INT305 Machine Learning
Lecture 10
k-Means and EM Algorithm

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Today

- Today's lecture:
  - First, introduce K-means, a simple algorithm for clustering, i.e. grouping data points into clusters
  - Then, we will reformulate clustering as a latent variable model, apply the EM algorithm

Clustering

- Sometimes the data form clusters, where samples within a cluster are similar to each other, and samples in different clusters are dissimilar:
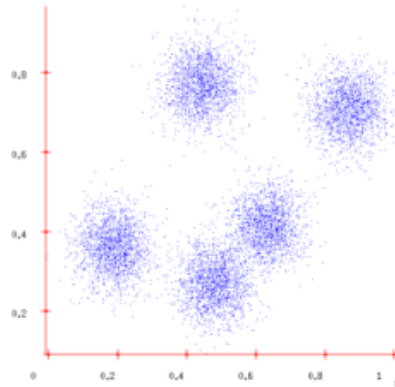

- Such a distribution is multimodal, since it has multiple modes, or regions of high probability mass.
- Grouping data points into clusters, with no observed labels, is called clustering. It is an unsupervised learning technique.
- E.g. clustering machine learning papers based on topic (deep learning, Bayesian models, etc.)
  - ▶ But topics are never observed (unsupervised).

Clustering
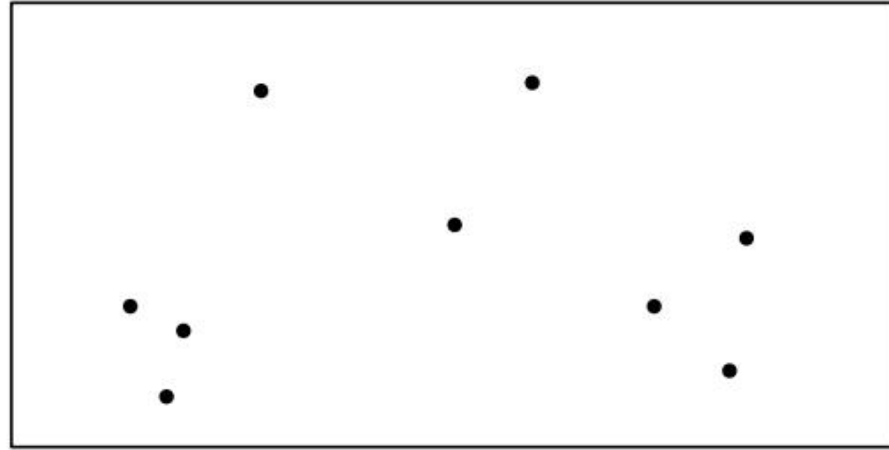
- Assume the data $\{\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(N)}\}$ lives in a Euclidean space, $\mathbf{x}^{(n)} \in \mathbb{R}^D$.

- Assume each data point belongs to one of K clusters

- Assume the data points from same cluster are similar, i.e. close in Euclidean distance.

- How can we identify those clusters (data points that belong to each cluster)? Let's formulate as an optimization problem.

K-means Objective
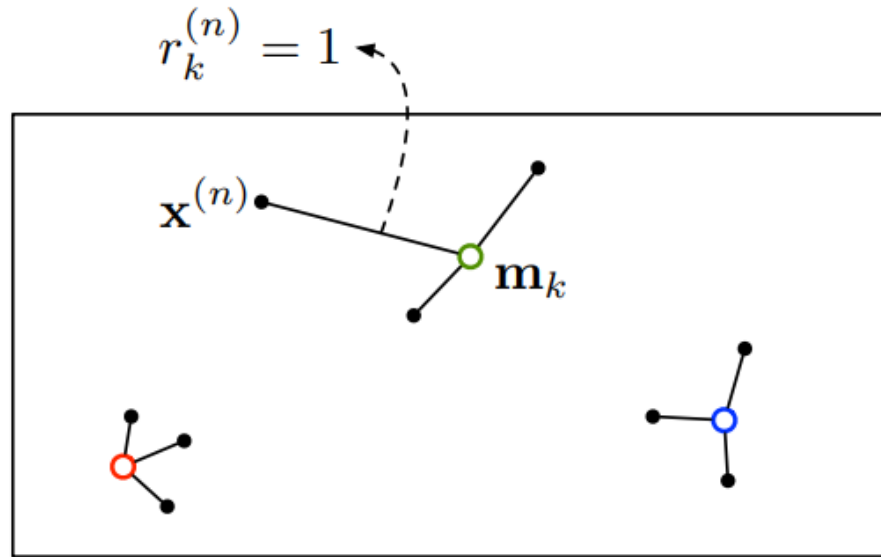
K-means Objective: Find cluster centers $\{\mathbf{m}_k\}_{k=1}^K$ and assignments $\{\mathbf{r}^{(n)}\}_{n=1}^N$ to minimize the sum of squared distances of data points $\{\mathbf{x}^{(n)}\}$ to their assigned centers.

- Data sample $n = 1, .., N$: $\mathbf{x}^{(n)} \in \mathbb{R}^D$ (observed),

- Cluster center $k = 1, .., K$: $\mathbf{m}_k \in \mathbb{R}^D$ (not observed),

- Responsibilities: Cluster assignment for sample n:
  $\mathbf{r}^{(n)} \in \mathbb{R}^K$ 1-of-K encoding (not observed)

K-means Objective

- **K-means Objective**: Find cluster centers $\{\mathbf{m}_k\}_{k=1}^K$ and assignments $\{\mathbf{r}^{(n)}\}_{n=1}^N$ to minimize the sum of squared distances of data points $\{\mathbf{x}^{(n)}\}$ to their assigned centers.

  - ▶ Data sample $n = 1, .., N$: $\mathbf{x}^{(n)} \in \mathbb{R}^D$ (observed),
  - ▶ Cluster center $k = 1, .., K$: $\mathbf{m}_k \in \mathbb{R}^D$ (not observed),
  - ▶ Responsibilities: Cluster assignment for sample n: $\mathbf{r}^{(n)} \in \mathbb{R}^K$ 1-of-K encoding (not observed)

- Mathematically:

$$\min_{\{\mathbf{m}_k\},\{\mathbf{r}^{(n)}\}} J(\{\mathbf{m}_k\}, \{\mathbf{r}^{(n)}\}) = \min_{\{\mathbf{m}_k\},\{\mathbf{r}^{(n)}\}} \sum_{n=1}^N \sum_{k=1}^K r_k^{(n)} \|\mathbf{m}_k - \mathbf{x}^{(n)}\|^2$$

where $r_k^{(n)} = \mathbb{I}[\mathbf{x}^{(n)}$ is assigned to cluster $k]$, i.e., $\mathbf{r}^{(n)} = [0, .., 1, .., 0]^\top$

- Finding an optimal solution is an NP-hard problem!

K-means Objective

- Optimization problem:

$$\min_{\{\mathbf{m}_k\},\{\mathbf{r}^{(n)}\}} \sum_{n=1}^{N} \underbrace{\sum_{k=1}^{K} r_k^{(n)} ||\mathbf{m}_k - \mathbf{x}^{(n)}||^2}_{\substack{\text{distance between } x^{(n)} \\ \text{and its assigned cluster center}}}$$

- Since $r_k^{(n)} = \mathbb{I}[\mathbf{x}^{(n)}$ is assigned to cluster $k]$, i.e., $\mathbf{r}^{(n)} = [0, .., 1, .., 0]^\top$

- inner sum is over K terms but only one of them is non-zero.

- E.g. say sample $\mathbf{x}^{(n)}$ is assigned to cluster $k = 3$, then

$$\mathbf{r}^n = [0, 0, 1, 0, ...]$$

$$\sum_{k=1}^{K} r_k^{(n)} ||\mathbf{m}_k - \mathbf{x}^{(n)}||^2 = ||\mathbf{m}_3 - \mathbf{x}^{(n)}||^2$$

How to optimize? Alternating Minimization

Optimization problem:

$$\min_{\{\mathbf{m}_k\},\{\mathbf{r}^{(n)}\}} \sum_{n=1}^{N} \sum_{k=1}^{K} r_k^{(n)} ||\mathbf{m}_k - \mathbf{x}^{(n)}||^2$$

- Problem is hard when minimizing jointly over the parameters $\{\mathbf{m}_k\}, \{\mathbf{r}^{(n)}\}$

- But note that if we fix one and minimize over the other, then it becomes easy.

- Doesn't guarantee the same solution!

Optimization problem:

$$\min_{\{\mathbf{m}_k\},\{\mathbf{r}^{(n)}\}} \sum_{n=1}^{N} \sum_{k=1}^{K} r_k^{(n)} ||\mathbf{m}_k - \mathbf{x}^{(n)}||^2$$

- Note:
  - ▶ If we fix the centers $\{\mathbf{m}_k\}$ then we can easily find the optimal assignments $\{\mathbf{r}^{(n)}\}$ for each sample n

$$\min_{\mathbf{r}^{(n)}} \sum_{k=1}^{K} r_k^{(n)} ||\mathbf{m}_k - \mathbf{x}^{(n)}||^2$$

  - ▶ Assign each point to the cluster with the nearest center

$$r_k^{(n)} = \begin{cases} 1 & \text{if } k = \arg\min_j ||\mathbf{x}^{(n)} - \mathbf{m}_j||^2 \\ 0 & \text{otherwise} \end{cases}$$

  - ▶ E.g. if $\mathbf{x}^{(n)}$ is assigned to cluster $\hat{k}$,

$$\mathbf{r}^{(n)} = \underbrace{[0, 0, ..., 1, ..., 0]}_{\text{Only } \hat{k}\text{-th entry is 1}}^{\top}$$

Alternating Minimization

- Likewise, if we fix the assignments $\{\mathbf{r}^{(n)}\}$ then can easily find optimal centers $\{\mathbf{m}_k\}$

 - ▶ Set each cluster's center to the average of its assigned data points: For $l = 1, 2, ..., K$

$$0 = \frac{\partial}{\partial \mathbf{m}_l} \sum_{n=1}^{N} \sum_{k=1}^{K} r_k^{(n)} ||\mathbf{m}_k - \mathbf{x}^{(n)}||^2$$

$$= 2 \sum_{n=1}^{N} r_l^{(n)} (\mathbf{m}_l - \mathbf{x}^{(n)}) \implies \mathbf{m}_l = \frac{\sum_n r_l^{(n)} \mathbf{x}^{(n)}}{\sum_n r_l^{(n)}}$$

- Let's alternate between minimizing $J(\{\mathbf{m}_k\}, \{\mathbf{r}^{(n)}\})$ with respect to $\{\mathbf{m}_k\}$ and $\{\mathbf{r}^{(n)}\}$

- This is called alternating minimization

K-means Algorithm

High level overview of algorithm:

- Initialization: randomly initialize cluster centers

- The algorithm iteratively alternates between two steps:

  ▶ Assignment step: Assign each data point to the closest cluster
  ▶ Refitting step: Move each cluster center to the mean of the data assigned to it
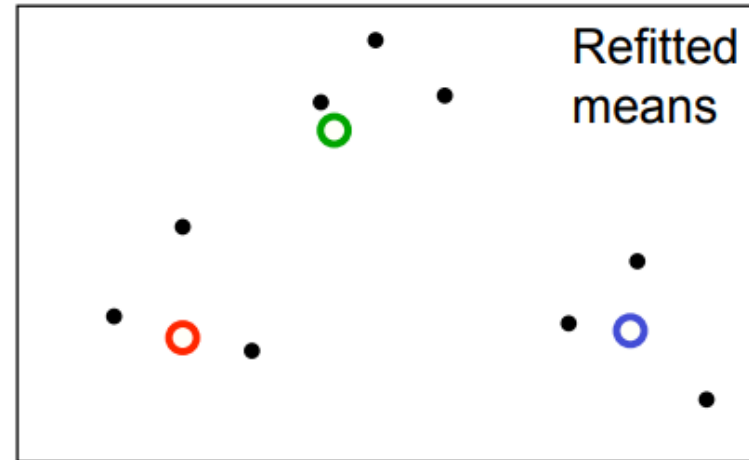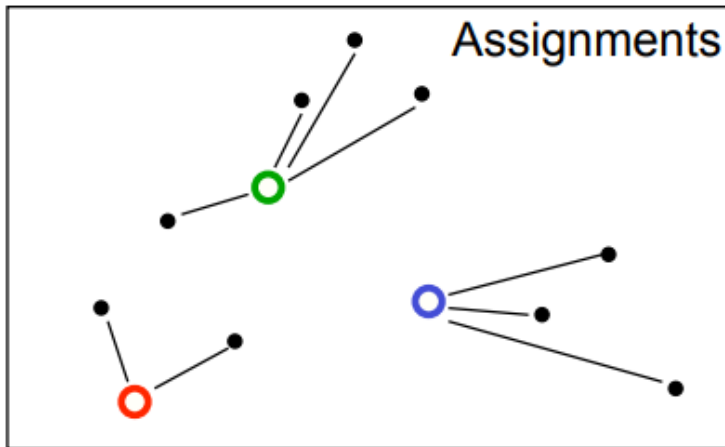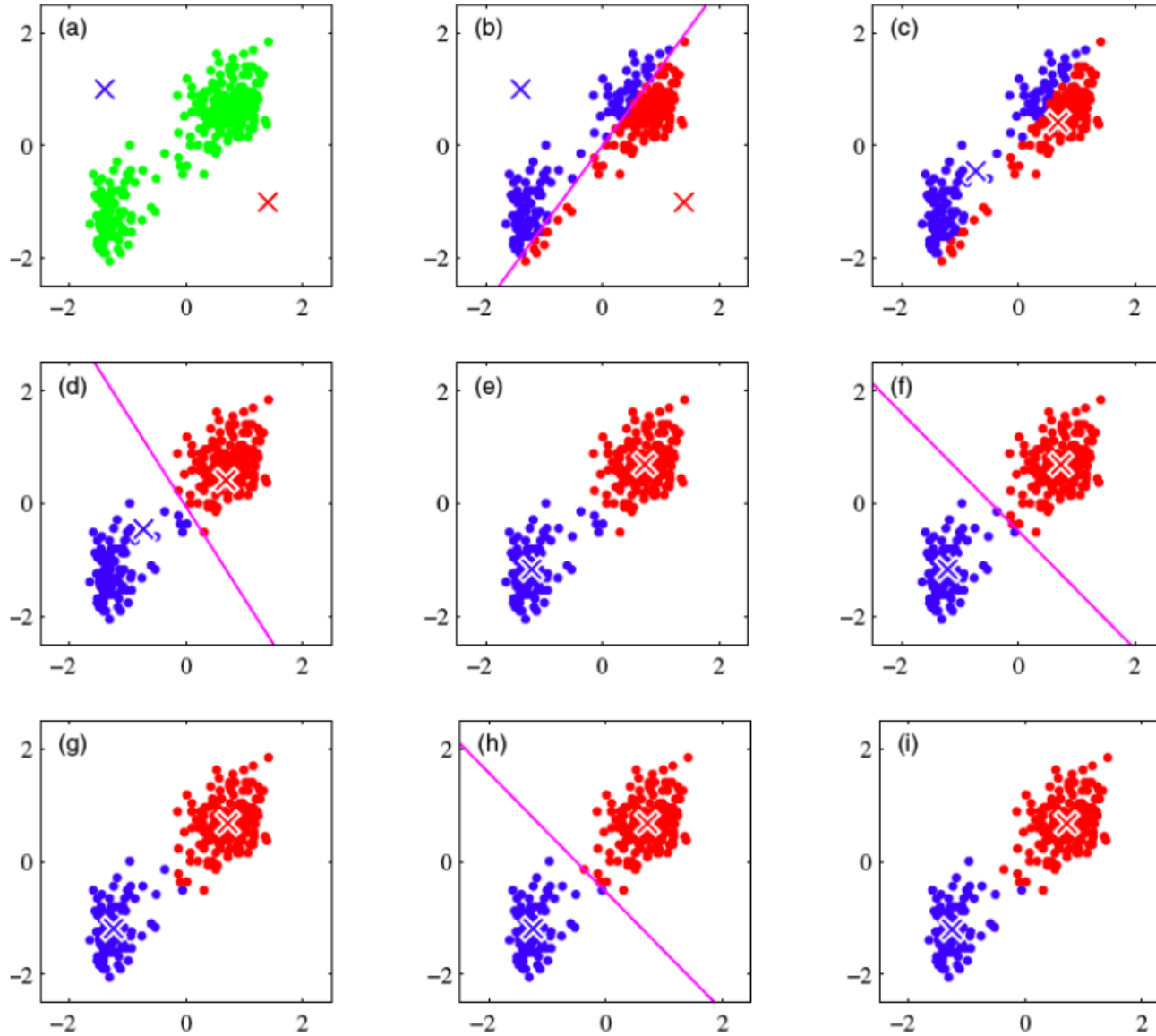
K-means Algorithm

Figure from Bishop

The K-means Algorithm

- Initialization: Set K cluster means $\mathbf{m}_1, \ldots, \mathbf{m}_K$ to random values

- Repeat until convergence (until assignments do not change):

  ▶ Assignment: Optimize J w.r.t. $\{\mathbf{r}\}$: Each data point $\mathbf{x}^{(n)}$ assigned to nearest center

  $$\hat{k}^{(n)} = \arg\min_k ||\mathbf{m}_k - \mathbf{x}^{(n)}||^2$$

  and Responsibilities (1-hot or 1-of-K encoding)

  $$r_k^{(n)} = \mathbb{I}[\hat{k}^{(n)} = k] \quad \text{for} \quad k = 1, .., K$$

  ▶ Refitting: Optimize J w.r.t. $\{\mathbf{m}\}$: Each center is set to mean of data assigned to it

  $$\mathbf{m}_k = \frac{\sum_n r_k^{(n)} \mathbf{x}^{(n)}}{\sum_n r_k^{(n)}}.$$
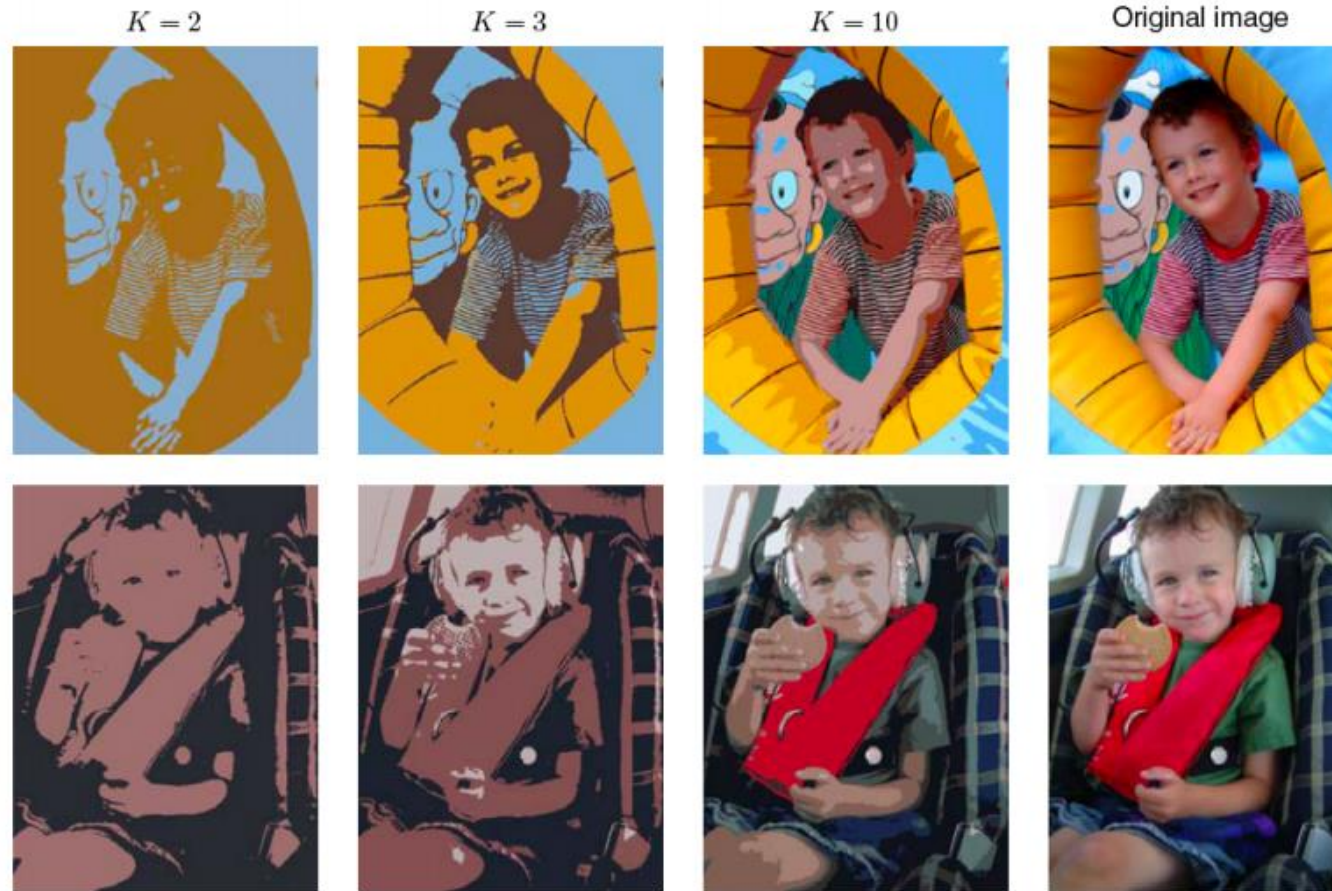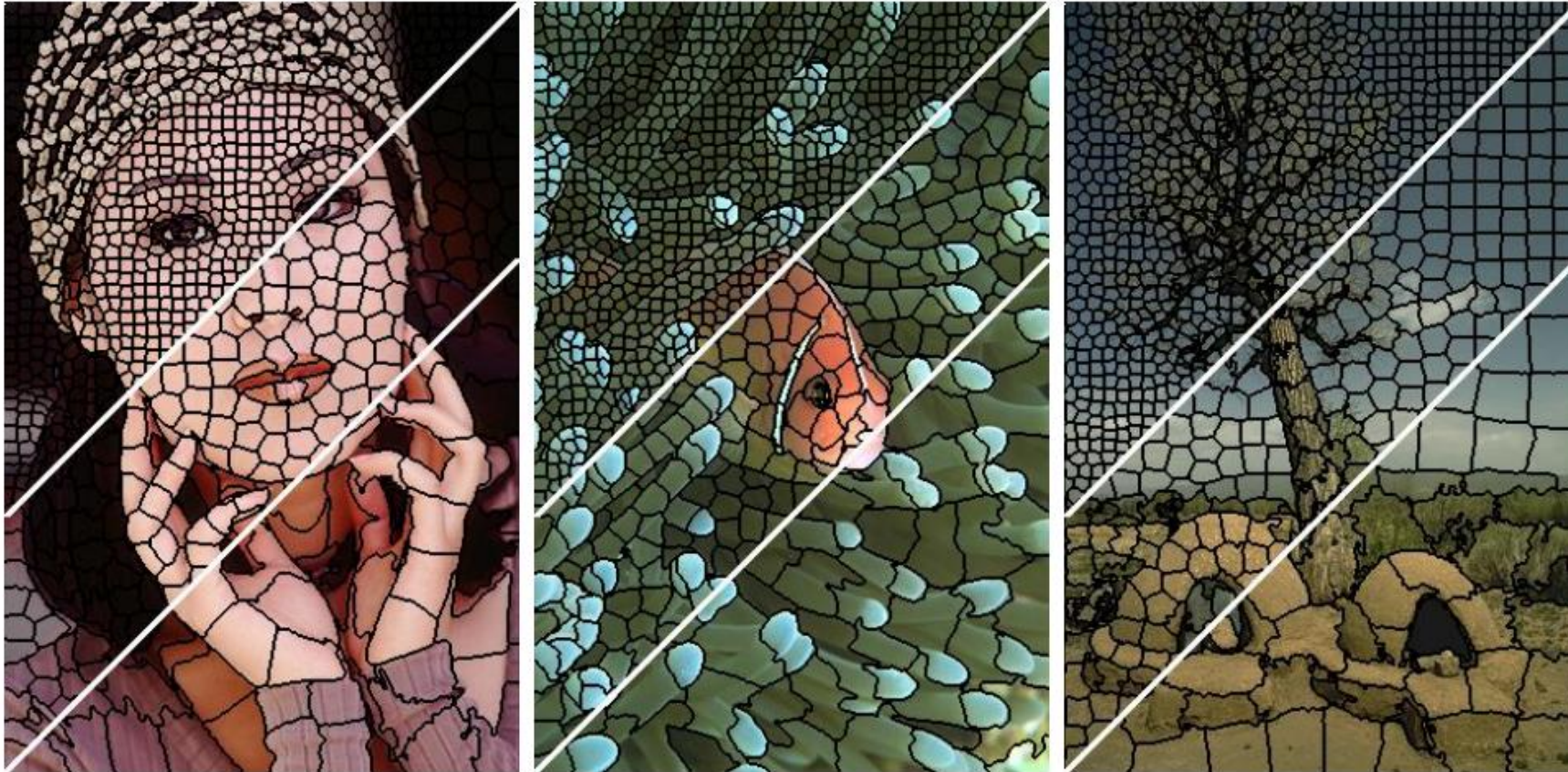
K-means for Vector Quantization

Figure from Bishop

- Given image, construct "dataset" of pixels represented by their RGB pixel intensities
- Run k-means, replace each pixel by its cluster center

K-means for Image Segmentation

- Given image, construct "dataset" of pixels, represented by their RGB pixel intensities and grid locations

- Run k-means (with some modifications) to get superpixels

Questions about K-means

- Why does update set $\mathbf{m}_k$ to mean of assigned points?

- What if we used a different distance measure?

- How can we choose the best distance?

- How to choose K?

- Will it converge?

Why K-means Converges

- K-means algorithm reduces the cost at each iteration.
  - ▶ Whenever an assignment is changed, the sum squared distances J of data points from their assigned cluster centers is reduced.
  - ▶ Whenever a cluster center is moved, J is reduced.
- Test for convergence: If the assignments do not change in the assignment step, we have converged (to at least a local minimum).
- This will always happen after a finite number of iterations, since the number of possible cluster assignments is finite
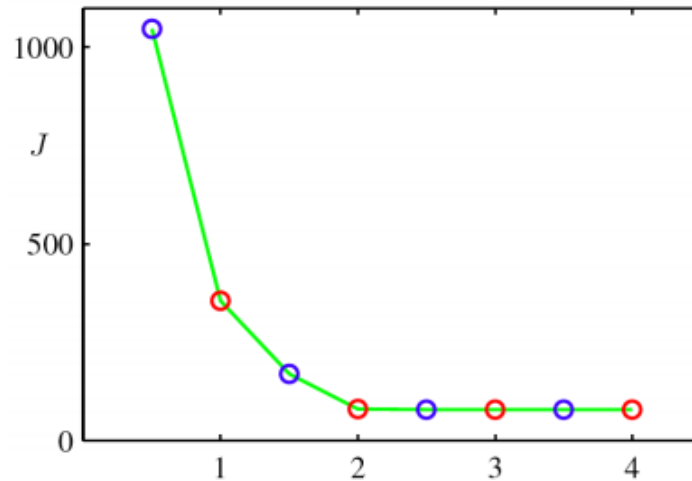

- K-means cost function after each assignment step (blue) and refitting step (red). The algorithm has converged after the third refitting step.

Local Minima

- The objective J is non-convex (so coordinate descent on J is not guaranteed to converge to the global minimum)

- There is nothing to prevent k-means getting stuck at local minima.

- We could try many random starting points

A bad local optimum
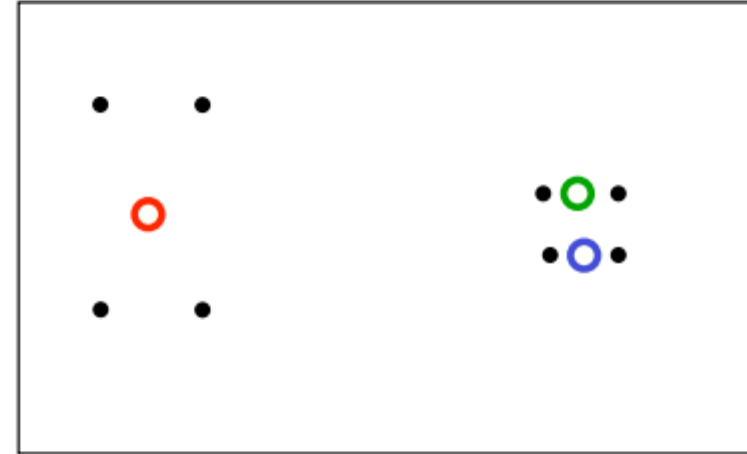
Soft K-means

- Instead of making hard assignments of data points to clusters, we can make soft assignments. One cluster may have a responsibility of .7 for a datapoint and another may have a responsibility of .3.

    ▸ Allows a cluster to use more information about the data in the refitting step.
    ▸ How do we decide on the soft assignments?
    ▸ We already saw this in multi-class classification:
        ▸ 1-of-K encoding vs softmax assignments

Soft K-means Algorithm

- Initialization: Set K means $\{\mathbf{m}_k\}$ to random values

- Repeat until convergence (measured by how much J changes):

  ▶ Assignment: Each data point n given soft "degree of assignment" to each cluster mean k, based on responsibilities

  $$r_k^{(n)} = \frac{\exp[-\beta\|\mathbf{m}_k - \mathbf{x}^{(n)}\|^2]}{\sum_j \exp[-\beta\|\mathbf{m}_j - \mathbf{x}^{(n)}\|^2]}$$

  $$\implies \mathbf{r}^{(n)} = \mathrm{softmax}(-\beta\{\|\mathbf{m}_k - \mathbf{x}^{(n)}\|^2\}_{k=1}^K)$$

  ▶ Refitting: Model parameters, means, are adjusted to match sample means of datapoints they are responsible for:

  $$\mathbf{m}_k = \frac{\sum_n r_k^{(n)}\mathbf{x}^{(n)}}{\sum_n r_k^{(n)}}$$

Questions about Soft K-means

Some remaining issues

- How to set β?

- Clusters with unequal weight and width?

These aren't straightforward to address with K-means. Instead, in the sequel, we'll reformulate clustering using a generative model.

A Generative View of Clustering

- Next: probabilistic formulation of clustering

- An obvious approach is to imagine that the data was produced by a generative model

  ▸ Then we adjust the model parameters using maximum likelihood i.e. to maximize the probability that it would produce exactly the data we observed

The Generative Model

- We'll be working with the following generative model for data $\mathcal{D}$
- Assume a datapoint $\mathbf{x}$ is generated as follows:
  - ▶ Choose a cluster z from $\{1, \ldots, K\}$ such that $p(z = k) = \pi_k$
  - ▶ Given z, sample $\mathbf{x}$ from a Gaussian distribution $\mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_z, \mathbf{I})$
- Can also be written:

$$p(z = k) = \pi_k$$

$$p(\mathbf{x}|z = k) = \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_k, \mathbf{I})$$
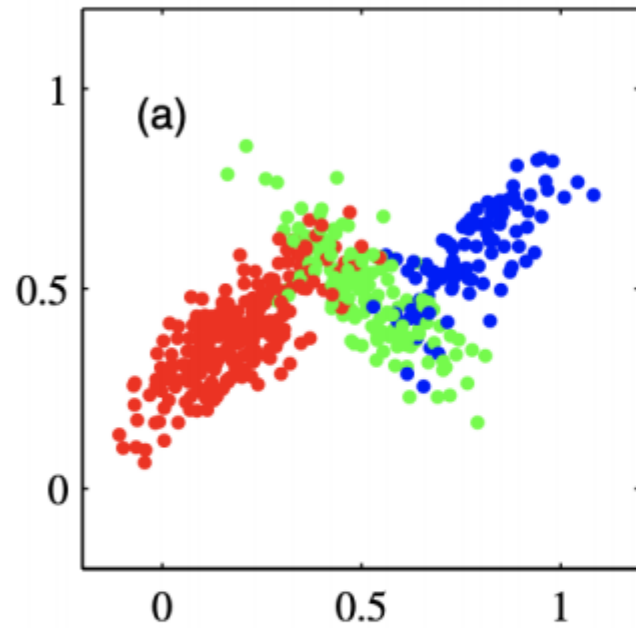
Clusters from Generative Model

- This defines joint distribution $p(z, \mathbf{x}) = p(z)p(\mathbf{x}|z)$ with parameters $\{\pi_k, \boldsymbol{\mu}_k\}_{k=1}^{K}$

- The marginal of $\mathbf{x}$ is given by $p(\mathbf{x}) = \sum_z p(z, \mathbf{x})$

- $p(z = k|\mathbf{x})$ can be computed using Bayes rule

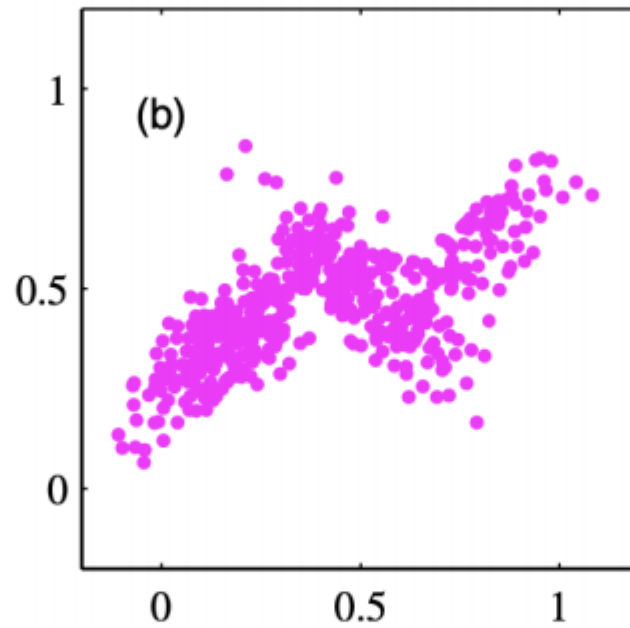$$p(z = k|\mathbf{x}) = \frac{p(\mathbf{x} \,|\, z = k)p(z = k)}{p(\mathbf{x})}$$

and tells us the probability $\mathbf{x}$ came from the kth cluster
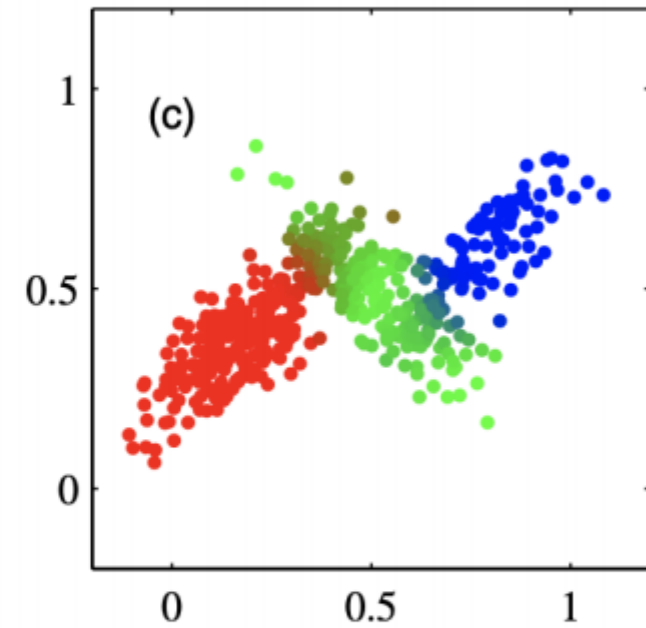
The Generative Model

- 500 points drawn from a mixture of 3 Gaussians.

a) Samples from $p(\mathbf{x} \mid z)$ b) Samples from the marginal $p(\mathbf{x})$ c) Responsibilities $p(z \mid \mathbf{x})$

Maximum Likelihood with Latent Variables

- How should we choose the parameters $\{\pi_k, \boldsymbol{\mu}_k\}_{k=1}^K$?
- Maximum likelihood principle: choose parameters to maximize likelihood of observed data
- We don't observe the cluster assignments z, we only see the data $\mathbf{x}$
- Given data $\mathcal{D} = \{\mathbf{x}^{(n)}\}_{n=1}^N$, choose parameters to maximize:

$$\log p(\mathcal{D}) = \sum_{n=1}^N \log p(\mathbf{x}^{(n)})$$

- We can find $p(\mathbf{x})$ by marginalizing out z:

$$p(\mathbf{x}) = \sum_{k=1}^K p(z = k, \mathbf{x}) = \sum_{k=1}^K p(z = k)p(\mathbf{x}|z = k)$$

Gaussian Mixture Model (GMM)

What is $p(\mathbf{x})$?

$$p(\mathbf{x}) = \sum_{k=1}^{K} p(z = k) p(\mathbf{x}|z = k) = \sum_{k=1}^{K} \pi_k \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_k, \mathbf{I})$$

- This distribution is an example of a Gaussian Mixture Model (GMM), and π_k are known as the mixing coefficients

- In general, we would have different covariance for each cluster, i.e., $p(\mathbf{x}\,|\,z = k) = \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$. For this lecture, we assume $\boldsymbol{\Sigma}_k = \mathbf{I}$ for simplicity.
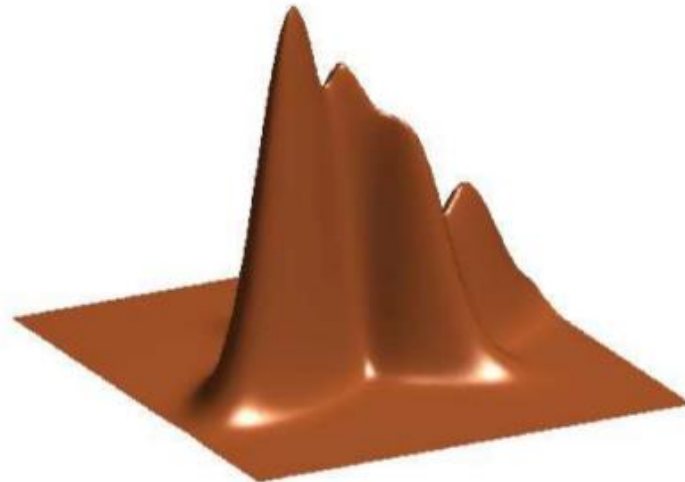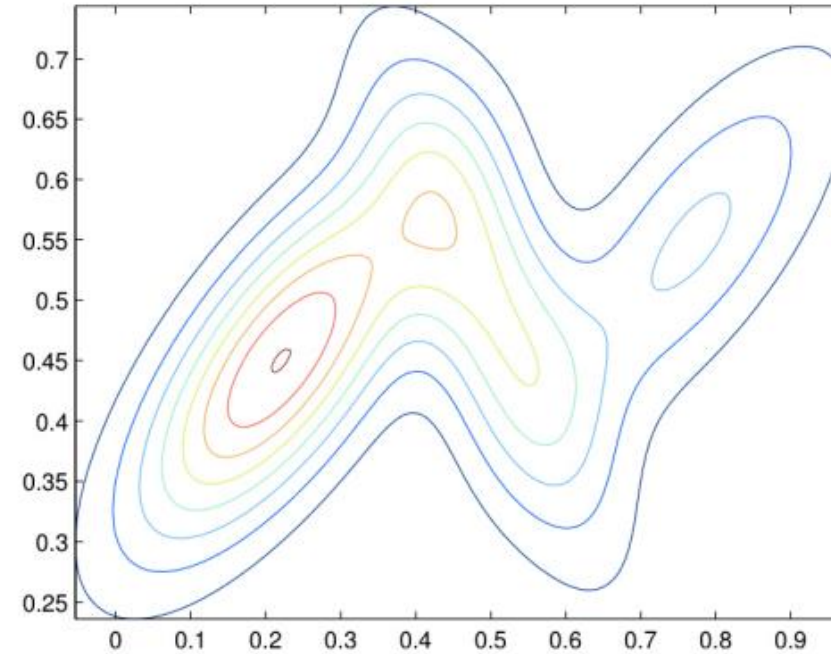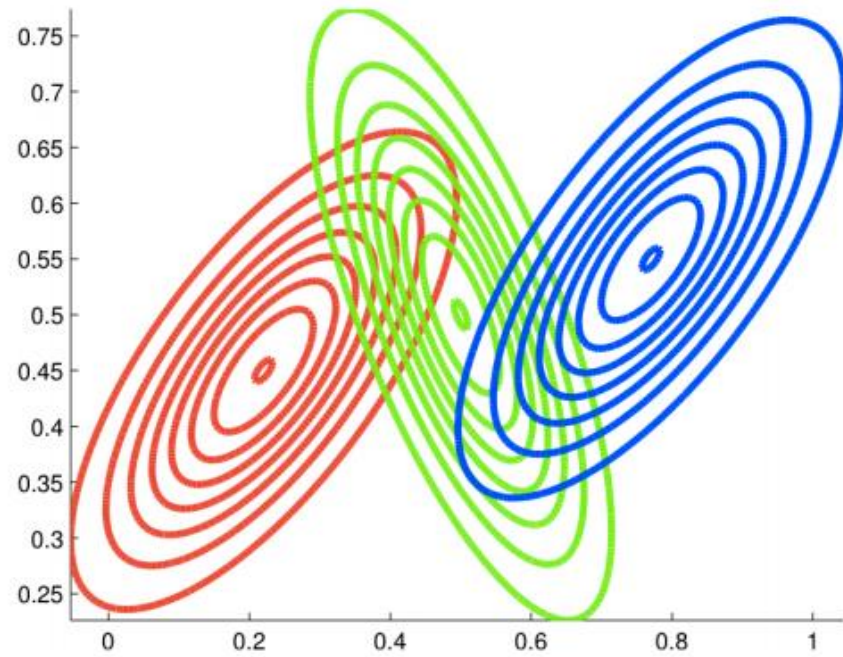
Visualizing a Mixture of Gaussians– 1D Gaussians

- If you fit a Gaussian to data:
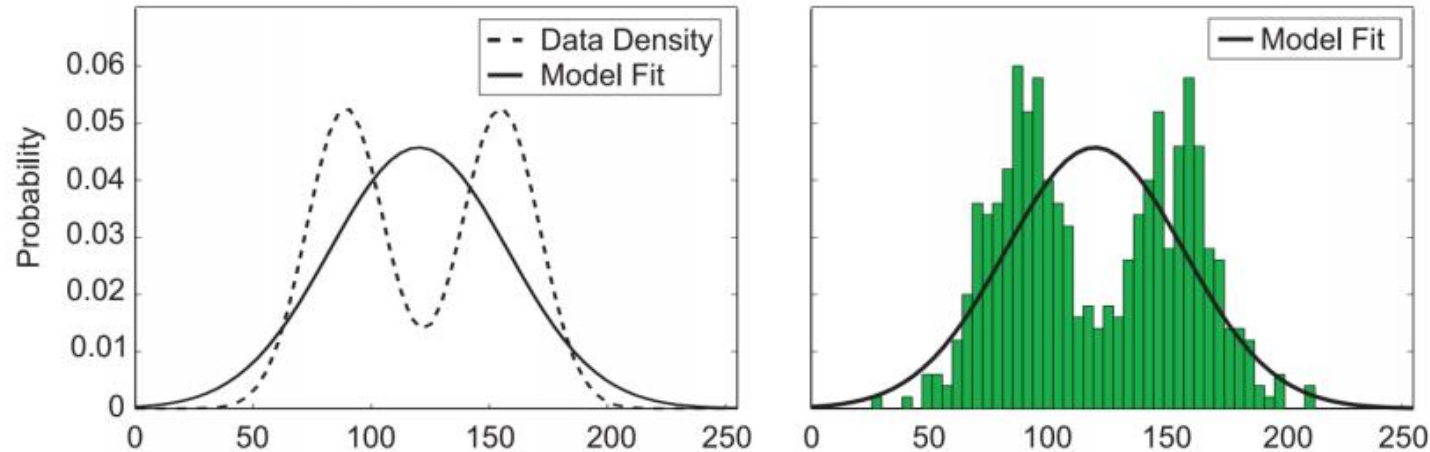
- Now, we are trying to fit a GMM (with $K = 2$ in this example):

[Slide credit: K. Kutulakos]
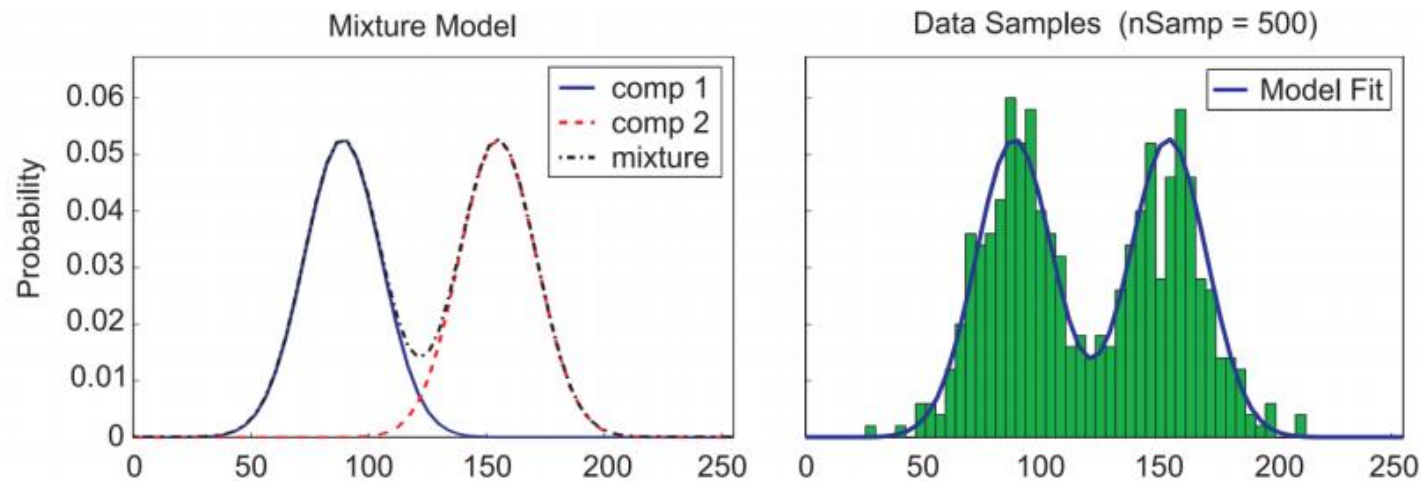
Visualizing a Mixture of Gaussians– 2D Gaussians

Fitting GMMs: Maximum Likelihood

Maximum likelihood objective:

$$\log p(\mathcal{D}) = \sum_{n=1}^{N} \log p(\mathbf{x}^{(n)}) = \sum_{n=1}^{N} \log \left(\sum_{k=1}^{K} \pi_k \mathcal{N}(\mathbf{x}^{(n)} | \boldsymbol{\mu}_k, \mathbf{I}) \right)$$

- How would you optimize this w.r.t. parameters $\{\pi_k, \boldsymbol{\mu}_k\}$?

  ▶ No closed form solution when we set derivatives to 0
  ▶ Difficult because sum inside the log

- One option: gradient ascent. Can we do better?

- Can we have a closed form update?

Maximum Likelihood

- **Observation: if we knew** $z^{(n)}$ for every $\mathbf{x}^{(n)}$, (i.e. our dataset was $\mathcal{D}_{\text{complete}} = \{(z^{(n)}, \mathbf{x}^{(n)})\}_{n=1}^{N})$ the maximum likelihood problem is easy:

$$
\log p(\mathcal{D}_{\text{complete}}) = \sum_{n=1}^{N} \log p(z^{(n)}, \mathbf{x}^{(n)})
$$

$$
= \sum_{n=1}^{N} \log p(\mathbf{x}^{(n)} | z^{(n)}) + \log p(z^{(n)})
$$

$$
= \sum_{n=1}^{N} \sum_{k=1}^{K} \mathbb{I}[z^{(n)} = k] \left(\log \mathcal{N}(\mathbf{x}^{(n)} | \boldsymbol{\mu}_k, \mathbf{I}) + \log \pi_k \right)
$$

Maximum Likelihood

$$\log p(\mathcal{D}_{\text{complete}}) = \sum_{n=1}^{N} \sum_{k=1}^{K} \mathbb{I}[z^{(n)} = k] \left(\log \mathcal{N}(\mathbf{x}^{(n)} | \boldsymbol{\mu}_k, \mathbf{I}) + \log \pi_k \right)$$

- We have been optimizing something similar for Naive bayes classifiers
- By maximizing $\log p(\mathcal{D}_{\text{complete}})$, we would get this:

$$\hat{\boldsymbol{\mu}}_k = \frac{\sum_{n=1}^{N} \mathbb{I}[z^{(n)} = k] \mathbf{x}^{(n)}}{\sum_{n=1}^{N} \mathbb{I}[z^{(n)} = k]} = \text{class means}$$

$$\hat{\pi}_k = \frac{1}{N} \sum_{n=1}^{N} \mathbb{I}[z^{(n)} = k] = \text{class proportions}$$

Maximum Likelihood

- We haven't observed the cluster assignments $z^{(n)}$, but we can compute $p(z^{(n)}|\mathbf{x}^{(n)})$ using Bayes rule

- Conditional probability (using Bayes rule) of z given $\mathbf{x}$

$$
\begin{aligned}
p(z = k|\mathbf{x}) &= \frac{p(z = k)p(\mathbf{x}|z = k)}{p(\mathbf{x})} \\
&= \frac{p(z = k)p(\mathbf{x}|z = k)}{\sum_{j=1}^{K} p(z = j)p(\mathbf{x}|z = j)} \\
&= \frac{\pi_k \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_k, \mathbf{I})}{\sum_{j=1}^{K} \pi_j \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_j, \mathbf{I})}
\end{aligned}
$$

Maximum Likelihood

$$\log p(\mathcal{D}_{\text{complete}}) = \sum_{n=1}^{N} \sum_{k=1}^{K} \mathbb{I}[z^{(n)} = k](\log \mathcal{N}(\mathbf{x}^{(n)}|\boldsymbol{\mu}_k, \mathbf{I}) + \log \pi_k)$$

- We don't know the cluster assignments $\mathbb{I}[z^{(n)} = k]$, but we know their expectation $\mathbb{E}[\mathbb{I}[z^{(n)} = k] \mid \mathbf{x}^{(n)}] = p(z^{(n)} = k|\mathbf{x}^{(n)})$.

- If we plug in $r_k^{(n)} = p(z^{(n)} = k|\mathbf{x}^{(n)})$ for $\mathbb{I}[z^{(n)} = k]$, we get:

$$\sum_{n=1}^{N} \sum_{k=1}^{K} r_k^{(n)}(\log \mathcal{N}(\mathbf{x}^{(n)}|\boldsymbol{\mu}_k, \mathbf{I}) + \log \pi_k)$$

- This is still easy to optimize! Solution is similar to what we have seen:

$$\hat{\boldsymbol{\mu}}_k = \frac{\sum_{n=1}^{N} r_k^{(n)} \mathbf{x}^{(n)}}{\sum_{n=1}^{N} r_k^{(n)}} \qquad \hat{\pi}_k = \frac{\sum_{n=1}^{N} r_k^{(n)}}{N}$$

- Note: this only works if we treat $r_k^{(n)} = \frac{\pi_k \mathcal{N}(\mathbf{x}^{(n)}|\boldsymbol{\mu}_k, \mathbf{I})}{\sum_{j=1}^{K} \pi_j \mathcal{N}(\mathbf{x}^{(n)}|\boldsymbol{\mu}_j, \mathbf{I})}$ as fixed.

How Can We Fit a Mixture of Gaussians?

- This motivates the Expectation-Maximization algorithm, which alternates between two steps:

  1. **E-step**: Compute the posterior probabilities $r_k^{(n)} = p(z^{(n)} = k | \mathbf{x}^{(n)})$ given our current model - i.e. how much do we think a cluster is responsible for generating a datapoint.

  2. **M-step**: Use the equations on the last slide to update the parameters, assuming $r_k^{(n)}$ are held fixed- change the parameters of each Gaussian to maximize the probability that it would generate the data it is currently responsible for.
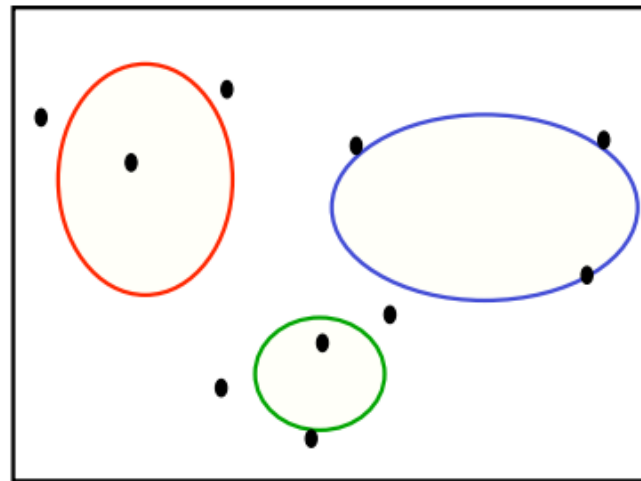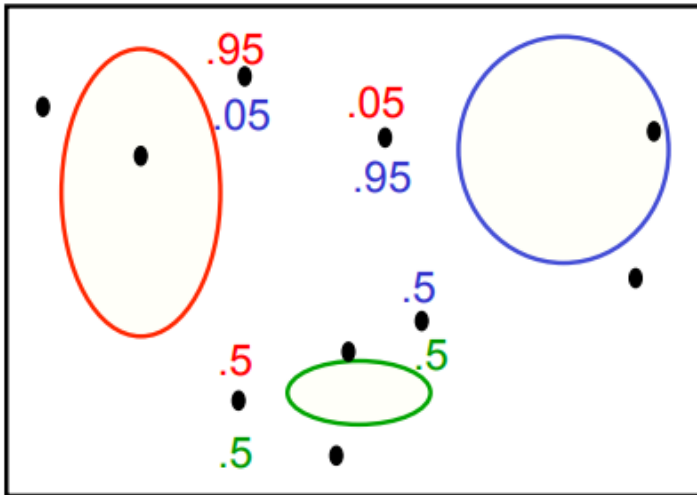
EM Algorithm for GMM

- Initialize the means $\hat{\boldsymbol{\mu}}_k$ and mixing coefficients $\hat{\pi}_k$

- Iterate until convergence:

  - E-step: Evaluate the responsibilities $r_k^{(n)}$ given current parameters

  $$r_k^{(n)} = p(z^{(n)} = k | \mathbf{x}^{(n)}) = \frac{\hat{\pi}_k \mathcal{N}(\mathbf{x}^{(n)} | \hat{\boldsymbol{\mu}}_k, \mathbf{I})}{\sum_{j=1}^{K} \hat{\pi}_j \mathcal{N}(\mathbf{x}^{(n)} | \hat{\boldsymbol{\mu}}_j, \mathbf{I})} = \frac{\hat{\pi}_k \exp\{-\frac{1}{2}\|\mathbf{x}^{(n)} - \hat{\boldsymbol{\mu}}_k\|^2\}}{\sum_{j=1}^{K} \hat{\pi}_j \exp\{-\frac{1}{2}\|\mathbf{x}^{(n)} - \hat{\boldsymbol{\mu}}_j\|^2\}}$$

  - M-step: Re-estimate the parameters given current responsibilities

  $$\hat{\boldsymbol{\mu}}_k = \frac{1}{N_k} \sum_{n=1}^{N} r_k^{(n)} \mathbf{x}^{(n)}$$

  $$\hat{\pi}_k = \frac{N_k}{N} \quad \text{with} \quad N_k = \sum_{n=1}^{N} r_k^{(n)}$$

  - Evaluate log likelihood and check for convergence

  $$\log p(\mathcal{D}) = \sum_{n=1}^{N} \log \left(\sum_{k=1}^{K} \hat{\pi}_k \mathcal{N}(\mathbf{x}^{(n)} | \hat{\boldsymbol{\mu}}_k, \mathbf{I}) \right)$$

What just happened: A review

- The maximum likelihood objective $\sum_{n=1}^{N} \log p(\mathbf{x}^{(n)})$ was hard to optimize

- The complete data likelihood objective was easy to optimize:

$$\sum_{n=1}^{N} \log p(z^{(n)}, \mathbf{x}^{(n)}) = \sum_{n=1}^{N} \sum_{k=1}^{K} \mathbb{I}[z^{(n)} = k](\log \mathcal{N}(\mathbf{x}^{(n)} | \boldsymbol{\mu}_k, \mathbf{I}) + \log \pi_k)$$

- We don't know $z^{(n)}$'s (they are latent), so we replaced $\mathbb{I}[z^{(n)} = k]$ with responsibilities $r_k^{(n)} = p(z^{(n)} = k | \mathbf{x}^{(n)})$.

- That is: we replaced $\mathbb{I}[z^{(n)} = k]$ with its expectation under $p(z^{(n)} | \mathbf{x}^{(n)})$ (E-step).

What just happened: A review

- We ended up with the expected complete data log-likelihood:

$$\sum_{n=1}^{N} \mathbb{E}_{p(z^{(n)}|\mathbf{x}^{(n)})}[\log p(z^{(n)}, \mathbf{x}^{(n)})] = \sum_{n=1}^{N}\sum_{k=1}^{K} r_k^{(n)} \left(\log \mathcal{N}(\mathbf{x}^{(n)}|\boldsymbol{\mu}_k, \mathbf{I}) + \log \pi_k \right)$$

  which we maximized over parameters $\{\pi_k, \boldsymbol{\mu}_k\}_k$ (M-step)
- The EM algorithm alternates between:
  - ▶ The E-step: computing the $r_k^{(n)} = p(z^{(n)} = k|\mathbf{x}^{(n)})$ (i.e. expectations $\mathbb{E}[\mathbb{I}[z^{(n)} = k]|\mathbf{x}^{(n)}])$ given the current model parameters $\pi_k, \boldsymbol{\mu}_k$
  - ▶ The M-step: update the model parameters $\pi_k, \boldsymbol{\mu}_k$ to optimize the expected complete data log-likelihood

Relation to k-Means

- The K-Means Algorithm:
    1. Assignment step: Assign each data point to the closest cluster
    2. Refitting step: Move each cluster center to the average of the data assigned to it
- The EM Algorithm:
    1. E-step: Compute the posterior probability over z given our current model
    2. M-step: Maximize the probability that it would generate the data it is currently responsible for.