10-11学年冬季学期《偏微分方程》期末试卷及答案

January 4, 2012

一.（20分）判别下列二阶线性偏微分方程

$$\frac{\partial^2 u}{\partial x^2} + 2\cos x \frac{\partial^2 u}{\partial x \partial y} - \sin^2 x \frac{\partial^2 u}{\partial y^2} - sinx \frac{\partial u}{\partial y} = 0$$

的类型，并求出满足条件

$$u|_{y=\sin x} = x, \quad \left.\frac{\partial u}{\partial y}\right|_{y=\sin x} = 1$$

的解。

解．$\Delta = \cos^2 x + \sin^2 x = 1 > 0$, 方程为双曲型方程。
特征方程为$(dy)^2 - 2cosxdydx - \sin^2 x(dx)^2 = 0$, 解得特征线为

$$y - x - \sin x = C_1, \quad y + x - sinx = C_2.$$

令

$$\xi = y - x - \sin x, \quad \eta = y + x - \sin x, \ u(x, y) = u(\xi, \eta),$$

则由原方程可得

$$-4\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

一.（20分）判别下列二阶线性偏微分方程

$$\frac{\partial^2 u}{\partial x^2} + 2\cos x \frac{\partial^2 u}{\partial x \partial y} - \sin^2 x \frac{\partial^2 u}{\partial y^2} - sinx \frac{\partial u}{\partial y} = 0$$

的类型，并求出满足条件

$$u|_{y=\sin x} = x, \quad \left.\frac{\partial u}{\partial y}\right|_{y=\sin x} = 1$$

的解。

解. $\Delta = \cos^2 x + \sin^2 x = 1 > 0$, 方程为**双曲型**方程。

特征方程为$(dy)^2 - 2cosxdydx - \sin^2 x(dx)^2 = 0$, 解得特征线为

$$y - x - \sin x = C_1, \quad y + x - sinx = C_2.$$

令

$$\xi = y - x - \sin x, \quad \eta = y + x - \sin x, \; u(x, y) = u(\xi, \eta),$$

则由原方程可得

$$-4\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

一.（20分）判别下列二阶线性偏微分方程

$$\frac{\partial^2 u}{\partial x^2}+2\cos x\frac{\partial^2 u}{\partial x\partial y}-\sin^2 x\frac{\partial^2 u}{\partial y^2}-sinx\frac{\partial u}{\partial y}=0$$

的类型，并求出满足条件

$$u|_{y=\sin x}=x,\quad \left.\frac{\partial u}{\partial y}\right|_{y=\sin x}=1$$

的解。

解. $\Delta=\cos^2 x+\sin^2 x=1>0$, 方程为**双曲型**方程。

特征方程为$(dy)^2-2cosxdydx-\sin^2 x(dx)^2=0$, 解得特征线为

$$y-x-\sin x=C_1,\quad y+x-sinx=C_2.$$

令

$$\xi=y-x-\sin x,\quad \eta=y+x-\sin x,\ u(x,y)=u(\xi,\eta),$$

则由原方程可得

$$-4\frac{\partial^2 u}{\partial\xi\partial\eta}=0$$

一.（20分）判别下列二阶线性偏微分方程

$$\frac{\partial^2 u}{\partial x^2} + 2\cos x \frac{\partial^2 u}{\partial x \partial y} - \sin^2 x \frac{\partial^2 u}{\partial y^2} - sinx \frac{\partial u}{\partial y} = 0$$

的类型，并求出满足条件

$$u|_{y=\sin x} = x, \quad \left.\frac{\partial u}{\partial y}\right|_{y=\sin x} = 1$$

的解。

解. $\Delta = \cos^2 x + \sin^2 x = 1 > 0$, 方程为**双曲型**方程。

特征方程为$(dy)^2 - 2cosxdydx - \sin^2 x(dx)^2 = 0$, 解得特征线为

$$y - x - \sin x = C_1, \quad y + x - sinx = C_2.$$

令

$$\xi = y - x - \sin x, \quad \eta = y + x - \sin x, \; u(x, y) = u(\xi, \eta),$$

则由原方程可得

$$-4\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

一.（20分）判别下列二阶线性偏微分方程

$$\frac{\partial^2 u}{\partial x^2} + 2\cos x \frac{\partial^2 u}{\partial x \partial y} - \sin^2 x \frac{\partial^2 u}{\partial y^2} - sinx \frac{\partial u}{\partial y} = 0$$

的类型，并求出满足条件

$$u|_{y=\sin x} = x, \quad \left.\frac{\partial u}{\partial y}\right|_{y=\sin x} = 1$$

的解。

解. $\Delta = \cos^2 x + \sin^2 x = 1 > 0$, 方程为**双曲型**方程。
特征方程为$(dy)^2 - 2cosxdydx - \sin^2 x(dx)^2 = 0$, 解得特征线为

$$y - x - \sin x = C_1, \quad y + x - sinx = C_2.$$

令

$$\xi = y - x - \sin x, \quad \eta = y + x - \sin x, \ u(x, y) = u(\xi, \eta),$$

则由原方程可得

$$-4\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

从而有

$$u = F(\xi) + G(\eta) = F(y - x - \sin x) + G(y + x - \sin x)$$

F和G是二个任意可微函数。由初值条件得

$$F(-x) + G(x) = x, \quad F'(-x) + G'(x) = 1$$

$$\Rightarrow F(-x) + G(x) = x, \quad -F(-x) + G(x) = x + C$$

$$\Rightarrow F(x) = -\frac{1}{2}C, G(x) = x + \frac{1}{2}C$$

$$\Rightarrow u(x, y) = y + x - \sin x.$$

从而有

$$u = F(\xi) + G(\eta) = F(y - x - \sin x) + G(y + x - \sin x)$$

F和G是二个任意可微函数。由初值条件得

$$F(-x) + G(x) = x, \quad F'(-x) + G'(x) = 1$$

$$\Rightarrow F(-x) + G(x) = x, \quad -F(-x) + G(x) = x + C$$

$$\Rightarrow F(x) = -\frac{1}{2}C, G(x) = x + \frac{1}{2}C$$

$$\Rightarrow u(x, y) = y + x - \sin x.$$

从而有

$$u = F(\xi) + G(\eta) = F(y - x - \sin x) + G(y + x - \sin x)$$

F和G是二个任意可微函数。由初值条件得

$$F(-x) + G(x) = x, \quad F'(-x) + G'(x) = 1$$

$$\Rightarrow F(-x) + G(x) = x, \quad -F(-x) + G(x) = x + C$$

$$\Rightarrow F(x) = -\frac{1}{2}C, G(x) = x + \frac{1}{2}C$$

$$\Rightarrow u(x, y) = y + x - \sin x.$$

从而有

$$u = F(\xi) + G(\eta) = F(y - x - \sin x) + G(y + x - \sin x)$$

F和G是二个任意可微函数。由初值条件得

$$F(-x) + G(x) = x, \quad F'(-x) + G'(x) = 1$$

$$\Rightarrow F(-x) + G(x) = x, \quad -F(-x) + G(x) = x + C$$

$$\Rightarrow F(x) = -\frac{1}{2}C, G(x) = x + \frac{1}{2}C$$

$$\Rightarrow u(x, y) = y + x - \sin x.$$

从而有

$$u = F(\xi) + G(\eta) = F(y - x - \sin x) + G(y + x - \sin x)$$

F和G是二个任意可微函数。由初值条件得

$$F(-x) + G(x) = x, \quad F'(-x) + G'(x) = 1$$

$$\Rightarrow F(-x) + G(x) = x, \quad -F(-x) + G(x) = x + C$$

$$\Rightarrow F(x) = -\frac{1}{2}C, G(x) = x + \frac{1}{2}C$$

$$\Rightarrow u(x, y) = y + x - \sin x.$$

二.（20分）(1). 已知函数e^{-x^2}的Fourier变换是$\sqrt{\pi}e^{-\frac{\lambda^2}{4}}$，求函数$e^{-Ax^2}$的逆变换, 其中常数$A>0$.

(2). 利用等式$\int_{-\infty}^{+\infty}|f(x)|^2\,dx=\frac{1}{2\pi}\int_{-\infty}^{+\infty}|F(f)(\lambda)|^2\,d\lambda$（其中$F(f)$表示函数$f$的Fourier变换）及Fourier变换证明：下列初值问题

$$\begin{cases}\frac{\partial^2 u}{\partial t^2}-\frac{\partial^4 u}{\partial x^4}=0, & x\in(-\infty,+\infty), t>0\\ u|_{t=0}=\varphi(x), & \frac{\partial u}{\partial t}|_{t=0}=\psi(x)\end{cases}$$

存在有限能量解$u(t,x)$（即存在依赖于函数φ和ψ的常数M使得$\int_{-\infty}^{+\infty}|u(t,x)|^2\,dx\le M<+\infty$的充分必要条件是$\psi(x)=\frac{\partial^2\varphi(x)}{\partial x^2}$。

(3). 在条件$\psi(x)=\frac{\partial^2\varphi(x)}{\partial x^2}$下求出上述初值问题的有限能量解$u(t,x)$。

解. (1). $F^{-1}[e^{-Ax^2}](\lambda)=\frac{1}{2\sqrt{\pi A}}e^{-\frac{\lambda^2}{4A}}$.

二.（20分）(1). 已知函数e^{-x^2}的Fourier变换是$\sqrt{\pi}e^{-\frac{\lambda^2}{4}}$，求函数$e^{-Ax^2}$的逆变换，其中常数$A>0$.

(2). 利用等式$\int_{-\infty}^{+\infty}|f(x)|^2\,dx=\frac{1}{2\pi}\int_{-\infty}^{+\infty}|F(f)(\lambda)|^2\,d\lambda$（其中$F(f)$表示函数$f$的Fourier变换）及Fourier变换证明：下列初值问题

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}-\frac{\partial^4 u}{\partial x^4}=0, & x\in(-\infty,+\infty), t>0 \\ u|_{t=0}=\varphi(x), & \frac{\partial u}{\partial t}|_{t=0}=\psi(x) \end{cases}$$

存在有限能量解$u(t,x)$（即存在依赖于函数φ和ψ的常数M使得$\int_{-\infty}^{+\infty}|u(t,x)|^2\,dx\le M<+\infty$的充分必要条件是$\psi(x)=\frac{\partial^2\varphi(x)}{\partial x^2}$。

(3). 在条件$\psi(x)=\frac{\partial^2\varphi(x)}{\partial x^2}$下求出上述初值问题的有限能量解$u(t,x)$。

解. (1). $F^{-1}[e^{-Ax^2}](\lambda)=\frac{1}{2\sqrt{\pi A}}e^{-\frac{\lambda^2}{4A}}$.

二.（20分）(1). 已知函数e^{-x^2}的Fourier变换是$\sqrt{\pi}e^{-\frac{\lambda^2}{4}}$，求函数$e^{-Ax^2}$的逆变换, 其中常数$A>0$.

(2). 利用等式$\int_{-\infty}^{+\infty}|f(x)|^2\,dx=\frac{1}{2\pi}\int_{-\infty}^{+\infty}|F(f)(\lambda)|^2\,d\lambda$（其中$F(f)$表示函数$f$的Fourier变换）及Fourier变换证明：下列初值问题

$$\begin{cases}\frac{\partial^2 u}{\partial t^2}-\frac{\partial^4 u}{\partial x^4}=0, & x\in(-\infty,+\infty), t>0\\ u|_{t=0}=\varphi(x), & \frac{\partial u}{\partial t}|_{t=0}=\psi(x)\end{cases}$$

存在有限能量解$u(t,x)$（即存在依赖于函数φ和ψ的常数M使得$\int_{-\infty}^{+\infty}|u(t,x)|^2\,dx\le M<+\infty$的充分必要条件是$\psi(x)=\frac{\partial^2\varphi(x)}{\partial x^2}$。

(3). 在条件$\psi(x)=\frac{\partial^2\varphi(x)}{\partial x^2}$下求出上述初值问题的有限能量解$u(t,x)$。

解. (1). $F^{-1}[e^{-Ax^2}](\lambda)=\frac{1}{2\sqrt{\pi A}}e^{-\frac{\lambda^2}{4A}}$.

二.（20分）(1). 已知函数e^{-x^2}的Fourier变换是$\sqrt{\pi}e^{-\frac{\lambda^2}{4}}$，求函数$e^{-Ax^2}$的逆变换，其中常数$A>0$.

(2). 利用等式$\int_{-\infty}^{+\infty}|f(x)|^2\,dx=\frac{1}{2\pi}\int_{-\infty}^{+\infty}|F(f)(\lambda)|^2\,d\lambda$（其中$F(f)$表示函数$f$的Fourier变换）及Fourier变换证明：下列初值问题

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}-\frac{\partial^4 u}{\partial x^4}=0, & x\in(-\infty,+\infty), t>0 \\ u|_{t=0}=\varphi(x), & \frac{\partial u}{\partial t}|_{t=0}=\psi(x) \end{cases}$$

存在有限能量解$u(t,x)$（即存在依赖于函数φ和ψ的常数M使得$\int_{-\infty}^{+\infty}|u(t,x)|^2\,dx\le M<+\infty$的充分必要条件是$\psi(x)=\frac{\partial^2\varphi(x)}{\partial x^2}$。

(3). 在条件$\psi(x)=\frac{\partial^2\varphi(x)}{\partial x^2}$下求出上述初值问题的有限能量解$u(t,x)$。

解. (1). $F^{-1}[e^{-Ax^2}](\lambda)=\frac{1}{2\sqrt{\pi A}}e^{-\frac{\lambda^2}{4A}}$.

(2). 记函数$u(t,x)$、$\varphi(x)$和$\psi(x)$关于变量x的Fourier变换为$\hat{u}$、$\hat{\varphi}$及$\hat{\psi}$ $\Rightarrow$

$$\begin{cases} \dfrac{\partial^2 \hat{u}}{\partial t^2} - \lambda^4 \hat{u} = 0 \\ \hat{u}|_{t=0} = \hat{\varphi}, \qquad \dfrac{\partial \hat{u}}{\partial t}|_{t=0} = \hat{\psi} \end{cases}$$

解得

$$\hat{u}(t,\lambda) = C_1 e^{-\lambda^2 t} + C_2 e^{\lambda^2 t}$$

其中

$$C_1 = \frac{1}{2}\left(\hat{\varphi}(\lambda) - \frac{\hat{\psi}(\lambda)}{\lambda^2}\right), \; C_2 = \frac{1}{2}\left(\hat{\varphi}(\lambda) + \frac{\hat{\psi}(\lambda)}{\lambda^2}\right).$$

注意到$\int_{-\infty}^{+\infty} |u(t,x)|^2\, dx \le M < +\infty \Rightarrow C_2 = 0 \Rightarrow$

$$0 = \lambda^2 \hat{\varphi}(\lambda) + \hat{\psi}(\lambda) = F[-\frac{\partial^2 \varphi}{\partial x^2} + \psi](\lambda) \Rightarrow \psi(x) = \frac{\partial^2 \varphi(x)}{\partial x^2}$$

(2). 记函数$u(t,x)$、$\varphi(x)$和$\psi(x)$关于变量x的Fourier变换为$\hat{u}$、$\hat{\varphi}$及$\hat{\psi}$ $\Rightarrow$

$$\begin{cases} \dfrac{\partial^2 \hat{u}}{\partial t^2} - \lambda^4 \hat{u} = 0 \\ \hat{u}|_{t=0} = \hat{\varphi}, \qquad \dfrac{\partial \hat{u}}{\partial t}|_{t=0} = \hat{\psi} \end{cases}$$

解得

$$\hat{u}(t,\lambda) = C_1 e^{-\lambda^2 t} + C_2 e^{\lambda^2 t}$$

其中

$$C_1 = \frac{1}{2}\left(\hat{\varphi}(\lambda) - \frac{\hat{\psi}(\lambda)}{\lambda^2}\right),\ C_2 = \frac{1}{2}\left(\hat{\varphi}(\lambda) + \frac{\hat{\psi}(\lambda)}{\lambda^2}\right).$$

注意到$\int_{-\infty}^{+\infty} |u(t,x)|^2\, dx \le M < +\infty \Rightarrow C_2 = 0 \Rightarrow$

$$0 = \lambda^2 \hat{\varphi}(\lambda) + \hat{\psi}(\lambda) = F[-\frac{\partial^2 \varphi}{\partial x^2} + \psi](\lambda) \Rightarrow \psi(x) = \frac{\partial^2 \varphi(x)}{\partial x^2}$$

(2). 记函数$u(t,x)$、$\varphi(x)$和$\psi(x)$关于变量x的Fourier变换为$\hat{u}$、$\hat{\varphi}$及$\hat{\psi}$ $\Rightarrow$

$$\begin{cases} \frac{\partial^2 \hat{u}}{\partial t^2} - \lambda^4 \hat{u} = 0 \\ \hat{u}\big|_{t=0} = \hat{\varphi}, \qquad \frac{\partial \hat{u}}{\partial t}\big|_{t=0} = \hat{\psi} \end{cases}$$

解得

$$\hat{u}(t,\lambda) = C_1 e^{-\lambda^2 t} + C_2 e^{\lambda^2 t}$$

其中

$$C_1 = \frac{1}{2}\left(\hat{\varphi}(\lambda) - \frac{\hat{\psi}(\lambda)}{\lambda^2}\right), \quad C_2 = \frac{1}{2}\left(\hat{\varphi}(\lambda) + \frac{\hat{\psi}(\lambda)}{\lambda^2}\right).$$

注意到$\int_{-\infty}^{+\infty} |u(t,x)|^2\, dx \le M < +\infty \Rightarrow C_2 = 0 \Rightarrow$

$$0 = \lambda^2 \hat{\varphi}(\lambda) + \hat{\psi}(\lambda) = F[-\frac{\partial^2 \varphi}{\partial x^2} + \psi](\lambda) \Rightarrow \psi(x) = \frac{\partial^2 \varphi(x)}{\partial x^2}$$

(2). 记函数$u(t,x)$、$\varphi(x)$和$\psi(x)$关于变量x的Fourier变换为$\hat{u}$、$\hat{\varphi}$及$\hat{\psi}$ $\Rightarrow$

$$\begin{cases} \frac{\partial^2 \hat{u}}{\partial t^2} - \lambda^4 \hat{u} = 0 \\ \hat{u}|_{t=0} = \hat{\varphi}, \qquad \frac{\partial \hat{u}}{\partial t}|_{t=0} = \hat{\psi} \end{cases}$$

解得

$$\hat{u}(t,\lambda) = C_1 e^{-\lambda^2 t} + C_2 e^{\lambda^2 t}$$

其中

$$C_1 = \frac{1}{2}\left(\hat{\varphi}(\lambda) - \frac{\hat{\psi}(\lambda)}{\lambda^2}\right), \; C_2 = \frac{1}{2}\left(\hat{\varphi}(\lambda) + \frac{\hat{\psi}(\lambda)}{\lambda^2}\right).$$

注意到$\int_{-\infty}^{+\infty} |u(t,x)|^2 \, dx \leq M < +\infty \Rightarrow C_2 = 0 \Rightarrow$

$$0 = \lambda^2 \hat{\varphi}(\lambda) + \hat{\psi}(\lambda) = F[-\frac{\partial^2 \varphi}{\partial x^2} + \psi](\lambda) \Rightarrow \psi(x) = \frac{\partial^2 \varphi(x)}{\partial x^2}$$

(2). 记函数$u(t,x)$、$\varphi(x)$和$\psi(x)$关于变量x的Fourier变换为$\hat{u}$、$\hat{\varphi}$及$\hat{\psi}$ $\Rightarrow$

$$\begin{cases} \frac{\partial^2 \hat{u}}{\partial t^2} - \lambda^4 \hat{u} = 0 \\ \hat{u}|_{t=0} = \hat{\varphi}, \qquad \frac{\partial \hat{u}}{\partial t}|_{t=0} = \hat{\psi} \end{cases}$$

解得

$$\hat{u}(t,\lambda) = C_1 e^{-\lambda^2 t} + C_2 e^{\lambda^2 t}$$

其中

$$C_1 = \frac{1}{2}\left(\hat{\varphi}(\lambda) - \frac{\hat{\psi}(\lambda)}{\lambda^2}\right), \; C_2 = \frac{1}{2}\left(\hat{\varphi}(\lambda) + \frac{\hat{\psi}(\lambda)}{\lambda^2}\right).$$

注意到$\int_{-\infty}^{+\infty} |u(t,x)|^2 \, dx \leq M < +\infty \Rightarrow C_2 = 0 \Rightarrow$

$$0 = \lambda^2 \hat{\varphi}(\lambda) + \hat{\psi}(\lambda) = F[-\frac{\partial^2 \varphi}{\partial x^2} + \psi](\lambda) \Rightarrow \psi(x) = \frac{\partial^2 \varphi(x)}{\partial x^2}$$

(3). 由（2）得$\hat{u}(t,\lambda)=\hat{\varphi}(\lambda)e^{-\lambda^2 t}$,再由(1) 以及Fourier变换的性质得

$$u(t,x)=\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty}\varphi(\xi)e^{-\frac{(x-\xi)^2}{4t}}d\xi.$$

(3). 由（2）得$\hat{u}(t,\lambda)=\hat{\varphi}(\lambda)e^{-\lambda^2 t}$,再由(1) 以及Fourier变换的性质得

$$u(t,x)=\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty}\varphi(\xi)e^{-\frac{(x-\xi)^2}{4t}}d\xi.$$

三.（20分）利用对称延拓法求下列半无界初边值问题:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}-\dfrac{\partial^2 u}{\partial x^2}=x, \quad x>0, t>0 \\ \dfrac{\partial u}{\partial x}\Big|_{x=0}=2 \\ u\big|_{t=0}=-\frac{1}{6}x^3, \quad \dfrac{\partial u}{\partial t}\Big|_{t=0}=0 \end{cases}$$

解. 取函数$w(x)$满足$-\dfrac{d^2w(x)}{dx^2}=x, \quad \dfrac{\partial w}{\partial x}\Big|_{x=0}=2\Rightarrow$

$$w(x)=-\frac{1}{6}x^3+2x$$

令$v(t,x)=u(t,x)-w(x),\Rightarrow$

$$\begin{cases} \dfrac{\partial^2 v}{\partial t^2}-\dfrac{\partial^2 v}{\partial x^2}=0, \quad x>0, t>0 \\ \dfrac{\partial v}{\partial x}\Big|_{x=0}=0 \\ v\big|_{t=0}=-2x, \quad \dfrac{\partial v}{\partial t}\Big|_{t=0}=0 \end{cases}$$

三.（20分）利用对称延拓法求下列半无界初边值问题:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = x, & x > 0, t > 0 \\ \dfrac{\partial u}{\partial x}\Big|_{x=0} = 2 \\ u\Big|_{t=0} = -\frac{1}{6}x^3, & \dfrac{\partial u}{\partial t}\Big|_{t=0} = 0 \end{cases}$$

解. 取函数$w(x)$满足$-\dfrac{d^2 w(x)}{dx^2} = x, \quad \dfrac{\partial w}{\partial x}\Big|_{x=0} = 2 \Rightarrow$

$$w(x) = -\frac{1}{6}x^3 + 2x$$

令$v(t,x) = u(t,x) - w(x), \Rightarrow$

$$\begin{cases} \dfrac{\partial^2 v}{\partial t^2} - \dfrac{\partial^2 v}{\partial x^2} = 0, & x > 0, t > 0 \\ \dfrac{\partial v}{\partial x}\Big|_{x=0} = 0 \\ v\Big|_{t=0} = -2x, & \dfrac{\partial v}{\partial t}\Big|_{t=0} = 0 \end{cases}$$

三.（20分）利用对称延拓法求下列半无界初边值问题:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = x, & x > 0, t > 0 \\ \frac{\partial u}{\partial x}|_{x=0} = 2 \\ u|_{t=0} = -\frac{1}{6}x^3, & \frac{\partial u}{\partial t}|_{t=0} = 0 \end{cases}$$

解. 取函数$w(x)$满足$-\frac{d^2 w(x)}{dx^2} = x, \quad \frac{\partial w}{\partial x}|_{x=0} = 2 \Rightarrow$

$$w(x) = -\frac{1}{6}x^3 + 2x$$

令$v(t,x) = u(t,x) - w(x), \Rightarrow$

$$\begin{cases} \frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} = 0, & x > 0, t > 0 \\ \frac{\partial v}{\partial x}|_{x=0} = 0 \\ v|_{t=0} = -2x, & \frac{\partial v}{\partial t}|_{t=0} = 0 \end{cases}$$

三.（20分）利用对称延拓法求下列半无界初边值问题:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = x, & x > 0, t > 0 \\ \frac{\partial u}{\partial x}|_{x=0} = 2 \\ u|_{t=0} = -\frac{1}{6}x^3, & \frac{\partial u}{\partial t}|_{t=0} = 0 \end{cases}$$

解. 取函数$w(x)$满足$-\frac{d^2 w(x)}{dx^2} = x, \quad \frac{\partial w}{\partial x}|_{x=0} = 2 \Rightarrow$

$$w(x) = -\frac{1}{6}x^3 + 2x$$

令$v(t,x) = u(t,x) - w(x), \Rightarrow$

$$\begin{cases} \frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} = 0, & x > 0, t > 0 \\ \frac{\partial v}{\partial x}|_{x=0} = 0 \\ v|_{t=0} = -2x, & \frac{\partial v}{\partial t}|_{t=0} = 0 \end{cases}$$

利用对称延拓法（偶延拓）我们先讨论初值问题

$$\begin{cases} \dfrac{\partial^2 U}{\partial t^2} - \dfrac{\partial^2 U}{\partial x^2} = 0, & x \in (-\infty, +\infty), t > 0 \\ U|_{t=0} = -2|x|, & \dfrac{\partial U}{\partial t}|_{t=0} = 0 \end{cases}$$

解得$U(t,x) = -|x-t| - |x+t|$, 从而

$$v(t,x) = U(t,x)|_{x\geq 0} = \begin{cases} -2x, & x \geq t > 0, \\ -2t, & 0 \leq x < t. \end{cases}$$

$$u(t,x) = v(t,x) + w(x) = \begin{cases} -\frac{1}{6}x^3, & x \geq t > 0, \\ -\frac{1}{6}x^3 + 2x - 2t, & 0 \leq x < t. \end{cases}$$

利用对称延拓法（偶延拓）我们先讨论初值问题

$$\begin{cases} \frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} = 0, & x \in (-\infty, +\infty), t > 0 \\ U|_{t=0} = -2|x|, & \frac{\partial U}{\partial t}|_{t=0} = 0 \end{cases}$$

解得$U(t,x) = -|x-t| - |x+t|$, 从而

$$v(t,x) = U(t,x)|_{x\geq 0} = \begin{cases} -2x, & x \geq t > 0, \\ -2t, & 0 \leq x < t. \end{cases}$$

$$u(t,x) = v(t,x) + w(x) = \begin{cases} -\frac{1}{6}x^3, & x \geq t > 0, \\ -\frac{1}{6}x^3 + 2x - 2t, & 0 \leq x < t. \end{cases}$$

利用对称延拓法（偶延拓）我们先讨论初值问题

$$\begin{cases} \frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} = 0, & x \in (-\infty, +\infty), t > 0 \\ U|_{t=0} = -2|x|, & \frac{\partial U}{\partial t}|_{t=0} = 0 \end{cases}$$

解得$U(t,x) = -|x-t| - |x+t|$, 从而

$$v(t,x) = U(t,x)|_{x\geq 0} = \begin{cases} -2x, & x \geq t > 0, \\ -2t, & 0 \leq x < t. \end{cases}$$

$$u(t,x) = v(t,x) + w(x) = \begin{cases} -\frac{1}{6}x^3, & x \geq t > 0, \\ -\frac{1}{6}x^3 + 2x - 2t, & 0 \leq x < t. \end{cases}$$

利用对称延拓法（偶延拓）我们先讨论初值问题

$$\begin{cases} \frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} = 0, & x \in (-\infty, +\infty), t > 0 \\ U|_{t=0} = -2|x|, & \frac{\partial U}{\partial t}|_{t=0} = 0 \end{cases}$$

解得$U(t,x) = -|x-t| - |x+t|$, 从而

$$v(t,x) = U(t,x)|_{x\geq 0} = \begin{cases} -2x, & x \geq t > 0, \\ -2t, & 0 \leq x < t. \end{cases}$$

$$u(t,x) = v(t,x) + w(x) = \begin{cases} -\frac{1}{6}x^3, & x \geq t > 0, \\ -\frac{1}{6}x^3 + 2x - 2t, & 0 \leq x < t. \end{cases}$$

四.（20分）利用分离变量法求解:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 1,\ y > 0 \\ \dfrac{\partial u}{\partial x}\Big|_{x=0} = 0,\ \dfrac{\partial u}{\partial x}\Big|_{x=1} = 0 \\ u|_{y=0} = 1 - x, \quad \lim_{y\to+\infty} u = 0 \end{cases}$$

解. 令$u(x,y) = X(x)Y(y)$代入方程和边值条件

$$\begin{cases} X''(x) + \lambda X(x) = 0, \\ X'(0) = 0,\ X'(1) = 0 \end{cases} \qquad Y''(y) - \lambda Y(y) = 0.$$

得

$$\lambda_0 = 0,\ X_0(x) = c_0,$$

$$Y_0(y) = a_0 y + b_0,\ u_0(x,y) = A_0 y + B_0$$

$$\lambda_n = n^2\pi^2,\ X_n(x) = c_n \cos n\pi x,$$

$$Y_n(y) = a_n e^{n\pi y} + b_n e^{-n\pi y},\ n = 1, 2, \cdots$$

$$u_n(x,y) = \left(A_n e^{n\pi y} + B_n e^{-n\pi y}\right) \cos n\pi x,\ n = 1, 2, \cdots$$

四.（20分）利用分离变量法求解:

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 1,\ y > 0 \\ \frac{\partial u}{\partial x}\Big|_{x=0} = 0,\ \frac{\partial u}{\partial x}\Big|_{x=1} = 0 \\ u\big|_{y=0} = 1 - x, \quad \lim_{y\to+\infty} u = 0 \end{cases}$$

解. 令$u(x,y) = X(x)Y(y)$代入方程和边值条件

$$\begin{cases} X''(x) + \lambda X(x) = 0, \\ X'(0) = 0,\ X'(1) = 0 \end{cases} \qquad Y''(y) - \lambda Y(y) = 0.$$

得

$$\lambda_0 = 0,\ X_0(x) = c_0,$$

$$Y_0(y) = a_0 y + b_0,\ u_0(x,y) = A_0 y + B_0$$

$$\lambda_n = n^2\pi^2,\ X_n(x) = c_n \cos n\pi x,$$

$$Y_n(y) = a_n e^{n\pi y} + b_n e^{-n\pi y},\ n = 1, 2, \cdots$$

$$u_n(x,y) = \left(A_n e^{n\pi y} + B_n e^{-n\pi y}\right)\cos n\pi x,\ n = 1, 2, \cdots$$

四.（20分）利用分离变量法求解:

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 1,\ y > 0 \\ \frac{\partial u}{\partial x}\Big|_{x=0} = 0,\ \frac{\partial u}{\partial x}\Big|_{x=1} = 0 \\ u\big|_{y=0} = 1 - x, \quad \lim_{y\to+\infty} u = 0 \end{cases}$$

解. 令$u(x,y) = X(x)Y(y)$代入方程和边值条件

$$\begin{cases} X''(x) + \lambda X(x) = 0, \\ X'(0) = 0,\ X'(1) = 0 \end{cases} \qquad Y''(y) - \lambda Y(y) = 0.$$

得

$$\lambda_0 = 0,\ X_0(x) = c_0,$$

$$Y_0(y) = a_0 y + b_0,\ u_0(x,y) = A_0 y + B_0$$

$$\lambda_n = n^2\pi^2,\ X_n(x) = c_n \cos n\pi x,$$

$$Y_n(y) = a_n e^{n\pi y} + b_n e^{-n\pi y},\ n = 1, 2, \cdots$$

$$u_n(x,y) = \left(A_n e^{n\pi y} + B_n e^{-n\pi y}\right)\cos n\pi x,\ n = 1, 2, \cdots$$

四.（20分）利用分离变量法求解:

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 1,\ y > 0 \\ \frac{\partial u}{\partial x}\Big|_{x=0} = 0,\ \frac{\partial u}{\partial x}\Big|_{x=1} = 0 \\ u\big|_{y=0} = 1 - x, \quad \lim_{y\to+\infty} u = 0 \end{cases}$$

解. 令$u(x,y) = X(x)Y(y)$代入方程和边值条件

$$\begin{cases} X''(x) + \lambda X(x) = 0, \\ X'(0) = 0,\ X'(1) = 0 \end{cases} \qquad Y''(y) - \lambda Y(y) = 0.$$

得

$$\lambda_0 = 0,\ X_0(x) = c_0,$$

$$Y_0(y) = a_0 y + b_0,\ u_0(x,y) = A_0 y + B_0$$

$$\lambda_n = n^2\pi^2,\ X_n(x) = c_n \cos n\pi x,$$

$$Y_n(y) = a_n e^{n\pi y} + b_n e^{-n\pi y},\ n = 1, 2, \cdots$$

$$u_n(x,y) = \left(A_n e^{n\pi y} + B_n e^{-n\pi y}\right)\cos n\pi x,\ n = 1, 2, \cdots$$

取

$$u(x,y) = A_0 y + B_0 + \sum_{n=1}^{\infty} \left(A_n e^{n\pi y} + B_n e^{-n\pi y}\right) \cos n\pi x$$

由初值条件$u|_{y=0} = 1 - x$和$\lim_{y\to+\infty} u = 0$得

$$B_0 + \sum_{n=1}^{\infty} (A_n + B_n) \cos n\pi x = 1 - x, \; A_n = 0 \, (n = 0, 1, 2, \cdots).$$

$$\Rightarrow A_n = 0 \, (n = 0, 1, 2, \cdots),$$

$$B_0 = \frac{1}{2}, \; B_n = \frac{2}{(n\pi)^2} \left(1 - (-1)^n\right) \, (n = 1, 2, \cdots).$$

$$\Rightarrow u(x,y) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(n\pi)^2} \left(1 - (-1)^n\right) e^{-n\pi y} \cos n\pi x.$$

取

$$u(x,y)=A_0y+B_0+\sum_{n=1}^{\infty}\left(A_ne^{n\pi y}+B_ne^{-n\pi y}\right)\cos n\pi x$$

由初值条件$u|_{y=0}=1-x$和$\lim_{y\to+\infty}u=0$得

$$B_0+\sum_{n=1}^{\infty}(A_n+B_n)\cos n\pi x=1-x,\ A_n=0\,(n=0,1,2,\cdots).$$

$$\Rightarrow A_n=0\,(n=0,1,2,\cdots),$$

$$B_0=\frac{1}{2},\ B_n=\frac{2}{(n\pi)^2}\left(1-(-1)^n\right)\ (n=1,2,\cdots).$$

$$\Rightarrow u(x,y)=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{2}{(n\pi)^2}\left(1-(-1)^n\right)e^{-n\pi y}\cos n\pi x.$$

取

$$u(x,y)=A_0y+B_0+\sum_{n=1}^{\infty}\left(A_ne^{n\pi y}+B_ne^{-n\pi y}\right)\cos n\pi x$$

由初值条件$u|_{y=0}=1-x$和$\lim_{y\to+\infty}u=0$得

$$B_0+\sum_{n=1}^{\infty}(A_n+B_n)\cos n\pi x=1-x,\ A_n=0\,(n=0,1,2,\cdots).$$

$$\Rightarrow A_n=0\,(n=0,1,2,\cdots),$$

$$B_0=\frac{1}{2},\ B_n=\frac{2}{(n\pi)^2}\left(1-(-1)^n\right)\,(n=1,2,\cdots).$$

$$\Rightarrow u(x,y)=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{2}{(n\pi)^2}\left(1-(-1)^n\right)e^{-n\pi y}\cos n\pi x.$$

取

$$u(x,y) = A_0 y + B_0 + \sum_{n=1}^{\infty} \left(A_n e^{n\pi y} + B_n e^{-n\pi y} \right) \cos n\pi x$$

由初值条件$u|_{y=0} = 1 - x$和$\lim_{y\to+\infty} u = 0$得

$$B_0 + \sum_{n=1}^{\infty} (A_n + B_n) \cos n\pi x = 1 - x,\ A_n = 0\,(n = 0, 1, 2, \cdots).$$

$$\Rightarrow A_n = 0\,(n = 0, 1, 2, \cdots),$$

$$B_0 = \frac{1}{2},\ B_n = \frac{2}{(n\pi)^2} \left(1 - (-1)^n\right)\ (n = 1, 2, \cdots).$$

$$\Rightarrow u(x,y) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(n\pi)^2} \left(1 - (-1)^n\right) e^{-n\pi y} \cos n\pi x.$$

取

$$u(x,y)=A_0y+B_0+\sum_{n=1}^{\infty}\left(A_ne^{n\pi y}+B_ne^{-n\pi y}\right)\cos n\pi x$$

由初值条件$u|_{y=0}=1-x$和$\lim_{y\to+\infty}u=0$得

$$B_0+\sum_{n=1}^{\infty}(A_n+B_n)\cos n\pi x=1-x,\ A_n=0\,(n=0,1,2,\cdots).$$

$$\Rightarrow A_n=0\,(n=0,1,2,\cdots),$$

$$B_0=\frac{1}{2},\ B_n=\frac{2}{(n\pi)^2}\left(1-(-1)^n\right)\ (n=1,2,\cdots).$$

$$\Rightarrow u(x,y)=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{2}{(n\pi)^2}\left(1-(-1)^n\right)e^{-n\pi y}\cos n\pi x.$$

五.（20分）利用分离变量法求解:

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x} = te^x \sin(2\pi x), \quad 0 < x < 1,\ t > 0 \\ u|_{x=0} = 0,\ u|_{x=1} = 0 \\ u|_{t=0} = e^x \sin(5\pi x) \end{cases}$$

解. 令$u(x,y) = X(x)T(t)$代入相应地齐次线性方程和边值条件得本征值问题

$$\begin{cases} X''(x) - 2X'(x) + \lambda X(x) = 0, \\ X(0) = 0,\ X(1) = 0 \end{cases}$$

解得本征值和本征函数分别为$(n = 1, 2, \cdots)$

$$\lambda_n = 1 + n^2\pi^2,\ X_n(x) = a_n e^x \sin n\pi x.$$

(按本征函数展开法求解)取

$$u(t,x) = \sum_{n=1}^{\infty} T_n(t) e^x \sin n\pi x,$$

代入初边值问题得

五.（20分）利用分离变量法求解:

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x} = te^x \sin(2\pi x), \quad 0 < x < 1,\ t > 0 \\ u|_{x=0} = 0,\ u|_{x=1} = 0 \\ u|_{t=0} = e^x \sin(5\pi x) \end{cases}$$

解. 令$u(x,y) = X(x)T(t)$代入相应地齐次线性方程和边值条件得本征值问题

$$\begin{cases} X''(x) - 2X'(x) + \lambda X(x) = 0, \\ X(0) = 0,\ X(1) = 0 \end{cases}$$

解得本征值和本征函数分别为$(n = 1, 2, \cdots)$

$$\lambda_n = 1 + n^2\pi^2,\ X_n(x) = a_n e^x \sin n\pi x.$$

(按本征函数展开法求解)取

$$u(t,x) = \sum_{n=1}^{\infty} T_n(t) e^x \sin n\pi x,$$

代入初边值问题得

五.（20分）利用分离变量法求解:

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x} = te^x \sin(2\pi x), \quad 0 < x < 1,\ t > 0 \\ u|_{x=0} = 0,\ u|_{x=1} = 0 \\ u|_{t=0} = e^x \sin(5\pi x) \end{cases}$$

解. 令$u(x,y) = X(x)T(t)$代入相应地齐次线性方程和边值条件得本征值问题

$$\begin{cases} X''(x) - 2X'(x) + \lambda X(x) = 0, \\ X(0) = 0,\ X(1) = 0 \end{cases}$$

解得本征值和本征函数分别为$(n = 1, 2, \cdots)$

$$\lambda_n = 1 + n^2\pi^2,\ X_n(x) = a_n e^x \sin n\pi x.$$

(按本征函数展开法求解)取

$$u(t,x) = \sum_{n=1}^{\infty} T_n(t) e^x \sin n\pi x,$$

代入初边值问题得

五.（20分）利用分离变量法求解:

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x} = te^x \sin(2\pi x), & 0 < x < 1,\ t > 0 \\ u|_{x=0} = 0,\ u|_{x=1} = 0 \\ u|_{t=0} = e^x \sin(5\pi x) \end{cases}$$

解. 令$u(x,y) = X(x)T(t)$代入相应地齐次线性方程和边值条件得本征值问题

$$\begin{cases} X''(x) - 2X'(x) + \lambda X(x) = 0, \\ X(0) = 0,\ X(1) = 0 \end{cases}$$

解得本征值和本征函数分别为$(n = 1, 2, \cdots)$

$$\lambda_n = 1 + n^2\pi^2,\ X_n(x) = a_n e^x \sin n\pi x.$$

(按本征函数展开法求解)取

$$u(t,x) = \sum_{n=1}^{\infty} T_n(t) e^x \sin n\pi x,$$

代入初边值问题得

五.（20分）利用分离变量法求解:

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x} = te^x \sin(2\pi x), \quad 0 < x < 1,\ t > 0 \\ u|_{x=0} = 0,\ u|_{x=1} = 0 \\ u|_{t=0} = e^x \sin(5\pi x) \end{cases}$$

解. 令$u(x,y) = X(x)T(t)$代入相应地齐次线性方程和边值条件得本征值问题

$$\begin{cases} X''(x) - 2X'(x) + \lambda X(x) = 0, \\ X(0) = 0,\ X(1) = 0 \end{cases}$$

解得本征值和本征函数分别为$(n = 1, 2, \cdots)$

$$\lambda_n = 1 + n^2\pi^2,\ X_n(x) = a_n e^x \sin n\pi x.$$

(按本征函数展开法求解)取

$$u(t,x) = \sum_{n=1}^{\infty} T_n(t) e^x \sin n\pi x,$$

代入初边值问题得

$$
\begin{cases}
\sum_{n=1}^{\infty}\left[T_n'(t)+(1+n^2\pi^2)T_n(t)\right]e^x\sin n\pi x = te^x\sin 2\pi x, \\
\sum_{n=1}^{\infty}T_n(0)e^x\sin n\pi x = 3e^x\sin 5\pi x.
\end{cases}
$$

即

$$
\begin{cases}
T_n'(t)+(1+n^2\pi^2)T_n(t)=\begin{cases} t, & n=2 \\ 0, & n\neq 2 \end{cases} \\
T_n(0)=\begin{cases} 3, & n=5 \\ 0, & n\neq 5 \end{cases}
\end{cases}
$$

解得

$$
T_2(t)=\frac{1}{1+4\pi^2}t-\frac{1}{(1+4\pi^2)^2}+\frac{e^{-(1+4\pi^2)t}}{(1+4\pi^2)^2},
$$

$$
T_5(t)=3e^{-(1+25\pi^2)t}, \qquad T_n(t)=0,\ (n\neq 2,5).
$$

从而

$$
u(t,x)=\left[\frac{1}{1+4\pi^2}t-\frac{1}{(1+4\pi^2)^2}+\frac{e^{-(1+4\pi^2)t}}{(1+4\pi^2)^2}\right]e^x\sin 2\pi x
$$

$$
+3e^{-(1+25\pi^2)t}e^x\sin 5\pi x.
$$

$$\begin{cases} \sum_{n=1}^{\infty} \left[T_n'(t) + (1+n^2\pi^2)T_n(t)\right] e^x \sin n\pi x = te^x \sin 2\pi x, \\ \sum_{n=1}^{\infty} T_n(0)e^x \sin n\pi x = 3e^x \sin 5\pi x. \end{cases}$$

即

$$\begin{cases} T_n'(t) + (1+n^2\pi^2)T_n(t) = \begin{cases} t, & n=2 \\ 0, & n\neq 2 \end{cases} \\ T_n(0) = \begin{cases} 3, & n=5 \\ 0, & n\neq 5 \end{cases} \end{cases}$$

解得

$$T_2(t) = \frac{1}{1+4\pi^2}t - \frac{1}{(1+4\pi^2)^2} + \frac{e^{-(1+4\pi^2)t}}{(1+4\pi^2)^2},$$

$$T_5(t) = 3e^{-(1+25\pi^2)t}, \qquad T_n(t) = 0, \ (n \neq 2, 5).$$

从而

$$u(t,x) = \left[\frac{1}{1+4\pi^2}t - \frac{1}{(1+4\pi^2)^2} + \frac{e^{-(1+4\pi^2)t}}{(1+4\pi^2)^2}\right] e^x \sin 2\pi x$$

$$+3e^{-(1+25\pi^2)t} e^x \sin 5\pi x.$$

$$\begin{cases} \sum_{n=1}^{\infty} \left[T_n'(t) + (1+n^2\pi^2)T_n(t)\right] e^x \sin n\pi x = te^x \sin 2\pi x, \\ \sum_{n=1}^{\infty} T_n(0)e^x \sin n\pi x = 3e^x \sin 5\pi x. \end{cases}$$

即

$$\begin{cases} T_n'(t) + (1+n^2\pi^2)T_n(t) = \begin{cases} t, & n=2 \\ 0, & n \neq 2 \end{cases} \\ T_n(0) = \begin{cases} 3, & n=5 \\ 0, & n \neq 5 \end{cases} \end{cases}$$

解得

$$T_2(t) = \frac{1}{1+4\pi^2}t - \frac{1}{(1+4\pi^2)^2} + \frac{e^{-(1+4\pi^2)t}}{(1+4\pi^2)^2},$$

$$T_5(t) = 3e^{-(1+25\pi^2)t}, \qquad T_n(t) = 0,\ (n \neq 2, 5).$$

从而

$$u(t,x) = \left[\frac{1}{1+4\pi^2}t - \frac{1}{(1+4\pi^2)^2} + \frac{e^{-(1+4\pi^2)t}}{(1+4\pi^2)^2}\right] e^x \sin 2\pi x$$

$$+3e^{-(1+25\pi^2)t} e^x \sin 5\pi x.$$

$$\begin{cases} \sum_{n=1}^{\infty} \left[T_n'(t) + (1+n^2\pi^2) T_n(t) \right] e^x \sin n\pi x = te^x \sin 2\pi x, \\ \sum_{n=1}^{\infty} T_n(0) e^x \sin n\pi x = 3e^x \sin 5\pi x. \end{cases}$$

即

$$\begin{cases} T_n'(t) + (1+n^2\pi^2) T_n(t) = \begin{cases} t, & n = 2 \\ 0, & n \neq 2 \end{cases} \\ T_n(0) = \begin{cases} 3, & n = 5 \\ 0, & n \neq 5 \end{cases} \end{cases}$$

解得

$$T_2(t) = \frac{1}{1+4\pi^2} t - \frac{1}{(1+4\pi^2)^2} + \frac{e^{-(1+4\pi^2)t}}{(1+4\pi^2)^2},$$

$$T_5(t) = 3e^{-(1+25\pi^2)t}, \qquad T_n(t) = 0, \ (n \neq 2, 5).$$

从而

$$u(t,x) = \left[\frac{1}{1+4\pi^2} t - \frac{1}{(1+4\pi^2)^2} + \frac{e^{-(1+4\pi^2)t}}{(1+4\pi^2)^2} \right] e^x \sin 2\pi x$$

$$+3e^{-(1+25\pi^2)t} e^x \sin 5\pi x.$$

考前答疑：2012年1月8日下午13：30—16：00，东1A-302