Note

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Contents

Abstract

1 Basic properties

Let $\nu = g\,d\mu$. Then

$$\int f\,d\nu = \int fg\,d\mu$$

Let $\nu = \mu_{x,r}$. Then

$$\int f\,d\nu = \int f(\frac{y-x}{r})\,d\mu(y) = \int (s_r \circ \tau_x) f\,d\mu$$

$$\int f\,d\mu = \int f(ry+x)\,d\nu(y) = \int (\tau_{x/r} \circ s_{1/r}) f\,d\nu$$

where $s_r f(y) \coloneqq f(y/r)$ and $\tau_x f(y) \coloneqq f(y-x)$. One can define s_r, τ_x for measures, and write as

$$\int f\,d(\tau_x \circ s_r)\mu = \int (s_r \circ \tau_x) f\,d\mu$$

2 Chapter 2

2.1 Proof of Prop 2.2 (Lebesgue differentiation theorem)

Using the Besicovitch Differentiation of Measures (Theorem 2.10), we prove the Proposition 2.2, which states: If μ is locally finite Borel regular measure and $f \in L^1(\mu)$, then for μ-a.e. x, f is Lebesgue continuous at x with respect to μ.

To begin with, consider the following basic lemmas:

Lemma 2.1. *Let μ be any measure, f be a measurable function. If $\int_A f\,d\mu = 0$ for all measurable A then $f = 0$ for μ-a.e.*

To prove this, notice that $f = f\chi_{\{f>0\}} + f\chi_{\{f\le 0\}}$ Using the assumption, one can easily show that $f\chi_{\{f>0\}} = 0$ and $f\chi_{\{f\le 0\}} = 0$.

Now prove the Proposition 2.2:

Proof. Let $q \in \mathbb{Q}$ and $\nu_q := |f - q|\, d\mu$. One can easily check that ν_q is locally finite Borel regular measure. By the Besicovitch Differentiation of Measures, the limit

$$h(x) = \lim_{r\downarrow 0} \frac{\nu_q(B_r(x))}{\mu(B_r(x))}$$

exists for μ-a.e. $x \in \operatorname{supp}(\mu)$, and $\nu_q = h\, d\mu$ (one can easily check that $\nu_q \llcorner E$ vanishes, i.e. zero meausure). By the Lemma 1.1, $h = |f - q|$, which is

$$\lim_{r\downarrow 0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y) - q|\, d\mu(y) = |f(x) - q|$$

for μ-a.e. x.
Let $D_q := \{x \in \mathbb{R}^n : \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y) - q|\, d\mu(y) = |f(x) - q|\}$. By above, $\mu(\mathbb{R}^n \setminus D_q) = 0$ for any $q \in \mathbb{Q}$. Let $D := \bigcap_{q\in\mathbb{Q}} D_q$ which satisfies $\mu(\mathbb{R}^n \setminus D) = 0$ (where $\mathbb{Q}$ being countable is important here).
Now, for each $x \in D$,

$$\lim_{r\downarrow 0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y) - q|\, d\mu(y) = |f(x) - q|$$

which holds for any $q \in \mathbb{Q}$.
We see that

$$\begin{aligned}
&\limsup_{r\downarrow 0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y) - f(x)|\, d\mu(y) \\
\le &\limsup_{r\downarrow 0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y) - q|\, d\mu(y) + \limsup_{r\downarrow 0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |q - f(x)|\, d\mu(y) \\
&= 2|f(x) - q|
\end{aligned}$$

for any $q \in \mathbb{Q}$. By the density of $\mathbb{Q}$,

$$\lim_{r\downarrow 0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y) - f(x)|\, d\mu(y) = 0$$

for $x \in D$. ∎

Remark. We can always switch the order of universal quantifiers, i.e. $\forall x \in A, \forall y \in X, P(x,y)$ is equivalent to $\forall y \in X, \forall x \in A, P(x,y)$. But, we cannot always switch the order if we had a.e. instead, i.e. $\forall x \in A$, for a.e. $y \in X, P(x,y)$ may not be equivalent to for a.e. $y \in X$, $\forall x \in A, P(x,y)$. One can check that the second statement is stronger than the first. Problem with the first statement is that the a.e. set where P holds depends on x. But when A is countable, we can switch, which is what we did in the proof with $A = \mathbb{Q}$.

3 Chapter 3

We would like to prove Theorem 3.1:
Let μ be a locally finite Borel measure, $\alpha \ge 0$, and E be a Borel set with $\mu(E) > 0$. Suppose that

$$0 < \theta_*(\mu, x) = \theta^*(\mu, x) < \infty \quad \text{for all } x \in E.$$

Then α is an integer.

3.1 Proof of Prop 3.4

Proposition 3.4 states that:
Let μ be as in Theorem 3.1. Then for μ-a.e. $x \in E$,

$$\varnothing \neq \mathrm{Tan}_\alpha(\mu, x) \subseteq \{\theta^\alpha(\mu, x)\nu : \nu \in \mathcal{U}^\alpha(\mathbb{R}^n)\}$$

Proof. $\mathrm{Tan}_\alpha(\mu, x) \neq \varnothing$ follows immediately from the compactness. Show that $\mathrm{Tan}_\alpha(\mu, x) \subseteq \theta^\alpha(\mu, x)\mathcal{U}^\alpha(\mathbb{R}^n)$ Let $\nu \in \mathrm{Tan}_\alpha(\mu, x)$ and pick $r_i \downarrow 0$ so that $\nu^i := \frac{\mu_{x,r_i}}{r_i^\alpha} \xrightarrow{*} \nu$.
Proof of $0 \in \mathrm{supp}\,(\nu)$: For any $\rho > 0$,

$$0 < \omega_\alpha \theta_*(\mu, x) = \liminf_{r \downarrow 0} \frac{\mu(B_r(x))}{r^\alpha} \leq \liminf_{j \to \infty} \frac{\mu(B_{\rho r_j}(x))}{(\rho r_j)^\alpha} \leq \limsup_{j \to \infty} \frac{\mu(B_{\rho r_j}(x))}{(\rho r_j)^\alpha} = \rho^{-\alpha} \limsup_{j \to \infty} \frac{\mu_{x,r_j}(B_\rho(0))}{r_j^\alpha}$$

$$\leq \rho^{-\alpha} \limsup_{j \to \infty} \frac{\mu_{x,r_j}(\overline{B}_\rho(0))}{r_j^\alpha} \leq \rho^{-\alpha} \nu(\overline{B}_\rho(0))$$

Hence, $\theta_*(\nu, 0) \geq \theta_*(\mu, x) > 0$ by the continuity of measures. Note that we only used $\theta_*(\mu, x) > 0$ to prove this.

To begin with, we make a naive attempt. Let $y \in \mathrm{supp}\,(\nu)$. By Prop 2.7, there is at most countable set S so that

$$\nu^m(B_\rho(y)) \to \nu(B_\rho(y))$$

which is

$$\frac{\mu(B_{\rho r_m}(x + r_m y))}{r_m^\alpha} \to \nu(B_\rho(y))$$

Also, by the existence of density $\theta(\mu, x)$,

$$\frac{\mu(B_{\rho r_m}(x))}{r_m^\alpha} \to \theta(\mu, x)\omega_\alpha \rho^\alpha$$

If we can show that

$$\left| \frac{\mu(B_{\rho r_m}(x + r_m y))}{r_m^\alpha} - \frac{\mu(B_{\rho r_m}(x))}{r_m^\alpha} \right| \to 0$$

then we are done.
The proof in the note proves the above limit with some modifications. It is important to understand (1) why $E^{i,j,k}$ are defined (2) why we consider $\nu \in \mathrm{Tan}_\alpha(\mu \llcorner E^{i,j,k}, x)$ instead of $\nu \in \mathrm{Tan}_\alpha(\mu, x)$ and (3) why we consider $x \in F_1$.

For positive integers i, j, k, define

$$E^{i,j,k} := \{x \in E : \frac{(j-1)\omega_\alpha}{i} \leq \frac{\mu(B_r(x))}{r^\alpha} \leq \frac{(j+1)\omega_\alpha}{i} \quad \text{for all } r \leq 1/k\}$$

Since $\theta_*(\mu, x) = \theta^*(\mu, x)$ for $x \in E$, for each i,

$$E = \bigcup_{j,k} E^{i,j,k}$$

Claim: for each positive integers i, j, k, and for μ-a.e. $x \in E^{i,j,k}$

$$|\nu(B_r(y)) - \theta^\alpha(\mu, x)\omega_\alpha r^\alpha| \leq \frac{2\omega_\alpha r^\alpha}{i} \quad \text{for every } y \in \mathrm{supp}\,(\nu) \text{ and } r > 0$$

for all $\nu \in \mathrm{Tan}_\alpha(\mu \llcorner E^{i,j,k}, x)$.

By the locality of tangent meausure, $\mathrm{Tan}_\alpha(\mu \llcorner E^{i,j,k}, x) = \mathrm{Tan}_\alpha(\mu, x)$ for μ-a.e. $x \in E^{i,j,k}$. Moreover, since positive integers is countable, we can switch "for all positive integers i, j, k" and "for μ-a.e. $x \in E^{i,j,k}$". To

sum up, the claim becomes:
For μ-a.e. $x \in E$ and $\nu \in \mathrm{Tan}_\alpha(\mu, x)$,

$$\nu(B_r(y)) = \theta^\alpha(\mu, x)\omega_\alpha r^\alpha \quad \text{for every } y \in \mathrm{supp}\,(\nu) \text{ and } r > 0$$

This implies that $\nu/\theta^\alpha(\mu, x)$ is an α-uniform measure.

Proof of claim:
For simplicity, let $F := E^{i,j,k}$ and

$$F_1 := \{x \in F : \lim_{r\downarrow 0} \frac{\mu(B_r(x) \setminus F)}{r^\alpha} = 0\}$$

For μ-a.e. $x \in \mathbb{R}^n$,

$$\lim_{r\downarrow 0} \frac{\mu(B_r(x) \setminus F)}{\mu(B_r(x))} = \lim_{r\downarrow 0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} \chi_{\mathbb{R}^n \setminus F}\, d\mu = \chi_{\mathbb{R}^n \setminus F}(x)$$

Since $\theta^*(\mu, x) < \infty$, it follows that

$$\lim_{r\downarrow 0} \frac{\mu(B_r(x) \setminus F)}{r^\alpha} = \lim_{r\downarrow 0} \frac{1}{r^\alpha} \int_{B_r(x)} \chi_{\mathbb{R}^n \setminus F}\, d\mu = \chi_{\mathbb{R}^n \setminus F}(x) = 0$$

Thus, $\mu(F \setminus F_1) = 0$ and so it suffices to prove the claim with $x \in F_1$.
Let $\nu \in \mathrm{Tan}_\alpha(\mu \llcorner F, x)$ and $r_i \downarrow 0$ so that $\nu^i := (\mu \llcorner F)_{x,r_i}/r_i^\alpha \overset{*}{\rightharpoonup} \nu$. Fix $y \in \mathrm{supp}\,(\nu)$.
We start by making a naive attempt, which will be remedied soon. By Prop 2.7, there is a countable set S such that

$$\lim_{m\to\infty} \nu^m(B_r(y)) = \nu(B_r(y)) \quad \text{for all } r \in \mathbb{R}^+ \setminus S$$

Rewriting this limit,

$$\lim_{m\to\infty} \frac{\mu(B_{rr_m}(x_m) \cap F)}{r_m^\alpha} = \nu(B_r(y)) \quad \text{for all } r \in \mathbb{R}^+ \setminus S$$

where $x_m := x + r_m y$.
Since $x_m = x + r_m y \to x$ (here, it is important that $|x_m - x| \le Cr_m$ for some large C) and $x \in F_1$, one can show that

$$\lim_{m\to\infty} \frac{\mu(B_{rr_m}(x_m) \cap F)}{r_m^\alpha} = \lim_{m\to\infty} \frac{\mu(B_{rr_m}(x_m))}{r_m^\alpha}$$

and so

$$\lim_{m\to\infty} \frac{\mu(B_{rr_m}(x_m))}{r_m^\alpha} = \nu(B_r(y)) \quad \text{for all } r \in \mathbb{R}^+ \setminus S$$

By definition,

$$\lim_{m\to\infty} \frac{\mu(B_{rr_m}(x))}{r_m^\alpha} = \theta(\mu, x)\omega_\alpha r^\alpha \quad \text{for all } r > 0$$

If we had $x_m \in F$ for all m, then

$$\left| \frac{\mu(B_{rr_m}(x_m) \cap F)}{r_m^\alpha} - \frac{\mu(B_{rr_m}(x))}{r_m^\alpha} \right| \le \frac{2\omega_\alpha r^\alpha}{i}$$

and by taking limit as $m \to \infty$, we get the desired inequality for $r \in \mathbb{R}^+ \setminus S$. Then, since S is countable, for every $r \in S$, we have $s_\ell \in \mathbb{R}^+ \setminus S$ so that $s_\ell \uparrow r$. By the continuity of measures, we get the desired inequality for all $r > 0$.
Thus, to end this proof, it suffices to pick $x_m \in F$ satisfying $|x_m - x| \le Cr_m$ for some large C. In particular, we show that there is a sequence (x_m) in F so that

$$y_m := \frac{x_m - x}{r_m} \to y$$

This follows from $y \in \mathrm{supp}\,(\nu)$, Prop 2.7, and the fact that if $\mu(S) > 0$), then $S \neq \varnothing$. ■

3.2 Lemma 3.7

3.3 Proof of Lemma 3.8

Lemma 3.8 says: If $\alpha \in (0, n)$ and $\mu \in \mathcal{U}^\alpha(\mathbb{R}^n)$ and $\operatorname{supp}(\mu) \subseteq \{x_1 \geq 0\}$, then

$$\operatorname{supp}(\tilde{\nu}) \subseteq \{x_1 = 0\} \quad \text{for all } \tilde{\nu} \in \operatorname{Tan}_\alpha(\nu, 0)$$

Proof. Let $r_i \downarrow 0$ and $\frac{\nu_{0,r_i}}{r_i^\alpha} \stackrel{*}{\rightharpoonup} \tilde{\nu}$. Assume the following estimate

$$|\langle b(r), y\rangle| \leq C(\alpha)|y|^2$$

for $y \in \operatorname{supp}(\nu) \cap B_{2r}(0)$.
If $b(r) = 0$ for all r, then we are done. Suppose $b(r_0) \neq 0$ for some $r_0 > 0$. Let $z \in \operatorname{supp}(\tilde{\nu})$. Then there exists $y_i \in \operatorname{supp}(\nu)$ such that $\frac{y_i}{r_i} \to z$ (here, we are taking a subsequence of r_i, but for convenience, we keep using r_i). Notice that $y_i \to 0$ and so $y_i \in B_{2r_0}(0)$ for any large i. Then

$$|\langle b(r_0), y_i\rangle| \leq C(\alpha)|y_i|^2$$

which is

$$|\langle b(r_0), \frac{y_i}{r_i}\rangle| \leq C(\alpha)\frac{|y_i|^2}{r_i} \to 0$$

Hence, $\operatorname{supp}(\tilde{\nu}) \subseteq b(r_0)^\perp$, i.e. $\operatorname{supp}(\tilde{\nu})$ is contained in a hyperplane. Since $\operatorname{supp}(\tilde{\nu}) \subseteq \{x_1 \geq 0\}$, it follows that $\operatorname{supp}(\tilde{\nu}) \subseteq \{x_1 = 0\}$. ■