Curtis Thompson

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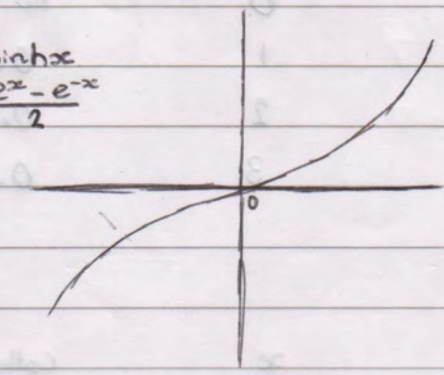
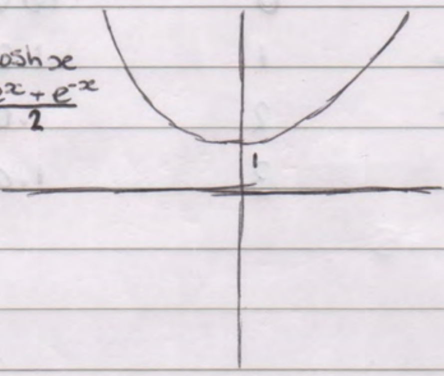
Hyperbolic Functions

$f(x) = \sinh x$
$= \frac{e^x - e^{-x}}{2}$



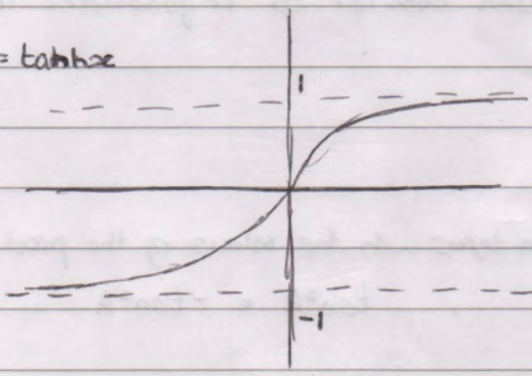
x	$\sinh x$
0	0
1	1.18
$\ln(1+\sqrt{2})$	1
$\ln(2+\sqrt{5})$	2

$f(x) = \cosh x$
$= \frac{e^x + e^{-x}}{2}$



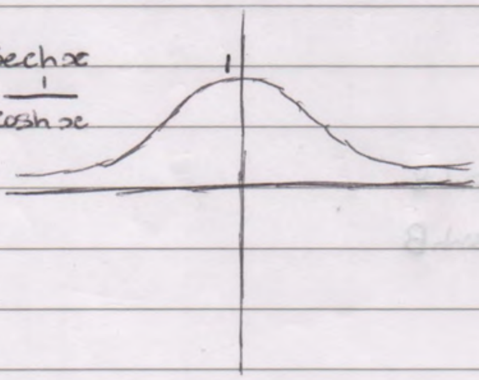
x	$\cosh x$
0	1
1	1.54
no value	0
	2

$f(x) = \tanh x$



x	$\tanh x$
0	0
1	0.76
2	0.96
3	1.00

$f(x) = \operatorname{sech} x$
$= \frac{1}{\cosh x}$



x	$\operatorname{sech} x$
0	1
1	0.65
2	0.27
3	0.01

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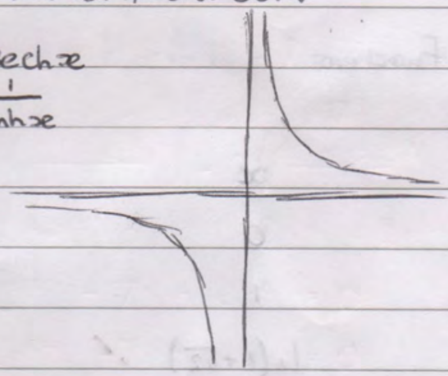
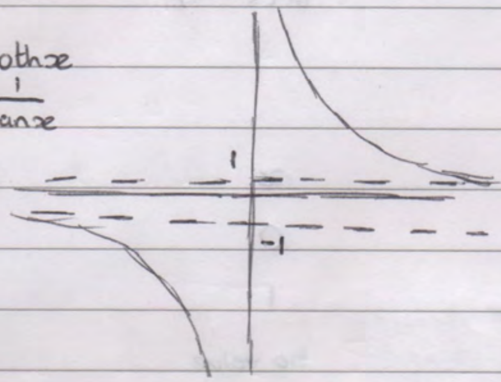
$f(x) = \operatorname{cosech} x$
$= \frac{1}{\sinh x}$



x	$\operatorname{cosech} x$
0	no value
1	0.85
2	0.28
3	0.01

$f(x) = \coth x$
$= \frac{1}{\tanh x}$



x	$\coth x$
0	no value
1	1.31
2	1.04
3	1.01

Hyperbolic Identities

- Hyperbolic identities can be written similar to trigonometric identities by using Osborn's Rule:
  - Replace $\sin A$ with $\sinh A$
  - Replace $\cos A$ with $\cosh A$
  - Replace the product of two sin terms with the minus of the product e.g. $\sin A \sin B \rightarrow -\sinh A \sinh B$, $\tan^2 A = -\tan^2 A$

$\cosh^2 A - \sinh^2 A = 1$

$\operatorname{sech}^2 A = 1 - \tanh^2 A$

$\operatorname{cosech}^2 A = \coth^2 A - 1$

$\sinh(A \pm B) = \sinh A \cosh B \pm \cosh A \sinh B$

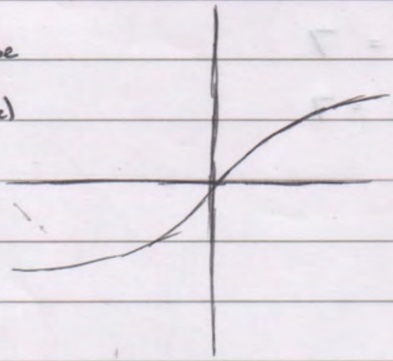
$\cosh(A \pm B) = \cosh A \cosh B \pm \sinh A \sinh B$

$\tanh(A \pm B) = \frac{\tanh A \pm \tanh B}{1 \mp \tanh A \tanh B}$

Further Hyperbolic Functions

$f(x) = \text{arsinh}\,x$

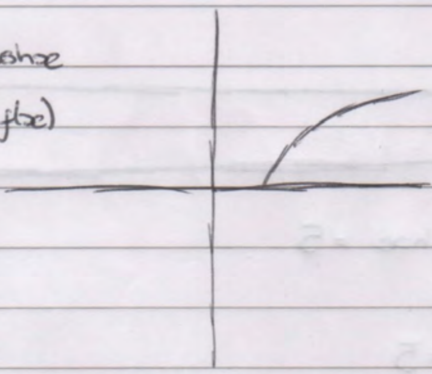
$x = \sinh f(x)$



$\text{arsinh}\,x = \ln\left(x + \sqrt{x^2+1}\right)$

$f(x) = \text{arcosh}\,x$

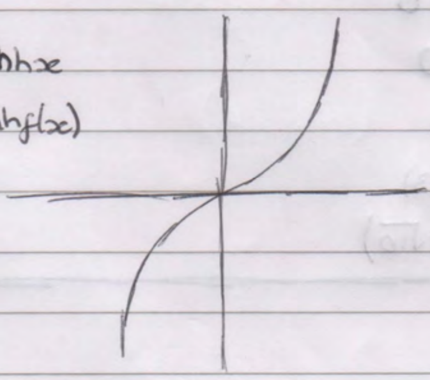
$x = \cosh f(x)$



$\text{arcosh}\,x = \ln\left(x + \sqrt{x^2-1}\right)$

$f(x) = \text{artanh}\,x$

$x = \tanh f(x)$



$\text{artanh}\,x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

Function	Inverse
sinh	arsinh
cosh	arcosh
tanh	artanh$x$
sech	arsech$x$
cosech	arcosech x
coth	arcoth$x$

e.g. Solve $6\sinh x - 2\cosh x = 7$

$$6\sinh x - 2\cosh x = 7$$

$$6\left(\frac{e^x - e^{-x}}{2}\right) - 2\left(\frac{e^x + e^{-x}}{2}\right) = 7$$

$$3e^x - 3e^{-x} - e^x - e^{-x} = 7$$

$$2e^x - 7 - e^{-x} = 0$$

$$2e^{2x} - 7e^x - 1 = 0$$

$$(2e^x + 1)(e^x - 4) = 0$$

$$e^x = -\tfrac{1}{2} \qquad e^x = 4$$

$$\therefore e^x = 4$$

$$x = \ln 4$$

e.g. Solve $2\cosh^2 x - 5\sinh x = 5$

$$2\cosh^2 x - 5\sinh x = 5$$

$$2(1 + \sinh^2 x) - 5\sinh x = 5$$

$$2\sinh^2 x - 5\sinh x - 3 = 0$$

$$(2\sinh x + 1)(\sinh x - 3) = 0$$

$$\sinh x = -\tfrac{1}{2}, \quad \sinh x = 3$$

$$x = \operatorname{arsinh}(-\tfrac{1}{2}), \ \operatorname{arsinh}(3)$$

$$= \ln\left(-\tfrac{1}{2} + \tfrac{\sqrt{5}}{2}\right), \ \ln(3 + \sqrt{10})$$

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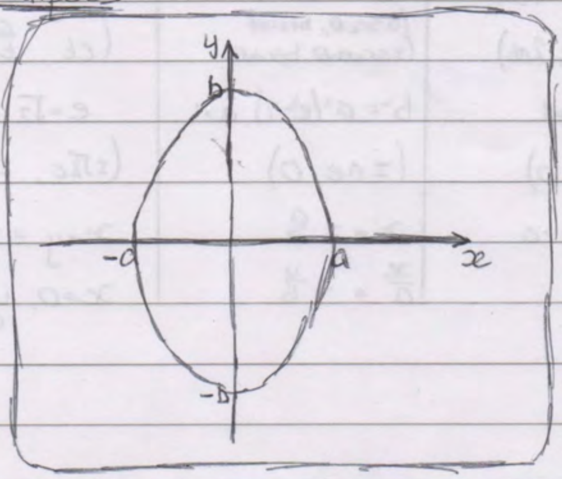
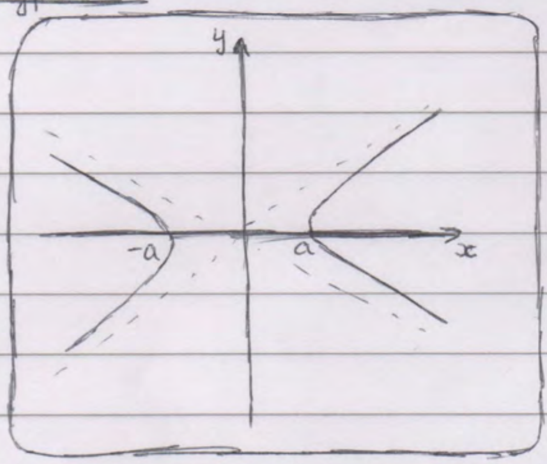
Coordinate Systems

Ellipses



- Equation : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
- General Point : $(a\cos\theta,\ b\sin\theta)$
- Tangent : $ay\sin\theta + bx\cos\theta = ab$
- Normal : $by\cos\theta = ax\sin\theta + (b^2 - a^2)\cos\theta\sin\theta$
- Eccentricity : $e < 1$ given by $b^2 = a^2(1 - e^2)$
- Foci : $(\pm ae, 0)$
- Directrices : $x = \pm \frac{a}{e}$

Hyperbola



- Equation : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
- General Point : $(a\cosh\theta,\ b\sinh\theta)$
  $(a\sec\theta,\ b\tan\theta)$
- Tangent : $ay\sinh\theta + ab = bx\cosh\theta$
- Normal : $by + a\sin\theta x = (a^2 + b^2)\tan\theta$
- Eccentricity : $e > 1$ given by $b^2 = a^2(e^2 - 1)$
- Foci : $(\pm ae, 0)$
- Directrices : $x = \pm \frac{a}{e}$
- Asymptotes : $\frac{x}{a} = \pm \frac{y}{b}$

Eccentricity

- Eccentricity is a constant for ratio $\frac{\text{distance to focus}}{\text{distance to directrix}}$ for any point
- $e = \frac{PF}{PD}$
- If $0 < e < 1$ the point describes an ellipses
- If $e = 1$ the point describes a parabola
- If $e > 1$ the point describes a hyperbola
- For an ellipses the major axis containing the foci ($a > b$ is x axis, $b > a$ is y axis)
- Foci will be $(\pm ae, 0)$ or $(0, \pm be)$
- Corresponding directrices will be $x = \pm \frac{a}{e}$ or $y = \pm \frac{b}{e}$

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Summary

Property	Ellipses	Parabola	Hyperbola	Rectangular Hyperbola
Standard Form	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$y^2=4ax$	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$xy=c^2$
Coordinates	$(a\cos\theta, b\sin\theta)$	$(at^2, 2at)$	$(a\sec\theta, b\tan\theta)$ $(\pm a\cosh\theta, b\sinh\theta)$	$\left(ct, \frac{c}{t}\right)$
Eccentricity	$b^2=a^2(1-e^2)$, $e<1$	$e=1$	$b^2=a^2(e^2-1)$, $e>1$	$e=\sqrt{2}$
Foci	$(\pm ae, 0)$	$(a,0)$	$(\pm ae, 0)$	$(\pm\sqrt{2}c, \pm\sqrt{2}c)$
Directricies	$x=\pm\frac{a}{e}$	$x=-a$	$x=\pm\frac{a}{e}$	$x+y=\pm\sqrt{2}c$
Asymptotes			$\frac{x}{a}=\pm\frac{y}{b}$	$x=0$, $y=0$

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Differentiation

$\frac{d}{dx}(\sinh x) = \cosh x$

$\frac{d}{dx}(\cosh x) = \sinh x$

$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

$\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$

$\frac{d}{dx}(\operatorname{cosech} x) = -\coth x \operatorname{cosech} x$

$\frac{d}{dx}(\operatorname{sech} x) = -\tanh x \operatorname{sech} x$

$\frac{d}{dx}(\operatorname{arsinh} x) = \frac{1}{\sqrt{x^2+1}}$

$\frac{d}{dx}(\operatorname{arcosh} x) = \frac{1}{\sqrt{x^2-1}}$

$\frac{d}{dx}(\operatorname{artanh} x) = \frac{1}{1-x^2}$

$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$

e.g. Differentiate $\sinh x$

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right)$$
$$= \frac{e^x + e^{-x}}{2}$$
$$= \cosh x$$

e.g. Differentiate $x^3 \cosh 2x$

$$\frac{d}{dx}(x^3 \cosh 2x) = (x^3)(2\sinh 2x) + (3x^2)(\cosh 2x)$$
$$= 2x^3 \sinh 2x + 3x^2 \cosh 2x$$
$$= x^2(2x \sinh 2x + 3\cosh 2x)$$

e.g. Differentiate $\operatorname{arsinh} x$

$\frac{d}{dx}(\operatorname{arsinh} x)$

$y = \operatorname{arsinh} x$

$\sinh y = x$

$\cosh y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\cosh y}$

$= \frac{1}{\sqrt{x^2+1}}$

$\cosh^2 y - \sinh^2 y = 1$

$\cosh y = \sqrt{\sinh^2 y + 1}$

$= \sqrt{x^2+1}$

e.g. Find the turning points of $5\cosh x - 3\sinh x$

$y = 5\cosh x - 3\sinh x$

$\frac{dy}{dx} = 5\sinh x - 3\cosh x$

$\frac{d^2y}{dx^2} = 5\cosh x - 3\sinh x$

$5\sinh x - 3\cosh x = 0$

$5\sinh x = 3\cosh x$

$\frac{\sinh x}{\cosh x} = \frac{3}{5}$

$\tanh x = \frac{3}{5}$

$x = \ln 2$

$y = 5\left(\frac{e^{\ln 2} + e^{-\ln 2}}{2}\right) - 3\left(\frac{e^{\ln 2} - e^{-\ln 2}}{2}\right)$

$= \frac{5}{2}\left(\frac{5}{2}\right) - \frac{3}{2}\left(\frac{3}{2}\right)$

$= \frac{25-9}{4}$

$= 4$

$\therefore \frac{d^2y}{dx^2} > 0$ $\therefore$ minimum turning point

$\therefore$ Minimum turning point at $(\ln 2, 4)$

FP3 Integration

$\int \sinh x \, dx = \cosh x$

$\int \cosh x \, dx = \sinh x$

$\int \operatorname{sech}^2 x \, dx = \tanh x$

$\int \operatorname{cosech}^2 x \, dx = -\coth x$

$\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x$

$\int \operatorname{cosech} x \coth x \, dx = -\operatorname{cosech} x$

$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x$

$\int \frac{1}{1+x^2} \, dx = \arctan x$

$\int \frac{1}{\sqrt{1+x^2}} \, dx = \operatorname{arsinh} x$

$\int \frac{1}{\sqrt{x^2-1}} \, dx = \operatorname{arcosh} x$

Substitutions

	①	②
$\sqrt{1-x^2}$	① $x = \sin\theta$	② $x = \tanh\theta$
$1+x^2$	① $x = \tan\theta$	② $x = \sinh\theta$
$\sqrt{1+x^2}$	① $x = \sinh\theta$	② $x = \tan\theta$
$\sqrt{x^2-1}$	① $x = \cosh\theta$	② $x = \sec\theta$

- Try using ① first, then try ② if it does not work

e.g. use substitution $x = a\tan\theta$ to show $\int \frac{1}{a^2+x^2} \, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

$x = a\tan\theta \qquad \theta = \tan^{-1}\left(\frac{x}{a}\right)$

$\frac{dx}{d\theta} = a\operatorname{sech}^2\theta \qquad dx = a\operatorname{sech}^2\theta \, d\theta$

$\int \frac{1}{a^2+a^2\tan^2\theta} \, a\operatorname{sech}^2\theta \, d\theta$

$= \frac{1}{a}\int \frac{\operatorname{sech}^2\theta}{1+\tan^2\theta} \, d\theta$

$= \frac{1}{a}\int \frac{\operatorname{sech}^2\theta}{\operatorname{sech}^2\theta} \, d\theta$

$= \frac{1}{a}\int 1 \, d\theta$

$= \frac{1}{a}\theta + c$

$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

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Reciprocal of Quadratics

- $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right)$
- $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$
- $\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \text{arsinh}\left(\frac{x}{a}\right)$
- $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \text{arcosh}\left(\frac{x}{a}\right)$
- $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)$
- $\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)$

e.g. $\int \frac{1}{x^2-8x+8}$

$x^2-8x+8 = (x-4)^2-8$

let $u = x-4$, $x = u+4$

$\therefore x^2-8x+8 = u^2-8$

$$\int \frac{1}{x^2-8x+8}\,dx = \int \frac{1}{u^2-8}\,du$$
$$= \frac{1}{4\sqrt{2}}\ln\left(\frac{u-2\sqrt{2}}{u+2\sqrt{2}}\right)$$
$$= \frac{1}{4\sqrt{2}}\ln\left(\frac{x-4-2\sqrt{2}}{x-4+2\sqrt{2}}\right)$$

e.g. $\int \frac{1}{\sqrt{x^2-4x-12}}$

$x^2-4x-12 = (x-2)^2-16$

let $u = x-2$

$\therefore x^2-4x-12 = u^2-16$

$$\int \frac{1}{\sqrt{x^2-4x-12}}\,dx = \int \frac{1}{\sqrt{u^2-16}}\,du$$
$$= \text{arcosh}\left(\frac{u}{4}\right)$$
$$= \text{arcosh}\left(\frac{x-2}{4}\right)$$

Further Integration

Reduction Formula

e.g. Given $I_n = \int x^n e^x \, dx$ Show $I_n = x^n e^x - n I_{n-1}$ and find $\int x^4 e^x \, dx$.

$u = x^n \qquad v = e^x$

$u' = n x^{n-1} \qquad v' = e^x$

$$\int x^n e^x \, dx = x^n e^x - \int n x^{n-1} e^x \, dx$$
$$= x^n e^x - n \int x^{n-1} e^x \, dx$$
$$= x^n e^x - n I_{n-1}$$

$$\int x^4 e^x \, dx = x^4 e^x - 4 I_3$$
$$= x^4 e^x - 4x^3 e^x + 12 I_2$$
$$= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24 I_1$$
$$= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24x e^x + 24e^x + c$$

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Length of Arc

- $S = \int_{x_a}^{x_b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$
- $S = \int_{y_a}^{y_b} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$
- $S = \int_{t_a}^{t_b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$

e.g. Find length of arc of curve $y = \frac{1}{3} x^{\frac{3}{2}}$ from origin to $x = 12$

$$y = \frac{1}{3} x^{\frac{3}{2}}$$
$$\frac{dy}{dx} = \frac{1}{2} x^{\frac{1}{2}}$$

$$S = \int_0^{12} \sqrt{1 + \left(\tfrac{1}{2} x^{\frac{1}{2}}\right)^2} \, dx$$
$$= \int_0^{12} \sqrt{1 + \tfrac{1}{4} x} \, dx$$
$$= \int_0^{12} \left(1 + \tfrac{1}{4} x\right)^{\frac{1}{2}} \, dx$$
$$= \left[\left(\tfrac{1}{3}(x+4)\right)^{\frac{3}{2}} \right]_0^{12}$$
$$= \frac{1}{3} (16)^{\frac{3}{2}}$$
$$= \frac{64}{3}$$

Surface Area of Revolution

- $S = \int 2\pi y \frac{ds}{dx} \, dx$ $(x\text{-axis})$
- $S = \int 2\pi y \, dx$ $(x\text{-axis})$
- $S = 2\pi \int_{x_a}^{x_b} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$ $(x\text{-axis})$
- $S = 2\pi \int_{t_a}^{t_b} y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$ $(x\text{-axis})$
- $S = 2\pi \int_{y_a}^{y_b} x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$ $(y\text{-axis})$
- $S = 2\pi \int_{x_a}^{x_b} x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$ $(y\text{-axis})$
- $S = 2\pi \int_{t_a}^{t_b} x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$ $(y\text{-axis})$

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Vectors

Vector Product

- Scalar product : $a.b = |a||b|\cos\theta$
- Vector product : $a \times b = |a||b|\sin\theta\,\hat{n}$
- $\underline{a} \times \underline{b} = (a_2b_3 - a_3b_2)\underline{i} + (a_3b_1 - a_1b_3)\underline{j} + (a_1b_2 - a_2b_1)\underline{k}$
  $= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \underline{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \underline{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \underline{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$
- Area of triangle ABC $= \frac{1}{2}|\vec{AB} \times \vec{AC}|$
- Area of parallelogram ABCD $= |\vec{AB} \times \vec{AD}|$
- Triple Scalar Product: $\underline{a}\cdot(\underline{b} \times \underline{c})$
- Volume of parallelepiped $= |\underline{a}\cdot(\underline{b} \times \underline{c})|$
- Volume of tetrahedron $= |\frac{1}{6}\underline{a}\cdot(\underline{b} \times \underline{c})|$

e.g. Find area of triangle ABC where position vectors of A, B and C are $(4\underline{i} - 2\underline{j} + \underline{k})$, $(-12\underline{i} + 14\underline{j} + \underline{k})$ and $(-4\underline{i} - 2\underline{j} + \underline{k})$

$\vec{AB} = -16\underline{i} + 16\underline{j}$

$\vec{AC} = -8\underline{i}$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ -16 & 16 & 0 \\ -8 & 0 & 0 \end{vmatrix}$$

$$= \underline{i}(16\times0 - 0\times0) - \underline{j}(-16\times0 - -8\times0) + \underline{k}(-16\times0 - -8\times16)$$

$$= 128\,\underline{k}$$

$\therefore$ area $= \frac{1}{2}|\vec{AB} \times \vec{AC}|$

$= \frac{1}{2}|128\underline{k}|$

$= \underline{\underline{64}}$

Vector Equations of a Line

- $\underline{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ $\qquad \underline{r} = \underline{a} + \lambda \underline{b}$
- $\left(\underline{r} - \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right) \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = 0$ $\qquad (\underline{r} - \underline{a}) \times \underline{b} = 0$
- $\dfrac{x - a_1}{b_1} = \dfrac{y - a_2}{b_2} = \dfrac{z - a_3}{b_3} = \lambda$

Vector Equations of Planes

- $\underline{r} \cdot \underline{n} = p$
- $\underline{r} = \underline{a} + \lambda \underline{b} + \mu \underline{c}$
- $ax + by + cz + d = 0$

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Matrices

Transposing a Matrix

- Transposing a matrix means interchanging the rows and columns
- The first row becomes the first column, second row becomes the second column, and so on
- If $A = A^T$ then the matrix is symetrical

e.g. Transposing $\begin{pmatrix} 3 & 2 & -3 \\ 1 & 8 & -7 \\ -6 & 4 & 0 \end{pmatrix}$
gives $\begin{pmatrix} 3 & 1 & 6 \\ 2 & 8 & 4 \\ -3 & -7 & 0 \end{pmatrix}$

- An $m \times n$ matrix becomes $n \times m$ when transposed

Determinant of a 3x3 Matrix

- The determinant of a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
- The determinant of a 3×3 matrix $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ is
$a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

e.g. Find the determinant of $\begin{pmatrix} 1 & 2 & 1 \\ 4 & -7 & 2 \\ -2 & 3 & 0 \end{pmatrix}$

$(1)\begin{vmatrix} -7 & 2 \\ 3 & 0 \end{vmatrix} - (2)\begin{vmatrix} 4 & 2 \\ -2 & 0 \end{vmatrix} + (1)\begin{vmatrix} 4 & -7 \\ -2 & 3 \end{vmatrix}$

$= (-7 \times 0) - (2 \times 3) \quad -2[(4 \times 0) - (2 \times -2)] \quad + (4 \times 3) - (-7 \times -2)$

$= -6 \quad -8 + 12 - 14$

$= -16$

Find the Minor of an Element

- The minor of an element in a 3×3 matrix is the determinant of the remaining 2×2 matrix when the row and column of an element are crossed out

e.g. Find the minor of 5 in the matrix $\begin{pmatrix} 5 & 11 & -3 \\ 14 & 1 & 2 \\ 2 & 3 & 7 \end{pmatrix}$

$\begin{pmatrix} 5 & 11 & -3 \\ 14 & 1 & 2 \\ 2 & 3 & 7 \end{pmatrix}$

minor $= (1)(7) - (3)(2)$

$= 1$

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Finding the Inverse of a 3x3 Matrix

- There are five steps to finding a 3x3 inverse matrix:
  1. Find the determinant, det M
  2. Find the matrix of minors, M_m
  3. Find the cofactor matrix of M_m by changing signs, C
  4. ~~Find~~ Transpose C, C^T
  5. Write out the complete inverse, $M^{-1} = \frac{1}{\det M} C^T$

e.g. Find the inverse of $M = \begin{pmatrix} 12 & 1 & 3 \\ 6 & -6 & -18 \\ 3 & 14 & -22 \end{pmatrix}$

$$\det M = 12\begin{vmatrix} -6 & -18 \\ 14 & -22 \end{vmatrix} - \begin{vmatrix} 6 & -18 \\ 3 & -22 \end{vmatrix} + 3\begin{vmatrix} 6 & -6 \\ 3 & 14 \end{vmatrix}$$
$$= 4992$$

$$M_m = \begin{pmatrix} 384 & -78 & 102 \\ -64 & -273 & 165 \\ 0 & -234 & -78 \end{pmatrix}$$

$$C = \begin{pmatrix} 384 & 78 & 102 \\ 64 & -273 & -165 \\ 0 & 234 & -78 \end{pmatrix}$$

$$C^T = \begin{pmatrix} 384 & 64 & 0 \\ 78 & -273 & 234 \\ 102 & -165 & -78 \end{pmatrix}$$

$$\therefore M^{-1} = \frac{1}{4992}\begin{pmatrix} 384 & 64 & 0 \\ 78 & -273 & 234 \\ 102 & -165 & -78 \end{pmatrix}$$

- For the cofactor matrix you change signs in the following pattern:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

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Eigenvectors

- Eigenvector of matrix A Satisfies $Ax = \lambda x$
- λ is eigenvalue (scalar)
- Characteristic Equation : $\det(A - \lambda I) = 0$

e.g. Find the eigenvalues of $\begin{pmatrix} 2 & 5 \\ -1 & -4 \end{pmatrix}$

$$A - \lambda I = \begin{pmatrix} 2 & 5 \\ -1 & -4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$
$$= \begin{pmatrix} 2-\lambda & 5 \\ -1 & -4-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (2-\lambda)(-4-\lambda) - (5)(-1)$$
$$= -8 + 2\lambda + \lambda^2 + 5$$
$$= \lambda^2 + 2\lambda - 3$$

$$\lambda^2 + 2\lambda - 3 = 0$$
$$(\lambda - 1)(\lambda + 3) = 0$$

$\therefore$ eigenvalues are 1 and -3

e.g. Find the eigenvector of $\begin{pmatrix} 2 & 5 \\ -1 & -4 \end{pmatrix}$ for eigenvalue of -3

$$\begin{pmatrix} 2 & 5 \\ -1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -3\begin{pmatrix} x \\ y \end{pmatrix}$$
$$\begin{pmatrix} 2x + 5y \\ -x - 4y \end{pmatrix} = \begin{pmatrix} -3x \\ -3y \end{pmatrix}$$

$2x + 5y = -3x \qquad \therefore y = -x$

$-x - 4y = -3y \qquad \therefore y = -x$

Let $x = 1$, then $y = -1$

$\therefore$ eigenvector is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

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Diagonal Matrix

- If $MM^T = I$ then M is an orthogonal matrix
- Also $M^{-1} = M^T$
- Scalar product of any two column vector in matrix is 0
- $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ is 2×2 diagonal matrix
- $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ is 3×3 diagonal matrix
- To reduce Symmetric matrix, A, to diagonal matrix, D:
  1. Find normalised eigenvectors of A
  2. Form matrix P, columns of normalised eigenvectors of A
  3. Find P^T
  4. $D = P^T A P$
- $P^T = P^{-1}$ so $P^T A P = P^{-1} A P = D$