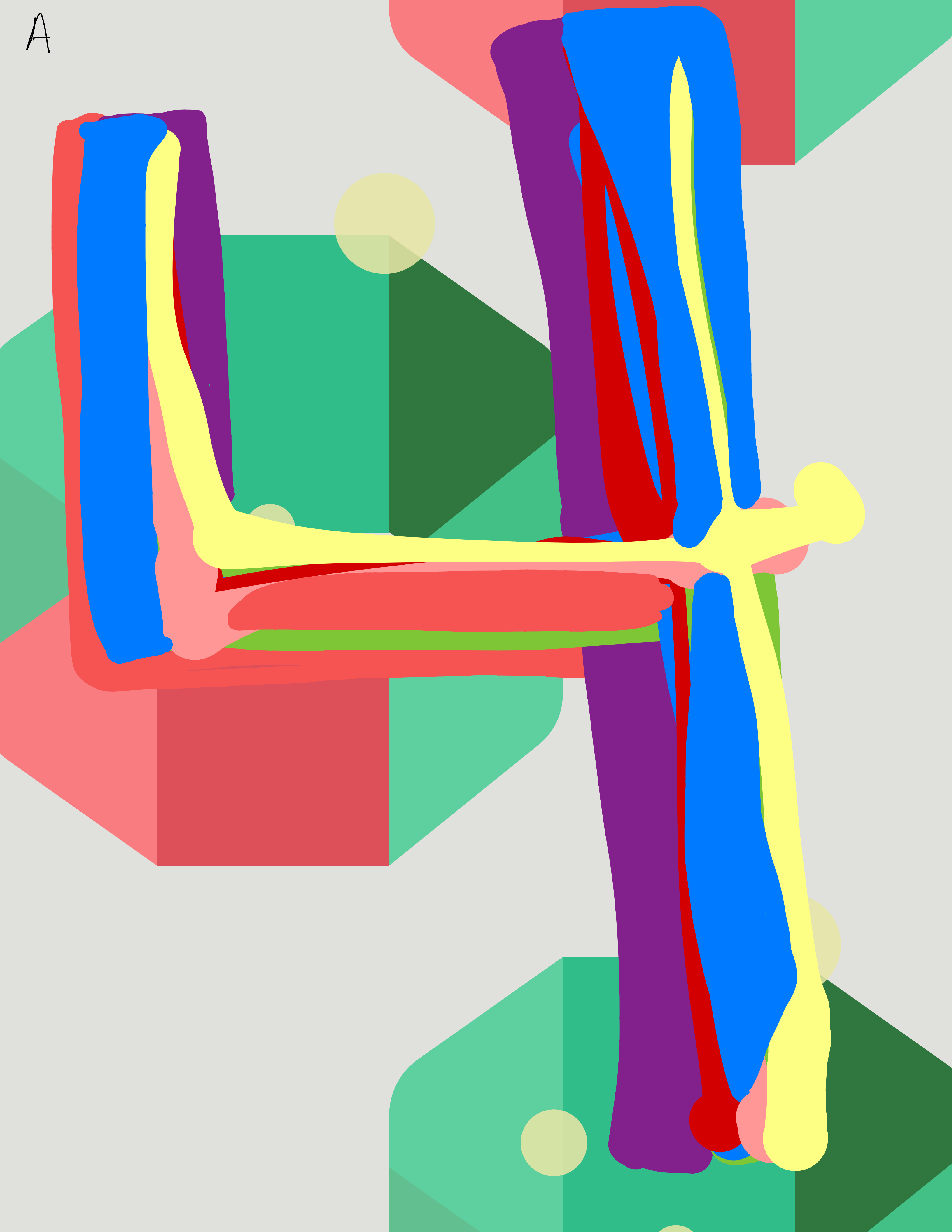
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Section 2.1 : Matrix Operations

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

Topics and Objectives

Topics

We will cover these topics in this section.

1. Identity and zero matrices
2. Matrix algebra (sums and products, scalar multiplies, matrix powers)
3. Transpose of a matrix

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. **Apply** matrix algebra, the matrix transpose, and the zero and identity matrices, to **solve** and **analyze** matrix equations.

Week	Dates	Lecture	Studio	Lecture	Studio	Lecture
1	1/8 – 1/12	1.1	WS1.1	1.2	WS1.2	1.3
2	1/15 – 1/19	Break	WS1.3	1.4	WS1.4	1.5
3	1/22 – 1/26	1.7	WS1.5,1.7	1.8	WS1.8	1.9
4	1/29 – 2/2	1.9,2.1	WS1.9,2.1	**Exam 1**, Review	Cancelled	2.2

Section 2.1 : Matrix Operations

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Definitions: Zero and Identity Matrices

1. A **zero matrix** is any matrix whose every entry is zero.

$$0_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad 0_{2\times 1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2. The $n \times n$ **identity matrix** has ones on the main diagonal, otherwise all zeros.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: any matrix with dimensions $n \times n$ is **square**. Zero matrices need not be square, identity matrices must be square.

Sums and Scalar Multiples

Suppose $A \in \mathbb{R}^{m\times n}$, and $a_{i,j}$ is the element of A in row i and column j.

1. If A and B are $m \times n$ matrices, then the elements of $A + B$ are $a_{i,j} + b_{i,j}$.
2. If $c \in \mathbb{R}$, then the elements of cA are $ca_{i,j}$.

For example, if

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + c\begin{bmatrix} 7 & 4 & 7 \\ 0 & 0 & k \end{bmatrix} = \begin{bmatrix} 15 & 10 & 17 \\ 4 & 5 & 16 \end{bmatrix}$$

What are the values of c and k?

$$\begin{bmatrix} 1+7c & 2+4c & 3+7c \\ 4 & 5 & 6+ck \end{bmatrix} = \begin{bmatrix} 15 & 10 & 17 \\ 4 & 5 & 16 \end{bmatrix}$$

$1+7c = 15$
$7c = 14$
$c = 2$

$6+2k = 16$
$2k = 10$
$k = 5$

Properties of Sums and Scalar Multiples

Scalar multiples and matrix addition have the expected properties.

If $r, s \in \mathbb{R}$ are scalars, and A, B, C are $m \times n$ matrices, then

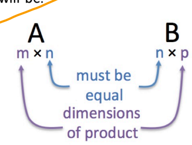
1. $A + 0_{m\times n} = A$ Additive Identity
2. $(A+B)+C = A+(B+C)$ Associative Property
3. $r(A+B) = rA + rB$
4. $(r+s)A = rA + sA$] Distributive Properties
5. $r(sA) = (rs)A$ Associative Property for Scalar Multiplication

Matrix Multiplication

Definition
Let A be a $m \times n$ matrix, and B be a $n \times p$ matrix. The product is AB a $m \times p$ matrix, equal to

$$AB = A\begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_p \end{bmatrix} = \begin{bmatrix} A\vec{b}_1 & \cdots & A\vec{b}_p \end{bmatrix}$$

Note: the dimensions of A and B determine whether AB is defined, and what its dimensions will be.



$A\vec{x}$ = a linear combination of the colum of A.

$A\vec{b}_1$ is a linear combination of column of A.

Ex
$$\underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} \cdot \underset{3\times 2}{\begin{bmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{bmatrix}} = \underset{2\times 2}{\begin{bmatrix} 50 & ? \\ 122 & ? \end{bmatrix}}$$

match

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 4 \end{bmatrix} + 8\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 9\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 50 \\ 122 \end{bmatrix}$$

Properties of Matrix Multiplication

Let A, B, C be matrices of the sizes needed for the matrix multiplication to be defined, and A is a $m \times n$ matrix.

1. (Associative) $(AB)C = A(BC)$
2. (Left Distributive) $A(B + C) = AB + AC$
3. (Right Distributive) $\cdots$ $(A+B)C = AC + BC$
4. (Identity for matrix multiplication) $I_m A = AI_n$

Warnings:

1. (non-commutative) In general, $AB \neq BA$.
2. (non-cancellation) $AB = AC$ does not mean $B = C$.
3. (Zero divisors) $AB = 0$ does not mean that either $A = 0$ or $B = 0$.

2) in $\mathbb{R}$ $\quad 0(3) = 0(2)$

but $3 \neq 2$

2) in $\mathbb{R}^{m \times n}$, A could be nonzero.

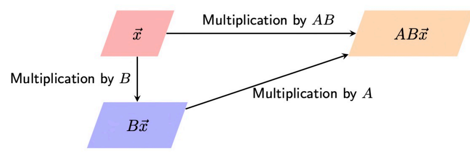
The Associative Property

The associative property is $(AB)C = A(BC)$. If $C = \vec{x}$, then

$$(AB)\vec{x} = A(B\vec{x})$$

Like function composition $f(g(x))$

Schematically:



The matrix product $AB\vec{x}$ can be obtained by either: multiplying by matrix AB, or by multiplying by B then by A. This means that matrix multiplication corresponds to **composition of the linear transformations**.

Proof of the Associative Law

Let A be $m \times n$, $B = \begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_p \end{bmatrix}$ a $n \times p$ and $C = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$ a $p \times 1$ matrix. Then,

$$BC = \underbrace{c_1\vec{b}_1 + \cdots + c_p\vec{b}_p}_{\text{lin combin of cols of } B}$$

So

$$\begin{aligned} A(BC) &= A(c_1\vec{b}_1 + \cdots + c_p\vec{b}_p) \\ &= c_1 A\vec{b}_1 + \cdots + c_p A\vec{b}_p && \text{(multiply by } A \text{ is linear)} \\ &= \begin{bmatrix} A\vec{b}_1 & \cdots & A\vec{b}_p \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix} && \text{(lin combin of cols of } AB\text{)} \\ &= (AB)C. \end{aligned}$$

Example

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Give an example of a 2×2 matrix B that is non-commutative with A.

$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$AB \overset{?}{\neq} BA$

$AB = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$

$BA = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & \\ 3 & \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

The Transpose of a Matrix

A^T is the matrix whose columns are the rows of A.

Example

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 2 & 0 \end{bmatrix}}_{2\times 5}^T = \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 0 \\ 4 & 2 \\ 5 & 6 \end{bmatrix}}_{5 \times 2}$$

Properties of the Matrix Transpose

1. $(A^T)^T =$ A
2. $(A+B)^T =$ $A^T + B^T$
3. $(rA)^T =$ rA^T
4. $(AB)^T =$ $B^T A^T$

Matrix Powers

For any $n \times n$ matrix and positive integer k, A^k is the product of k copies of A.

$$A^k = AA\ldots A$$

Example: Compute C^8.

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$C^8 = C \cdot C \cdot C \cdot C \cdot C \cdot C \cdot C \cdot C$$

Because C is diagonal

$$C^8 = \begin{bmatrix} 1^8 & 0 & 0 \\ 0 & 2^8 & 0 \\ 0 & 0 & 2^8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 256 & 0 \\ 0 & 0 & 256 \end{bmatrix}$$

Example

Define $B^T : 3\times 2$

$$A = \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}}_{2\times 2}, \quad B = \underbrace{\begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 8 \end{bmatrix}}_{2\times 3}, \quad C = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 9 \end{bmatrix}}_{3\times 3}$$

Which of these operations are defined, and what is the result?

1. AB defined
2. $3C$ defined
3. $A + 3C$ not defined
4. $B^T A$ defined
5. C^3 defined
6. CB^T defined

$B^2 = B \cdot B$ not defined

AB^T not defined

2x2 3x2

Additional Example (if time permits)

True or false:

1. For any I_n and any $A \in \mathbb{R}^{n\times n}$, $(I_n + A)(I_n - A) = I_n - A^2$. True

$$I_n \cdot I_n - I_n A + A I_n - A \cdot A$$
$$= I_n - A + A - A^2$$
$$= I_n - A^2$$

2. For any A and B in $\mathbb{R}^{n\times n}$, $(A+B)^2 = A^2 + B^2 + 2AB$. False

$$(A+B)^2 = (A+B)(A+B)$$
$$= A^2 + \underbrace{AB + BA}_{\neq 2AB \text{ in general.}} + B^2$$

2.1 Exercises

In Exercises 1 and 2, compute each matrix sum or product if it is defined. If an expression is undefined, explain why. Let

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

1. $-2A,\ B - 2A,\ AC,\ CD$

2. $A + 2B,\ 3C - E,\ CB,\ EB$

In the rest of this exercise set and in those to follow, you should assume that each matrix expression is defined. That is, the sizes of the matrices (and vectors) involved "match" appropriately.

3. Let $A = \begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix}$. Compute $3I_2 - A$ and $(3I_2)A$.

4. Compute $A - 5I_3$ and $(5I_3)A$, when

$$A = \begin{bmatrix} 9 & -1 & 3 \\ -8 & 7 & -3 \\ -4 & 1 & 8 \end{bmatrix}.$$

In Exercises 5 and 6, compute the product AB in two ways: (a) by the definition, where $A\mathbf{b}_1$ and $A\mathbf{b}_2$ are computed separately, and (b) by the row–column rule for computing AB.

5. $A = \begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -4 \\ -2 & 1 \end{bmatrix}$

6. $A = \begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix}$

7. If a matrix A is 5×3 and the product AB is 5×7, what is the size of B?

8. How many rows does B have if BC is a 3×4 matrix?

9. Let $A = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}$. What value(s) of k, if any, will make $AB = BA$?

10. Let $A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix}$, and $C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$. Verify that $AB = AC$ and yet $B \neq C$.

11. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. Compute AD and DA. Explain how the columns or rows of A change when A is multiplied by D on the right or on the left. Find a 3×3 matrix B, not the identity matrix or the zero matrix, such that $AB = BA$.

12. Let $A = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}$. Construct a 2×2 matrix B such that AB is the zero matrix. Use two different nonzero columns for B.

Exercises 15–24 concern arbitrary matrices A, B, and C for which the indicated sums and products are defined. Mark each statement True or False **(T/F)**. Justify each answer.

15. **(T/F)** If A and B are 2×2 with columns $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{b}_1, \mathbf{b}_2$, respectively, then $AB = [\mathbf{a}_1\mathbf{b}_1 \quad \mathbf{a}_2\mathbf{b}_2]$.

16. **(T/F)** If A and B are 3×3 and $B = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3]$, then $AB = [A\mathbf{b}_1 + A\mathbf{b}_2 + A\mathbf{b}_3]$.

17. **(T/F)** Each column of AB is a linear combination of the columns of B using weights from the corresponding column of A.

18. **(T/F)** The second row of AB is the second row of A multiplied on the right by B.

19. **(T/F)** $AB + AC = A(B + C)$

20. **(T/F)** $A^T + B^T = (A + B)^T$

21. **(T/F)** $(AB)C = (AC)B$

22. **(T/F)** $(AB)^T = A^T B^T$

23. **(T/F)** The transpose of a product of matrices equals the product of their transposes in the same order.

24. **(T/F)** The transpose of a sum of matrices equals the sum of their transposes.

25. If $A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$ and $AB = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}$, determine the first and second columns of B.

26. Suppose the first two columns, $\mathbf{b}_1$ and $\mathbf{b}_2$, of B are equal. What can you say about the columns of AB (if AB is defined)? Why?

27. Suppose the third column of B is the sum of the first two columns. What can you say about the third column of AB? Why?

28. Suppose the second column of B is all zeros. What can you say about the second column of AB?

29. Suppose the last column of AB is all zeros, but B itself has no column of zeros. What can you say about the columns of A?

30. Show that if the columns of B are linearly dependent, then so are the columns of AB.

31. Suppose $CA = I_n$ (the $n \times n$ identity matrix). Show that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Explain why A cannot have more columns than rows.

32. Suppose $AD = I_m$ (the $m \times m$ identity matrix). Show that for any $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution. [*Hint:* Think about the equation $AD\mathbf{b} = \mathbf{b}$.] Explain why A cannot have more rows than columns.

33. Suppose A is an $m \times n$ matrix and there exist $n \times m$ matrices C and D such that $CA = I_n$ and $AD = I_m$. Prove that $m = n$ and $C = D$. [*Hint:* Think about the product CAD.]

Exam 1 Review

(b) (2 points) For what value(s) of h is the following set of vectors linearly dependent?

$$\left\{\begin{pmatrix}1\\1\\h\end{pmatrix}, \begin{pmatrix}1\\h\\1\end{pmatrix}, \begin{pmatrix}-1\\0\\h\end{pmatrix}\right\}$$

$A\vec{x} = \vec{0}$

$$\begin{bmatrix}1 & 1 & -1\\1 & h & 0\\h & 1 & h\end{bmatrix} \underset{R_3 - hR_1 \to R_3}{\overset{R_2 - R_1 \to R_2}{\sim}} \begin{bmatrix}1 & 1 & -1\\0 & h-1 & 1\\0 & 1-h & 2h\end{bmatrix}$$

If $h-1$ and $1-h$ are both 0, then 2nd column is free.
So if $\boxed{h=1}$ vectors are linearly dependent.

$$\overset{R_3 + R_2 \to R_3}{\sim} \begin{bmatrix}1 & 1 & -1\\0 & h-1 & 1\\0 & 0 & 2h+1\end{bmatrix}$$

If $2h+1 = 0$, the the third column is free,

$$2h+1=0$$
$$2h=-1$$
$$\boxed{h=-\tfrac{1}{2}}$$

$h = 1, -\frac{1}{2}$ would make the vectors linearly dependent.

$T: \mathbb{R}^n \to \mathbb{R}^m$ A: $m \times n$ matrix

One-to-one: A linear transformation $T(\vec{x}) = A\vec{x}$ is 1-to-1 iff:

- A has a pivot in every column.
- For every $\vec{b} \in \mathbb{R}^m$, the equation $T(\vec{x}) = \vec{b}$ has at most one solution.
- $T(\vec{x}) = \vec{0}$ has exactly one solution.

Fact: If $T: \mathbb{R}^n \to \mathbb{R}^m$ and $n > m$, then T cannot be 1-to-1.

$T: \mathbb{R}^n \to \mathbb{R}^m$, A is $m \times n$ matrix

onto: A linear transformation $T(\vec{x}) = A\vec{x}$ is onto iff:

- A has a pivot in every row.
- For every $\vec{b} \in \mathbb{R}^m$, the equation $T(\vec{x}) = \vec{b}$ has at least one solution.

Fact: If $T: \mathbb{R}^n \to \mathbb{R}^m$ and $n < m$, then T cannot be onto.

Linear Transformation: T is a function so that

$$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$$
$$T(r\vec{u}) = r\,T(\vec{u})$$

Range is the set of outputs of T.

$$\text{Range} = \{ T(\vec{x}) \mid \text{when } x \in \mathbb{R}^n \}$$

$\text{Span}\{\vec{v}_1, \dots, \vec{v}_n\}$ is the set of all linear combinations of $\vec{v}_1, \dots, \vec{v}_n$.

$$\text{Span}\{\underbrace{\vec{v}_1, \dots, \vec{v}_n}_{\text{These are not unique}}\} = \{ c_1\vec{v}_1 + \dots + c_n\vec{v}_n \mid c_1, \dots, c_n \in \mathbb{R} \}.$$

$$\text{Span}\{\vec{v}_1, \vec{v}_2\} \stackrel{?}{=} \text{Span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_1 - \vec{v}_2\}$$

$$\vec{v}_1 = \frac{1}{2}(\vec{v}_1 + \vec{v}_2) + \frac{1}{2}(\vec{v}_1 - \vec{v}_2)$$

$\vec{v}_1$ lives here

$$\vec{v}_2 = \frac{1}{2}(\vec{v}_1 + \vec{v}_2) - \frac{1}{2}(\vec{v}_1 - \vec{v}_2)$$

$$\underbrace{\text{Span}\{\vec{v}_1, \vec{v}_2\}}_{\text{If } \vec{v}_1, \vec{v}_2 \text{ are lin ind, then not equal}} \stackrel{?}{=} \text{Span}\{\overbrace{\vec{v}_1 + \vec{v}_2, 2\vec{v}_1 + 2\vec{v}_2}^{1\text{-dim}}\}$$

1 T/F Questions

Question 1 For each statement, mark the statement *true* if it's *always* true, or *false* otherwise.

true	false	
◯	◯	Given the linear $T\colon \mathbb{R}^n \to \mathbb{R}^m$ where $m = n$ and $T(\boldsymbol{x}) = \boldsymbol{b}$ has a solution for every $\boldsymbol{b} \in \mathbb{R}^m$, T is one-to-one.
◯	◯	If $A \in \mathbb{R}^{m\times n}$ and $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \ldots, \boldsymbol{b}_p\}$ is the set of vectors from $\mathbb{R}^m$ such that for any vector $\boldsymbol{b}_i$ in the set, $A\boldsymbol{x} = \boldsymbol{b}_i$ has a solution, then $A\boldsymbol{x} = \tilde{\boldsymbol{b}}$, where $\tilde{\boldsymbol{b}} \in \text{Span}\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \ldots, \boldsymbol{b}_p\}$, also has a solution.
◯	◯	If a matrix has a mixture of pivot and non-pivot columns, then the non-pivot columns can be expressed as a linear combination of the pivot columns.
◯	◯	For any two matrices $A, B \in \mathbb{R}^{n\times n}$, where $A \neq B$, $AB \neq BA$.
◯	◯	If the columns of a 16×42 matrix A span all of $\mathbb{R}^{16}$, then there exists 26 free variables in the equation $A\boldsymbol{x} = \boldsymbol{b}$.
◯	◯	It is possible that a set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ in $\mathbb{R}^3$ form a linearly indpenendent set but not span all of $\mathbb{R}^3$.
◯	◯	There exists a vector $\boldsymbol{b} \in \mathbb{R}^2$ such that the solution set of the equation $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boldsymbol{b}$ is the x_3 axis.

2 Possible/Impossible Questions

Question 2 Mark each statement as *possible* if it could ever be true, or *impossible* otherwise.

possible	impossible	
◯	◯	A linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ that is one-to-one and its standard matrix has a pivot in every column.
◯	◯	A linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ that is onto but not one-to-one.
◯	◯	A linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ that is not onto and not one-to-one.
◯	◯	A linearly dependent set made up of two vectors, but one of them is not a scalar multiple of the other.
◯	◯	A matrix $A \in \mathbb{R}^{m\times n}$ such that the equation $A\boldsymbol{x} = \boldsymbol{b}$ has at least one solution for an infinite number of vectors $\boldsymbol{b} \in \mathbb{R}^m$ and also the equation $A\boldsymbol{x} = \boldsymbol{d}$ has no solutions for an infinte number of vectors $\boldsymbol{d} \in \mathbb{R}^m$.
◯	◯	A matrix $A \in \mathbb{R}^{m\times n}$ with $m < n$ and A has a pivot in every column.
◯	◯	A matrix A where the equation $A\boldsymbol{x} = \boldsymbol{b}$ has multiple solutions but only the trivial solution is a solution to the equation $A\boldsymbol{x} = \boldsymbol{0}$.
◯	◯	A set of linearly independent vectors from $\mathbb{R}^n$ $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ where $\boldsymbol{v}_3 \in \text{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

3 Short Answer Questions

Question 3 For each question, provide your work and answer.

1. Consider the linear transformation T_1 defined by $T_1(x) = (2x, 0, x)$, where the domain of T_1 is $\mathbb{R}$.
   (a) What is the codomain of the transformation?
   (b) What is the range of the transformation?

2. Now consider the transformation T_2 where $T_2(x_1, x_2) = (2x_1, x_2, x_1 + 2x_2)$.
   (a) What is the standard matrix of the transformation?
   (b) Find a vector $\boldsymbol{x}$ such that $T_2(\boldsymbol{x}) = \boldsymbol{b}$ where $\boldsymbol{b} = \begin{bmatrix} 6 \\ -1 \\ 1 \end{bmatrix}$.
   (c) Is T_2 one-to-one? Is it onto?

3. Construct a matrix $A \in \mathbb{R}^{3\times 4}$ where $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}$ are nontrivial solutions to the equation $A\boldsymbol{x} = \boldsymbol{0}$.
   (*Note: A* cannot be the zero matrix.)

4. Construct two augmented matrices $[A \mid \boldsymbol{b}]$ and $[C \mid \boldsymbol{d}]$ in RREF such that $A \in \mathbb{R}^{3\times 3}$ and $C \in \mathbb{R}^{4\times 3}$ and the solution corresponding to both systems has $\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ as the only solution.

5. Find the standard matrix of the transformation that rotates every vector in $\mathbb{R}^2$ by an angle of $\pi/2$ radians clockwise and reflects everything across the line defined by $x_1 = x_2$.

6. Provide a nontrivial solution to the equation $T(\boldsymbol{x}) = \boldsymbol{0}$ given that for the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$,
$$T\left(\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) \neq \boldsymbol{0}.$$

$T\left(\begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}\right)$

6. $T\left(\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}\right) - T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \vec{0}$

4. A : 3 eqns, 3 variables

C : 4 eqns, 3 variables

$\vec{b} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$

$$\begin{cases} x + y + z = 4 \\ 2x + y + z = 5 \\ 3x + 2y + z = 7 \end{cases}$$

C

$\Big\{$ Copy the 3

$3x + y + z = 6$

In-Class Midterm 1 Review, Math 1554

Week	Dates	Lecture	Studio	Lecture	Studio	Lecture
1	1/8 – 1/12	1.1	WS1.1	1.2	WS1.2	1.3
2	1/15 – 1/19	Break	WS1.3	1.4	WS1.4	1.5
3	1/22 – 1/26	1.7	WS1.5,1.7	1.8	WS1.8	1.9
4	1/29 – 2/2	1.9,2.1	WS1.9,2.1	**Exam 1**, Review	Cancelled	2.2

1. Consider the matrix A and vectors $\vec{b}_1$ and $\vec{b}_2$.

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}, \quad \vec{b}_1 = \begin{pmatrix} -2 \\ -4 \end{pmatrix}, \quad \vec{b}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

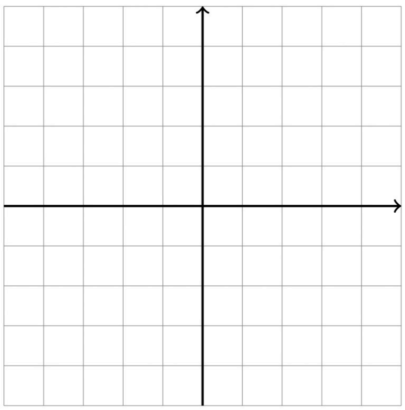
If possible, on the grids below, draw

(i) the two vectors and the span of the columns of A,

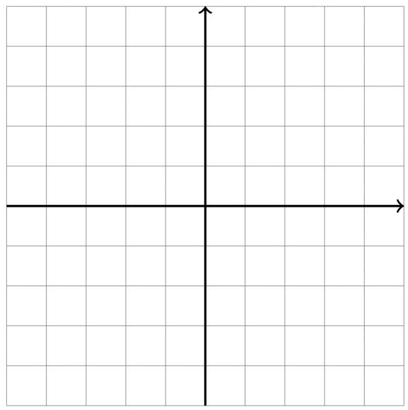
(ii) the solution set of $A\vec{x} = \vec{b}_1$.

(iii) the solution set of $A\vec{x} = \vec{b}_2$.

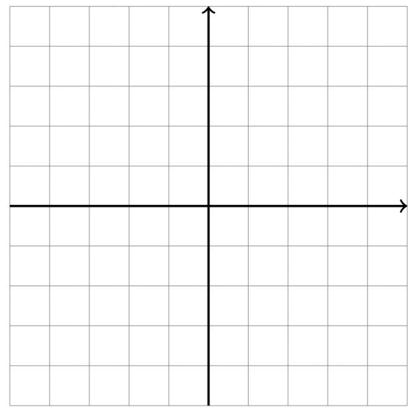
(i) $\vec{b}_1$, $\vec{b}_2$, column span



ii) solution set $Ax = \vec{b}_1$



iii) solution set $Ax = \vec{b}_2$



2. Indicate **true** if the statement is true, otherwise, indicate **false**. For the statements that are false, give a counterexample.

	true	false	counterexample
a) If $A \in \mathbb{R}^{M\times N}$ has linearly dependent columns, then the columns of A cannot span $\mathbb{R}^M$.	○	○	
b) If there are some vectors $\vec{b} \in \mathbb{R}^M$ that are not in the range of $T(\vec{x}) = A\vec{x}$, then there cannot be a pivot in every row of A.	○	○	
c) If the transform $\vec{x} \to A\vec{x}$ projects points in $\mathbb{R}^2$ onto a line that passes through the origin, then the transform cannot be one-to-one.	○	○	

3. If possible, write down an example of a matrix with the following properties. If it is not possible to do so, write *not possible*.

(a) A linear system that is homogeneous and has no solutions.

(b) A standard matrix A associated to a linear transform, T. Matrix A is in RREF, and $T_A : \mathbb{R}^3 \to \mathbb{R}^4$ is one-to-one.

(c) A 3×7 matrix A, in RREF, with exactly 2 pivot columns, such that $A\vec{x} = \vec{b}$ has exactly 5 free variables.

4. Consider the linear system $A\vec{x} = \vec{b}$, where

$$A = \begin{pmatrix} 1 & 0 & 7 & 0 & -5 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

(a) Express the augmented matrix $(\, A \,|\, \vec{b} \,)$ in RREF.

(b) Write the set of solutions to $A\vec{x} = \vec{b}$ in parametric vector form. Your answer must be expressed as a vector equation.

Section 2.2 : Inverse of a Matrix

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

"Your scientists were so preoccupied with whether or not they could, they didn't stop to think if they should."

- Spielberg and Crichton, Jurassic Park, 1993 film

The algorithm we introduce in this section **could** be used to compute an inverse of an $n \times n$ matrix. At the end of the lecture we'll discuss some of the problems with our algorithm and why it can be difficult to compute a matrix inverse.

Topics and Objectives

Topics

We will cover these topics in this section.

1. Inverse of a matrix, its algebraic properties, and its relation to solving systems of linear equations.
2. Elementary matrices and their role in calculating the matrix inverse.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Apply the formal definition of an inverse, and its algebraic properties, to solve and analyze linear systems.
2. Compute the inverse of an $n \times n$ matrix, and use it to solve linear systems.
3. Construct elementary matrices.

Motivating Question

Is there a matrix, A, such that $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} A = I_3$?

Section 2.2 : Inverse of a Matrix

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

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Topics and Objectives

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Course Schedule

Cancellations due to inclement weather will likely result in cancelling review lectures and possibly moving through course material at a faster pace.

		Mon	Tue	Wed	Thu	Fri
Week	**Dates**	**Lecture**	**Studio**	**Lecture**	**Studio**	**Lecture**
1	1/6 – 1/10	1.1	WS1.1	1.2	WS1.2	1.3
2	1/13 – 1/17	1.4	WS1.3,1.4	1.5	WS1.5	1.7
3	1/20 – 1/24	Break	WS1.7	1.8	WS1.8	1.9
4	1/27 – 1/31	2.1	WS1.9,2.1	**Exam 1**, Review	Cancelled	2.2
5	2/3 – 2/7	2.3	WS2.2,2.3	2.4,2.5	WS2.4	2.5
6	2/10 – 2/14	2.8	WS2.5,2.8	2.9,3.1	WS2.9	3.2
7	2/17 – 2/21	3.3	WS3.1-3.3	4.9	WS4.9	5.1
8	2/24 – 2/28	5.2	WS5.1,5.2	**Exam 2**, Review	Cancelled	5.3
9	3/3 – 3/7	5.3	WS5.3	5.5	WS5.5	6.1
10	3/10 – 3/14	6.1,6.2	WS6.1	6.2	WS6.2	6.3
11	3/17 – 3/21	Break	Break	Break	Break	Break
12	3/24 – 3/28	6.4	WS6.3	6.4,6.5	WS6.4	6.5
13	3/31 – 4/4	6.6	WS6.5,6.6	**Exam 3**, Review	Cancelled	PageRank
14	4/7 – 4/11	7.1	WSPageRank	7.2	WS7.1,7.2	7.3
15	4/14 – 4/18	7.3,7.4	WS7.3	7.4	WS7.4	7.4
16	4/21 – 4/22	Last lecture	Last Studio	Reading Period		
17	4/28 – 5/2	Final Exams: **MATH 1554 Common Final Exam Tuesday, April 29th at 6:00pm**				

The Matrix Inverse

Definition

$A \in \mathbb{R}^{n\times n}$ is **invertible** (or **non-singular**) if there is a $C \in \mathbb{R}^{n\times n}$ so that

$$AC = CA = I_n.$$

If there is, we write $C = A^{-1}$.

Ex In $\mathbb{R}$ Can we find x so that $5x = 1$?

$\Rightarrow x = \frac{1}{5} = 5^{-1}$

The Inverse of a 2×2 Matrix

There's a formula for computing the inverse of a 2×2 matrix.

Theorem

The 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non-singular if and only if $ad - bc \neq 0$, and then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example

State the inverse of the matrix below.

$$A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2(-7) - (-3)(5)}\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$$

$$= \frac{1}{1}\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}.$$

$$AA^{-1} = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}\begin{bmatrix} -7 \\ 3 \end{bmatrix} = -7\begin{bmatrix} 2 \\ -3 \end{bmatrix} + 3\begin{bmatrix} 5 \\ -7 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}\begin{bmatrix} -5 \\ 2 \end{bmatrix} = -5\begin{bmatrix} 2 \\ -3 \end{bmatrix} + 2\begin{bmatrix} 5 \\ -7 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Exercise: Check

Assume $ad - bc \neq 0$

$$\frac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = I_2.$$

The Matrix Inverse

Theorem

$A \in \mathbb{R}^{n\times n}$ has an inverse if and only if for all $\vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ has a unique solution. And, in this case, $\vec{x} = A^{-1}\vec{b}$.

Important: In applications, the entries of A are given in terms of units, say deflection per unit load. Then A^{-1} is given in terms of load per unit deflection. (Always keep units in mind in applications.)

Example

Solve the linear system.

$$3x_1 + 4x_2 = 7$$
$$5x_1 + 6x_2 = 7$$

in $\mathbb{R}$

$5x = 10$

$5^{-1}(5x) = 5^{-1}(10)$

$1x = 2$

in $\mathbb{R}^n$ $\quad A\vec{x} = \vec{b}$

if A^{-1} exists,

$A^{-1}(A\vec{x}) = A^{-1}(\vec{b})$

$I_n\vec{x} = A^{-1}\vec{b}$

$\vec{x} = A^{-1}\vec{b}$

$3x_1 + 4x_2 = 7$

$5x_1 + 6x_2 = 7$

$$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

Properties of the Matrix Inverse

A and B are invertible $n \times n$ matrices.

1. $(A^{-1})^{-1} = A$
2. $(AB)^{-1} = B^{-1}A^{-1}$ (Non-commutative!)
3. $(A^T)^{-1} = (A^{-1})^T$

Example

True or false: $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$. True

Pf 1 $(ABC)^{-1} = (A(BC))^{-1} \overset{②}{=} (BC)^{-1}A^{-1} \overset{②}{=} (C^{-1}B^{-1})A^{-1} = C^{-1}B^{-1}A^{-1}$

Pf 2 $\underbrace{(C^{-1}B^{-1}A^{-1})}_{\text{behaves like } (ABC)^{-1}}(ABC) = C^{-1}B^{-1}A^{-1}ABC = C^{-1}B^{-1}IBC$
$= C^{-1}B^{-1}BC$
$= C^{-1}C = I.$

Is $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$ invertible?

Yes, because $3(6) - (5)(4) \neq 0$.

$$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}^{-1} = \frac{1}{3(6)-5(4)}\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix}$$

$$\cancel{\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}^{-1}\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}^{-1}\begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

$$I_2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\frac{1}{2}\left(7\begin{bmatrix} 6 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -4 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -7 \\ 7 \end{bmatrix}.$$

An Algorithm for Computing A^{-1}

If $A \in \mathbb{R}^{n\times n}$, and $n > 2$, how do we calculate A^{-1}? Here's an algorithm we can use:

1. Row reduce the augmented matrix $(A \mid I_n)$
2. If reduction has form $(I_n \mid B)$ then A is invertible and $B = A^{-1}$. Otherwise, A is not invertible.

Example

Compute the inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$.

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{\sim}{\text{Swap } R_1 \,\&\, R_2}\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{matrix} R_1 - 3R_3 \to R_1 \\ \sim \\ R_2 - 2R_3 \to R_2 \end{matrix}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & 1 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

A^{-1}

$$AA^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & -3 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} = I \checkmark$$

Why Does This Work?

We can think of our algorithm as simultaneously solving n linear systems:

$$A\vec{x}_1 = \vec{e}_1 \rightarrow \vec{x}_1 = A^{-1}\vec{e}_1$$
$$A\vec{x}_2 = \vec{e}_2$$
$$\vdots$$
$$A\vec{x}_n = \vec{e}_n$$

Each column of A^{-1} is $A^{-1}\vec{e}_i = \vec{x}_i$.

There's another explanation, which uses elementary matrices.

$$\left[A \mid \vec{e}_1 \ \vec{e}_2 \ \cdots \ \vec{e}_n\right]$$

$$\downarrow$$

$$\left[\vec{e}_1 \cdots \vec{e}_n \mid A^{-1}\right]$$

Elementary Matrices

An elementary matrix, E, is one that differs by I_n by one row operation.

Recall our elementary row operations:

1. swap rows
2. multiply a row by a non-zero scalar
3. add a multiple of one row to another

We can represent each operation by a matrix multiplication with an **elementary matrix**.

Every elementary matrix is invertible!

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \xrightarrow{\text{Swap } R_1 \,\&\, R_2} \begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = 1\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{3R_1} \begin{bmatrix} 3 & 6 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 3 & 4 \end{bmatrix}$$

Example

Suppose

$$E\begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 + 2R_1 \to R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

By inspection, what is E? How does it compare to I_3?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2+2R_1 \to R_2} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E$$

Check:

$$E\begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$$

Theorem

Returning to understanding why our algorithm works, we apply a sequence of row operations to A to obtain I_n:

$$(E_k \cdots E_3 E_2 E_1)A = I_n$$

Thus, $E_k \cdots E_3 E_2 E_1$ is the inverse matrix we seek.

Our algorithm for calculating the inverse of a matrix is the result of the following theorem.

> **Theorem**
> Matrix A is invertible if and only if it is row equivalent to the identity. In this case, the any sequence of elementary row operations that transforms A into I, applied to I, generates A^{-1}.

$$A \sim I_n$$

$$\underbrace{(E_k \cdots E_2 E_1)}_{E}A = I_n$$

$$EA = I_n$$

$$\Rightarrow \boxed{E = A^{-1}}$$

$$\left[\, A \mid I_n \,\right] \longrightarrow \left[\, I_n \mid A^{-1} \,\right]$$

$$\left[\, E_k \cdots E_2 E_1 A \mid E_k \cdots E_2 E_1 I_n \,\right]$$

Using The Inverse to Solve a Linear System

- We could use A^{-1} to solve a linear system,

$$A\vec{x} = \vec{b}$$

We would calculate A^{-1} and then:

$$\vec{x} = A^{-1}\vec{b}$$

- As our textbook points out, A^{-1} is seldom used: computing it can take a very long time, and is prone to numerical error.
- So why did we learn how to compute A^{-1}? Later on in this course, we use elementary matrices and properties of A^{-1} to derive results.
- A recurring theme of this course: just because we **can** do something a certain way, doesn't that we **should**.

(b) (2 points) For what value(s) of h is the following set of vectors linearly dependent?

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ h \end{pmatrix}, \begin{pmatrix} 1 \\ h \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ h \end{pmatrix} \right\}$$

Find the inverses of the matrices in Exercises 1–4.

1. $\begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$ 2. $\begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix}$

3. $\begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix}$ 4. $\begin{bmatrix} 3 & -4 \\ 7 & -8 \end{bmatrix}$

5. Use the inverse found in Exercise 1 to solve the system

$$\begin{aligned} 8x_1 + 6x_2 &= 2 \\ 5x_1 + 4x_2 &= -1 \end{aligned}$$

6. Use the inverse found in Exercise 3 to solve the system

$$\begin{aligned} 8x_1 + 5x_2 &= -9 \\ -7x_1 - 5x_2 &= 11 \end{aligned}$$

7. Let $A = \begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$, and $\mathbf{b}_4 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.
   a. Find A^{-1}, and use it to solve the four equations $A\mathbf{x} = \mathbf{b}_1$, $A\mathbf{x} = \mathbf{b}_2$, $A\mathbf{x} = \mathbf{b}_3$, $A\mathbf{x} = \mathbf{b}_4$
   b. The four equations in part (a) can be solved by the *same* set of row operations, since the coefficient matrix is the same in each case. Solve the four equations in part (a) by row reducing the augmented matrix $[A \ \ \mathbf{b}_1 \ \mathbf{b}_2 \ \mathbf{b}_3 \ \mathbf{b}_4]$.

8. Use matrix algebra to show that if A is invertible and D satisfies $AD = I$, then $D = A^{-1}$.

In Exercises 9 and 10, mark each statement True or False. Justify each answer.

9. a. In order for a matrix B to be the inverse of A, both equations $AB = I$ and $BA = I$ must be true.
   b. If A and B are $n \times n$ and invertible, then $A^{-1}B^{-1}$ is the inverse of AB.
   c. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ab - cd \neq 0$, then A is invertible.
   d. If A is an invertible $n \times n$ matrix, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for *each* $\mathbf{b}$ in $\mathbb{R}^n$.
   e. Each elementary matrix is invertible.

10. a. A product of invertible $n \times n$ matrices is invertible, and the inverse of the product is the product of their inverses in the same order.
    b. If A is invertible, then the inverse of A^{-1} is A itself.
    c. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad = bc$, then A is not invertible.
    d. If A can be row reduced to the identity matrix, then A must be invertible.
    e. If A is invertible, then elementary row operations that reduce A to the identity I_n also reduce A^{-1} to I_n.

11. Let A be an invertible $n \times n$ matrix, and let B be an $n \times p$ matrix. Show that the equation $AX = B$ has a unique solution $A^{-1}B$.

12. Let A be an invertible $n \times n$ matrix, and let B be an $n \times p$ matrix. Explain why $A^{-1}B$ can be computed by row reduction:

If $[A \ \ B] \sim \cdots \sim [I \ \ X]$, then $X = A^{-1}B$.

If A is larger than 2×2, then row reduction of $[A \ \ B]$ is much faster than computing both A^{-1} and $A^{-1}B$.

13. Suppose $AB = AC$, where B and C are $n \times p$ matrices and A is invertible. Show that $B = C$. Is this true, in general, when A is not invertible?

14. Suppose $(B - C)D = 0$, where B and C are $m \times n$ matrices and D is invertible. Show that $B = C$.

15. Suppose A, B, and C are invertible $n \times n$ matrices. Show that ABC is also invertible by producing a matrix D such that $(ABC)\,D = I$ and $D\,(ABC) = I$.

16. Suppose A and B are $n \times n$, B is invertible, and AB is invertible. Show that A is invertible. [*Hint:* Let $C = AB$, and solve this equation for A.]

17. Solve the equation $AB = BC$ for A, assuming that A, B, and C are square and B is invertible.

18. Suppose P is invertible and $A = PBP^{-1}$. Solve for B in terms of A.

19. If A, B, and C are $n \times n$ invertible matrices, does the equation $C^{-1}(A + X)B^{-1} = I_n$ have a solution, X? If so, find it.

Find the inverses of the matrices in Exercises 29–32, if they exist. Use the algorithm introduced in this section.

29. $\begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix}$ 30. $\begin{bmatrix} 5 & 10 \\ 4 & 7 \end{bmatrix}$

31. $\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$ 32. $\begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$

33. Use the algorithm from this section to find the inverses of

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Let A be the corresponding $n \times n$ matrix, and let B be its inverse. Guess the form of B, and then prove that $AB = I$ and $BA = I$.

34. Repeat the strategy of Exercise 33 to guess the inverse of $A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 0 & & 0 \\ 1 & 2 & 3 & & 0 \\ \vdots & & & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix}$. Prove that your guess is correct.

35. Let $A = \begin{bmatrix} -2 & -7 & -9 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{bmatrix}$. Find the third column of A^{-1}

38. Let $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$. Construct a 4×2 matrix D using only 1 and 0 as entries, such that $AD = I_2$. Is it possible that $CA = I_4$ for some 4×2 matrix C? Why or why not?