行波法

行波法,也称为特线法,是求解偏微分方程的一种重要方法。

- 从物理学来说,该方法反映了波沿着特征线传播这一事实。
- 从数学来说，先求微分方程的通解，再根据定解条件得到满足要求的特解。
- 该方法的不足之处是：只能用它来求解波传播方程的初值问题。

无界弦的自由振动

例. 无界弦的自由振动问题可归结为

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, -\infty < x < +\infty, t > 0 \\ u(x,t)|_{t=0} = \phi(x), \ \frac{\partial u}{\partial t}|_{t=0} = \psi(x). \end{cases}$$

解. 特征方程为$(dx)^2 - a^2(dt)^2 = 0 \Rightarrow (dx - adt)(dx + adt) = 0$
$\Rightarrow dx - adt = 0, \ dx + adt = 0, \Rightarrow$特征线族为

$$x - at = C_1, x + at = C_2.$$

作变量变换$\xi = x - at, \eta = x + at, u(x,t) = u(\xi,\eta)$

$$u_x = u_\xi + u_\eta, \ u_t = -au_\xi + au_\eta$$
$$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}, \ u_{tt} = a^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta})$$
$$\text{原方程} \Rightarrow -4a^2 u_{\xi\eta} = 0 \Rightarrow u_{\xi\eta} = 0 \Rightarrow u_\xi(\xi,\eta) = f_1(\xi)$$
$$\Rightarrow u(\xi,\eta) = \int f_1(\xi)d\xi + f_2(\eta) = F(\xi) + G(\eta),$$

由$\xi = x - at, \eta = x + at$得$u(x, y) = F(x - at) + G(x + at)$,

$$\text{初值条件} \Rightarrow \quad u(x,t)\,|_{t=0} = F(x) + G(x) = \phi(x),$$
$$\frac{\partial u}{\partial t}\,|_{t=0} = a[-F'(x) + G'(x)] = \psi(x)$$
$$\Rightarrow \quad F(x) + G(x) = \phi(x),$$
$$-F(x) + G(x) = \frac{1}{a}\int_{x_0}^{x} \psi(s)ds + C$$
$$\Rightarrow \quad F(x) = \frac{1}{2}\phi(x) - \frac{1}{2a}\int_{x_0}^{x} \psi(s)ds - \frac{C}{2},$$
$$G(x) = \frac{1}{2}\phi(x) + \frac{1}{2a}\int_{x_0}^{x} \psi(s)ds + \frac{C}{2}$$
$$u(x,t) = \frac{\phi(x-at) + \phi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(s)ds.$$

称为达朗倍尔公式。

达朗倍尔公式: $u(x,t) = \frac{\phi(x-at)+\phi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at}\psi(s)ds$

例. 求解方程 $\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}\,(-\infty < x < +\infty)$ 满足初始条件

$$u(x,t)|_{t=0} = x,\ \frac{\partial u}{\partial t}|_{t=0} = 1$$

解. 利用达朗倍尔公式直接计算

$$\begin{aligned} u(x,y) &= \frac{(x-2t)+(x+2t)}{2} + \frac{1}{4}\int_{x-2t}^{x+2t} ds \\ &= x+t. \end{aligned}$$

例. 求解偏微分方程 $u_{tt}-3u_{xt}-4u_{xx}=0$ 满足初始条件

$$u(x,t)|_{t=0}=x,\ \frac{\partial u}{\partial t}|_{t=0}=1$$

解. 特征方程为$(dx)^2+3dxdt-4(dt)^2=0$,特征线为$x-t=C_1$, $x+4t=C_2$。作变换$\xi=x-t,\ \eta=x+4t$,

$$u_x=u_\xi+u_\eta,\ u_t=-u_\xi+4u_\eta,\ u_{xx}=u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta},$$
$$u_{tt}=u_{\xi\xi}-8u_{\xi\eta}+16u_{\eta\eta}, u_{xt}=-u_{\xi\xi}+3u_{\xi\eta}+4u_{\eta\eta},$$

$$\text{原方程}\Leftrightarrow -25u_{\xi\eta}=0$$
$$\Rightarrow u(\xi,\eta)=\int f_1(\xi)d\xi+f_2(\eta)=F(\xi)+G(\eta)$$
$$\Rightarrow u(x,y)=u(\xi,\eta)=F(x-t)+G(x+4t).$$

例. 求偏微分方程 $u_{tt}-3u_{xt}-4u_{xx}=0$ 满足初始条件 $u(x,t)|_{t=0}=x,\ \frac{\partial u}{\partial t}|_{t=0}=1$,

初值问题的解为: $u(x,y)=u(\xi,\eta)=F(x-t)+G(x+4t)$

$$\begin{aligned}\text{初值条件}\quad &\Rightarrow F(x)+G(x)=x,\ -F'(x)+4G'(x)=1\\ &\Rightarrow F(x)+G(x)=x,\ -F(x)+4G(x)=x+C\\ &\Rightarrow F(x)=\frac{4}{5}x-\frac{1}{5}C,\ G(x)=\frac{2}{5}x+\frac{1}{5}C.\end{aligned}$$

所以初值问题的解为

$$u(x,y)=F(x-t)+G(x+4t)=\frac{6}{5}x+\frac{4}{5}t.$$

无界弦的强迫振动问题

例. 无界弦的强迫振动问题可归结为

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t), -\infty < x < +\infty, t > 0 \\ u(x,t)|_{t=0} = \phi(x), \ \frac{\partial u}{\partial t}|_{t=0} = \psi(x). \end{cases}$$

利用叠加原理，可解为下列二个定解问题的解之和:

$$(I) \begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \hookleftarrow \text{齐次线性方程} \\ u(x,t)|_{t=0} = \phi(x), \ \frac{\partial u}{\partial t}|_{t=0} = \psi(x). \hookleftarrow \text{非齐次初始条件} \end{cases}$$

$$(II) \begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t), \hookleftarrow \text{非齐次线性方程} \\ u(x,t)|_{t=0} = 0, \ \frac{\partial u}{\partial t}|_{t=0} = 0. \hookleftarrow \text{齐次初始条件} \end{cases}$$

由达朗贝尔公式知定解问题(I)的解是

$$u_1(x,t) = \frac{\phi(x-at) + \phi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(s)ds.$$

定解问题(II):

$$(II)\begin{cases}\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}+f(x,t),-\infty<x<+\infty,t>0\\ u(x,t)|_{t=0}=0,\ \frac{\partial u}{\partial t}|_{t=0}=0.\end{cases}$$

利用**齐次化原理**(Duhamel's原理):
设 $w=w(x,t,\tau)$(其中τ是一个参数）是初值问题

$$\begin{cases}\frac{\partial^2 w}{\partial t^2}=a^2\frac{\partial^2 w}{\partial x^2},-\infty<x<+\infty,t>0\\ w(x,t,\tau)|_{t=0}=0,\ \frac{\partial w}{\partial t}|_{t=0}=f(x,\tau)\end{cases}$$

的解，则$u(x,t)=\int_0^t w(x,t-\tau,\tau)d\tau$ 是初值问题(II)的解。

无界弦的强迫振动问题

利用齐次化原理来求问题 (II) 的解。达朗倍尔公式，

$$w(x,t,\tau)=\frac{1}{2a}\int_{x-at}^{x+at}f(\xi,\tau)d\xi$$

$$u_2(x,t)=\int_0^t w(x,t-\tau,\tau)d\tau=\frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}f(\xi,\tau)d\xi d\tau.$$

无界弦的强迫振动问题的解为

$$\begin{aligned}u(x,t)\quad&=\quad\frac{\phi(x-at)+\phi(x+at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi\\&+\frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}f(\xi,\tau)d\xi d\tau.\end{aligned}$$

称为 **一维非齐次波动方程解的 Kirchoff 公式**。

齐次化原理的证明

$u(x,t) = \int_0^t w(x, t-\tau, \tau) d\tau$

由 $u(x,t)$ 的表达式有 $u(x,t)|_{t=0} = 0$,

$$
\begin{aligned}
u_t(x,t) &= w(x,0,t) + \int_0^t \frac{\partial}{\partial t} w(x, t-\tau, \tau) d\tau \\
&= \int_0^t \frac{\partial}{\partial t} w(x, t-\tau, \tau) d\tau, \\
u_t(x,t)|_{t=0} &= [w(x,0,t) + \int_0^t \frac{\partial}{\partial t} w(x, t-\tau, \tau) d\tau]_{t=0} = 0.
\end{aligned}
$$

齐次化原理的证明

$$
\begin{aligned}
u_{tt}(x,t) &= \frac{\partial}{\partial t} w(x,0,\tau)|_{\tau=t} + \int_0^t \frac{\partial^2}{\partial t^2} w(x,t-\tau,\tau)d\tau \\
&= f(x,t) + a^2 \int_0^t \frac{\partial^2}{\partial x^2} w(x,t-\tau,\tau)d\tau \\
&= f(x,t) + a^2 \frac{\partial^2}{\partial x^2} \int_0^t w(x,t-\tau,\tau)d\tau = f(x,t) + a^2 u_{xx}.
\end{aligned}
$$

因此

$$
u(x,t) = \int_0^t w(x,t-\tau,\tau)d\tau
$$

是初值问题(II)的解。

无界弦的强迫振动问题

例. 求解方程

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + e^x, -\infty < x < +\infty \\ u(x,t)|_{t=0} = x, \ \frac{\partial u}{\partial t}|_{t=0} = 1 \end{cases}$$

解法1 特征方程是$(dx)^2 = (dt)^2 \Rightarrow (dx-dt)(dx+dt) = 0$
$\Rightarrow x - t = C_1, x + t = C_2$.令 $\xi = x - t, \eta = x + t, \Rightarrow u_{\xi\eta} = -\frac{1}{4}e^{\frac{1}{2}(\xi+\eta)} \Rightarrow u_\eta = -\frac{1}{2}e^{\frac{1}{2}(\xi+\eta)} + f_1(\eta)$
$\Rightarrow u = -e^{\frac{1}{2}(\xi+\eta)} + F(\xi) + G(\eta) = -e^x + F(x-t) + G(x+t)$,

初始条件

$$\Rightarrow -e^x + F(x) + G(x) = x, -F'(x) + G'(x) = 1$$

$$\Rightarrow F(x) + G(x) = x + e^x, -F(x) + G(x) = x + C$$

$$\Rightarrow F(x) = \frac{1}{2}e^x - C/2, G(x) = x + \frac{1}{2}e^x + C/2$$

$$\Rightarrow u = -e^x + x + t + \frac{1}{2}e^{x+t} + \frac{1}{2}e^{x-t}.$$

例. 求解方程

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2} + e^x, -\infty < x < +\infty \\ u(x,t)|_{t=0} = x, \ \dfrac{\partial u}{\partial t}|_{t=0} = 1 \end{cases}$$

解法 2 直接利用公式计算

$$\begin{aligned} u(x,t) &= \frac{\phi(x-at)+\phi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)d\xi \\ &\quad + \frac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)d\xi d\tau \\ &= x + t + \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} e^{\xi} d\xi d\tau \\ &= -e^x + x + t + \frac{1}{2}e^{x+t} + \frac{1}{2}e^{x-t} \end{aligned}$$

无界弦的强迫振动问题

例. 求解方程

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + e^x, -\infty < x < +\infty \\ u(x,t)|_{t=0} = x, \ \frac{\partial u}{\partial t}|_{t=0} = 1 \end{cases}$$

解法 3 利用线性叠加原理。注意到方程有一个特解 $U(x,t) = -e^x$,记 $v = u(x,t) - U(x,t)$, 则 v 满足

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial x^2}, \ v(x,t)|_{t=0} = x + e^x, \ \frac{\partial v}{\partial t}|_{t=0} = 1.$$

$$\begin{aligned} v(x,t) &= \frac{\phi(x-t)+\phi(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \psi(s)ds \\ &= x + t + \frac{1}{2}e^{x+t} + \frac{1}{2}e^{x-t} \\ u(x,t) &= v + U = -e^x + x + t + \frac{1}{2}e^{x+t} + \frac{1}{2}e^{x-t}. \end{aligned}$$

一维弦振动方程解的物理意义

一维自由弦振动方程

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, -\infty < x < +\infty, t > 0 \\ u(x,t)|_{t=0} = \phi(x),\ \frac{\partial u}{\partial t}|_{t=0} = \psi(x). \end{cases}$$

解为 $u(x,t) = F(x-at) + G(x+at)$.

- 右行波—$F(x-at)$; 左行波—$G(x+at)$,波速—a;

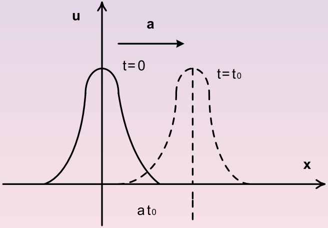
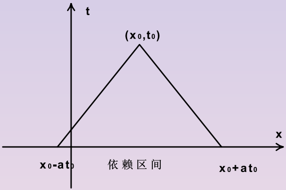


Figure: 右行波

一维弦振动方程解的物理意义

- (x_0, t_0) 的 **依赖区间**—$[x_0 - at_0, x_0 + at_0]$;



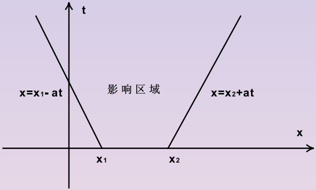


Figure: 依赖区间　　　　影响区域

- 区间 $[x_1, x_2]$**的影响区域**:

$$\{(x,t): \quad x_1 - at \le x \le x_2 + at,\ t > 0\}$$

一维弦振动方程解的物理意义

- 区间 $[x_1, x_2]$的 **决定区域**:

$$\{(x,t): \quad x_1 + at \le x \le x_2 - at\}$$

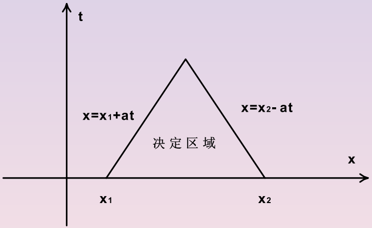


Figure: 决定区域

反射波法

反射波法,也称为对称延拓法,是求解半无界初边值问题的重要方法之一(另外一种重要方法是Laplace 变换法)。本节我们通过研究半无界弦振动问题来介绍反射波法。

端点固定的半无界弦振动

例. 端点固定的半无界弦振动的可描述为

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + f(x,t),\ x \geq 0,\ t \geq 0 \\ u|_{x=0} = \nu(t), \quad t \geq 0 \\ u(x,t)|_{t=0} = \phi(x),\ \frac{\partial u}{\partial t}|_{t=0} = \psi(x), \end{cases}$$

我们把它分解为几步来讨论。
第一步：边界条件化为齐次。 记 $v = u(x,t) - \nu(t)$,

$$\begin{cases} \frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial x^2} + f(x,t) - \nu''(t), 0 < x < +\infty \\ v|_{x=0} = 0, \quad t \geq 0 \\ v(x,t)|_{t=0} = \phi(x) - \nu(0),\ \frac{\partial v}{\partial t}|_{t=0} = \psi(x) - \nu'(0), \end{cases}$$

第二步：半无界弦的振动转化为无界弦的振动。把非齐次项和初始条件关于 x 作奇延拓

$$F(x,t)=\begin{cases} f(x,t)-\nu''(t),\ x\geq 0 \\ -f(-x,t)+\nu''(t),\ x<0 \end{cases}$$

$$\Phi(x)=\begin{cases} \phi(x)-\nu(0),\ x\geq 0 \\ -\phi(-x)+\nu(0),\ x<0 \end{cases}$$

$$\Psi(x)=\begin{cases} \psi(x)-\nu'(0),\ x\geq 0 \\ -\psi(-x)+\nu'(0),\ x<0 \end{cases}$$

端点固定的半无界弦振动

记 $V(x,t)$ 为初边值问题

$$\begin{cases} \dfrac{\partial^2 V}{\partial t^2} = \dfrac{\partial^2 V}{\partial x^2} + F(x,t), \ -\infty < x < +\infty, \ t > 0 \\ V(x,t)|_{t=0} = \Phi(x), \ \frac{\partial V}{\partial t}|_{t=0} = \Psi(x) \end{cases}$$

的解。函数 $F(x,t)$, $\Phi(x)$ 和 $\psi(x)$关于 x 是奇函数$\Rightarrow$ $V(x,t)$ 关于x是奇函数$\Rightarrow$ $V(x,t)|_{x=0} = 0.$

- 限制函数 $V(x,t)|_{x\geq 0, t\geq 0}$ 满足

$$\begin{cases} \dfrac{\partial^2 v}{\partial t^2} = \dfrac{\partial^2 v}{\partial x^2} + f(x,t) - \nu''(t), 0 < x < +\infty \\ v|_{x=0} = 0, \quad t \geq 0 \\ v(x,t)|_{t=0} = \phi(x) - \nu(0), \frac{\partial v}{\partial t}|_{t=0} = \psi(x) - \nu'(0) \end{cases}$$

- $u(x,t)=V(x,t)+\nu(t)$在区域$\{(x,t):\ x\geq 0, t\geq 0\}$内满足

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+f(x,t),\ x\geq 0,\ t\geq 0 \\ u|_{x=0}=\nu(t), \quad t\geq 0 \\ u(x,t)|_{t=0}=\phi(x),\ \frac{\partial u}{\partial t}|_{t=0}=\psi(x), \end{cases}$$

端点自由的半无界弦振动

例. 端点自由的半无界弦振动的可描述为

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2} + f(x,t),\ x \geq 0,\ t \geq 0 \\ u_x|_{x=0} = \nu(t), \quad t \geq 0 \\ u(x,t)|_{t=0} = \phi(x),\ \dfrac{\partial u}{\partial t}|_{t=0} = \psi(x), \end{cases}$$

第一步：边界条件化为齐次。 记 $v = u(x,t) - x\nu(t)$,

$$\begin{cases} \dfrac{\partial^2 v}{\partial t^2} = \dfrac{\partial^2 v}{\partial x^2} + f(x,t) - x\nu''(t),\ x \geq 0,\ t \geq 0 \\ v_x|_{x=0} = 0, \quad t \geq 0 \\ v(x,t)|_{t=0} = \phi(x) - x\nu(0),\ \dfrac{\partial v}{\partial t}|_{t=0} = \psi(x) - x\nu'(0) \end{cases}$$

第二步：半无界弦的振动转化为无界弦的振动。把非齐次项和初始条件关于 x 作偶延拓

$$F(x,t)=\begin{cases} f(x,t)-x\nu''(t),\ x\geq 0\\ f(-x,t)+x\nu''(t),\ x<0\end{cases}$$

$$\Phi(x)=\begin{cases} \phi(x)-x\nu(0),\ x\geq 0\\ \phi(-x)+x\nu(0),\ x<0\end{cases}$$

$$\Psi(x)=\begin{cases} \psi(x)-x\nu'(0),\ x\geq 0\\ \psi(-x)+x\nu'(0),\ x<0\end{cases}$$

记$V(x,t)$ 为初边值问题

$$\begin{cases} \dfrac{\partial^2 V}{\partial t^2}=\dfrac{\partial^2 V}{\partial x^2}+F(x,t),\ -\infty<x<+\infty,\ t>0\\ V(x,t)|_{t=0}=\Phi(x),\ \dfrac{\partial V}{\partial t}|_{t=0}=\Psi(x)\end{cases}$$

的解。$V(x,t)$ 关于 x是偶函数，因此$V_x(x,t)|_{x=0}=0$.

端点自由的半无界弦振动

- $V(x,t)|_{x\geq 0,t\geq 0}$ 满足

$$\begin{cases} \frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial x^2} + f(x,t) - x\nu''(t),\ x \geq 0,\ t \geq 0 \\ v_x|_{x=0} = 0, \quad t \geq 0 \\ v(x,t)|_{t=0} = \phi(x) - x\nu(0),\ \frac{\partial v}{\partial t}|_{t=0} = \psi(x) - x\nu'(0) \end{cases}$$

- $u(x,t) = V(x,t) + x\nu(t)$在区域$\{(x,t):\ x \geq 0, t \geq 0\}$内满足

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + f(x,t),\ x \geq 0,\ t \geq 0 \\ u_x|_{x=0} = \nu(t), \quad t \geq 0 \\ u(x,t)|_{t=0} = \phi(x),\ \frac{\partial u}{\partial t}|_{t=0} = \psi(x), \end{cases}$$