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Complex Numbers

Imaginary and Complex Numbers

- A term containing i is complex
- i is equal to $\sqrt{-1}$
- A complex number, z, is $a+bi$ where a and b are real
- The complex conjugate, z^*, would then be $a-bi$

$i = \sqrt{-1}$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

Addition, Subtraction and Multiplication

- You deal with the real and imaginary parts seperately when adding or Subtracting

e.g. Find $(2+4i) + (6-i)$
$= (2+6) + (4i-i)$
$= 8+3i$

e.g. Find $(7-3i) - (2+2i)$
$= (7-2) + (-3i-2i)$
$= 5-5i$

- Multiplication works similarly, just expand the brackets then add
  - Remember to replace i^2 with -1, i^3 with $-i$ and i^4 with 1

e.g. Find $(2+i)(6+2i)$
$= 12 + 10i + 2i^2$
$= 12 + 10i - 2$
$= 10 + 10i$

Division

e.g. $\frac{2+2i}{3+i} = \frac{(2+2i)(3-i)}{(3+i)(3-i)}$
$= \frac{8+4i}{10}$
$= \frac{8}{10} + \frac{4}{10}i$

- To divide you must make the denominator real
- This can be done by multiplying numerator and denominator by the complex conjugate
- Then simplify

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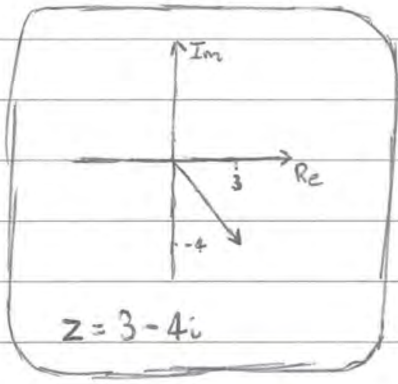
Argand Diagrams

- Argand diagrams can be used to show complex numbers
- $z = x + yi$ has the point (x, y)
- The real axis is x, the imaginary axis is y



Modulus-Argument Form

- The modulus of a complex number is written as $|z|$ or r
- If $z = a + bi$ then $|z| = \sqrt{a^2 + b^2}$
- The modulus must always be positive

- The argument of a complex number is written as $\arg z$ or θ
- The argument must be $-\pi < \theta \leq \pi$
- The standard equation is $\tan\theta = \frac{b}{a}$ if $z = a + bi$
- The specific equation depends on what quadrant of an argand diagram the complex number is in

$\pi - \tan^{-1}\left(\frac{b}{a}\right)$	$\tan^{-1}\left(\frac{b}{a}\right)$
$\tan^{-1}\left(\frac{b}{a}\right) - \pi$	$-\tan^{-1}\left(\frac{b}{a}\right)$

- Modulus-Argument form for $z = a + bi$ is $z = r(\cos\theta + i\sin\theta)$
- $|z_1 z_2| = |z_1||z_2|$
- $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

e.g. Write $3+4i$ in mod-arg form

$r = \sqrt{3^2 + 4^2}$

$= 5$

$\theta = \tan^{-1}\left(\frac{4}{3}\right)$

$= 0.93$

$\therefore z = 5(\cos 0.93 + i\sin 0.93)$

Numerical Solutions of Equations

Interval Bisection

- Requires an interval to do
- Uses the midpoint of interval to half it
- Can be repeated quickly for better accuracy

e.g. A root of $x^3 - 7x + 2 = 0$ lies in interval $[2,3]$. Use interval bisection twice to find a more accurate interval.

$f(2) = -4$
$f(3) = 8$

bisector is 2.5
$f(2.5) = 0.125$
new interval is $[2, 2.5]$

bisector is 2.25
$f(2.25) = -2.359...$
new interval is $[2.25, 2.5]$

Linear Interpolation

- Requires an interval to do
- Uses similar triangles to find an approximation of root
- For interval $[a, b]$ equation is $x_1 = \frac{a|f(b)| + b|f(a)|}{|f(a)| + |f(b)|}$

e.g. Use linear interpolation with interval $[2,3]$ to find an approximation of root of $x^3 - 7x + 2 = 0$

$f(2) = -4 \qquad f(3) = 8$

$x_1 = \frac{2 \times 8 + 3 \times 4}{8 + 4}$

$= 2.33$ (3sf)

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Newton - Raphson Method

- Uses a formula to find an approximation for a root
- $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
- Can sometimes generate a worse approximation
- Cannot be used if $f'(x_n) = 0$
- Does not require an interval only one approximation

e.g. Use the Newton-Raphson method to find an approximation of the root of $x^4 + x^2 = 80$, using $x = 3$ as a first approximation.

$f(x) = x^4 + x^2 - 80$

$f(3) = 10$

$f'(x) = 4x^3 + 2x$

$f'(3) = 114$

$x_1 = 3 - \frac{10}{114}$

$= 2.9122807\ldots$

$= 2.91$ (3sf)

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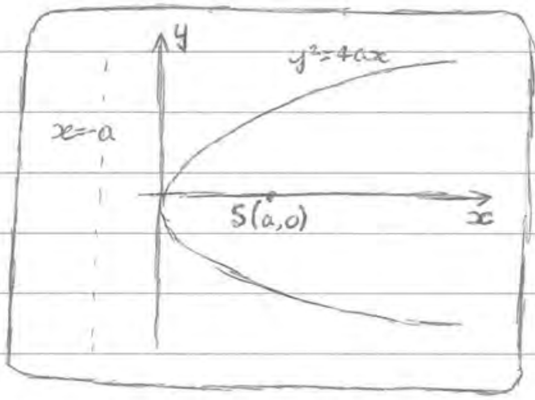
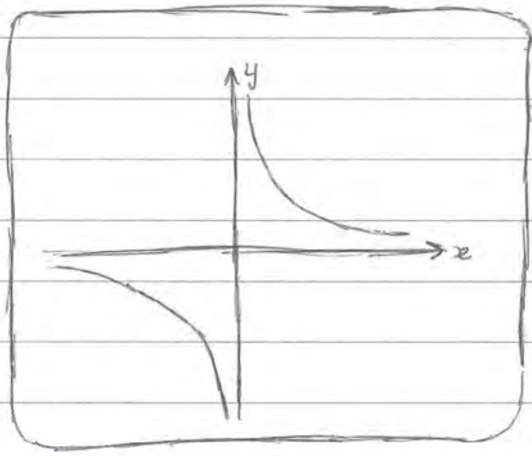
Coordinate Systems

Parabola



- A parabola is a locus of points where each point is the same distance from a fixed point (focus) and a fixed straight line (directrix)
- General equation is $y^2 = 4ax$
- General point is $P(at^2, 2at)$
- Focus is at $(a, 0)$
- Directrix has equation $x = -a$
- Vertex at $(0, 0)$
- a is always a positive constant

Rectangular Hyperbola



- General equation is $xy = c^2$
- General point is $P\left(ct, \frac{c}{t}\right)$
- Two asymptotes are $x=0$ (y axis) and $y=0$ (x axis)
- c is always a positive constant

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Matrices

$$\begin{pmatrix} 3 & 4 & 0 & 1 \\ 2 & -3 & 2 & 1 \\ -1 & 8 & 6 & 1 \end{pmatrix}$$

← a row (~~column~~)

↑ a column

this is a 3×4 matrix

$$\begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ 8 & 1 \end{pmatrix} = \begin{pmatrix} 2+4 & 1+0 \\ 3+8 & 0+1 \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 11 & 1 \end{pmatrix}$$

- A matrix is an array of numbers
- An $n \times m$ matrix has n rows and m columns
- When adding or subtracting you just add or subtract the corresponding elements
- Matrices must have the same dimensions for addition or subtraction

Multiplication of Matrices

- Multiply the elements on a row of the left matrix with the corresponding element in a column on a right matrix and sum the answers for that row and column
- Where the row and column cross is where the new element goes
- Columns on left matrix must equal rows on right matrix
- Dimensions: $(n \times m) \times (m \times k) = (n \times k)$

e.g. $$\begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 7 & 4 \end{pmatrix} = \begin{pmatrix} 6\times3 + 7\times-1 & 6\times1 - 1\times4 \\ 2\times3 + 3\times7 & 2\times1 + 3\times4 \end{pmatrix} = \begin{pmatrix} 11 & 2 \\ 27 & 14 \end{pmatrix}$$

Inverse of a 2×2 Matrix

- The determinant of a matrix, $\det M$, is equal to $ad - bc$
- For matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the inverse $M^{-1} = \frac{1}{\det M}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
- If $\det M = 0$ then M is singular and there is no inverse

e.g. $M = \begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix}$

$\det M = (3\times2) - (-4\times-1)$

$= 2$

$M^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$

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Transformations

- (x,y) coordinates can be written in a 2×1 matrix $\begin{pmatrix} x \\ y \end{pmatrix}$
- An $n\times2$ can then be used to represent a transformation
- Multiply transformation by coordinates to get new coordinates
- Transformation is written before coordinates
- For a shape or object that has been transformed:

  Area of Image = Area of Object $\times$ $|\det M|$

e.g. $(2,1)$ is transformed by the matrix $\begin{pmatrix} 4 & -2 \\ -1 & 2 \end{pmatrix}$, find the new coordinates

$$\begin{pmatrix} 4 & -2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$$

$\therefore$ new coordinates are $(6,0)$

Matrices Transformations

The important transformations in FP1 are:

- Rotation about $(0,0)$ of angles that are multiples of $45°$
- Enlargement centre $(0,0)$ of scale factor k
- Reflection in axes or lines $y = \pm x$
- Identity $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and performs no transformation

Enlargement

- Always state the scale factor, k and that the centre of enlargement is $(0,0)$
- Enlargements are always in the form $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ where k is the scale factor.

  e.g. $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ an enlargement of scale factor 3 about the centre $(0,0)$

Reflection

- Always state the mirror line (e.g. $y = x$)
- Reflections have a negative determinant
- The four main reflections are:

$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ y-axis, $x = 0$

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ x-axis, $y = 0$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $y = x$

$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ $y = -x$

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Rotation

- Always state the angle, direction and centre (0,0)
- The general formula for anticlockwise is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

θ	$\cos\theta$
0	1
45	$\frac{1}{\sqrt{2}}$
90	0
135	$-\frac{1}{\sqrt{2}}$
180	-1
225	$-\frac{1}{\sqrt{2}}$
270	0
315	$\frac{1}{\sqrt{2}}$

θ	$\sin\theta$
0	0
45	$\frac{1}{\sqrt{2}}$
90	1
135	$\frac{1}{\sqrt{2}}$
180	0
225	$-\frac{1}{\sqrt{2}}$
270	-1
315	$-\frac{1}{\sqrt{2}}$

General Notes

- Given any matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

  $M\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$

  $M\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$

  The first column is the image of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

  The second column is the image of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

- If a shape is transformed by a matrix it's new area will equal old area × matrix determinant

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Series

$$\sum_{r=1}^{n} r = \frac{n}{2}(n+1)$$

$$\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^3 = \frac{n^2}{4}(n+1)^2$$

Combining Series

- The sums of series above can be combined to solve more complicated series
- Always factorise out terms like n and $(n+1)$

e.g. $$\sum_{r=1}^{n}(r^3+2r^2+7r-18) = \sum_{r=1}^{n} r^3 + 2\sum_{r=1}^{n} r^2 + 7\sum_{r=1}^{n} r - 18\sum_{r=1}^{n} 1$$

$$= \frac{n^2}{4}(n+1)^2 + \frac{n}{3}(n+1)(2n+1) + \frac{7n}{2}(n+1) - 18n$$

$$= \frac{n}{12}\left[3n(n+1)^2 + 4(2n+1)(n+1) + 42(n+1) - 18\right]$$

$$= \frac{n}{12}\left[3n^3 + 6n^2 + 3n + 12n^2 + 18n + 6 + 42n + 42 - 18\right]$$

$$= \frac{n}{2}\left[3n^3 + 18n^2 + 63n + 30\right]$$

$$= \frac{3n}{2}\left[n^3 + 6n^2 + 21n + 10\right]$$

Finding Values For Series

- For $\sum_{r=a}^{n} f(r)$ rewrite it as $\sum_{r=1}^{n} f(r) - \sum_{r=1}^{a-1} f(r)$
- If the number on the bottom is (a) then one of the series will be $(a-1)$

e.g. $$\sum_{r=5}^{10}(r^3+2r^2+7r-18) = \sum_{r=1}^{10}(r^3+2r^2+7r-18) - \sum_{r=1}^{4}(r^3+2r^2+7r-18)$$

$$= 15(10^3 + 6(10)^2 + 21(10) + 10) - 6(4^3 + 6(4)^2 + 21(4) + 10)$$

$$= 27300 - 1524$$

$$= 25776$$

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Proof by Induction

- Method to prove a formula for $n \in \mathbb{Z}^+$
- There are four steps to it:
  1. Prove for $n=1$
  2. Assume true for $n=k$
  3. Prove true for $n=k+1$
  4. Summarise
- "If it is true for $n=k$ then it is true for $n=k+1$. Since it is true for $n=1$ by induction it must be true for all $n \in \mathbb{Z}^+$"

e.g. Prove for $n \in \mathbb{Z}^+$ that $\sum_{r=1}^{n}(2r-1) = n^2$

for $n=1$

$$LHS = \sum_{r=1}^{1}(2r-1) = 1 \qquad RHS = (1)^2 = 1$$

$LHS = RHS \quad \therefore$ true for $n=1$

assume true for $n=k \quad \therefore \sum_{r=1}^{k}(2r-1) = k^2$

then for $n=k+1$

$$\sum_{r=1}^{k+1}(2r-1) = \sum_{r=1}^{k}(2r-1) + [2(k+1)-1]$$

$$= k^2 + 2k + 1$$

$$= (k+1)^2$$

$\therefore$ true for $n=k+1$

If true for $n=k$ then true for $n=k+1$. Since true for $n=1$ by induction it is true for all $n \in \mathbb{Z}^+$

e.g. Prove $9^n - 1$ is divisible by 8 for $n \in \mathbb{Z}^+$

for $n=1$

$9^1 - 1 = 8$, divisible by 8 $\therefore$ true for $n=1$

assume true for $n=k$, $9^k - 1$ divisible by 8

then for $n=k+1$

$$\begin{aligned} f(k+1) - f(k) &= 9^{k+1} - 1 - 9^k + 1 \\ &= 9^{k+1} - 9^k \\ &= 9(9^k) - 9^k \\ &= 8(9^k) \quad \therefore \text{divisible by } 8 \end{aligned}$$

$$f(k+1) = f(k) + 8(9^k)$$

$f(k)$ and $8(9^k)$ divisible by 8 $\therefore$ true for $n=k+1$

If true for $n=k$ then true for $n=k+1$. Since true for $n=1$ by induction it is true for all $n \in \mathbb{Z}^+$

e.g. Prove $\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^n = \begin{pmatrix} 1 & 1-2^n \\ 0 & 2^n \end{pmatrix}$ for $n \in \mathbb{Z}^+$

for $n=1$

$$LHS = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^1 = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \qquad RHS = \begin{pmatrix} 1 & 1-2^1 \\ 0 & 2^1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$$

$LHS = RHS$ $\therefore$ true for $n=1$

assume true for $n=k$ $\therefore \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^k = \begin{pmatrix} 1 & 1-2^k \\ 0 & 2^k \end{pmatrix}$

then for $n=k+1$

$$\begin{aligned} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^{k+1} &= \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^k \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1-2^k \\ 0 & 2^k \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1+2(1-2^k) \\ 0 & 2(2^k) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1-2^{k+1} \\ 0 & 2^{k+1} \end{pmatrix} \end{aligned}$$

$\therefore$ true for $n=k+1$

If true for $n=k$ then true for $n=k+1$, Since true for $n=1$ by induction it is true for $n \in \mathbb{Z}^+$