FP2 : Inequalities

- To solve an inequality equation:
  - Find the critical values
  - Sketch a graph of the equations
  - Write the inequality answers using $>, <, \geq$ and $\leq$
- If there are fractions containing x in the denominator, both sides must be multiplied by that denominator squared

e.g. Find the set of values of x for which $\frac{x+1}{2x-3} < \frac{1}{x-3}$

$$\frac{x+1}{2x-3} < \frac{1}{x-3}$$

$$(x+1)(2x-3)(x-3)^2 < (x-3)(2x-3)^2$$

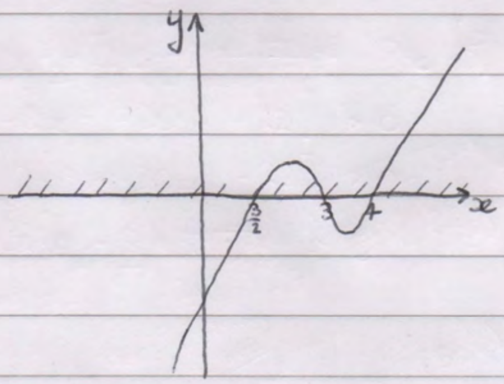
$$(x+1)(2x-3)(x-3)^2 - (x-3)(2x-3)^2 < 0$$

$$(x-3)(2x-3)[(x+1)(x-3) - (2x-3)] < 0$$

$$(x-3)(2x-3)[x^2 - 4x] < 0$$

$$4(x-3)(2x-3)(x-4) < 0$$

$\therefore$ critical values, $x = 3, \frac{3}{2}, 4$



$\therefore x < \frac{3}{2}$

$\underline{3 < x < 4}$

e.g. Solve the inequality $|x^2 - 7| < 3x + 3$

$x^2 - 7 = 3x + 3$

$x^2 - 3x - 10 = 0$

$(x-5)(x+2) = 0$

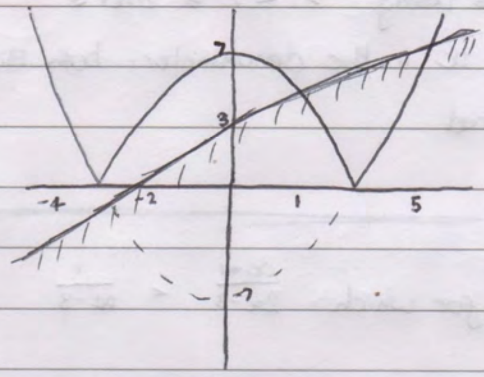
$\therefore x = -2, 5$

$7 - x^2 = 3x + 3$

$x^2 + 3x - 4 = 0$

$(x+4)(x-1) = 0$

$\therefore x = -4, 1$



$\therefore 1 < x < 5$

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FP2: Method of Differences

- The method of differences allows some sums of series to be expressed simply
- It relies on cancelling out terms that appear several times
- You must find a pattern of repeating terms which can be cancelled, and then add the remaining terms together

e.g. Find $\sum_{r=1}^{n}\left[\frac{1}{r}-\frac{1}{r+1}\right]$ using the method of differences

when $r=1$ $\quad \frac{1}{1}-\cancel{\frac{1}{2}}$

$r=2$ $\quad \cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}$

$r=3$ $\quad \cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}$

...

$r=n-1$ $\quad \cancel{\frac{1}{n-1}}-\cancel{\frac{1}{n}}$

$r=n$ $\quad \cancel{\frac{1}{n}}-\frac{1}{n+1}$

$$\therefore \sum_{r=1}^{n}\left[\frac{1}{r}-\frac{1}{r+1}\right] = 1-\frac{1}{n+1}$$
$$= \frac{n+1}{n+1}-\frac{1}{n+1}$$
$$= \underline{\underline{\frac{n}{n+1}}}$$

e.g. Given $\sum_{r=1}^{n}\frac{2}{(r+1)(r+3)} = \frac{n(5n+13)}{6(n+2)(n+3)}$ find $\sum_{r=21}^{30}\frac{2}{(r+1)(r+3)}$ to 5 dp

$$\sum_{r=21}^{30} = \sum_{r=1}^{30} - \sum_{r=1}^{20}$$
$$= \frac{30(150+13)}{6(32)(33)} - \frac{20(100+13)}{6(22)(23)}$$
$$= \frac{815}{1056} - \frac{565}{759}$$
$$= \frac{665}{24288}$$
$$= 0.02738 \text{ (5dp)}$$

e.g. Express $\frac{1}{(r+2)(r+3)}$ in partial fractions and hence find $\sum_{r=1}^{n} \frac{1}{(r+2)(r+3)}$

$$\frac{1}{(r+2)(r+3)} = \frac{A}{r+2} + \frac{B}{r+3}$$

$$1 = A(r+3) + B(r+2)$$

let $r=-3$, $1 = -B$

let $r=-2$, $1 = A$

$$\therefore \frac{1}{(r+2)(r+3)} = \frac{1}{r+2} - \frac{1}{r+3}$$

when $r=1$ $\quad \frac{1}{3} - \cancel{\frac{1}{4}}$

$r=2$ $\quad \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}$

$r=3$ $\quad \cancel{\frac{1}{5}} - \cancel{\frac{1}{6}}$

...

$r=n-1$ $\quad \cancel{\frac{1}{n+1}} - \cancel{\frac{1}{n+2}}$

$r=n$ $\quad \cancel{\frac{1}{n+2}} - \frac{1}{n+3}$

$$\therefore \sum_{r=1}^{n} \frac{1}{(r+2)(r+3)} = \frac{1}{3} - \frac{1}{n+3}$$

$$= \frac{n+3-3}{3(n+3)}$$

$$= \frac{n}{3(n+3)}$$

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Complex Numbers

Forms of Complex Numbers

- Complex numbers are typically written as $z = x + iy$
- Modulus-Argument form is $z = r(\cos\theta + i\sin\theta)$
- r is modulus $(r^2 = x^2 + y^2)$
- θ is argument $(\theta = \tan^{-1}(\frac{y}{x}))$
- Exponential form is $z = re^{i\theta}$

Multiplication and Division

- $|z_1 z_2| = |z_1||z_2|$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
- $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$
- $z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$
  $= r_1 r_2 e^{i(\theta_1 + \theta_2)}$
- $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$
  $= \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

e.g. Express $3(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}) \times 4(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12})$ in $x + iy$ form

$$3(\cos\tfrac{5\pi}{12} + i\sin\tfrac{5\pi}{12}) \times 4(\cos\tfrac{\pi}{12} + i\sin\tfrac{\pi}{12}) = (3)(4)\left[\cos\left(\tfrac{5\pi}{12} + \tfrac{\pi}{12}\right) + i\sin\left(\tfrac{5\pi}{12} + \tfrac{\pi}{12}\right)\right]$$
$$= 12\left(\cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2}\right)$$
$$= 12(0 + i)$$
$$= 12i$$

e.g. Simplify $\dfrac{4e^{i\frac{\pi}{3}}}{2e^{i\frac{\pi}{6}}}$

$$\frac{4e^{i\frac{\pi}{3}}}{2e^{i\frac{\pi}{6}}} = \frac{4}{2} e^{i\left(\frac{\pi}{3} - \frac{\pi}{6}\right)}$$
$$= 2e^{i\frac{\pi}{6}}$$

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De Moivre's Theorem

- States that $z^n = [r(\cos\theta + i\sin\theta)]^n$
  $= r^n(\cos n\theta + i\sin n\theta)$

  for any integer n

e.g. Simplify $\dfrac{(\cos 2\theta + i\sin 2\theta)^7}{(\cos 4\theta + i\sin 4\theta)^3}$

$$\frac{(\cos 2\theta + i\sin 2\theta)^7}{(\cos 4\theta + i\sin 4\theta)^3} = \frac{\cos 14\theta + i\sin 14\theta}{\cos 12\theta + i\sin 12\theta}$$

$$= \cos 2\theta + i\sin 2\theta$$

- Can be used with binomial expansion to find $\cos n\theta$ and $\sin n\theta$

e.g. Express $\cos 3\theta$ in terms of $\cos\theta$

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

$$= \cos^3\theta + i3\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$

$$\cos 3\theta = \mathrm{Re}[(\cos\theta + i\sin\theta)^3]$$

$$= \cos^3\theta - 3\cos\theta\sin^2\theta$$

$$= \cos^3\theta - 3\cos\theta(1 - \cos^2\theta)$$

$$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$$

$$= 4\cos^3\theta - 3\cos\theta$$

- Also $z^n + \frac{1}{z^n} = 2\cos n\theta$ and $z^n - \frac{1}{z^n} = 2i\sin n\theta$ are used to find $\cos^n\theta$ and $\sin^n\theta$

e.g. Express $\sin^4\theta$ in the form $a\cos 4\theta + b\cos 2\theta + c$

$$\left(z - \frac{1}{z}\right)^4 = (2i\sin\theta)^4 = 16\sin^4\theta$$

$$= z^4 - 4z^3\frac{1}{z} + 6z^2\frac{1}{z^2} - 4z\frac{1}{z^3} + \frac{1}{z^4}$$

$$= z^4 - 4z^2 + 6 - 4\frac{1}{z^2} + \frac{1}{z^4}$$

$$= \left(z^4 + \frac{1}{z^4}\right) - 4\left(z^2 + \frac{1}{z^2}\right) + 6$$

$$16\sin^4\theta = 2\cos 4\theta - 8\cos 2\theta + 6$$

$$\sin^4\theta = \frac{1}{8}\cos 4\theta - \frac{1}{2}\cos 2\theta + \frac{3}{8}$$

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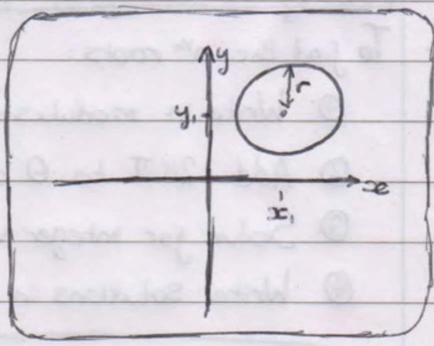
Complex Numbers and Loci

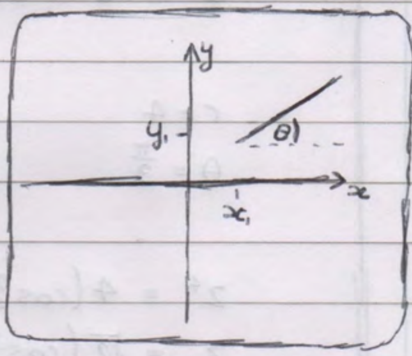
$|z - z_1| = r$ (circles)

- $|z - z_1| = r$ represents a circle
- r is radius
- $z_1 = x_1 + iy_1$
  - (x_1, y_1) is centre
- $\leq$ for inside circle
- $\geq$ for outside circle



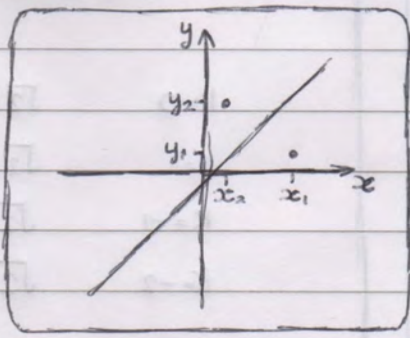
$\arg(z - z_1) = \theta$ (half-line)

- θ is angle made with real(x)-axis
- $z_1 = x_1 + iy_1$
  - (x_1, y_1) is start point
- $\leq$ for closer to real(x)-axis
- $\geq$ for further from real(x)-axis



$|z - z_1| = |z - z_2|$ (Perpendicular bisector)

- $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$
  - line equidistant from (x_1, y_1) and (x_2, y_2)
- $\geq$ for closer to z_2
- $\leq$ for closer to z_1



W-Plane Transformations

- $w = z + a + ib$ represents translation with vector $\binom{a}{b}$
- $w = kz$ represents enlargement of scale factor k about $(0,0)$
- $w = kz + a + ib$ represents enlargement followed by translation

Nth Roots

Finding Nth Roots of a Complex Number

- To find the n^{th} roots:
  1. Write in modulus-argument form
  2. Add $2k\pi$ to θ and divide by n
  3. Solve for integer values of k where $-\pi < \theta \leq \pi$
  4. Write solutions in modulus-argument or exponential form

e.g. Solve the equation $z^4 - 2 - 2\sqrt{3}\,i = 0$

$$z^4 - 2 - 2\sqrt{3}\,i = 0$$

$$z^4 = 2 + 2\sqrt{3}\,i$$

$$r = 4$$

$$\theta = \frac{\pi}{3}$$

$$z^4 = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$z = \sqrt{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{\frac{1}{4}}$$

$$= \sqrt{2}\left(\cos\left(\frac{\pi}{3} + 2k\pi\right) + i\sin\left(\frac{\pi}{3} + 2k\pi\right)\right)^{\frac{1}{4}}$$

$$= \sqrt{2}\left[\cos\left(\frac{\pi}{12} + \frac{k\pi}{2}\right) + i\sin\left(\frac{\pi}{12} + \frac{k\pi}{2}\right)\right]$$

$k=0$ $\quad \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

$k=1$ $\quad \sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$

$k=-1$ $\quad \sqrt{2}\left(\cos\frac{-5\pi}{12} + i\sin\frac{-5\pi}{12}\right)$

$k=-2$ $\quad \sqrt{2}\left(\cos\frac{-11\pi}{12} + i\sin\frac{-11\pi}{12}\right)$

$$\therefore z = \sqrt{2}\,e^{i\frac{\pi}{12}},\ \sqrt{2}\,e^{i\frac{7\pi}{12}},\ \sqrt{2}\,e^{-i\frac{5\pi}{12}},\ \sqrt{2}\,e^{-i\frac{11\pi}{12}}$$

First Order Differential Equations

Type 1

- When $\frac{dy}{dx} = f(x)\,g(y)$ it can be written as $\int \frac{1}{g(y)}\,dy = \int f(x)\,dx + c$

e.g. Find the general solution of $\frac{dy}{dx} = 4\frac{x^3}{y^2}$

$$\int y^2\,dy = 4\int x^3\,dx + c$$
$$\frac{y^3}{3} = x^4 + c$$
$$y^3 = 3(x^4 + c)$$
$$y = \sqrt[3]{3(x^4 + c)}$$

Type 2

- $f(x)\frac{dy}{dx} + f'(x)\,y = \frac{d}{dx}\left(f(x)y\right)$

e.g. Find the general solution of $x\frac{dy}{dx} + y = \cos x$

$$x\frac{dy}{dx} + y = \cos x$$
$$\frac{d}{dx}(x \cdot y) = \cos x$$
$$x \cdot y = \sin x + c$$
$$y = \frac{\sin x + c}{x}$$

Type 3

- For $\frac{dy}{dx} + Py = Q$ where P and Q are functions of x the integrating factor is $e^{\int P\,dx}$
- Multiplying by the integrating factor allows it to be solved

e.g. Find the general solution of $\frac{dy}{dx} + 2y = e^x$

$$\text{I.F. is } e^{\int 2\,dx} = e^{2x}$$
$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = e^x$$
$$\frac{d}{dx}(e^{2x} \cdot y) = e^x$$
$$e^{2x} \cdot y = e^x + c$$
$$y = \frac{e^x + c}{e^{2x}}$$

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Type 4

- Substitutions can be used to simplify equations for solving
- The substitution will always be given in an exam
- Where substituting with z, write equation in terms of z and x. Solve and then rewrite with y

e.g. Find the general solution of $\frac{dy}{dx} + xy = xy^2$ using the substitution $z = y^{-1}$

$$y = z^{-1}$$
$$\frac{dy}{dx} = -\frac{1}{z^2}\frac{dz}{dx}$$

$$-\frac{1}{z^2}\frac{dz}{dx} + \frac{1}{z}x = xz^{-2}$$
$$-\frac{dz}{dx} + zx = x$$
$$\frac{dz}{dx} - zx = -x$$

I.F. is $e^{\int -x\,dx} = e^{-\frac{x^2}{2}}$

$$e^{-\frac{x^2}{2}}\frac{dz}{dx} - zxe^{-\frac{x^2}{2}} = -e^{-\frac{x^2}{2}}x$$
$$\frac{d}{dx}\left(z \cdot e^{-\frac{x^2}{2}}\right) = -e^{-\frac{x^2}{2}}x$$
$$z \cdot e^{-\frac{x^2}{2}} = e^{-\frac{x^2}{2}} + c$$
$$z = 1 + ce^{\frac{x^2}{2}}$$

$$y = \frac{1}{1 + ce^{\frac{x^2}{2}}}$$

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Second Order Differential Equations

Auxiliary Equation

- Can be done when $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$
- Replace $\frac{d^2y}{dx^2}$ with m^2, $\frac{dy}{dx}$ with m and y with 1
- Solve to find roots α and β
- General Solution then depends on whether roots are real or complex

Roots	Form
2 real, distinct root (α and β)	$Ae^{\alpha x} + Be^{\beta x}$
2 real, equal roots (α)	$(A + Bx)e^{\alpha x}$
2 imaginary roots ($\pm i\omega$)	$A\cos\omega x + B\sin\omega x$
2 complex roots ($p \pm iq$)	$e^{px}(A\cos qx + B\sin qx)$

e.g. $\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 25y = 0$

$$m^2 + 10m + 25 = 0$$
$$(m+5)^2 = 0$$
$$\therefore m = -5$$
$$\therefore y = (A + Bx)^{-5x}$$

- A and B are arbitrary constants
- Thes forms a complementary function

Particular Integral

- When $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$ a substitution must be used to find a particular integral
- y = complementary function + particular integral
- A particular integral cannot be part of the complementary function (then use λx over λ)

Form of $f(x)$	Form of particular integral
k	λ
kx	$\lambda + \mu x$
kx^2	$\lambda + \mu x + \nu x^2$
ke^{px}	λe^{px}
$m\cos\omega x + n\sin\omega x$	$\lambda\cos\omega x + \mu\sin\omega x$

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e.g. Find the general solution of $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 3$

$m^2 - 2m = 0$

$m(m-2) = 0$

$\therefore m = 0, 2$

$\therefore$ C.F. : $y = A + Be^{2x}$

λ cannot be P.I. as part of C.F. so use λx

$y = \lambda x$

$\frac{dy}{dx} = \lambda$

$\frac{d^2y}{dx^2} = 0$

$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 3$

$0 - 2\lambda = 3$

$\lambda = -\frac{3}{2}$

$\therefore$ P.I. is $-\frac{3}{2}x$

General Solution: $y = A + Be^{2x} - \frac{3}{2}x$

- If values for x, y, $\frac{dy}{dx}$ or $\frac{d^2y}{dx^2}$ are given the constants can be found to give a specific solution (usually by differentiating the general solution and solving simultaneous equations)

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Maclaurin Series

- Can be used to express some functions of x as infinite series in ascending powers of x
- $f(0), f'(0), f''(0), \ldots, f^{(r)}(0)$ must all have finite values
- Maclaurin Series:

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \ldots + f^{(r)}(0)\frac{x^r}{r!} + \ldots$$

e.g. Find the Maclaurin expansion for $\sin x$ up to x^7

$f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$f''''(x) = \sin x \qquad f''''(0) = 0$

$$\therefore \sin x = x + \frac{(-1)x^3}{3!} + \frac{1x^5}{5!} + \frac{(-1)x^7}{7!} + \ldots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

e.g. Find the first three non-zero terms in the expansion of $\ln\frac{\sqrt{1+2x}}{1-3x}$

$$\ln\frac{\sqrt{1+2x}}{1-3x} = \ln\sqrt{1+2x} - \ln(1-3x)$$

$$= \tfrac{1}{2}\ln(1+2x) - \ln(1-3x)$$

$$\tfrac{1}{2}\ln(1+2x) = \tfrac{1}{2}\left[2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \ldots\right]$$

$$= x - x^2 + \tfrac{4}{3}x^3 - \ldots$$

$$\ln(1-3x) = (-3x) - \frac{(-3x)^2}{2} + \frac{(-3x)^3}{3} - \ldots$$

$$= -3x - \tfrac{9}{2}x^2 - 9x^3 \ldots$$

$$\therefore \ln\frac{\sqrt{1+2x}}{1-3x} = \left(x - x^2 + \tfrac{4}{3}x^3\right) - \left(-3x - \tfrac{9}{2}x^2 - 9x^3\right)$$

$$= 4x + \tfrac{7}{2}x^2 + \tfrac{31}{3}x^3 \ldots$$

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Taylor Series

- Improvement on Maclaurin series to find $f(x)$ close to $x=a$
- Taylor Series:

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + \dots + f^{(r)}(a)\frac{(x-a)^r}{r!} + \dots$$

$$f(x+a) = f(a) + f'(a)x + f''(a)\frac{x^2}{2!} + \dots + f^{(r)}(a)\frac{x^r}{r!} + \dots$$

e.g. Express $\tan(x + \frac{\pi}{4})$ as series in ascending powers of x up to x^3

$f(x) = \tan x$ $\quad f(\frac{\pi}{4}) = 1$

$f'(x) = \sec^2 x$ $\quad f'(\frac{\pi}{4}) = 2$

$f''(x) = 2\sec^2 x \tan x$ $\quad f''(\frac{\pi}{4}) = 4$

$f'''(x) = 2\sec^4 x + 2\sec^2 x \tan^2 x$ $\quad f'''(\frac{\pi}{4}) = 16$

$$\therefore \tan(x + \tfrac{\pi}{4}) = (1) + (2)x + \frac{(4)x^2}{2!} + \frac{(16)x^3}{3!} + \dots$$
$$= 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \dots$$

e.g. Use the Taylor Series to find a series solution up to x^3 of $\frac{d^2y}{dx^2} = y - \sin x$ where $x=0$, $y=1$, $\frac{dy}{dx} = 2$

$$\frac{d^2y}{dx^2} = 1 - \sin 0$$
$$= 1$$

$$\frac{d^3y}{dx^3} = \frac{dy}{dx} - \cos x$$
$$= 2 - \cos 0$$
$$= 1$$

$$\therefore y = y_0 + x\left(\frac{dy}{dx}\right)_0 + \frac{x^2}{2!}\left(\frac{d^2y}{dx^2}\right)_0 + \frac{x^3}{3!}\left(\frac{d^3y}{dx^3}\right)_0 + \dots$$
$$= 1 + x(2) + \frac{(1)x^2}{2} + \frac{(1)x^3}{6} + \dots$$
$$= 1 + 2x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$$

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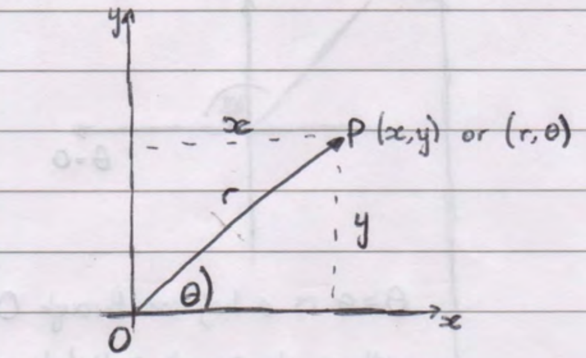
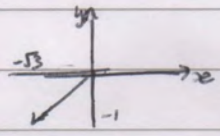
FP2: Polar Coordinates



$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$r^2 = x^2 + y^2$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

- The origin, O, is also known as the pole
- The positive x-axis is known as the initial line
- θ is usually written in radians unless asked otherwise

e.g. Express $(-\sqrt{3}, -1)$ in polar coordinates



$r = \sqrt{(-\sqrt{3})^2 + (-1)^2} = 2$

$\theta = \tan^{-1}\left(\frac{-1}{-\sqrt{3}}\right) = -\frac{\pi}{6}, \frac{7\pi}{6}, \ldots$

$\therefore$ the point has polar coordinates $\left(2, \frac{7\pi}{6}\right)$

e.g. Express $\left(8, \frac{2\pi}{3}\right)$ in cartesian form

$x = r\cos\theta$
$= 8\cos\frac{2\pi}{3}$
$= -4$

$y = r\sin\theta$
$= 8\sin\frac{2\pi}{3}$
$= 4\sqrt{3}$

$\therefore$ coordinates are $(-4, 4\sqrt{3})$

e.g. Express $r^2 = \sin 2\theta$ as a cartesian equation

$r^2 = 2\sin\theta\cos\theta$

$r^4 = 2r\sin\theta\, r\cos\theta$

$(x^2 + y^2)^2 = 2xy$

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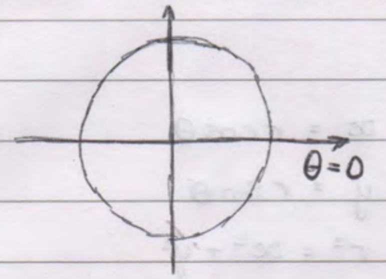
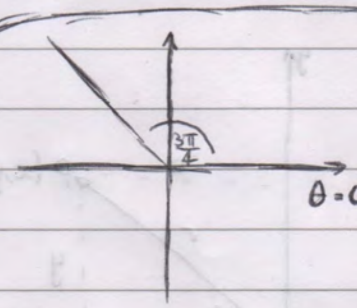
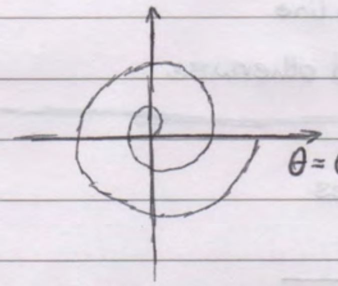
Sketching Curves



$r=a$ is a circle, centre O and radius a



$\theta=\alpha$ is a half-line through O with angle α with initial line



$r=a\theta$ is a spiral starting at O

Area of Sectors

- Area of a sector is given by $A=\frac{1}{2}\int_\alpha^\beta r^2\,d\theta$

e.g. Find the area enclosed by $r=a(1+\cos\theta)$ between 0 and π

$$\text{Area} = \frac{1}{2}\int_0^\pi [a(1+\cos\theta)]^2\,d\theta$$
$$= \frac{a^2}{2}\int_0^\pi (1+\cos\theta)^2\,d\theta$$
$$= \frac{a^2}{2}\int_0^\pi (1+2\cos\theta+\cos^2\theta)\,d\theta$$
$$= \frac{a^2}{2}\int_0^\pi \left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos2\theta\right)d\theta$$
$$= \frac{a^2}{2}\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin2\theta\right]_0^\pi$$
$$= \frac{a^2}{2}\left[\frac{3}{2}\pi\right]$$
$$= \frac{3a^2\pi}{4}$$

- To find a tangent parallel to initial line set $\frac{dy}{d\theta}=0$
- To find a tangent perpendicular to initial line set $\frac{dx}{d\theta}=0$