions:

$$U(r)=0\begin{cases} z=0 \\ z=d \end{cases}$$

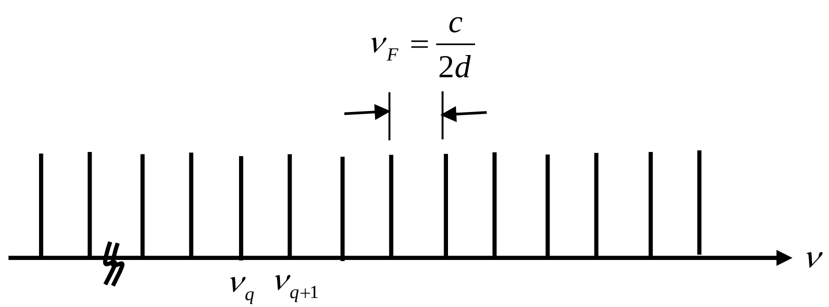


So that the general solution is standing wave:

$$U(\vec{r})=A\sin kz$$

With boundary condition, we have $kd=q\pi$

q is integer.



$$k_q=\frac{q\pi}{d} \qquad \because \quad k=2\pi\big/\lambda=\frac{2\pi\nu}{c}$$

Resonance frequencies

$$\nu_q=q\frac{c}{2d},q=1,2,\ldots,$$

$$\nu_F\equiv\nu_q-\nu_{q-1}=\frac{c}{2d}$$

The resonance wavelength is: $$\lambda_q = \frac{c}{\nu_q} = 2d/q$$

The length of the resonator, $d = q\,\lambda_q/2$, is an integer number of half wavelength

Attention: $c = c_0 / n$ Where *n* is the refractive index in the resonator

Resonator Modes as Traveling Waves

A mode of the resonator: is a self-reproducing wave, i.e., a wave that reproduces itself after a single round trip , The phase shift imparted by a single round trip of propagation (a distance 2d) must therefore be a multiple of 2π.

$$\varphi = k2d = \frac{4\pi n}{\lambda_0} d = \frac{4\pi \nu}{c} d \equiv q2\pi \qquad q = 1,2,3,\ldots$$

Density of Modes (1D)

The density of modes *M(ν),* which is the number of modes per unit frequency per unit length of the resonator, is

$$M(\nu)=\frac{4}{c}$$

For 1D resonator

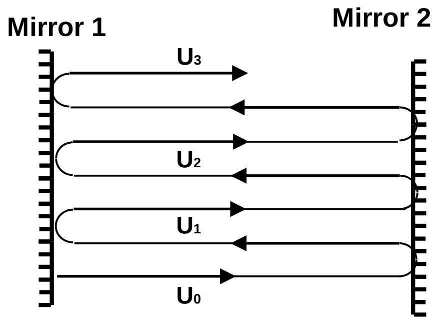
The number of modes in a resonator of length *d* within the frequency interval *Δν* is:

$$\frac{4}{c}d\Delta\nu$$

This represents the number of degrees of freedom for the optical waves existing in the resonator, i.e., the number of independent ways in which these waves may be arranged.

Losses and Resonance Spectral Width

The magnitude ratio of two consecutive phasors is the round-trip amplitude attenuation factor *r* introduced by the two mirror reflections and by absorption in the medium. Thus:



$$U_1 = hU_0 = \gamma e^{-i\varphi}U_0 = \gamma e^{-i\frac{4\pi nd}{\lambda}}U_0 = \gamma e^{-i2kd}U_0$$

So that, the sum of the sequential reflective light with field of

$$U = U_0 + U_1 + U_2 + U_3 + \ldots = U_0(1 + h + h^2 + h^3 + \ldots) = U_0 / (1-h)$$

$$I = |U|^2 = |U_0|^2 / \left|1-\gamma e^{-i\varphi}\right|^2 = I_0 / (1+\gamma^2 - 2\gamma\cos\varphi) = I_0 / \left[(1-\gamma)^2 + 4\gamma\sin^2\left(\varphi/2\right)\right]$$

finally, we have

$$I = \frac{I_{\max}}{1+(2F/\pi)^2\sin^2(\varphi/2)}, \qquad I_{\max} = \frac{I_0}{(1-\gamma)^2}$$

$$F = \frac{\pi\gamma^{1/2}}{1-\gamma}$$

Finesse of the resonator

The spectral peak width

$$I = \frac{I_{\max}}{1 + (2F/\pi)^2 \sin^2(\varphi/2)}$$

$$\frac{1}{2} I_{\max} = \frac{I_{\max}}{1 + (2F/\pi)^2 \sin^2(\varphi/2)}$$

$$(2F/\pi)^2 \sin^2(\varphi_{\lambda_1\lambda_0}/2) = 1$$

$$\sin(\varphi_{\lambda_1\lambda_0}/2) = \pi/2F$$

$$\varphi_{\lambda_1\lambda_0} = \pi/F$$

Full width half maximum is $2\varphi_{\lambda_1\lambda_0} = 2\pi/F = \Delta\varphi$

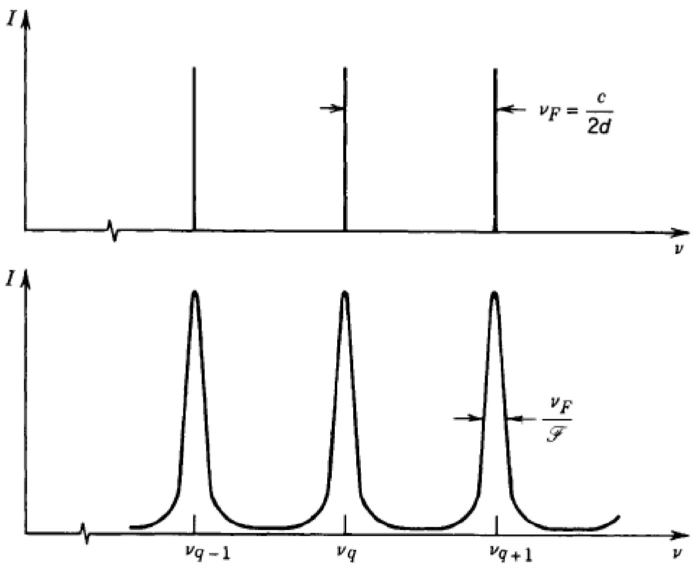
$\because$ $\varphi = \dfrac{4\pi\nu d}{c}$ So that $\delta\nu = \left(c/_{4\pi d}\right)\Delta\varphi = \nu_F/_F$

The resonance spectral peak has a full width of half maximum (FWHM):

$$\delta\nu = \left(c/4\pi d\right)\Delta\varphi = \nu_F / F$$

Due to $\varphi = \dfrac{4\pi\nu d}{c}$ We have $\boxed{I = \dfrac{I_{\max}}{1+(2F/\pi)^2 \sin^2(\pi\nu/\nu_F)}}$ $I_{\min} = \dfrac{I_{\max}}{1+(2F/\pi)^2}$

where $\nu_F = c/2d$ $\nu = \nu_q = q\nu_F, q = 1, 2, \ldots,$



$$\nu_F = \frac{c}{2d}$$

$$\boxed{\delta\nu = \frac{\nu_F}{F}}$$

Spectral response of Fabry-Perot Resonator

The intensity *I* is a periodic function of φ with period 2π. The dependence of *I* on ν, which is the spectral response of the resonator, has a similar periodic behavior since $\varphi = 4\pi\nu d/c$ is proportional to ν. This resonance profile:

$$I = \frac{I_{\max}}{1+(2F/\pi)^2 \sin^2(\pi\nu/\nu_F)}$$

The maximum $I = I_{max}$, is achieved at the resonance frequencies

$$\nu = \nu_q = q\nu_F, q = 1, 2, \ldots,$$

whereas the minimum value

$$I_{\min} = \frac{I_{\max}}{1+(2F/\pi)^2}$$

The FWHM of the resonance peak is

$$\delta\nu = \frac{c}{4\pi d}\Delta\varphi = \frac{\nu_F}{F}$$

Sources of Resonator Loss

- Absorption and scattering loss during the round trip: *exp (-2α_sd)*
- Imperfect reflectance of the mirror: R_1, R_2

$$\gamma^2 = R_1 R_2 \exp(-2\alpha_s d)$$

Defineding that

$$\gamma^2 = \exp(-2\alpha_r d)$$

we get:

α_r is an effective overall distributed-loss coefficient, which is used generally in the system design and analysis

$$\alpha_r = \alpha_s + \frac{1}{2d} \ln \frac{1}{R_1 R_2}$$

$$\alpha_r = \alpha_s + \frac{1}{2d} \ln \frac{1}{R_1 R_2} \equiv \alpha_s + \alpha_{m1} + \alpha_{m2}$$

$$\alpha_{m1} = \frac{1}{2d} \ln \frac{1}{R_1} \qquad \alpha_{m1} = \frac{1}{2d} \ln \frac{1}{R_1}$$

- If the reflectance of the mirrors is very high, approach to 1, so that

$$R_1 \approx 1 \approx R_2 \equiv R$$

- The above formula can approximate as

$$\alpha_{m1} \approx \frac{1-R_1}{2d} \approx \frac{1-R_2}{2d} \approx \alpha_{m2} \approx \frac{1-R}{2d}$$

$$\alpha_r = \alpha_s + \frac{1}{2d}\ln\frac{1}{R_1R_2} \equiv \alpha_s + \alpha_{m1} + \alpha_{m2} \Rightarrow \alpha_r \approx \alpha_s + \frac{1-R}{d}$$

The finesse ***F*** can be expressed as a function of the effective loss coefficient α_r,

$$F = \frac{\pi\exp(-\alpha_r d/2)}{1-\exp(-\alpha_r d)}$$

Because $\alpha_r d<<1$, so that $exp(-\alpha_r d)=1-\alpha_r d$, we have:

$$F \approx \frac{\pi}{\alpha_r d}$$

The finesse is inversely proportional to the loss factor $\alpha_r d$

Photon Lifetime of Resonator

The relationship between the resonance linewidth and the resonator loss may be viewed as a manifestation of the time-frequency uncertainty relation. Form the linewidth of the resonator, we have

$$\delta\nu \approx \frac{c/2d}{\pi/\alpha_r d} = \frac{c\alpha_r}{2\pi}$$

Because α_r is the loss per unit length, $c\alpha_r$ is the loss per unit time, so that we can Defining the characteristic decay time as the resonator lifetime or photon lifetime

$$\tau_p = \frac{1}{c\alpha_r}$$

The resonance line broadening is seen to be governed by the decay of optical energy arising from resonator losses

$$\delta\nu = \frac{1}{2\pi\tau_p}$$

The Quality Factor Q

The quality factor Q is often used to characterize electrical resonance circuits and microwave resonators, for optical resonators, the Q factor may be determined by percentage of that stored energy to the loss energy per cycle:

$$Q = \frac{2\pi(storedenergy)}{energylosspercycle}$$

Large Q factors are associated with low-loss resonators

For a resonator of loss at the rate $c\alpha_r$ (per unit time), which is equivalent to the rate $c\alpha_r / \nu_0$ (per cycle), so that

$$Q = 2\pi\left[1/(c\alpha_r/\nu_0)\right] \quad \because \quad \delta\nu = c\alpha_r/2\pi \quad \Longrightarrow \quad \boxed{Q = \frac{\nu_0}{\delta\nu}}$$

The quality factor is related to the resonator lifetime (photon lifetime)

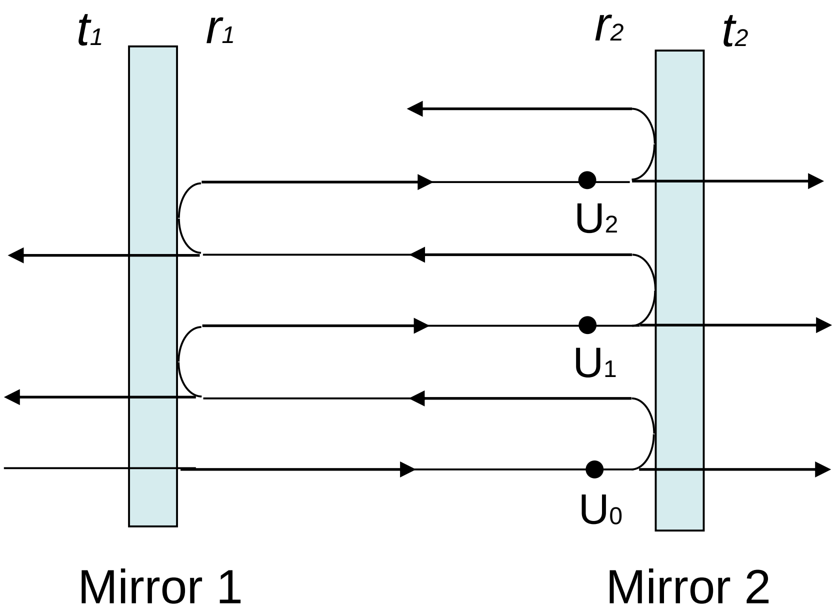
$$\because \quad \tau_p = 1/c\alpha_r = \frac{1}{2\pi\delta\nu} \qquad \boxed{Q = 2\pi\nu_0\tau_p}$$

The quality factor is related to the finesse of the resonator by $\boxed{Q = \frac{\nu_0}{\nu_F}F}$

- In summary, three parameters are convenient for characterizing the losses in an optical resonator:
  - the finesse $\mathscr{F}$
  - the loss coefficient α_r (cm^{-1}),
  - FWHM $\Delta\nu=\frac{\nu_F}{\mathscr{F}}$
  - photon lifetime $\tau_p = 1/c\alpha_r$, (seconds)
  - quality factor Q=$\frac{\nu_0}{\nu_F}\mathscr{F}$

B. The Resonator as a Spectrum Analyzer

Transmission of a plane wave across a planar-mirror resonator (Fabry-Perot etalon)



$$T(\nu) = {I_t}/{I}$$

$$T(\nu) = \frac{T_{\max}}{1+(2F/\pi)^2 \sin^2(\pi\nu/\nu_F)}$$

Where:

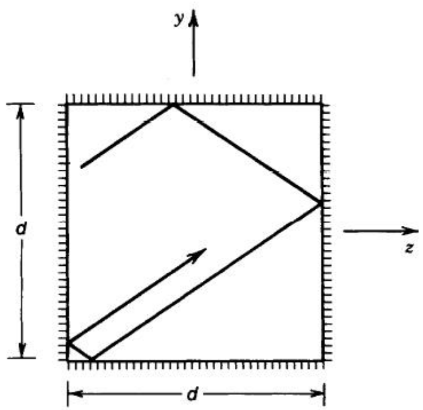
$$T_{\max} = \frac{|t|^2}{(1-\gamma)^2}, t = t_1 t_2, \gamma = \gamma_1\gamma_2 \qquad F = \frac{\pi\nu^{1/2}}{1-\gamma}$$

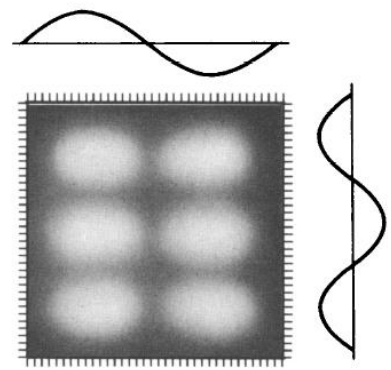
The change of the length of the cavity will change the resonance frequency

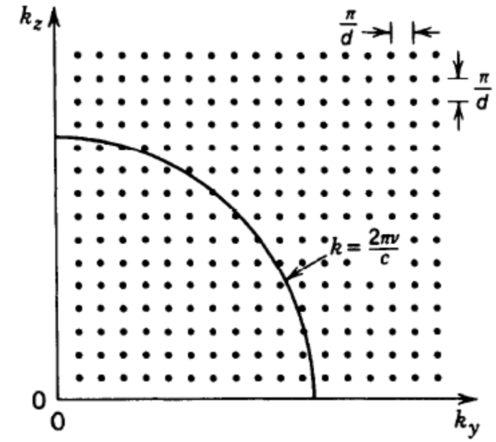
$$\Delta\nu_q = -\left({qc}/{2d^2}\right)\Delta d = {-\nu_q \Delta d}/{d}$$

C. Two- and Three-Dimensional Resonators

- Two-Dimensional Resonators







$$k_y = \frac{q_y \pi}{d}, k_z = \frac{q_z \pi}{d}, q_y = 1, 2, \ldots, q_z = 1, 2, \ldots,$$

$$k^2 = k_y^2 + k_z^2 = (\frac{2\pi\nu}{c})^2$$

- Mode density

the number of modes per unit frequency per unit surface of the resonator

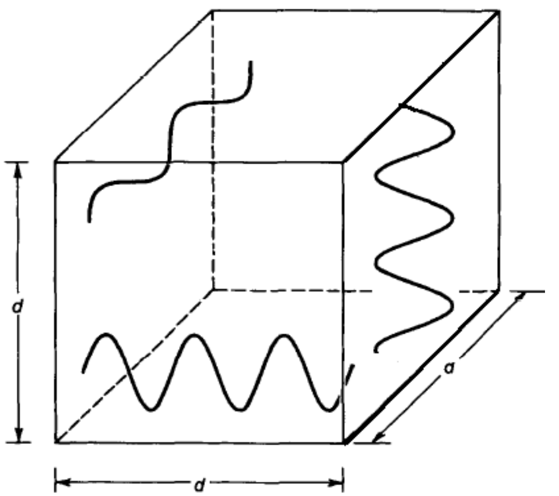
The mode number between k$\in$ (0,v) is

$$N = \frac{k\ space}{surface\ per\ mode} \times 2 = \frac{\pi\left(\frac{2\pi\upsilon}{c}\right)^2/4}{(\frac{\pi}{d})^2} \times 2 = \frac{2\pi\upsilon^2 d^2}{c^2}$$

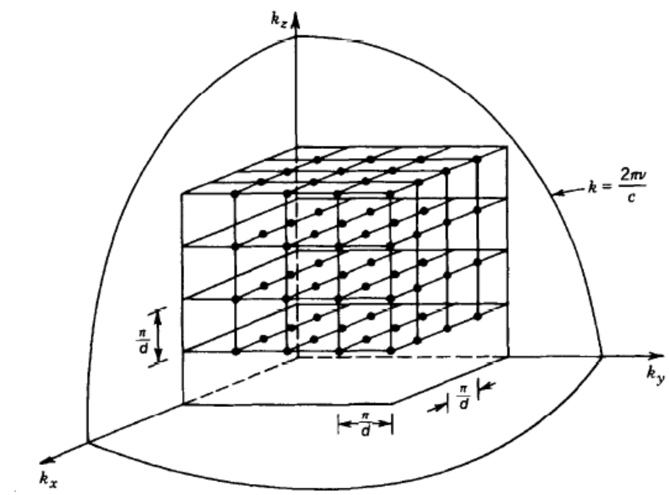
$$M(\nu) = \frac{4\pi\nu}{c^2}$$

Three-Dimensional Resonators

Physical space resonator



Wave vector space



$$k_x = \frac{q_x \pi}{d}, k_y = \frac{q_y \pi}{d}, k_z = \frac{q_z \pi}{d}, q_x, q_y, q_z = 1, 2, \ldots,$$

$$k^2 = k_x^2 + k_y^2 + k_z^2 = (\frac{2\pi\nu}{c})^2$$

Mode density

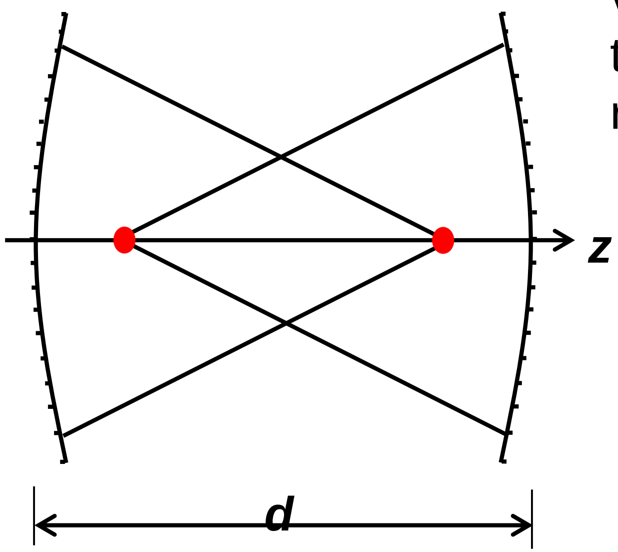
$$M(\nu) = \frac{8\pi\nu^2}{c^3}$$

The number of modes lying in the frequency interval between 0 and *v* corresponds to the number of points lying in the volume of the positive octant of a sphere of radius *k* in the k diagram

Optical resonators and stable condition

- A. Ray Confinement of spherical resonators

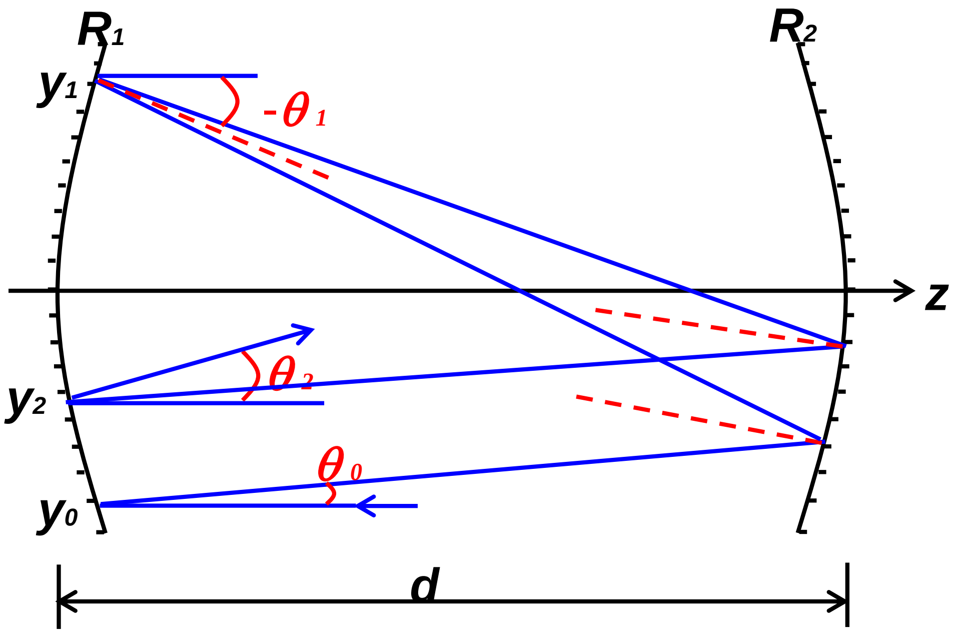
The rule of the sign: concave mirror ($R < 0$), convex ($R > 0$). The planar-mirror resonator is $R_1 = R_2 = \infty$



The matrix-optics methods introduced which are valid only for paraxial rays, are used to study the trajectories of rays as they travel inside the resonator

B. Stable condition of the resonator

For paraxial rays, where all angles are small, the relation between $(y_{m+1},\ \theta_{m+1})$ and $(y_m,\ \theta_m)$ is linear and can be written in the matrix form



$$\begin{bmatrix} y_{m+1} \\ \theta_{m+1} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_m \\ \theta_m \end{bmatrix}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{2}{R_1} & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{2}{R_2} & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

reflection from a mirror of radius R_1

reflection from a mirror of radius R_2

propagation a distance d through free space

$$A = 1 + {}^{2d}\!/\!{}_{R_2}$$
$$B = 2d(1 + {}^{d}\!/\!{}_{R_2})$$
$$C = {}^{2}\!/\!{}_{R_1} + {}^{2}\!/\!{}_{R_2} + {}^{4d}\!/\!{}_{R_1 R_2}$$
$$D = {}^{2d}\!/\!{}_{R_1} + ({}^{2d}\!/\!{}_{R_1} + 1)({}^{2d}\!/\!{}_{R_1} + 1)$$

$$\det|M| = Ad - BC = 1 = F^2$$
$$y_m = y_{\max}\sin(m\phi + \phi_0)$$
$$b = (A+D)/2 = 2\left(1+\frac{d}{R_1}\right)\left(1+\frac{d}{R_2}\right) - 1$$

It the way is harmonic, we need $\phi = cos^{-1} b$ must be real, that is

$$|b| \le 1 \qquad |b| = |(A+D)/2| = \left|2\left(1+\frac{d}{R_1}\right)\left(1+\frac{d}{R_2}\right) - 1\right| \le 1$$

for $g_1=1+d/R_1$; $g_2=1+d/R_2$

$$\boxed{0 \le \left(1+\frac{d}{R_1}\right)\left(1+\frac{d}{R_2}\right) \le 1} \Rightarrow \boxed{0 \le g_1 g_2 \le 1}$$

For a resonator is in conditionally stable, there will be:

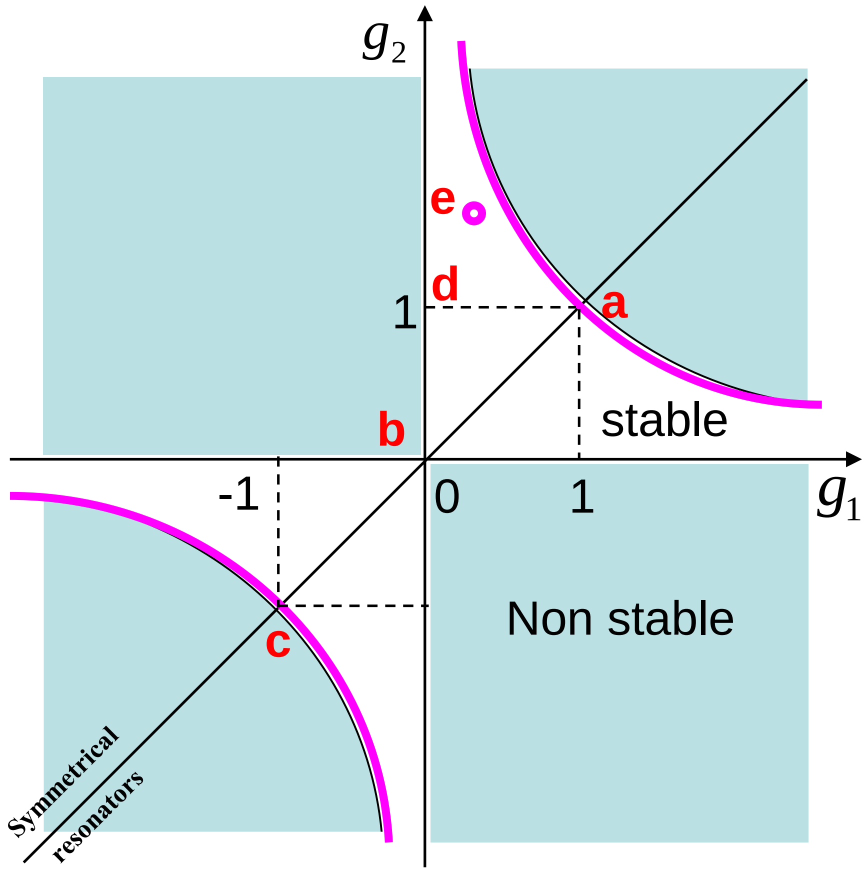
$$0 \le \left(1+\frac{d}{R_1}\right)\left(1+\frac{d}{R_2}\right) \le 1$$

$$0 \le g_1 g_2 \le 1$$

In summary, the confinement condition for paraxial rays in a spherical-mirror resonator, constructed of mirrors of radii R_1,R_2 seperated by a distance d, is $0 \le g_1 g_2 \le 1$, where $g_1=1+d/R_1$ and $g_2=1+d/R_2$

For the concave R is negative, for the convex R is positive

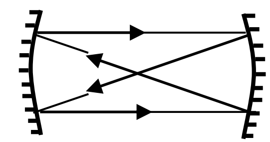
Stable and unstable resonators



a. Planar

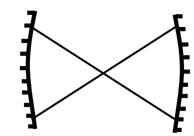
($R_1= R_2=\infty$)

b. Symmetrical confocal



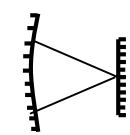
($R_1= R_2=-d$)

c. Symmetrical concentric



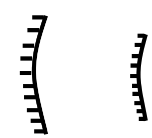
($R_1= R_2=-d/2$)

d. confocal/planar



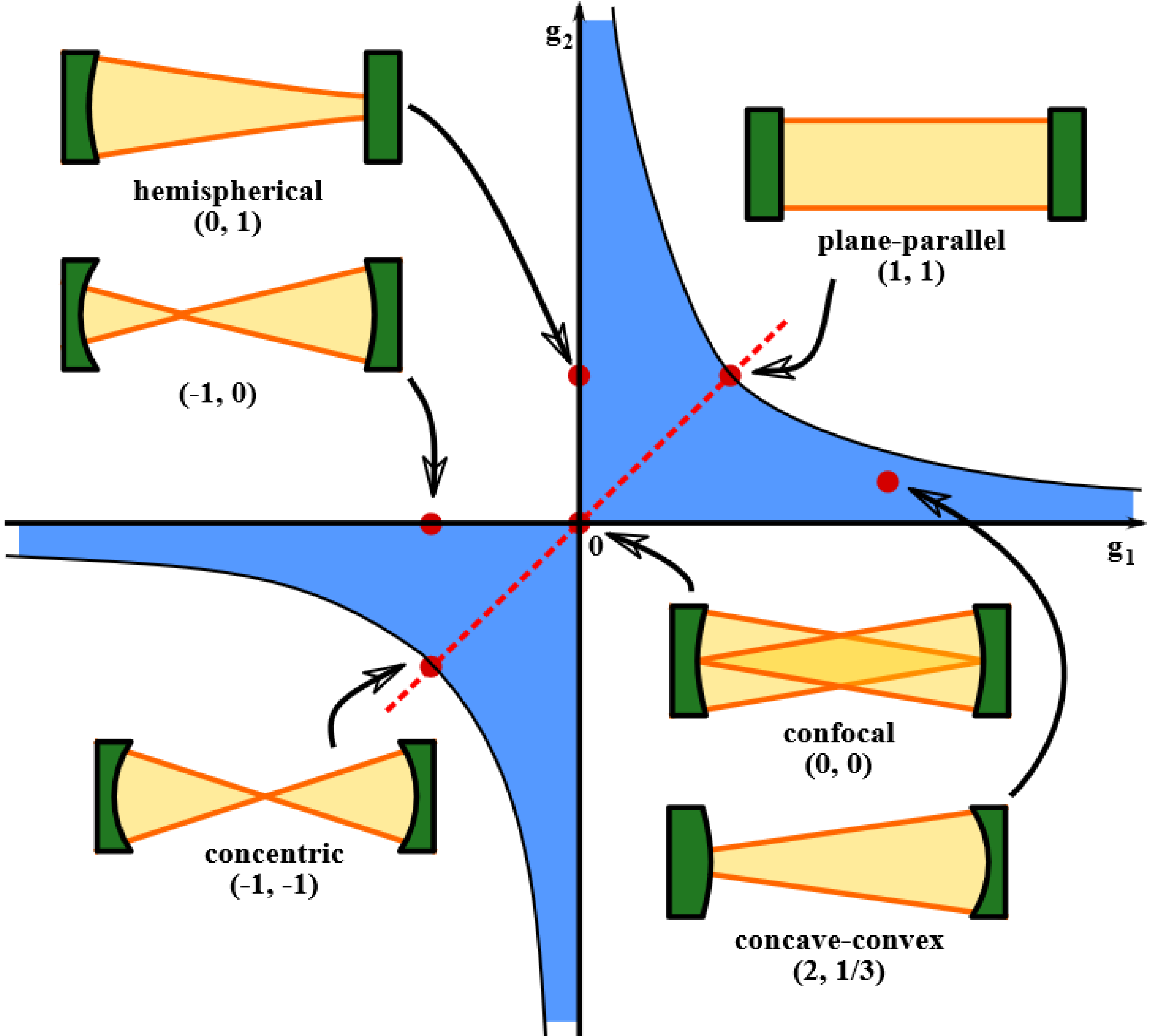
($R_1= -d, R_2=\infty$)

e. concave/convex



($R_1<0, R_2>0$)

d/(-R) = 0, 1, and 2, corresponding to planar, confocal, and concentric resonators


g2
hemispherical
(0, 1)
(-1, 0)
plane-parallel
(1, 1)
0
g1
confocal
(0, 0)
concentric
(-1, -1)
concave-convex
(2, 1/3)

The stable properties of optical resonators

Crystal state resonators

$$g_1 g_2 = 0 \quad or \quad g_1 g_2 = 1$$

Stable

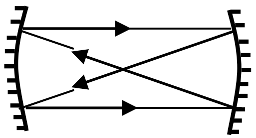
unstable

a. Planar

($\boldsymbol{R_1 = R_2 = \infty}$)

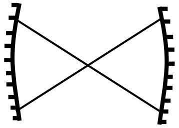
b. Symmetrical confocal

($\boldsymbol{R_1 = R_2 = -d}$)



c. Symmetrical concentric

($\boldsymbol{R_1 = R_2 = -d/2}$)

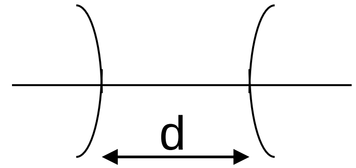


Unstable resonators

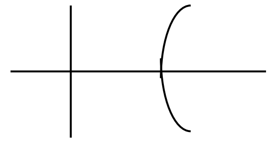
Unstable cavity corresponds to the high loss

$$g_1 g_2 < 0 \quad or \quad g_1 g_2 > 1$$

a. Biconvex resonator

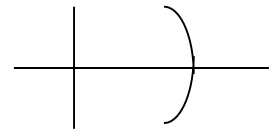


b. plan-convex resonator



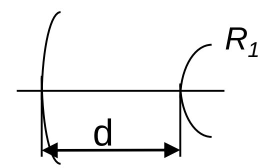
c. Some cases in plan-concave resonator

When $R_2 < d$, unstable



d. Some cases in concave-convex resonator

When $R_1 < d$ and $R_1 + R_2 = R_1 - |R_2| > d$

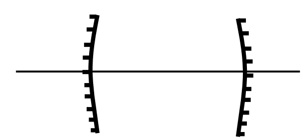


e. Some cases in biconcave resonator

$$g_1 g_2 = (1 + d/R_1)(1 + d/R_2) < 0$$

$$g_1 g_2 = (1 + d/R_1)(1 + d/R_2) > 1$$

$$|R_1 + R_2| < d$$



Applications of optical resonator

- Spectral analyzer
- Light wave tracker
- Energy storage
- Standing wave generator
- Optical delay line
- Optical feedback
- Group velocity of resonant frequency $v_g = \frac{d\omega}{dk}$
- Loss detector
- Sensitive interferometer
- Beam directional

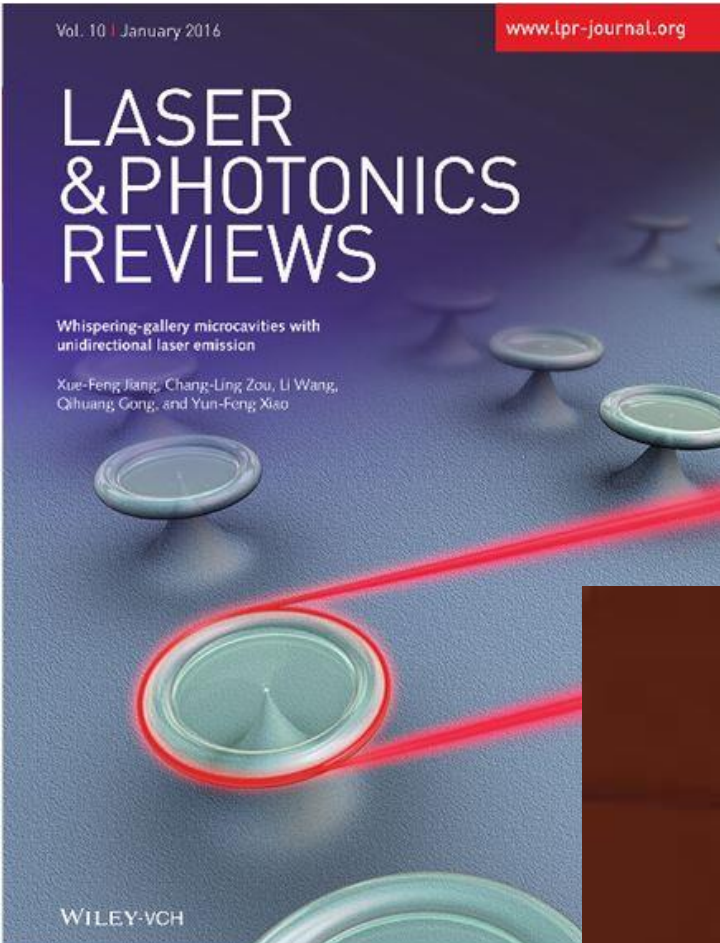
Cavity = Resonator

Home works

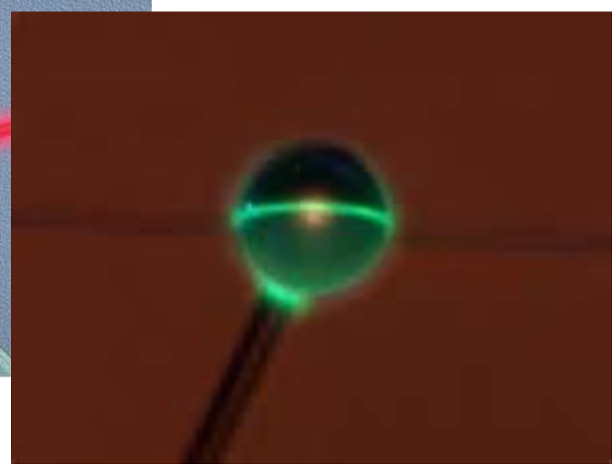
1. Resonance Frequencies of a Resonator with an Etalon. (a) Determine the spacing between adjacent resonance frequencies in a resonator constructed of two parallel planar mirrors separated by a distance d = 15 cm in air (n = 1). (b) A transparent plate of thickness d, = 2.5 cm and refractive index n = 1.5 is placed inside the resonator and is tilted slightly to prevent light reflected from the plate from reaching the mirrors. Determine the spacing between the resonance frequencies of the resonator.
2. Semiconductor lasers are often fabricated from crystals whose surfaces are cleaved along crystal planes. These surfaces act as reflectors and therefore serve as the resonator mirrors. Consider a crystal with refractive index n = 3.6 placed in air (n = 1). The light reflects between two parallel surfaces separated by the distance d = 0.2 mm. Determine the spacing between resonance frequencies v_f, the overall distributed loss coefficient α_r, the finesse , and the spectral width $\triangle$v. Assume that the loss coefficient α_s= 1 cm^{-1}.
3. What time does it take for the optical energy stored in a resonator of finesse = 100, length d = 50 cm, and refractive index n = 1, to decay to one-half of its initial value?

11， 12， 13, 16, 17,18，1, 2 ***at the page 40 to 42***

Whispering gallery mode



A optical disk of diameter 100 micron with refractive index of 1.46 in the air, please calculate the resonant frequencies in visible region, if there is some toxic gas appears (with refractive index of 1.40), what is the change of the resonant frequency?



2.3 Gaussian waves and its characteristics

The Gaussian beam is named after the great mathematician ***Karl Friedrich Gauss*** (1777- 1855)

A. Gaussian beam

The electromagnetic wave propagation is under the way of Helmholtz equation

$$\nabla^2 U + k^2 U = 0$$

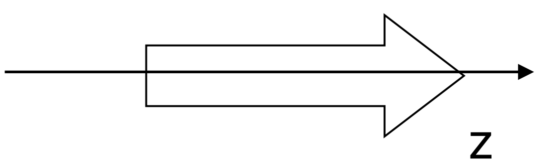
Normally, a plan wave (in z direction) will be

$$U = U_0 \exp\{-i(\omega t + \mathrm{k}\cdot\mathrm{r})\} = U_0 \exp(-ikz)\exp(-i\omega t)$$

When amplitude is not constant, the wave is

$$U = A(x, y, z)\exp(-ikz)\exp(-i\omega t)$$

An axis symmetric wave in the amplitude



$$U = A(\rho, z)\exp(-ikz)\exp(-i\omega t)$$

frequency $\omega = 2\pi\nu$

Wave vector $k = \dfrac{2n\pi}{\lambda}$

Paraxial Helmholtz equation

Substitute the *U* into the Helmholtz equation we have:

$$\nabla_T^2 A - i2k\frac{\partial A}{\partial z} = 0$$

where

$$\nabla^2{}_T = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

One simple solution is paraboloidal wave:

$$A(\vec{r}) = \frac{A_1}{z}\exp(-jk\frac{\rho^2}{2z})$$

$$\rho^2 = x^2 + y^2$$

The equation has the other solution,

$$\nabla_T^2 A - i2k\frac{\partial A}{\partial z} = 0 \Rightarrow A(\vec{r}) = \frac{A_0}{q(z)} exp\left[-ik\frac{\rho^2}{2q(z)}\right], \cdots q(z) = z + iz_0$$

q ***parameter***

Using relation: $$\frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{\pi W^2(z)}$$

which is Gaussian wave:

$$U(\vec{r}) = A_0\frac{W_0}{W(z)}\exp[-\frac{\rho^2}{W^2(z)}]\exp[-ikz - ik\frac{\rho^2}{2R(z)} + i\xi(z)]$$

where $W(z) = W_0[1+(\frac{z}{z_0})^2]^{1/2}$

$$W_0 = (\frac{\lambda z_0}{\pi})^{1/2} = W(z)\big|_{z=0} = W(0)$$

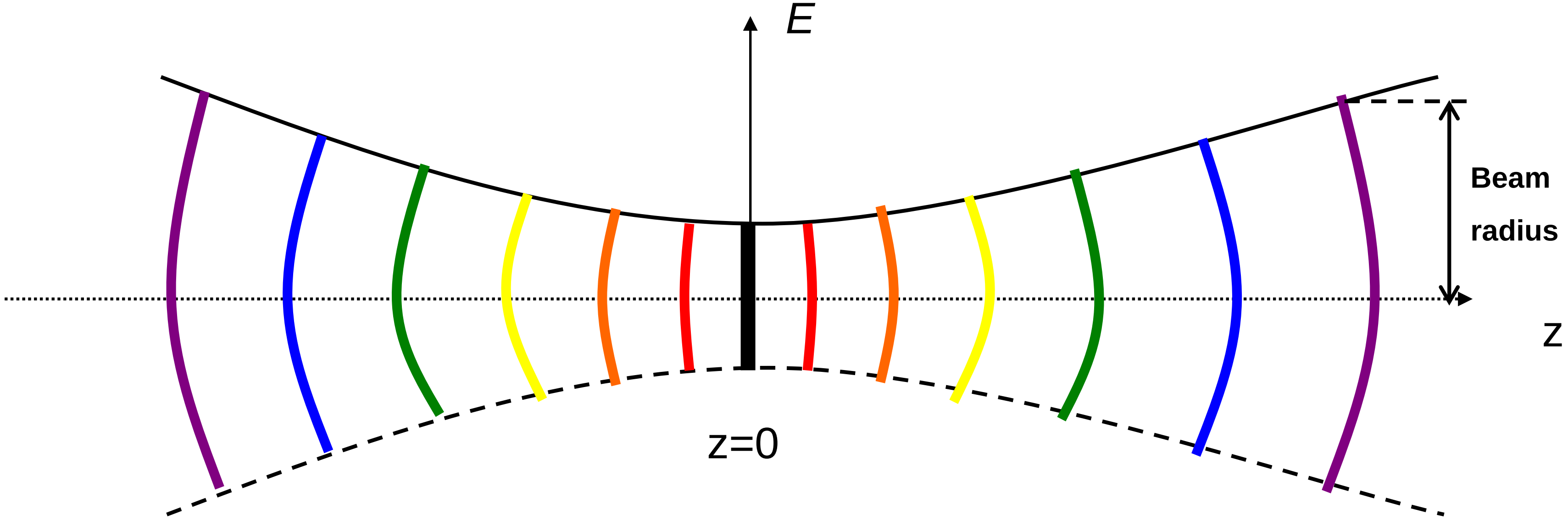
$$R(z) = z[1+(\frac{z_0}{z})^2]$$

z_0 is ***Rayleigh range***

$$\xi(z) = \tan^{-1}\frac{z}{z_0}$$

$$q(z) = z + iz_0$$

Gaussian Beam



Electric field of Gaussian wave propagates in z direction

$$E(x,y,z) = \frac{A_0}{W(z)} \exp[\frac{-(x^2+y^2)}{W^2(z)}] \cdot \exp[-ik(\frac{x^2+y^2}{2R(z)} + z) + i\xi(z)]$$

Physical meaning of parameters

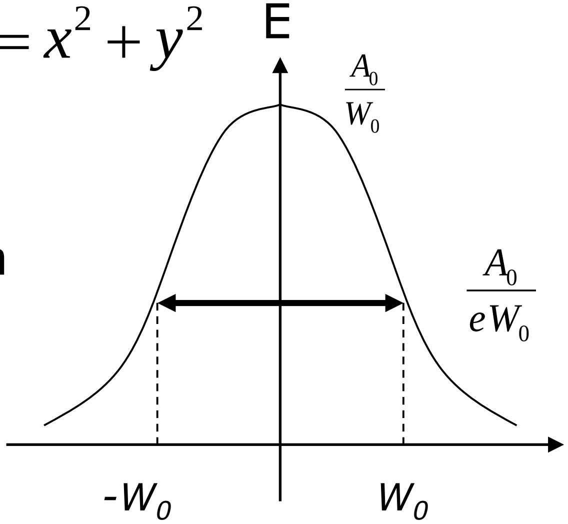
- Beam width at z $\boxed{W(z) = W_0[1+(\frac{z}{z_0})^2]^{1/2}}$

$$z_0 = \frac{\pi W_0^2}{\lambda}$$

- Waist width $W_0 = W(0)$
- Radii of wave front at z $R(z) = z[1+(\frac{\pi W_0^2}{\lambda z})^2] = \boxed{z[1+(\frac{z_0}{z})^2]}$
- Phase factor $\xi(z) = \arctan\frac{\lambda z}{\pi W_0^2} = \boxed{tg^{-1}\frac{z}{z_0}}$

Gaussian beam at z=0

$$E(x,y,0)=\frac{A_0}{W_0}\exp[-\frac{r^2}{W_0^2}]$$ where $$r^2=x^2+y^2$$



Beam width:

$$W(z)=W_0[1+(\frac{z}{z_0})^2]^{1/2}$$ will be minimum

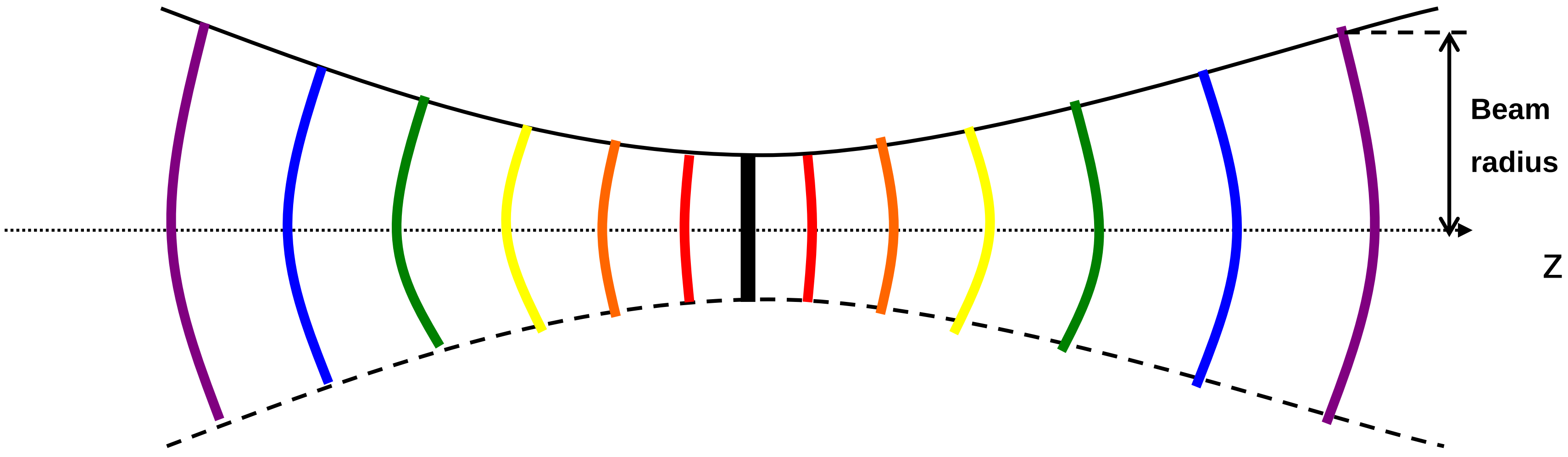
wave front

$$\lim_{z\to 0} R(z)=\lim_{z\to 0}\left\{z\left[1+\left(\frac{\pi W_0^2}{\lambda z}\right)^2\right]\right\}=\infty$$

at z=0, the wave front of Gaussian beam is a plan surface, but the electric field is Gaussian form

***W_0* is the waist half width**

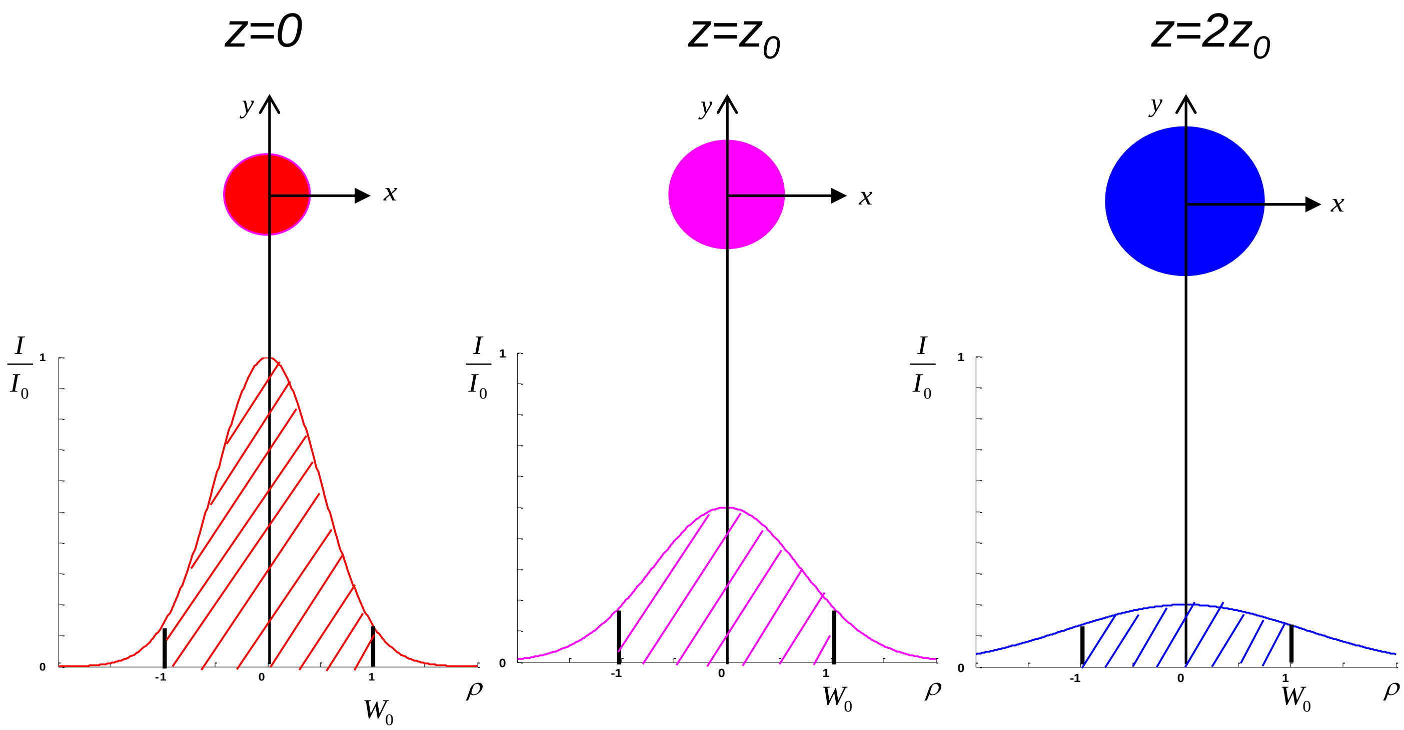
B. The characteristics of Gaussian beam



Gaussian beam is a axis symmetrical wave, at z=0 phase is plan and the intensity is Gaussian form, at the other z, it is Gaussian spherical wave.

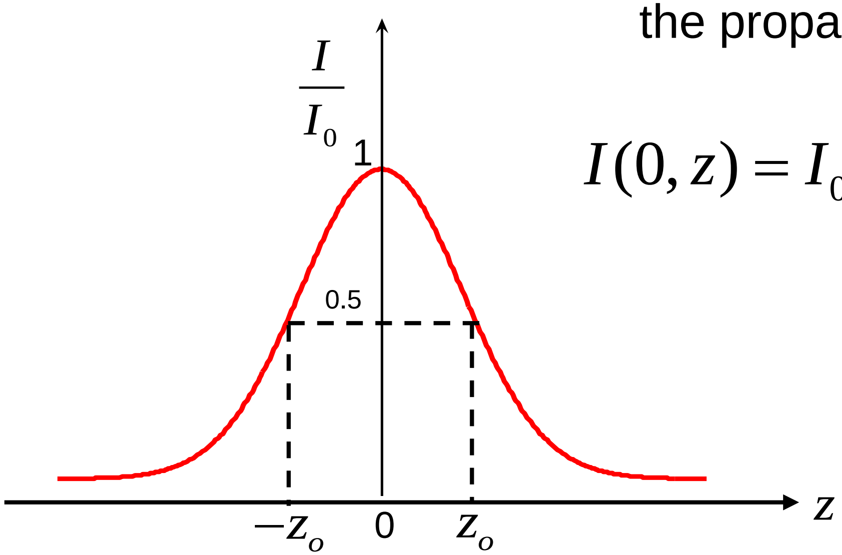
Intensity of Gaussian beam

- Intensity of Gaussian beam $$I(\rho,z)=I_0[\frac{W_0}{W(z)}]^2\exp[-\frac{\rho^2}{W^2(z)}]$$



The normalized beam intensity as a function of the radial distance at different axial distances

On the beam axis (ρ = 0) the intensity



Variation of axial intensity as the propagation length z

$$I(0,z) = I_0[\frac{W_0}{W(z)}]^2 = \frac{I_0}{1+(\frac{z}{z_0})^2}$$

z_0 is ***Rayleigh range***

The normalized beam intensity I/I_0 at points on the beam axis (ρ=0) as a function of z

$$z_0 = \frac{\pi W_0^2}{\lambda}$$

Power of the Gaussian beam

The power of Gaussian beam is calculated by the integration of the optical intensity over a transverse plane

$$P = \frac{1}{2} I_0 \pi W_0^2$$

So that we can express the intensity of the beam by the power

$$I(\rho, z) = \frac{2P}{\pi W^2(z)} \exp[-\frac{2\rho^2}{W^2(z)}]$$

The ratio of the power carried within a circle of radius ρ. in the transverse plane at position z to the total power is

$$\frac{1}{P}\int_0^{\rho_0} I(\rho, z) 2\pi\rho d\rho = 1 - \exp[-\frac{2\rho_0^2}{W^2(z)}]$$

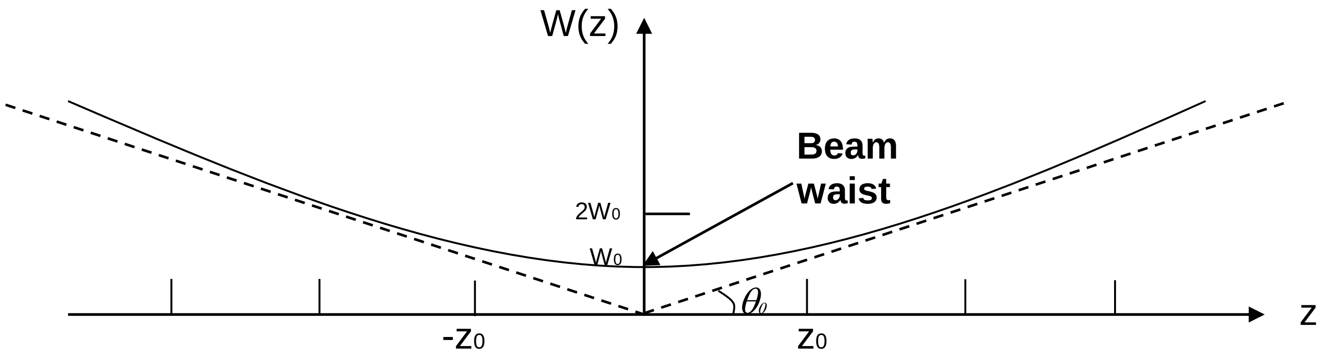
Beam Radius

$$W(z) = W_0[1+(\frac{z}{z_0})^2]^{1/2}$$

$$W(z) \approx \frac{W_0}{z_0} z = \theta_0 z$$

$$\because \quad z_0 = \frac{\pi W_0^2}{\lambda}$$

$$\theta_0 = \frac{\lambda}{\pi W_0}$$



The beam radius *W(z)* has its minimum value W_0 at the waist (*z=0*) reaches $\sqrt{2}W_0$ at $z=\pm z_0$ and increases linearly with z for large z.

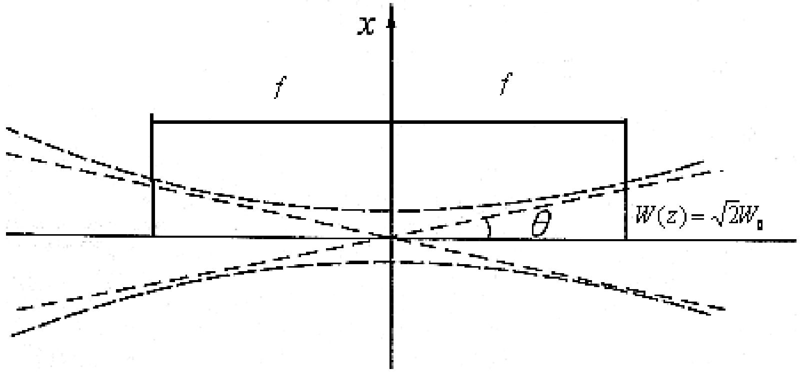
Beam Divergence

$$2\theta = 2\frac{dW(z)}{dz} = \frac{2\lambda z}{\pi W_0}[(\frac{\pi^2 W_0^2}{\lambda})^2 + z^2]^{-\frac{1}{2}}$$

$$\theta_0 = \frac{\lambda}{\pi W_0}$$

The characteristics of divergence angle

- z=0， $2\theta=0$
- $z=\pi W_0^2/\lambda = z_0 \quad 2\theta=\sqrt{2}\lambda/\pi W_0$
- $z\rightarrow\infty \qquad 2\theta = \dfrac{2\lambda}{\pi W_0}$ or $2\theta = \lim\limits_{x\to\infty}\dfrac{2W(z)}{z}$



z_0 is ***Rayleigh range***

Define *$f=z_0$ as the* ***<u>confocal parameter of Gaussian beam</u>***

$$f = z_0 = \frac{\pi W_0^2}{\lambda}$$

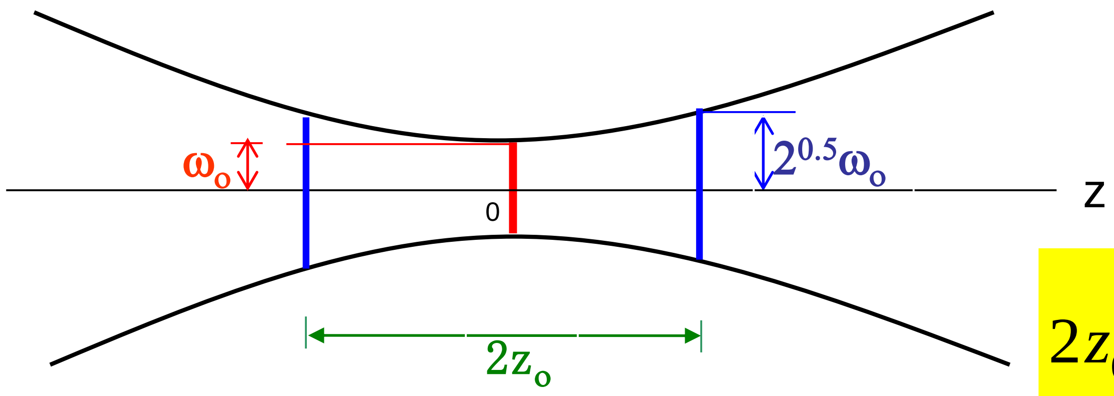
The physical means of f： the half distance between two section of width

$W(z)=\sqrt{2}W_0$

$$2\theta = \lim_{z\to\infty}\frac{2W(z)}{z} = \lim_{z\to\infty}\frac{2\sqrt{\frac{f\lambda}{\pi}(1+\frac{z^2}{f^2})}}{z} = 2\sqrt{\frac{\lambda}{f\pi}}$$

Depth of Focus

Since the beam has its minimum width at z = 0, it achieves its best focus at the plane z = 0. In either direction, the beam gradually grows "out of focus." The axial distance within which the beam radius lies within a factor $2^{0.5}$ of its minimum value (i.e., its area lies within a factor of 2 of its minimum) is known as the depth of focus or confocal parameter



$$2z_0 = \frac{2\pi W_0^2}{\lambda} = 2f$$

The depth of focus of a Gaussian beam.

Phase of Gaussian beam

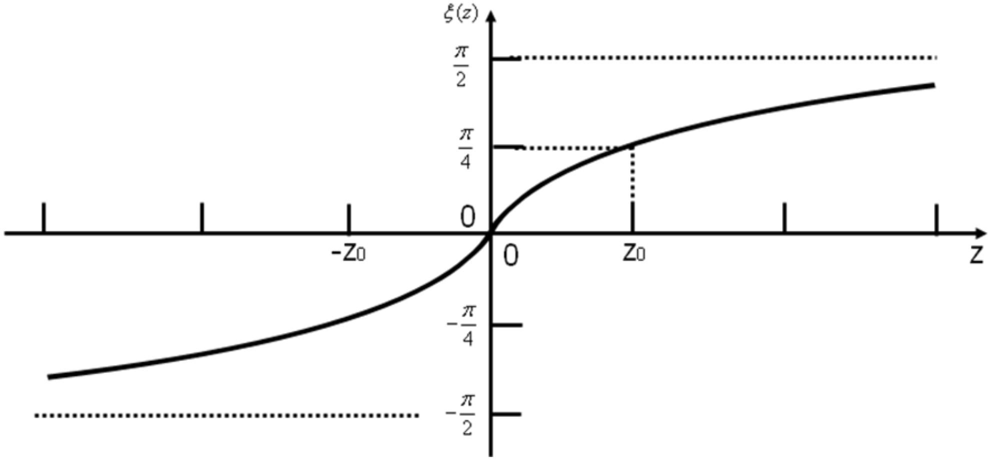
$$\xi(z) = \arctan\frac{\lambda z}{\pi W_0^2} = tg^{-1}\frac{z}{z_0}$$

The phase of the Gaussian beam is,

$$\varphi(\rho, z) = kz - \xi(z) + \frac{k\rho^2}{2R(z)}$$

On the beam axis (p = 0) the phase

$$\varphi(0, z) = kz - \xi(z)$$



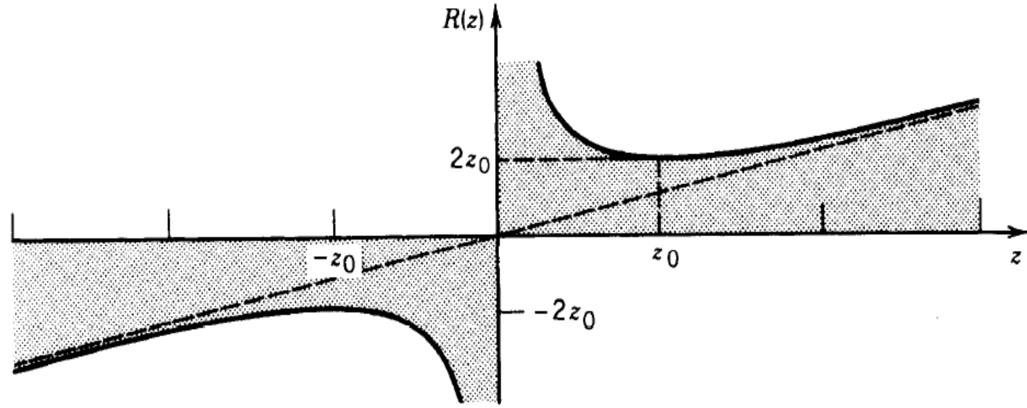
kz Phase of plan wave

$\xi(z)$ an excess delay of the wavefront in comparison with a plane wave or a spherical wave

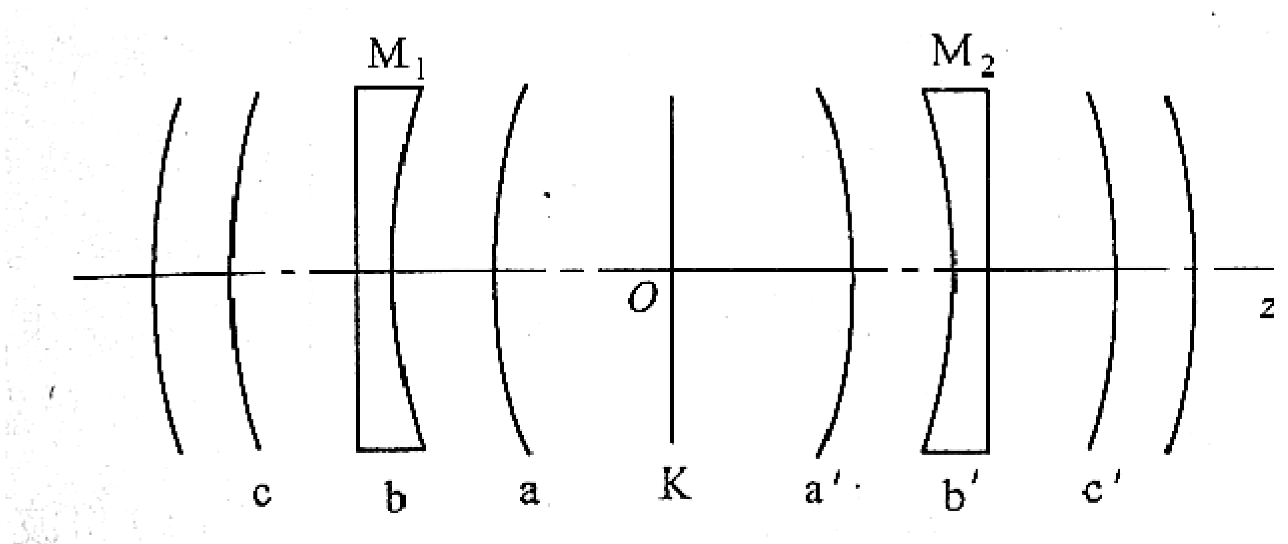
The excess delay is –π/2 at z=-∞, and π/2 at z= ∞

The total accumulated excess retardation as the wave travels from z = -∞ to z =∞is π. This phenomenon is known as the ***Guoy effect***.

Wavefront



$$R(z) = z[1 + (\frac{\pi W_0^{\,2}}{\lambda z})^2] = \left| z + \frac{f^{\,2}}{z} \right|$$



Confocal field and its equal phase front

Parameters Required to Characterize a Gaussian Beam

How many parameters are required to describe a plane wave, a spherical wave, and a Gaussian beam?

- The plane wave is completely specified by its complex amplitude and direction.
- The spherical wave is specified by its amplitude and the location of its origin.
- The Gaussian beam is characterized by more parameters- its peak amplitude the parameter A, its direction (the beam axis), the location of its waist, and one additional parameter: the waist radius W_0 or the Rayleigh range z_o,

Parameter used to describe a Gaussian beam

- ***q-parameter*** is sufficient for characterizing a Gaussian beam of known peak amplitude and beam axis

$$\frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{\pi W^2(z)} \quad \rightarrow \quad \frac{1}{q(z)} = \frac{1}{z + iz_0}$$

$$\Rightarrow \quad q(z) = z + iz_0$$

If the complex number $q(z) = z + iz_0$, is known, the distance z to the beam waist and the Rayleigh range z_0. are readily identified as the real and imaginary parts of $q(z)$.

the real part of $q(z)$ z is the beam waist place

the imaginary parts of $q(z)$ z_0 is the Rayleigh range

B. HERMITE - GAUSSIAN BEAMS

The self-reproducing waves exist in the resonator, and resonating inside of spherical mirrors, plan mirror or some other form paraboloidal wavefront mirror, are called the modes of the resonator

There exists higher order modes, caused by the limitation in beam diameter

Hermite - Gaussian Beam Complex Amplitude

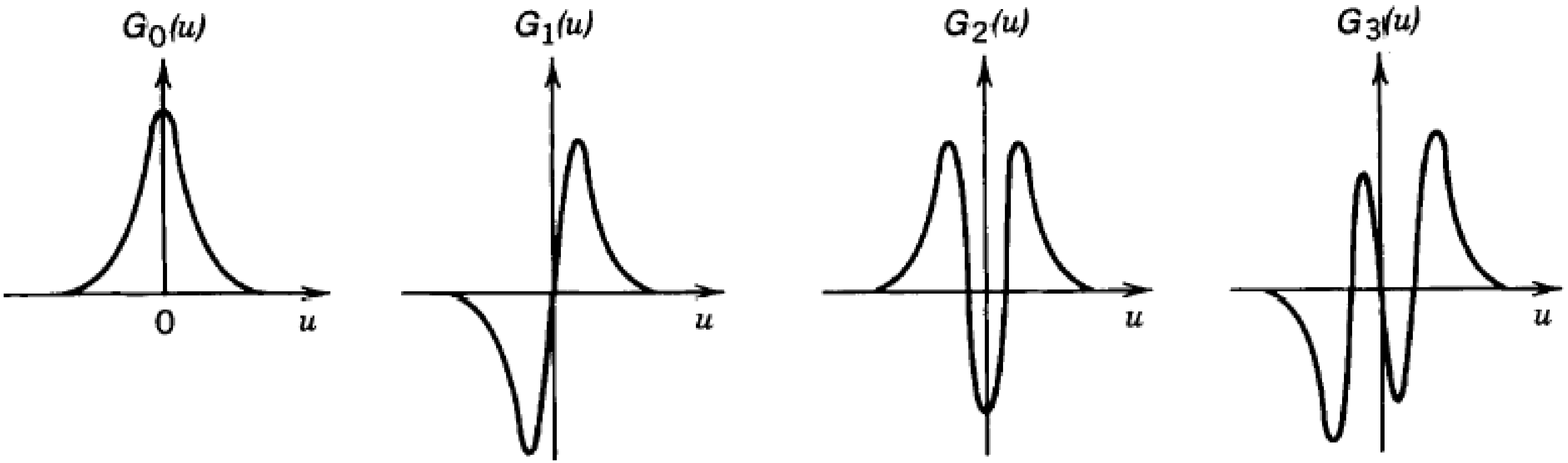
$$U_{l,m}(x,y,z) = A_{l,m}[\frac{W_0}{W(z)}]G_l[\frac{\sqrt{2}x}{W(z)}]G_m[\frac{\sqrt{2}y}{W(z)}]\times\exp[-jkz - jk\frac{x^2+y^2}{2R(z)} + j(l+m+1)\zeta(z)]$$

where $$G_l(u) = H_l(u)\exp(\frac{-u^2}{2}), \qquad l = 0,1,2,\ldots,$$

is known as the **Hermite-Gaussian function** of order *l*, and $A_{l,m}$ is a constant

Hermite-Gaussian beam of order *(l, m).*

The Hermite-Gaussian beam of order (0, 0) is the Gaussian beam.

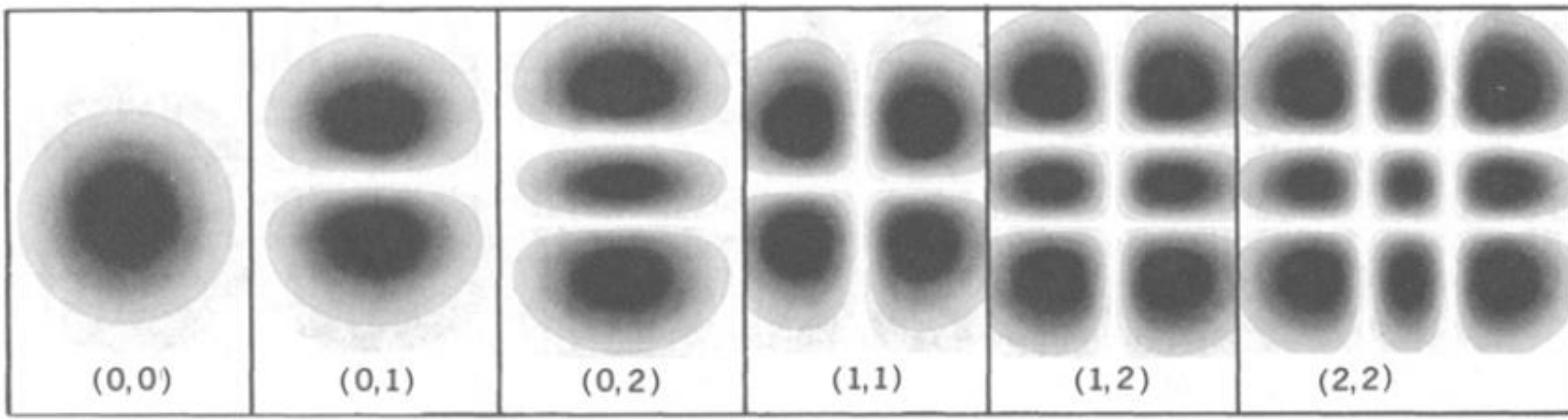


$H_0(u) = 1$, the Hermite-Gaussian function of order O, the Gaussian function.
$G_1(u) = 2u\ exp(-u^2/2)$ is an odd function,
$G_2(u) = (4u^2 - 2)\ exp(-u^2/2)$ is even,
$G_3(u) = (8u^3 - 12u)exp(-u^2/2)$ is odd,

Intensity Distribution

The optical intensity of the *(l, m)* Hermite-Gaussian beam is

$$I_{l,m}(x,y,z) = \left|A_{l,m}\right|^2 \left[\frac{W_0}{W(z)}\right]^2 G_l^2\left[\frac{\sqrt{2}x}{W(z)}\right] G_m^2\left[\frac{\sqrt{2}y}{W(z)}\right]$$



Beam quality: $\mathbb{M}^2$ factor

- The measure of the quality of an optical beam is the deviation of its profile from Gaussian form.

- $$\frac{the\ waist\text{–}diameter\text{–}divergence\ porduct}{waist\text{–}diameter\text{–}divergence\ of\ a\ Gaussian\ beam} = \frac{2W_m 2\theta_m}{2W_0 2\theta_0} = \frac{W_m \theta_m}{\lambda/\pi}$$

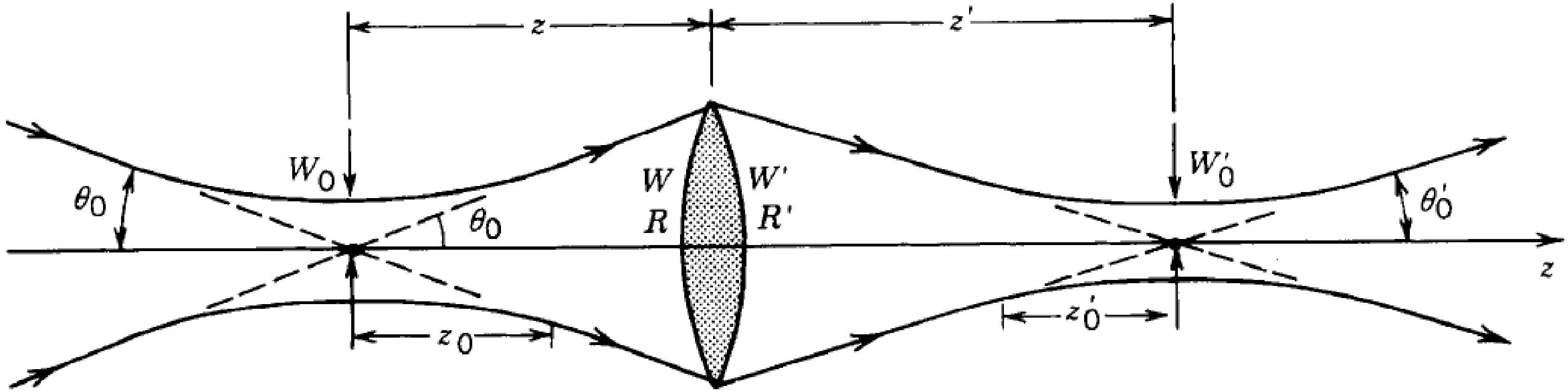
- For Gaussian beam, M2=1
- M2 factor use to express the deviation of the beam from diffraction limit, the bigger M2, the diffraction more important.
- High quality laser M2 approach to 1 or 1.1

- High quality beam $M^2<1.1$
- Ion laser M^2 used to 1.1~1.3
- TEM00 diode laser 1.1~ 1.7
- High energy multimode laser M^2> 3~4

C. TRANSMISSION THROUGH OPTICAL COMPONENTS

a). Transmission Through a Thin Lens

For a thin lens, the transmittance function is proportional to $\exp(ik\rho^2/2f)$



Phase +phase induce by lens must equal to the back phase

$$\boxed{kz + k\frac{\rho^2}{2R} - \zeta} - k\frac{\rho^2}{2f} = kz + k\frac{\rho^2}{2R'} - \zeta \quad \Rightarrow \quad \frac{1}{R'} = \frac{1}{R} - \frac{1}{f} \quad \Rightarrow \quad \frac{1}{R} - \frac{1}{R'} = \frac{1}{f}$$

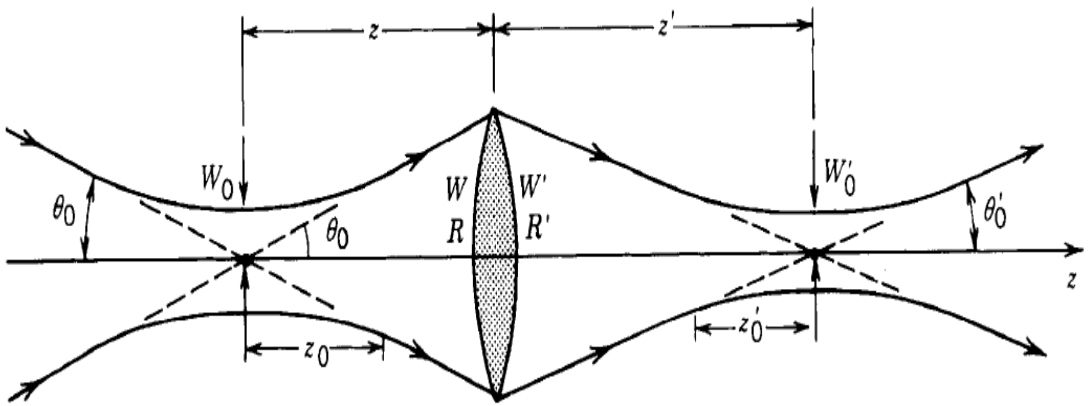
Notes:

R is positive since the wavefront of the incident beam is diverging and R' is negative since the wavefront of the transmitted beam is converging.

In the thin lens transform, we have

$$\begin{cases} W = W' \\ \dfrac{1}{R'} = \dfrac{1}{R} - \dfrac{1}{f} \end{cases}$$

$$R' = \frac{Rf}{f - R}$$



If we know W_0, z, f

we can get R', and then using

$$\begin{cases} W_0'^2 = W^2[1+(\dfrac{\pi W^2}{\lambda R'})^2]^{-1} \\ -z' = R'[1+(\dfrac{\lambda R'}{\pi W^2})^2]^{-1} \end{cases}$$

We get z_0'

The minus sign is due to the waist lies to the right of the lens.

$$W_0' = \frac{W}{[1+(\pi W^2/\lambda R')^2]^{1/2}} \qquad -z' = \frac{R'}{1+(\pi R'/\lambda W^2)^2}$$

because $R = z[1+(z_0/z)^2]$ *and* $W = W_0[1+(z/z_0)^2]^{1/2}$

Waist radius $$W_0' = MW_0$$

Waist location $$(z'-f) = M^2(z-f)$$

Depth of focus $$2z_0' = M^2(2z_0)$$

Divergence angle $$2\theta' = 2\theta_0 / M$$

The beam waist is magnified by M, the beam depth of focus is magnified by M², and the angular divergence is minified by the factor M.

,

magnification $$M = \frac{M_r}{(1+r^2)^{1/2}}$$ where $$r = \frac{z_0}{z-f}$$

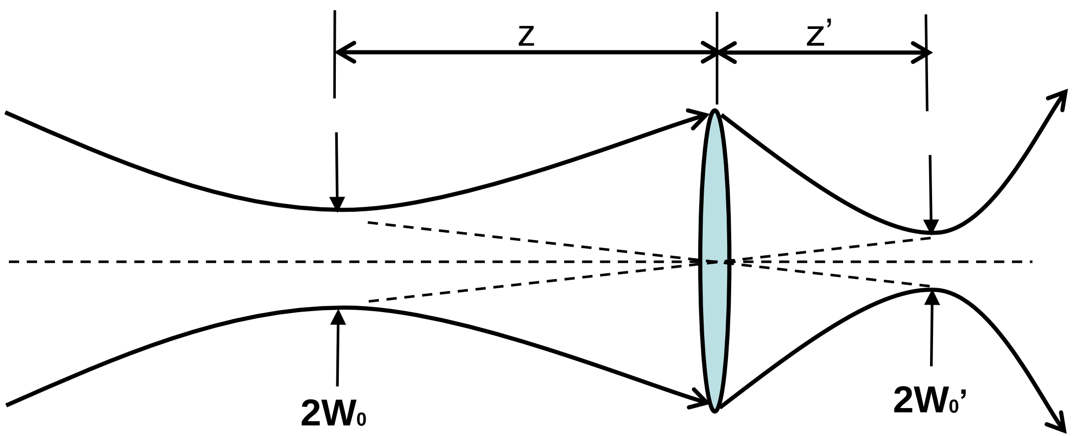
$$M_r = \left|\frac{f}{z-f}\right|$$

The formulas for lens transformation

Limit of Ray Optics

Consider the limiting case in which *(z - f)* >>z_o, so that the lens is well outside the depth of focus of the incident beam, The beam may then be approximated by a spherical wave, thus

$$r = \frac{z_0}{z-f} \to 0 \qquad and \qquad M \approx M_r$$



$$W_0' \approx M W_0$$

$$\frac{1}{z'} + \frac{1}{z} \approx \frac{1}{f}$$

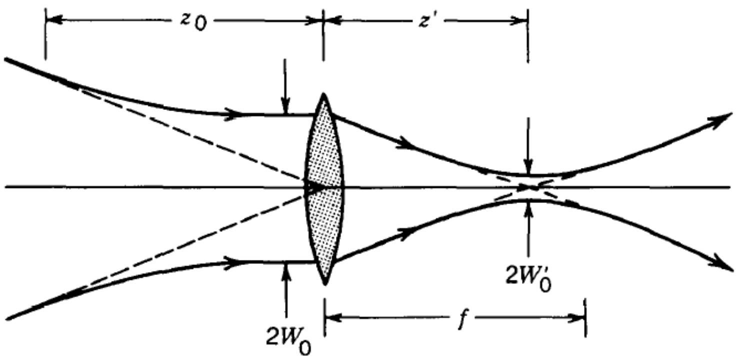
Imaging relation

$$M \approx M_r = \left| \frac{f}{z-f} \right|$$

The magnification factor Mr is that based on ray optics. provides that *M* < *Mr,* the maximum magnification attainable is the ray-optics magnification Mr.

b). Beam Shaping

Beam Focusing



If a lens is placed at the waist of a Gaussian beam, so *z=0*, then

$$\because \quad M = \frac{1}{[1+(z_0/f)^2]^{1/2}}$$

$$W_0' = \frac{W_0}{[1+(z_0/f)^2]^{1/2}}$$

$$z' = \frac{f}{1+(f/z_0)^2}$$

If the depth of focus of the incident beam $2z_0$, is much longer than the focal length *f* of the lens, then $W_0' = (f/z_o)W_o$. Using $z_0 = \pi W_0^2/\lambda$, we obtain

$$W_0' \approx \frac{\lambda}{\pi W_0} f = \theta_0 f \qquad z' \approx f$$

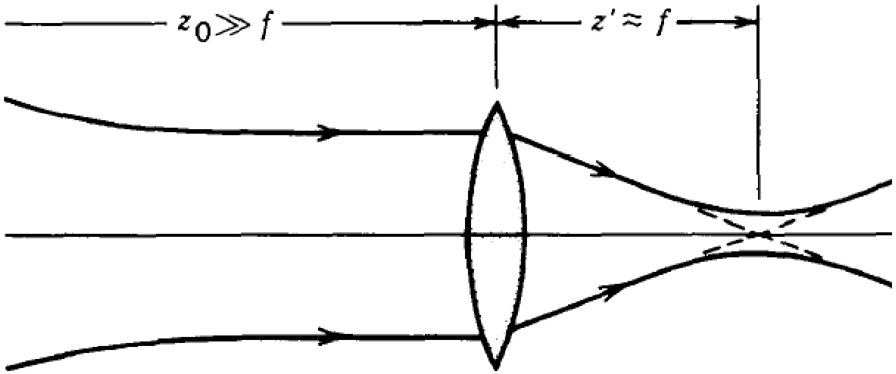
The transmitted beam is then focused at the lens' focal plane as would be expected for parallel rays incident on a lens. This occurs because the incident Gaussian beam is well approximated by a plane wave at its waist. **The spot size expected from ray optics is zero**

In laser scanning, laser printing, and laser fusion, it is desirable to generate the smallest possible spot size, this may be achieved by use of the shortest possible wavelength, the widest incident beam, and the shortest focal length. Since the lens should intercept the incident beam, its diameter D must be at least $2W_0$. Assuming that D = $2W_o$, the diameter of the focused spot is given by

$$2W_0' \approx \frac{4}{\pi}\lambda F_{\#}$$

$$F_{\#} = \frac{f}{D}$$

where F# is the F-number of the lens. A microscope objective with small F-number is often used.



Focus of Gaussian beam

$$W'^2_0 = \frac{W_0^2}{(1-\frac{z_1}{f})+(\frac{W_0^2}{\lambda f})^2}$$

- For given f, W'^2_0 changes as

when $z_1 < f$ W'^2_0 decreases as z decreases

$z_1 = 0$ W_0' reaches minimum, and M<1, for f>0, it is focal effect

when $z = f$ W'_0 reaches maximum, when $\frac{\pi W_0^2}{\lambda} > f$, it will be focus

when $z_1 > f$, W_0' increases as z increases

when $z_1 >> f$ the bigger z, smaller f, better focus

Beam collimate

locations of the waists of the incident and transmitted beams, z and z' are

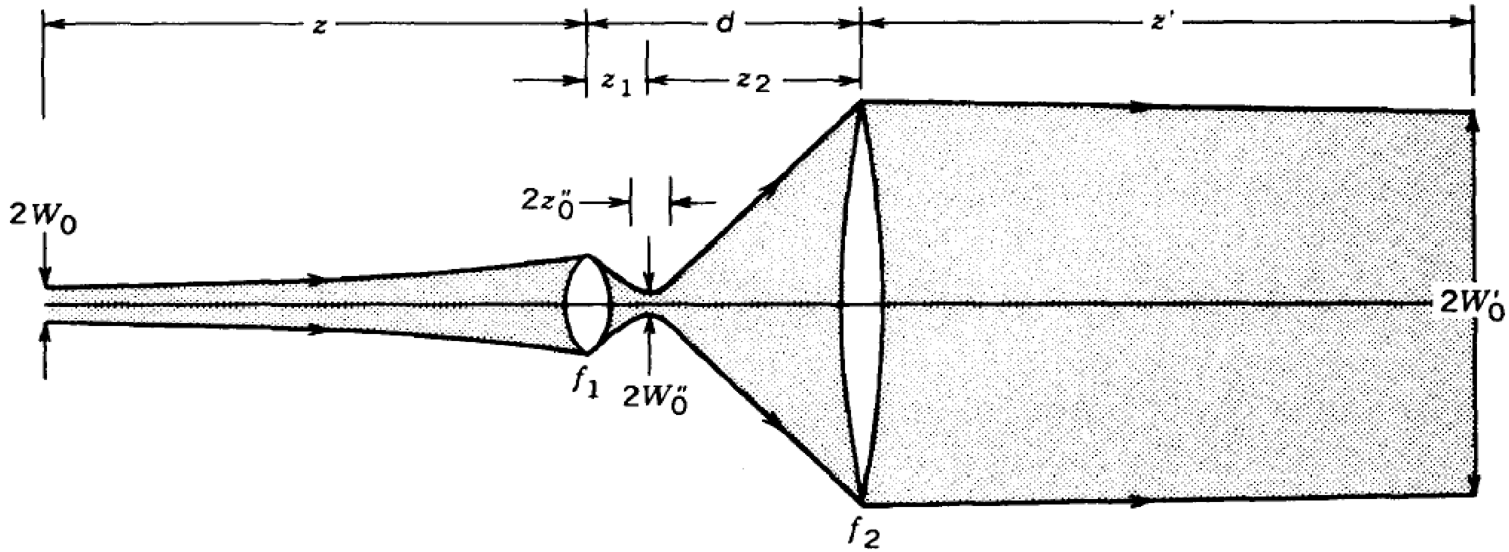
$$\frac{z'}{f}-1=\frac{z/f-1}{(z/f-1)^2+(z_0/f)^2}$$

The beam is collimated by making the location of the new waist z' as distant as possible from the lens.

This is achieved by

- the smallest ratio z_0/f
- z=f

Beam expanding



A Gaussian beam is expanded and collimated using two lenses of focal lengths fi and f2,

Assuming that $f_1 << z$ and $z - f_1 >> z_0$, determine the optimal distance *d* between the lenses such that the distance z’ to the waist of the final beam is as large as possible.

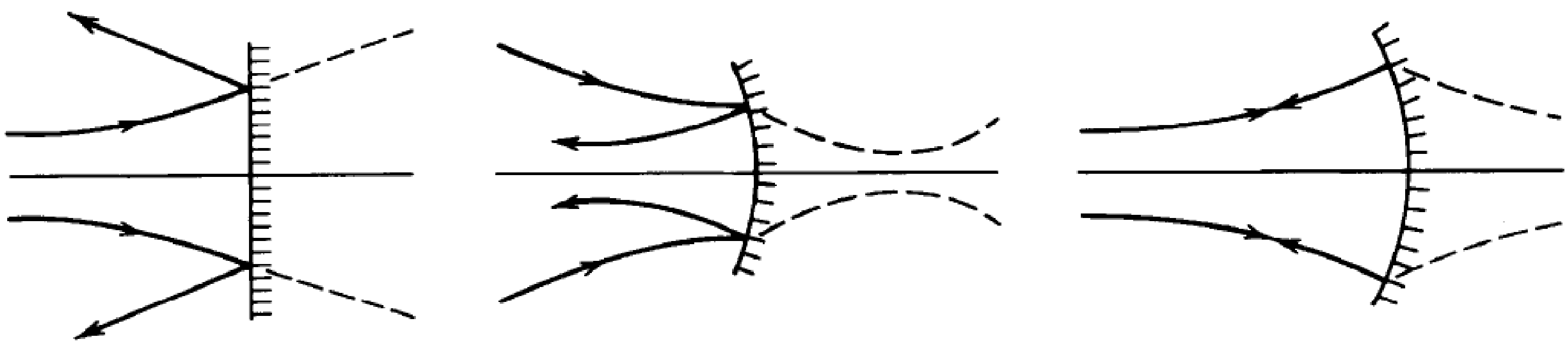
overall magnification $M = W_0' / W_o$

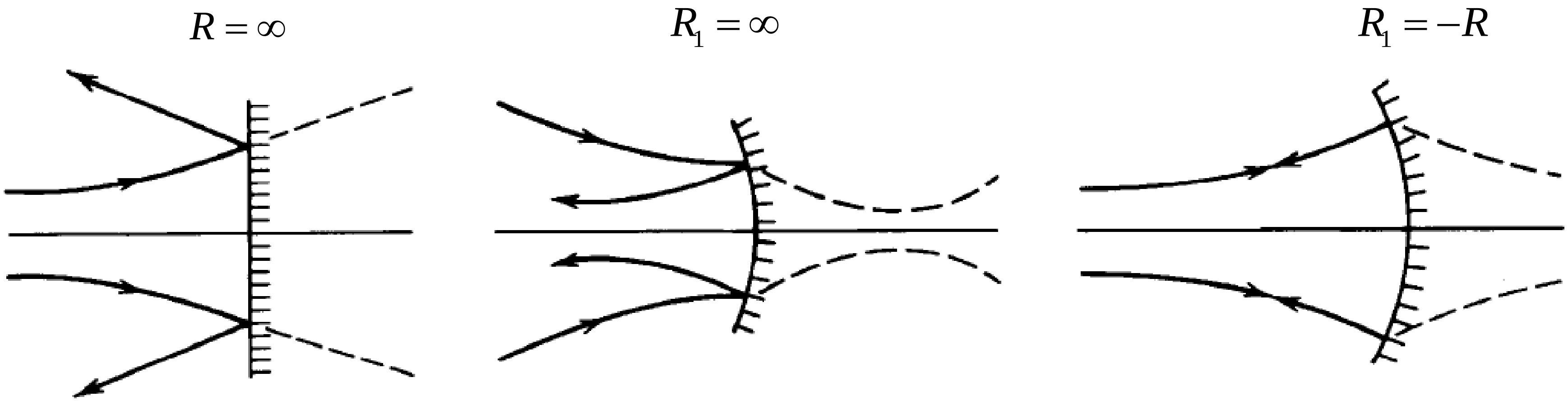
C). Reflection from a Spherical Mirror

Reflection of a Gaussian beam of curvature R_1 from a mirror of curvature R:

$$W_2 = W_1 \qquad \frac{1}{R_2} = \frac{1}{R_1} + \frac{2}{R} \qquad f = -R/2.$$

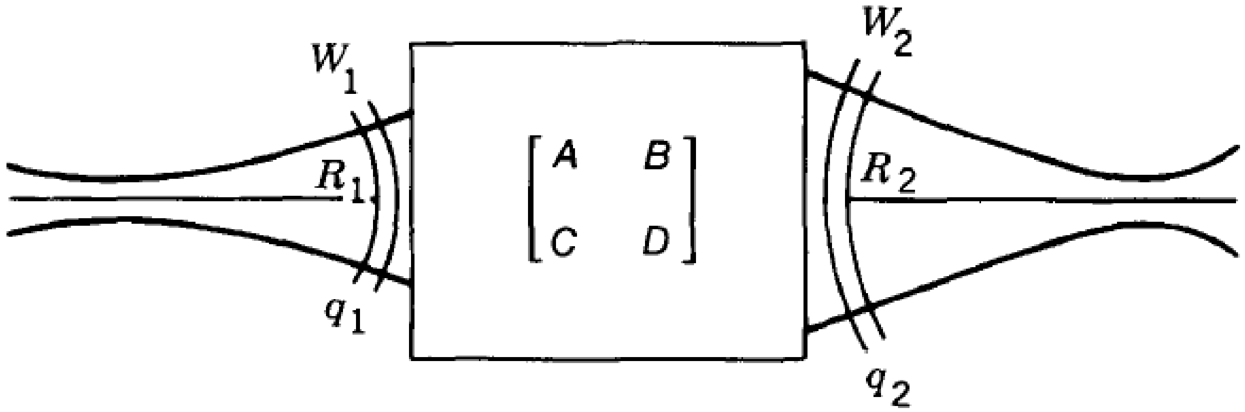
R > 0 for convex mirrors and R < 0 for concave mirrors,





- If the mirror is planar, i.e., R =∞, then R_2= R_1, so that the mirror reverses the direction of the beam without altering its curvature
- If R_1= ∞, i.e., the beam waist lies on the mirror, then R_2= R/2. If the mirror is concave (R < 0), R_2 < 0, so that the reflected beam acquires a negative curvature and the wavefronts converge. The mirror then focuses the beam to a smaller spot size.
- If R_1= -R, i.e., the incident beam has the same curvature as the mirror, then R_2= R. The wavefronts of both the incident and reflected waves coincide with the mirror and the wave retraces its path. This is expected since the wavefront normals are also normal to the mirror, so that the mirror reflects the wave back onto itself. the mirror is concave (R < 0); the incident wave is diverging (R_1 > 0) and the reflected wave is converging (R_2< 0).

d). Transmission Through an Arbitrary Optical System



An optical system is completely characterized by the matrix M of elements (A, B, C, D) ray-transfer matrix relating the position and inclination of the transmitted ray to those of the incident ray

The q-parameters, q_1 and q_2, of the incident and transmitted Gaussian beams at the input and output planes of a paraxial optical system described by the (A, B, C, D) matrix are related by

ABCD law

The q-parameters, q_1 and q_2, of the incident and transmitted Gaussian beams at the input and output planes of a par-axial optical system described by the (A, B, C, D) matrix are related by

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D}$$

Because the q parameter identifies the width W and curvature R of the Gaussian beam, this simple law, called the ABCD law

Invariance of the ABCD Law to Cascading

If the ABCD law is applicable to each of two optical systems with matrices M_i =(A_i, B_i, C_i, D_i), i = 1,2,…, it must also apply to a system comprising their cascade (a system with matrix $M = M_1M_2$).

The key points of this chepter

- Resonator conditions: stable and resonance
- The characters to describe the resonator
- Cavity
- Gaussian beam and characters of it. Z0
- Propagation of Gaussian beam in optical system
- Gaussian beam in cavity

- Design the beam properties and design the laser cavity!

Home work 2

- Exercises in page 41，no：4,6,7,9,10
- From the relation of q parameter

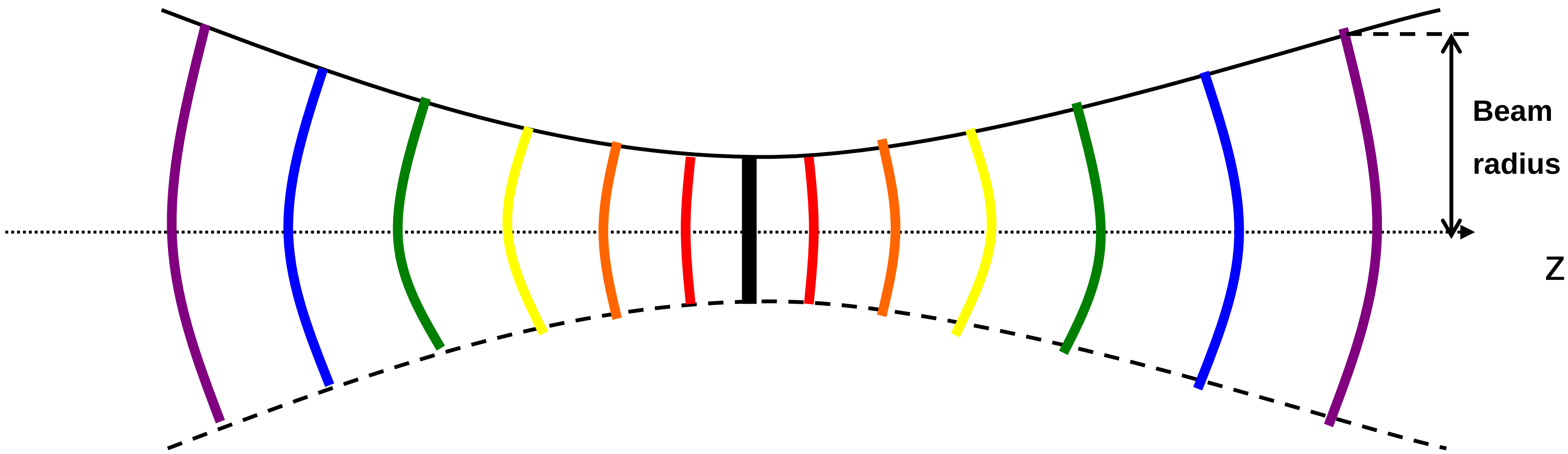
$$\frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{\pi W^2(z)}$$

prove that $q(z) = z + iz_0$

2.4 Gaussian beam in Spherical-Mirror Resonators

A. Gaussian Modes

- Gaussian beams are modes of the spherical-mirror resonator; Gaussian beams provide solutions of the Helmholtz equation under the boundary conditions imposed by the spherical-mirror resonator



a Gaussian beam is a circularly symmetric wave whose energy is confined about its axis (the z axis) and whose wavefront normals are paraxial rays

Gaussian beam intensity: $$I = I_0 \left[\frac{W_0}{W(z)}\right]^2 e^{-\frac{2(x^2+y^2)}{W^2(z)}} e^{-i\left[k\left(z+\frac{x^2+y^2}{2R}\right)-tg^{-1}\frac{z}{z_0}\right]}$$

The Rayleigh range z_0 $$z_0 = \frac{\pi W_0^2}{\lambda}$$

where z_0 is the distance called **Rayleigh range**, at which the beam wavefronts are most curved or we usually called **confocal prrameter**

Beam width $$W(z) = W_0[1+(\frac{z}{z_0})^2]^{1/2}$$

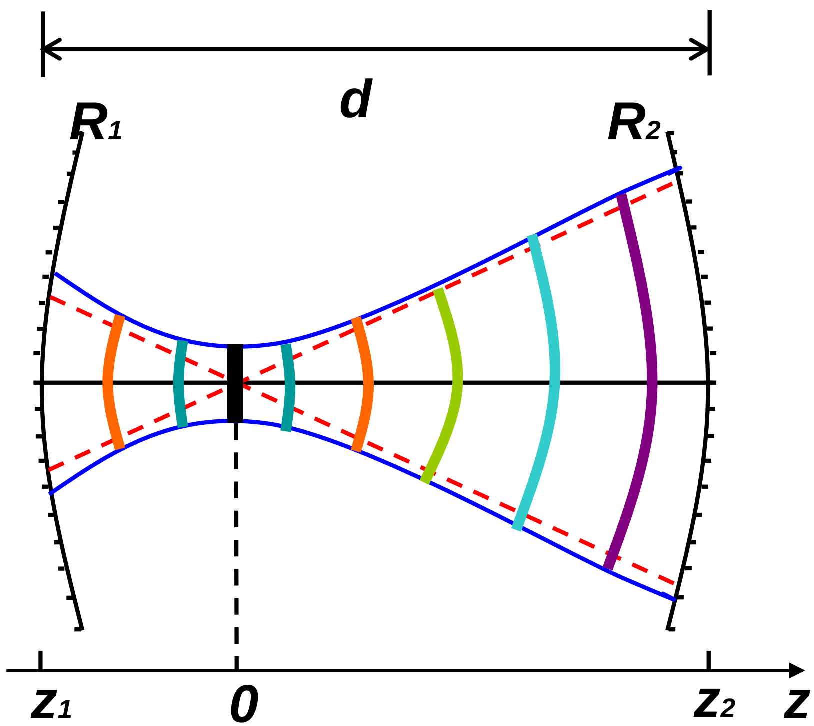
minimum value W_0 at the beam waist *(z = 0)*.

The radius of curvature $$R(z) = z + \frac{z_0^2}{z}$$

$$R = R(z) = z\left[1+\left(\frac{z}{z_0}\right)^2\right] = z_0\left(\frac{z}{z_0}+\frac{z_0}{z}\right) = z + \frac{z_0{}^2}{z}$$

Beam waist $$W_0 = \sqrt{\frac{\lambda z_0}{\pi}}$$

B. Gaussian Mode of a Symmetrical Spherical-Mirror Resonator



$$\begin{cases} z_2 = z_1 + d \\ R_1 = z_1 + \dfrac{z_0^2}{z_1} \\ -R_2 = z_2 + \dfrac{z_0^2}{z_2} \end{cases}$$

$$\begin{cases} z_1 = \dfrac{-d(R_2+d)}{R_2+R_1+2d}, z_2 = z_1 + d \\ z_0^2 = \dfrac{-d(R_1+d)(R_2+d)(R_2+R_1+2d)}{(R_2+R_1+2d)^2} \end{cases}$$

the beam radii at the mirrors

$$W_i = W_0[1+(\frac{z_i}{z_0})^2]^{1/2}, i = 1, 2.$$

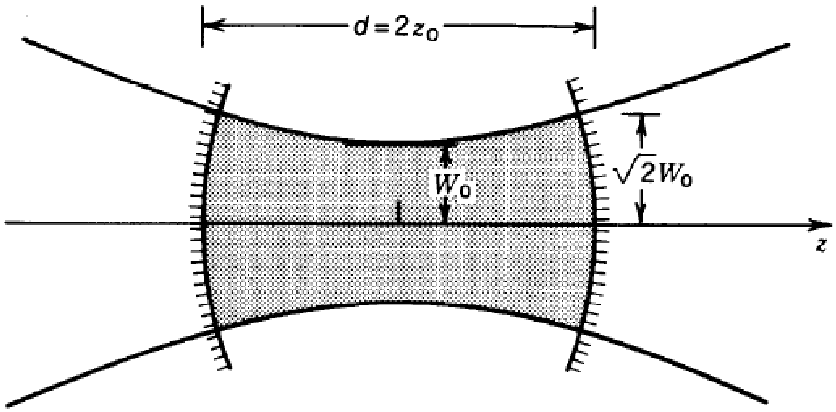
For relation: $$z_0^2 = \frac{-d(R_1+d)(R_2+d)(R_2+R_1+2d)}{(R_2+R_1+2d)^2}$$

- An imaginary value of z_0 signifies that the Gaussian beam is in fact a paraboloidal wave, which is an unconfined solution,
- for a confined solution z_0 must be real. it is not difficult to show that the condition $z_0^2 > 0$ is equivalent to

$$0 \le (1+\frac{d}{R_1})(1+\frac{d}{R_2}) \le 1$$

Gaussian Mode of a Symmetrical Spherical-Mirror Resonator

Symmetrical resonators with concave mirrors that is $R_1 = R_2 = -|R|$ so that $z_1 = -d/2$, $z_2 = d/2$. Thus the beam center lies at the center



$$z_0 = \frac{d}{2}\left(2\frac{|R|}{d} - 1\right)^{1/2}$$

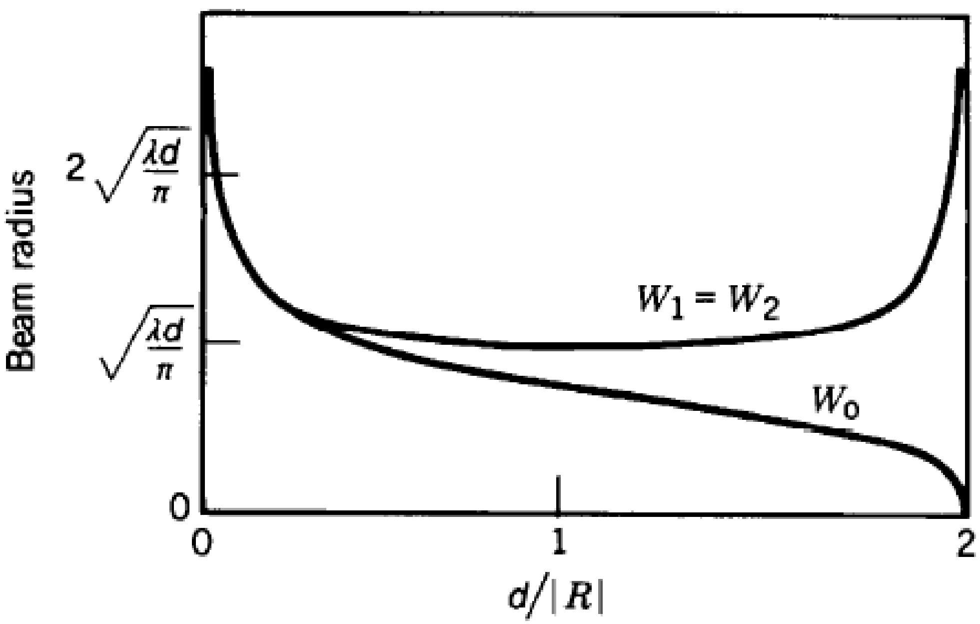
$$W_0^2 = \frac{\lambda d}{2\pi}\left(2\frac{|R|}{d} - 1\right)^{1/2}$$

$$W_1^2 = W_2^2 = \frac{\lambda d / \pi}{\{(d/|R|)[2-(d/|R|)]\}^{1/2}}$$

The confinement condition becomes

$$0 \le \frac{d}{|R|} \le 2$$

Given a resonator of fixed mirror separation *d*, we now examine the effect of increasing mirror curvature (increasing $d/|R|$) on the beam radius at the waist W_0, and at the mirrors $W_l = W_2$.



As $d/|R|$ increases, W_0 decreases until it vanishes for the concentric resonator ($d/|R| = 2$); at this point $W_1 = W_2 = \infty$

The radius of the beam at the mirrors has its minimum value, $W_l = W_2 = (\lambda d/\pi)^{1/2}$, when $d/|R| = 1$

$$z_0 = \frac{d}{2} \qquad W_0 = \left(\frac{\lambda d}{2\pi}\right)^{1/2}$$

$$W_1 = W_2 = \sqrt{2} W_0$$

C. Resonance Frequencies of a Gaussian beam

The phase of a Gaussian beam, $$\varphi(x,y,z) = kz - tg^{-1}(z/z_0) + \frac{k(x^2+y^2)}{2R(z)}$$

At the locations of the mirrors z1 and z2 on the optical aixs (x2+y2=0), we have,

$$\varphi(0,z_2) - \varphi(0,z_1) = k(z_2 - z_1) - [\zeta(z_2) - \zeta(z_1)] = kd - \Delta\zeta$$ where $$\varsigma(z) = tg^{-1}\left(\frac{z}{z_0}\right)$$

As the traveling wave completes a round trip between the two mirrors, therefore, its phase changes by $$2kz - 2\Delta\varsigma$$

For the resonance, the phase must be in condition $$2kz - 2\Delta\varsigma = 2q\pi, \qquad q = 1, 2, 3...$$

If we consider the plane wave resonance frequency $$k = 2\pi\nu/c \quad and \quad \nu_F = c/2d$$

We have $$\nu_q = q\nu_F + \frac{\Delta\zeta}{\pi}\nu_F$$

Spherical-Mirror Resonator Resonance Frequencies (Gaussian Modes)

$$\nu_q = q\nu_F + \frac{\Delta\zeta}{\pi}\nu_F$$

1. The frequency spacing of adjacent modes is $V_F = c/2d$, which is the same result as that obtained for the planar-mirror resonator.
2. For spherical-mirror resonators, this frequency spacing is independent of the curvatures of the mirrors.
3. The second term in the fomula, which does depend on the mirror curvatures, simply represents a displacement of all resonance frequencies.

For Hermite gaussian mode it may be more complicate

Hermite - Gaussian Modes

Hermite-Gaussian is one resolution for Helmholtz equation
An entire family of solutions, the Hermite-Gaussian family, exists. Although a Hermite-Gaussian beam of order (l, m) has the same wavefronts as a Gaussian beam, its amplitude distribution differs . It follows that the entire family of Hermite-Gaussian beams represents modes of the spherical-mirror resonator

$$U_{l,m}(x,y,z) = A_{l,m}[\frac{W_0}{W(z)}]G_l[\frac{\sqrt{2}x}{W(z)}]G_m[\frac{\sqrt{2}y}{W(z)}]\times\exp[-jkz - jk\frac{x^2+y^2}{2R(z)} + j(l+m+1)\zeta(z)]$$

$$\varphi(0,z) = kz-(l+m+1)\zeta(z)$$

$$2kd - 2(l+m+1)\Delta\zeta = 2\pi q, q = 0, \pm1, \pm2, \ldots,$$

Spherical mirror resonator Resonance Frequencies
(Hermite -Gaussian Modes)

$$\nu_{l,m,q} = q\nu_F + (l+m+1)\frac{\Delta\zeta}{\pi}\nu_F$$

Longitudinal or axial modes: different q and same indices (l, m) the intensity will be the same

Transverse modes: The indices (l, m) label different means different spatial intensity dependences

$$\nu_{l,m,q} = q\nu_F + (l + m + 1)\frac{\Delta\zeta}{\pi}\nu_F$$

- Longitudinal modes corresponding to a given transverse mode (l, m) have resonance frequencies spaced by $\nu_F = c/2d$, i.e., $\nu_{l,m,q} - \nu_{l',m',q} = \nu_F$.
- Transverse modes, for which the sum of the indices $l + m$ is the same, have the same resonance frequencies.
- Two transverse modes (l, m), (l', m') corresponding mode q frequencies spaced

$$\nu_{l,m,q} - \nu_{l',m',q} = [(l+m) - (l'+m')]\frac{\Delta\zeta}{\pi}\nu_F$$

*E. Finite Apertures and Diffraction Loss

Since the resonator mirrors are of finite extent, a portion of the optical power escapes from the resonator on each pass. An estimate of the power loss may be determined by calculating the fractional power of the beam that is not intercepted by the mirror. That is the **finite apertures effect** and this effect will cause **diffraction loss**.

For example:

If the Gaussian beam with radius W and the mirror is circular with radius a and $a= 2W$, each time there is a small fraction, $exp(- 2a^2/ W^2) = 3.35 \times 10^{-4}$, of the beam power escapes on each pass.

Higher-order transverse modes suffer greater losses since they have greater spatial extent in the transverse plane.

- In the resonator, the mirror transmission and any aperture limitation will induce loss
- The aperture induce loss is due to diffraction loss, and the loss depend mainly on the diameters of laser beam, the aperture place and its diameter
- We can used Fresnel number N to represent the relation between the size of light beam and the aperture, and use N to represent the loss of resonator.

Diffraction loss

The Fresnel number N_F

$$N_F = \frac{a^2}{d\lambda} = \frac{a^2}{2z\ \lambda} = \frac{a^2}{\pi W^2}$$

Attention: the W here is the beam width in the mirror, a is the dia. of mirror

Physical meaning：the ratio of the accepting angle *(a/d)* (form one mirror to the other of the resonator)to diffractive angle of the beam *(λ/a)* .

The higher Fresnel number corresponds to a smaller loss

- N is the maximum number of trip that light will propagate in side resonator without escape.
- 1/N represent each round trip the ratio of diffraction loss to the total energy

- Symmetric confocal resonator

$$\frac{a_1^2}{\pi W_1^2} = \frac{a_2^2}{\pi W_2^2} = N_F$$

- For general stable concave mirror resonator, the Fresnel number for two mirrors are:

$$N_{F1} = \frac{a_1^2}{\pi W_1^2} = \frac{a_1^2}{d\lambda}[\frac{g_1}{g_2}(1 - g_1 g_2)]^{\frac{1}{2}}$$

$$N_{F2} = \frac{a_2^2}{\pi W_2^2} = \frac{a_2^2}{d\lambda}[\frac{g_2}{g_1}(1 - g_1 g_2)]^{\frac{1}{2}}$$

2.5 The other cavities and beams

- Bessel beam

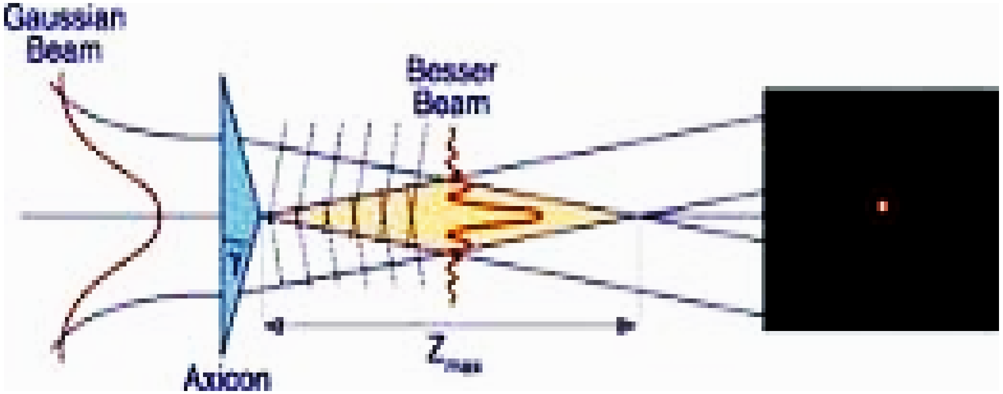
one resolution of wave equation:

$$A(x,y) = A_m J_m(k_T \rho)\exp(im\phi)$$

for m=0 basic Bessel beam is

$$A(r) = A_0 J_0(k_T \rho)\exp(i\beta z)$$

so that wave front normal are all parallel to z axis, no diffaction



Airy beam

- An **Airy beam** is a non-diffracting waveform which gives the appearance of curving as it travels.

$$\Phi(\xi,\,s) = \mathrm{Ai}(\,s - (\xi/2)^2)\exp(i(\,s\xi/2) - i(\xi^3/12))$$

Ai(x) is the Airy function. $\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right)\,dt.$

Φ is the electric field envelope, represents a dimensionless traverse coordinate
s is an arbitrary traverse scale, ξ is a normalized propagation distance