按固有函数展开法

例. 有热源的有界杆的热传导问题:一端温度恒为零，另一端绝热的有界杆，产生热量的强度为 $f(x,t)$ 的热传导问题可归结为

$$\begin{cases} \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t), \; 0 < x < \ell, \; t > 0 \\ u(x,t)|_{x=0} = 0, \; \frac{\partial u(x,t)}{\partial x}|_{x=\ell} = 0, \\ u(x,t)|_{t=0} = \phi(x), 0 < x < \ell. \end{cases}$$

按固有函数展开法

第一步: 求出满足相应地齐次方程和齐次边界条件的可写为 $u(x,t) = X(x)T(t)$（分离变量）的所有非零解.
把 $u(x,t) = X(x)T(t)$ 代入相应地齐次方程 $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$,

$$X(x)T'(t) = a^2 X''(x)T(t) \Rightarrow \frac{X''(x)}{X(x)} = \frac{T'(t)}{a^2 T(t)} = -\lambda,$$

函数 $X(x)$ 和 $T(t)$ 分别满足

$$X''(x) + \lambda X(x) = 0, \ T'(t) + \lambda a^2 T(t) = 0.$$

边界条件 $\Rightarrow X(0)T(t) = 0, \ X'(\ell)T(t) = 0 \Rightarrow X(0) = X'(\ell) = 0.$

$X(x)$和T 满足

$$\begin{cases} X''(x) + \lambda X(x) = 0, 0 < x < \ell \\ X(0) = X'(\ell) = 0. \end{cases}, \ T' + \lambda a^2 T = 0$$

按固有函数展开法

第二步： 求出固有值问题$\begin{cases} X''(x) + \lambda X(x) = 0, \ 0 < x < \ell \\ X(0) = X'(\ell) = 0. \end{cases}$ 的所有的固有值和本征函数。

- 当 $\lambda < 0$ 时，通解为$X(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}$,

  $$X(0) = X'(\ell) = 0 \Rightarrow \begin{cases} C_1 + C_2 = 0 \\ C_1\sqrt{-\lambda}e^{\sqrt{-\lambda}\ell} + C_2\sqrt{-\lambda}e^{-\sqrt{-\lambda}\ell} = 0. \end{cases}$$

  解得 $C_1 = C_2 = 0 \Rightarrow X(x) = 0$ 只有零解。
- 当 $\lambda = 0$ 时，通解为$X(x) = C_1 x + C_2, X(0) = X'(\ell) = 0 \Rightarrow C_2 = 0, \ C_1 = 0 \Rightarrow X(x) = 0$ 只有零解。

按固有函数展开法

- 当 $\lambda > 0$ 时，通解为$X(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x, X(0) = X'(\ell) = 0$

$$\Rightarrow C_1 = 0, \quad -C_1\sqrt{\lambda}\sin\sqrt{\lambda}\ell + C_2\sqrt{\lambda}\cos\sqrt{\lambda}\ell = 0.$$

$$\Rightarrow C_1 = 0, C_2\cos\sqrt{\lambda}\ell = 0, (C_2 \neq 0) \Rightarrow \cos\sqrt{\lambda}\ell = 0$$

$$\Rightarrow \sqrt{\lambda}\ell = (n - 1/2)\pi, \; n = 1, 2, 3, \cdots$$

$$\Rightarrow \lambda_n = \left(\frac{(n-1/2)\pi}{\ell}\right)^2, \; n = 1, 2, 3, \cdots$$

$$\Rightarrow X_n(x) = C_n \sin\frac{(n-1/2)\pi x}{\ell}, \; n = 1, 2, 3, \cdots$$

按固有函数展开法

注意： 当 $T_n(t)$ 满足方程$T'(t)+\lambda_n a^2 T(t)=0$, 即$T_n(t)=a_n e^{\frac{(n-1/2)^2\pi^2 a^2}{\ell^2}t}(n=1,2,3,\cdots)$时， $u_n(x,t)=X_n(x)T_n(t)$满足齐次方程和边界条件

$$\begin{cases} \frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2},\ 0<x<\ell,\ t>0 \\ u(x,t)|_{x=0}=0,\ \frac{\partial u(x,t)}{\partial x}|_{x=\ell}=0, \end{cases}$$

它们的叠加$u(x,t)=\sum_{n=1}^{\infty}T_n(t)X_n(x)$不可能满足非齐次方程

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}+f(x,t),\ 0<x<\ell,\ t>0.$$

按固有函数展开法

但是 对于任何函数 $T_n(t)$, 表达式

$$u(x,t)=\sum_{n=1}^{\infty}T_n(t)X_n(x)=\sum_{n=1}^{\infty}T_n(t)\sin\frac{(n-1/2)\pi x}{\ell}$$

总能满足边界条件

$$u(x,t)|_{x=0}=0,\ \frac{\partial u(x,t)}{\partial x}|_{x=\ell}=0,$$

哪么我们能否找到适当的函数 $T_n(t)$ 使得它能满足非齐次方程和初始条件

$$\begin{cases}\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}+f(x,t),\ 0<x<\ell,\ t>0\\ u(x,t)|_{t=0}=\phi(x),\ 0<x<\ell.\end{cases}\ ?$$

上述的想法类似常微分方程中的 **“常数变易法”**

按固有函数展开法

$$\begin{cases} \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t), \ 0 < x < \ell, \ t > 0 \\ u(x,t)|_{x=0} = 0, \ \frac{\partial u(x,t)}{\partial x}|_{x=\ell} = 0, \\ u(x,t)|_{t=0} = \phi(x), 0 < x < \ell. \end{cases}$$

第三步: 利用“常数变易法”求出满足非齐次方程和初始条件的解。由第二步我们得到固有函数

$$X_n(x) = C_n \sin \frac{(n-1/2)\pi x}{\ell}, \ n = 1, 2, 3, \cdots$$

利用这些固有函数构造

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{(n-1/2)\pi x}{\ell},$$

对于任意函数$T_n(t)$满足它满足边界条件。适当选取 $T_n(t)$ 使它还满足非齐次方程和初始条件.

按固有函数展开法

方程+初始条件 $\Rightarrow$

$$\begin{cases} \sum_{n=1}^{\infty}\left(T_n'(t)+(\frac{(n-1/2)\pi a}{\ell})^2T_n(t)\right)\sin\frac{(n-1/2)\pi x}{\ell}=f(x,t), \\ \sum_{n=1}^{\infty}T_n(0)\sin\frac{(n-1/2)\pi x}{\ell}=\phi(x), \end{cases}$$

把$f(x,t)$和$\phi(x)$按固有函数展开

$$\begin{cases} f(x,t)=\sum_{n=1}^{\infty}f_n(t)\sin\frac{(n-1/2)\pi x}{\ell}, \\ \phi(x)=\sum_{n=1}^{\infty}\phi_n\sin\frac{(n-1/2)\pi x}{\ell}, \end{cases}$$

$$\begin{cases} f_n(t)=\frac{2}{\ell}\int_0^{\ell}f(x,t)\sin\frac{(n-1/2)\pi x}{\ell}dx, \\ \phi_n=\frac{2}{\ell}\int_0^{\ell}\phi(x)\sin\frac{(n-1/2)\pi x}{\ell}dx,\ n=1,2,3,\cdots \end{cases}$$

$T_n(t)$ 满足

$$\begin{cases} T_n'(t)+(\frac{(n-1/2)\pi a}{\ell})^2T_n(t)=f_n(t),\ t>0 \\ T_n(0)=\phi_n. \end{cases}$$

按固有函数展开法

$$T_n(t) = e^{-\left(\frac{(n-1/2)\pi a}{\ell}\right)^2 t}\left[\phi_0 + \int_0^t e^{\left(\frac{(n-1/2)\pi a}{\ell}\right)^2 \tau} f_n(\tau) d\tau.\right]$$

把 $T_n(t)$ 代入

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin\frac{(n-1/2)\pi x}{\ell}$$

得到有热源的有界杆的热传导问题

$$\begin{cases} \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t),\ 0 < x < \ell,\ t > 0 \\ u(x,t)|_{x=0} = 0,\ \frac{\partial u(x,t)}{\partial x}|_{x=\ell} = 0, \\ u(x,t)|_{t=0} = \phi(x), 0 < x < \ell. \end{cases}$$

的解。

按固有函数展开法

例. 有强迫项的有限长的弦，二端在位移方向作自由运动，弦的振动可描述为

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t), 0 < x < \ell, \ t > 0 \\ \frac{\partial u}{\partial x}|_{x=0} = 0, \ \frac{\partial u}{\partial x}|_{x=\ell} = 0, \\ u(x,t)|_{t=0} = \phi(x), \ \frac{\partial u}{\partial t}|_{t=0} = \psi(x), 0 < x < \ell. \end{cases}$$

按固有函数展开法

第一步： 求出满足相应地齐次方程和边界条件的可写为 $u(x,t)=X(x)T(t)$（分离变量）的所有非零解。

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}\Rightarrow X(x)T''(t)=a^2X''(x)T(t)$$

$$\Rightarrow\quad \frac{X''(x)}{X(x)}=\frac{T''(t)}{a^2T(t)}=-\lambda,$$

边界条件 $\Rightarrow X'(0)T(t)=0,\ X'(\ell)T(t)=0\Rightarrow X'(0)=0,\ X'(\ell)=0$

因此固有值问题

$$X''(x)+\lambda X(x)=0\,(0<x<\ell),\ X'(0)=0,\ X'(\ell)=0.$$

按固有函数展开法

第二步: 求出固有值问题

$$X''(x)+\lambda X(x)=0\,(0<x<\ell),\ X'(0)=0,\ X'(\ell)=0.$$

的所有的本征函数。

- 当 $\lambda<0$ 时，通解为$X(x)=C_1e^{\sqrt{-\lambda}x}+C_2e^{-\sqrt{-\lambda}x}$,

$$\begin{cases} X'(0)=0, \\ X'(\ell)=0 \end{cases} \Rightarrow \begin{cases} \sqrt{-\lambda}C_1-\sqrt{-\lambda}C_2=0 \\ \sqrt{-\lambda}C_1e^{\sqrt{-\lambda}\ell}-\sqrt{-\lambda}C_2e^{-\sqrt{-\lambda}\ell}=0. \end{cases}$$

$\Rightarrow C_1=C_2=0$, 只有零解。

- 当 $\lambda=0$ 时,通解为$X(x)=C_1x+C_2$,
$X'(0)=0,\ X'(\ell)=0\Rightarrow C_1=0,C_2$ 为任意非零常数。 $\Rightarrow\lambda_0=0$是固有值,$X_0(x)=C_0$为固有函数。

按固有函数展开法

- 当 $\lambda > 0$ 时，通解为$X(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$.

$$X'(0) = 0,\ X'(\ell) = 0$$

$$\Rightarrow C_2 = 0,\ -\sqrt{\lambda}C_1 \sin\sqrt{\lambda}\ell + \sqrt{\lambda}C_2 \cos\sqrt{\lambda}\ell = 0.$$

$$\Rightarrow C_2 = 0,\ C_1 \sin\sqrt{\lambda}\ell = 0, (C_1 \neq 0) \Rightarrow \sin\sqrt{\lambda}\ell = 0$$

$$\Rightarrow \lambda_n = \frac{n^2\pi^2}{\ell^2},\ X_n(x) = C_n \cos\frac{n\pi x}{\ell},\ n = 1, 2, 3, \cdots.$$

综合:所有固有值和固有函数为

$$\lambda_n = \frac{n^2\pi^2}{\ell^2},\ X_n(x) = C_n \cos\frac{n\pi x}{\ell},\ n = 0, 1, 2, 3, \cdots$$

按固有函数展开法

第三步： 利用“常数变易法”求出满足非齐次方程和初始条件的解。

$$u(x,t)=\sum_{n=0}^{\infty}T_n(t)\cos\frac{n\pi x}{\ell}$$

满足边界条件$\frac{\partial u}{\partial x}|_{x=0}=0$, $\frac{\partial u}{\partial x}|_{x=\ell}=0$.求$T_n(t)$使得$u$ 满足非齐次方程和初始条件

$$\begin{cases}\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}+f(x,t),\ 0<x<\ell,\ t>0\\ u(x,t)|_{t=0}=\phi(x),\ \frac{\partial u}{\partial t}|_{t=0}=\psi(x),0<x<\ell.\end{cases}$$

$$\Rightarrow\begin{cases}\sum_{n=0}^{\infty}\left(T_n''(t)+(\frac{n\pi a}{\ell})^2T_n(t)\right)\cos\frac{n\pi x}{\ell}=f(x,t),\\ \sum_{n=0}^{\infty}T_n(0)\cos\frac{n\pi x}{\ell}|_{t=0}=\phi(x),0<x<\ell\\ \sum_{n=0}^{\infty}\frac{n\pi}{\ell}T_n'(t)\cos\frac{n\pi x}{\ell}|_{t=0}=\psi(x),0<x<\ell.\end{cases}$$

按固有函数展开法

把函数 $f(x,t)$ 和 $\phi(x)$ 和 $\psi(x)$ 关于固有函数系展开

$$\begin{cases} f(x,t) = \sum_{n=0}^{\infty} f_n(t) \cos \frac{n\pi x}{\ell}, \ 0 < x < \ell, \ t > 0 \\ \phi(x) = \sum_{n=0}^{\infty} \phi_n \cos \frac{n\pi x}{\ell}, \ \phi(x) = \sum_{n=0}^{\infty} \psi_n \cos \frac{n\pi x}{\ell}, 0 < x < \ell. \end{cases}$$

对于 $n = 1, 2, 3, \cdots,$

$$f_0(t) = \frac{1}{\ell} \int_0^{\ell} f(x,t) dx, \ f_n(t) = \frac{2}{\ell} \int_0^{\ell} f(x,t) \cos \frac{n\pi x}{\ell} dx$$

$$\phi_0 = \frac{1}{\ell} \int_0^{\ell} \phi(x) dx, \ \phi_n = \frac{2}{\ell} \int_0^{\ell} \phi(x) \cos \frac{n\pi x}{\ell} dx$$

$$\psi_0 = \frac{1}{\ell} \int_0^{\ell} \psi(x) dx, \ \psi_n = \frac{2}{\ell} \int_0^{\ell} \psi(x) \cos \frac{n\pi x}{\ell} dx$$

按固有函数展开法

$$\begin{cases} \sum_{n=0}^{\infty}\left(T_n''(t)+\left(\frac{n\pi a}{\ell}\right)^2 T_n(t)\right)\cos\frac{n\pi x}{\ell} = f(x,t), \\ \sum_{n=0}^{\infty} T_n(0)\cos\frac{n\pi x}{\ell}|_{t=0} = \phi(x), 0<x<\ell \\ \sum_{n=0}^{\infty} \frac{n\pi}{\ell} T_n'(t)\cos\frac{n\pi x}{\ell}|_{t=0} = \psi(x), 0<x<\ell. \end{cases}$$

$$\begin{cases} f(x,t)=\sum_{n=0}^{\infty} f_n(t)\cos\frac{n\pi x}{\ell},\ 0<x<\ell,\ t>0 \\ \phi(x)=\sum_{n=0}^{\infty}\phi_n\cos\frac{n\pi x}{\ell},\ \phi(x)=\sum_{n=0}^{\infty}\psi_n\cos\frac{n\pi x}{\ell}, 0<x<\ell. \end{cases}$$

$$\Rightarrow \begin{cases} T_n''(t)+\left(\frac{n\pi a}{\ell}\right)^2 T_n(t)=f_n(t),\ t>0 \\ T_n(0)=\phi_n,\ T_n'(0)=\psi_n. \end{cases}$$

如何求解上述常微分方程?

按固有函数展开法

$$\begin{cases} T_n''(t) + (\frac{n\pi a}{\ell})^2 T_n(t) = f_n(t), \\ T_n(0) = \phi_n,\ T_n'(0) = \psi_n. \end{cases}$$

分解为

$$(I) \begin{cases} w_n''(t) + (\frac{n\pi a}{\ell})^2 w_n(t) = 0, \\ w_n(0) = \phi_n,\ w_n'(0) = \psi_n. \end{cases}$$

$$(II) \begin{cases} h_n''(t) + (\frac{n\pi a}{\ell})^2 h_n(t) = f_n(t), \\ h_n(0) = 0,\ h_n'(0) = 0. \end{cases}$$

问题（I）的解为

$$w_0(t) = \phi_0 + \psi_0 t,$$

$$w_n(t) = \phi_n \cos\frac{n\pi a}{\ell} t + \psi_n \frac{\ell}{n\pi a} \sin\frac{n\pi a}{\ell} t,\ n = 1, 2, \cdots .$$

问题 (II)如何求解?

按固有函数展开法

我们用 **齐次化原理**: 求解常微分方程

$$\begin{cases} y''(t) + ay'(t) + by(t) = f(t) \\ y(0) = y'(0) = 0 \end{cases}$$

其中 a, b 为常数。先求解齐次常微分方程

$$\begin{cases} y''(t) + ay'(t) + by(t) = 0 \\ y(0) = 0,\ y'(0) = f(\tau) \end{cases}$$

的解 $w = w(t, \tau)$, 则原方程的解为

$$y(t) = \int_0^t w(t - \tau, \tau) d\tau$$

按固有函数展开法

我们用 **齐次化原理**求解问题(II):

$$(II)\begin{cases} h_n''(t)+(\frac{n\pi a}{\ell})^2 h_n(t)=f_n(t),\ t>0 \\ h_n(0)=0,\ h_n'(0)=0. \end{cases}$$

它的齐次化方程为

$$\begin{cases} v_n''(t)+(\frac{n\pi a}{\ell})^2 v_n(t)=0,\ t>0 \\ v_n(0)=0,\ v_n'(0)=f_n(\tau). \end{cases}$$

解为

$$v_0(t)=f_0(\tau)t$$

$$v_n(t)=f_n(\tau)\frac{\ell}{n\pi a}\sin\frac{n\pi a}{\ell}t,\ n=1,2,\cdots$$

$$\Rightarrow h_0(t)=\int_0^t f_0(\tau)(t-\tau)d\tau$$

按固有函数展开法

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} + f(x,t), 0 < x < \ell,\ t > 0 \\ \frac{\partial u}{\partial x}|_{x=0} = 0,\ \frac{\partial u}{\partial x}|_{x=\ell} = 0, \\ u(x,t)|_{t=0} = \phi(x),\ \frac{\partial u}{\partial t}|_{t=0} = \psi(x), 0 < x < \ell. \end{cases}$$

的解是

$$u(x,t) = \sum_{n=0}^{\infty} T_n(t)\cos\frac{n\pi x}{\ell}$$

$$\begin{aligned} T_0(t) &= \phi_0 + \psi_0 t + \int_0^t f_0(\tau)(t-\tau)d\tau \\ T_n(t) &= \phi_n\cos\frac{n\pi a}{\ell}t + \psi_n\frac{\ell}{n\pi a}\sin\frac{n\pi a}{\ell}t \\ &\quad + \int_0^t f_n(\tau)\frac{\ell}{n\pi a}\sin\frac{n\pi a(t-\tau)}{\ell}d\tau (n = 1, 2, \cdots) \end{aligned}$$

非齐次边界条件的定解问题

想法: 找一个适当替换，将边界条件化为齐次边界条件。
例. 有强迫项的有限长的弦，二端的位移已知，弦的振动可描述为

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t),\ 0 < x < \ell,\ t > 0 \\ u|_{x=0} = \nu(t),\ u|_{x=\ell} = \mu(t), \\ u(x,t)|_{t=0} = \phi(x),\ \frac{\partial u}{\partial t}|_{t=0} = \psi(x), 0 < x < \ell. \end{cases}$$

取 $w(x,t) = \frac{\ell - x}{\ell}\nu(t) + \frac{x}{\ell}\mu(t)$,$w(x,t)$ 满足

$$w|_{x=0} = \nu(t),\ w|_{x=\ell} = \mu(t)$$

记 $U(x,t) = u(x,t) - w(x,t)$,

非齐次边界条件的定解问题

$U(x,t)$ 满足

$$\begin{cases} \frac{\partial^2 U}{\partial t^2} = a^2 \frac{\partial^2 U}{\partial x^2} + f(x,t) - (\frac{\ell - x}{\ell}\nu''(t) + \frac{x}{\ell}\mu''(t)), 0 < x < \ell,\ t > 0 \\ U|_{x=0} = 0,\ U|_{x=\ell} = 0,\ \Leftrightarrow \text{齐次边界条件} \\ U(x,t)|_{t=0} = \phi(x) - (\frac{\ell - x}{\ell}\nu(0) + \frac{x}{\ell}\mu(0)), 0 < x < \ell \\ \frac{\partial U}{\partial t}|_{t=0} = \psi(x) - (\frac{\ell - x}{\ell}\nu'(0) + \frac{x}{\ell}\mu'(0)), 0 < x < \ell. \end{cases}$$

满足要求的函数 $w(x,t)$ 有很多，如

$$w(x,t) = \nu(t) + \sin(\frac{x}{\ell})[\mu(t) - \nu(t)].$$

事实上，只要 $w(x,t)$ 满足

$$w|_{x=0} = \nu(t),\ w|_{x=\ell} = \mu(t)$$

的已知函数都是允许的。

非齐次边界条件的定解问题

那么能否找到合适的变换,把方程和边界条件化为齐次方程和齐次边界条件?这时$w(x,t)$需要满足

$$\begin{cases} \frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2} + f(x,t), \\ u|_{x=0} = \nu(t),\ u|_{x=\ell} = \mu(t), \end{cases}$$

较难得到$w(x,t)$.**但是** 当$f(x,t)=f(x)$,$\nu(t)=A$和$\mu(t)=B(t$无关)时,可以找到一个仅仅与x有关的函数$w(x)$

$$\begin{cases} \frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2} + f(x,t), \\ u|_{x=0} = \nu(t),\ u|_{x=\ell} = \mu(t), \end{cases} \Rightarrow \begin{cases} \frac{\partial^2 w}{\partial x^2} = -\frac{1}{a^2} f(x), \\ u|_{x=0} = A,\ u|_{x=\ell} = B, \end{cases}$$

方程与边界条件$\Rightarrow$齐次方程和齐次边界条件

非齐次边界条件的定解问题

例. 求下列定解问题

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2} + A,\ 0 < x < \ell,\ t > 0 \\ u|_{x=0} = 0,\ u|_{x=\ell} = B, \\ u(x,t)|_{t=0} = 0,\ \frac{\partial u}{\partial t}|_{t=0} = 0, 0 < x < \ell. \end{cases}$$

其中 A 和 B 为给定的常数。

$$\begin{cases} \frac{\partial^2 w}{\partial x^2} = -\frac{A}{a^2}, \\ u|_{x=0} = 0,\ u|_{x=\ell} = B, \end{cases} \Rightarrow w(x,t) = -\frac{A}{2a^2}x^2 + (\frac{A\ell}{2a^2} + \frac{B}{\ell})x$$

记 $U(x,t) = u(x,t) - w(x,t)$,那么

$$\begin{cases} \dfrac{\partial^2 U}{\partial t^2} = a^2 \dfrac{\partial^2 U}{\partial x^2}, & \Leftrightarrow \text{齐次方程} \\ U|_{x=0} = 0,\ U|_{x=\ell} = 0, & \Leftrightarrow \text{齐次边界条件} \\ U(x,t)|_{t=0} = \frac{A}{2a^2}x^2 - (\frac{A\ell}{2a^2} + \frac{B}{\ell})x, 0 < x < \ell \\ \frac{\partial U}{\partial t}|_{t=0} = 0, 0 < x < \ell. \end{cases}$$

非齐次边界条件的定解问题

分离变量法$\Rightarrow$

$$
\begin{aligned}
U(x,t) &= \sum_{n=1}^{\infty}\left(C_n\cos(\frac{n\pi at}{\ell}) + D_n\sin(\frac{n\pi at}{\ell})\right)\sin(\frac{n\pi x}{\ell}) \\
U(x,t)|_{t=0} &= \frac{A}{2a^2}x^2 - (\frac{A\ell}{2a^2}+\frac{B}{\ell})x = \sum_{n=1}^{\infty} C_n\sin(\frac{n\pi x}{\ell}) \\
\frac{\partial U}{\partial t}|_{t=0} &= 0 = \sum_{n=1}^{\infty} D_n\frac{n\pi}{\ell}\sin(\frac{n\pi x}{\ell})
\end{aligned}
$$

$$
\Rightarrow D_n = 0,\quad
\begin{aligned}
C_n &= \tfrac{2}{\ell}\int_0^{\ell}\left[\tfrac{A}{2a^2}x^2 - (\tfrac{A\ell}{2a^2}+\tfrac{B}{\ell})x\right]\sin(\tfrac{n\pi x}{\ell})dx \\
&= -\frac{2A\ell^2}{a^2n^3\pi^3} + (-1)^n\frac{2}{n\pi}\left(\frac{A\ell^2}{a^2n^2\pi^2}+B\right)
\end{aligned}
$$

因此原来初边值问题的解为

$$
u(x,t) = -\frac{A}{2a^2}x^2 + (\frac{A\ell}{2a^2}+\frac{B}{\ell})x + \sum_{n=1}^{\infty} C_n\cos(\frac{n\pi at}{\ell})\sin(\frac{n\pi x}{\ell}).
$$