static magnetic fields

Introduction

- Electric force $\vec{F} = q\vec{E}(N)$
- Magnetic force $\vec{F}_m = q\vec{u} \times \vec{B}(N)$
- Elecmagnetic fore $\vec{F} = q(\vec{E} + \vec{u} \times \vec{B})$

(N) ~ Lorentz's force equation

6.2 Fundamental Postulates of Magnetostatics in Free Space

Free space

- Static Electric Field

$$\vec{\nabla} \cdot \vec{D} = \rho$$

$$\vec{\nabla} \times \vec{E} = 0$$

- Static Magnetic Field

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J}$$

$\vec{\nabla} \cdot \vec{J} = 0$ Steady current

$$\mu_o = 4\pi \times 10^{-7} (\text{Henry}/m)$$

Permeability of free space

6.2 Fundamental Postulates of Magnetostatics in Free Space

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \Rightarrow \qquad \oint_s \vec{B} \cdot d\vec{s} = 0$$

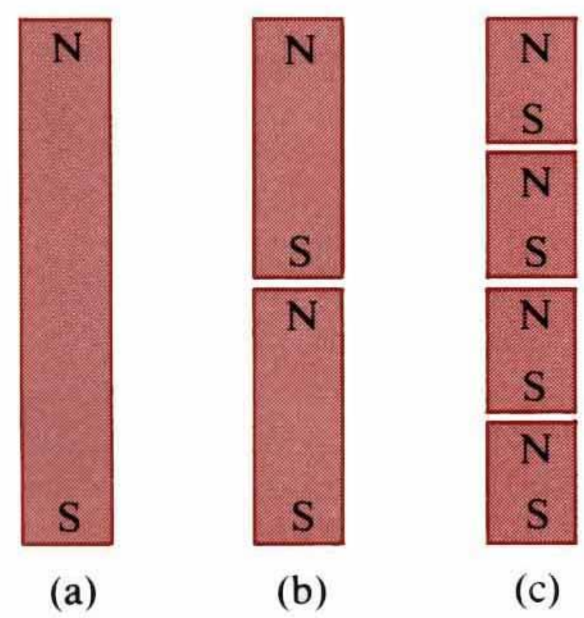


FIGURE 6–1
Successive division of a bar magnet.

- No magnetic flow sources
- Magnetic flux lines always close
- Low of conservation of magnetic flux

- Each magnets has a north pole south
- Magnetic poles cannot be isolated

6.2 Fundamental Postulates of Magnetostatics in Free Space

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J} \Rightarrow \int_s (\vec{\nabla} \times \vec{B}) \cdot d\vec{s} = \mu_o \int_s \vec{J} \cdot d\vec{s}$$

$$\oint_c \vec{B} \cdot d\vec{\ell} = \mu_o I$$ Ampere's circuital law

summary

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \oint_s \vec{B} \cdot d\vec{s} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J} \qquad \oint_c \vec{B} \cdot d\vec{\ell} = \mu_o I$$

6-3 Vector Magnetic Potential

$$\boxed{\vec{\nabla}\cdot\vec{B}=0} \Rightarrow \vec{B}=\vec{\nabla}\times\vec{A},$$ $\vec{A}$: magnetic potential [Vector]

c.f $$\boxed{\vec{\nabla}\times\vec{E}=0} \Rightarrow \vec{E}=-\vec{\nabla}\phi,$$ ϕ : electric potential [Scalar]

$$\vec{\nabla}\times\vec{B}=\mu_o\vec{J}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\vec{\nabla}^2\vec{A}=\mu_o\vec{J}$$

$$\boxed{\vec{\nabla}\cdot\vec{A}\;=0} \quad\Rightarrow\quad \boxed{\vec{\nabla}^2\vec{A}=-\mu_o\vec{J}}$$

Coulomb gauge

Vector

Poisson's equation

6-3 Vector Magnetic Potential

In Cartesian coordinates,

Vector

$$\begin{cases} \vec{\nabla}^2 A_x = -\mu_o J_x \\ \vec{\nabla}^2 A_y = -\mu_o J_y \\ \vec{\nabla}^2 A_z = -\mu_o J_z \end{cases} \Rightarrow A_x = \frac{\mu_o}{4\pi}\int_{u'} \frac{J_x}{r} du' \Rightarrow \boxed{\vec{A} = \frac{\mu_o}{4\pi}\int_{u'} \frac{\vec{J}}{r} du' (Wb/m)}$$

$$c.f. \qquad \vec{\nabla}^2 \phi = -\frac{\rho}{\varepsilon_o} \Rightarrow \phi = \frac{\mu_o}{4\pi}\int_{u'} \frac{\rho}{r} du'$$

6-3 Vector Magnetic Potential

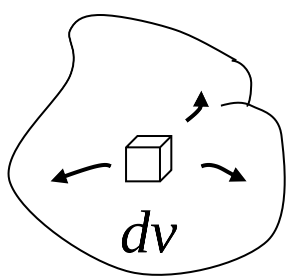
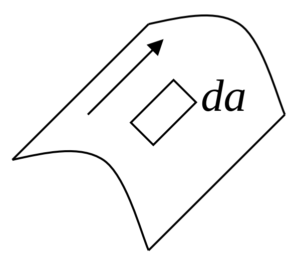
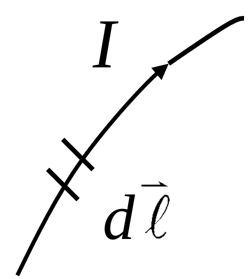
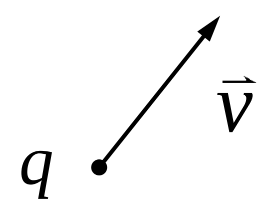
Magnetic Flux Φ through a given area S which is bounded by contour C

$$\Phi = \int_s \vec{B} \cdot d\vec{s} \qquad (Web)$$

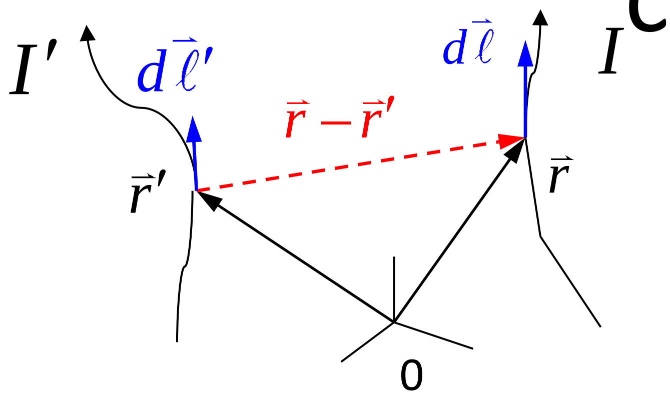
$$\Phi = \int_s (\vec{\nabla} \times \vec{A}) \cdot d\vec{s} = \oint_c \vec{A} \cdot d\vec{\ell} \qquad (Web)$$

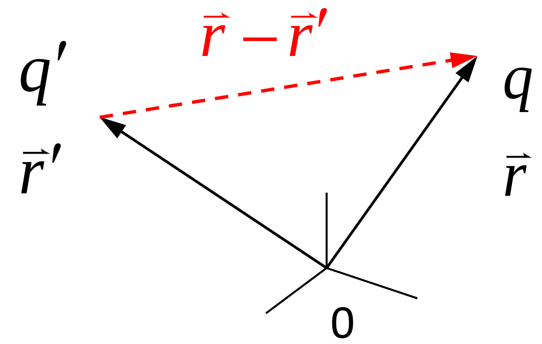
6-4 Biot-Savart Law and applications

Magnetic: $\boxed{\text{Vector source}}$

	3-dim	2-dim	1-dim	0-dim
Current distribution	Volume current density	Surface current density	current	
	$\vec{j}\left[\dfrac{coul}{\sec\cdot m^2}\right]$	$\vec{j}_s\left[\dfrac{coul}{\sec\cdot m}\right]$	$I\left[\dfrac{coul}{\sec}\right]$	$q[coul]$
Current element	$\vec{j}dv\left[\dfrac{coul}{\sec}\cdot m\right]$	$\vec{j}_s da\left[\dfrac{coul}{\sec}\cdot m\right]$	$Id\vec{\ell}\left[\dfrac{coul}{\sec}\cdot m\right]$	$q\vec{v}\left[\dfrac{coul}{\sec}\cdot m\right]$
				

Biot-Savart Law：[Valid in steady current]





$$d\vec{F}=\frac{\mu_o}{4\pi}Id\vec{\ell}\times\left[I'd\vec{\ell}'\times\frac{(\vec{r}-\vec{r}')}{\left|\vec{r}-\vec{r}'\right|^3}\right]$$

c.f.

$$\vec{F}_q=\frac{1}{4\pi\varepsilon_o}qq'\frac{\vec{r}-\vec{r}'}{\left|\vec{r}-\vec{r}'\right|^3}$$

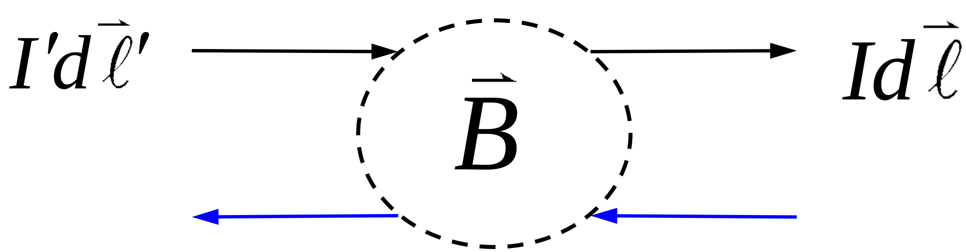
$$[\mu_o]\cdot v^2=\frac{1}{[\varepsilon_0]}$$

Biot-Sarvart Law：

$$d\vec{F} = Id\vec{\ell} \times d\vec{B}$$

$$\boxed{d\vec{B} = \frac{\mu_o}{4\pi} I' d\vec{\ell}' \times \frac{(\vec{r} - \vec{r}')}{\left|\vec{r} - \vec{r}'\right|^3}}$$

Action at a distance ：$\vec{B}$ field



$$\oint_s \vec{B}\cdot d\vec{a} = 0$$

Gauss thm.

$$\vec{\nabla}_{\vec{r}} \cdot \vec{B}(\vec{r}) = 0$$

特殊式

$$\vec{B} = \vec{\nabla}_{\vec{r}} \times \vec{A}(\vec{r})$$

$$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi}\int_{a.s.} \frac{\vec{j}(\vec{r}\,')}{\left|\vec{r}-\vec{r}\,'\right|} dv'$$

Biot-Sarvart Law

$$\vec{B}(\vec{r}) = \frac{\mu_o}{4\pi}\int_{a.s.} \vec{j}(\vec{r}\,')\times\frac{(\vec{r}-\vec{r}\,')}{\left|\vec{r}-\vec{r}\,'\right|^3} dv'$$

$$\vec{\nabla}^2_{\vec{r}}\vec{A}(\vec{r}) = -\mu_o \vec{j}$$

Poission Eqe.

$$\oint_p \vec{B}\cdot d\vec{\ell} = \mu_o \int_s \vec{j}\cdot d\vec{a}$$

Stoke Thm.

$$\vec{\nabla}_{\vec{r}} \times \vec{B}(\vec{r}) = \mu_o \vec{j}(\vec{r})$$

Ampere’s Law

$$\vec{B}(\vec{r})=\frac{\mu_o}{4\pi}\int_{a.s.}\vec{j}(\vec{r}')\times\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}dv'$$

$$(1)\vec{\nabla}_{\vec{r}}\cdot\vec{B}(\vec{r})=0$$

PF: $$\vec{\nabla}_{\vec{r}}\cdot\vec{B}=\frac{\mu_o}{4\pi}\vec{\nabla}_{\vec{r}}\cdot\left[\int_{a.s.}\vec{j}(\vec{r}')\times\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}dv'\right]$$

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B})=(\vec{\nabla}\times\vec{A})\cdot\vec{B}-(\vec{\nabla}\times\vec{B})\cdot\vec{A}$$

$$=\frac{\mu_o}{4\pi}\int_{a.s.}\vec{\nabla}_{\vec{r}}\cdot\left[\vec{j}(\vec{r}')\times\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\right]dv'$$

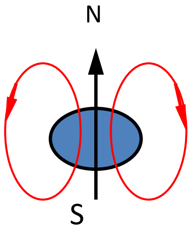
$$=\frac{\mu_o}{4\pi}\int_{a.s.}\left\{\underbrace{\left[\vec{\nabla}_{\vec{r}}\times\vec{j}(\vec{r}')\right]}_{=0}\cdot\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}-\left[\underbrace{\vec{\nabla}_{\vec{r}}\times\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}}_{=0}\right]\cdot\vec{j}(\vec{r}')\right\}dv'$$

$$=0$$

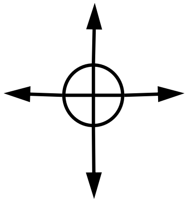
$$\vec{\nabla}\times\left(\frac{\vec{e}_r}{r^2}\right)=0$$

C.F

$\vec{\nabla}\cdot\vec{B}=0$ 封閉磁迴路



$\vec{\nabla}\cdot\vec{E}=\dfrac{\rho}{\varepsilon_o}$ 孤立電單極



From $\quad \vec{\nabla}_{\vec{r}} \cdot \vec{B}(\vec{r}) = 0$

$$(2)\vec{B} = \vec{\nabla}_{\vec{r}} \times \vec{A}(\vec{r}) \qquad \vec{A} = \frac{\mu_o}{4\pi}\int_{a.s.} \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}\,dv'$$

$$\vec{B} = \frac{\mu_o}{4\pi}\int_{a.s.} \vec{j}(\vec{r}') \times \frac{(\vec{r}-\vec{r}')}{\underline{|\vec{r}-\vec{r}'|^3}}\,dv'$$

$$= -\vec{\nabla}_{\vec{r}}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

$$= \frac{\mu_o}{4\pi}\int_{a.s.} \left\{ \vec{\nabla}_{\vec{r}} \times \left(\frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|} \right) - \frac{1}{|\vec{r}-\vec{r}'|} \cancelto{0}{\left(\vec{\nabla}_{\vec{r}} \times \vec{j}(\vec{r}')\right)} \right\} dv'$$

$$= \vec{\nabla}_{\vec{r}} \times \left\{ \frac{\mu_o}{4\pi}\int_{a.s.} \left(\frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|} \right) dv' \right\}$$

$$\boxed{\begin{aligned} &\vec{\nabla} \times (f\vec{A}) \\ &= (\vec{\nabla} f) \times \vec{A} + f(\vec{\nabla} \times \vec{A}) \end{aligned}}$$

$$= \vec{\nabla}_{\vec{r}} \times \vec{A}(\vec{r})$$

其中 $\quad \vec{A}(\vec{r}) = \dfrac{\mu_o}{4\pi}\displaystyle\int_{a.s.} \left(\frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|} \right) dv'$

From $\vec{\nabla}_{\vec{r}} \times \vec{A}(\vec{r}) = \vec{B}(\vec{r}) \Rightarrow \begin{cases} \vec{\nabla}^2 \vec{A}(\vec{r}) = -\mu_o \vec{j}(\vec{r}) \\ \vec{\nabla} \times \vec{B}(\vec{r}) = \mu_o \vec{j}(\vec{r}) \\ \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{A}(\vec{r}) \end{cases}$

Steady current $\vec{\nabla} \cdot \vec{j}(\vec{r}) = 0$

Static magnetic field

$c.f.$ $\vec{\nabla}^2 \phi = -\frac{\rho}{\varepsilon_o}$ $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_o}$

$$pf: \quad \vec{\nabla}_{\vec{r}} \times [\vec{\nabla}_{\vec{r}} \times \vec{A}(\vec{r})] = \vec{\nabla}_{\vec{r}}[\underline{\vec{\nabla}_{\vec{r}} \cdot \vec{A}(\vec{r})}] - (\vec{\nabla}_{\vec{r}} \cdot \vec{\nabla}_{\vec{r}})\vec{A}(\vec{r})$$

其中：$\underline{\vec{\nabla}_{\vec{r}} \cdot \vec{A}(\vec{r})} = \vec{\nabla}_{\vec{r}} \cdot \left\{ \frac{\mu_o}{4\pi} \int_{a.s.} \frac{\vec{j}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} dv' \right\} = \frac{\mu_o}{4\pi} \int_{a.s.} \underline{\vec{\nabla}_{\vec{r}} \cdot \left[\frac{\vec{j}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} \right]} dv'$

$$= \frac{\mu_o}{4\pi} \int_{a.s.} \left\{ \vec{j}(\vec{r}\,') \cdot \vec{\nabla}_{\vec{r}} \left(\frac{1}{|\vec{r} - \vec{r}\,'|} \right) + \frac{1}{|\vec{r} - \vec{r}\,'|} \underline{\left[\vec{\nabla}_{\vec{r}} \cdot \vec{j}(\vec{r}\,') \right]} \right\} dv'$$

Steady state $\vec{\nabla} \cdot \vec{j} = 0$

$$= \frac{\mu_o}{4\pi} \oint_{s \to \infty} \frac{\vec{j}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} \cdot d\vec{a}\,' = 0 \uparrow$$

$$\because \lim_{r \to \infty} \frac{1}{|\vec{r} - \vec{r}\,'|} = 0$$

$\vec{\nabla} \cdot \vec{A} = 0$ Coulomb Gauge

$$\vec{\nabla}_{\vec{r}} \cdot \left[\frac{\vec{j}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \right] = \vec{\nabla}_{\vec{r}} \left(\frac{1}{|\vec{r}-\vec{r}\,'|} \right) \cdot \vec{j}(\vec{r}\,') + \frac{1}{|\vec{r}-\vec{r}\,'|} \left[\vec{\nabla}_{\vec{r}} \cdot \vec{j}(\vec{r}\,') \right]$$

$$\vec{\nabla}_{\vec{r}} \times \vec{B}(\vec{r}) = -(\vec{\nabla}_{\vec{r}} \cdot \vec{\nabla}_{\vec{r}})\vec{A}(\vec{r}) = -\vec{\nabla}_{\vec{r}}^2 \vec{A}(\vec{r})$$

$$\vec{\nabla}_{\vec{r}}^2 \vec{A}(\vec{r}) = \vec{\nabla}_{\vec{r}}^2 \left\{ \frac{\mu_o}{4\pi} \int_{a.s.} \frac{\vec{j}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} dv' \right\} = \frac{\mu_o}{4\pi} \int_{a.s.} \vec{\nabla}_{\vec{r}}^2 \left(\frac{\vec{j}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \right) dv'$$

$$= -\mu_o \vec{j}(\vec{r})$$

$$= \vec{j}(\vec{r}\,') \vec{\nabla}_{\vec{r}}^2 \left(\frac{1}{|\vec{r}-\vec{r}\,'|} \right)$$

$$= \vec{\nabla}^2 \left(\frac{1}{r} \right) = -4\pi\delta^3(\vec{r})$$

$$\vec{\nabla}^2 \left(\frac{1}{|\vec{r}-\vec{r}\,'|} \right) = -4\pi\delta^3(\vec{r}-\vec{r}\,')$$

$\vec{\nabla}_{\vec{r}} \times \vec{B}(\vec{r}) = \mu_o \vec{j}(\vec{r})$: Ampere' Law

$\vec{\nabla}_{\vec{r}}^2 \vec{A}(\vec{r}) = -\mu_o \vec{j}(\vec{r})$: Poission Equ.

$$\oint_s \vec{B}(\vec{r})\cdot d\vec{a}=0$$

$$pf\ :\vec{\nabla}\cdot\vec{B}(\vec{r})=0$$

Gauss $\int_v \vec{\nabla}\cdot\vec{B}(\vec{r})dv=0$

Thm.

$$\oint_s \vec{B}(\vec{r})\cdot d\vec{a}=0$$

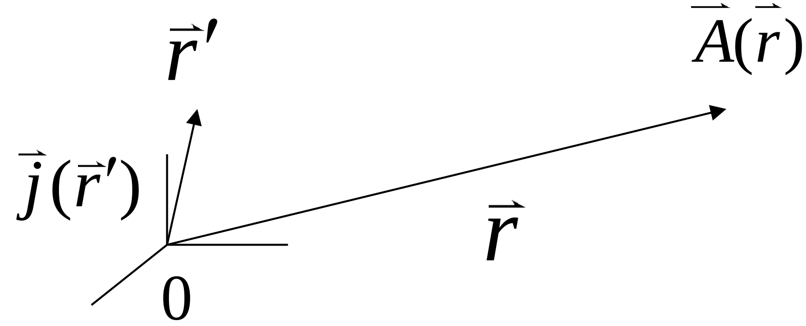
$$\oint_C \vec{B}(\vec{r})\cdot d\vec{\ell}=\mu_o\int_s \vec{j}(\vec{r})\cdot d\vec{a}$$

$$pf\ :\vec{\nabla}\cdot\vec{B}(\vec{r})=\mu_o\,\vec{j}(\vec{r})$$

$$\int_s[\vec{\nabla}\cdot\vec{B}(\vec{r})]\cdot d\vec{a}=\mu_o\int\vec{j}(\vec{r})\cdot d\vec{a}$$

$$\oint_C \vec{B}(\vec{r})\cdot d\vec{\ell}=\mu_o\int\vec{j}(\vec{r})\cdot d\vec{a}$$

6-5 Magnetic Dipole



$$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi}\int_v \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}dv'$$

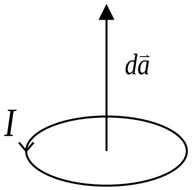
$$= \underbrace{\frac{\mu_o}{4\pi}\frac{\int_v \vec{j}(\vec{r}')dv'}{r}}_{2^0 \text{ pole } = 0} + \underbrace{\frac{\mu_o}{4\pi}\frac{\int_v \vec{j}(\vec{r}')(\vec{r}'\cdot\hat{a}_r')dv'}{r^2}}_{2' \text{ pole } = 0} + \ldots$$

$$\nabla\cdot\vec{j}(\vec{r}) = 0$$

$$\int_v \vec{j}(\vec{r}')(\vec{r}'\cdot\hat{a}_r')dv' \equiv \vec{m}\times\hat{a}_r$$

$$\vec{m} = \frac{1}{2}\int_v \vec{r}'\times\vec{j}(\vec{r}')dv'$$

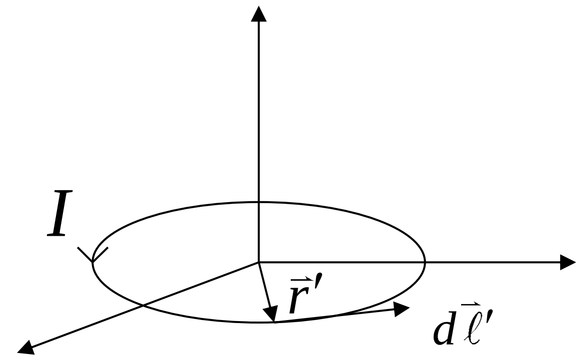
$$\vec{m} = Id\vec{a}$$



$$\vec{m} = \frac{1}{2}\int_{dp} \vec{r}\,' \times \vec{j}(\vec{r}\,')dv'$$

$$= \frac{1}{2}\int_{dp} \vec{r}\,' \times (Id\vec{\ell}') = I\,\frac{1}{2}\int_{dp} \vec{r}\,' \times d\vec{\ell}'$$

$$= Id\vec{a}$$

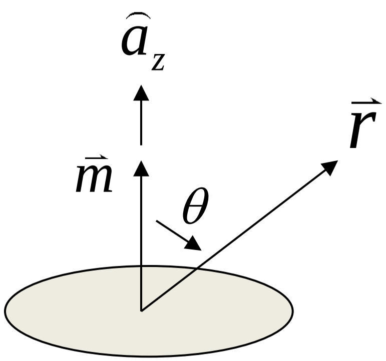


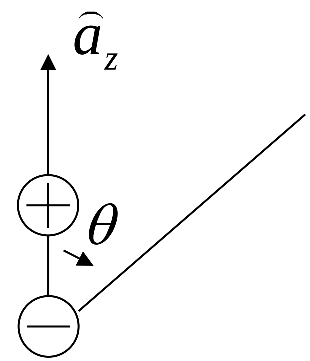
$$d\vec{a} = \frac{1}{2}\vec{r}' \times d\vec{\ell}'$$

$$\vec{A} = \frac{\mu_o}{4\pi}\frac{\vec{m}\times\hat{a}_r}{r^2} = \frac{\mu_o}{4\pi}\frac{m\sin\theta}{r^2}\hat{a}_\varphi$$

c.f.

$$\phi = \frac{1}{4\pi\varepsilon_o}\frac{\vec{P}\cdot\vec{a}_r}{r^2} = \frac{1}{4\pi\varepsilon_o}\frac{P\cos\theta}{r^2}$$





$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{\mu_o}{4\pi} m \frac{1}{r^2 \sin\theta} \begin{vmatrix} \widehat{a}_r & r\widehat{a}_\theta & r\sin\theta\widehat{a}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ 0 & 0 & r\sin\theta\frac{\sin\theta}{r^2} \end{vmatrix}$$

$$= \frac{\mu_o}{4\pi} m \frac{1}{r^2 \sin\theta} \left[\widehat{a}_r \frac{\partial}{\partial \theta}(\frac{\sin^2\theta}{r}) - r\widehat{a}_\theta \frac{\partial}{\partial r}(\frac{\sin^2\theta}{r}) \right]$$

$$= \frac{\mu_o}{4\pi} m \frac{\widehat{a}_r 2\cos\theta + \widehat{a}_\theta \sin\theta}{r^3}$$

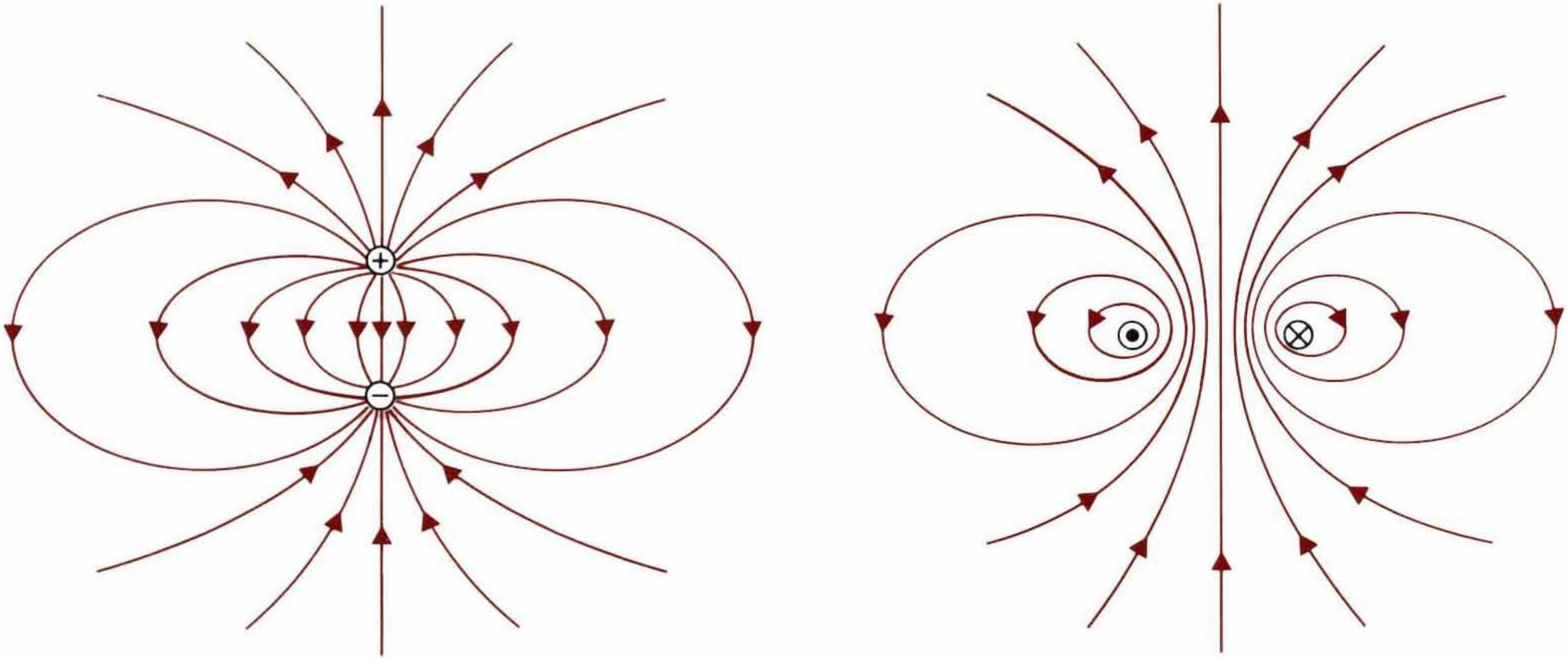
$$\frac{\partial}{\partial \theta}(\frac{\sin^2\theta}{r}) = \frac{2\sin\theta\cos\theta}{r}$$

$$\frac{\partial}{\partial r}(\frac{\sin^2\theta}{r}) = -\frac{\sin^2\theta}{r^2}$$

c.f.

$$\vec{E} = -\vec{\nabla}\phi = \frac{1}{4\pi\varepsilon_o}(-1)P\left[\widehat{a}_r \frac{\partial}{\partial r}(\frac{\cos\theta}{r^2}) + \widehat{a}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}(\frac{\cos\theta}{r^2}) \right]$$

$$= \frac{P}{4\pi\varepsilon_o}\left[\widehat{a}_r \frac{2\cos\theta}{r^3} + \widehat{a}_\theta \frac{\sin\theta}{r^3} \right]$$



(a) Electric dipole.

(b) Magnetic dipole.

FIGURE 6–9
Electric field lines of an electric dipole and magnetic flux lines of a magnetic dipole.

Scalar Magnetic Potential

$$\vec{\nabla}\times\vec{B}=\mu_o\vec{J} \qquad if \quad \vec{J}=0 \quad \vec{\nabla}\times\vec{B}=0$$

$$\vec{B}=-\mu_o\vec{\nabla}\phi_m, \qquad \phi_m$$: Scalar Magnetic Potential

$$\phi_{m2}-\phi_{m1}=-\int_{p_1}^{p_2}\frac{1}{\mu_o}\vec{B}\cdot d\vec{\ell}\Rightarrow\phi_m=\frac{1}{4\pi}\int_{v'}\frac{\rho_m}{r}dv'$$

(not physical)

$$\vec{m}=q_m\vec{d}=\hat{a}_n IS$$

$$\phi_m=\frac{\vec{m}\cdot\hat{a}_r}{4\pi r^2}$$

$$if \quad \vec{J}\neq 0, \qquad \vec{B}$$: Non conservative (path dependent)

6-6 Magnetization and Equivalent Current Density

	Components	source
Conductor	free electron	$\vec{j}_f = \sigma \vec{E}$
Non-conductor	polarized ion	$\vec{j}_m = \vec{\nabla} \times \vec{M}$

$$\overrightarrow{M} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} m_k}{\Delta v} \left(A/m \right)$$

$$\vec{\mathrm{A}}(\vec{\mathrm{r}}) = \frac{\mu_o}{4\pi} \int_v \overrightarrow{M}(\vec{r}') \times \underbrace{\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}}_{= \vec{\nabla}_r \left(\frac{-1}{|\vec{r}-\vec{r}'|}\right) = \vec{\nabla}_{r'} \left(\frac{1}{|\vec{r}-\vec{r}'|}\right)} dv'$$

$$\overrightarrow{M}(\vec{r}') \times \vec{\nabla}_{r'} \left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

$$= \vec{\nabla}_{r'} \times \left[\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) (-\overrightarrow{M}(\vec{r}')) \right] + \frac{1}{|\vec{r}-\vec{r}'|} \left[\vec{\nabla}_{r'} \times \overrightarrow{M}(\vec{r}') \right]$$

$$\vec{\nabla} \times (f \vec{A}) = \vec{\nabla} f \times \vec{A} + f (\vec{\nabla} \times \vec{A})$$

$$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi}\int_{v'} \vec{\nabla}_{\vec{r}'} \times \left[\frac{-\overrightarrow{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right] dv' + \frac{\mu_o}{4\pi}\int_{v'} \frac{\vec{\nabla}_{\vec{r}'} \times \overrightarrow{M}(\vec{r}')}{|\vec{r}-\vec{r}'|} dv'$$

$$= \frac{\mu_o}{4\pi}\oint_{S'} da' \times \left[\frac{-\overrightarrow{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right] + \frac{\mu_o}{4\pi}\int_{v'} \frac{\vec{\nabla}_{\vec{r}'} \times \overrightarrow{M}(\vec{r}')}{|\vec{r}-\vec{r}'|} dv'$$

$$= \frac{\mu_o}{4\pi}\oint_{S'} da' \frac{\overrightarrow{M}(\vec{r}') \times \vec{n}'}{|\vec{r}-\vec{r}'|} + \frac{\mu_o}{4\pi}\int_{v'} \frac{\vec{\nabla}_{\vec{r}'} \times \overrightarrow{M}(\vec{r}')}{|\vec{r}-\vec{r}'|} dv'$$

$$= \frac{\mu_o}{4\pi}\oint_{S'} da' \frac{\vec{j}_{ms}}{|\vec{r}-\vec{r}'|} + \frac{\mu_o}{4\pi}\int_{v'} \frac{\vec{j}_m}{|\vec{r}-\vec{r}'|} dv'$$

$$\left[\begin{array}{l} \vec{j}_m = \vec{\nabla} \times \overrightarrow{M} \left({}^{A}\!/\!{}_{m^2} \right) \\ \vec{j}_{ms} = \overrightarrow{M} \times \widehat{a}_n \left({}^{A}\!/\!{}_{m} \right) \end{array}\right]$$

c.f. $\rho_p = -\vec{\nabla} \cdot \vec{\mathrm{P}}; \quad \rho_{sp} = \widehat{a}_n \cdot \vec{\mathrm{P}}$

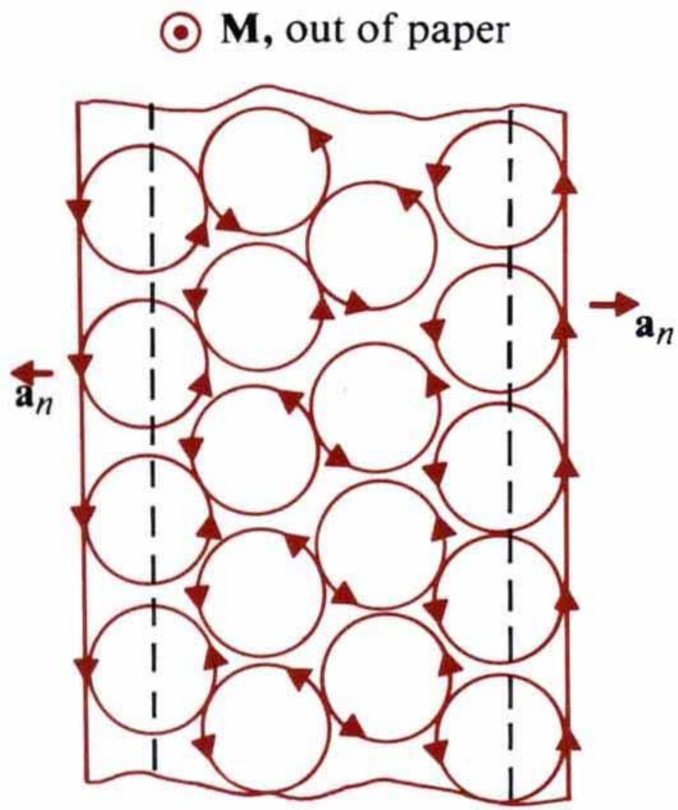


FIGURE 6–10
A cross section of a magnetized material.

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_o(\vec{j}_f + \vec{j}_m)$$

$$= \mu_o(\vec{j}_f + \vec{\nabla} \times \vec{M})$$

$$\vec{\nabla} \times \left[\frac{1}{\mu_o} \vec{B} - \vec{M} \right] = \vec{j}_f$$

$$\vec{\nabla} \times \vec{H} = \vec{j}_f$$

$$\therefore \vec{H} = \frac{1}{\mu_o} \vec{B} - \vec{M} = \frac{1}{\mu_o \mu_r} \vec{B}$$

C.F.

$$\vec{\nabla} \times \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_o}(\rho_f + \rho_p)$$

$$= \frac{1}{\varepsilon_o}(\rho_f - \vec{\nabla} \cdot \vec{P})$$

$$\vec{\nabla} \cdot \left[\varepsilon_o \vec{E} + \vec{P} \right] = \rho_f$$

$$\vec{\nabla} \cdot \vec{D} = \rho_f$$

$$\vec{D} = \varepsilon_o \vec{E} + \vec{P}$$

Static Magnetic

Source : $\vec{j}_f \,, \qquad \mu$ (permeativity)

$$\downarrow \; \vec{\nabla} \times \vec{H} = \vec{j}_f$$

Conductor : $\vec{H} \;\to\; \vec{B} \;\to\; \Phi_m \;\to\; L$

$$\vec{B} = \mu \vec{H} \qquad \Phi_m = \int \vec{B} \cdot d\vec{s} \qquad \frac{1}{L} = \frac{I_f}{\Phi_m}$$

Magnetic material :

$$\begin{tikzcd}
\vec{j}_m & & \vec{j}_{ms} \\
& \vec{M} \arrow[ul, "\vec{\nabla} \times \vec{M}"'] \arrow[ur, "\vec{M} \times \vec{a}_n"] \arrow[dl, "-\vec{\nabla} \cdot \vec{M}"] \arrow[dr, "\vec{M} \cdot \vec{a}_n"'] & \\
\rho_m & & \rho_{ms}
\end{tikzcd}$$

$$d\phi_m = \frac{\vec{M} \cdot \hat{a}_r}{4\pi r^2}$$

$$\phi_m = \frac{1}{4\pi} \int_{v'} \frac{\vec{M} \cdot \hat{a}_r}{r^2} dv'$$

$$= \frac{1}{4\pi} \oint_{s'} \frac{\vec{M} \cdot \hat{a}_n}{r} ds' + \frac{1}{4\pi} \int_{v'} \frac{-\left(\vec{\nabla} \times \vec{M}\right)}{r} dv'$$

$$\rho_{ms} = \vec{M} \cdot \hat{a}_n \;;\; \rho_m = -\vec{\nabla} \cdot \vec{M}$$

◎ $\vec{j}_{ms}$, $\vec{j}_m$, $\vec{A} = \frac{\mu_0}{4\pi}\left[\oint_{s'} \frac{\vec{j}_{ms}}{\left|\vec{r} - \vec{r}\,'\right|} da' + \int_{v'} \frac{\vec{j}_m}{\left|\vec{r} - \vec{r}\,'\right|} dv'\right]$, $\vec{B} = \vec{\nabla} \times \vec{A}$

◎ ρ_{ms} , ρ_m , $\phi_m = \frac{1}{4\pi}\left[\oint_{s'} \frac{\rho_{ms}}{\left|\vec{r} - \vec{r}\,'\right|} da' + \int_{v'} \frac{\rho_m}{\left|\vec{r} - \vec{r}\,'\right|} dv'\right]$, $\vec{H} = -\vec{\nabla}\phi_m \Rightarrow \vec{B} = \frac{1}{\mu}\vec{H}$

6-7 Magnetic Field Intensity and Relative Permeability

$$\vec{\nabla}\times\vec{H}=\vec{J}_f$$

$$\left(\vec{H}=\frac{B}{\mu_0}-\vec{M}\right)$$

$$\int_s\left(\vec{\nabla}\times\vec{H}\right)\cdot d\vec{S}=\int_s\vec{J}\cdot d\vec{S}$$

$$\oint\vec{H}\cdot d\ell=I$$

Ampere's circuital law

$$\vec{B}=\mu_0\left(1+\chi_m\right)\vec{H}$$

$$=\mu_0\mu_r\vec{H}$$

$$\boxed{\vec{H}=\frac{1}{\mu}\vec{B}}\quad ;\quad \boxed{u_r=1+\chi_m=\frac{\mu}{\mu_0}}$$

relative permeability

Electrostatics	Magnetostatics
E	**B**
D	**H**
ϵ	$\frac{1}{\mu}$
P	$-$**M**
ρ	**J**
V	**A**
$\cdot$	$\times$
$\times$	$\cdot$

6-8 Magnetic Circuits

Electric circuit : Voltage / Current source ; V, I, …

Magnetic circuit : Transformer / Generator / Motor …

$\vec{\nabla} \cdot \vec{B} = 0$

$\vec{\nabla} \times \vec{H} = \vec{J}$; closed path c to enclose N turns of I

$\oint_c \vec{H} \cdot d\vec{l} = NI = V_m$ (m.m.f) magnetomotive force [Amp]

Magnetic Flux $\Phi \approx B_f S$; S : cross-section

$$B_f = \frac{\mu_0 \mu N I_o}{\mu_0 \left(2\pi r_o - l_g\right) + \mu l_g} = \frac{N I_o}{\left(\frac{2\pi r_o - l_g}{\mu}\right) + \frac{l_g}{\mu_o}}$$

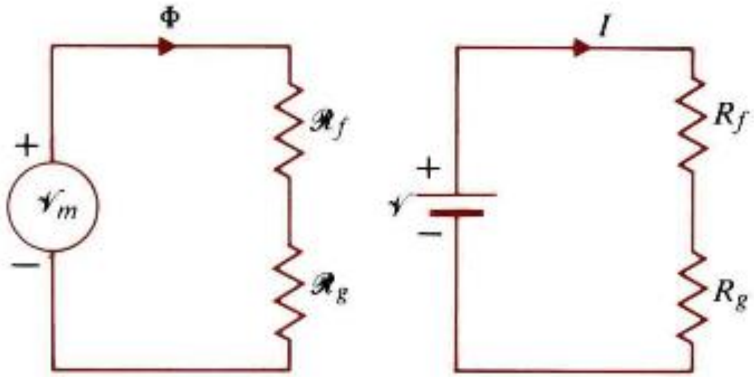
$$\Phi = B_f \cdot S = \frac{N I_o}{\left(\frac{2\pi r_o - l_g}{\mu S}\right) + \frac{l_g}{\mu_o S}} = \frac{V_m}{R_f + R_g}$$

$R_f = \frac{2\pi r_o - l_g}{\mu S} = \frac{l_f}{\mu S}; l_f = 2\pi r_o - l_g$: length of ferromagnetic core.

$R_g = \frac{l_g}{\mu_o S}$: Reluctance $\begin{cases} R_f & \text{:ferromagnetic core} \\ R_g & \text{:air gap} \end{cases}$

Analog to : [Electric circuit]

$$I = \frac{v}{R_f + R_g}$$



(a) Magnetic circuit. (b) Electric circuit.

FIGURE 6–14
Equivalent magnetic circuit and analogous electric circuit for toroidal coil with air gap in Fig. 6–13.

Magnetic circuit

$$\Phi = \frac{V_m}{R_f + R_g}; R = \frac{l}{\mu S}$$

mmf $V_m(= NI)$

mag. flux Φ

reluctance R

Permeability μ

Electric circuit

$$I = \frac{v}{R_f + R_g}; R = \frac{l}{\sigma S}$$

emf V

electric current , I

resistance , R

conductivity , σ

An exact analysis of magnetic circuits is difficult

◎Leakage Fluxes ◎Fringing effect ◎ $\overline{\mathrm{B}} = \mu(\vec{\mathrm{B}},\vec{\mathrm{H}})\vec{\mathrm{H}}$

2 conditions must be satisfied

$$\left.\begin{aligned} &\mathrm{H_g l_g} + \mathrm{H_f l_f} = \mathrm{NI_o} \\ &\mathrm{B_f} = \mathrm{B_g} = \mu_o \mathrm{H_g} \end{aligned}\right\} \Rightarrow \mathrm{B_f} + \mu_o \frac{\mathrm{l_f}}{\mathrm{l_g}} \mathrm{H_f} = \frac{\mu_o}{\mathrm{l_g}} \mathrm{NI_o}$$

Similar to

Kirchhoff’s voltage Law

$$\sum_j \mathrm{N_j I_j} = \sum_k \mathrm{R_k} \Phi_k$$

Kirchhoff’s current Law

$$\sum_j \Phi_j = 0 \qquad \vec{\nabla} \bullet \vec{\mathrm{B}} = 0$$

6-9 Behavior of Magnetic Materials

$$\vec{M} = \chi_m \vec{H}\,,\ \chi_m : \text{magnetic susceptibility}$$

$$\vec{H} = \frac{1}{\mu}\vec{B}\,,\ \mu_r = 1 + \chi_m = \frac{\mu}{\mu_o}$$

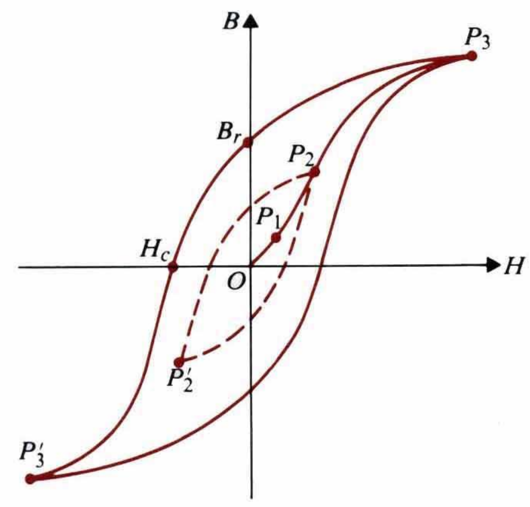


FIGURE 6–17
Hysteresis loops in the B–H plane for ferromagnetic material.

◎Diamagetic: $\mu_r \le 1$ (χ_m : small negative number)

◎Paramagnetic: $\mu_r \ge 1$ (χ_m : small positive number)

◎Ferromagnetic: $\mu_r >> 1$ (χ_m : large positive number)

6-10 Boundary Conditions for Magnetostatic Field

$$B_{1n} = B_{2n} \Longrightarrow \mu_1 H_{1n} = \mu_2 H_{2n}$$

$$a_{n2} \times \left(H_1 - H_2 \right) = J_s$$

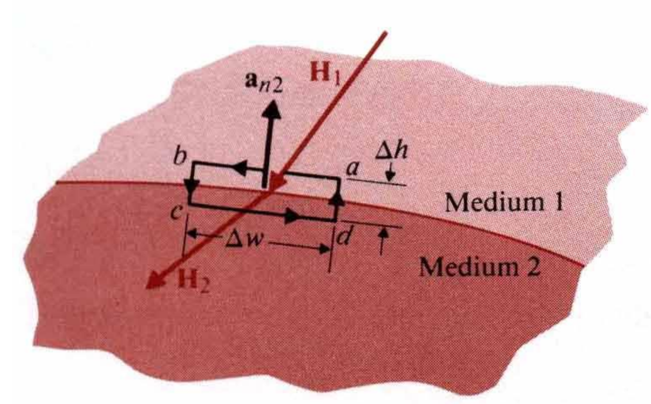


FIGURE 6–19
Closed path about the interface of two media for determining the boundary condition of H_t.

6-11 Inductances & Inductors

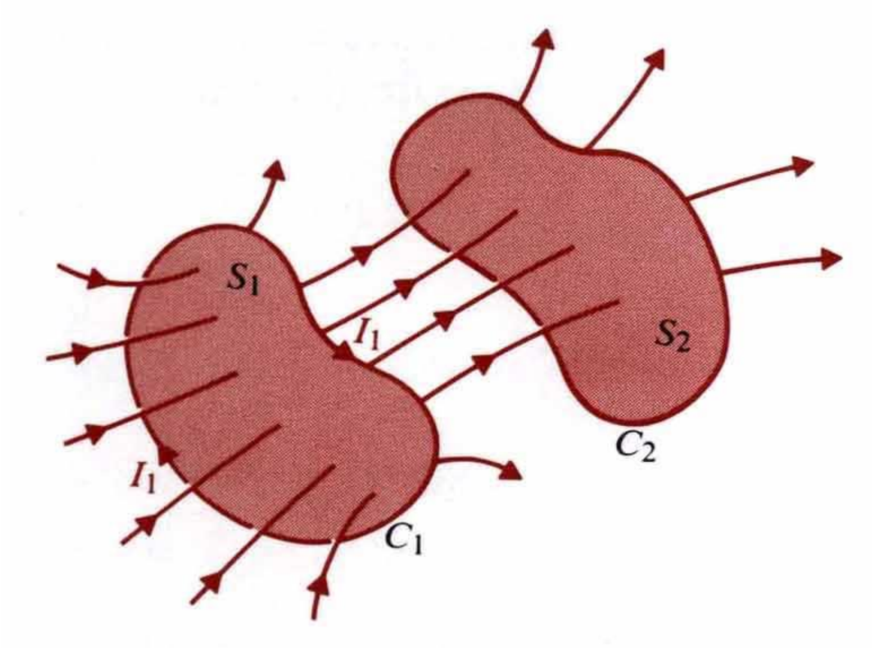


FIGURE 6–22
Two magnetically coupled loops.

Mutual flux $\Phi_{12} = \int_{S_2} \vec{B}_1 \cdot d\vec{S}_2$

$$\Phi_{12} = L_{12} I_1$$

L_{12} : mutual inductance

between loops C_1 and C_2

If loop C_2 has N_2 turns ,

$$\Lambda_{12} = N_2 \Phi_{12}$$

Generalizes to

$$\Lambda_{12} = L_{12} I_1$$

$$\boxed{L_{12} = \frac{\Lambda_{12}}{I_1}} \Rightarrow \boxed{L_{12} = \frac{d\Lambda_{12}}{dI_1} \ (H)}$$

Some of $\vec{B}$ produced by I_1 links only with C_1 loop itself, not with C_2

$$\Lambda_{11} = N_1 \Phi_{11} > N_1 \Phi_{12}$$

Self inductance of C_1 loop

$$\boxed{L_{11} = \frac{\Lambda_{11}}{I_1}} \Longrightarrow \boxed{L_{11} = \frac{d\Lambda_{11}}{dI_1}}$$

Procedure for Finding Inductance

1. Appropriate coordinate system

2. Find

3. $$\vec{B} = \frac{\mu_0}{4\pi} \int_{v'} \vec{J}(\vec{r}') \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} dv'$$

$$\Phi = \int_S \vec{B} \cdot d\vec{S}$$

4. $$\Lambda = N\Phi$$

5. $$L = \frac{\Lambda}{I}$$

6-12 Magnetic Energy

Loop 1 $V_1 = L_1 \frac{di_1}{dt}$

$$W_1 = \int V_1 \; i_1 dt$$

$$= L_1 \int_0^{I_1} i_1 di_1$$

$$= \frac{1}{2} L_1 I_1^2 = \frac{1}{2} \Phi_1 L_1$$

Loop 2 : C_1 & C_2

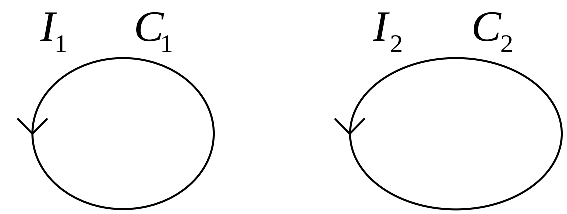
$$W_{21} = \int V_{21} I_1 dt$$

$$= L_{21} I_1 \int_0^{I_2} di_2$$

$$= L_{21} I_1 I_2$$

Similary

$$W_{22} = \frac{1}{2} L_2 I_2^2$$



Total work at C_2

$$W_m = \frac{1}{2} L I^2$$

$$W_2 = W_1 + W_{12} + W_{22}$$

$$= \frac{1}{2} L_1 I_1^2 + L_1 I_1 I_2 + \frac{1}{2} L_2 I_2^2$$

$$= \frac{1}{2} \sum_{j=1}^{2} \sum_{k=1}^{2} L_{jk} I_j I_k$$

Generalizing I_1, I_2, I_3, … I_N,

$$W_m = \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} L_{jk} I_j I_k$$

Consider Kth loop of N coupled loops

$$dW_k = V_k i_k dt$$
$$= i_k d\varphi_k$$

$$V_k = \frac{d\varphi_k}{dt}$$

Magnetic energy

$$dW_m = \sum_{k=1}^{N} dW_k = \sum_{k=1}^{N} i_k d\varphi_k$$

Total magnetic energy $\quad i_k = \alpha I_k \qquad \phi_k = \alpha \Phi_k$

$$W_m = \int dW_m = \sum_{k=1}^{N} I_k \Phi_k \int_0^1 \alpha\, d\alpha$$
$$= \frac{1}{2}\sum_{k=1}^{N} I_k \Phi_k$$

$$W_m = \frac{1}{2}\sum_{k=1}^{N} I_k \Phi_k$$

$$\Phi_k = \sum_{j=1}^{N} L_{jk} I_j$$

$$W_m = \frac{1}{2}\int_{v'} (\overrightarrow{H} \cdot \overrightarrow{B}) dv'$$

$$\overrightarrow{H} = \frac{\overrightarrow{B}}{\mu}$$

$$\boxed{W_m = \frac{1}{2}\int_{v'} \frac{B^2}{\mu} dv'}$$

or

$$\boxed{W_m = \frac{1}{2}\int_{v'} \mu H^2 dv'}$$

c.f.

$$W_e = \frac{1}{2}\int_{v'} (\overrightarrow{E} \cdot \overrightarrow{D}) dv'$$

$$W_e = \frac{1}{2}\int_{v'} \varepsilon E^2 dv' = \frac{1}{2}\int_{v'} \frac{D^2}{\varepsilon} dv'$$

Magnetic energy density Wm

$$W_m = \int_{v'} W_m dv'$$

$$W_m = \frac{1}{2}\overrightarrow{H} \cdot \overrightarrow{B} = \frac{B^2}{2\mu} = \frac{1}{2}\mu H^2$$

$$\boxed{L = \frac{2W_m}{I^2}}$$