Problem 1. 求解方程

$$
\begin{cases}
u_{tt} - a^2 u_{xx} = 0, & x > 0, t > 0, \\
u\big|_{t=0} = \varphi(x), & x > 0, \\
u_t\big|_{t=0} = \psi(x), & x > 0, \\
u\big|_{x=0} = \eta(t), & t \geq 0 \ \textit{and} \ \varphi(0) = \eta(0).
\end{cases}
$$

证明. 首先做变换$v(x,t) := u(x,t) - \eta(t)$，则$v$满足

$$
\begin{cases}
v_{tt} - a^2 v_{xx} = -\eta''(t) =: F(x,t), & x > 0, t > 0, \\
v\big|_{t=0} = \varphi(x) - \eta(0) =: \Phi(x), & x > 0, \\
v_t\big|_{t=0} \psi(x) - \eta'(0) =: \Psi(x), & x > 0, \\
v\big|_{x=0} = 0, & t \geq 0.
\end{cases}
$$

然后做奇延拓，

$$
F(x,t) = \begin{cases}
-\eta''(t), & x \geq 0; \\
\eta''(t), & x < 0.
\end{cases}
$$

$$
\Phi(x) = \begin{cases}
\varphi(x) - \eta(0), & x \geq 0, \\
-\varphi(-x) + \eta(0), & x < 0;
\end{cases}
\Psi(x) = \begin{cases}
\psi(x) - \eta'(0), & x \geq 0, \\
-\psi(-x) + \eta'(0), & x < 0.
\end{cases}
$$

则此时，由Kirchhoff公式（见[LCP]的24页），

$$
u(x,t) = \frac{1}{2}\left(\Phi(x+at) + \Phi(x-at)\right) + \frac{1}{2a}\int_{x-at}^{x+at} \Psi(\xi) d\xi + \frac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} F(\xi,\tau) d\xi d\tau.
$$

当$x - at \geq 0$，

$$
\begin{aligned}
u(x,t) = v(x,t) + \eta(t) =& \frac{1}{2}\left(\varphi(x+at) + \varphi(x-at) - 2\eta(0)\right) + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi) - \eta'(0) d\xi \\
&+ \frac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} -\eta''(\tau) d\xi d\tau + \eta(t);
\end{aligned}
$$

当$x - at < 0$，

$$
\begin{aligned}
u(x,t) = v(x,t) + \eta(t) =& \frac{1}{2}\left(\varphi(x+at) - \varphi(at-x)\right) + \frac{1}{2a}\int_{at-x}^{x+at} (\psi(\xi) - \eta'(0)) d\xi \\
&+ \frac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} F(\xi,\tau) d\xi d\tau + \eta(t) \\
=& \frac{1}{2}\left(\varphi(x+at) - \varphi(at-x)\right) + \frac{1}{2a}\int_{at-x}^{x+at} (\psi(\xi) - \eta'(0)) d\xi \\
&- \eta'(t)(x+at) + 2\eta'(0)x + \eta'(t - \frac{x}{a})(at - x) + \eta(t)
\end{aligned}
$$

$\square$

References

[LCP] 李胜宏陈仲慈潘祖梁. 数学物理方程[M]. 浙江大学出版社, 2008.