Morse theory

JUNDING HUANG

Imperial College London

Imperial College London

Introduction

Manifold is a crucial concept in many areas of geometry and sometimes they are complicated and unimaginable. Then it is time to introduce Morse theory, which is a powerful method to study manifold by capturing manifolds in terms of Euclidean space. It allows us to conduct **calculus** on manifolds to analyze **topological structure** on them. The ultimate goal of this poster is to demonstrate the proof of Morse lemma, which is a gateway theorem of Morse theory.

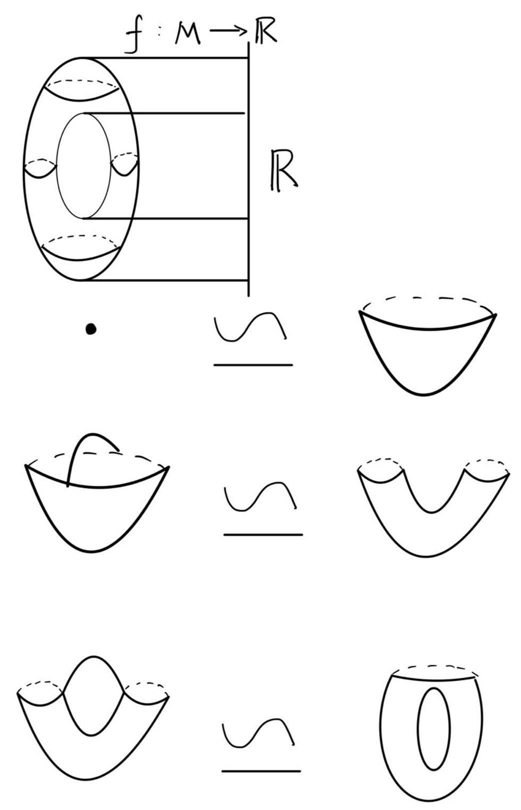

Figure 1: homotopy equivalence of the torus

Smooth Manifold(C^∞)

Definition 1.1 Let M be a set. An **n-dimensional smooth atlas** on M is a collection of triples $(U_\alpha, V_\alpha, \varphi_\alpha)$ such that

- $U_\alpha \subset M$; $V_\alpha \subset \mathbb{R}^n$ is open
- $\bigcup_{\alpha\in I} U_\alpha = M$
- Each $\varphi_\alpha : U_\alpha \to V_\alpha$ is a bijection
- The composition $\varphi_\beta \circ \varphi_\alpha^{-1}|_{\varphi_\alpha(U_\alpha\cap U_\beta)} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is a smooth map for all ordered pairs (α, β)

The number n is called the dimension of M, the maps φ_α are called coordinate charts, the compositions $\varphi_\beta \circ \varphi_\alpha^{-1}$ are called transition maps or change of coordinates.

Definition 1.2 M is called a smooth n -dimensional manifold if

1. M has an n -dimensional smooth atlas;
2. M is Hausdorff;
3. M is second-countable.

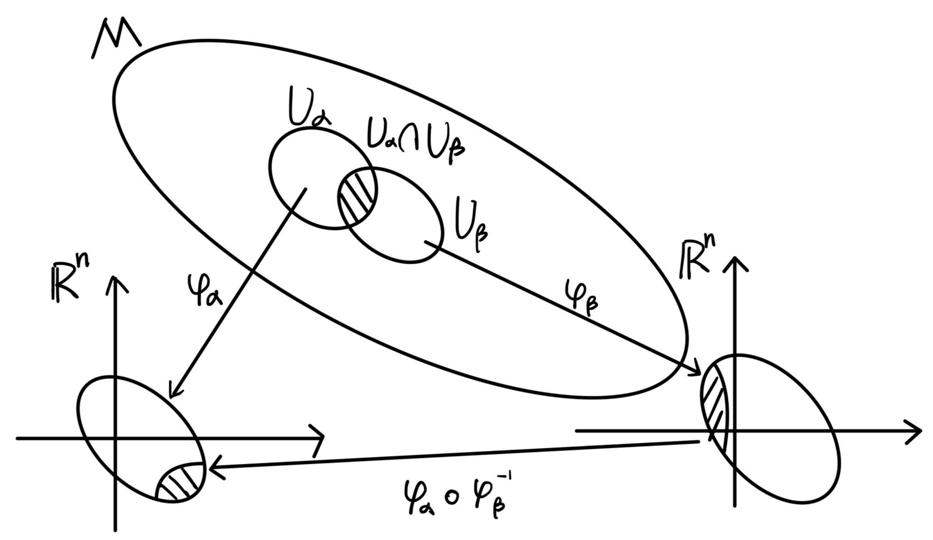

Figure 2: Compatible charts

Preliminary work for Morse lemma

Lemma 2.1 Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Suppose $f(0) = 0$ then there exist smooth functions $g_1, \cdots, g_n : \mathbb{R}^n \to \mathbb{R}$ with $g_i(0) = \frac{\partial f(0)}{\partial x_i}$ such that

$$f(x_1, \cdots, x_n) = \sum_{i=1}^{n} x_i g_i(x_1, \cdots, x_n)$$

Theorem 2.2 (Inverse function theorem). Let $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a $C^{k\geq 1}$ function on Ω. If $detDF(a) \neq 0$ at some $a \in \Omega$, then there exists an open set $U \subset \Omega$ containing a such that $\widetilde{F} := F|_U : U \to F(U)$ is bijective, $F(U)$ is open in $\mathbb{R}^n$, and the inverse map $F^{-1} : F(U) \to U$ is C^k

Lemma 2.3 The degeneracy of a critical point p is independent of the choice of local coordinate

Morse lemma[1]

Lemma 2.4 :(Morse lemma) Let p be non degenerate critical point of f with index λ. Then there is a local coordinate system $Y : V \subset \mathbb{R}^n \to U_p$ in a neighbourhood U_p of p with $0 \in V$ and $Y(0) = p$ such that the identity

$$(f \circ Y)(y_1, y_2, \cdots, y_n) = f(p) - y_1^2 - \cdots - y_\lambda^2 + y_{\lambda+1}^2 + \cdots + y_n^2$$

Proof: We will conduct the proof by an inductive argument. Without loss of generality, assume that $f(p) = 0$. Choose a local coordinate system $X : V_0 \subset \mathbb{R}^n \to U_p$ in a neighborhood U_p of p such that $X(0) = p$ and and $\frac{\partial^2 f}{\partial x_1^2}(p) \neq 0$. Since $f(p) = (f \circ X)(0) = 0$ and $(f \circ X) \in C^\infty$, by **Lemma 2.1**,

$$(f \circ X)(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n} x_i g_i(x_1, x_2, \cdots, x_n)$$

and satisfy

$$g_i(0) = \frac{\partial (f \circ X)}{\partial x_i}(0), \quad \forall i = 1, 2, \cdots, n$$

Using **Lemma 2.1** again for each g_i, we have

$$g_i(x_1, x_2, \cdots, x_n) = \sum_{j=1}^{n} x_j h_{ij}(x_1, x_2, \cdots, x_n),$$

with $h_{ij}(0) = \frac{\partial g_i}{\partial x_j}(0)$, for any $j = 1, 2, \cdots, n$ Hence

$$\begin{aligned}(f \circ X)(x_1, x_2, \cdots, x_n) &= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j h_{ij}(x_1, x_2, \cdots, x_n) &(1)\\ &= \sum_{i=1}^{n} x_i^2 H_{ii}(x_1, x_2, \cdots, x_n) + 2\sum_{i<j} x_i x_j H_{ij}(x_1, x_2, \cdots, x_n), &(2)\end{aligned}$$

where $H_{ij} = \frac{1}{2}(h_{ij} + h_{ji}) = H_{ji}$. By assumption $h_{11}(0, \cdots, 0) \neq 0$. Then h_{11} is not zero in a neighborhood around origin. Call this neighborhood V_1

We define a new first coordinate y_1 by

$$y_1 = \sqrt{|H_{11}|}\left(x_1 + \sum_{j=2}^{n} x_j \frac{H_{1j}}{H_{11}}\right)$$

and the determinant of the Jacobian matrix of this transformation is not 0 then by **inverse function theorem**, $(y_1, x_2, \cdots, x_n)$ are local coordinate on a smaller neighborhood $\widetilde{V_1} \subset V_1$.
Therefore,

$$f \circ X = \pm y_1^2 + \sum_{j=2}^{n} x_j^2 H_{jj}^{(1)} + 2\sum_{2\leq i<j} x_i x_j H_{ij}^{(1)}$$

where y_1 term is positive if $H_{11} > 0$ and negative if $H_{11} < 0$. The rest of the term depends on the local coordinate x_i where $i \neq 1$ By induction from $k = 1$ to $k = n$, we can eliminate the i^{th} coordinate x_i by

$$y_i = \sqrt{|H_{ii}^{(i)}|}\left(x_i + \sum_{j=i+1}^{n} x_j \frac{H_{ij}^{(i)}}{H_{ii}^{(i)}}\right)$$

Finally, we will obtain a quadratic form, hence finishing the proof of Morse lemma

Simple Corollary of Morse lemma

Corollary 3.1: Let $f : M \to \mathbb{R}$ be a smooth function on a smooth manifold M. A non-degenerate critical point of a smooth function is isolated. In particular, if f is a Morse function and M is compact, then f has finitely many critical points.

Proof: Directly apply Morse lemma to f then the proof is done. Now suppose that M is compact. If the set of critical points were infinite, it would have a limit point. By continuity of df , such a point would also be a critical point which is not isolated, which is a contradiction.

Application of Morse Lemma(Reeb's theorem)[2][3]

Theorem 3.2: Let $f : M \to \mathbb{R}$ be a Morse function on a compact smooth manifold M of dimension n with exactly two critical points. Then M is homeomorphic to S^n

Proof: Let p and q be two critical points of f then p and q must be minimum and maximum points of f since M is compact. Let p be the minima and q be the maxima. By **Morse lemma**, the index of p is 0. If the index of p is $\lambda \neq 0$ then there exist a $X : V \subset \mathbb{R}^n \to U_p$ with $0 \in V$ and $X(0) = p$ such that

$$f \circ X = f(p) - \sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{n} x_i^2$$

We have $(\delta, 0, \cdots, 0) \in V$ for some $\delta > 0$ and so $f \circ X(\delta, 0, \cdots, 0) = f(p) - \delta < f(p)$ which is contradiction since f takes minimum at p.

Without loss of generality, we assume that $f(p) = 0$ and $f(q) = 1$. Therefore, f can be expressed in terms of coordinate systems $(x_1, \cdots, x_n)$ in a neighborhood U_p of p and $(y_1, \cdots, y_n)$ in a neighborhood of U_q of q

$$f = \begin{cases} x_1^2 + x_2^2 + \cdots + x_n^2 \\ 1 - y_1^2 - y_2^2 - \cdots - y_n^2 \end{cases}$$

We choose an ϵ such that $0 \leq \sum_{i=1}^n x_i^2 \leq \epsilon$ and $1 - \epsilon \leq 1 - \sum_{i=1}^n y_i^2 \leq 1$, then the sets $f^{-1}([0, \epsilon])$ and $f^{-1}([1-\epsilon, 1])$ are diffeomorphic to closed n-disks D_p^n and D_q^n respectively. By fundamental theorem of Morse theory, $f^{-1}([0, \epsilon]) = M^\epsilon$ is diffeomorphic to $f^{-1}([0, 1-\epsilon]) = M^{1-\epsilon}$. Hence, $M = f^{-1}([0, 1-\epsilon]) \cup f^{-1}([1-\epsilon, 1])$ is diffeomorphic to $D_p^n \cup D_q^n$ which is the union of two closed n-disks glued along their boundary. Since $D_p^n \cup D_q^n$ is diffeomorphic to S^n. We can conclude that M is homeomorphic to S^n

References

[1] Augustin Banyaga and David E. Hurtubise.
A proof of the morse-bott lemma.
Expositiones Mathematicae, 22(4):365–373, 2004.

[2] José Luis González.
Morse theory and application, 2017.

[3] Amy Hua.
An introductory treatment of morse theory on manifolds.
2010.

jh1722@ic.ac.uk