Chapter 7 Time-Varying Fields and Maxwell's Equations

7.1 Introduction

Fundamental relations for electrostatic and magnetostatic models

Fundamental Relations	Electrostatic Model	Magnetostatic Model
Governing equations	$\nabla \times \mathbf{E} = 0$ $\nabla \cdot \mathbf{D} = \rho$	$\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{H} = \mathbf{J}$
Constitutive relations (linear and isotropic media)	$\mathbf{D} = \epsilon \mathbf{E}$	$\mathbf{H} = \frac{1}{\mu} \mathbf{B}$

- In the static case (non-time-varying), electric vectors ***E*** and ***D*** are independent with magnetic vectors ***B*** and ***H***.
- In a conductive medium, electric and magnetic fields may both exist and form an ***electromagnetostatic*** field.

7.2 Faraday's Law of Electromagnetic Induction

Faraday's Law (1831): the quantitative relationship between the induced emf and the rate of change of magnetic flux linking a conducting loop.

Fundamental postulate for Electromagnetic Induction:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

- It applies to every point in the space.
- ***E*** is nonconservative in a region of time-varying magnetic flux density.

The integral form over a surface:

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}.$$

7.2.1 A Stationary Circuit in a Time-varying Magnetic Field

For a stationary circuit with a contour *C* and surface *S*, we have

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s}. \qquad \text{(7-3)}$$

If we define,

$$\mathscr{V} = \oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = \text{emf induced in circuit with contour } C \qquad \text{(V)}$$

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = \text{magnetic flux crossing surface } S \qquad \text{(Wb)},$$

■ Faraday's Law of Electromagnetic Induction

$$\mathcal{V} = -\frac{d\Phi}{dt} \qquad \text{(V)}.$$

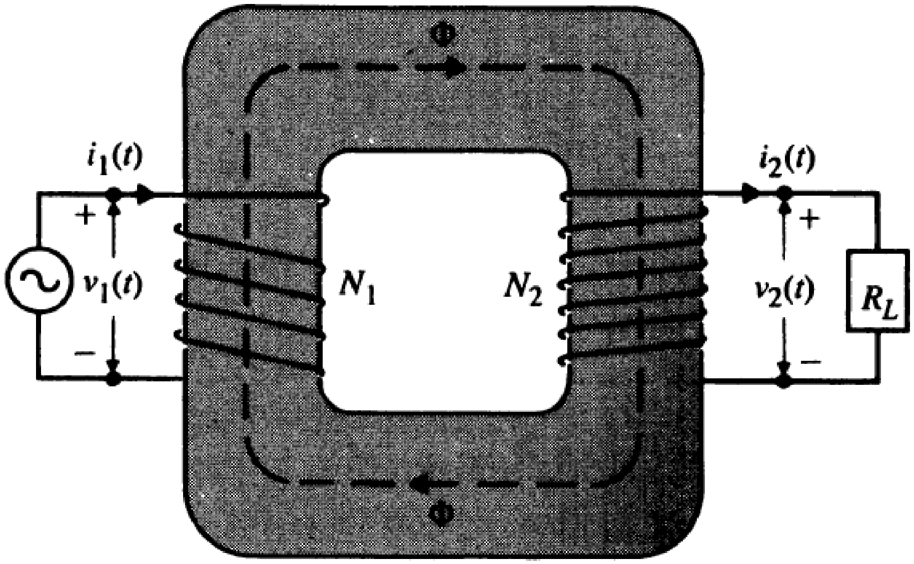
- ***The electromotive force induced in a stationary closed circuit is equal to the negative rate of increase of the magnetic flux linking the circuit.***
- ***Lenz's Law***: the induced current in the closed loop in such a direction as to oppose the change in the linking magnetic flux.
- ***Transformer emf***: the emf induced in a stationary loop caused by a time-varying magnetic field.

7.2.2 Transformers

Transformer: an alternating-current (ac) device that transforming voltages, currents and impedances.

For closed magnetic circuit, we have

$$N_1 i_1 - N_2 i_2 = \mathscr{R}\Phi,$$



R: reluctance of the magnetic circuit
magnetic flux
Φ: Magnetic flux.

- The induced mmf in the secondary circuit, $N_2 i_2$, opposes the flow of the magnetic flux created by the mmf in the primary circuit, $N_1 i_1$.

For $\mathscr{R} = \dfrac{\ell}{\mu S}$ (*l*: length of the magnetic core, *S*: cross section, μ: permeability)

We have: $$N_1 i_1 - N_2 i_2 = \frac{\ell}{\mu S}\Phi.$$

a) Ideal transformer (assume $\mu \rightarrow \infty$)

$$\Rightarrow \boxed{\frac{i_1}{i_2} = \frac{N_2}{N_1}.}$$

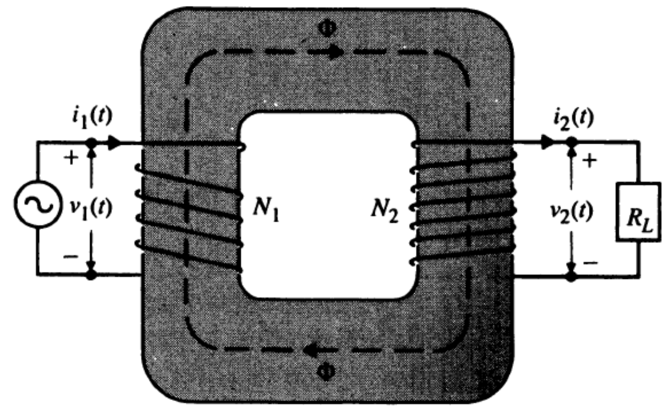
Faraday's law

$$v_1 = N_1 \frac{d\Phi}{dt} \quad v_2 = N_2 \frac{d\Phi}{dt} \quad \Rightarrow \quad \boxed{\frac{v_1}{v_2} = \frac{N_1}{N_2}}$$

- The ratio of current is inversely proportional to the ratio of the number of turns.
- The ratio of voltage is proportional the the ratio of the number of turns.

- **Effective load $(R_1)_{eff}$** seen by the source:

$$(R_1)_{\text{eff}} = \frac{v_1}{i_1} = \frac{(N_1/N_2)v_2}{(N_2/N_1)i_2}$$



Or, $$\boxed{(R_1)_{\text{eff}} = \left(\frac{N_1}{N_2}\right)^2 R_L}$$

R_L: load of the secondary windings

- **Impedance transformation** for a sinusoidal source $v(t)$ and a load impedance Z_l:

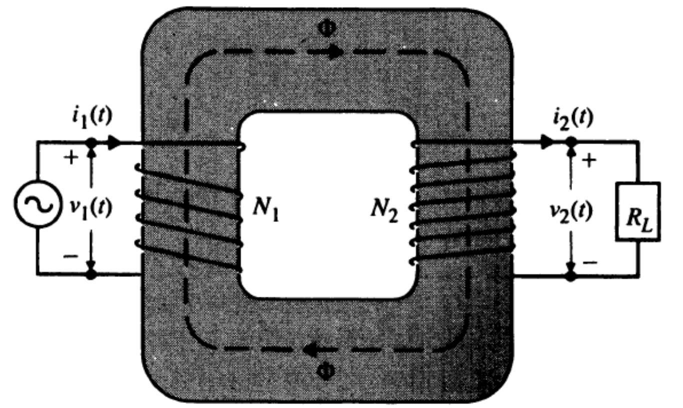
$$\boxed{(Z_1)_{\text{eff}} = \left(\frac{N_1}{N_2}\right)^2 Z_L.}$$

b) Real transformer.

We can write the magnetic flux linkage as

$$\Lambda_1 = N_1\Phi = \frac{\mu S}{\ell}(N_1^2 i_1 - N_1 N_2 i_2),$$

$$\Lambda_2 = N_2\Phi = \frac{\mu S}{\ell}(N_1 N_2 i_1 - N_2^2 i_2).$$



According to Faraday's law, we have

$$\boxed{\begin{aligned} v_1 &= L_1\frac{di_1}{dt} - L_{12}\frac{di_2}{dt}, \\ v_2 &= L_{12}\frac{di_1}{dt} - L_2\frac{di_2}{dt}, \end{aligned}}$$

Where,

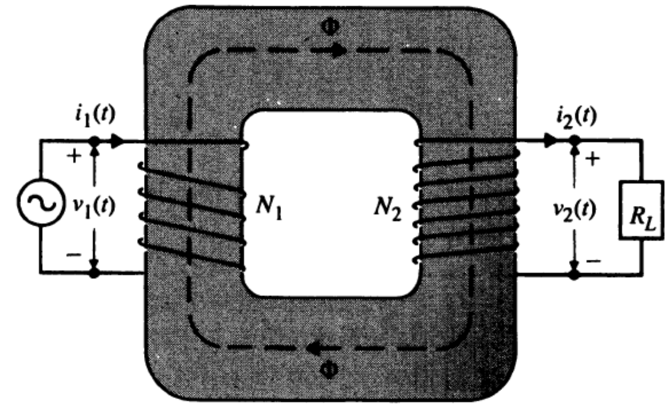
$$L_1 = \frac{\mu S}{\ell}N_1^2,$$ (Self-inductance of primary windings)

$$L_2 = \frac{\mu S}{\ell}N_2^2,$$ (Self-inductance of secondary windings)

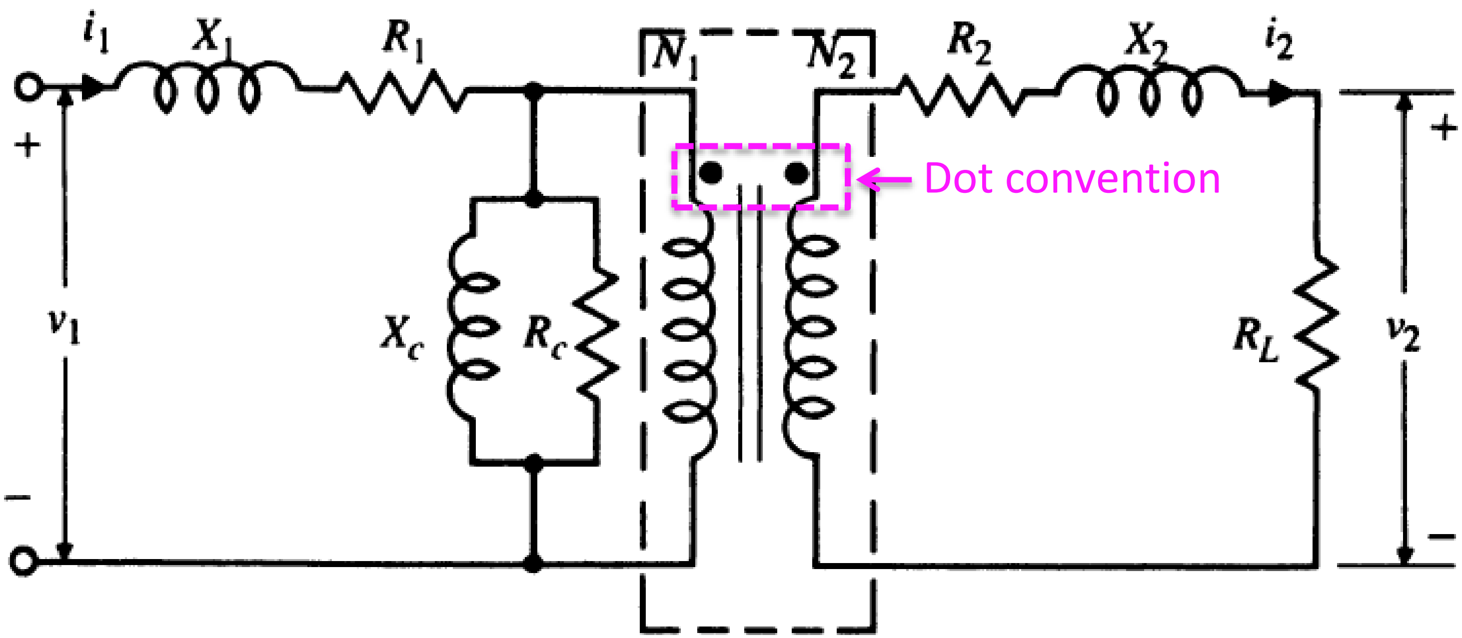
$$L_{12} = \frac{\mu S}{\ell}N_1 N_2$$ (mutual-inductance)

For real transformers, $L_{12} = k\sqrt{L_1 L_2},\qquad k < 1$ (k: coefficient of coupling)

Real-life conditions for real transformers:
Flux leakage, nonlinear inductance, nonzero winding resistance, hysteresis and eddy-current losses

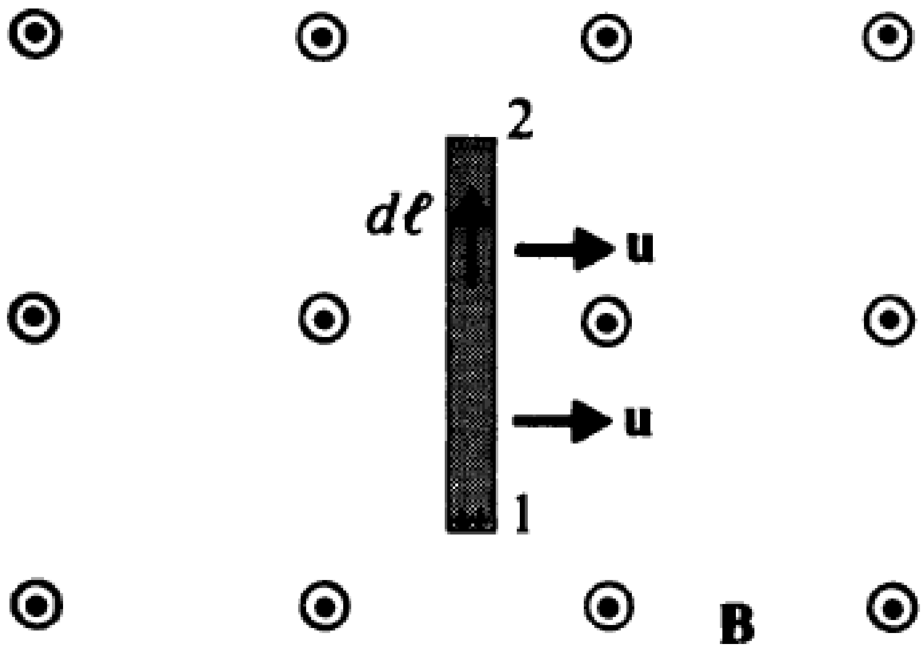


Equivalent circuit



X_1,X_2: leakage inductive reactances, R_c: power loss due to hysteresis and eddy-current effects, X_c: nonlinear inductive reactance

7.2.3 A Moving Conductor in A Static Magnetic Field



The forces applied to the free charges in a moving conductor:

(1) Magnetic force

$$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$$

(2) Colombian force

u: velocity of the conductor in magnetic field ***B***

At equilibrium, which is rapidly reached, the net force on the free charges in the moving conductor is **zero**.

Flux cutting emf or motional emf

For an observe moving with the inductor, the magnetic force per unit charge can be regarded as an induced electric field acting along the conductor and producing a voltage

$$V_{21} = \int_1^2 (\mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell}$$

For a closed circuit *C*, then the emf generated around the circuit is

Flux cutting emf or ***motional emf***

$$\mathcal{V}' = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell} \qquad \text{(V)}.$$

Note: only the part of the circuit that moves in a direction not parallel to ***B*** will contribute to *V'*.

7.2.4 A Moving Circuit in A Time-varying Magnetic Field

- **Lorentz's force** (for a charge *q* moves in an electric field ***E*** and a magnetic field ***B***):

$$\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

For an observer moving with *q*, there is no apparent motion, and the force on *q* can be interpreted by an electric field ***E'***

$$\mathbf{E}' = \mathbf{E} + \mathbf{u} \times \mathbf{B}$$

Or

$$\mathbf{E} = \mathbf{E}' - \mathbf{u} \times \mathbf{B}.$$

■ **Faraday's law** in a time varying magnetic field

When a conducting circuit with contour *C* and surface *S* moves with a velocity *u*, we have the following form

$$\underbrace{\oint_C \mathbf{E}' \cdot d\boldsymbol{\ell}} = \underbrace{-\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}} + \underbrace{\oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell}} \qquad \text{(7-34)}$$

emf induced in the moving frame of reference

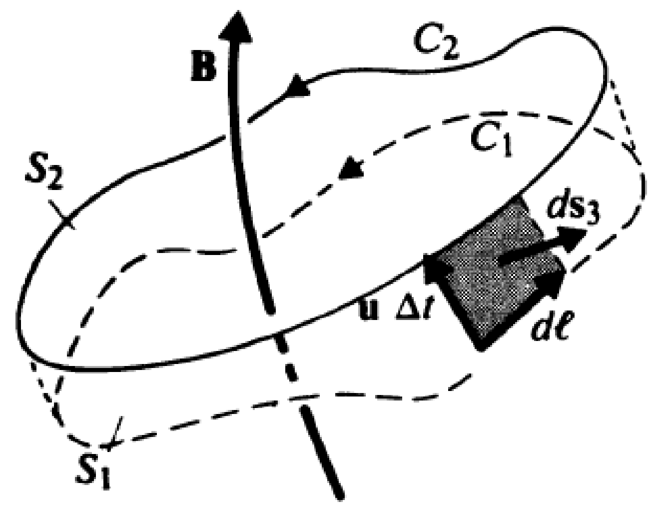
Transformer emf

Motional emf

The division of the induced emf between transformer and motional parts depends on the chosen frame of reference.

A moving circuit in a time-varying magnetic field

From time t to $t+\Delta t$, consider a contour C moves from C_1 to C_2 in a changing magnetic field in an arbitrary manner.

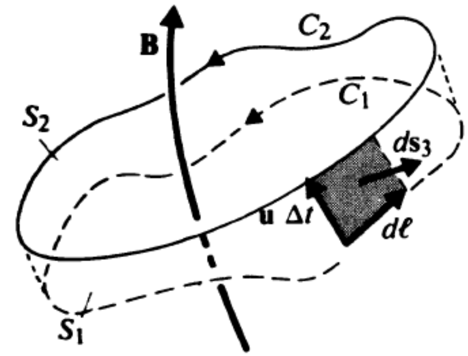


Time-varying change of magnetic flux through the contour

$$\frac{d\Phi}{dt} = \frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s}$$

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{S_2}\mathbf{B}(t+\Delta t)\cdot d\mathbf{s}_2 - \int_{S_1}\mathbf{B}(t)\cdot d\mathbf{s}_1\right] \quad (7\text{-}35)$$

Express $\boldsymbol{B}(t+\Delta t)$ by Taylor's expansion,



$$\mathbf{B}(t+\Delta t) = \mathbf{B}(t) + \frac{\partial \mathbf{B}(t)}{\partial t}\Delta t + \underbrace{\text{H.O.T.}}_{\text{High-order terms}} \qquad (7\text{-}36)$$

Substituting Eq. (7-36) in Eq. (7-35) yields

$$\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s} = \int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s} + \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{S_2}\mathbf{B}\cdot d\mathbf{s}_2 - \int_{S_1}\mathbf{B}\cdot d\mathbf{s}_1 + \text{H.O.T.}\right],$$

$$d\mathbf{s}_3 = d\boldsymbol{\ell}\times\mathbf{u}\,\Delta t, \qquad \int_V \nabla\cdot\mathbf{B}\,dv = \int_{S_2}\mathbf{B}\cdot d\mathbf{s}_2 - \int_{S_1}\mathbf{B}\cdot d\mathbf{s}_1 + \int_{S_3}\mathbf{B}\cdot d\mathbf{s}_3$$

$$\int_{S_2}\mathbf{B}\cdot d\mathbf{s}_2 - \int_{S_1}\mathbf{B}\cdot d\mathbf{s}_1 = -\Delta t\oint_C (\mathbf{u}\times\mathbf{B})\cdot d\boldsymbol{\ell}$$

$$\boxed{\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s} = \int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s} - \oint_C (\mathbf{u}\times\mathbf{B})\cdot d\boldsymbol{\ell}}$$

(identical as the negative of the right of Eq. (7-34))

If we designate

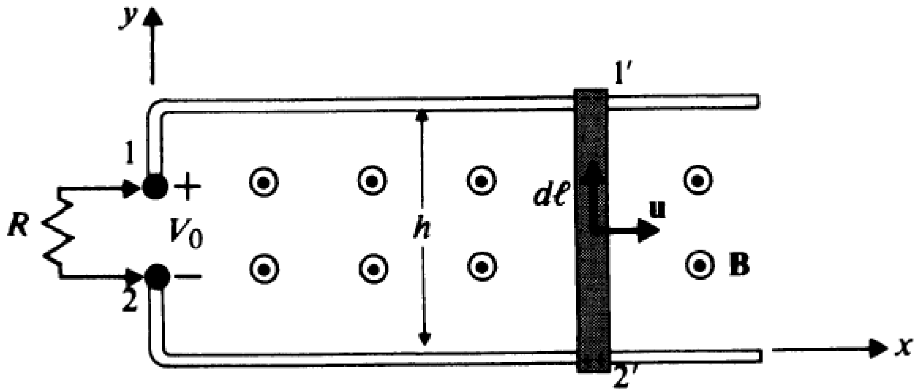
$$\mathscr{V}' = \oint_C \mathbf{E}' \cdot d\boldsymbol{\ell}$$
$$= \text{emf induced in circuit } C \text{ measured in the moving frame,}$$

$$\oint_C \mathbf{E}' \cdot d\boldsymbol{\ell} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} + \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell}$$

$$\boxed{\begin{aligned} \mathscr{V}' &= -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{s} \\ &= -\frac{d\Phi}{dt} \qquad (\text{V}), \end{aligned}}$$

- *V'* reduces to *V* for a stationary circuit;
- Faraday's law applies to both stationary and moving circuit, thus for a general case;

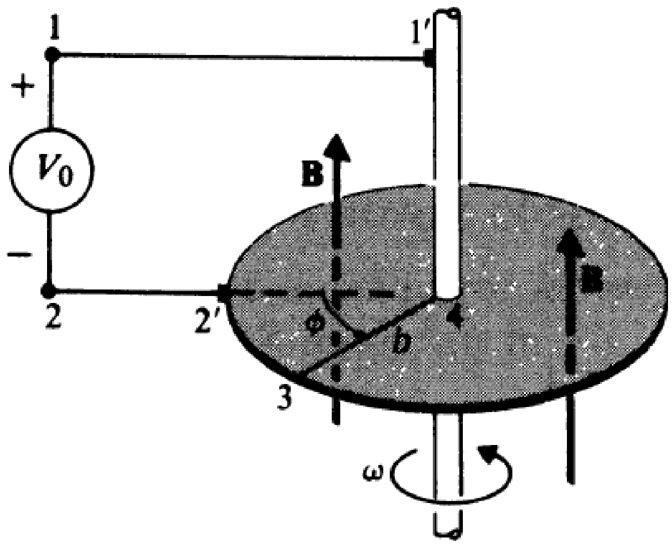
Example 7-2: find the voltage of a sliding metal bar over conducting rails



$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = B_0(hut)$$

$$V_0 = -\frac{d\Phi}{dt} = -uB_0h \qquad \text{(V)},$$

Example 7-3: find the voltage in Faraday disk generator



$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = B_0 \int_0^b \int_0^{\omega t} r\, d\phi\, dr$$

$$= B_0(\omega t)\frac{b^2}{2}$$

$$V_0 = -\frac{d\Phi}{dt} = -\frac{\omega B_0 b^2}{2},$$

7.3 Maxwell's Equations

A time-varying magnetic field induces an electric field and from Table 7-1:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (7\text{-}47a)$$

$$\nabla \times \mathbf{H} = \mathbf{J}, \quad (7\text{-}47b)$$

$$\nabla \cdot \mathbf{D} = \rho, \quad (7\text{-}47c)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (7\text{-}47d)$$

Question;
Satisfy these equations at time-varying conditions?

In addition we know the principle of conservation of charge must be satisfied. We have

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}. \quad (7\text{-}48)$$

Taking the divergence of Eq. (7-47b), we have

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J}, \qquad \text{(7-49)}$$

In time varying case, $\nabla \cdot \mathbf{J} = -\dfrac{\partial \rho}{\partial t}$.

In general, not true

For the consistence of Eqs. (7-47a,b,c, and d) to Eq. (7-48), a term $\partial\rho/\partial t$ must be added at the right side of Eq. (7-49)

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t}. \qquad \text{(7-50)}$$

$\nabla \cdot \mathbf{D} = \rho,$

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.} \qquad \text{(7-52)}$$

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.}$$

- Time-varying electric field will generate magnetic field

- $\partial \mathbf{D}/\partial t$ is necessary for the consistence with the principle of conservation of charge

$\partial \mathbf{D}/\partial t$ *: Displacement current density*

- One of major contributions of Jame Clerk Maxwell (1831-1879)

Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t},$$

$$\nabla \cdot \mathbf{D} = \rho,$$

$$\nabla \cdot \mathbf{B} = 0.$$

J: density of free current (= convection current + conduction current)

ρ: volume density of free charge

- Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t},$$

$$\nabla \cdot \mathbf{D} = \rho,$$

$$\nabla \cdot \mathbf{B} = 0.$$

- Charge conservation equation

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}.$$

- Lorentz equation

$$\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

All macroscopic electromagnetic phenomena could be explained by the above three equations.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t},$$

$$\nabla \cdot \mathbf{D} = \rho,$$

$$\nabla \cdot \mathbf{B} = 0.$$

- Not all independent
- 12 unknowns: ***E***, ***D***, ***B***, ***H***
- *12 scalar equations*
- **Constituent relations**

$$\mathbf{D} = \epsilon \mathbf{E}$$

$$\mathbf{H} = \mathbf{B}/\mu,$$

7.3.1 Integral form of Maxwell's equations

In a physical environment we must deal with finite objects of specified shape and boundaries.

Take the surface integral of both sides of the curl equations over an open surface *S* with a contour *C* and apply Stokes's theorem.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \Rightarrow \quad \oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \Rightarrow \quad \oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{s}.$$

Taking the volume integral of both sides of the divergence equations over a a volume *V* with a closed surface *S* and using divergence theorem

$$\nabla \cdot \mathbf{D} = \rho, \quad \Longrightarrow \quad \oint_S \mathbf{D} \cdot d\mathbf{s} = \int_V \rho \, dv$$

$$\nabla \cdot \mathbf{B} = 0. \quad \Longrightarrow \quad \oint_S \mathbf{B} \cdot d\mathbf{s} = 0$$

(no isolated magnetic charge)

Maxwell's equations

Differential Form	Integral Form	Significance
$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\dfrac{d\Phi}{dt}$	Faraday's law
$\nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$	$\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I + \int_S \dfrac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s}$	Ampère's circuital law
$\nabla \cdot \mathbf{D} = \rho$	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$	No isolated magnetic charge

7.4 Potential Functions

Vector magnetic potential ***A***:

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \text{(T)}. \longleftarrow \nabla \cdot \mathbf{B} = 0$$

Substitute the above equation into Faradays' law

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A})$$

or

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0. \longrightarrow \boxed{\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V,}$$

From the above, we obtain

$$\boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \qquad (\mathrm{V/m}).} \qquad (7.57)$$

- In the static case, $\partial \mathbf{A}/\partial t = 0$, and Eq. (7.57) reduces to $\mathbf{E} = -\nabla V$.
- For time-varying fields, ***E*** depends on both *V* and ***A***.
- ***E*** and ***B*** are coupled.

In the static or quasi-static state we have the solutions of Poisson's equations of Eqs. (4-6) and (6-21)

$$V = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho}{R}\, dv', \qquad \mathbf{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{J}}{R}\, dv'.$$

Substitute Eqs. (7-55) and (7-57) into Eq. (7-53b) and make use of the constitutive relations $\mathbf{H} = \mathbf{B}/\mu$ and $\mathbf{D} = \epsilon\mathbf{E}$. We have

$$\nabla \times \nabla \times \mathbf{A} = \mu\mathbf{J} + \mu\epsilon \frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial \mathbf{A}}{\partial t}\right),$$

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} = \mu\mathbf{J} - \nabla\left(\mu\epsilon \frac{\partial V}{\partial t}\right) - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2}$$

or

$$\nabla^2\mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu\mathbf{J} + \nabla\left(\nabla \cdot \mathbf{A} + \mu\epsilon \frac{\partial V}{\partial t}\right). \quad (7.61)$$

We take the liberty to choose the divergence of ***A***

$$\nabla \cdot \mathbf{A} + \mu\epsilon \frac{\partial V}{\partial t} = 0, \qquad (7.62)$$

Lorentz conditions or Lorentz gauge for the potentials

Use the above relations, Eq. (7.61) could be rewritten as

$$\boxed{\nabla^2 \mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}.} \qquad (7.63)$$

Nonhomogeneous wave equations for vector potential A

A corresponding equation for V can be obtained by substitute Eq. (7-57) into Eq. (7-53c)

$$-\nabla \cdot \epsilon\left(\nabla V + \frac{\partial \mathbf{A}}{\partial t}\right) = \rho,$$

For constant *ε* and use Eq. (7-62), we have

$$\boxed{\nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon},}$$

Nonhomogeneous wave equations for scalar potential V

Lorentz condition in Eq. (7.62) uncouples wave equations for ***A*** and *V*.

7.5 Electromagnetic Boundary Conditions

It is necessary to know the boundary conditions for ***E, D, H,*** and ***B*** in order to solve the electromagnetic problems in contiguous regions.

Methodologies to acquire the boundary equations

- For **curl equations**, apply the integral form to a flat closed path at a boundary with top and bottom sides in the two touching media yielding the boundary conditions for ***the tangential components***
- For **divergence equations**, apply the integral form to a shallow pillbox at an interface with top and bottom surface yielding the boundary conditions for ***the normal components***.

Boundary condition for tangential *E*

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \Leftrightarrow \oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d\Phi}{dt} \Rightarrow \boxed{E_{1t} = E_{2t}}$$

1. ***The tangential component of an* E *field is continuous across an interface.***

Boundary condition for tangential *H*

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \Leftrightarrow \oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s} \Rightarrow \boxed{\mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s}$$

2. ***The tangential component of an* H *field is discontinuous across an interface where a surface current exists, the amount of discontinuity being determined by Eq. (7–66b).***

- Boundary condition for normal ***D***

$$\nabla \cdot \mathbf{D} = \rho \Leftrightarrow \oint_S \mathbf{D} \cdot d\mathbf{s} = Q \Rightarrow \boxed{\mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s}$$

3. *The normal component of a* D *field is discontinuous across an interface where a surface charge exists, the amount of discontinuity being determined by Eq. (7–66c).*

- Boundary condition for normal ***B***

$$\nabla \cdot \mathbf{B} = 0 \Leftrightarrow \oint_S \mathbf{B} \cdot d\mathbf{s} = 0 \Rightarrow \boxed{B_{1n} = B_{2n}}$$

4. *The normal component of a* B *field is continuous across an interface.*

Note: The boundary equations are not completely independent.

Interface between two lossless linear media

$$\rho_s = 0 \text{ and } \mathbf{J}_s = 0$$

$$E_{1t} = E_{2t} \rightarrow \frac{D_{1t}}{D_{2t}} = \frac{\epsilon_1}{\epsilon_2}$$

$$H_{1t} = H_{2t} \rightarrow \frac{B_{1t}}{B_{2t}} = \frac{\mu_1}{\mu_2}$$

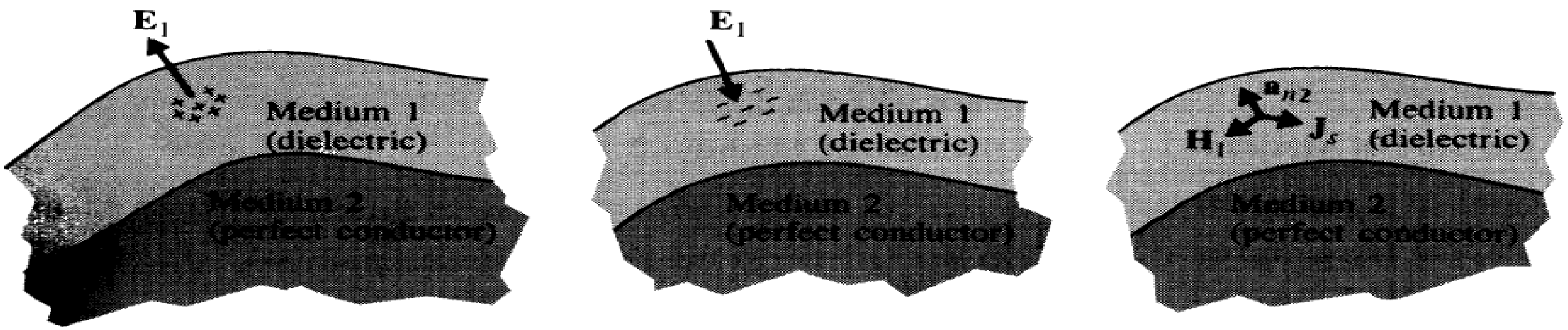
$$D_{1n} = D_{2n} \rightarrow \epsilon_1 E_{1n} = \epsilon_2 E_{2n}$$

$$B_{1n} = B_{2n} \rightarrow \mu_1 H_{1n} = \mu_2 H_{2n}$$

Interface between a dielectric and a perfect conductor

Boundary Conditions between a Dielectric (Medium 1) and a Perfect Conductor (Medium 2) (Time-Varying Case)

On the Side of Medium 1	On the Side of Medium 2
$E_{1t} = 0$	$E_{2t} = 0$
$\mathbf{a}_{n2} \times \mathbf{H}_1 = \mathbf{J}_s$	$H_{2t} = 0$
$\mathbf{a}_{n2} \cdot \mathbf{D}_1 = \rho_s$	$D_{2n} = 0$
$B_{1n} = 0$	$B_{2n} = 0$



(a) (b) (c)

7.6 Wave Equations and their Solutions

For given charge and current distribution,

$$\nabla^2 \mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}.$$

$$-\nabla \cdot \epsilon \left(\nabla V + \frac{\partial \mathbf{A}}{\partial t} \right) = \rho$$

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

7.6.1 Solutions of Wave Equations for Potentials

- Assume an point charge at time *t*, $\rho(t)\Delta v'$, at the origin of a spherical coordinate and out of the source region we have

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right)-\mu\epsilon\frac{\partial^2 V}{\partial t^2}=0.$$

Introduce a new variable

$$V(R,t)=\frac{1}{R}U(R,t),$$

$$\frac{\partial^2 U}{\partial R^2}-\mu\epsilon\frac{\partial^2 U}{\partial t^2}=0. \qquad (7\text{-}73)$$

One-dimensional homogeneous wave equation

Any twice differential function of $(t - R\sqrt{me})$ or of $(t + R\sqrt{me})$ is a solution of Eq. (7-73). But here we only select a function of $(t - R\sqrt{me})$ for causality. Hence we have

$$U(R, t) = f(t - R\sqrt{\mu\epsilon}).$$

Wave travelling in the positive R direction with a velocity $1/\sqrt{me}$

As we see, the function at $R + \Delta R$ at a later time $t + \Delta t$ is

$$U(R + \Delta R, t + \Delta t) = f[t + \Delta t - (R + \Delta R)\sqrt{\mu\epsilon}] = f(t - R\sqrt{\mu\epsilon}).$$

Thus the function retains its form if $\Delta t = \Delta R\sqrt{\mu\epsilon} = \Delta R/u$, where $u = 1/\sqrt{\mu\epsilon}$ is the ***velocity of propagation***, a characteristic of the medium. From Eq. (7–72) we get

$$V(R, t) = \frac{1}{R} f(t - R/u). \quad \text{(7-75)}$$

To determine what the specific function $f(t - R/u)$ must be, we note from Eq. (3–47) that for a static point charge $\rho(t)\,\Delta v'$ at the origin,

$$\Delta V(R) = \frac{\rho(t)\,\Delta v'}{4\pi\epsilon R}. \tag{7–76}$$

Comparison of Eqs. (7–75) and (7–76) enables us to identify

$$\Delta f(t - R/u) = \frac{\rho(t - R/u)\,\Delta v'}{4\pi\epsilon}.$$

Electric potential due to a charge distribution over a volume *V'*:

$$\boxed{V(R, t) = \frac{1}{4\pi\epsilon}\int_{V'} \frac{\rho(t - R/u)}{R}\,dv' \qquad \text{(V).}}$$

Retarded scalar potential

Magnetic vector potential due to a current distribution over a volume *V*' (could be processed in the same way):

$$\boxed{\mathbf{A}(R, t) = \frac{\mu}{4\pi} \int_{V'} \frac{\mathbf{J}(t - R/u)}{R}\, dv' \qquad (\mathrm{Wb/m}).}$$

Retarded vector potential

- The electric and magnetic fields derived from A and V will be functions of $(t - R\sqrt{\mu\varepsilon})$ and therefore retarded in time.
- It takes for electromagnetic waves to travel and to be felt at a distance.

7.6.2 Source-free Wave Equations

For source-free regions (ρ=0 and J=0) in simple (linear, isotropic and homogenous) non-conducting (σ=0) medium, Maxwell's equations are reduced to

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}, \qquad (7\text{–}79a)$$

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t}, \qquad (7\text{–}79b)$$

$$\nabla \cdot \mathbf{E} = 0, \qquad (7\text{–}79c)$$

$$\nabla \cdot \mathbf{H} = 0. \qquad (7\text{–}79d)$$

Take the curl of Eq. (7-79a) and use Eq. (7-79b)

$$\nabla \times \nabla \times \mathbf{E} = -\mu \frac{\partial}{\partial t}(\nabla \times \mathbf{H}) = -\mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Now $\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} = -\nabla^2\mathbf{E}$ because of Eq. (7–79c). Hence we have

$$\nabla^2\mathbf{E} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0;$$

$$\Big\downarrow \; u = 1/\sqrt{\mu\epsilon},$$

$$\boxed{\nabla^2\mathbf{E} - \frac{1}{u^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0.}$$

Similarly,

$$\boxed{\nabla^2\mathbf{H} - \frac{1}{u^2}\frac{\partial^2 \mathbf{H}}{\partial t^2} = 0.}$$

Homogeneous vector wave equations
(could be decomposed into three one-dimensional homogeneous scalar wave equations.)

7.7 Time-Harmonic Fields

- Arbitrary periodic time functions can be expanded into Fourier series of harmonic sinusoidal components
- Transient nonperiodic functions can be expressed as Fourier integrals
- Sinusoidal time variations of source functions will produce sinusoidal variations of ***E*** and ***H*** with the same frequency
- Electrodynamic fields can be determined in terms of those caused by the various frequency components of an arbitrary time-varying source function.
- The principle of superposition

7.7.1 The Use of Phasors- A Review

A sinusoidal quantity is defined by three parameters: amplitude, frequency and phase. For example,

$$i(t) = I \cos(\omega t + \phi), \tag{7-83}$$

It is not convenient to work directly with an instantaneous expressions such as the cosine function when differentiation or integration of *i*(*t*) are involved.

Example: a series **RLC** circuit with an applied voltage $e(t) = E \cos \omega t$

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = e(t). \tag{7–84}$$

If we write $i(t)$ as in Eq. (7–83), Eq. (7–84) yields

$$I\left[-\omega L \sin(\omega t + \phi) + R\cos(\omega t + \phi) + \frac{1}{\omega C}\sin(\omega t + \phi)\right] = E\cos\omega t. \tag{7–85}$$

Complicated mathematical manipulation is required.

It is much simpler to use exponential functions by writing the applied voltage as

$$\begin{aligned} e(t) &= E \cos \omega t = \mathscr{R}e[(Ee^{j0})e^{j\omega t}] \\ &= \mathscr{R}e(E_s e^{j\omega t}) \end{aligned} \tag{7–86}$$

and $i(t)$ in Eq. (7–83) as

$$\begin{aligned} i(t) &= \mathscr{R}e[(Ie^{j\phi})e^{j\omega t}] \\ &= \mathscr{R}e(I_s e^{j\omega t}), \end{aligned} \tag{7–87}$$

where $\mathscr{R}e$ means "the real part of." In Eqs. (7–86) and (7–87),

$$E_s = Ee^{j0} = E \tag{7–88a}$$

$$I_s = Ie^{j\phi} \tag{7–88b}$$

Phasors : contains amplitude and phase but independent of it

Now,

$$\frac{di}{dt} = \mathcal{R}e(j\omega I_s e^{j\omega t}), \tag{7–89}$$

$$\int i\,dt = \mathcal{R}e\left(\frac{I_s}{j\omega}\, e^{j\omega t}\right). \tag{7–90}$$

Substitution of Eqs. (7–86) through (7–90) in Eq. (7–84) yields

$$\left[R + j\left(\omega L - \frac{1}{\omega C}\right)\right] I_s = E_s, \tag{7–91}$$

- *I_s* can be solved very easily
- $i(t) = \mathrm{Re}(I_s e^{iWt})$

7.7.2 Time-Harmonic Electromagnetics

For a time-harmonic E field referring to cosωt can be written as

$$\mathbf{E}(x, y, z, t) = \mathscr{R}e[\mathbf{E}(x, y, z)e^{j\omega t}], \qquad (7\text{-}93)$$

where $\mathbf{E}(x, y, z)$ is a ***vector phasor*** that contains information on direction, magnitude, and phase. Phasors are, in general, complex quantities. From Eqs. (7–93), (7–87), (7–89), and (7–90) we see that, if $\mathbf{E}(x, y, z, t)$ is to be represented by the vector phasor $\mathbf{E}(x, y, z)$, then $\partial\mathbf{E}(x, y, z, t)/\partial t$ and $\int \mathbf{E}(x, y, z, t)\,dt$ would be represented by vector phasors $j\omega\mathbf{E}(x, y, z)$ and $\mathbf{E}(x, y, z)/j\omega$, respectively. Higher-order differentiations and integrations with respect to t would be represented by multiplications and divisions, respectively, of the phasor $\mathbf{E}(x, y, z)$ by higher powers of $j\omega$.

Time-harmonic Maxwell's equations in a simple (linear, isotropic and homogeneous) medium

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t},$$
$$\nabla \cdot \mathbf{D} = \rho,$$
$$\nabla \cdot \mathbf{B} = 0.$$

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H},$$
$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega\epsilon\mathbf{E},$$
$$\nabla \cdot \mathbf{E} = \rho/\epsilon,$$
$$\nabla \cdot \mathbf{H} = 0.$$

- Almost exclusively deal with time-harmonic fields (and therefore with phasors)
- Phase quantities are not functions of t;
- Any quantity containing j must necessarily be a phasor.

Time-harmonic wave equations for scalar potential *V* and vector potential ***A***

non-homogenous Helmholtz equations

$$\nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon}, \quad \longrightarrow \quad \nabla^2 V + k^2 V = -\frac{\rho}{\epsilon}$$

$$\nabla^2 \mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu\mathbf{J}. \quad \longrightarrow \quad \nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu\mathbf{J},$$

Where **wavenumber:** $k = \omega\sqrt{\mu\epsilon} = \frac{\omega}{u}$

and Lorentz condition $\nabla \cdot \mathbf{A} + j\omega\mu\epsilon V = 0.$

The phasor solutions (retarded) for the potential equations:

$$\boxed{V(R) = \frac{1}{4\pi\epsilon}\int_{V'} \frac{\rho e^{-jkR}}{R}\,dv' \qquad \text{(V)},} \qquad (7\text{-}99)$$

$$\boxed{\mathbf{A}(R) = \frac{\mu}{4\pi}\int_{V'} \frac{\mathbf{J} e^{-jkR}}{R}\,dv' \qquad \text{(Wb/m).}} \qquad (7\text{-}100)$$

If $kR \ll 1$, equations (7-99) and (7-100) then simplify to be the static expressions in Eqs. (7-58) and (7-59).

<u>Formal procedure to determine E and H due harmonic charges and currents:</u>

1. Find phasors $V(R)$ and $\mathbf{A}(R)$ from Eqs. (7–99) and (7–100).
2. Find phasors $\mathbf{E}(R) = -\nabla V - j\omega\mathbf{A}$ and $\mathbf{B}(R) = \nabla \times \mathbf{A}$.
3. Find instantaneous $\mathbf{E}(R, t) = \mathscr{R}_e[\mathbf{E}(R)e^{j\omega t}]$ and $\mathbf{B}(R, t) = \mathscr{R}_e[\mathbf{B}(R)e^{j\omega t}]$ for a cosine reference.

7.7.3 Source-free Fields in Simple Media

$$\boxed{\rho = 0, \mathbf{J} = 0, \sigma = 0,}$$

$$\begin{aligned}
\nabla \times \mathbf{E} &= -j\omega\mu\mathbf{H}, \\
\nabla \times \mathbf{H} &= \mathbf{J} + j\omega\epsilon\mathbf{E}, \\
\nabla \cdot \mathbf{E} &= \rho/\epsilon, \\
\nabla \cdot \mathbf{H} &= 0.
\end{aligned}$$

$$\begin{aligned}
\nabla \times \mathbf{E} &= -j\omega\mu\mathbf{H}, \\
\nabla \times \mathbf{H} &= j\omega\epsilon\mathbf{E}, \\
\nabla \cdot \mathbf{E} &= 0, \\
\nabla \cdot \mathbf{H} &= 0.
\end{aligned}$$

Homogenous vector wave equations

$$\nabla^2\mathbf{E} - \frac{1}{u^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = 0.$$

$$\nabla^2\mathbf{H} - \frac{1}{u^2}\frac{\partial^2\mathbf{H}}{\partial t^2} = 0.$$

Homogenous vector Helmholtz's equations

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = 0$$

$$\nabla^2\mathbf{H} + k^2\mathbf{H} = 0,$$

EXAMPLE 7–7 Show that if (**E**, **H**) are solutions of source-free Maxwell's equations in a simple medium characterized by ϵ and μ, then so also are ($\mathbf{E}'$, $\mathbf{H}'$), where

$$\mathbf{E}' = \eta\mathbf{H} \tag{7–107a}$$

(Principle of duality)

$$\mathbf{H}' = -\frac{\mathbf{E}}{\eta}. \tag{7–107b}$$

In the above equations, $\eta = \sqrt{\mu/\epsilon}$ is called the ***intrinsic impedance*** of the medium.

Solution We prove the statement by taking the curl and the divergence of $\mathbf{E}'$ and $\mathbf{H}'$ and using Eqs. (7–104a, b, c, and d):

$$\begin{aligned}\nabla \times \mathbf{E}' &= \eta(\nabla \times \mathbf{H}) = \eta(j\omega\epsilon\mathbf{E}) \\ &= -j\omega\epsilon\eta^2\left(-\frac{\mathbf{E}}{\eta}\right) = -j\omega\mu\mathbf{H}'\end{aligned} \tag{7–108a}$$

$$\begin{aligned}\nabla \times \mathbf{H}' &= -\frac{1}{\eta}(\nabla \times \mathbf{E}) = -\frac{1}{\eta}(-j\omega\mu\mathbf{H}) \\ &= j\omega\mu\frac{1}{\eta^2}(\eta\mathbf{H}) = j\omega\epsilon\mathbf{E}'\end{aligned} \tag{7–108b}$$

$$\nabla \cdot \mathbf{E}' = \eta(\nabla \cdot \mathbf{H}) = 0 \tag{7–108c}$$

$$\nabla \cdot \mathbf{H}' = -\frac{1}{\eta}(\nabla \cdot \mathbf{E}) = 0. \tag{7–108d}$$

Equations (7–108a, b, c, and d) are source-free Maxwell's equations in $\mathbf{E}'$ and $\mathbf{H}'$.

In a conducting medium (σ≠0), equation (7-104b) should be changed into

$$\nabla \times \mathbf{H} = (\sigma + j\omega\epsilon)\mathbf{E} = j\omega\left(\epsilon + \frac{\sigma}{j\omega}\right)\mathbf{E} = j\omega\epsilon_c\mathbf{E}$$

Complex permittivity: $\epsilon_c = \epsilon - j\frac{\sigma}{\omega} \quad \text{(F/m)}.$

In general case, $\epsilon_c = \epsilon' - j\epsilon'' \quad \text{(F/m)},$

-**Damping loss** due to the inertial property of charged particles;
-**Ohmic loss** in metal or semiconductor (good conductor: $\sigma >> \omega\varepsilon$)

Similarly, $\mu = \mu' - j\mu''.$

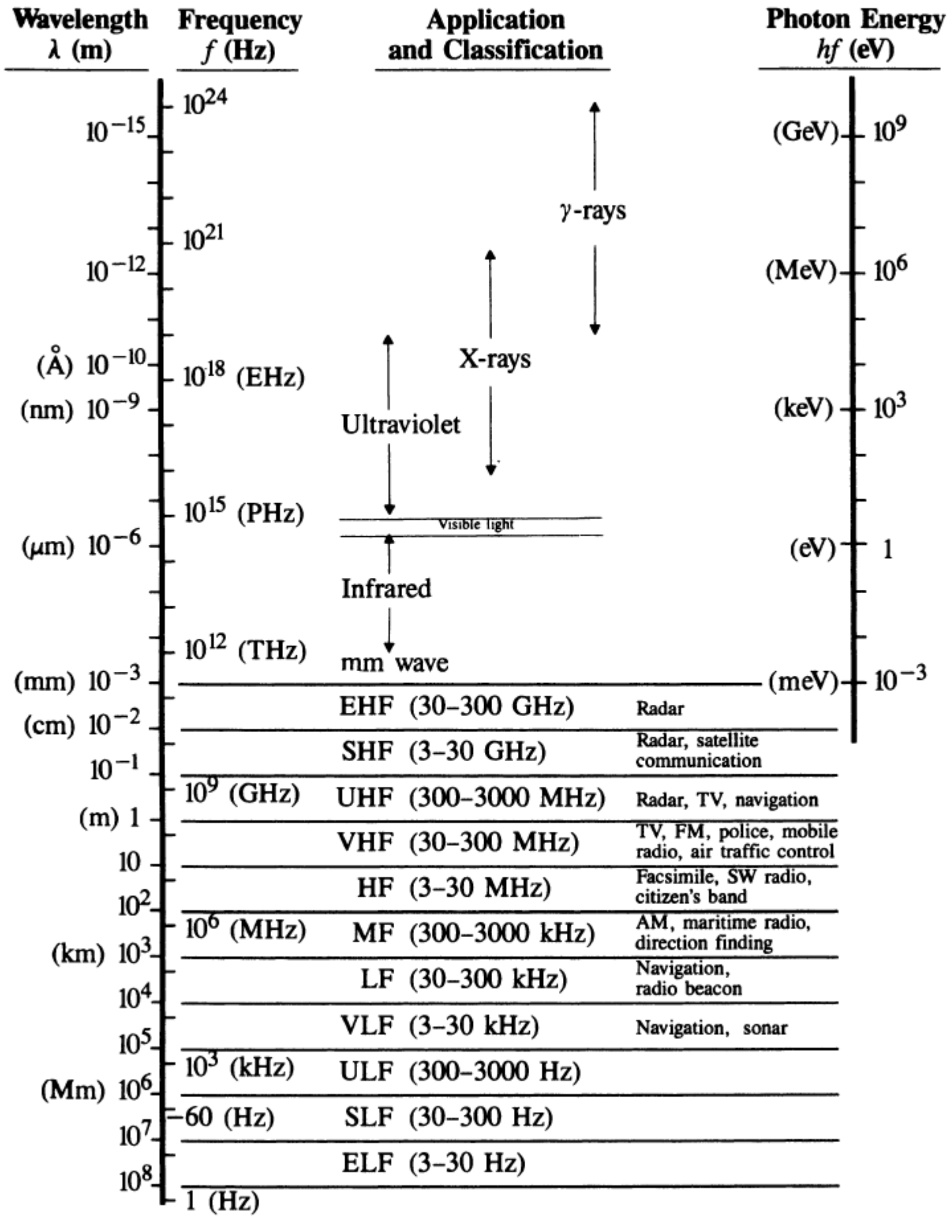
Complex wavenumber:

$$k_c = \omega\sqrt{\mu\epsilon_c} = \omega\sqrt{\mu(\epsilon' - j\epsilon'')}$$

Loss tangent (δ_c-loss angle):

$$\tan\delta_c = \frac{\epsilon''}{\epsilon'} \cong \frac{\sigma}{\omega\epsilon}.$$

7.7.4 The Electromagnetic Spectrum



Band Designations for Microwave Frequency Ranges

Old†	New	Frequency Ranges (GHz)
Ka	K	26.5–40
K	K	20–26.5
K	J	18–20
Ku	J	12.4–18
X	J	10–12.4
X	I	8–10
C	H	6–8
C	G	4–6
S	F	3–4
S	E	2–3
L	D	1–2
UHF	C	0.5–1