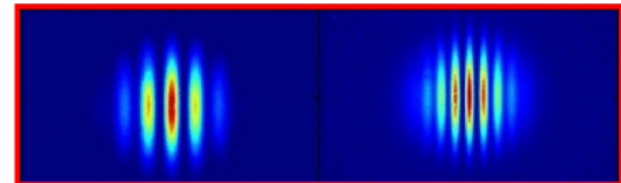
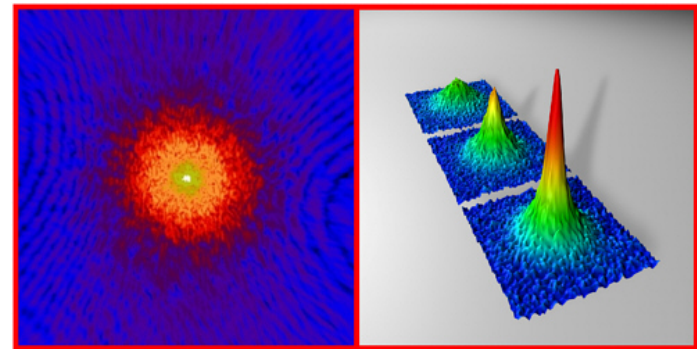
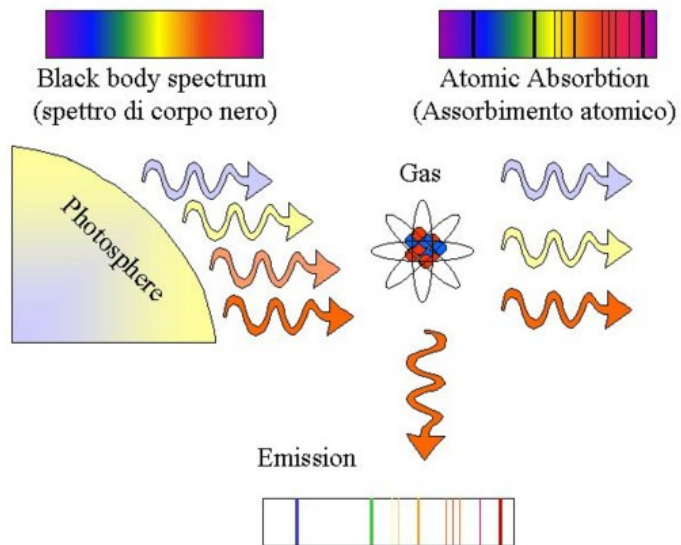


# CHAPTER 2 PHOTONS AND ATOMS



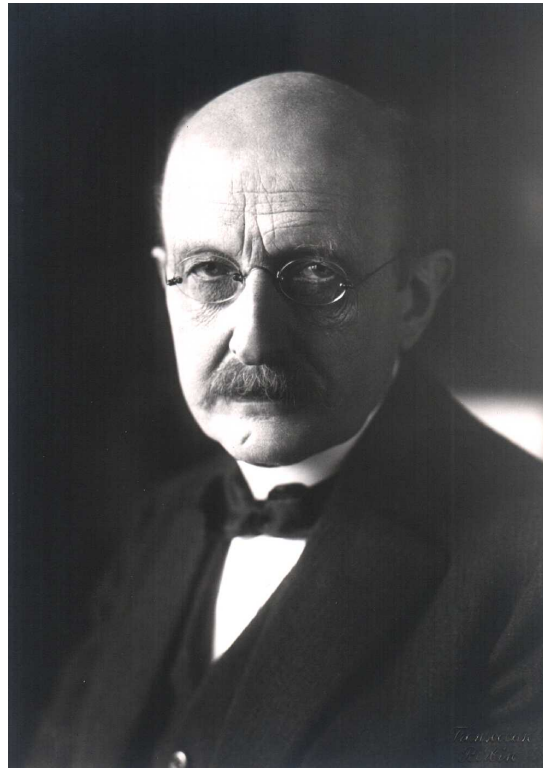
# Outlines

- THE PHOTON
  - Photon Energy, Photon Position, Photon Momentum
  - Photon Polarization, Photon Interference, Photon Time
  - Photon Streams (Mean Photon Flux)
- ATOMS, MOLECULES, AND SOLIDS
  - A. Energy Levels
  - B. Occupation of Energy Levels in Thermal Equilibrium
- INTERACTIONS OF PHOTONS WITH ATOMS
  - A. Interaction of Single-Mode Light with an Atom
  - B. Spontaneous Emission
  - C. Stimulated Emission and Absorption
  - D. Line Broadening

# The Photon



Erwin Schrodinger



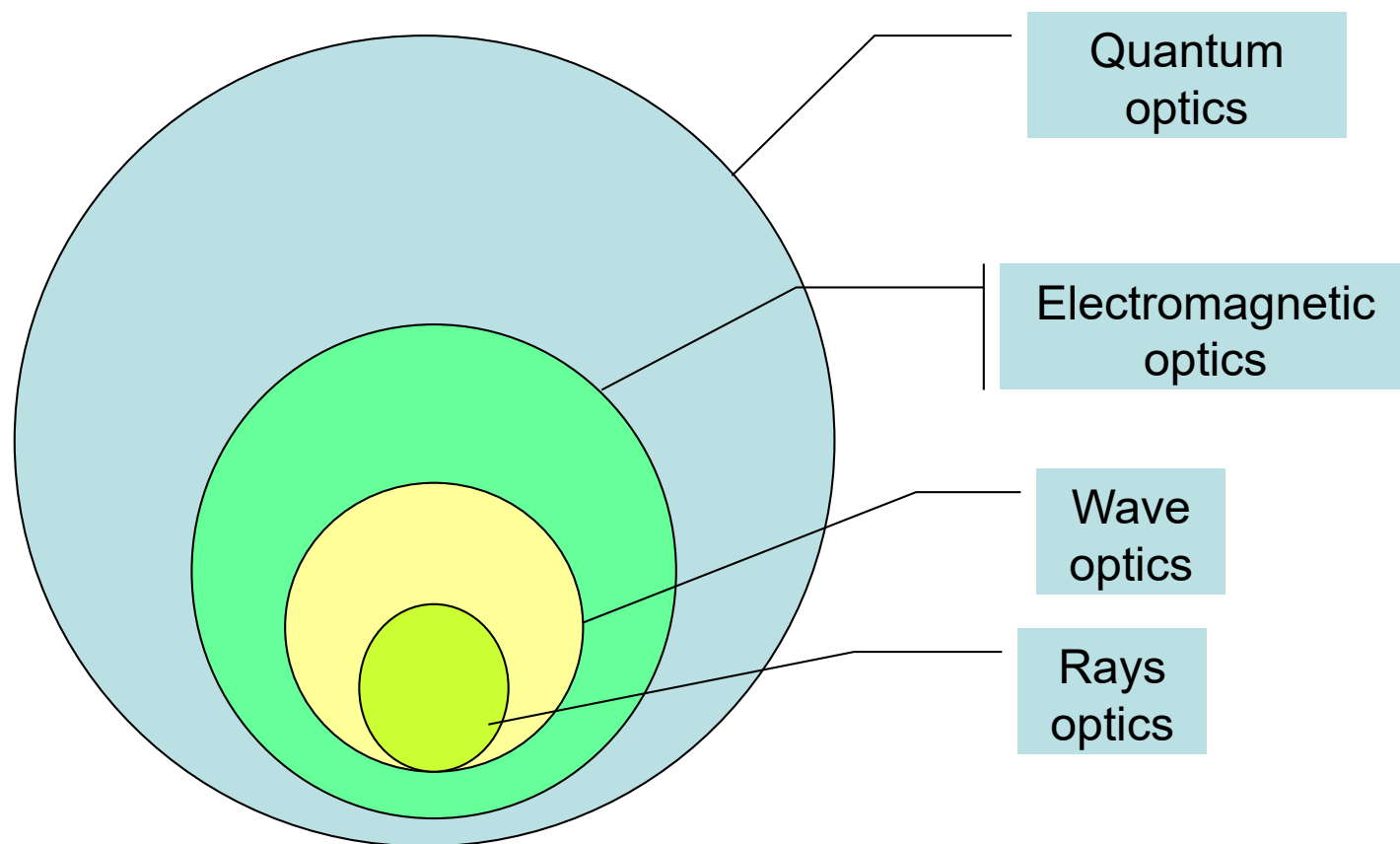
Max Plank



Werner Heisenberg

- Quantum electromagnetic theory is known as **quantum electrodynamics** (QED).
- Quantum electrodynamics is more general than classical electrodynamics and it is today accepted as a theory that is useful for explaining virtually all known optical phenomena. For optical phenomena, this theory is also referred to as **quantum optics**.
- it is possible to derive many of the quantum-mechanical properties of light and its interaction with matter by supplementing electromagnetic optics with a few simple relationships drawn from QED that represent the corpuscularity, localization, and fluctuations of electromagnetic fields and energy. This set of rules, which we call **photon optics**, permits us to deal with optical phenomena that are beyond the reach of classical theory, while retain classical optics as a limiting case.

# The relation among the optical theories



# The Photon

- Light consists of particles called photons.
- zero rest mass  $m_0=0$
- electromagnetic energy
- momentum.
- intrinsic angular momentum (or spin)--- its polarization properties.
- travels at the speed of light (in vacuum  $c_0$ )

# Photon energy

$$E = h\nu = \hbar\omega \quad \lambda\nu = c \quad E(\text{eV}) = \frac{1.24}{\lambda(\mu\text{m})}$$

Where  $h=6.63 \times 10^{-34}$  Js --- Planck constant and  $\hbar=h/2\pi$

For a resonator, A mode containing zero photons **nevertheless** carries an energy

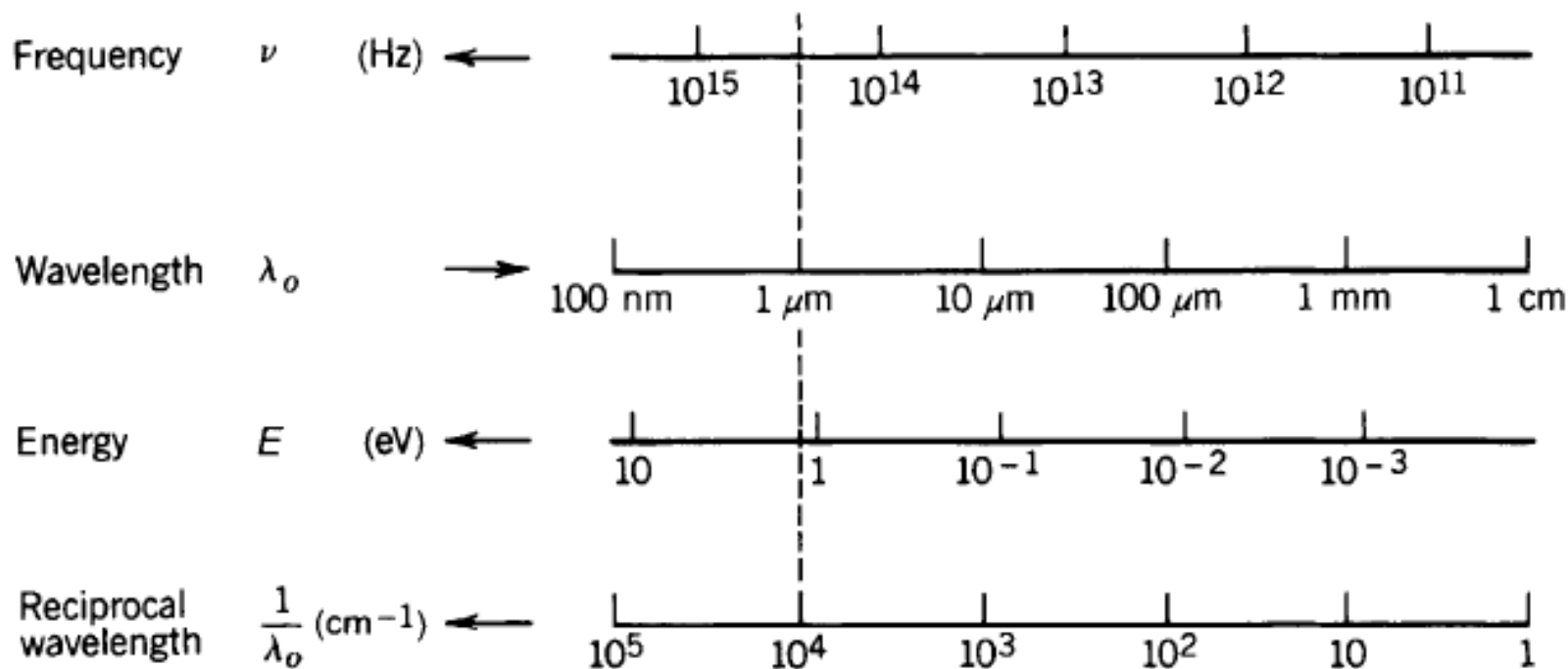
$$E_0 = \frac{1}{2}h\nu$$

called the **zero-point energy**. In most experiments the zero-point energy is not directly observable. The presence of the zero-point energy can, however, be manifested in subtle ways when matter is exposed to static fields. It plays a crucial role in the process of spontaneous emission from an atom,

When it carries  $n$  photons, therefore, the mode has total energy

$$E_n = (n + \frac{1}{2})h\nu, \quad n = 0, 1, 2, \dots$$

# Relation energy, wavelength frequency and wavenumber





# Photon Position

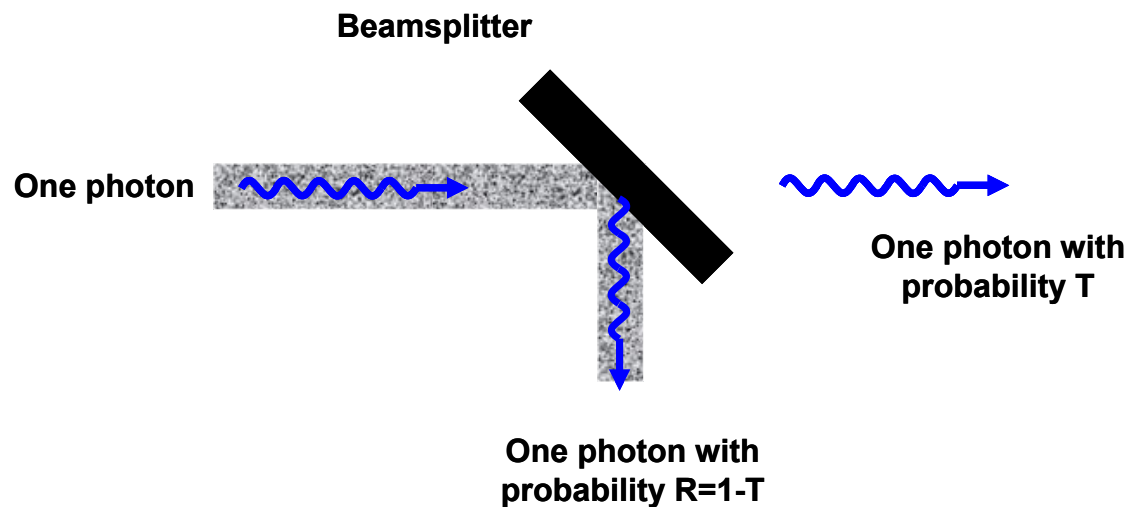
- The photon is more likely to be found at those locations where the intensity is high.
- The probability  $p(r)dA$  of observing a photon at a point  $r$  within an incremental area  $dA$ , at any time, is proportional to the local optical intensity  $I(r) \propto |U(r)|^2$ , i.e.

$$p(r)dA \propto I(r)dA$$

# Transmission of a Single Photon Through a Beamsplitter

The probability for a photon is transmitted is equal to the transmittance  $T$ . The probability that it is reflected is  $1 - T$ . Because, transmittance is a square of normalization of light intensity.

From a probability point of view, the problem is identical to that of flipping a coin.



# Photon Momentum

- The **momentum of a photon** is related to **the wavevector** of its associated wave function

A photon in a mode describes by a plane wave as

$$E(\mathbf{r}, t) = A \exp(-i\mathbf{k} \cdot \mathbf{r}) \exp(i2\pi\nu t) \hat{e}$$

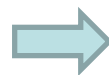
Has a momentum vector

$$\mathbf{p} = \hbar \mathbf{k}$$

And the amplitude of the momentum is

$$p = \hbar k = \hbar \frac{2\pi}{\lambda} = \frac{h}{\lambda}$$

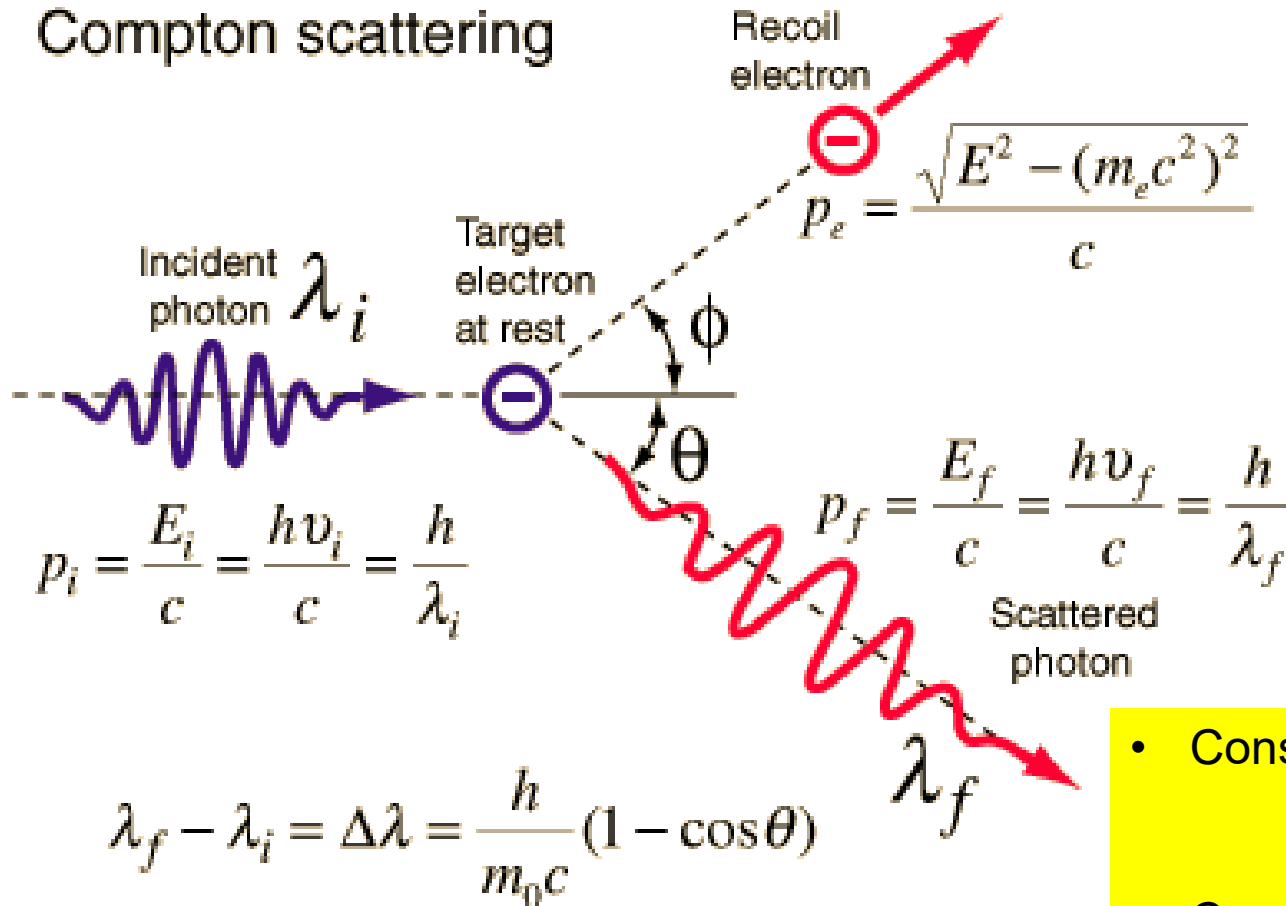
$$\text{Q} \quad k = \frac{2\pi\nu}{c}$$



$$p = h\nu / c$$

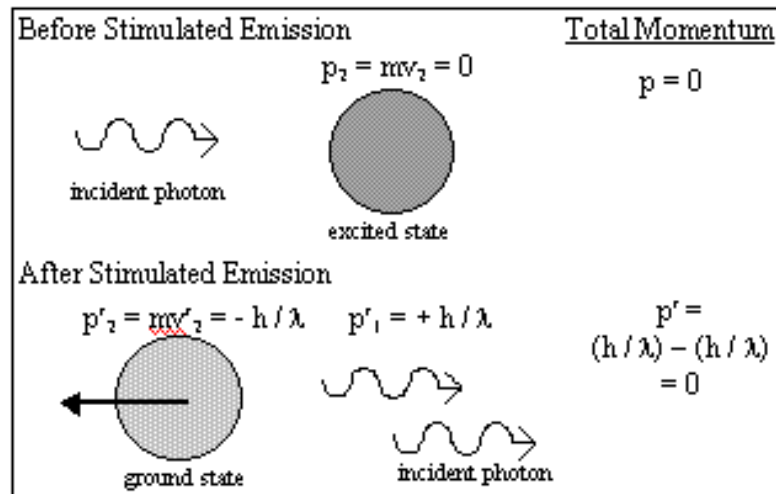
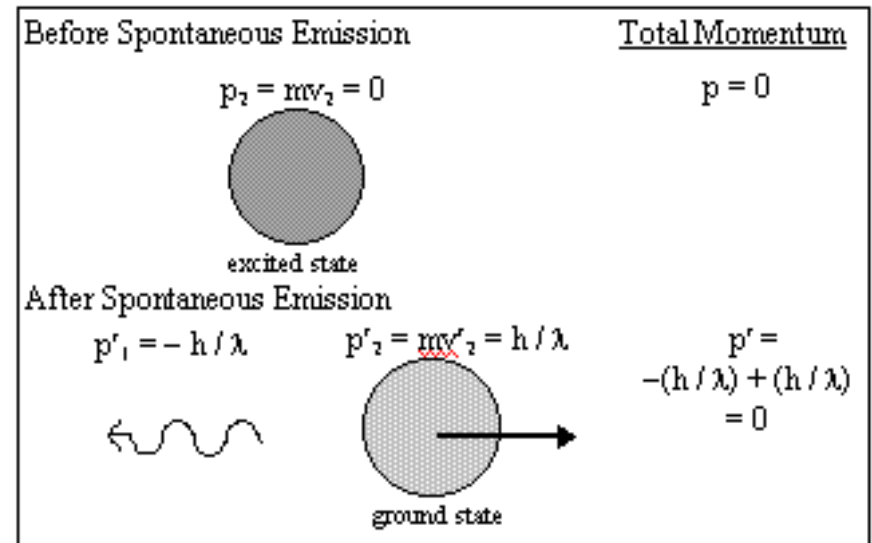
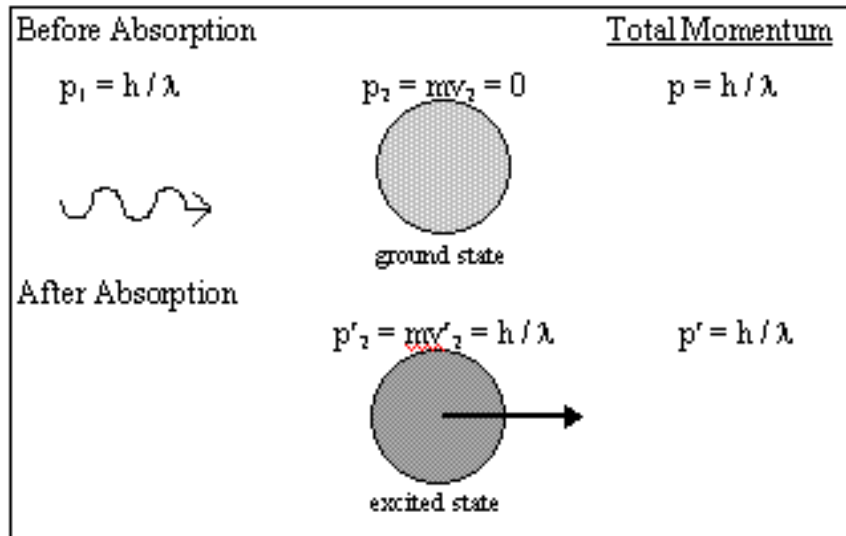
# Compton scattering

## Compton scattering



- Conservation energy
- Conservation momentum

# Conservation momentum



# Photon momentum and Energy

The momentum of a photon is given by:  $p = \frac{h}{\lambda} = mc$

where  $p$  is momentum,  $h$  is [Planck's constant](#),  $\lambda$  is [wavelength](#),  $m$  is mass, and  $c$  is speed of light in vacuum. This expression shows the wave-particle duality.

$$E = mc^2 = pc$$

is the mass-energy relationship where  $E$  is the energy. Then

$$p = \frac{E}{c}$$

# Momentum of a localized wave

In general case, for a complex wave  $U(r)\exp(i2\pi\nu t)$ , one can expand it into a sum of plane waves of different wave vectors in

$$\int A(\mathbf{k})\exp(-i\mathbf{k} \cdot \mathbf{r}) d\mathbf{k}$$

Therefore, the momentum of a photon described by an arbitrary complex wavefunction is uncertain.

$$\mathbf{p} = \hbar \mathbf{k}$$

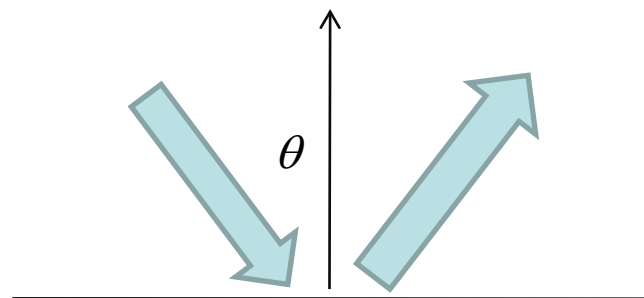
With probability proportional to  $|A(k)|^2$ , where  $A(k)$  is the amplitude of the plane wave Fourier component of  $U(r)$  with wave vector  $k$

# Radiation Pressure (or Photon Pressure)

- Because momentum is conserved, its association with a photon means that the emitting atom experiences a recoil of magnitude  $h\nu/c$ . Furthermore, the momentum associated with a photon can be transferred to objects of finite mass, giving rise to a force and causing mechanical motion.

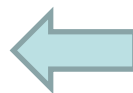
$$\Delta P_r = 2RI \cos^2 \theta / c$$

$$\Delta P_a = (1 - R)I \cos^2 \theta / c$$



The momentum change in unite time unite surface is

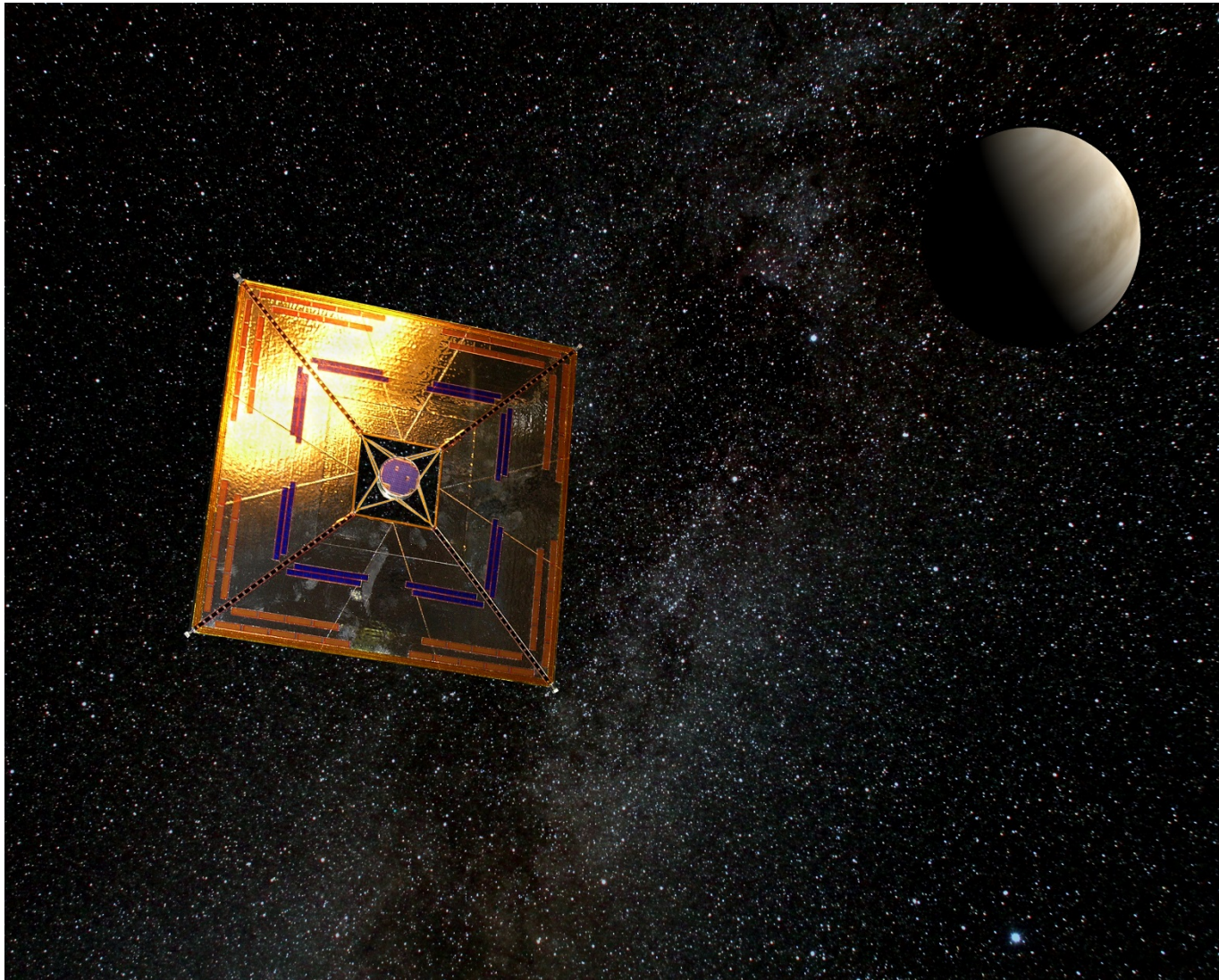
$$\Delta P = (1 + R)I \cos^2 \theta / c$$



Photon pressure



# Solar sail



# Photon Polarization

Light in a resonator, is characterized as a sum of modes of different frequencies, directions, and polarizations.

***The polarization of a photon is one kind of its mode.***

## Linear polarized light

$$E(\mathbf{r}, t) = (A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}}) \exp(-ikz) \exp(i2\pi\nu t)$$

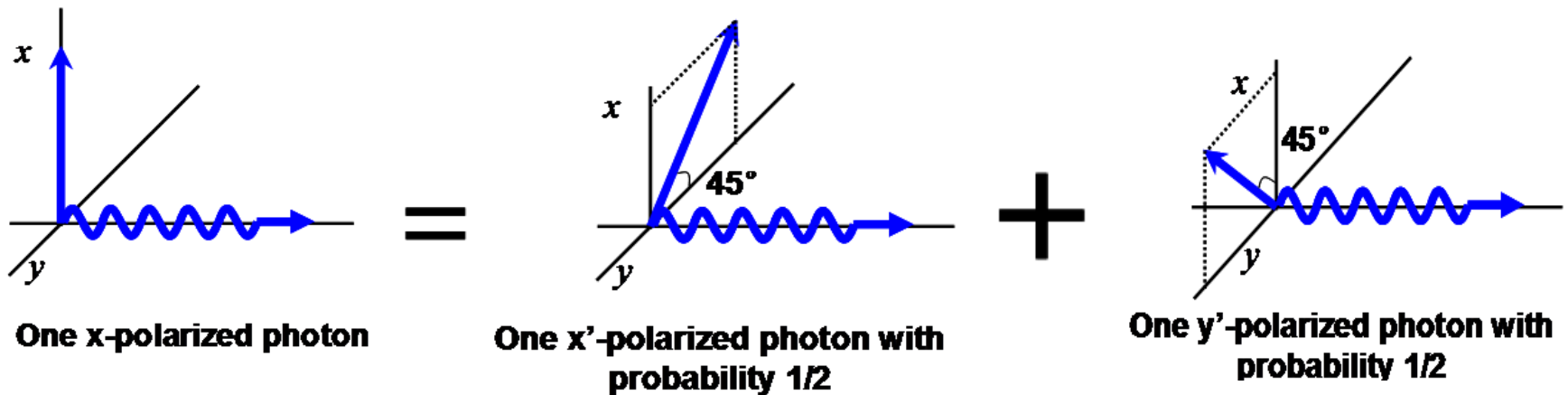
If we represent it in the coordinate system  $x'$ ,  $y'$ , which makes a 45° angle rotation with the initial coordinate system. Thus we can equally well view the field in photons polarized along the  $x'$  and  $y'$  directions,

$$E(\mathbf{r}, t) = (A_{x'} \hat{\mathbf{x}}' + A_{y'} \hat{\mathbf{y}}') \exp(-ikz) \exp(i2\pi\nu t)$$

where

$$A_{x'} = \frac{1}{\sqrt{2}}(A_x - A_y) \quad A_{y'} = \frac{1}{\sqrt{2}}(A_x + A_y)$$

- If we know that the x-polarized mode is occupied by a photon, and the y –polarized mode is empty.
- what can be said about the possibility of finding a photon polarized along the  $x'$ ,  $y'$
- The probabilities of finding a photon with x, y,  $x'$ , or  $y'$  polarization are proportional to the intensities  $|A_x|^2$ ,  $|A_y|^2$ ,  $|A_{x'}|^2$ , and  $|A_{y'}|^2$  respectively.
- For example  $|A_x|^2 = 1$ ,  $|A_y|^2 = 0$ , so that  $|A_{x'}|^2 = |A_{y'}|^2 = 1/2$



## • Circularly Polarized Photons

A modal expansion in terms of two circularly polarized plane-wave modes, one right-handed and one left-handed, can be used,

$$E(\mathbf{r}, t) = \left[ A_R \hat{\mathbf{e}}_R + A_L \hat{\mathbf{e}}_L \right] \exp(-ikz) \exp(i2\pi\nu t)$$

where  $\hat{\mathbf{e}}_R = \left( \frac{1}{\sqrt{2}} \right) (\hat{\mathbf{x}} + i\hat{\mathbf{y}})$        $\hat{\mathbf{e}}_L = \left( \frac{1}{\sqrt{2}} \right) (\hat{\mathbf{x}} - i\hat{\mathbf{y}})$

- These modes carry right-handed and left-handed circularly polarized photons.
- the **probabilities** of finding a photon with these polarizations are **proportional to the intensities**  $|A_R|^2$  and  $|A_L|^2$ .
- when a circularly polarized photon is passed through a linear polarizer, the probability of detecting it, **is 1/2**.

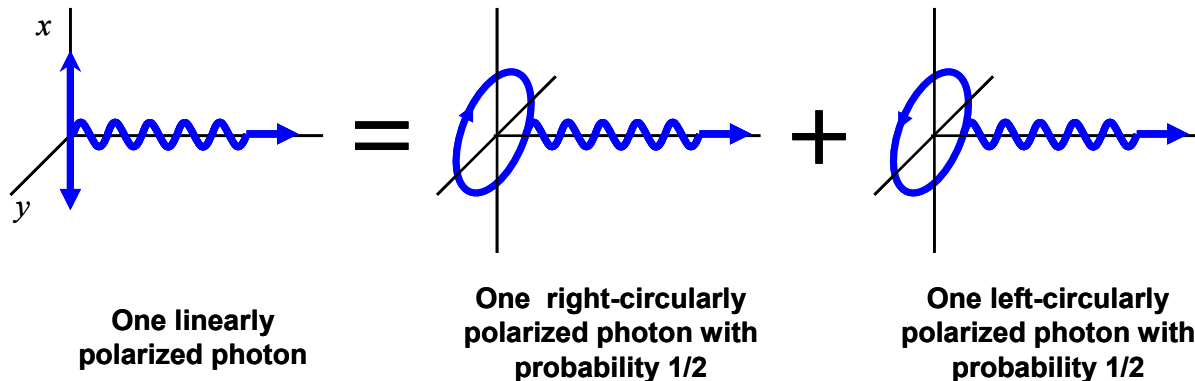
# Photon spin

Photons possess intrinsic angular momentum (spin).

The magnitude of the photon spin  $S = \pm \hbar$

- Right-handed (left-handed) circularly polarized photons have their spin vector parallel (antiparallel) to their momentum vector.
- Linearly polarized photons have an equal probability of exhibiting parallel and antiparallel spin.

Photons can transfer linear momentum to an object,  
circularly polarized photons can exert a torque on an object.



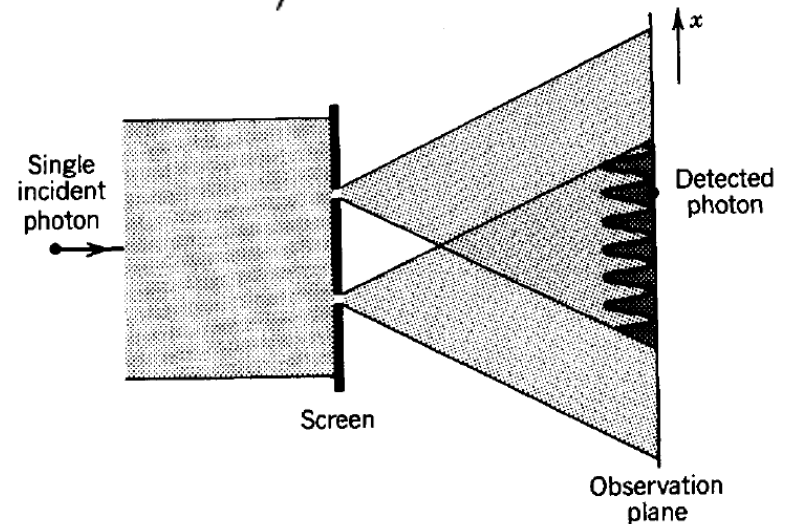
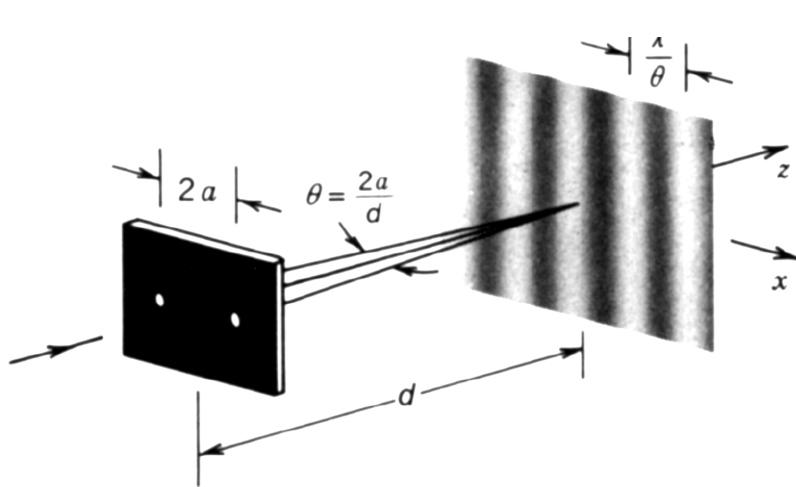


# Photon Interference

The intensity at the observation plane is calculated using electromagnetic (wave) optics and the result is converted to a probability density function that specifies the random position of the detected photon.

The interference arises from phase differences in the two paths.

$$I(x) = 2I_0 \left( 1 + \cos \frac{2\pi\theta x}{\lambda} \right)$$



# Photon Time

A photon in a monochromatic mode is equally likely to be detected at any time.

A general expansion may be made in terms of polychromatic modes (time-localized wavepackets).

The probability of detecting the photon described by the complex wavefunction  $U(\mathbf{r}, t)$ , at any position, in the incremental time interval between  $t$  and  $t + dt$ , is proportional to  $I(\mathbf{r}, t)dt \propto |U(\mathbf{r}, t)|^2 dt$ , photon time localization:

$$p(\mathbf{r}, t)dA dt \propto I(\mathbf{r}, t)dA dt \propto |U(\mathbf{r}, t)|^2 dA dt$$

# Heisenberg's uncertainty principle

The position of the electron can be determined to within the resolving power of the microscope, which is given by a formula from classical optics: Where  $\theta$  is the aperture angle of the microscope

$$\Delta x \sim \frac{\lambda}{\sin \theta}$$

since the photon is scattered anywhere within the aperture, the uncertainty of momentum transferred equals

$$\Delta p \sim p_{\text{photon}} \sin \theta = \frac{h}{\lambda} \sin \theta$$

So we have the product

$$\Delta x \Delta p \sim h$$



# Time-Energy Uncertainty

The frequency uncertainty is readily determined by Fourier expanding  $U(t)$  in terms of its harmonic components,

$$U(t) = \int_{-\infty}^{\infty} V(\nu) \exp(j2\pi\nu t) d\nu$$

The width  $\sigma_\nu$  of  $|V(\nu)|^2$  represents the spectral width. If  $\sigma_t$  is the rms width of the function  $|U(t)|^2$  (i.e., the power-rms width), then  $\sigma_\nu$  and  $\sigma_t$  must satisfy the ***duration-bandwidth reciprocity*** relation:

$$\sigma_\nu \sigma_t \geq 1/(4\pi) \qquad \sigma_w \sigma_t \geq 1/2$$

The energy of the photon  $h\nu$  then cannot be specified to an accuracy better than  $\sigma_E = h\sigma_\nu$ . It follows that the energy uncertainty of a photon, and the time during which it may be detected, must satisfy

$$\sigma_E \sigma_t \geq \hbar/2$$

***Time-energy uncertainty relation.***

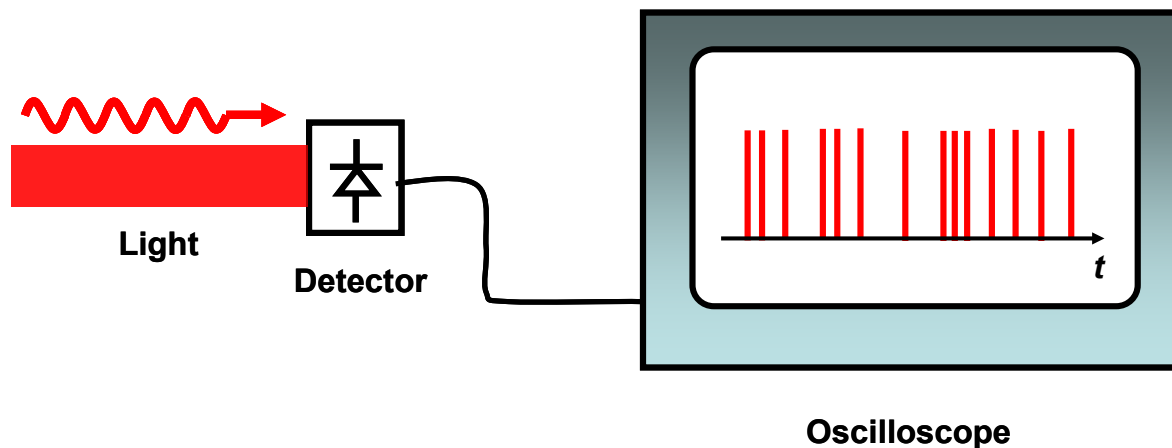
- A **monochromatic photon** ( $\sigma_v \rightarrow 0$ ) has an **eternal duration** within which it can be observed ( $\sigma_t \rightarrow \infty$ ). or ( $\sigma_\rho \rightarrow \infty$ )
- In contrast, a photon associated with an optical **wavepacket** is localized in time and is therefore **polychromatic** with a corresponding energy uncertainty.

# PHOTON STREAMS

The probability distribution obeyed by the photon number is governed by the quantum state of the mode, which is determined by the nature of the light source. Real photon streams often contain numerous propagating modes, each carrying a random number of photons.

The probability of detecting a photon in the incremental time interval between  $t$  and  $t + dt$  is proportional to the optical power  $P(t)$  at the time  $t$ .

$$\bar{n} = \frac{1}{e^{h\nu/k_B T} - 1}$$



# Mean Photon Flux

- Mean Photon-Flux Density

$$\phi(\mathbf{r}) = \frac{I(\mathbf{r})}{h\nu}$$

$$\phi(\mathbf{r}) = \frac{I(\mathbf{r})}{h\bar{\nu}}$$

- Mean Photon Flux

$$\Phi = \int_A \phi(\mathbf{r}) dA = P/h\bar{\nu}$$

$$P = \int_A I(\mathbf{r}) dA$$

- Mean Number of Photons

$$\bar{n} = \Phi T = E/h\nu$$

- Mean Photon momentum

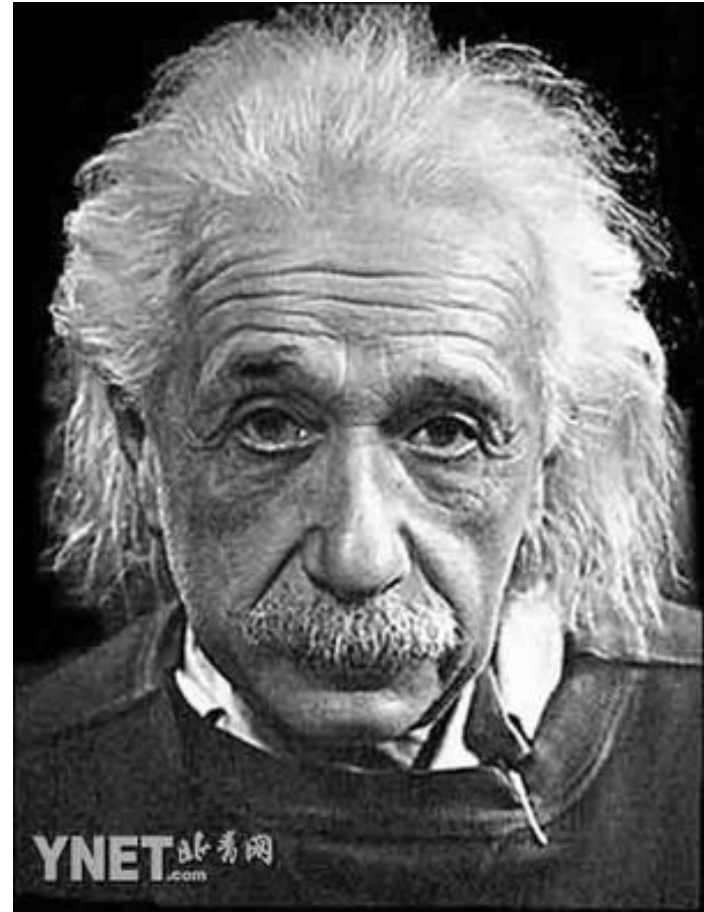
$$\bar{p} = \sum_i^m h\vec{k}_i = \sum_i^m \frac{h}{\bar{\lambda}} \cos \theta_i$$

Classical		Quantum	
Optical intensity	$I(\mathbf{r})$	Photon-flux density	$\phi(\mathbf{r}) = \frac{I(\mathbf{r})}{h\bar{\nu}}$
Optical power	$P$	Photon flux	$\Phi = \frac{P}{h\bar{\nu}}$
Optical energy	$E$	Photon number	$\bar{n} = \frac{E}{h\bar{\nu}}$

Classical	Quantum
$I_\nu$ (W/cm <sup>2</sup> -Hz)	$\phi_\nu = \frac{I_\nu}{h\nu}$ (photons/s-cm <sup>2</sup> -Hz)
$P_\nu$ (W/Hz)	$\Phi_\nu = \frac{P_\nu}{h\nu}$ (photons/s-Hz)
$E_\nu$ (J/Hz)	$\bar{n}_\nu = \frac{E_\nu}{h\nu}$ (photons/Hz)



**Niels Bohr**  
**Nobel prize 1922**



**Albert Einstein**  
**Nobel prize 1921**

### ***Thermal light:***

Light emitted from atoms, molecules, and solids, under conditions of **thermal equilibrium** and in the absence of other external energy sources.

### ***Luminescence light:***

Photon emission is induced by the presence of other **external sources of energy**, such as an external source of light, an electron current or a chemical reaction. The excited atoms can then emit non-thermal light.

## 2. ATOMS, MOLECULES AND SOLIDS

The behavior of a single nonrelativistic particle of mass  $m$  (e.g., an electron), with a potential energy  $V(\mathbf{r}, t)$ , is governed by a complex wavefunction  $\psi(\mathbf{r}, t)$  satisfying the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}, t) + V(\mathbf{r}, t)\psi(\mathbf{r}, t) = j\hbar\frac{\partial\psi(\mathbf{r}, t)}{\partial t}$$

If we simply to determine the allowed energy levels  $E$  of the particle in the absence of time-varying interactions, the wavefunction  $\psi(\mathbf{r}, t)$  may be expressed as:  $\Psi(\mathbf{r}, t) = \varphi(\mathbf{r}) \exp[i(E/\hbar)t]$ , where  $\varphi(\mathbf{r})$  satisfies the time independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\varphi(\mathbf{r}) + V(\mathbf{r})\varphi(\mathbf{r}) = E\varphi(\mathbf{r})$$



# Energy Levels of materials

## *Vibrational and Rotational Energy Levels of Molecules*

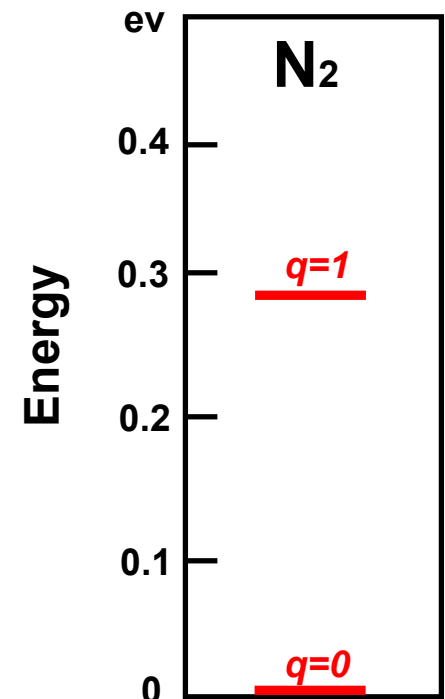
### **Vibrations of a Diatomic Molecule** (ex. $N_2, CO, HCl$ etc.)

The molecular vibrations take on the set of allowed energy levels appropriate for the quantum-mechanical harmonic oscillator

$$E_q = (q + \frac{1}{2})\hbar\omega$$

where  $\omega = (\kappa / m_r)^{1/2}$        $m_r = m_1 m_2 / (m_1 + m_2)$

Typical values of  $\hbar\omega$  lie between 0.05 and 0.5 eV, which corresponds to the photon in the infrared region

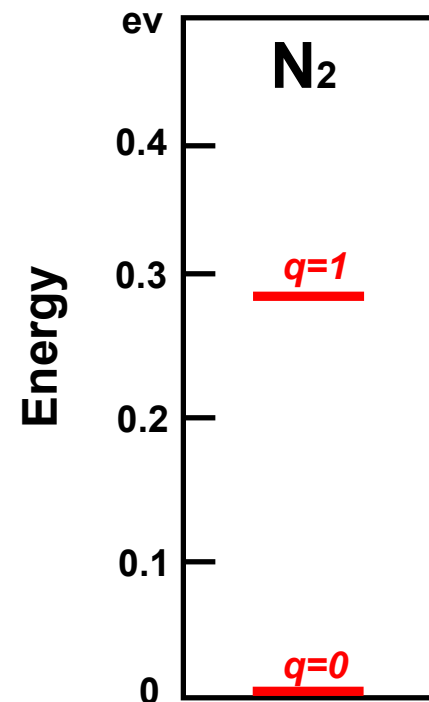


### ***Rotations of a Diatomic Molecule.***

The rotations of a diatomic molecule about its axes are similar to those of a rigid rotor with moment of inertia  $I$ . The rotational energy is quantized to the values

$$E_q = q(q+1) \frac{h^2}{2I}, \quad q = 0, 1, 2, \dots$$

Typical rotational energy levels are separated by values in the range 0.001 to 0.01 eV, so that the energy differences correspond to photons in the far infrared region



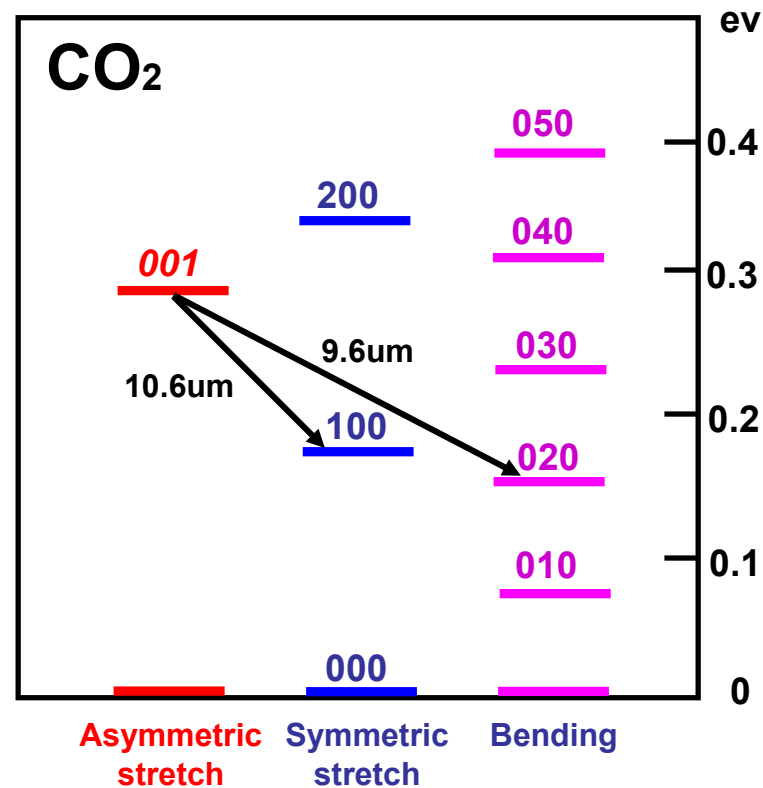
## Vibrations of the CO<sub>2</sub> Molecule

three kinds independent vibrations :

- asymmetric stretching (AS)
- symmetric stretching (SS)
- bending (B).

Each of these vibrational modes behaves like a harmonic oscillator. **But the frequencies  $\omega_{q1}$   $\omega_{q2}$   $\omega_{q3}$  are different.**

$$E_{q_1, q_2, q_3} = (q_1 + \frac{1}{2})h\omega_{q_1} : (q_2 + \frac{1}{2})h\omega_{q_2} : (q_3 + \frac{1}{2})h\omega_{q_3}$$



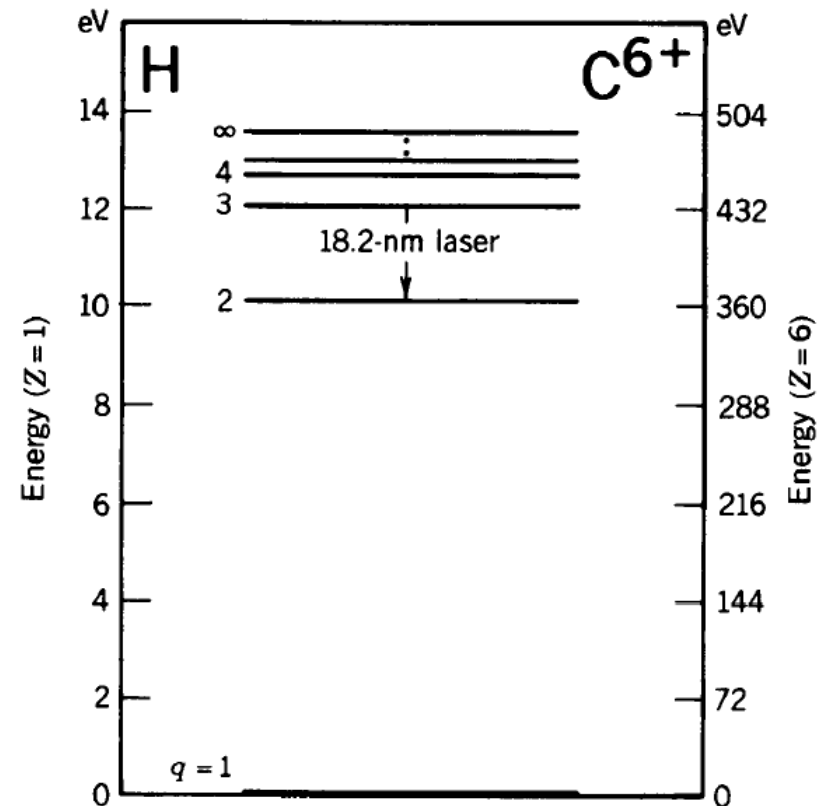
## Electron Energy Levels of Atoms and Molecules

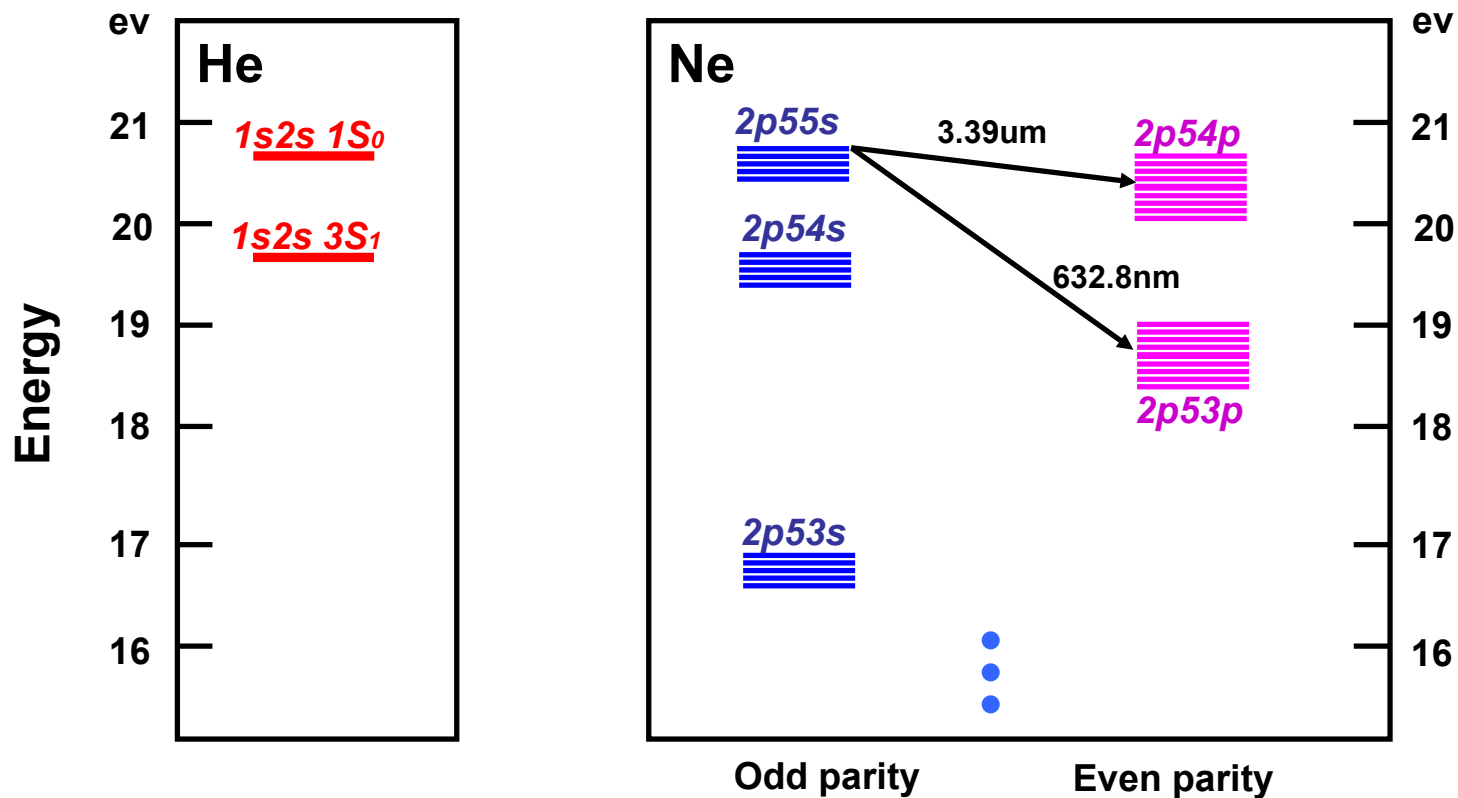
Isolated atom has a potential energy that derives from the Coulomb law of attraction between the proton and the electron

Isolate atom

$$E_q = -\frac{m_r Z^2 e^4}{2h^2 q^2}, q = 1, 2, 3, \dots,$$

where  $m_r$  is the reduced mass of the atom,  $e$  is the electron charge, and  $Z$  is the number of protons in the nucleus ( $Z = 1$  for hydrogen).





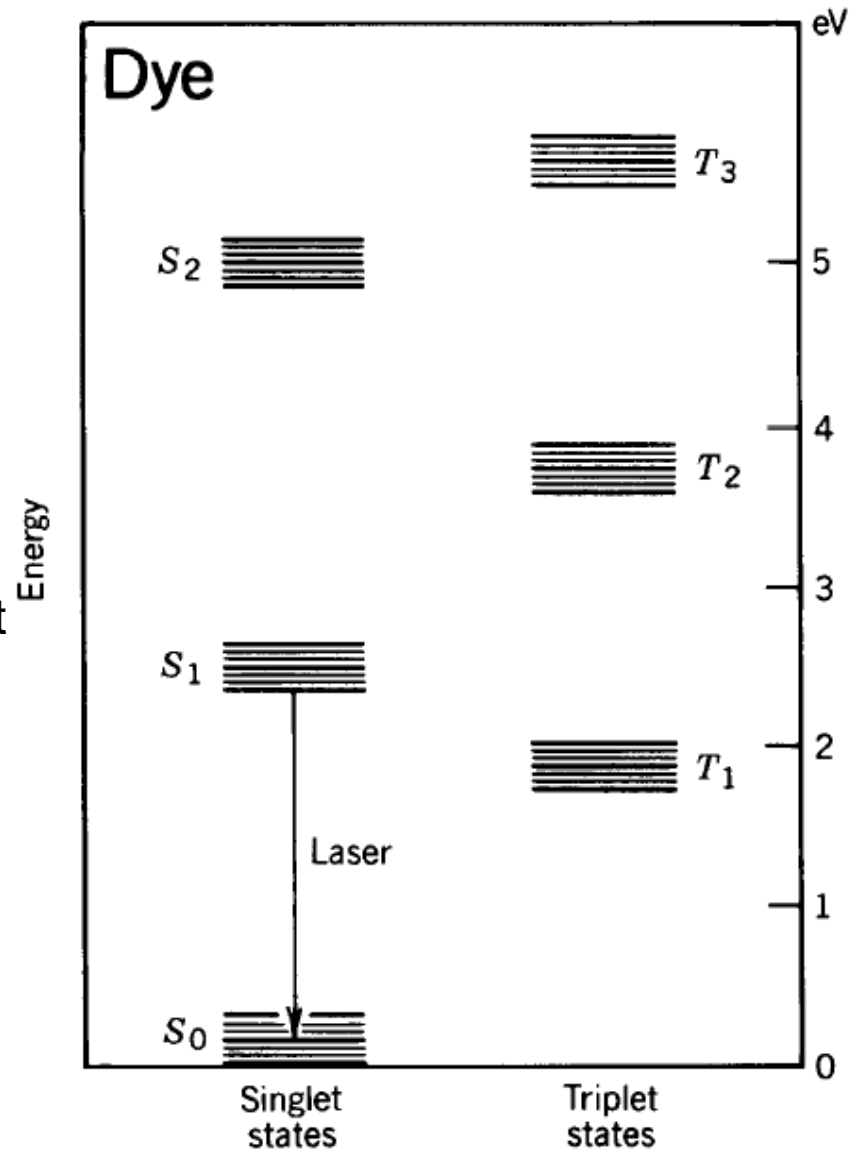
Some energy levels of He and Ne atoms. The Ne transitions marked by arrows correspond to photons of wavelengths  $3.39\ \mu\text{m}$  and  $632.8\ \text{nm}$ , as indicated. These transitions are used in He-Ne lasers

## Dye Molecules.

Organic dye molecules are large and complex. They may undergo electronic, vibrational, and rotational transitions so that they typically have many energy levels.

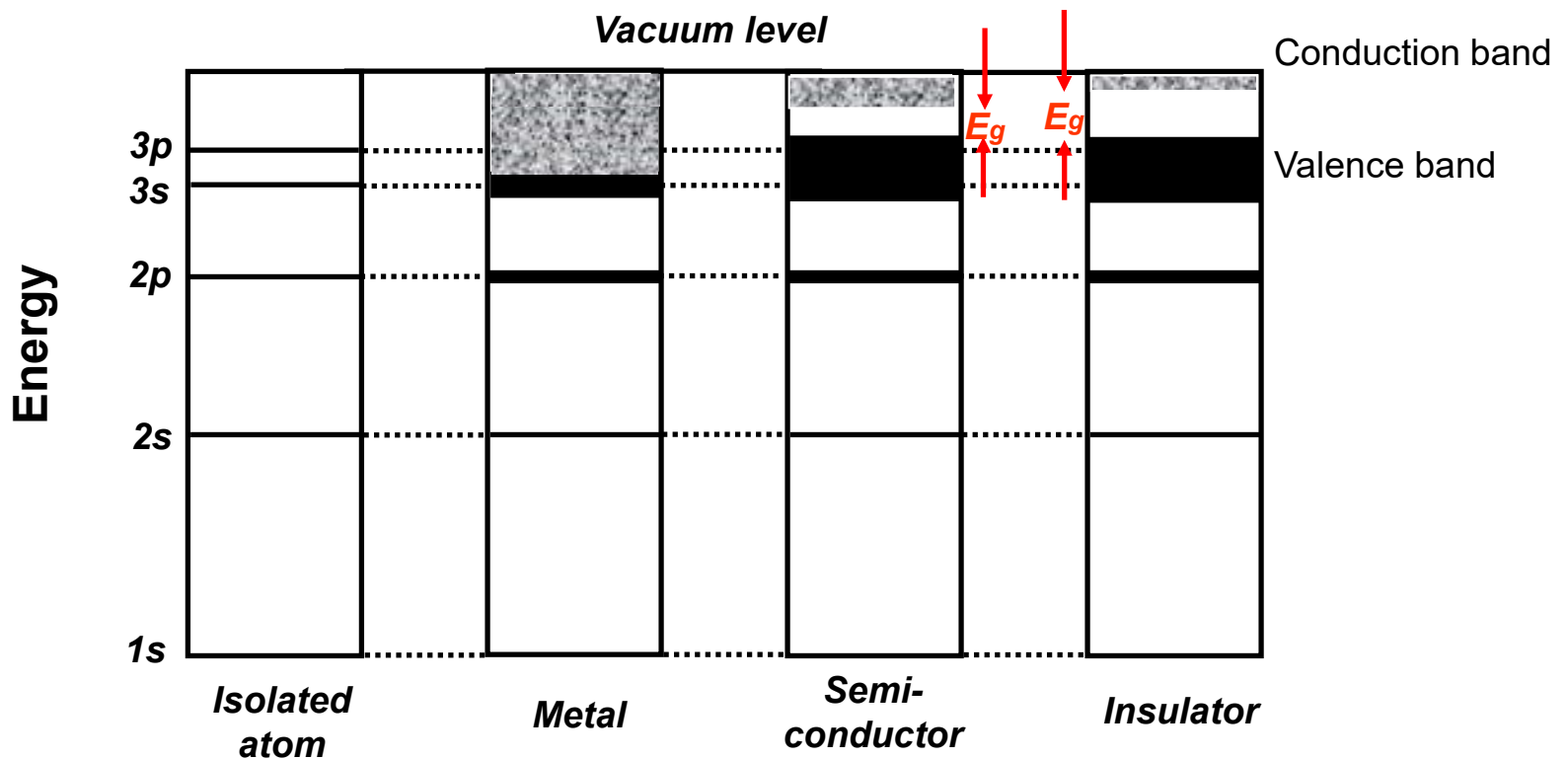
Levels exist in both singlet (S) and triplet (T) states.

- Singlet states have an excited electron whose spin is antiparallel to the spin of the remainder of the dye molecule;
- triplet states have parallel spins.



## Electron Energy Levels in Solids

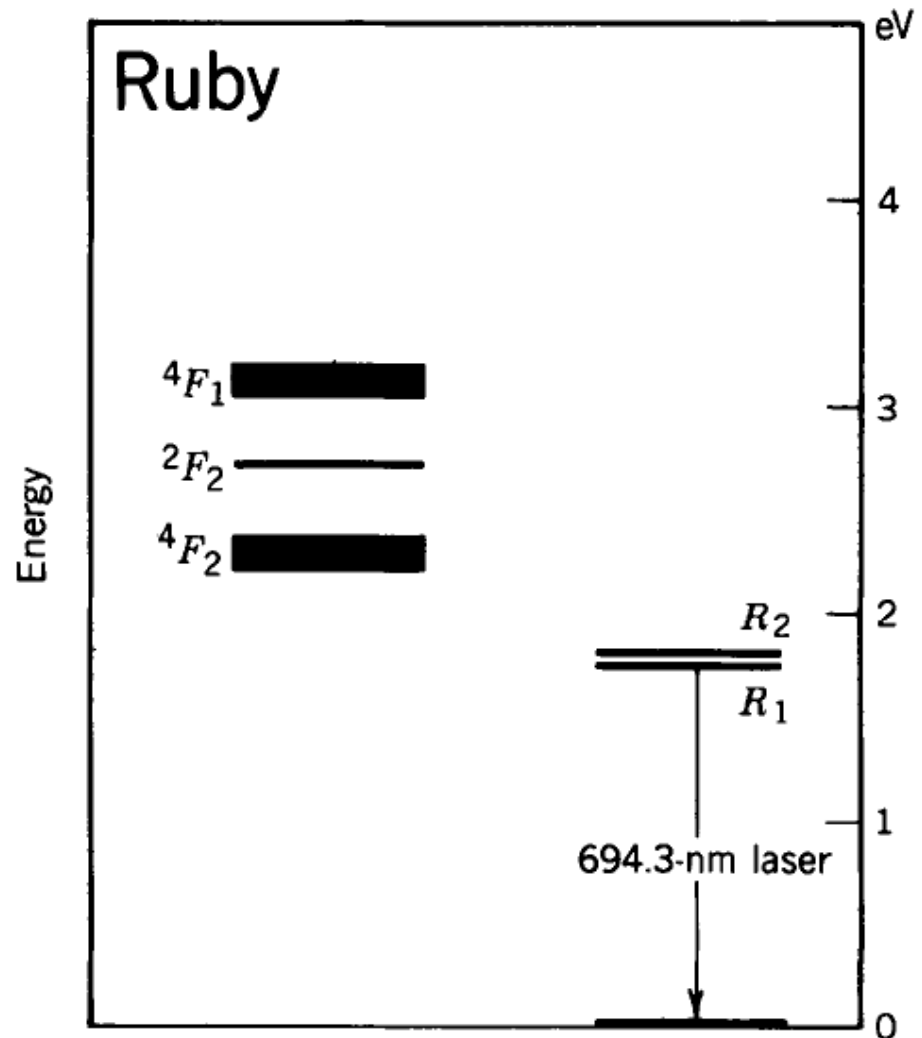
The energy levels of three generic solids with different electrical properties (metal, semiconductor, insulator) are shown in the figure



## Ruby

Ruby is an insulator. It is alumina (also known as sapphire,  $\text{Al}_2\text{O}_3$ ) in which a small fraction of the  $\text{Al}^{3+}$  ions are replaced by  $\text{Cr}^{3+}$  ions.

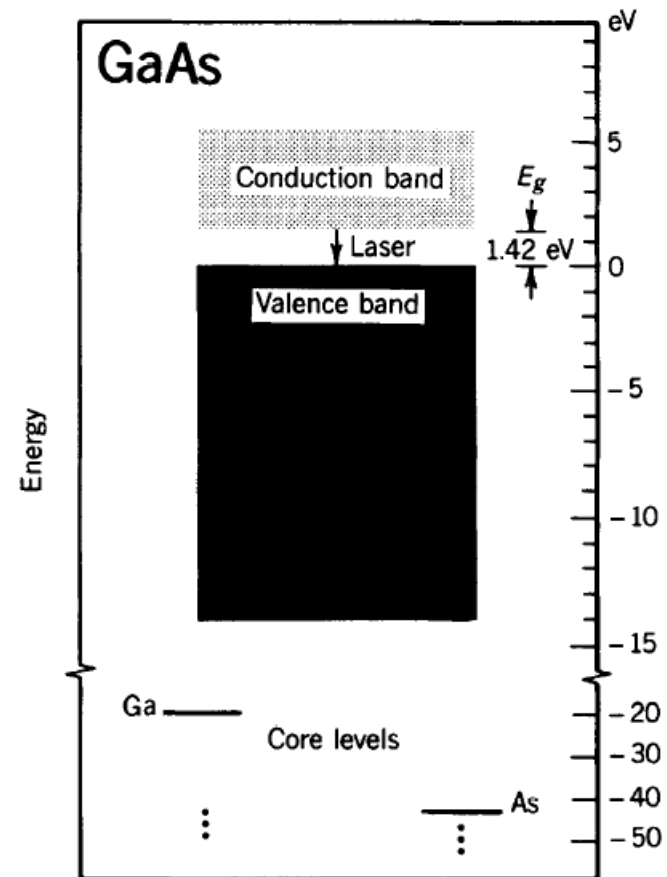
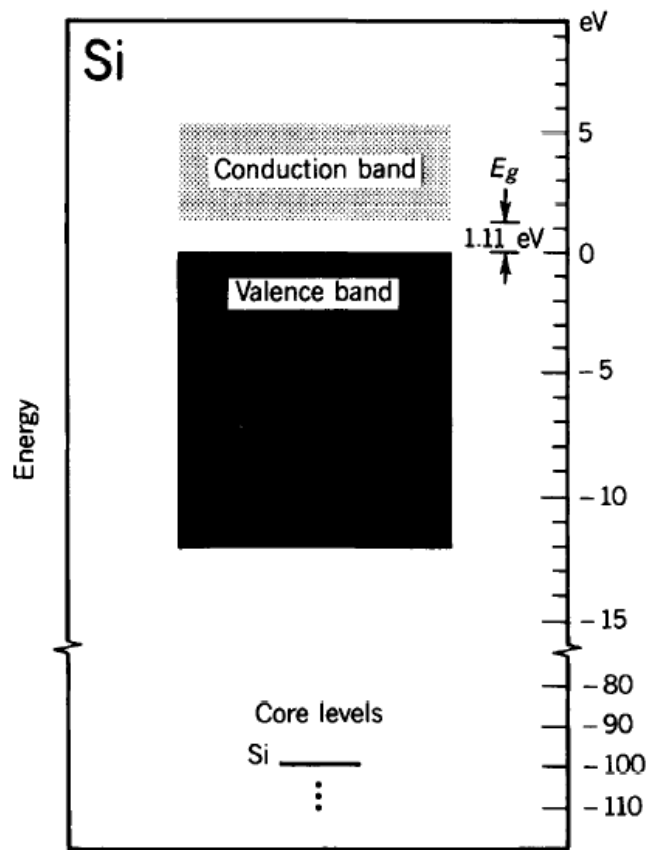
The green and violet absorption bands (indicated by the group-theory notations  $4F_2$  and  $4F_1$ , respectively) give the material its characteristic pink color.





## Semiconductors.

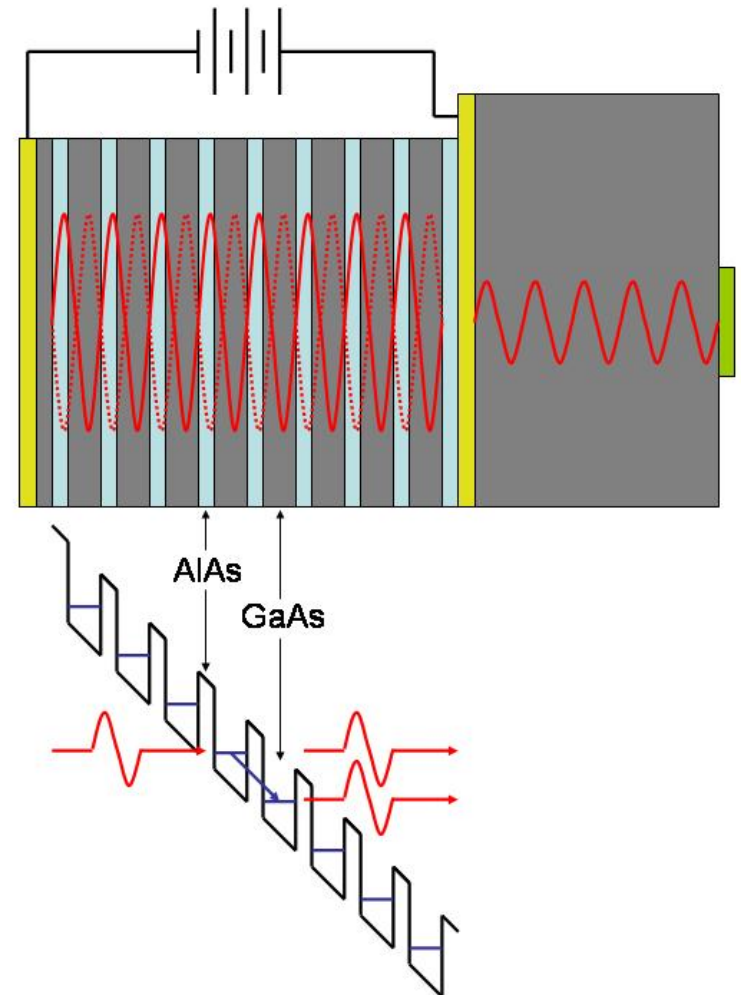
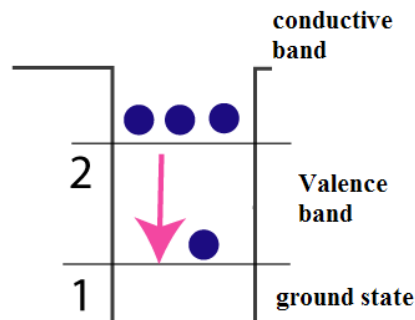
Semiconductors have closely spaced allowed electron energy levels that take the form of bands



## Quantum Wells and Superlattices (量子井与超晶格)

**Crystal-growth techniques**, such as molecular-beam epitaxy and vapor-phase epitaxy, can be used to grow materials with specially **designed band structures**. In semiconductor quantum-well structures, the energy bandgap is engineered to vary with position in a specified manner, leading to materials with unique electronic and optical properties.

Quantized energies in a single-crystal AlGaAs/GaAs multiquantum-well structure. The well widths can be arbitrary (as shown) or periodic.



## B. Occupation of Energy Levels in Thermal Equilibrium

### Boltzmann Distribution

In thermal equilibrium at temperature  $T$  (the atoms motion reaches a steady state that is invariant to time), the probability  $P(E_m)$  that an arbitrary atom is in energy level  $E_m$  is given by the **Boltzmann distribution**

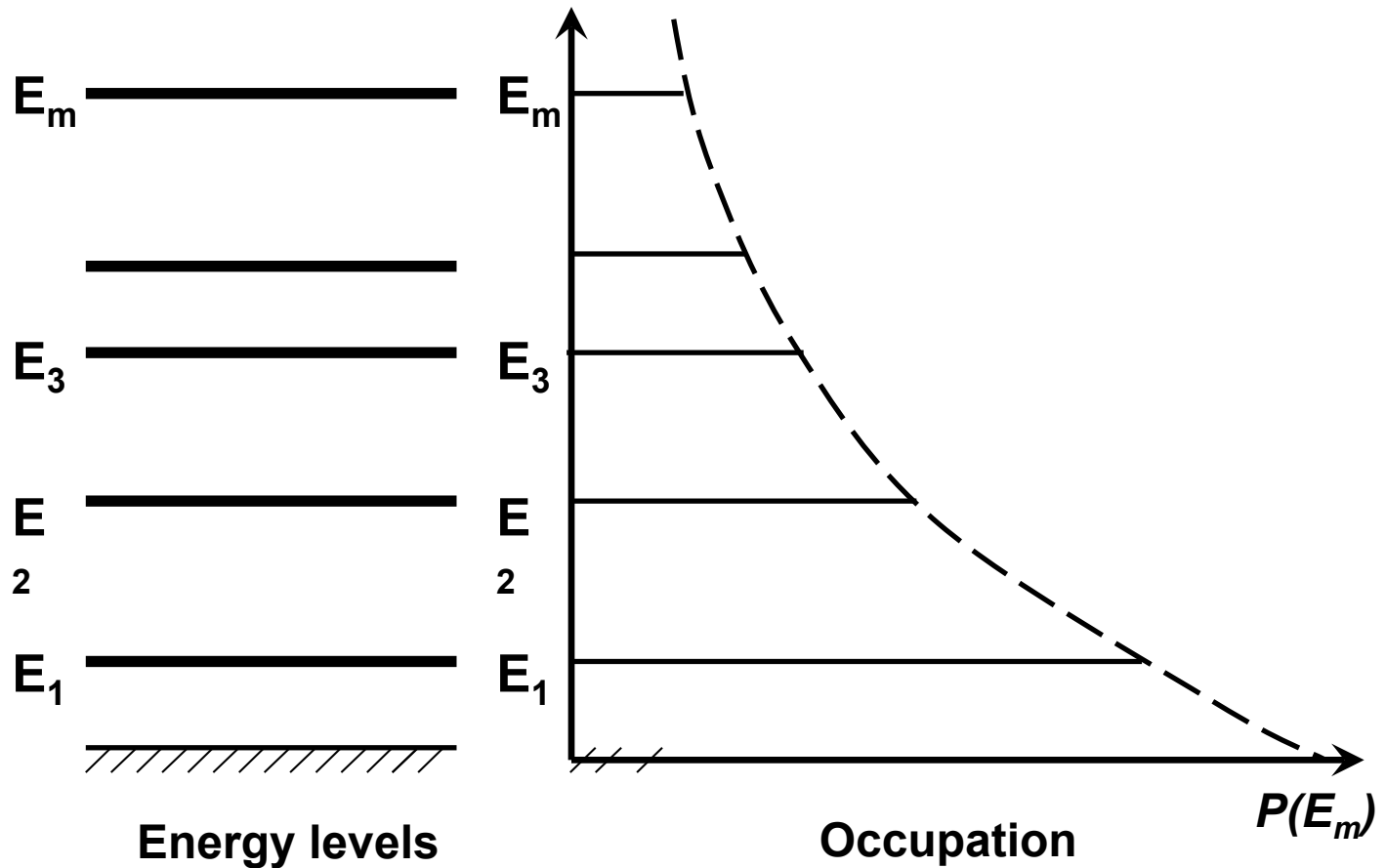
$$P(E_m) \propto \exp(-E_m / k_B T), \quad m = 1, 2, \dots,$$

$$P(E_m) \approx N_m / N$$

where  $N_m$  is the number of atoms occupying energy level  $E_m$ , and the population ratio is, on the average, is

$$\frac{N_2}{N_1} = \exp\left(-\frac{E_2 - E_1}{k_B T}\right)$$

where  $k_B$  is the Boltzmann constant



$$\frac{N_2}{N_1} = \exp\left(-\frac{E_2 - E_1}{k_B T}\right)$$

$$\frac{N_2}{N_1} = \exp\left(-\frac{E_2 - E_1}{k_B T}\right)$$

- At  $T = 0$  K, all atoms are in the lowest energy level (ground state).
- $T$  increases, the populations of the higher energy levels increase.
- Under equilibrium conditions, the population of a given energy level is always greater than that of a higher-lying level.
- **Population inversion** : A higher energy level has a greater population than a lower energy level, the basis for laser action

There are several different quantum states which can correspond to the same energy (e.g., different states of angular momentum).

To account for these degeneracies, we have

$$\frac{N_2}{N_1} = \frac{g_2}{g_1} \exp\left(-\frac{E_2 - E_1}{k_B T}\right)$$

The degeneracy parameters  $g_2$  and  $g_1$  represent the number of states corresponding to the energy levels  $E_2$  and  $E_1$ , respectively.

# Fermi-Dirac Distribution

## *Pauli exclusion principle:*

- Each state can be occupied by at most one electron.
- One state is either occupied or empty, the number of electrons  $N_m$  in state  $m$  is either 0 or 1.

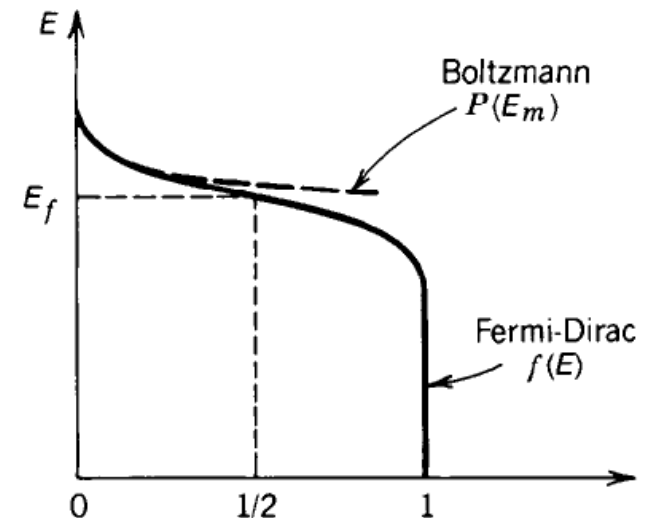
Electrons in a semiconductor obey a different occupation law. Since the atoms are located in close proximity to each other, the material must be treated as a single system within which the electrons are shared. So that very large number of energy levels exist and forming the **band**.

## Fermi-Dirac distribution

The probability that energy level  $E$  is occupied is given by the **Fermi-Dirac distribution**

$$f(E) = \frac{1}{\exp[(E - E_f)/k_B T] + 1}$$

where  $E_f$  is a constant called the Fermi energy. This distribution has a maximum value of unity, which indicates that the energy level  $E$  is definitely occupied.  $f(E)$  decreases monotonically as  $E$  increases, assuming the value  $1/2$  at  $E = E_f$ .



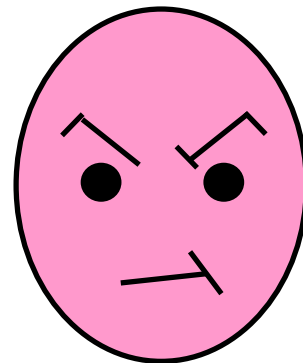
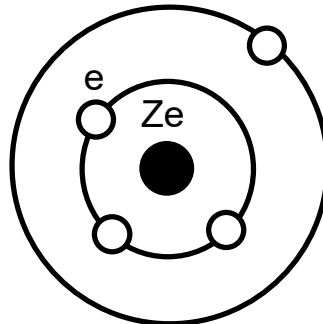
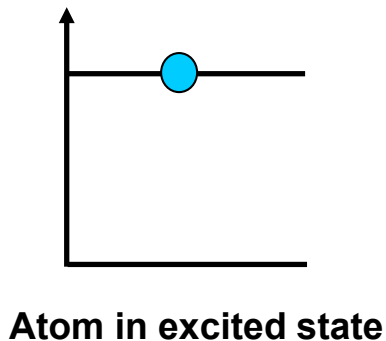
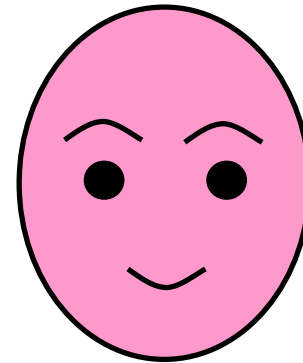
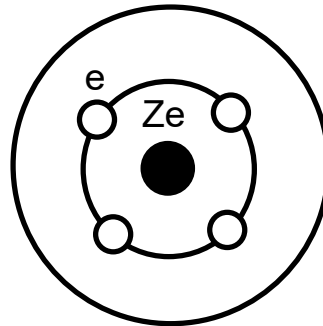
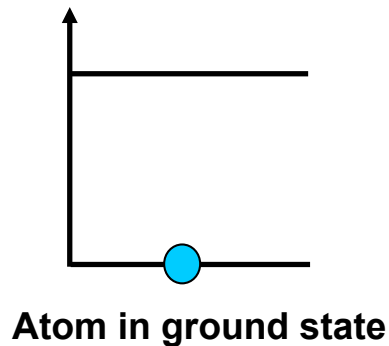
when  $E \gg E_f$ ,  $f(E)$  behaves like the **Boltzmann distribution**:

$$P(E) \propto \exp\left[-\frac{E - E_f}{k_B T}\right]$$

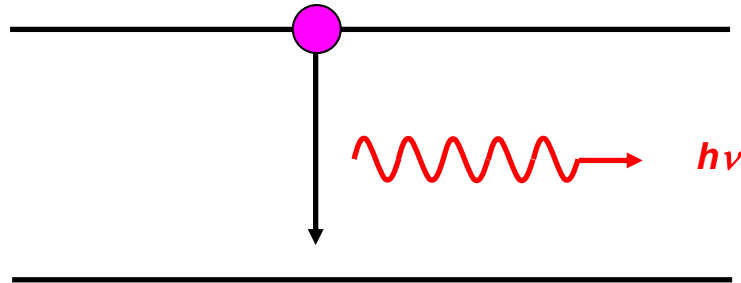


### 3. Interactions of photons with atoms

- Semi-classical view of atom excitations



# Spontaneous Emission 自发辐射



Spontaneous emission: a photon is generated into the mode of frequency  $\nu$  by an atomic transition from energy level 2 to level 1.

The photon energy:  $h\nu = E_2 - E_1$

**Probability Density** (rate) of Spontaneous Emission into a Single Prescribed Mode is

$$P_{sp} = \frac{c}{V} \sigma(\nu)$$

“probability density” signifies that the probability of an emission taking place in an incremental time interval

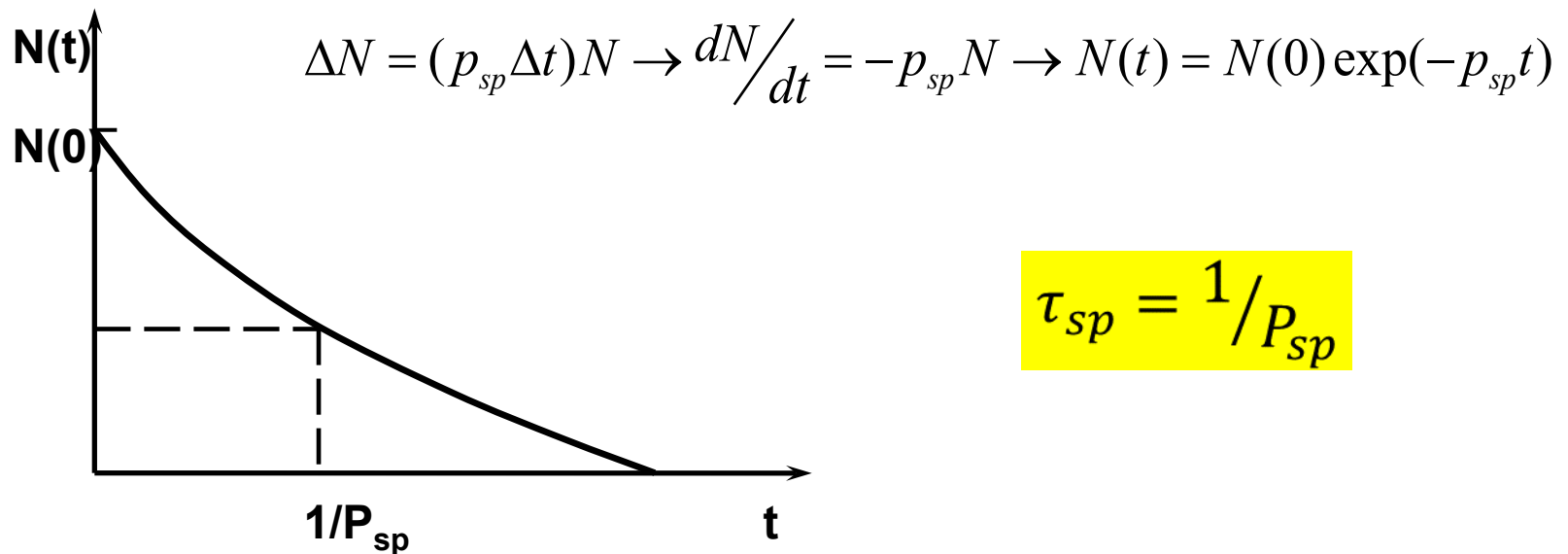
Unit is  $s^{-1}$

May be bigger than 1

The function  $\sigma(\nu)$  is a narrow function of  $\nu$  centered about the atomic resonance frequency  $\nu_0$ ; it is known as the **transition cross section**.

# Decay of the number of the excited atoms

The number of atoms in the excited state will be decreases Spontaneously as the time passes



Spontaneous emission into a single mode causes the number of excited atoms to decrease exponentially with time constant  $1/p_{sp}$

$$N(t) = N(0) \exp(-P_{sp} t)$$

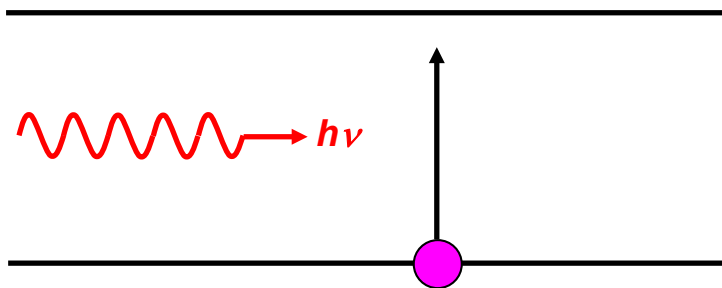
# Absorption

If the atom is initially in the lower energy level, the radiation mode contains a photon, the photon may be absorbed, thereby raising the atom to the upper energy level. The process is called **absorption**.

The probability density for the absorption

$$p_{ab} = \frac{c}{V} \sigma(\nu)$$

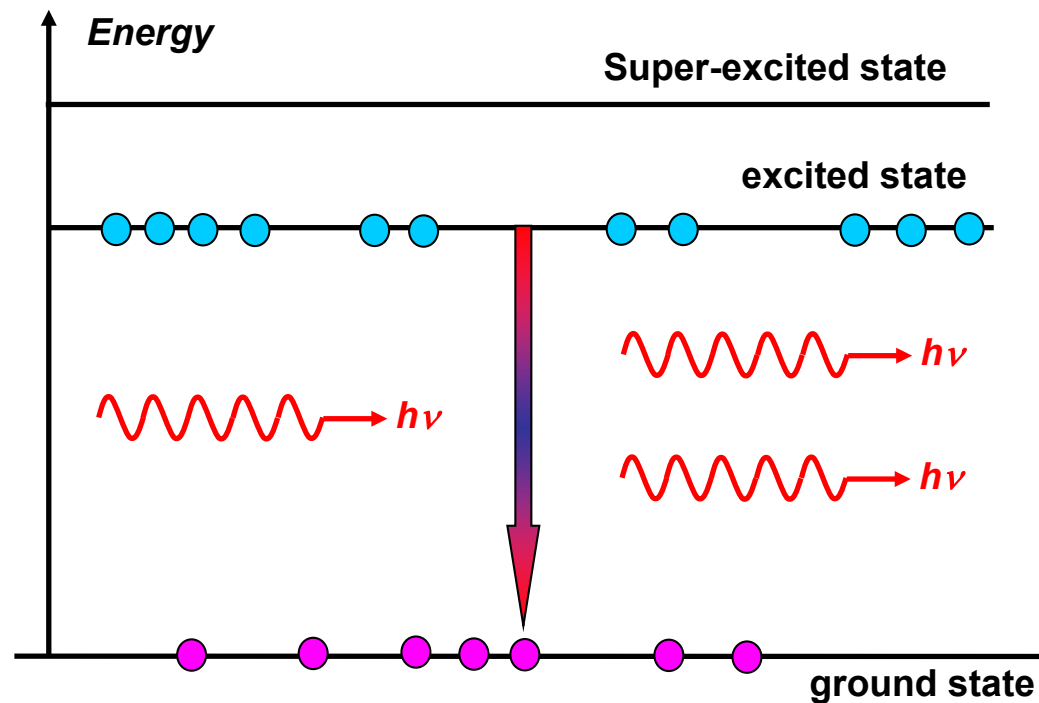
if there are  $n$  photons in the mode, the probability density that the atom absorbs one photon is  $n$  times greater



$$P_{ab} = n \frac{c}{V} \sigma(\nu)$$

# Stimulated Emission

if the atom is in the upper energy level and the mode contains a photon, the atom may be stimulated to emit another photon into the same mode. The presence of a photon in a mode of specified frequency, direction of propagation, and polarization stimulates the emission of a duplicate (“clone”) photon with precisely the same characteristics as the original photon



### Stimulated Emission

the probability density  $p_{st}$  that this process occurs in a cavity of volume  $V$  is governed by the same transition cross section,

$$p_{st} = \frac{c}{V} \sigma(\nu)$$

As in the case of absorption, if the mode originally carries  $n$  photons, probability density that the atom is stimulated to emit an additional photon is

$$P_{st} = n \frac{c}{V} \sigma(\nu)$$

After the emission, the radiation mode carries  $n + 1$  photons.

Since  $P_{st} = P_{ab}$ ,  
we use the notation  $W_i$  for the probability density of both stimulated emission and absorption.

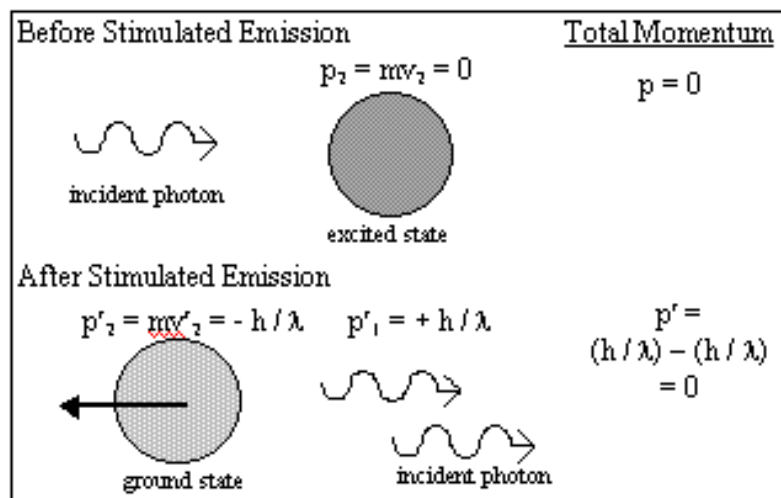
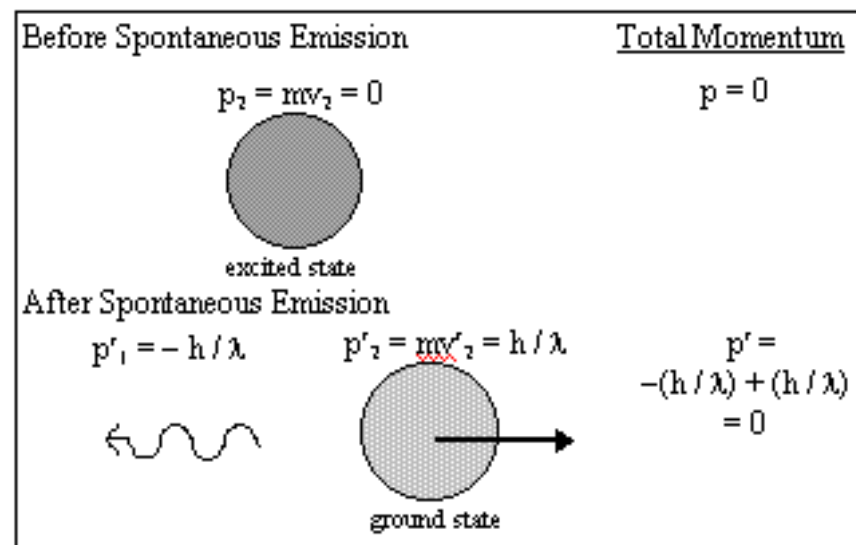
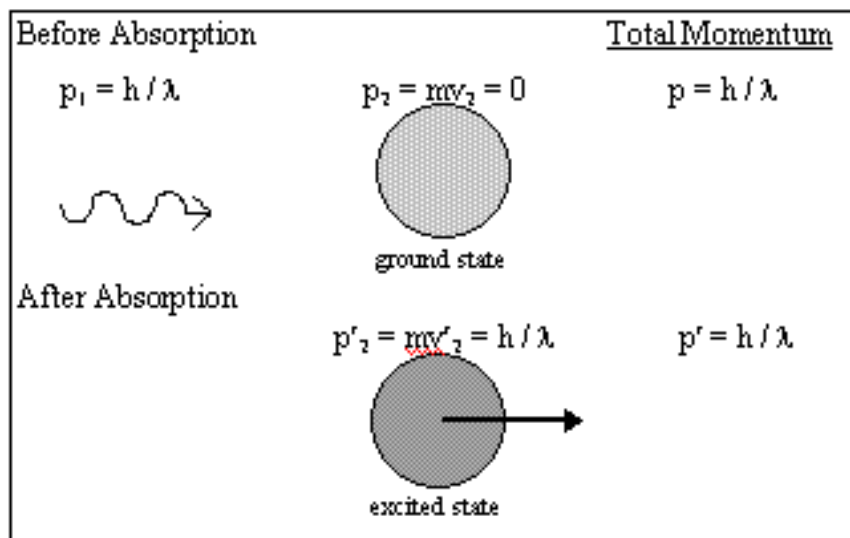
The photon emission is due to spontaneous emission and stimulated emission, the total probability density of the atom emitting a photon into the mode is :

$$P_{sp} + P_{st} = (n + 1)(c/V)\sigma(\nu)$$

from a quantum electrodynamic point of view, **spontaneous emission may be regarded as stimulated emission induced by the zero-point fluctuations of the mode.**

Because the zero-point energy is inaccessible for absorption,  $P_{ab}$  is proportional to  $n$  rather than to  $(n + 1)$ .

# Conservation momentum





Consider a single atom in the beam of a laser. Let the beam propagate in the positive x direction.

The total change in the x component of momentum of the atom given absorption, spontaneous emission, and stimulated emission can be obtained using the expression:

$$\Delta p_x = \sum_{i=1}^n \frac{h}{\lambda} + \sum_{i=1}^j \frac{h}{\lambda} \cos \theta_i - \sum_{i=1}^k \frac{h}{\lambda}$$

where n, j, and k represent integers such that  $n = j + k$  and represents a random angle between 0 and  $2\pi$ .

Because the momentum of spontaneous emission is:

$$\sum_{i=1}^j \frac{h}{\lambda} \cos \theta_i, \text{ for } j = 0, \text{ for } j \gg 1$$

In case of the rate of spontaneous emission is much greater than the rate of stimulated emission,  
and the stimulated emission term is negligible since  $j \gg k$ ,

then, After numerous absorptions, though:

$$\Delta p_x \cong n \frac{h}{\lambda}$$

## ***The Lineshape Function***

The **transition cross section**  $\sigma(\nu)$  specifies the character of the interaction of the atom with the radiation. Its area

$$S = \int_0^{\infty} \sigma(\nu) d\nu$$

which has units of  $\text{cm}^2\text{Hz}$ , is called the **transition strength** or **oscillator strength**, and represents the strength of the interaction.

***Lineshape function*** is defined as

$$g(\nu) = \sigma(\nu)/S$$

The transition cross section  $\sigma(\nu)$  is normalized by the transition strength, with units of  $\text{Hz}^{-1}$ , and has the same profile as  $\sigma(\nu)$

there are

$$\int_0^{\infty} g(\nu) d\nu = 1$$

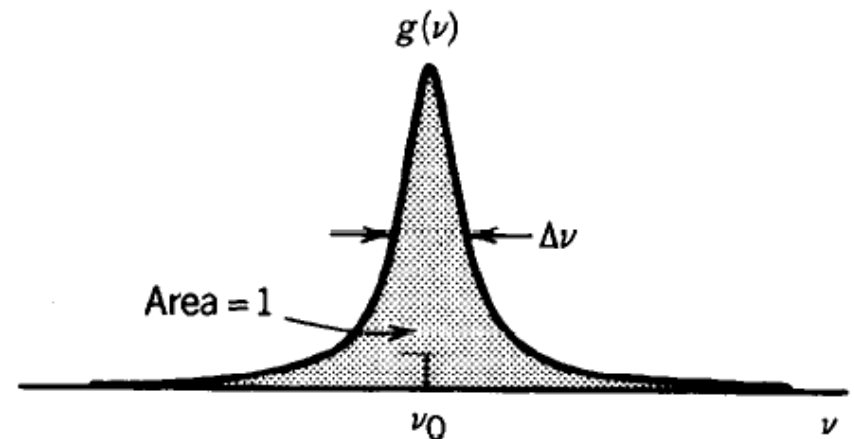
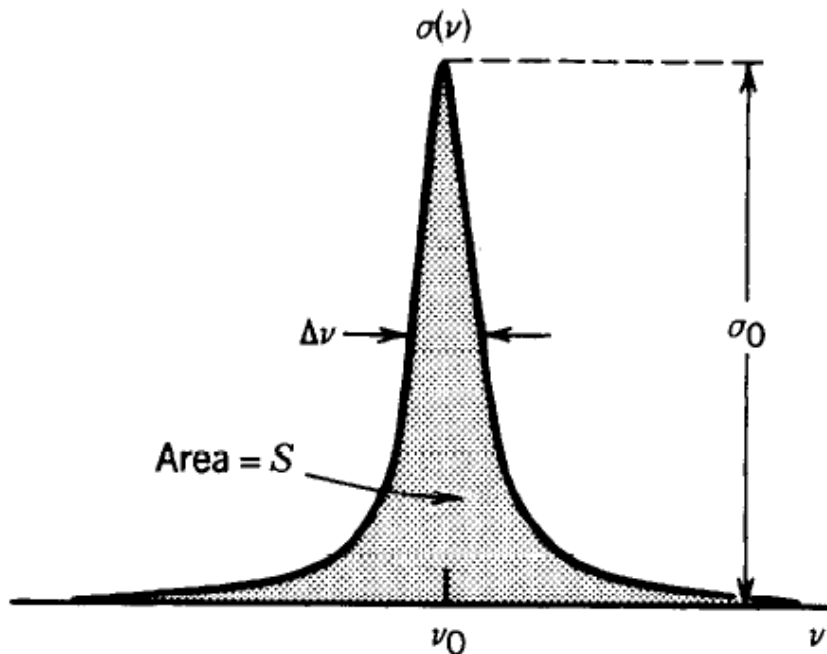
and

$$\sigma(\nu) = Sg(\nu)$$

The width of the function  $g(\nu)$  is known as the **transition linewidth**. The linewidth  $\Delta\nu$  is defined as the full width of the function  $g(\nu)$  at half its maximum value (FWHM).

$$\Delta\nu \propto \frac{1}{g(\nu_0)}$$

The function  $\sigma(\nu)$  is characterized by its height  $\sigma_0$ , width  $\Delta\nu$ , area  $S$ , and profile  $g(\nu)$



# Lineshape function $g(\nu)$

Lineshape is caused by the lifetime effect of the atom at the excited states.

The transition of the atom from excited state to low states creates lifetime, and the lifetime  $\tau$  of an energy level is related to the time uncertainty of the occupation of that level.

- if we assume the amplitude decay of an oscillation electron can be expressed as, an exponentially decaying harmonic function of time:

$$x(t) \sim e^{-t/2\tau} e^{2i\pi\nu_0 t}$$

- the Fourier transform of this amplitude, which has an energy that decays as  $e^{-t/\tau}$  (with time constant  $\tau$ ), is proportional to:

$$x(\nu) \sim 1/[1 + i4\pi(\nu - \nu_0)\tau]$$

- The self radiation is proportional to the amplitude square  $P(\nu) \sim \{x(\nu)\}^2$
- so that the lineshape function  $g(\nu)$  is

$$g(\nu) = P(\nu)/P \quad \longrightarrow \quad g(\nu) = \frac{\Delta\nu / 2\pi}{(\nu - \nu_0)^2 + (\Delta\nu / 2)^2}$$

## B. Spontaneous Emission

Total Spontaneous Emission into All Modes

Let's remember that the density of modes for a three-dimensional cavity is

$$M(\nu) = 8\pi\nu^2 / c^3$$

The probability density of spontaneous emission into a single prescribed mode must be weighted by the modal density.

$$P_{sp} = \int_0^\infty \left[ \frac{c}{V} \sigma(\nu) \right] [VM(\nu)] d\nu = c \int_0^\infty \sigma(\nu) M(\nu) d\nu \approx cM(\nu_0) \int_0^\infty \sigma(\nu) d\nu$$

so that we have:

$$P_{sp} = M(\nu_0)cS = \frac{8\pi S}{\lambda^2}$$

We define a time constant  $t_{sp}$  known as the **spontaneous lifetime** of the  $2 \rightarrow 1$  transition, such that:  $1/t_{sp} = P_{sp} = M(\nu_0)cS$ . Thus

$$P_{sp} = \frac{1}{t_{sp}}$$

Probability Density of Spontaneous  
Emission of One Photon into Any Mode

$$P_{sp} = \frac{1}{t_{sp}}$$

it is important to note, is ***independent of the cavity volume V.***

We can therefore express S as

$$S = \frac{\lambda^2}{8\pi t_{sp}}$$

the **transition strength** is determined from an experimental measurement of the spontaneous lifetime  $t_{sp}$ .

Typical values of  $t_{sp}$  are  $= 10^{-8}$  s for atomic transitions

## Relation Between the Transition Cross Section and the Spontaneous Lifetime

the **transition cross section** is related to the spontaneous lifetime and the lineshape function

$$\sigma(\nu) = Sg(\nu) \quad \Rightarrow \quad \sigma(\nu) = \frac{\lambda^2}{8\pi t_{sp}} g(\nu)$$

the transition cross section at the central frequency  $\nu_0$  is

$$\sigma_0 = \sigma(\nu_0) = \frac{\lambda^2}{8\pi t_{sp}} g(\nu_0)$$

because  $\Delta\nu \propto \frac{1}{g(\nu_0)}$

so that the peak transition cross section  $\sigma(\nu_0)$  is inversely proportional to the linewidth  $\Delta\nu$  for a given  $t_{sp}$ .

$$\sigma(\nu_0) \propto \frac{1}{\Delta\nu}$$



## C. Stimulated Emission and Absorption

### *Transitions Induced by Monochromatic Light*

A monochromatic light of frequency  $\nu$ , intensity  $I$ , and mean photon-flux density (photons/cm<sup>2</sup>-s): interact with an atom having a resonance frequency  $\nu_0$

$$\phi = \frac{I}{h\nu}$$

a volume in the form of a cylinder of area  $A$  and height  $c$  (the speed of light), the cylinder has a volume  $V=cA$ . The photon flux across the cylinder base is  $n=\phi A$  (photons per second), or

$$n = \phi \frac{V}{c}$$

because

$$P_{ab} = n \frac{c}{V} \sigma(\nu)$$

We have

$$W_i = P_{ab} = \phi \sigma(\nu)$$

$\phi \sigma(\nu)$  is the photon flux “captured” by the atom for the purpose of absorption or stimulated emission.

### ***Transitions in the Presence of Broadband Light***

An atom in a cavity of volume  $V$  illuminated by the multimode polychromatic light of spectral energy density  $\rho(\nu)$  (energy per unit bandwidth per unit volume) that is broadband in comparison with the atomic linewidth. The average number of photons in the  $\nu$  to  $\nu+d\nu$  band is  $\sim \rho(\nu)Vd\nu/\hbar\nu$ , each with a probability density  $(c/V)\sigma(\nu)$  of initiating an atomic transition, so that

$$W_i = \int_0^\infty \frac{\rho(\nu)V}{h\nu} \left[ \frac{c}{V} \sigma(\nu) \right] d\nu \approx \frac{\rho(\nu_0)}{h\nu_0} c \int_0^\infty \sigma(\nu) d\nu = \frac{\rho(\nu_0)}{h\nu_0} cS$$

Because  $S = \frac{\lambda^2}{8\pi t_{sp}}$  then  $W_i = \frac{\lambda^3}{8\pi h t_{sp}} \rho(\nu_0)$

where  $\lambda = c/\nu_0$  is the wavelength (in the medium) at the central frequency  $\nu_0$

Defining the mean number of photons per mode,

$$\bar{n} = \frac{\lambda^3}{8\pi h} \rho(\nu_0)$$

We have

$$W_i = \frac{\bar{n}}{t_{sp}}$$

The interpretation of  $\bar{n}$  follows from the ratio:

$$W_i/P_{sp} = \rho(\nu_0)/h\nu_0 M(\nu_0).$$

*The probability density  $W_i$  is a factor of  $\bar{n}$  greater than that for spontaneous emission since each of the modes contains an average of  $\bar{n}$  photons.*

# Einstein Coefficients

Einstein had proposed two coefficient A and B to describe the photon and atom interaction

The Einstein's A and B coefficients have relations as:

$$\begin{cases} P_{sp} = A \\ W_i = B\rho(\nu_0) \end{cases}$$

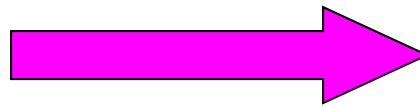
Using the above relation, we can have another form of A and B, as

$$\begin{cases} A = \frac{1}{t_{sp}} \\ B = \frac{\lambda^3}{8\pi h t_{sp}} \end{cases} \quad \Rightarrow \quad \frac{B}{A} = \frac{\lambda^3}{8\pi h}$$

$$P_{sp} = A$$

$$W_i = B\rho(\nu_0)$$

$$P_{sp} = \frac{1}{t_{sp}}$$



$$A = \frac{1}{t_{sp}}$$

$$B = \frac{\lambda^3}{8\pi h t_{sp}}$$

In quantum mechanics, the energy of an electron in an atom or molecule have discrete values. Namely  $E_n$  and  $E_m$ . The transition between these levels can be in three ways called respectively **absorption, spontaneous emission and stimulated emission**.

- **Absorption** is simply the excitation of electron to upper level. You may think it as an electromagnetic wave (photon) providing an electron harmonic oscillation with its altering fields. In fact this is obvious from our previous model on harmonic oscillation of dipoles.
- We also said before that this oscillation comes to an end after some time this is analogous to what we call **spontaneous emission**, after some time which is called relaxation time—analogueous to radiative lifetime- our electron returns to its original state.

- **Stimulated emission**, simply an extra photon comes and kicks a photon which is already at higher energy level and makes it emit a photon corresponding to energy of transition. In fact it is explained by Einstein as elastic collision because we observe no energy change of incoming photon only its momentum changes to opposite direction.

$$P_{st} = n \frac{c}{V} \sigma(\nu)$$

- All of these transitions are quantum mechanical and instantaneous, they do not occur gradually as in classical world and the only way to make instantaneous things meaningful for classical world is to work statistically.

## D. Lineshape Broadening

### *Lineshape $g(\nu)$ ----- Lifetime broadening*

Lifetime broadening is, in essence, a Fourier transform effect. The lifetime  $\tau$  of an energy level is related to the time uncertainty of the occupation of that level.

- if we assume the amplitude decay of an oscillation electron can be expressed as, an exponentially decaying harmonic function of time:

$$x(t) \sim e^{-t/2\tau} e^{2i\pi\nu_0 t}$$

- the Fourier transform of this amplitude, which has an energy that decays as  $e^{-t/\tau}$  (with time constant  $\tau$ ), is proportional to:

$$x(\nu) \sim 1/[1 + i4\pi(\nu - \nu_0)\tau] \quad \Rightarrow \quad \Delta\nu = 1/2\pi\tau$$

- The self radiation is proportional to the amplitude square  $P(\nu) \sim \{x(\nu)\}^2$  so that the lineshape function  $g(\nu)$  is

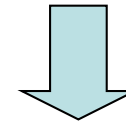
$$g(\nu) = P(\nu)/P \quad \Rightarrow \quad g(\nu) = \frac{\Delta\nu / 2\pi}{(\nu - \nu_0)^2 + (\Delta\nu / 2)^2}$$



## Lorentzian lineshape function

$$g(\nu) = \frac{\Delta\nu / 2\pi}{(\nu - \nu_0)^2 + (\Delta\nu / 2)^2}$$

where  $\nu_0 = (E_2 - E_1)/h$

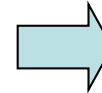


$$g(\nu_0) = 2/\pi\Delta\nu$$

$$g(\nu) = \frac{\Delta\nu / 2\pi}{(\nu - \nu_0)^2 + (\Delta\nu / 2)^2} = g(\nu_0) \frac{(\Delta\nu / 2)^2}{(\nu - \nu_0)^2 + (\Delta\nu / 2)^2}$$

The full width at half-maximum (FWHM) of the square magnitude of this Lorentzian function of frequency is

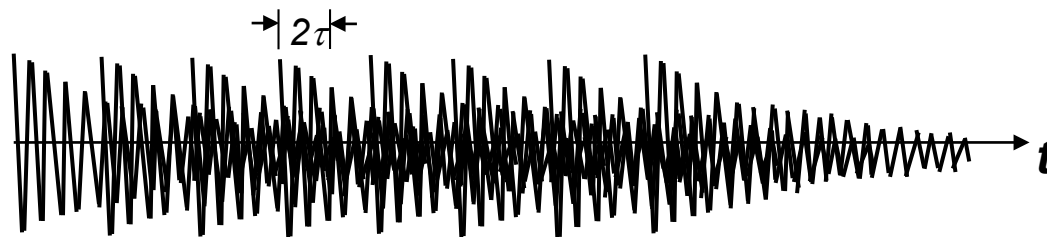
$$\Delta\nu = 1/2\pi\tau.$$



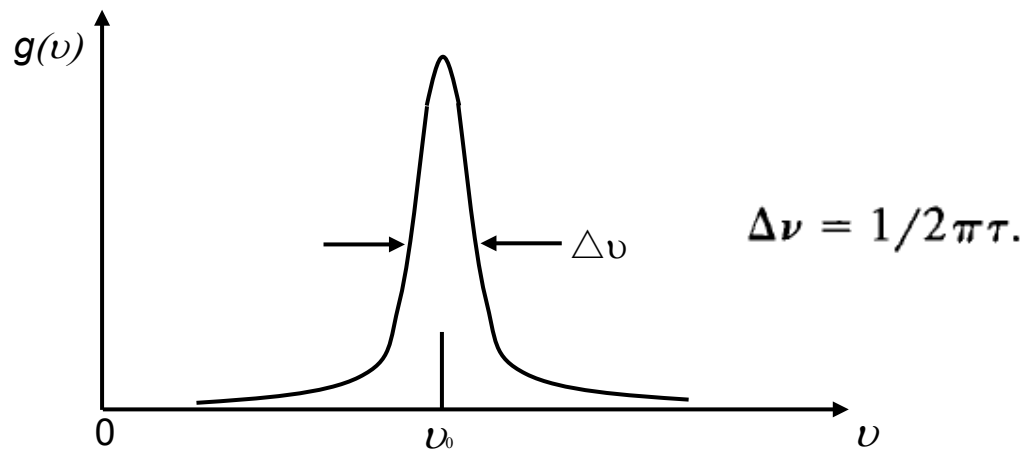
$$\tau = \frac{g(\nu_0)}{4}$$

This spectral uncertainty corresponds to an energy uncertainty :

$$\Delta E = h\Delta\nu = h/2\pi\tau.$$



Each of the photons emitted from the transition represents a wavepacket of central frequency  $\nu_0$  (the transition resonance frequency), with an exponentially decaying envelope of decay time  $2\tau$  (i.e., with energy decay time equal to the transition lifetime  $\tau$ ),



Wavepacket emissions at random times from a lifetime broadened atomic system with transition lifetime  $\tau$ . The light emitted has a Lorentzian power spectral density of width  $\Delta\nu=1/2\pi\tau$

**Life-time broadening**

For 2 level system, we have

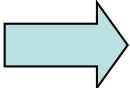
Because:  $\Delta E = h\Delta\nu$        $\Delta E = \Delta E_1 + \Delta E_2 = \frac{h}{2\pi} \left( \frac{1}{\tau_1} + \frac{1}{\tau_2} \right) = \frac{h}{2\pi} \frac{1}{\tau}$

So that:

$$\Delta\nu = \frac{1}{2\pi} \left( \frac{1}{\tau_1} + \frac{1}{\tau_2} \right)$$

called the ***lifetime-broadening linewidth***

$$\frac{1}{\tau} = \frac{1}{\tau_1} + \frac{1}{\tau_2}$$

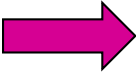
from  $g(\nu) = \frac{\Delta\nu/2\pi}{(\nu - \nu_0)^2 + (\Delta\nu/2)^2}$    $g(\nu_0) = 2/\pi\Delta\nu$

because  $\sigma_0 \equiv \sigma(\nu_0) = \frac{\lambda^2}{8\pi t_{sp}} g(\nu_0).$

So that we have

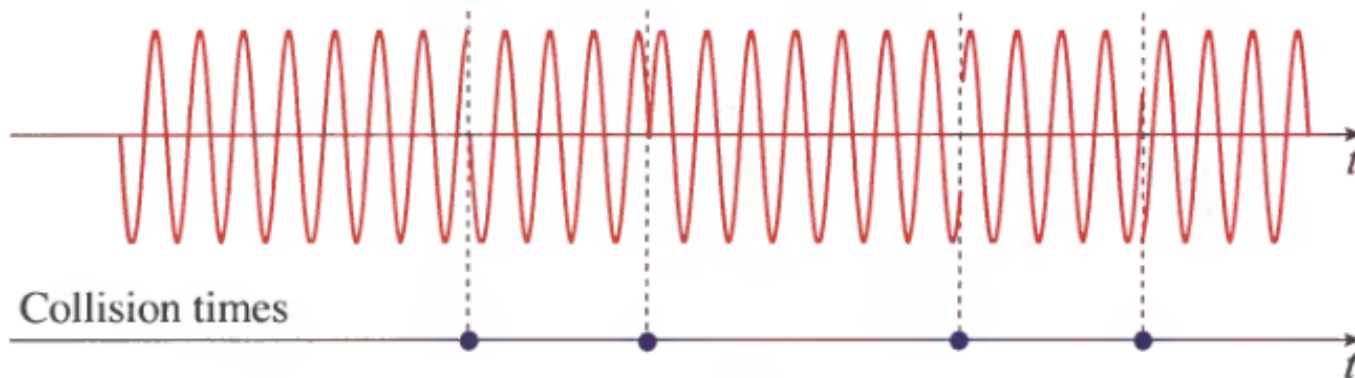
$$\sigma_0 = \frac{\lambda^2}{2\pi} \frac{1}{2\pi t_{sp} \Delta\nu}$$

The largest transition cross section occurs under ideal conditions when the decay is entirely radiative so that  $\tau_2 = t_{sp}$  and  $1/\tau_1 = 0$  (which is the case when level 1 is the ground state from which no decay is possible). Then  $\Delta\nu = 1/2\pi t_{sp}$  therefore we have

$\sigma_{\max}$    $\sigma_0 = \frac{\lambda^2}{2\pi}$

# Collision Broadening

Inelastic collisions, energy is exchanged, result in atomic transitions between energy levels. This contribution to the decay rates affects the lifetimes of all levels involved and hence the linewidth of the radiated field, as indicated above. Elastic collisions, on the other hand, do not involve energy exchange. Rather, they cause random phase shifts of the wavefunction associated with the energy level, which in turn results in a random phase shift of the radiated field at each collision time.



A sinewave interrupted at the rate  $f_{col}$  by random phase jumps has a Lorentzian spectrum of width  $\Delta \nu = f_{col}/\pi$

Collisions, they cause random phase shifts of the wavefunction associated with the energy level, which in turn results in a random phase shift of the radiated field at each collision time.

Collisions between atoms provide a source of such line broadening, exhibits spectral broadening.

The spectrum turns out to be Lorentzian, with width:

$$\Delta\nu = f_{col} / \pi,$$

where  $f_{col}$  is the collision rate (mean number of collisions per second).

Adding the linewidths arising from lifetime and collision broadening therefore results in an overall Lorentzian lineshape of linewidth

$$\Delta\nu = \frac{1}{2\pi} \left( \frac{1}{\tau_1} + \frac{1}{\tau_2} + 2f_{col} \right)$$

There are two kinds of line shape broadening

***homogeneous broadening***

***inhomogeneous broadening***

## ***Homogeneous broadening***

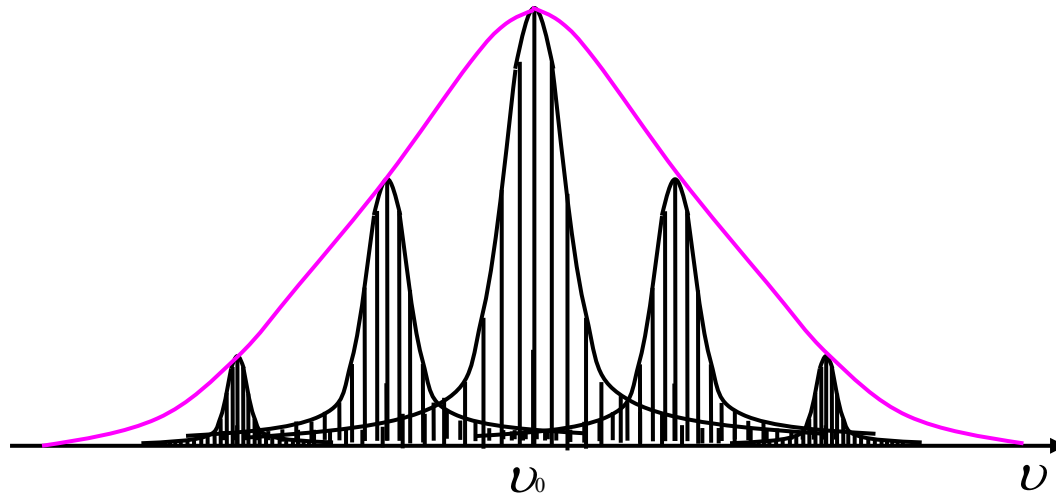
**Homogeneous broadening:** All of the atoms are assumed to be identical and to have identical lineshape functions.

Lifetime broadening and collision broadening are forms of **homogeneous broadening** that are exhibited by the atoms of a medium.

## Inhomogeneous Broadening

In many situations, the different atoms' constituting a medium have different lineshape functions or different center frequencies. In this case we can define an average lineshape function

$$\bar{g}(\nu) = \langle g_{\beta}(\nu) \rangle$$



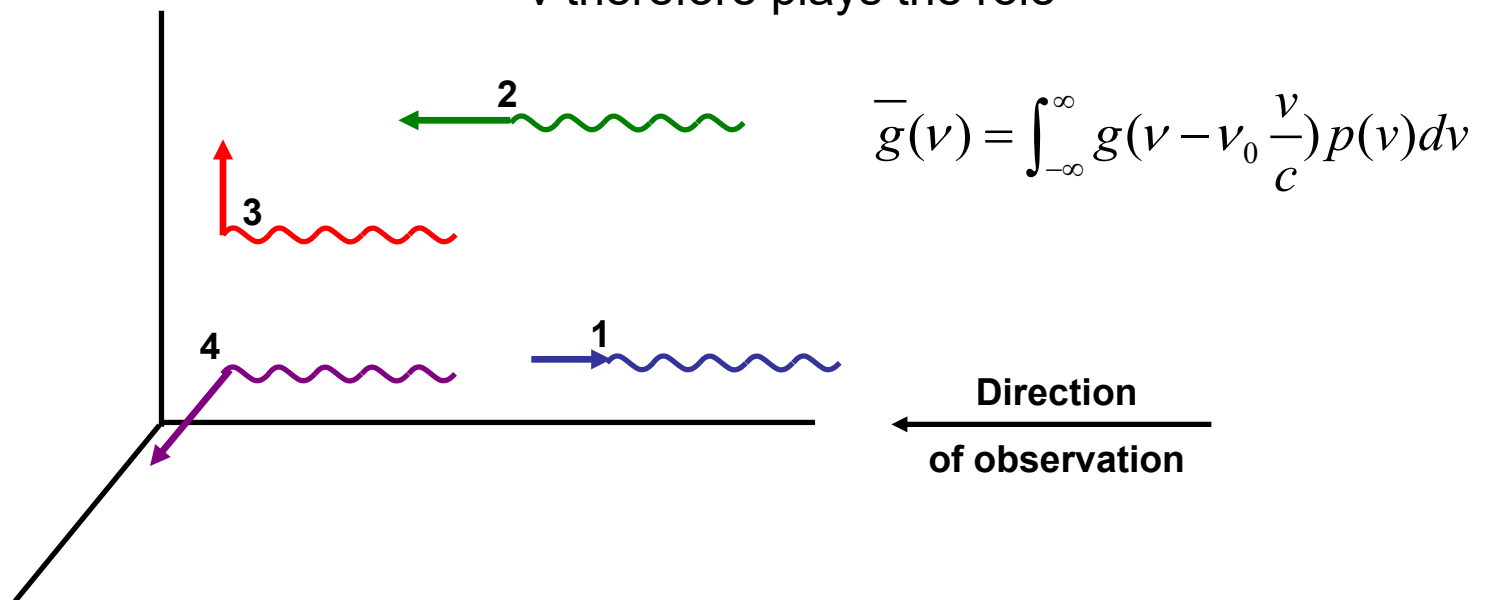
The average lineshape function of an inhomogeneously broadened collection of atoms



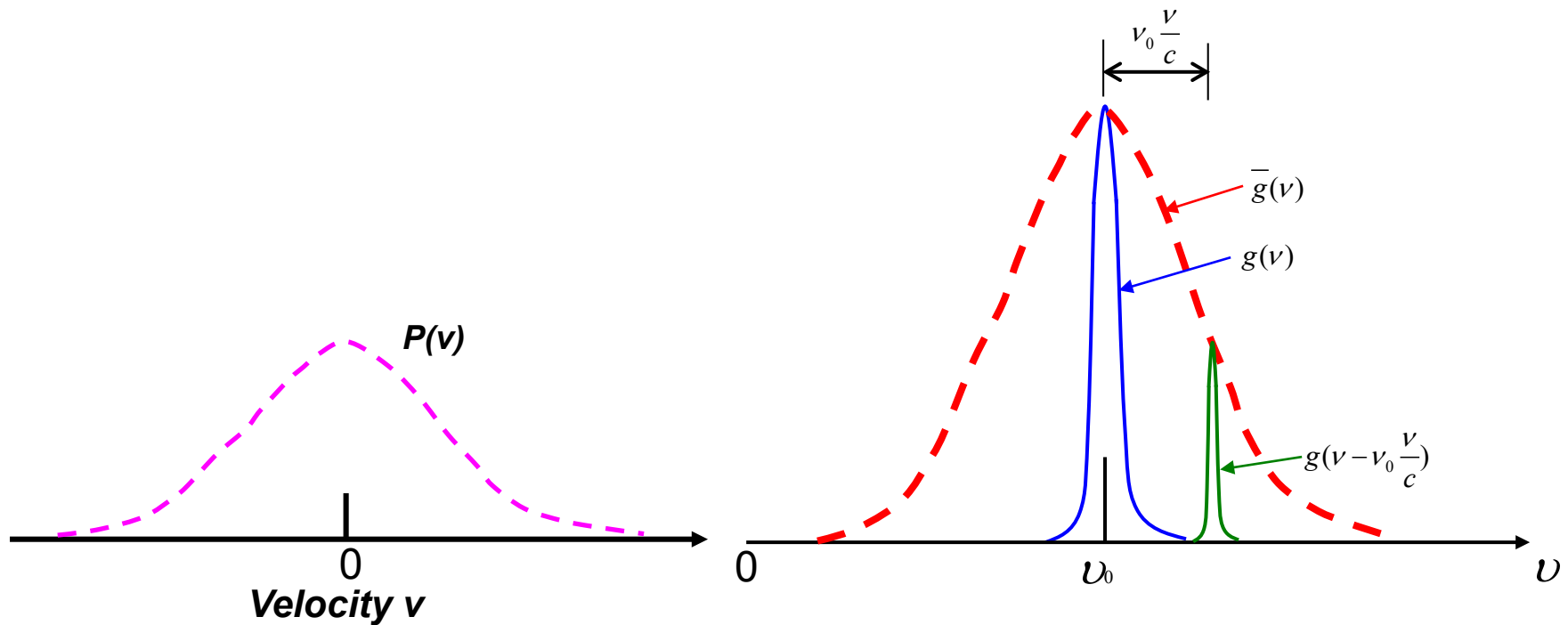
# Inhomogeneous Broadening

- Doppler effect

In the case of Doppler broadening, the velocity  $v$  therefore plays the role



One inhomogeneous broadening mechanism is Doppler broadening. As a result of the Doppler effect, an atom moving with velocity  $v$  along a given direction exhibits a spectrum that is shifted by the frequency  $\pm(v/c)\nu_0$ , where  $\nu_0$  is its central frequency,



The velocity distribution and average lineshape function of a Doppler-broadened atomic system.

$$\bar{g}(\nu) = \frac{1}{\sqrt{2\pi}\sigma_D} \exp\left[-\frac{(\nu - \nu_0)^2}{2\sigma_D^2}\right]$$

$$\sigma_D = \nu_0 \frac{\sigma_V}{c} = \frac{1}{\lambda} \left(\frac{k_B T}{M}\right)^{1/2}$$

$M$ : atom mass,  $T$  is temperature

The component of velocity  $v$  of atoms of a gas have a Gaussian probability density function

$$p(v) = \frac{1}{\sqrt{2\pi}\sigma_v} \exp\left(-\frac{v^2}{2\sigma_v^2}\right)$$

where  $\sigma_v^2 = k_B T / M$   $M$  is the atomic mass

if  $\Delta v \ll v_0 \sigma_v / c$  then

$$g(v) = \frac{1}{\sqrt{2\pi}\sigma_D} \exp\left[-\frac{(v - v_0)^2}{2\sigma_D^2}\right]$$

where

$$\sigma_D = v_0 \frac{\sigma_v}{c} = \frac{1}{\lambda} \left(\frac{k_B T}{M}\right)^{1/2}$$

The full-width half-maximum (FWHM) Doppler linewidth  $\Delta v_D = (8 \ln 2)^{1/2} \sigma_D \approx 2.35 \sigma_D$

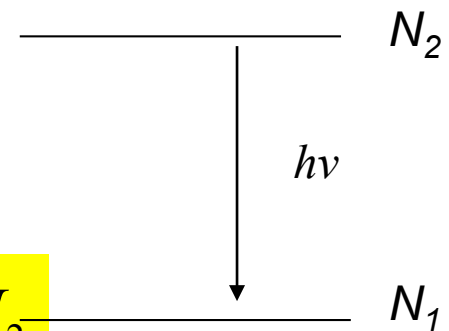
the maximum value of the transition cross section for the Gaussian lineshape is

$$\sigma_0 = \frac{\lambda^2}{8\pi} \left(\frac{4 \ln 2}{\pi}\right)^{1/2} \frac{1}{t_{sp} \Delta v_D} \approx 0.94 \frac{\lambda^2}{8\pi} \frac{1}{t_{sp} \Delta v_D}$$

## 4. Thermal light

Light emitted from atoms, molecules, and solids, under conditions of thermal equilibrium and in the absence of other external energy sources.

### Thermal Equilibrium Between Photons and Atoms



$$\frac{dN_2}{dt} = -\frac{N_2}{t_{sp}}$$

$$\frac{dN_2}{dt} = N_1 W_i = \frac{N_1 \bar{n}}{t_{sp}}$$

$$\frac{dN_2}{dt} = -\frac{N_2 \bar{n}}{t_{sp}}$$

Rate equation

$$\frac{dN_2}{dt} = -\frac{N_2}{t_{sp}} + \frac{\bar{n}N_1}{t_{sp}} - \frac{\bar{n}N_2}{t_{sp}}$$

In steady state  $\frac{dN_2}{dt} = 0$

we have 
$$\frac{N_2}{N_1} = \frac{\bar{n}}{1 + \bar{n}}$$

$$W_i = P_{sp} = P_{st} = W_i = \frac{\bar{n}}{t_{sp}}$$

where  $\bar{n}$  is the average number of photons per mode.

the atoms are in thermal equilibrium, their populations obey the Boltzmann distribution,

$$\frac{N_2}{N_1} = \exp\left(-\frac{E_2 - E_1}{k_B T}\right) = \exp\left(-\frac{h\nu}{k_B T}\right)$$

with

$$\frac{N_2}{N_1} = \frac{\bar{n}}{1 + \bar{n}}$$

We have: The average number of photons in a mode of frequency  $\nu$

$$\bar{n} = \frac{1}{\exp(h\nu / k_B T) - 1}$$

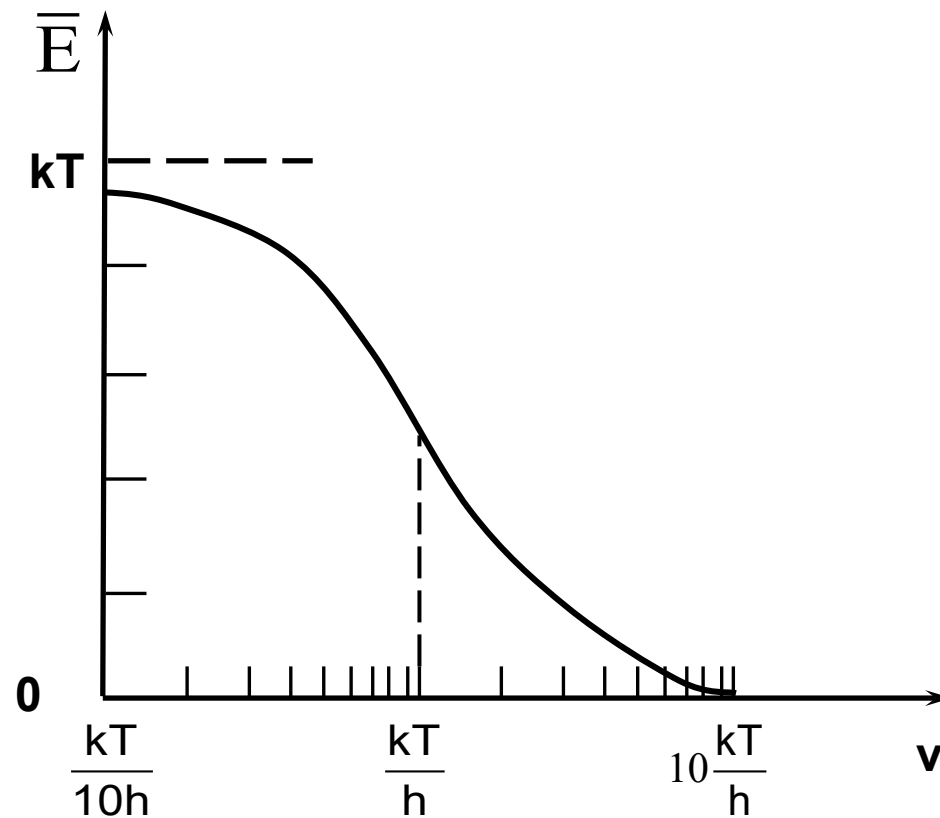
# Black-body Radiation

The average energy  $\bar{E}$  of a radiation mode is simply  $h\nu$ , so that

$$\bar{E} = \frac{h\nu}{\exp(h\nu / k_B T) - 1}$$

The spectral energy density (energy per unit bandwidth per unit cavity volume) gives rise to a  $\rho(\nu) = M(\nu)\bar{E}$ , that is : Multiplying this expression for the average energy per mode  $\bar{E}$ , by the modal density  $M(\nu) = 8\pi\nu^2/c^3$ , i.e.

$$\rho(\nu) = \frac{8\pi h\nu^3}{c^3} \frac{1}{\exp(h\nu / k_B T) - 1}$$



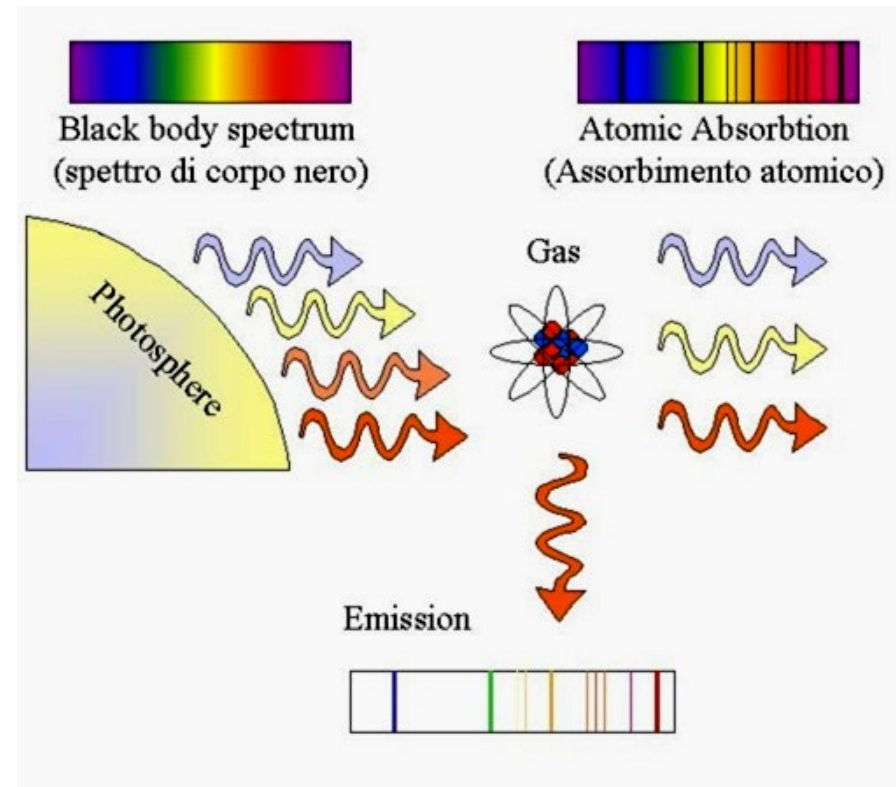
Semilogarithmic plot of the average energy  $\bar{E}$  of an electromagnetic mode in thermal equilibrium at temperature  $T$  as a function of the mode frequency  $\nu$ . At  $T=300\text{K}$ ,  $k_B T/h = 6.25\text{THz}$ , which corresponds to a wavelength of  $48\mu\text{m}$ .

## 2.4 LUMINESCENCE LIGHT

An applied external source of energy may cause an atomic or molecular system to undergo transitions to higher energy levels. In the course of decaying to a lower energy, the system may subsequently emit optical radiation.

□ Such “nonthermal” radiators are generally called luminescent radiators and the radiation process is called **luminescence**.

□ Luminescent radiators are classified according to the source of excitation energy





- **Cathodoluminescence** is caused by accelerated electrons that collide with the atoms of a target. An example is the cathode ray tube where electrons deliver their energy to a phosphor. The term betaluminescence is used when the fast electrons are the product of nuclear beta decay rather than an electron gun, as in the cathode-ray tube.
- **Photoluminescence** is caused by energetic optical photons. An example is the glow emitted by some crystals after irradiation by ultraviolet light. The term radioluminescence is applied when the energy source is x-ray or gamma-ray photons, or other ionizing radiation. Indeed, such high-energy radiation is often detected by the use of luminescent (scintillation) materials such as NaI, special plastics, or PbCO, in conjunction with optical detectors.
- **Chemiluminescence** provides energy through a chemical reaction. An example is the glow of phosphorus as it oxidizes in air. Bioluminescence, which characterizes the light given off by living organisms (e.g., fireflies and glowworms), provides another example of chemiluminescence.
- **Electroluminescence** results from energy provided by an applied electric field. An important example is injection electroluminescence, which occurs when electric current is injected into a forward-biased semiconductor junction diode. As injected electrons drop from the conduction band to the valence band, they emit photons. An example is the light-emitting diode (LED).
- **Sonoluminescence** is caused by energy acquired from a sound wave. The light emitted by water under irradiation by a strong ultrasonic beam is an example.

## Exercises

如果工作物质的某一跃迁波长为100nm，自发辐射跃迁几率 $A_{10}$ 等于 $10^6/s$ ，问：该跃迁的受激辐射爱因斯坦系数 $B_{10}$ 是多少？为了使受激跃迁几率比自发跃迁几率大3倍，腔内单色光能量密度应该为多少？

如果受激辐射爱因斯坦系数 $B_{10}=10^{19}m^3s^{-3}W^{-1}$ ，试求：波长为：6mm，600nm，60nm的各色光的自发辐射跃迁几率 $A_{10}$ 与自发辐射寿命

某一分子的能级E4到三个较低能级E1，E2，E3的自发辐射跃迁几率分别是： $A_{43}=5*10^7s^{-1}$ ， $A_{42}=1*10^7s^{-1}$ ， $A_{41}=3*10^7s^{-1}$ ，试求该分子的E4能级的自发辐射寿命 $\tau_{s4}$ 。如果 $\tau_4=\tau_{s4}$ ， $\tau_1=5*10^{-7}s$ ， $\tau_2=6*10^{-7}s$ ， $\tau_3=1*10^{-7}s$ ，对E4连续激发并达到稳态时，求能级上的粒子数密度比， $n_1/n_4, n_2/n_4, n_3/n_4$ 并指出在哪两个能级之间实现了粒子数反转。

**作业： p.114-116    1, 3, 4, 6, 8, 11, 13, 15**