第三讲 复变函数的积分

Thursday, September 27, 2018 7:28 AM

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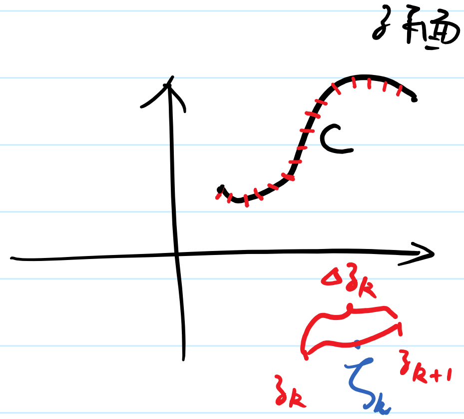


定义 C: $z = z(t)$, $\alpha \le t \le \beta$.

$\int_C f(z)dz$: 把 C 分割成 n 段.

$z_0, \cdots, z_n$

$z(t_0), \cdots, z(t_n)$, 记 $\Delta z_k = z_{k+1} - z_k$, $k = 0, \cdots, n-1$

$\zeta_k = z(\tau_k)$, $\tau_k \in (t_k, t_{k+1})$

如果 $\underbrace{\lim\limits_{\substack{n\to+\infty \\ \rho\to 0}} \sum\limits_{k=0}^{n-1} f(\zeta_k)\Delta z_k}$ 存在,

$|\Delta z_k| = \overset{\frown}{z_k z_{k+1}} = s_k \to 0$

$\rho = \max\limits_{0 \le k \le n-1} s_k$

则称该极限为 $f(z)$ 在 C 上的积分.

$$\int_C f(z)\,dz = s = \lim_{\substack{n\to\infty \\ \rho\to 0}} \sum_{k=0}^{n-1} f(\zeta_k)\Delta z_k$$

(?)

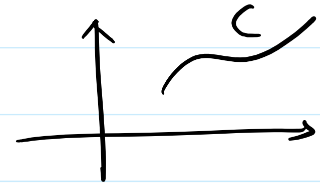
$$(\checkmark) = \lim_{\substack{n\to\infty \\ \rho\to 0}} \sum_{k=0}^{n-1} f(\zeta_k)\frac{z_{k+1}-z_k}{t_{k+1}-t_k}(t_{k+1}-t_k)$$

$$= \int_\alpha^\beta f(z(t))\, z'(t)\, dt$$

注: 1° C 封闭, $\oint_C f(z)dz$

2° C 是闭区间, $f(z)$ 是实函数, 则复积分退化为实积分.

Thm. $\underline{f(z)} = \underline{u(x,y)} + i\underline{v(x,y)}$



$$\int_C (u+iv)\,dz = \int_C (u\,dx - v\,dy) + i\left(\int_C v\,dx + u\,dy\right).$$

证明: $f(z) = u + iv$,

$\Delta z = \Delta x + i\Delta y$, $\zeta = \xi + i\eta$,

$$\int_C f(z)\,dz = \lim_{\substack{n\to\infty \\ \rho\to 0}} \sum_{k=0}^{n-1} f(\zeta_k)\Delta z_k = \sum_{k=0}^{n-1}\left[u(\xi_k,\eta_k) + i v(\xi_k,\eta_k)\right]\left[\Delta x_k + i\Delta y_k\right]$$

$$= \lim_{\substack{n\to\infty \\ \rho\to 0}} \sum_{k=0}^{n-1}\left[u(\xi_k,\eta_k)\Delta x_k - v(\xi_k,\eta_k)\Delta y_k\right] + i\left[u(\xi_k,\eta_k)\Delta y_k + v(\xi_k,\eta_k)\Delta x_k\right]$$

$$= \int_C u\,dx - v\,dy + i\left(\int_C u\,dy + v\,dx\right)$$

Thm $f(z)$在C上连续, C: $z(t)$, $t \in (\alpha, \beta)$

$$\int_C f(z)\,dz = \int_\alpha^\beta f(z(t))\,\underline{z'(t)\,dt}$$

证明: C: $z(t) = x(t) + iy(t)$, $t \in (\alpha, \beta)$.

$f(z) = u(x(t), y(t)) + i v(x(t), y(t))$

$$\int_C f(z)\,dz = \int_C u\,dx - v\,dy + i\int_C u\,dy + v\,dx$$

$$= \int_\alpha^\beta u(x(t),y(t))\,x'(t)\,dt - v(x(t),y(t))\,y'(t)\,dt$$

$$+ i\int_\alpha^\beta u(x(t),y(t))\,y'(t)\,dt + v(x(t),y(t))\,x'(t)\,dt$$

$$= \int_\alpha^\beta [u+iv][x'+iy']\,dt$$

$$= \int_\alpha^\beta f(z(t))\,z'(t)\,dt$$

复积分的性质：f, g 在 C 上可积

(1): $\int_C (\alpha f \pm \beta g) = \alpha\int_C f\,dz \pm \beta\int g\,dz$

(2): $\int_{C^-} f(z)\,dz = -\int_C f(z)\,dz$

(3): $C = C_1 + C_2$,

$$\int_{C_1+C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

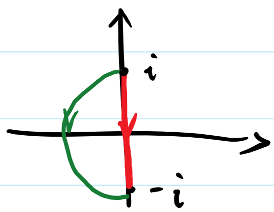
(4): $\left|\int_C f(z)\,dz\right| \le \int_C |f(z)|\,|dz|$

$$= \int_C |f(z)| \, ds$$

例：计算：$\int_C |z| \, dz$

C：(1)：$i \to -i$ 的直线段

(2)：$i \to -i$ 左半平面逆时针方向单位圆周.



解：(1)：C：$z(t) = it$，$t: 1 \to -1$.

$$\int_C f(z)\,dz = \int_1^{-1} |it| \, i \, dt = -2i\int_0^1 t\,dt$$

$$= -it^2\Big|_0^1 = -i$$

(2)：C：$z(t) = e^{it}$，$t: (\frac{\pi}{2}, \frac{3\pi}{2})$.

$$\int_C f(z)\,dz = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} |e^{it}| \, (e^{it})' \, dt$$

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 1 \, (ie^{it}) \, dt$$

$$= e^{it}\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = -2i$$

$$= e^{it}\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = -2i$$

例：计算：$\oint_C \frac{dz}{(z-z_0)^n}$，　$C: |z-z_0| = r > 0$. $(n\geq 1)$.

令 $z = z_0 + re^{i\theta}$, $\theta \in (0, 2\pi)$, $dz = rie^{i\theta}$

$(z-z_0)^n = (re^{i\theta})^n$

$$\oint_C \frac{dz}{(z-z_0)^n} = \int_0^{2\pi} \frac{rie^{i\theta}}{(re^{i\theta})^n} d\theta$$

$$= \frac{i}{r^{n-1}} \int_0^{2\pi} e^{i(1-n)\theta} d\theta$$

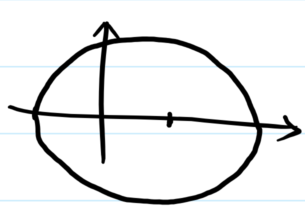
$$= \begin{cases} 2\pi i, & n = 1 \\ 0, & n \neq 1 \end{cases}$$

$$\oint_C \frac{1}{z-z_0} dz = 2\pi i,$$

例：证明：$\left| \oint_C \frac{z+1}{z-1} dz \right| \le 8\pi$.　$C: |z-1| = 2$.

证明：$\left| \oint \frac{z+1}{z-1} dz \right|$

$$\le \oint \frac{|z+1|}{|z-1|} |dz|$$



$|z-1| = 2$

$$\le \oint \frac{|z+1|}{|z-1|}|dz|$$

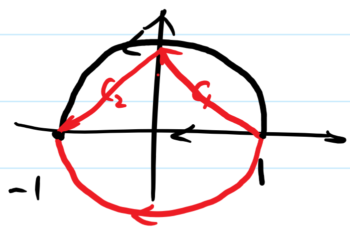
$$|z-1|=2$$

$$|z+1| = |z-1+2| \le |z-1|+2 \le 4$$

$$\le \oint \frac{4}{2}\, ds$$

$$\le 2\times \underline{2\pi\times 2} = 8\pi$$

例: $\int_C z^2\, dz$.



C_1: $1 \to -1$. (沿实轴)

C_2: $1 \to -1$. (上半单位圆周). $z = e^{i\theta}$, $\theta \in (0,\pi)$

$$\int_{C_1} z^2\, dz = \int_1^{-1} x^2\, dx = \frac{x^3}{3}\Big|_1^{-1} = \underline{-\frac{2}{3}}.$$

$$\int_{C_2} z^2\, dz = \int_0^{\pi} \left(e^{i\theta}\right)^2 i e^{i\theta}\, d\theta$$

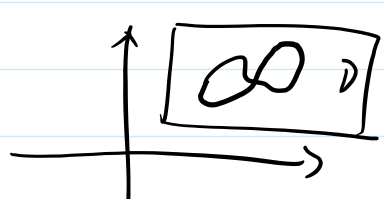
$$= i\int_0^{\pi} \underline{e^{i3\theta}\, d\theta} = i\times \frac{1}{3i} e^{i3\theta}\Big|_0^{\pi}$$

$$= \underline{-\frac{2}{3}}.$$

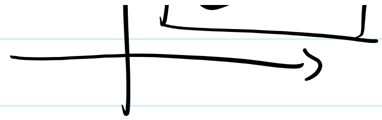
z^2的积分只与起、终点有关，与路径无关.

Thm: $f(z)$在单连通区域D内解析，C是D内简单闭曲线.

则: $\underline{\oint_C f(z) = 0}$



注: 1° C可以非简单.



注: 1° C 可以非简单.

2°: C 是D的边界: $\begin{cases} f(z)\text{在}D\text{内解析} \\ \text{但}f(z)\text{在}D\cup C\text{连续} \end{cases}$

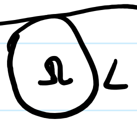
同样有: $\oint_C f(z)\,dz = 0$

证明: $f(z)$ 解析: $f(z) = u + iv$

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

$$\oint_C f(z)dz = \oint_C u\,dx - v\,dy + i\left(\oint_C u\,dy + v\,dx\right)$$

=



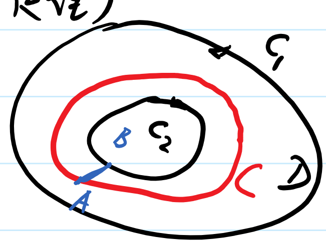
$$\boxed{\int_L P\,dx + Q\,dy = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy}$$ 格林公式.

$$= \iint\left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy + i\iint\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dx\,dy$$

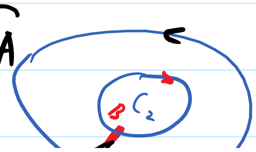
$$= 0 + 0 = 0$$

推广到多连通区域: (闭路变形定理)

$$\oint_C f(z) = \int_{C_2} f(z)\,dz.$$



证明: $\Gamma = C \cup \overset{\frown}{AB} \cup C_2^- \cup \overset{\frown}{BA}$

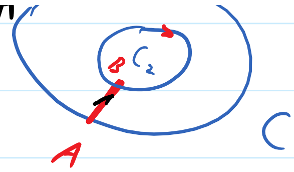


由Thm得

$\cup\, C_2^- \cup \widehat{BA}$

由Thm得

$$\oint_\Gamma f(z)\,dz = 0$$

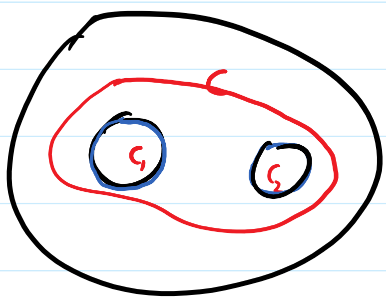


$$\oint_\Gamma = \int_{C\cup\widehat{AB}\cup C_2^-\cup\widehat{BA}} = \int_C + \cancel{\int_{\widehat{AB}}} + \int_{C_2^-} + \cancel{\int_{\widehat{BA}}}$$

$$\Rightarrow:\ \int_C + \int_{C_2^-} = 0$$

$$\Rightarrow:\ \int_C - \int_{C_2} = 0 \quad \Rightarrow:\ \int_C = \int_{C_2}$$

推广:



$$\int_C f(z)\,dz = \int_{C_1+C_2} f(z)\,dz$$

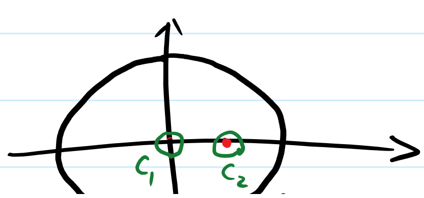
$$= \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

复合闭路定理.

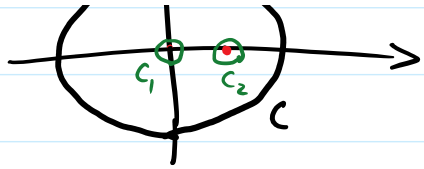
例: $\displaystyle\int_C \overset{f(z)}{\boxed{\frac{1}{z^2-z}}}\,dz,\quad C: |z|=2.$

$$= \frac{1}{z(z-1)}$$

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$



$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$



$$= \int_{C_1} \left(\frac{1}{z-1} - \frac{1}{z}\right)dz$$

C_1: $|z| = r$, $r \ll 1$.

C_2: $|z-1| = r$, $r \ll 1$.

$$+ \int_{C_2} \left(\frac{1}{z-1} - \frac{1}{z}\right)dz$$

$$= (0 - 2\pi i) + (2\pi i - 0) = 0$$

作业: P80:

1(1)(3), 2, 3, 7(2)(4)(6)

$$\int_{C_1 + C_2^-} f(z)\,dz = 0$$

$$\Downarrow$$

$$\left(\int_{C_1} - \int_{C_2}\right) f(z)\,dz \Rightarrow: \int_{C_1} = \int_{C_2}$$

$$F(z) = \int_{z_0}^{z} f(\xi)d\xi$$

称为 $f(z)$ 的原函数

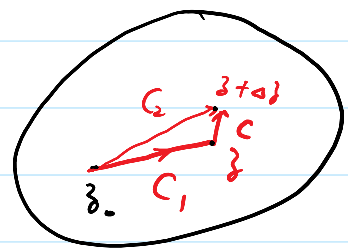
, $(F(z))' = f(z)$ (✓)

$$\int_{z_0}^{z_1} f(\xi)d\xi = F(z_1) - F(z_0) \quad (✓)$$

$$(F(z))' = \lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z}$$

$$F(z+\Delta z) = \int_{z_0}^{z+\Delta z} f(\xi)\,d\xi, \quad F(z) = \int_{z_0}^{z} f(\xi)\,d\xi.$$

$$\frac{1}{\Delta z}\left(\int_{C_2} f(\xi)\,d\xi - \int_{C_1} f(\xi)\,d\xi\right)$$



$$= \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(\xi)\,d\xi \xrightarrow{\Delta z \to 0} f(z)$$

$$\frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z)\,d\xi$$

$$\left| \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(\xi)\,d\xi - f(z) \right| = \left| \frac{1}{\Delta z}\int_{z}^{z+\Delta z} \big(f(\xi) - f(z)\big)\,d\xi \right|$$

$$\leq \frac{1}{|\Delta z|}\int_{C} |f(\xi) - f(z)|\,|d\xi|$$

f 解析，f 在 z 处连续，$\forall \varepsilon > 0, \exists \delta, |\xi - z| < \delta, |f(\xi) - f(z)| < \varepsilon$

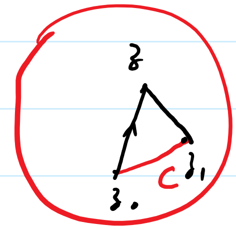
$$\leq \frac{1}{|\Delta z|} \cdot \varepsilon \int_{C} |d\xi|$$

$$\leq \varepsilon \to 0.$$

$$\Rightarrow: \lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} = f(z).$$

$\int^{z} f(\xi) \quad \int^{z} \ldots$

$$\int_{z_1}^{z} f(z) - \int_{z_0}^{z} f(z) = C.$$



(N-L公式)

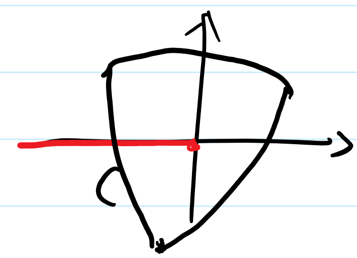
$f(z)$在D内解析, $F(z)$是$f(z)$的任一原函数.

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0).$$

P65.

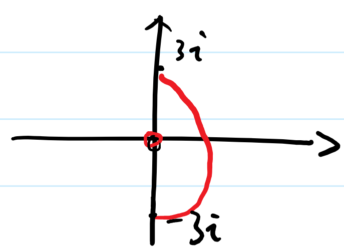
例: 在D内(单连通区域), $-\pi < \arg z < \pi$, 求 $f(z)=\frac{1}{z}$ 的原函数.

$$\int_{z_0}^{z} \frac{1}{z}\,dz = \ln z - \ln z_0$$



$\int_C \frac{1}{z^2}\,dz$, C: $|z|=3$, $\mathrm{Re}\,z>0$. 起点 $-3i$, 终点 $3i$.

$$= -\frac{1}{z}\Big|_{-3i}^{3i} = \frac{2}{3}i.$$



$\int \sqrt{z}\,dz$. 其中 $\sqrt{1}=-1$ 的那支.

$|z-1|=\frac{1}{2}$

单值分支上是解析的.

$\|$

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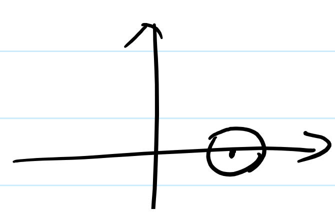

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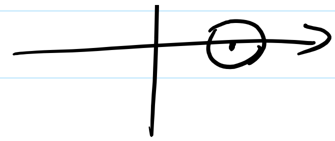
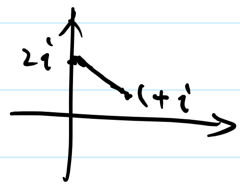


$\int_C \frac{1}{z}\,dz$. C:



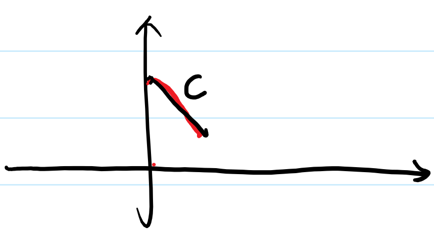
令 $z(t) = (1-t) + (1+t)i$. $t \in [0,1]$

$$\int_C \frac{1}{z}dz = \int_0^1 \frac{1}{(1-t)+(1+t)i}(-1+i)\,dt$$

$$= \int_0^1 \frac{d((i-1)t)}{(1+i)+(i-1)t} = \ln(1+i+(i-1)t)\Big|_0^1$$

$$= \ln 2i - \ln(1+i)$$

$$= \ln\frac{2i}{1+i} = \ln\frac{2i(1-i)}{2}$$

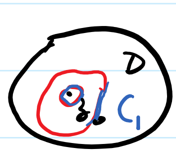


$$= \ln(1+i)$$

$$= \ln\sqrt{2} + i\frac{\pi}{4}$$

$$\int_{1+i}^{2i}\frac{1}{z}dz = \ln z\Big|_{1+i}^{2i} = \ln(2i) - \ln(1+i) = \ln\sqrt{2} + i\frac{\pi}{4}$$

Cauchy 积分公式.



$\int \left| f(z) \right|$.

$$\oint_C \boxed{\frac{f(z)}{z-z_0}} dz = ? \quad f(z) \text{ 解析}$$

$$\int_{C_1} \frac{f(z)}{z-z_0} dz = \int_{|z-z_0|=\delta} \frac{\boxed{f(z)}}{z-z_0} dz$$

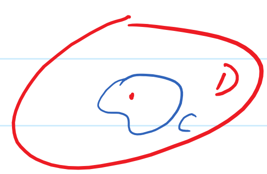
$$\approx f(z_0) \underbrace{\int_{|z-z_0|=\delta} \frac{1}{z-z_0} dz}$$

$$\approx 2\pi i f(z_0) \quad (\checkmark)$$

Thm: $f(z)$ 在 D 内处处解析，C 是任一闭合曲线

$$\int_C \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$$

或：$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - z} d\xi$



证明：

$f(z)$ 在z_0处解析 $\Rightarrow$ 连续：

$\forall \varepsilon > 0, \exists \delta$，$|z-z_0| < \delta$ 时，$|f(z)-f(z_0)| < \varepsilon$.

$$\int \frac{f(z)}{z-z_0} dz = \int \frac{f(z)}{\quad} dz$$

$$\underline{\int_C \frac{f(z)}{z-z_0}dz} = \int_{\underline{|z-z_0|=R<\delta}} \frac{f(z)}{z-z_0}dz$$

记为 K

$$= \underbrace{\int_K \frac{f(z)-f(z_0)}{z-z_0}dz}_{\downarrow\ 0} + \underbrace{\int_K \frac{f(z_0)}{z-z_0}dz}_{2\pi i f(z_0)}$$

$$\left|\int_K \frac{f(z)-f(z_0)}{z-z_0}dz\right| \le \int_K \frac{|f(z)-f(z_0)|}{\underline{|z-z_0|}}|dz|$$

$$\le \int_K \frac{\varepsilon}{R}ds$$

$$= \frac{\varepsilon}{\cancel{R}} \times 2\pi \cancel{R} = 2\pi\varepsilon \to 0$$

推论: C. $z = z_0 + Re^{i\theta}$. $dz = Re^{i\theta}\cdot i\,d\theta$

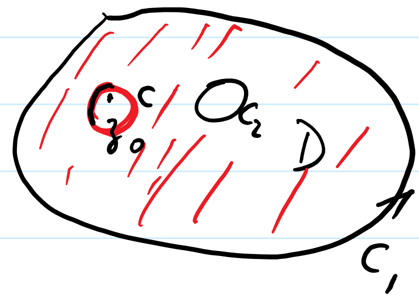
$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz = \frac{1}{2\pi \cancel{i}}\int_0^{2\pi} \frac{f(z_0+Re^{i\theta})}{\cancel{Re^{i\theta}}} \underline{\cancel{Re^{i\theta}}\cdot \cancel{i}\,d\theta}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(z_0+Re^{i\theta})\,d\theta$$

推广：

$f(z)$ 在 D内解析

$\frac{f(z)}{z-z_0}$



$\Gamma = C_1 \cup C_2^- \cup C^-$

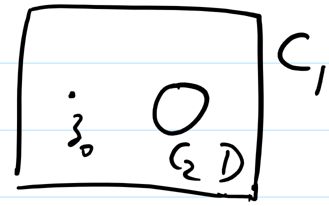
$$\int_\Gamma \frac{f(z)}{z-z_0} = 0$$

$$\Rightarrow \int_{C_1 \cup C_2^- \cup C^-} \boxed{\frac{f(z)}{z-z_0}} = 0$$

$$\Rightarrow \int_{C_1} - \int_{C_2} - \int_{C} \frac{f(z)}{z-z_0} = 0$$

$$\underbrace{\int_C \frac{f(z)}{z-z_0}}_{\parallel} = \left(\int_{C_1} - \int_{C_2}\right) \frac{f(z)}{z-z_0}\,dz$$

$2\pi i f(z_0)$



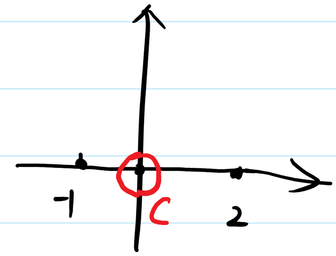
$$\Rightarrow f(z_0) = \frac{1}{2\pi i}\left(\int_{C_1} \frac{f(z)}{z-z_0}\,dz - \int_{C_2} \frac{f(z)}{z-z_0}\,dz\right)$$

$$\therefore \oint \cdots = 2\pi i \left(\int_{C_1} \frac{\cdots}{z-z_0} dz - \int_{C_2} \frac{\cdots}{z-z_0} dz \right)$$

例：计算积分 $I = \oint_C \frac{e^z}{z(z+1)(z-2)} dz$ 在 C： $|z| = r \quad (r \neq 0, 1, 2)$

1°： $0 < r < 1$，

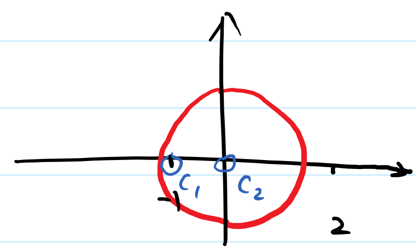
$$I = \oint \frac{\boxed{\frac{e^z}{(z+1)(z-2)}} = f(z)}{z-0} dz$$



$$= 2\pi i \times f(z)\Big|_{z=0}$$

$$= 2\pi i \times \frac{e^0}{(0+1)(0-2)} = -\pi i$$

2°： $1 < r < 2$



$$I = \oint_C \frac{\frac{e^z}{z-2}}{z(z+1)} dz$$

$$= \int_{C_1} \frac{\frac{e^z}{z-2}}{z(z+1)} dz + \int_{C_2} \frac{\frac{e^z}{z-2}}{z(z+1)} dz$$

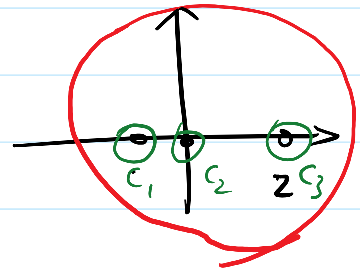
$$= \int_{C_1} \frac{\frac{e^z}{z(z-2)}}{z+1} dz + \int_{C_2} \frac{\frac{e^z}{(z+1)(z-2)}}{z} dz$$

$$= 2\pi i \left(\frac{e^z}{z(z-2)}\Big|_{z=-1} + \frac{e^z}{(z+1)(z-2)}\Big|_{z=0} \right)$$

2π .

$$= \frac{2\pi}{3e} i - \pi i$$

3°: $r > 2$,



$$I = \oint_{c_1+c_2+c_3} dz$$

$$= \oint_{c_1} + \oint_{c_2} + \oint_{c_3} \frac{\boxed{\frac{e^z}{z(z+1)}}}{z-2} dz$$

$$= \frac{2\pi}{3e} i - \pi i + 2\pi i \times \left. \frac{e^z}{z(z+1)} \right|_{z=2}$$

$$= \frac{2\pi i}{3e} - \pi i + \frac{e^2 \pi}{3} i$$

作业: 8 (2)(4), 9, 11