Field and Wave Electromagnetics

Solution of Electrostatic Problems

Chapter 4: Solutions of Electrostatic Problems

4.2 Poisson's and Laplaces's Equations

Maxwell Equation

$$\nabla \cdot \vec{D} = \rho \qquad \text{free charges}$$

$$\nabla \times \vec{E} = 0$$

$$\vec{E} = -\nabla V$$

In a <u>linear</u> and <u>isotropic</u> medium $\vec{D} = \varepsilon \vec{E}$

$$\nabla \cdot \varepsilon \vec{E} = \rho$$

$$\nabla \cdot (\varepsilon \nabla V) = -\rho$$

For a <u>homegeneous</u> medium: $\boxed{\nabla^2 V = -\dfrac{\rho}{\varepsilon}}$

Poisson's equation: $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = -\dfrac{\rho}{\varepsilon}$

$$\nabla^2 V = \nabla \cdot \nabla V = \left(\vec{a}_x \frac{\partial}{\partial x} + \vec{a}_y \frac{\partial}{\partial y} + \vec{a}_z \frac{\partial}{\partial z}\right) \cdot \left(\vec{a}_x \frac{\partial V}{\partial x} + \vec{a}_y \frac{\partial V}{\partial y} + \vec{a}_z \frac{\partial V}{\partial z}\right)$$

$$= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

4.2 Poisson's and Laplaces's Equations

Cylindrical coordinates:

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial V}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \Phi^2} + \frac{\partial^2 V}{\partial z^2}$$

Spherical coordinates:

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2\frac{\partial V}{\partial R}) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\frac{\partial V}{\partial \theta}) + \frac{1}{R^2 \sin^2\theta}\frac{\partial^2 V}{\partial \Phi^2}$$

no free charge **Laplance's equation**

$$\nabla^2 V = 0$$

electric field

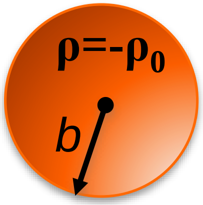
$$\vec{E} = -\nabla V$$

charge distribution on the conductor surfaces

$$\rho_s = \varepsilon \vec{E}_n$$

Determine the field both inside and outside a spherical cloud of electrons with a uniform volume charge density ρ=-ρ_0 (where ρ_0 is a positive quantity) for $0 \le R \le b$ and ρ=0 for *R>b* by solving Poisson's and Laplance's equation for *V*.

(1) Inside the cloud $0 \le R \le b$ $\rho = -\rho_0$



Poisson's equation $\nabla^2 V = -\frac{\rho}{\varepsilon}$

$$\frac{1}{R^2}\frac{d}{dR}(R^2\frac{dV_i}{dR}) = \frac{\rho_0}{\varepsilon_0} \qquad \frac{d}{dR}(R^2\frac{dV_i}{dR}) = \frac{\rho_0}{\varepsilon_0}R^2$$

$$\frac{dV_i}{dR} = \frac{\rho_0}{3\varepsilon_0}R + \frac{C_1}{R^2}.$$

The electric field intensity inside

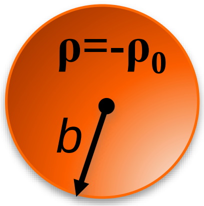
$$\vec{E}_i = -\nabla V_i = -\vec{a}_R(\frac{dV_i}{dR})$$

$$\vec{E}_i = -\vec{a}_R(\frac{\rho_0}{3\varepsilon_0})R$$

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2\frac{\partial V}{\partial R}) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial V}{\partial\theta}) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial\Phi^2}$$

Determine the field both inside and outside a spherical cloud of electrons with a uniform volume charge density ρ=-ρ_0 (where ρ_0 is a positive quantity) for $0 \le R \le b$ and ρ=0 for *R>b* by sloving Poisson's and Laplance's equation for *V*.

(2) Outside the cloud $b \le R$ $\rho = 0$



Laplace's equation $\nabla^2 V = 0$

$$\frac{1}{R^2}\frac{d}{dR}(R^2\frac{dV_o}{dR}) = 0$$

$$\frac{dV_0}{dR} = \frac{C_2}{R^2}$$

The electric field $\vec{E}_0 = -\nabla V_0 = -\vec{a}_R(\frac{dV_0}{dR}) = -\vec{a}_R(\frac{C_2}{R^2})$

The electric field continuity at *R=b* $\frac{C_2}{b^2} = \frac{\rho_0}{3\varepsilon_0}b$

$$\vec{E}_0 = -\vec{a}_R\frac{\rho_0 b^3}{3\varepsilon_0 R^2}$$

$$Q = -\rho_0\frac{4\pi}{3}b^3 \qquad \vec{E}_0 = \vec{a}_R\frac{Q}{4\pi\varepsilon_0 R^2}$$

4.3 Uniqueness of Electrostatic Solutions

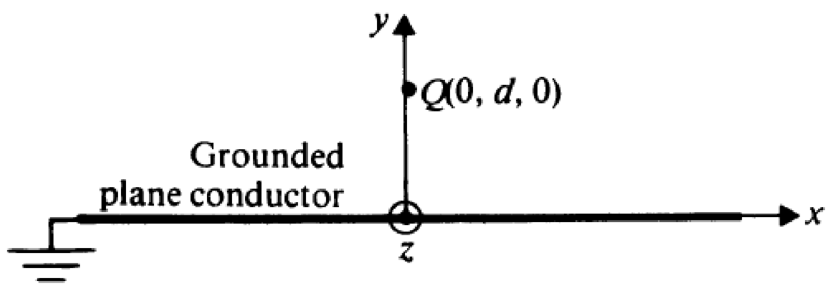
Uniqueness theorem

A solution of Possion's equation (of Lapalce's equation is a special case) that satisfies the given boundary conditions is a unique solution.

A solution of an electrostatic problem satisfying its boundary conditions is the only possible solution, irrespective of the method by with the solution is obtained.

4.4 Methods of Images
– point charge and conducting planes

Consider the case of a positive point charge, *Q* , located at a distance *d* above a large grounded (zero-potential) conducting plane
Find the potential at every point above the conducting plane (*y*>0).



The *V*(*x*,*y*,*z*) should satisfy the following conditions:
(1) At all points on the grounded conducting plane, *V*(*x*,0,*z*)=0.
(2) At points very close to *Q*, *V* approaches that of the point charge alone; that is as *R*→0

$$V \to \frac{Q}{4\pi\varepsilon_0 R}$$

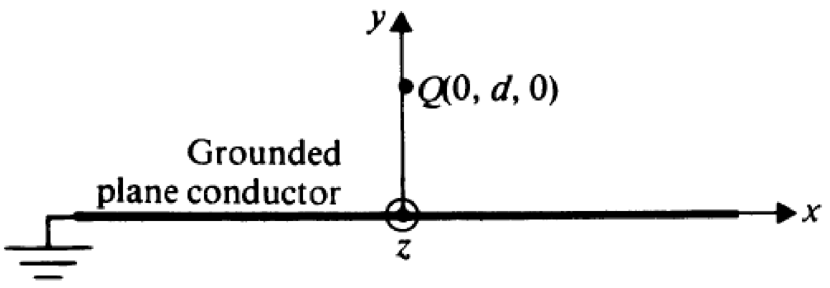
(3) At points very far from $Q(x \to \pm\infty, y \to +\infty, z \to \pm\infty)$, $V \to 0$.
(4) *V* is even with respect to the *x* and *z* coordinates:

$$V(x, y, z) = V(-x, y, z) \text{ and } V(x, y, z) = V(x, y, -z)$$

4.4 Methods of Images
– point charge and conducting planes

Positive charge Q at $y=d$ would induce negative charges on the surface of the conducting plane, resulting in a surface charge density ρ_s. Hence the potential at points above the conducting plane would be:

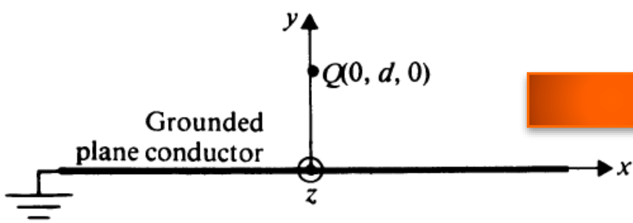
$$V(x,y,z)=\frac{Q}{4\pi\varepsilon_0\sqrt{x^2+(y-d)^2+z^2}}+\frac{1}{4\pi\varepsilon_0}\int_s\frac{\rho_s}{R_1}ds,$$



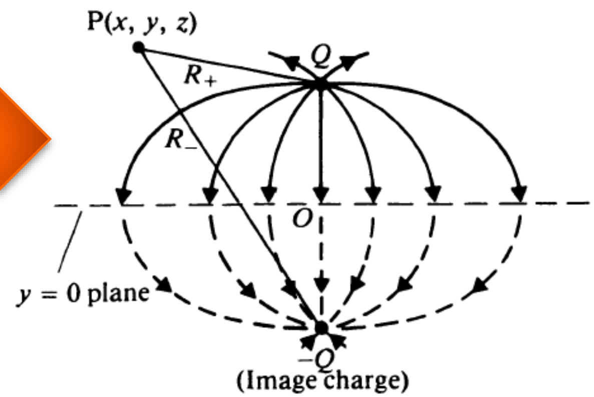
Trouble: how to determine the surface charge density ρ_s?

4.4 Methods of Images
– point charge and conducting planes

If we remove the conductor and replace it by an image point charge -*Q* at *y*=-*d*, then the potential at a point *P*(*x,y,z*) in the *y*>0 region is



(a) Physical arrangement.



(b) Image charge and field lines.

$$V(x,y,z)=\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_+}-\frac{1}{R_-}\right)$$

$$R_+=[x^2+(y-d)^2+z^2]^{\frac{1}{2}}$$

$$R_-=[x^2+(y+d)^2+z^2]^{\frac{1}{2}}$$

Note: The image charge should be located outside the region where the field is to be determined

4.4 Methods of Images
– linear charge and parallel conducting cylinder

Consider a line charge ρ_l (C/m) located at a distance d from the axis of a parallel, conducting, circular cylinder of radius a. Both the line charge and the conducting cylinder are assumed to be infinitely long. Determine the position of the image charge.

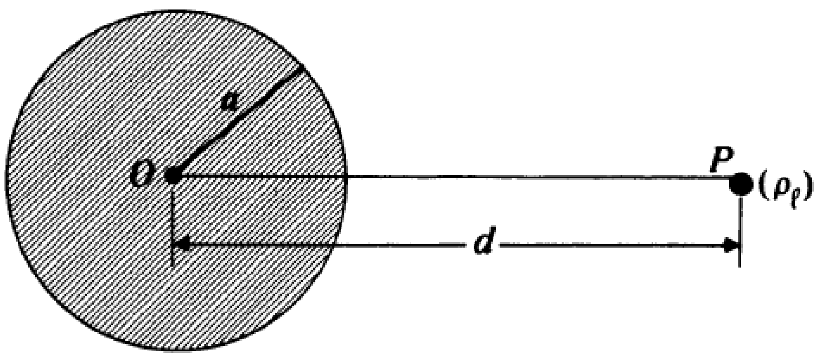
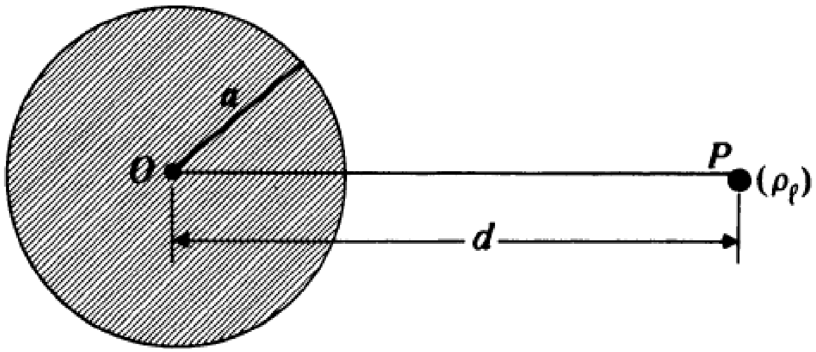
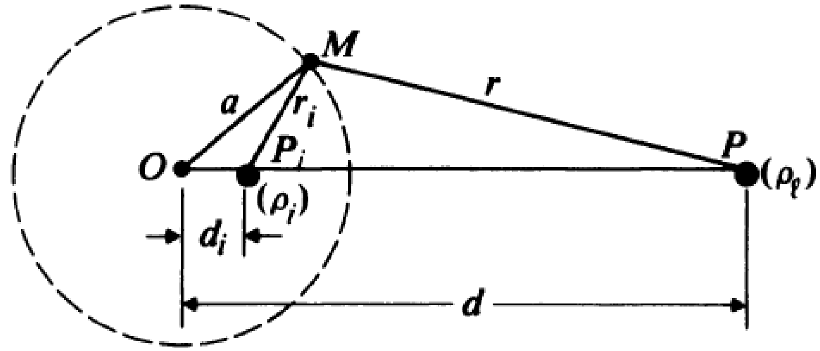


Image: (1) the cylindrical surface at $r=a$ an equipotential surface
→ the image must be a parallel line charge (ρ_i) inside the cylinder.
(2) symmetry with respect to the line OP
→ the image must lie somewhere along OP. Say at point P_i, which is at a distance d_i from the axis.

4.4 Methods of Images
– linear charge and parallel conducting cylinder





Determine: ρ_i and d_i

let us assume that $\rho_i = -\rho_l$

The electric potential at a distance r from a line charge of density ρ_l can be obtained by integrating the electric field intensity E:

$$V = -\int_{r_0}^{r} E_r dr = -\frac{\rho_l}{2\pi\varepsilon_0}\int_{r_0}^{r}\frac{1}{r}dr = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_0}{r}$$

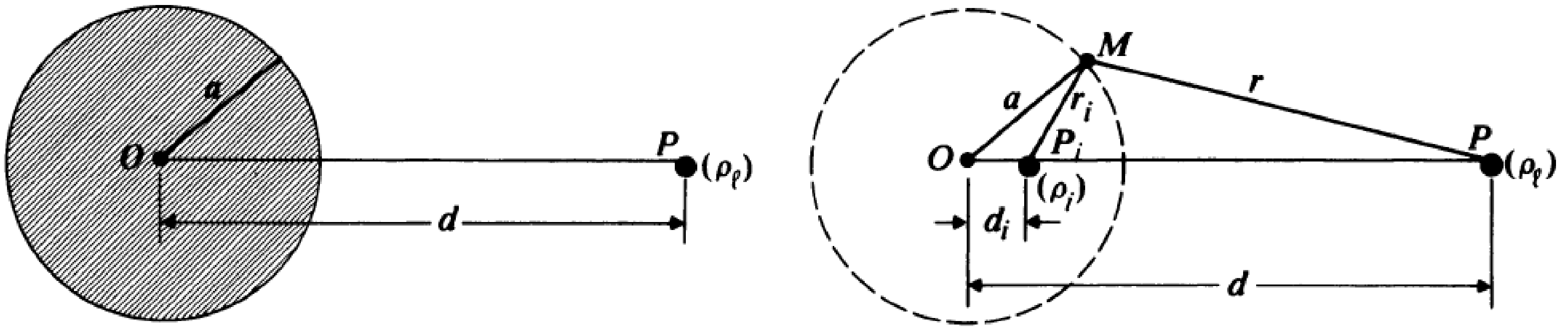
r_0 : zero potential

At any point M on the cylindrical surface, the potential:

$$V_M = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_0}{r} - \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_0}{r_i} = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_i}{r}$$

r_0 for both ρ_i and ρ_l

4.4 Methods of Images
– linear charge and parallel conducting cylinder



$$V_M = \frac{r_l}{2pe_0}\ln\frac{r_i}{r}$$

Equipotential

$$\frac{r_i}{r} = cons.$$

Triangles OMP$_i$ and OPM similar :

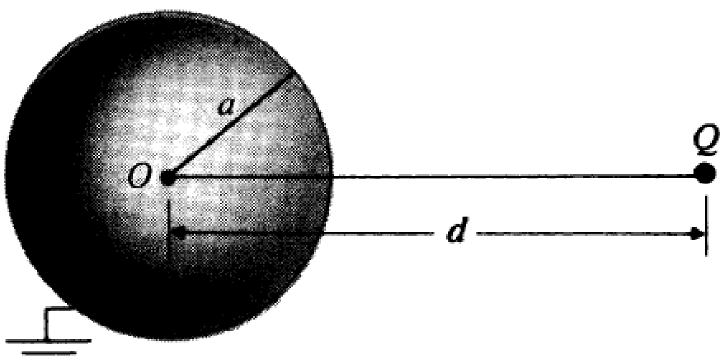
$$\frac{r_i}{r} = \frac{d_i}{a} = \frac{a}{d} = C$$

$$d_i = \frac{a^2}{d}$$

The image line charge ρ_i together with ρ_l, will make the dashed cylindrical surface equipotential.

4.4 Methods of Images
– point charge and conducting sphere

A point charge Q is at a distance d from the center of a grounded conducting sphere of radius a ($a<d$). Determined the charge distribution induced on the sphere and the total charge induced on the sphere.



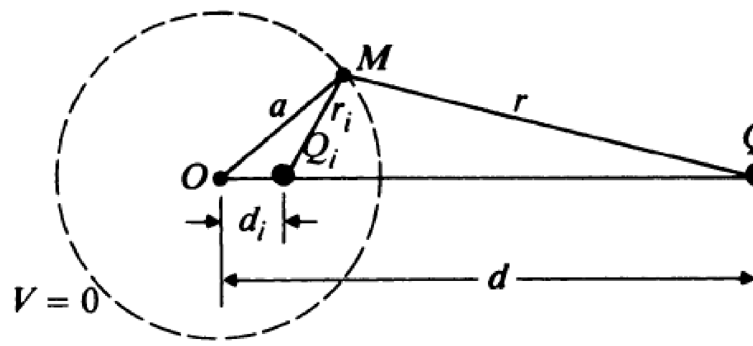


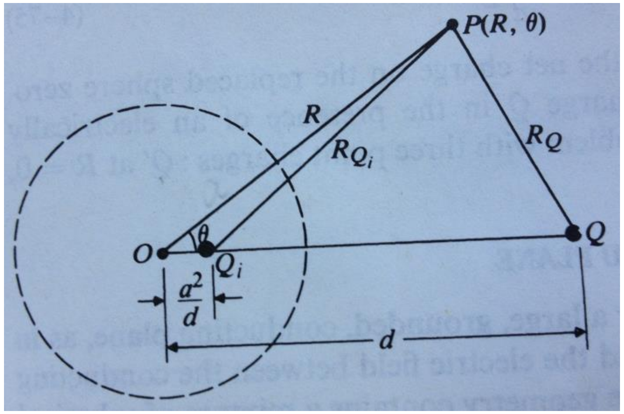
Image charge Q_i can be equal to $-Q$?

Triangles OMP_i and OPM are similar.

$$V_M = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \frac{Q_i}{r_i}\right) = 0 \rightarrow \frac{r_i}{r} = -\frac{Q_i}{Q} = cons. \rightarrow$$

$$\boxed{\begin{aligned} d_i &= \frac{a^2}{d} \\ Q_i &= -\frac{a}{d}Q \end{aligned}}$$

4.4 Methods of Images
– point charge and conducting sphere



The electric potential *V* at an arbitrary point

$$V(R,\theta)=\frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{R_Q}-\frac{\frac{a}{d}Q}{R_{Q_i}}\right)=\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_Q}-\frac{a}{dR_{Q_i}}\right)$$

The law of cosines $\quad R_Q=[R^2+d^2-2Rd\cos\theta]^{1/2}$

$$R_{Q_i}=[R^2+(\frac{a^2}{d})^2-2R(\frac{a^2}{d})\cos\theta]^{1/2}$$

4.4 Methods of Images
– point charge and conducting sphere

$$V(R,\theta)=\frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{R_Q}-\frac{\frac{a}{d}Q}{R_{Q_i}}\right)=\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_Q}-\frac{a}{dR_{Q_i}}\right)$$

The *R*-component of the electric field intensity E_R

$$E_R(R,\theta)=-\frac{\partial V(R,\theta)}{\partial R}$$

$$E_R(R,\theta)=\frac{Q}{4\pi\varepsilon_0}\left\{\frac{R-d\cos\theta}{(R^2+d^2-2Rd\cos\theta)^{3/2}}-\frac{a[R-(a^2/d)\cos\theta]}{d[R^2+(a^2/d)^2-2R(a^2/d)\cos\theta]^{3/2}}\right\}$$

(1) To find the induced surface charge on the sphere, we set *R*=*a*

$$\rho_s=\varepsilon_0 E_R(a,\theta)=-\frac{Q(d^2-a^2)}{4\pi a(a^2+d^2-2ad\cos\theta)^{3/2}}$$

Induced surface charge is negative and that its magnitude is maxmium at θ=0 and minimum θ=π.

4.4 Methods of Images
– point charge and conducting sphere

(2) The total charge induced on the sphere is obtained by integrating $\boldsymbol{\rho_s}$ over the surface of the sphere.

$$\text{Total-induced-charge} = \oint \rho_s ds = \int_0^{2\pi}\int_0^{\pi} \rho_s a^2 \sin\theta d\theta d\Phi = -\frac{a}{d}Q = Q_i.$$

4.5 Boundary-value problems in Cartesian Coordinates

Potential equation (free source): $\boxed{\nabla^2 V = 0}$

Method of images: free charges near conducting boundaries.

For problems consisting of a system of <u>conductors maintained at specified potentials</u> and <u>with no isolated free charges</u>, how to solve them?

Method of separation of variables: the solution can be expressed as a product of <u>three one-dimensional functions</u>, <u>each depending separately on one coordinate variable only</u>

Three types of boundary-value problem:

(1) Dirichlet problems: the potential value is specified everywhere on the boundaries;

(2) Neumann problems: the normal derivative of the potential is specified everywhere on the boundaries;

(3) Mixed boundary-value problems: the potential value is specified over some boundaries and the normal derivative of the potential is specified over the remaining ones;

4.5 Boundary-value problems in Cartesian Coordinates

Laplace's equation for *V* in Cartesian coordinates:

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0$$

Apply method of separation of variables, *V*(x,y,z) can be expressed as:

$$V(x,y,z)=X(x)Y(y)Z(z)$$

***X*(*x*),*Y*(*y*) and *Z*(*z*) are functions of only *x*, *y* and *z*, respectively.**

$$Y(y)Z(z)\frac{d^2X(x)}{dx^2}+X(x)Z(z)\frac{d^2Y(y)}{dy^2}+X(x)Y(y)\frac{d^2Z(z)}{dz^2}=0$$

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}+\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}+\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}=0$$

In order for Eq. to be satisfied for all values of *x*, *y*, *z*, each of the three terms must be a constant.

4.5 Boundary-value problems in Cartesian Coordinates

$$\frac{d}{dx}\left[\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}\right]=0$$

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}=-k_x^2$$

k_x^2 is a constant to be determined from the boundary conditions
k_x is imaginary, $-k_x^2$ is a positive real number
k_x is real, $-k_x^2$ is a negative real number

$$\frac{d^2X(x)}{dx^2}+k_x^2X(x)=0$$

$$\frac{d^2Y(y)}{dy^2}+k_y^2Y(y)=0$$

$$k_x^2+k_y^2+k_z^2=0$$

$$\frac{d^2Z(z)}{dz^2}+k_z^2Z(z)=0$$

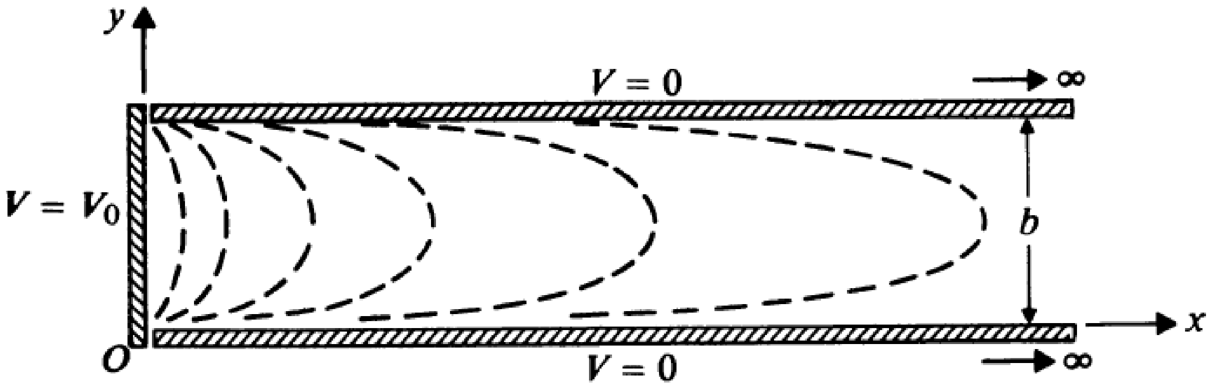
4.5 Boundary-value problems in Cartesian Coordinates

$$\frac{d^2 X(x)}{dx^2} + k_x^2 X(x) = 0$$

k_x^2	k_x	$X(x)$
0	0	$A_0 x + B_0$
+	k	$A_1 \sin kx + B_1 \cos kx$
−	jk	$C_2 e^{kx} + D_2 e^{-kx}$

4.5 Boundary-value problems in Cartesian Coordinates

Two grounded, semi-infinite, parallel-plane electrodes are separated by a distance *b*. A third electrode perpendicular to and insulated from both is maintained at a constant. Determine the potential distribution in the region enclosed by the electrodes.



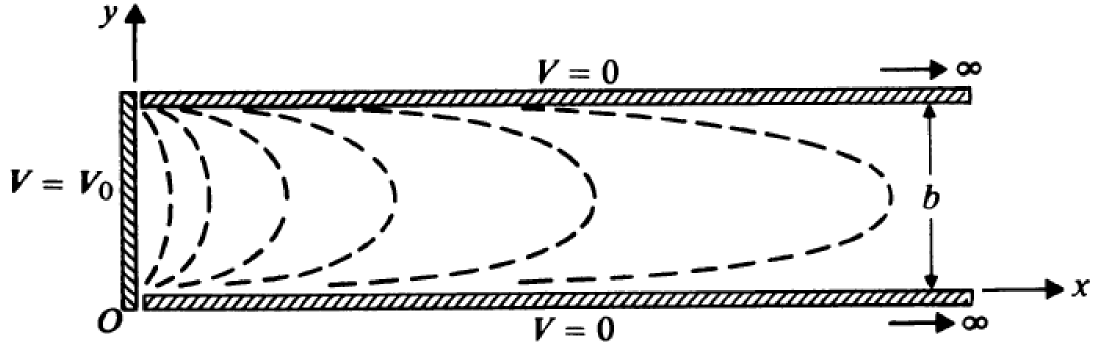
With *V* independent of *z* $V(x,y,z) = V(x,y)$

In the *x* direction $V(0,y) = V_0$ $V(\infty,y) = 0$

In the *y* direction $V(x,0) = 0$ $V(x,b) = 0$

$$k_z = 0 \qquad Z(z) = B_0$$

4.5 Boundary-value problems in Cartesian Coordinates



In the *x* direction $V(0,y)=V_0$ $V(\infty,y)=0$

In the *y* direction $V(x,0)=0$ $V(x,b)=0$

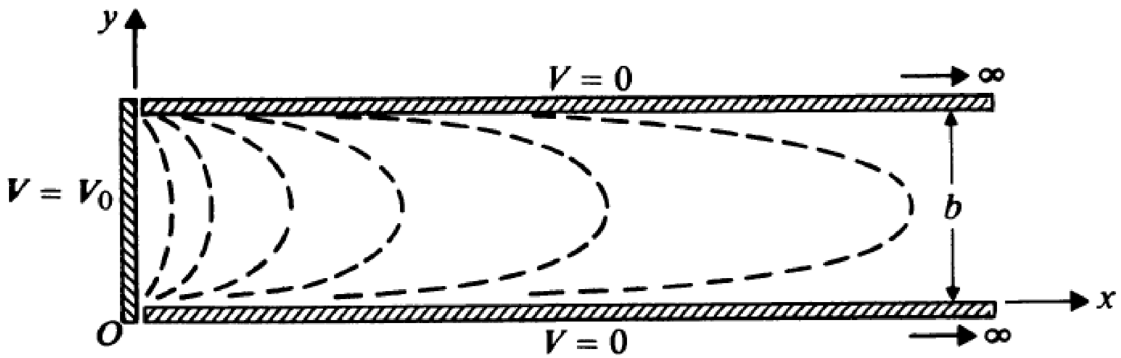
$$k_y^{\ 2}=-k_x^{\ 2}=k^2$$ ***k* real number, so *k*$_x$=*jk***

$$X(x)=D_2e^{-kx}$$

$$Y(y)=A_1\sin ky$$

$$V_n(x,y)=\left(B_0D_2A_1\right)e^{-kx}sinky=C_ne^{-kx}sinky$$

4.5 Boundary-value problems in Cartesian Coordinates



$$V_n\left(x,b\right) = C_n e^{-kx} sinkb = 0$$

Should be satisfied for all values of *x*, only if

$$k = \frac{n\pi}{b}, n = 1, 2, 3, \ldots$$

$$V_n(x,y) = C_n e^{-\frac{n\pi}{b}x} \sin\frac{n\pi}{b} y$$

$$V(0,y) = \sum_{n=1}^{\infty} V_n(0,y) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi}{b} y = V_0$$

$$C_n = \begin{cases} \dfrac{4V_0}{n\pi}, n - odd \\ 0, n - even \end{cases}$$

Solution of Electrostatic Problems

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Method of separation of variables: for problems consisting of a system of conductors maintained at specified potentials and with no isolated free charges.

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