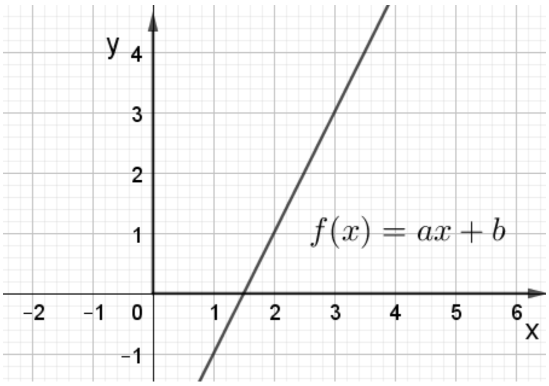
10. Transformation of functions

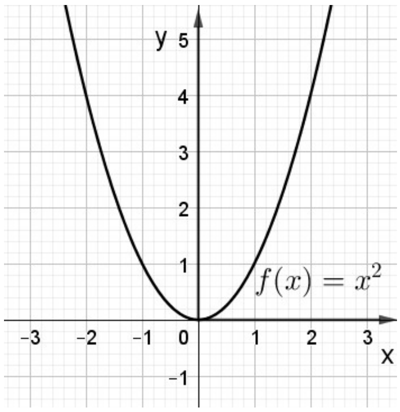
After studying the relation of the functional equation and the graph of several types of functions, we can now investigate the ways a function can be transformed. In particular we will focus on

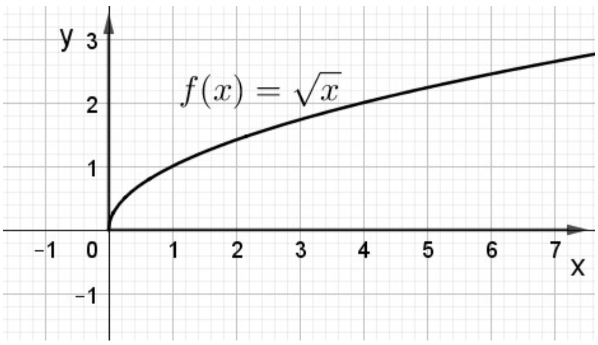
- vertical and horizontal translations
- reflections across the x- and the y-axis
- vertical and horizontal stretching

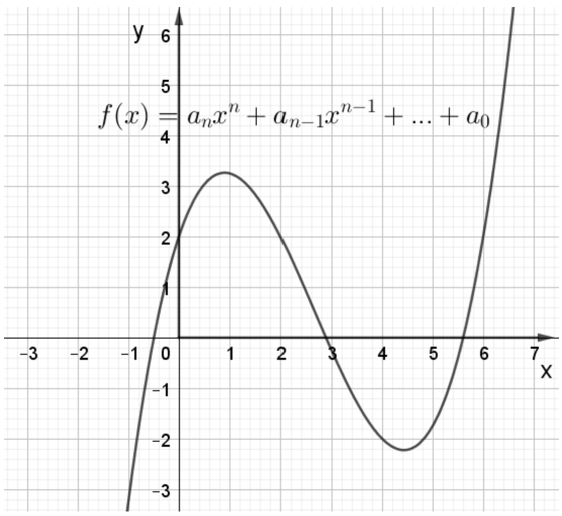
of any function $y = f(x)$.

As examples we will have a closer look at linear, quadratic, root and polynomial functions. The following diagrams exhibit these basic functions.









At this point you should be familiar with these basic types of functions. Knowing their properties and being able to draw these simple curves is essential to understanding the transformations we are going to apply on these functions.

10.1 Vertical and horizontal translations

We start with the simplest transformation, the translations. A vertical translation moves the functional graph along the y-axis and is illustrated in the left diagram below. The green curve corresponds to the quadratic function $f(x) = x^2$. The graph of this function is vertically shifted by adding or subtracting a number to the "parent" function f.

Vertical translation:

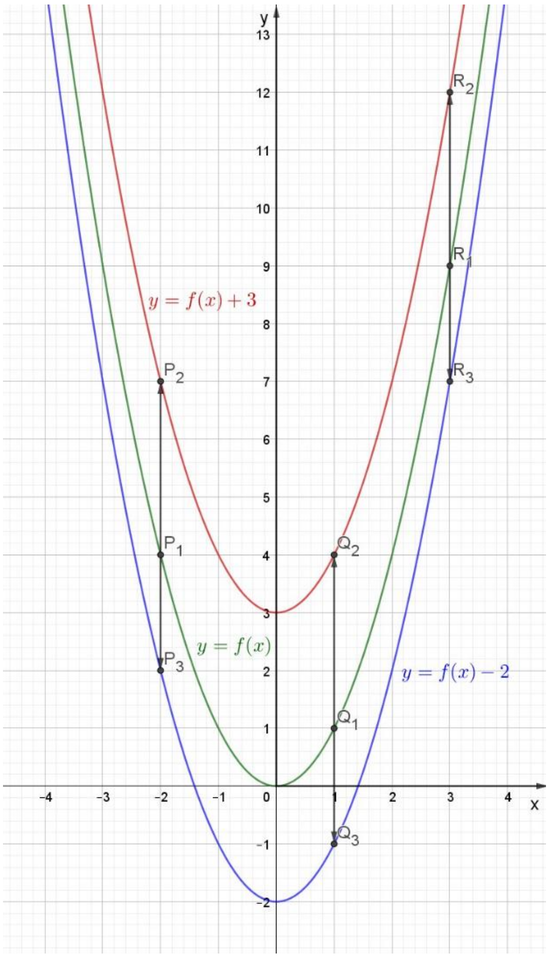
If $d > 0$, then

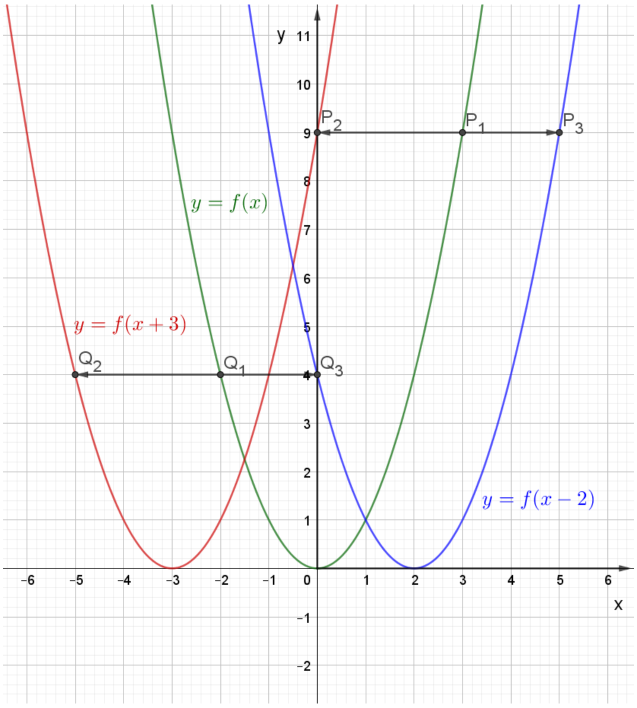
- The graph of $y = f(x) + d$ is obtained by shifting the graph of $y = f(x)$ *up* by d units.
- The graph of $y = f(x) - d$ is obtained by shifting the graph of $y = f(x)$ *down* by d units.

This vertical shift can easily be seen in the graph by comparing the y-coordinates of the points Q_1 and Q_2. The difference of the y-coordinates is exactly 3.

A bit trickier is the translation along the x-axis. This is illustrated in the right diagram.

As before, we have to add a number to the x-coordinate of the function f, but this time the direction of the shift opposes the sign of the coefficient. This can clearly be seen by comparing the x-coordinates of the points P_1 and P_2. The difference now is -3, although 3 has been added to the x-coordinate in the equation.





Horizontal translation:

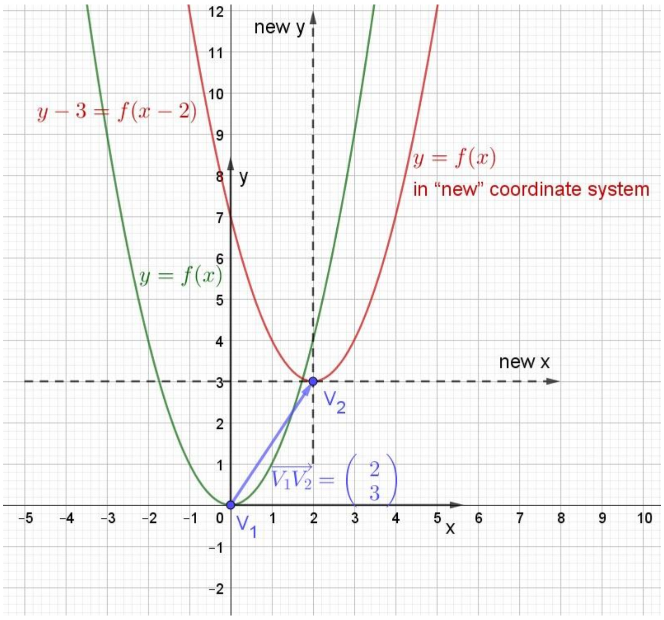
If $c > 0$, then

- The graph of $y = f(x + c)$ is obtained by shifting the graph of $y = f(x)$ by c units *to the left*.
- The graph of $y = f(x - c)$ is obtained by shifting the graph of $y = f(x)$ by c units *to the right*.

It has to be remarked that there seems to be a difference between the horizontal and the vertical translation as far as the sign of the coefficients c and d is concerned. This difference can easily be resolved, if we regard the shift along the y-axis in terms of the transformation of the functional equation in the y-coordinate:

$y = f(x) + d| - d$

$y - d = f(x)$



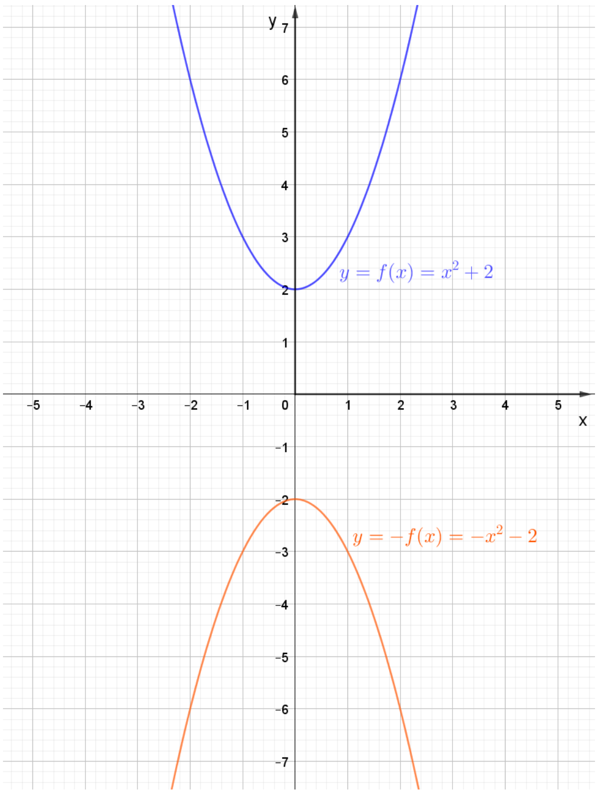
The shift of the curve in positive y-direction corresponds to a subtraction of the coefficient d from the y-coordinate.

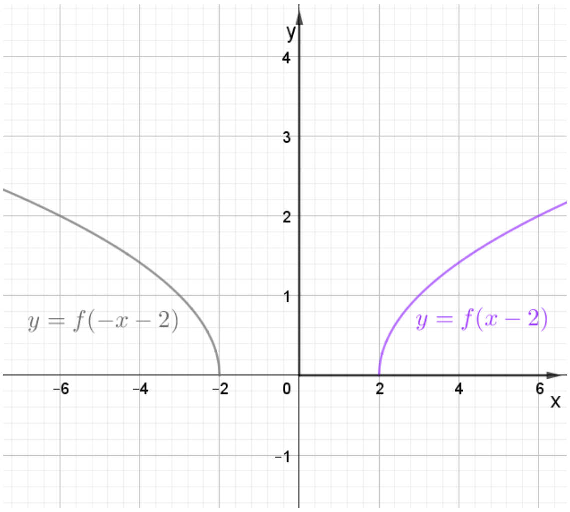
Furthermore, the horizontal and vertical translation can be regarded as a shift of the coordinate system instead of the functional graph, as is illustrated in the following diagram.

Instead of moving the graph we can also think of moving the origin of the coordinate system from $O(0|0)$ or V_1 to $V_2(2|3)$, which would be $O(0|0)$ in this "new" coordinate system. Thus, the equation of the function $y = f(x-2)+3$ or $y-3 = f(x-2)$ in the "old" system, would have the form $y = f(x)$ in the "new" system.

10.2 <u>Reflections</u>

From the discussion of symmetry of power- and polynomial functions it should be known how a functional graph can be reflected across the coordinate axes. Let us look at two examples.





The left diagram shows a reflection in the x-axis. Here, the whole functional term has to be multiplied by -1. The right diagram illustrates a reflection in the y-axis. Contrary to before, only the x-variable has to be multiplied by -1. How can this difference be understood? Does Maths in the end make no sense at all?

The answer has already been given in a short remark on the vertical translation. Instead of multiplying the term $f(x)$ by -1, the y-variable has to be multiplied by -1. This means that

$y = f(x) + 2$

To reflect this function across the y-axis, the y-coordinate is multiplied by -1:

$-y = f(x) + 2 | \cdot (-1)$

$y = -f(x) - 2$

So, the difference between handling the reflection in the two axes is resolved and Maths does make sense again…

Reflection in the coordinate axes:

- The graph of $y = -f(x)$ is obtained by reflecting the graph of the function $y = f(x)$ in the x-axis.
- The graph of $y = f(-x)$ is obtained by reflecting the graph of the function $y = f(x)$ in the y-axis.

10.3 Stretching and shrinking

All the transformations we applied to the graph of a function so far did only change the position of the curve. Now we are going to investigate how to change the shape of the graph of a function.

Firstly, we will investigate the vertical stretch. As with the vertical translation, we will have to change the y-coordinate, as the vertical axis, of the functional equation. The left diagram illustrates a vertical stretch of a polynomial function of 3rd degree. The factor of $a = 2$ stretches the y-coordinates of the graph by the factor 2, as can be seen by the coordinates of the points P_1 and Q_1. Please note also that the zeros of the function stay the same, as their y-coordinate is zero!

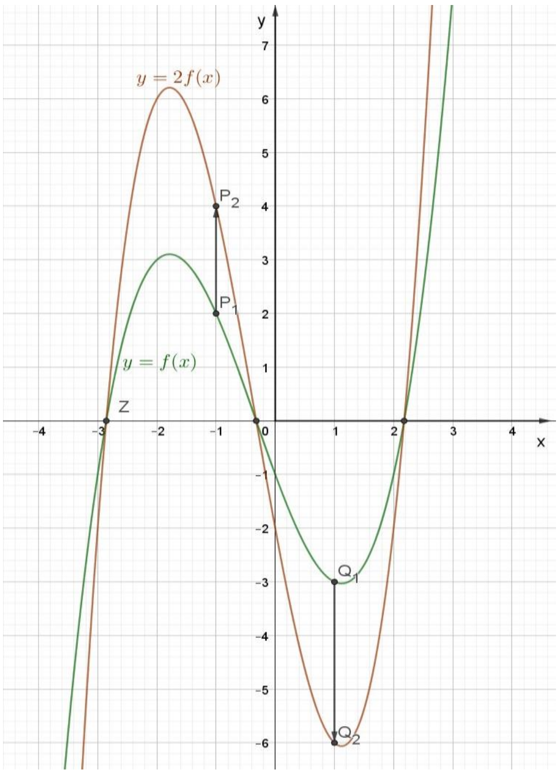
Vertical stretch:

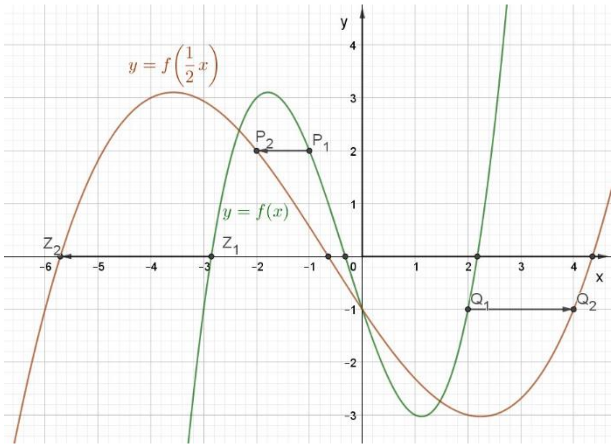
The graph of $y = af(x)$ is obtained by vertically *stretching* the graph of $y = f(x)$ by the factor a.

And of course, we see that stretching the graph of f corresponds either to a multiplication of $f(x)$ by a of a multiplication of y by $\frac{1}{a}$:

$y = a \cdot f(x) | : a$

$\frac{1}{a} \cdot y = f(x)$





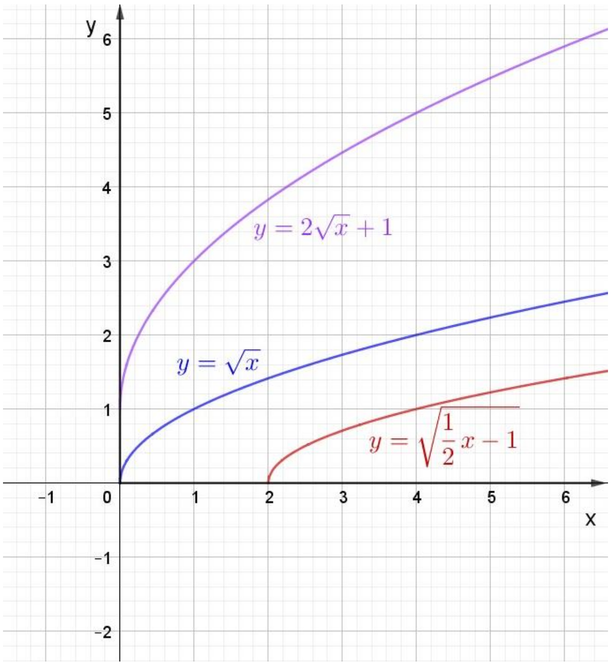
In the right diagram the horizontal stretch is shown. To stretch the graph by the factor 2, we have to use the coefficient $b = \frac{1}{2}$. This time the x-coordinates are multiplied by 2 and therefore the zeros change.

Horizontal stretch:

The graph of $y = f(bx)$ is obtained by horizontally *stretching* the graph of $y = f(x)$ by the factor $\frac{1}{b}$.

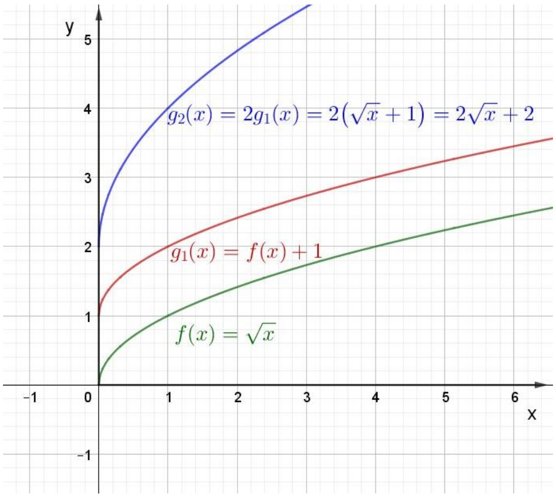
10.4 Combining transformations

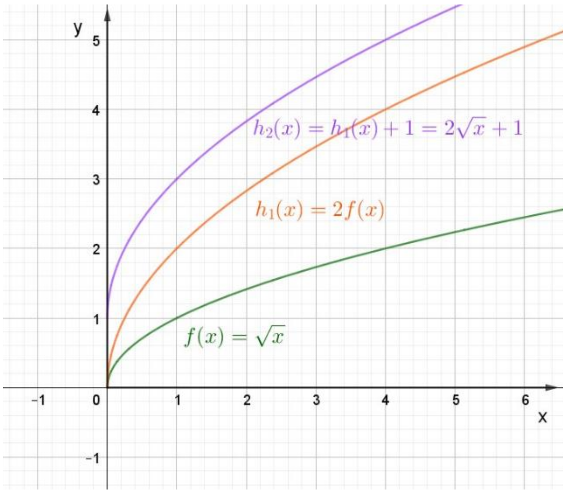
So far, we were just investigating single transformations. But what happens, if transformations are combined? The diagram on the right illustrates the problem!



What is special in this diagram? Let's have a closer look at the purple curve first. Obviously, the root function $y = \sqrt{x}$ (blue curve) has been stretched by two units along the y-axis and translated along the y-axis by one unit. This can be seen by the functional equation $y = 2\sqrt{x} + 1$. A similar transformation should be expected for the red curve. Its equation is $y = \sqrt{\frac{1}{2}x - 1}$. The coefficient $b = \frac{1}{2}$ stretches the curve by two units along the x-axis and the coefficient $c = -1$ shifts the curve by one unit along the x-axis. However, the curve has been shifted by two units!

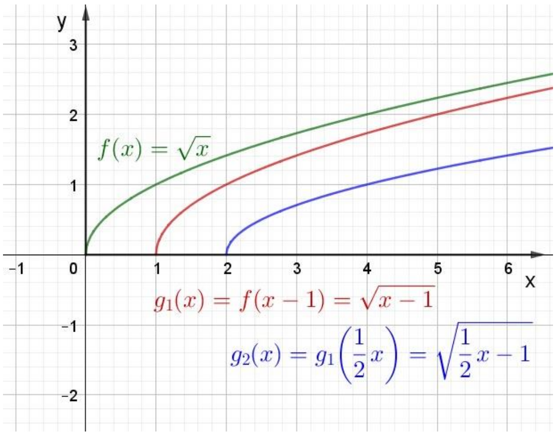
Before you resign and think that this is another evidence that Math doesn't make sense, we investigate this contradiction in detail. The following analysis will be done in two different ways, but with the same result. The first approach is the order of transformations, or more explicitly asked, does it make a difference whether a function is stretched first and then shifted or shifted first and then stretched? Have a look at the next two diagrams.

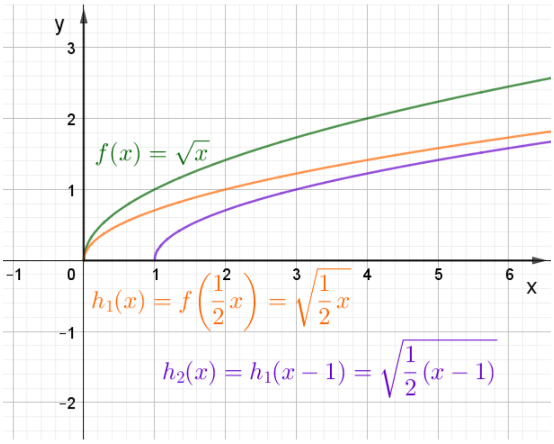




In the left diagram the root function $y = \sqrt{x}$ is shifted by one unit first and then stretched by the factor 2. As shown in the diagram the resulting curve has the equation $y = 2\sqrt{x} + 2$, because the coefficient $d = 1$ is multiplied by the stretching factor as well. In the right diagram the root function is stretched first and then shifted, which yields the expected result. Therefore, the order of operations does make a difference!

But to mess things up totally, we compare this result to stretching and shifting along the x-axis! The next two diagrams show the corresponding procedure.





As far as the order of operations is concerned, the function has to be stretched first and then shifted, as illustrated in the left diagram. But the functional equation, $y = \sqrt{\frac{1}{2}(x-1)} = \sqrt{\frac{1}{2}x - \frac{1}{2}}$, seems to be wrong. So, in order to stretch a function and shift it afterwards, the stretching factor also changes the shifting value. If a function has to be stretched by $\frac{1}{b}$ units along the x-axis and shifted by $-c$ units along the x-axis as well, the term to be used is $b\left(x + \frac{c}{b}\right)$.

In the second approach the different forms of the functional equations are explained. This has to be done by investigating the stretching and shifting algebraically once again. It has already been shown that the transformations along the y-axis actually have to be performed on the y-coordinate in the functional equation. Thus, we go backwards now:

$$y = 2\sqrt{x} + 1 \,|\, -1$$

$$y - 1 = 2\sqrt{x} \,|\, :2$$

$$\frac{1}{2}(y-1) = \sqrt{x}$$

So, to stretch the root function by the factor $a = 2$ and shift it by the coefficient $d = 1$, the y-coordinate has to modified to $\frac{1}{2}(y-1)$. If this transformation of the y-coordinate is compared to the correct transformation of the x-coordinate, $\frac{1}{2}(x-1)$, it is obvious that both transformations follow the same pattern.

10.5 Summary

	Transformation formula	Description
1.	$y = f(x) \pm d$	vertical translation by $\pm d$ units
	$y = f(x \pm c)$	horizontal translation by $\mp c$ units
2.	$y = -f(x)$	reflection across the x-axis
	$y = f(-x)$	reflection across the y-axis
3.	$y = af(x)$	vertical stretching by a units
	$y = f(bx)$	horizontal stretching by $\frac{1}{b}$ units
4.	$y = af(bx + c) + d$ $= af\left(b\left(x + \frac{c}{b}\right)\right) + d$	vertical stretching by a units, vertical shifting by d units, horizontal stretching by $\frac{1}{b}$ units and horizontal shifting by $-\frac{c}{b}$ units