Partial Fractions

- To split a fraction into partial fractions:
  - factorise denominator
  - write as unknown constants over each factor
  - multiply everything by each factor
  - use different x values to find unknowns

e.g. Split $\frac{6x-2}{(x-3)(x+1)}$ into partial fractions

$$\frac{6x-2}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$$

$$6x-2 = A(x+1) + B(x-3)$$

let $x = 3$

$$6(3) - 2 = A(3+1)$$
$$4A = 16$$
$$A = 4$$

let $x = -1$

$$6(-1) - 2 = B(-1-3)$$
$$-4B = -8$$
$$B = 2$$

$$\therefore \frac{6x-2}{(x-3)(x+1)} = \frac{4}{x-3} + \frac{2}{x+1}$$

- If there is a repeated factor, e.g. $(x-a)^2$, it can be written with two fractions $\frac{A}{x-a} + \frac{B}{(x-a)^2}$

e.g. Express $\frac{6x^2-29x-29}{(x+1)(x-3)^2}$ in partial fractions

$$\frac{6x^2-29x-29}{(x+1)(x-3)^2} = \frac{A}{x+1} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$$

$$6x^2-29x-29 = A(x-3)^2 + B(x+1)(x-3) + C(x+1)$$

let $x = 3$

$$6(3)^2 - 29(3) - 29 = C(3+1)$$

$$4C = -4$$

$$C = -1$$

let $x = -1$

$$6(-1)^2 - 29(-1) - 29 = A(-1-3)^2$$

$$16A = 6$$

$$A = \frac{6}{16} = \frac{3}{8}$$

let $x = 0$

$$6(0)^2 - 29(0) - 29 = A(-3)^2 + B(1)(-3) + C(1)$$

$$-29 = \frac{27}{8} - 3B - 1$$

$$3B = \frac{251}{8}$$

$$B = \frac{251}{24}$$

$$\therefore \frac{6x^2-29x-29}{(x+1)(x-3)^2} = \frac{3}{8(x+1)} + \frac{251}{24(x-3)} - \frac{1}{(x-3)^2}$$

Coordinate Geometry

- In parametric equations the x and y coordinates are expressed in terms of a third variable
- Can be used to simply express complex equations

e.g. a curve has parametric equation $x = \sin t + 2$, $y = \cos t - 3$. Find the cartesian equation.

$x = \sin t + 2 \quad \therefore \sin t = x - 2$

$y = \cos t - 3 \quad \therefore \cos t = y + 3$

$\sin^2 t + \cos^2 t = 1$

$(x-2)^2 + (y+3)^2 = 1$

e.g. $x = t - 1$, $y = 4 - t^2$, find the coordinates the curve meets the x-axis

$y = 4 - t^2$

$4 - t^2 = 0$

$t^2 = 4$

$t = \pm 2$

$x = t - 1$

$x = 2 - 1 \qquad x = -2 - 1$

$= 1 \qquad = -3$

$\therefore x = -3, 1$

- The area under a curve is given by $\int y \, dx$
- For parametric equations it is also $\int y \frac{dx}{dt} \, dt$

e.g. a curve has parametric equations $x = 2t - 5$, $y = 3t + 8$, find the area under the curve between $x = 0$ and $x = 4$

$x = 2t - 5$

$\frac{dx}{dt} = 2$

$$\text{area} = \int_0^4 (3t+8)(2) \, dt$$

$$= \int_0^4 6t + 16 \, dt$$

$$= \left[3t^2 + 16t\right]_0^4$$

$$= \underline{\underline{112}}$$

Binomial Expansion

- $(1+x)^n = 1 + nx + n(n-1)\frac{x^2}{2!} + n(n-1)(n-2)\frac{x^3}{3!} + \dots$
- A square root can be replaced with $\frac{1}{2}$ as a power
- $\frac{1}{(1+x)^n} = (1+x)^{-n}$

e.g. find the first four terms of $\frac{1}{(1+2x)^3}$

$$\begin{aligned}\frac{1}{(1+2x)^3} &= (1+2x)^{-3}\\ &= 1 + (-3)(2x) + (-3)(-4)\frac{(2x)^2}{2} + (-3)(-4)(-5)\frac{(2x)^3}{3!}\\ &= 1 - 6x + 24x^2 - 80x^3\end{aligned}$$

e.g. find the first three terms of $\sqrt{(1-3x)}$

$$\begin{aligned}\sqrt{(1-3x)} &= (1-3x)^{\frac{1}{2}}\\ &= 1 + \left(\tfrac{1}{2}\right)(-3x) + \left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right)\frac{(-3x)^2}{2!}\\ &= 1 - \frac{3}{2}x + \frac{3}{8}x^2\end{aligned}$$

- $(a+bx)^n$ can be expanded by taking out the factor a and using $(1+x)^n$

e.g. find the first four terms of $\frac{1}{(2+3x)^2}$

$$\begin{aligned}\frac{1}{(2+3x)^2} &= (2+3x)^{-2}\\ &= \left[2\left(1+\tfrac{3}{2}x\right)\right]^{-2}\\ &= \frac{1}{4}\left(1+\frac{3}{2}x\right)^{-2}\\ &= \frac{1}{4}\left[1 + (-2)\left(\tfrac{3}{2}x\right) + (-2)(-3)\frac{\left(\frac{3}{2}x\right)^2}{2!} + (-2)(-3)(-4)\frac{\left(\frac{3}{2}x\right)^3}{3!}\right]\\ &= \frac{1}{4}\left[1 - 3x + \frac{27}{4}x^2 - \frac{27}{2}x^3\right]\\ &= \frac{1}{4} - \frac{3}{4}x + \frac{27}{16}x^2 - \frac{27}{8}x^3\end{aligned}$$

Differentiation

Parametric Differentiation

- $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$
- if $\frac{dx}{dt}$ is flipped it becomes $\frac{dt}{dx}$ (e.g. $\frac{2}{3t-1}$ flipped is $\frac{3t-1}{2}$)

e.g. find the gradient when $t=2$ on the parametric curve

$x = t^3 + t \quad , \quad y = t^2 + 1$

$$\frac{dy}{dt} = 2t$$

$$\frac{dx}{dt} = 3t^2 + 1$$

$$\frac{dt}{dx} = \frac{1}{3t^2+1}$$

$$\frac{dy}{dx} = \frac{2t}{3t^2+1}$$

$$= \frac{4}{13}$$

Implicit Differentiation

- $\frac{d}{dx} y^n = n y^{n-1} \frac{dy}{dx}$
- differentiate y terms normally but multiply by $\frac{dy}{dx}$

e.g. Differentiate $2xy + x^2 + 2y^2 + \cos x = 4$

$$2xy + x^2 + 2y^2 + \cos x = 4$$

$$(2x)\left(\frac{dy}{dx}\right) + (2)(y) + 2x + 4y - \sin x = 0$$

$$2x\frac{dy}{dx} + 2y + 2x + 4y - \sin x = 0$$

$$2x\frac{dy}{dx} + 2x + 6y - \sin x = 0$$

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Differentiate Exponential Functions

- $\frac{d}{dx}(a^x) = a^x \ln a$
- the differential of e^x is e^x

Proof

$y = a^x$

$\ln y = \ln a^x$

$\ln y = x \ln a$

$\frac{1}{y}\frac{dy}{dx} = \ln a$

$\frac{dy}{dx} = y \ln a$

$= a^x \ln a$

e.g. Differentiate $9(3^x)$

$y = 9(3^x)$

$\frac{dy}{dx} = 9(3^x \ln 3)$

$= 3^2(3^x \ln 3)$

$= 3^{x+2} \ln 3$

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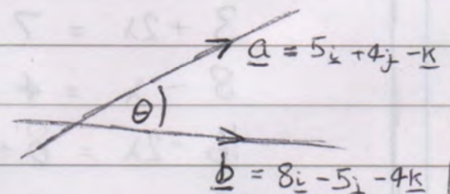
Vectors

- $\underline{i}$ is unit vector parallel to x-axis
- j is unit vector parallel to y-axis
- Vectors can be written as $a\underline{i} + b\underline{j}$ or $\begin{pmatrix} a \\ b \end{pmatrix}$
- vectors can also be named as two points $\overrightarrow{AB}$ or direction $\underline{a}$
- magnitude of $a\underline{i} + b\underline{j}$ is $\sqrt{a^2 + b^2}$
- for 3-dimensions, $\underline{K}$ is unit vector parallel to z-axis
- distance between two points is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Scalar Product

- $a \cdot b = |a||b| \cos\theta$
- if lines are perpendicular:
  - $a \cdot b = 0$
  - $|a||b| \cos\theta = 0$
- $a \cdot b = (i_1 i_2) + (j_1 j_2) + (k_1 k_2)$
- if lines are parallel $a \cdot b = |a||b|$
- to find angle between lines use the direction vectors
- Lines must intersect to have angle between them



$$\theta = \cos^{-1}\left(\frac{a \cdot b}{|a||b|}\right)$$
$$= \cos^{-1}\left(\frac{24}{\sqrt{42}\sqrt{105}}\right)$$
$$= \cos^{-1}(0.3614..)$$
$$= 68.8$$

Lines

- Equation of a line is given by $\underline{r} = \underline{a} + \lambda\underline{b}$
  - $\underline{a}$ is position vector
  - $\underline{b}$ is direction vector
  - λ is scalar parameter
- Can also be written as $\underline{r} = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} + \lambda\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$

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Check Whether Lines Intersect

- Equate lines
- Solve simultaneous equations for i and j
- Check that the k is also equal:
  - if equal then lines intersect
  - if not equal then lines are skew

e.g. Show that lines $\underline{r_1}$ and $\underline{r_2}$ intersect

$$\underline{r_1} = (3\underline{i} + 8\underline{j} - 2\underline{k}) + \lambda(2\underline{i} - \underline{j} + 3\underline{k})$$

$$\underline{r_2} = (7\underline{i} + 4\underline{j} + 3\underline{k}) + \mu(2\underline{i} + \underline{j} + 4\underline{k})$$

$$\begin{pmatrix} 3 + 2\lambda \\ 8 - \lambda \\ -2 + 3\lambda \end{pmatrix} = \begin{pmatrix} 7 + 2\mu \\ 4 + \mu \\ 3 + 4\mu \end{pmatrix}$$

$$3 + 2\lambda = 7 + 2\mu \quad ①$$

$$8 - \lambda = 4 + \mu \quad ②$$

$$16 - 2\lambda = 8 + 2\mu \quad ③$$

$$19 = 15 + 4\mu \quad ③+2$$

$$\mu = 1$$

$$\lambda = 3$$

$$-2 + 3(3) = 3 + 4(1)$$

$$-2 + 9 = 3 + 4$$

$$7 = 7$$

$\therefore$ lines do intersect

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Integration

- $\int x^n = \frac{x^{n+1}}{n+1} + c$
- $\int e^x = e^x + c$
- $\int \frac{1}{x} = \ln x + c$
- $\int \cos x = \sin x + c$
- $\int \sin x = -\cos x + c$
- $\int \sec^2 x = \tan x + c$
- $\int \operatorname{cosec} x \cot x = -\operatorname{cosec} x + c$
- $\int \operatorname{cosec}^2 x = -\cot x + c$
- $\int \sec x \tan x = \sec x + c$

e.g. $\int (4x+1)^3 = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)(4x+1)^4 + c$
$= \frac{1}{16}(4x+1)^4 + c$

e.g. $\int \frac{1}{3x+2} = \left(\frac{1}{3}\right)\left[\ln(3x+2)\right] + c$
$= \frac{1}{3}\ln(3x+2) + c$

e.g. $\int \operatorname{cosec}^2 3x = \left(\frac{1}{3}\right)(-\cot 3x) + c$
$= -\frac{1}{3}\cot 3x + c$

e.g. $\int -\operatorname{cosec} 2x \cot 2x = (-)\left(\frac{1}{2}\right)(-\operatorname{cosec} 2x) + c$
$= \frac{1}{2}\operatorname{cosec} 2x + c$

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Integration by Parts

- $\int u v' \, dx = uv - \int v u' \, dx$
- Sometimes integration by parts must be done twice

e.g. Integrate $x^2 \ln x$

$\int 2x^3 \ln x \, dx$

$u = \ln x \qquad v = x^2$

$u' = \frac{1}{x} \qquad v' = 2x$

$$\int x^2 \ln x \, dx = 2x \ln x - \int \tfrac{1}{x} x^2 \, dx$$
$$= 2x \ln x - \int x \, dx$$
$$= 2x \ln x - \frac{x^2}{2} + c$$

Areas and Volumes

- Area between $y = f(x)$ and x-axis:

  $\int_a^b y \, dx$
- Volume of revolution when $y = f(x)$ rotated about x-axis:

  $\pi \int_a^b y^2 \, dx$

e.g. $y = \sin 2x$ is rotated about the x-axis between $x = 0$ and $x = \frac{\pi}{2}$. Find the volume of revolution.

$$\text{Volume} = \pi \int_0^{\pi/2} \sin^2 2x \, dx$$
$$= \pi \int_0^{\pi/2} \tfrac{1}{2}(1 - \cos 4x) \, dx$$
$$= \frac{\pi}{2} \int_0^{\pi/2} 1 - \cos 4x \, dx$$
$$= \frac{\pi}{2} \left[x - \tfrac{1}{4} \sin 4x \right]_0^{\pi/2}$$
$$= \frac{\pi}{2} \left(\frac{\pi}{2} \right)$$
$$= \frac{\pi^2}{4}$$