第四讲

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常系数线性微分方程的解法

$$\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_n(x)y = f(x)$$

$$L[y] = f(x)$$

$L[y]=0$, $y = \sum_{i=1}^{n} c_i y_i(x)$, $y_i(x)$ 线性无关

· 常系数齐次 （ $p_i(x) = p_i$, $f=0$ ）

以 $n=2$ 为例.

$$\frac{d^2y}{dx^2} + p_1\frac{dy}{dx} + p_2 y = 0$$

猜测 $y = e^{\lambda x}$, 则: $y'' = \lambda^2 e^{\lambda x}$

$y' = \lambda e^{\lambda x}$

$$\lambda^2 e^{\lambda x} + p_1 \lambda e^{\lambda x} + p_2 e^{\lambda x} = 0$$

即: $(\lambda^2 + p_1\lambda + p_2)e^{\lambda x} = 0$

$\Rightarrow$: $\lambda^2 + p_1\lambda + p_2 = 0$. ← 记为特征方程

(1): $p_1^2 - 4p_2 > 0$, 则 λ 有两个不同实根（记 λ_1, λ_2）

$$y_1 = e^{\lambda_1 x}, \quad y_2 = e^{\lambda_2 x}$$

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} \end{vmatrix}$$

$e^{(\lambda_1+\lambda_2)x}$

$$= e \quad (\lambda_2-\lambda_1) \neq 0$$

$$\therefore\ y(x) = C_1 y_1 + C_2 y_2$$
$$= C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

(2) $\because\ p_1^2 - 4p_2 = 0$. 则 λ 有重根. $\lambda_1 = \lambda_2$.

$$y_1 = e^{\lambda_1 x}, \quad y_2 = \boxed{e^{\lambda_2 x}}\, u(x)$$

$$y_2'' + p_1 y_2' + p_2 y = \left(\lambda_2^2 e^{\lambda_2 x} u(x) + 2\lambda_2 u'(x) e^{\lambda_2 x} + e^{\lambda_2 x} u''(x)\right)$$
$$+ p_1\left(\lambda_2 e^{\lambda_2 x} u(x) + e^{\lambda_2 x} u'(x)\right)$$
$$+ p_2 e^{\lambda_2 x} u(x)$$

$$= e^{\lambda_2 x}\left[\lambda_2^2 + 2\lambda_2 u' + u'' + p_1\lambda_2 + p_1 u' + p_2 u\right]$$

$$= e^{\lambda_2 x}\left[u'' + \underbrace{2\lambda_2 u' + p_1 u'}_{\substack{\| \\ (2\lambda_2+p_1)u' \\ \| \\ 0}}\right] = 0$$

(根据 λ_2 是特征方程的根
$\lambda_1 + \lambda_2 = 2\lambda_2 = -p_1$

$\Rightarrow$: $u'' \equiv 0$ $\Rightarrow$: $u(x) = C_1 x + C_2$

$\therefore\ y_2(x) = x e^{\lambda_2 x}$.

$y_1(x) = e^{\lambda_1 x}$ 与 $y_2(x)$ 线性无关.

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{\lambda_1 x} & x e^{\lambda_2 x} \\ \lambda_1 e^{\lambda_1 x} & e^{\lambda_2 x} + x\lambda_2 e^{\lambda_2 x} \end{vmatrix} \neq 0$$

$$\therefore\ y(x) = C_1 y_1 + C_2 y_2$$
$$= C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_2 x} \quad (\lambda_1 = \lambda_2).$$

(3). $p_1^2 - 4p_2 < 0$ (有2个共轭的复根) $\lambda_1 = \alpha + \beta i$, $\lambda_2 = \alpha - \beta i$

$$y_1 = e^{\lambda_1 x} = e^{(\alpha+\beta i)x} = e^{\alpha x} e^{i\beta x} = e^{\alpha x}(\cos\beta x + i\sin\beta x)$$
$$y_2 = e^{\lambda_2 x} = e^{(\alpha-\beta i)x} = e^{\alpha x} e^{-i\beta x} = e^{\alpha x}(\cos\beta x - i\sin\beta x)$$

Euler 式:
$e^{ix} = \cos x + i\sin x$

$$\therefore\ y(x) = C_1 y_1 + C_2 y_2$$

记为 $\tilde{C}_1$

$e^{\cdot} = \cos x + i\sin x$

$$\therefore\ y(x) = C_1 y_1 + C_2 y_2$$

$$= \boxed{\frac{C_1+C_2}{2}} e^{\alpha x}\cos\beta x \quad \text{记为 } \tilde{C}_1$$

$$+ \boxed{\frac{C_1-C_2}{2} i}\, e^{\alpha x}\sin\beta x \quad \text{记为 } \tilde{C}_2$$

$$= \tilde{C}_1 \underline{e^{\alpha x}\cos\beta x}$$

$$+ \tilde{C}_2 \underline{e^{\alpha x}\sin\beta x}$$

$$= \tilde{C}_1 \tilde{y}_1 + \tilde{C}_2 \tilde{y}_2$$

2° $a \neq 0$ $y = c\, e^{-3x} \cos \ldots + c\, e^{-3x} \ldots$

2°: $a \neq 0$, $y = c_1 e^{-3x}\cos ax + c_2 e^{-3x}\sin ax$

• n阶常系数齐次线性

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1}y}{dx^{n-1}} + p_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots + p_n y = 0$$

令 $y = e^{\lambda x}$, $\frac{d^n y}{dx^n} = \lambda^n e^{\lambda x}$,

$\vdots$

$\frac{d^k y}{dx^k} = \lambda^k e^{\lambda x}$

$\vdots$

特征方程: $\lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + \cdots + p_n = 0$

假设n个根, 分别是 $\{\lambda_i\}_{i=1}^{n}$.

$y_i(x) = e^{\lambda_i x}$

如果有重根, $\underbrace{\lambda_i \text{ 是 } s \text{ 重根}}$, (s个)

$$\left\{ x^j e^{\lambda_i x} \right\}_{j=0}^{s-1}$$

例: 求 $y^{(3)} - 3y^{(2)} + 4y = 0$ 的通解.

解: $\lambda^3 - 3\lambda^2 + 4 = 0$

$$\begin{array}{r} \lambda^2 - 4\lambda + 4 \\ (\lambda+1) \overline{)\ \lambda^3 - 3\lambda^2 + 4} \\ \lambda^3 + \lambda^2 \\ \hline \end{array}$$

$\lambda^3 - 3\lambda^2 + 4$

$= (\lambda+1)(\lambda^2 - 4\lambda + 4)$

$= (\lambda+1)(\lambda-2)^2$

$$\begin{array}{r} \lambda^3+\lambda^2 \\ \hline -4\lambda^2+4 \\ -4\lambda^2-4\lambda \\ \hline 4\lambda+4 \\ 4\lambda+4 \\ \hline 0 \end{array}$$

$= (\lambda+1)(\lambda-2)^2$

$\therefore \lambda_1 = -1$

$\lambda_2 = \lambda_3 = 2$.

$\therefore y = C_1 e^{-x} + C_2 e^{2x} + C_3 (x e^{2x})$

例：求 $y = C_1 e^x + C_2 \cos 2x + C_3 \sin 2x$ 满足的方程.（首项系数为1）

解：$\lambda_1 = 1$， $\lambda_{2,3} = 0 \pm 2i$

$(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3) = 0$

$\lambda^3 - \lambda^2 + 4\lambda - 4 = 0$

$\therefore \quad y^{(3)} - y^{(2)} + 4y' - 4y = 0$

• $f(x) \neq 0$.（非齐次方程）

常系数线性

$L[y] = f(x)$. 的通解结构：

$$y = \sum_{i=1}^{n} C_i y_i(x) + y^*(x)$$

其中 $L[y_i] = 0$，$L[y^*] = f(x)$

问题：如何求 $y^*(x)$?

$\boxed{\text{以 } n=2 \text{ 为例}}$ 如果 $f(x)$ 是特殊形式. $f(x) = P_m(x)\, e^{\mu x}$

$\boxed{\text{以 } n=2 \text{ 为例}}$ 如果 $f(x)$ 是特殊形式：$f(x) = P_m(x)e^{\mu x}$

$y''+p_1y'+q_1y=f(x)$ 则 y^* 可以由待定系数法求出.

假设 $y^*(x) = Q(x)e^{\mu x}$ 其中（$Q(x)$ 是多项式）

代入方程得到：$(y^*)'' = Q''e^{\mu x}+2Q'\mu e^{\mu x}+Q(x)\mu^2e^{\mu x}$

$(y^*)' = Q'e^{\mu x}+\mu Qe^{\mu x}$

$$[Q''+2Q'\mu+Q\mu^2]\cancel{e^{\mu x}}+p_1[Q'+\mu Q]\cancel{e^{\mu x}}+q_1Q\cancel{e^{\mu x}} = P_m(x)\cancel{e^{\mu x}}$$

整理后可得：$Q''+(2\mu+p_1)Q'+(\mu^2+\mu p_1+q_1)Q = P_m(x)$

1°：如果 $\mu^2+\mu p_1+q_1\neq 0$（μ 不是特征方程的根）

Q 为 m 次多项式. $Q(x) = R_m(x)$（共有 $m+1$ 个系数）

2°：如果 $\mu^2+\mu p_1+q_1=0$（且 μ 是特征方程的单根）

即：$2\mu+p_1\neq 0$

则：$Q(x)$ 为 $m+1$ 次多项式. $Q''(x)+(2\mu+p_1)Q' = P_m(x)$

$Q(x) = xR_m(x)$

3°：如果 $\mu^2+\mu p_1+q_1=0$ 且 $2\mu+p_1=0$，即 $Q''(x) = P_m(x)$

$Q(x) = x^2R_m(x)$

综合：$Q(x) = x^kR_m(x)$ 其中 k 代表 μ 是特征方程根的重数

例：求 $y''+y=(x-2)e^{3x}$ 的通解.

例：求 $y''+y=(x-2)e^{3x}$ 的通解.

$$\boxed{\begin{array}{lll} y''+y=0 & & \\ \downarrow & & \\ \lambda^2+1=0 & y_1(x)=\cos x & y=C_1\cos x+C_2\sin x \\ \lambda_{1,2}=\pm i & y_2(x)=\sin x. & \end{array}}$$

$y^*=R_1(x)e^{3x}=(a_0x+a_1)e^{3x}$ 代入原方程.

$(y^*)''=0+2a_0\times 3e^{3x}+(a_0x+a_1)\times 3^2e^{3x}.$

$(y^*)'=a_0e^{3x}+3(a_0x+a_1)e^{3x}.$

左边 $=(y^*)''+(y^*)'=(12a_0x+6a_0+9a_1+a_0+3a_1)\cancel{e^{3x}}=(x-2)\cancel{e^{3x}}$

$$\Rightarrow:\ \left.\begin{array}{l} 12a_0=1 \\ 7a_0+12a_1=-2 \end{array}\right\}\Rightarrow:\ \begin{array}{l} a_0=\frac{1}{12} \\ a_1=\ldots \end{array}$$

例：$y''+py'+qy=0$ 的通解：$y=(C_1+C_2x)e^x=C_1e^x+C_2xe^x$

求：$y''+py'+qy=x$ 的特解．且满足 $y(0)=2$，$y'(0)=0$.

解：$y^*=Q(x)e^{0x}=Q(x)$

由于1是特征方程的二重根，则0不是根.

$(\lambda-1)^2=0$

$\lambda^2-2\lambda+1=0$

即：$p=-2$，$q=1$

$y^*=R_1(x)=Ax+B$ 代入方程.

$0+p\times A+q(Ax+B)=x.$

即：$-2A+(Ax+B)=x\ \Rightarrow:\ \begin{array}{l} A=1 \\ B=2. \end{array}$

$\therefore y^* = x+2.$

$\therefore y(x) = C_1 y_1 + C_2 y_2 + y^* = C_1 e^x + C_2 x e^x + x + 2.$

由于 $y(0) = C_1 + 0 + 2 = 2$

$y'(0) = C_1 + (C_2 + 0) + 1 = 0$ $\Big\} \Rightarrow$: $C_1 = 0$, $C_2 = -1$.

$\therefore y(x) = -xe^x + x + 2$

(二): $f(x) = e^{\mu x} \times P_m(x) \cos\beta x = Re\left(e^{(\mu+\beta i)x} P_m(x)\right)$

或 $f(x) = e^{\mu x} Q_e(x) \sin\beta x$

或者 $f(x) = e^{\mu x}(P_m \cos\beta x + Q_e(x) \sin\beta x).$

作业: P_{114}: 10, 12, 15

P_{115}: 20, 24.