Convex Shapes, Support Functions, and Fourier Representations

An Inquiry-Driven Research Book (Living Document)

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Contents

How to Use This Book

This is a living research book organized as a sequence of inquiries. Each chapter has: (i) a small number of **Inquiries** (what we want to understand), (ii) **Tasks** (proofs, computations, plots), (iii) **Tests** (sanity checks / invariants), (iv) a short **Synthesis** (what is now known, what remains uncertain).

Workflow. Write a short entry every time you (a) prove something, (b) run an experiment, (c) change a definition, or (d) update code.

Reproducibility. Every figure should have: data source, script path, parameters, and seed (if randomized).

Notation and Conventions

- A convex body is $K \subset \mathbb{R}^2$ compact, convex, with nonempty interior (unless stated otherwise).
- Unit direction $u(\theta) = (\cos\theta, \sin\theta)$, $\theta \in [0, 2\pi)$.
- Support function $h_K(\theta) = \sup_{x \in K} \langle x, u(\theta) \rangle$.
- Radial function (for star-shaped sets) $r(\theta)$ defined when appropriate.
- Fourier series convention: for periodic $f(\theta)$,

$$f(\theta) \approx \sum_{k=-K}^{K} c_k \mathrm{e}^{ik\theta}, \quad c_k = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \mathrm{e}^{-ik\theta} \, \mathrm{d}\theta.$$

Chapter 1

Support Functions: The Dual Geometry

1.1 Core definitions

Definition 1.1 (Support function)**.** Let $K \subset \mathbb{R}^2$ be convex and compact. Its support function is

$$h_K(\theta) = \max_{x \in K} \langle x, u(\theta) \rangle, \qquad u(\theta) = (\cos\theta, \sin\theta).$$

Inquiry 1.1 (What does $h(\theta)$ measure?)**.** Explain geometrically what $h_K(\theta)$ represents (supporting line, directional extent), and why it is a natural object for packing/collision.

Task 1.1 (Translation identity)**.** Prove that for any $v \in \mathbb{R}^2$,

$$h_{K+v}(\theta) = h_K(\theta) + \langle v, u(\theta) \rangle.$$

Hint 1.1. Use the definition and substitute $y = x + v$.

Task 1.2 (Rotation identity)**.** Let R_ϕ be rotation by ϕ. Prove:

$$h_{R_\phi K}(\theta) = h_K(\theta - \phi).$$

Test 1.1 (Numerical sanity checks)**.** Implement a routine to sample $h_K(\theta)$ for a polygon and verify **??** 1.1**??** 1.2 numerically to within tolerance.

1.2 Synthesis

What we now know.

- Translation acts as adding a first-harmonic linear functional.
- Rotation acts as phase shift on θ.

What remains unclear.

- Which geometric features are stable under Fourier truncation?
- How convexity constraints appear as inequalities on h.

Chapter 2

Fourier Representation of Support Functions

2.1 Sampling and FFT pipeline

Inquiry 2.1 (What does smoothness mean spectrally?)**.** Relate regularity of $h(\theta)$ (smooth vs corners) to decay of Fourier coefficients.

Task 2.1 (FFT experiment template)**.** Pick a shape K and produce: (i) K in physical space, (ii) plot of $h(\theta)$, (iii) log-magnitude spectrum $|c_k|$.

Test 2.1 (Rotation as phase shift in Fourier coefficients)**.** Rotate K by ϕ and verify c_k changes by factor $\mathrm{e}^{-ik\phi}$ (up to numerical error).

2.2 Notes / results

Chapter 3

Teaching Set: Circle, Ellipse, Square, Regular Polygons

3.1 Circle

Inquiry 3.1 (Why is the circle spectrally sparse?)**.** Explain why only the $k = 0$ mode remains for $h(\theta) \equiv R$.

3.2 Ellipse

Inquiry 3.2 (Anisotropy in Fourier space)**.** Explain the qualitative structure of Fourier coefficients for an ellipse and how it differs from a circle.

3.3 Square and corners

Inquiry 3.3 (Corners as high-frequency content)**.** Explain and illustrate Gibbs-like artifacts when truncating Fourier series of a square support function.

Chapter 4

Constant Width, Radial Functions, and Convex Hull

4.1 Constant width

Inquiry 4.1 (Constant width constraint)**.** Show (or verify) the identity

$$h(\theta) + h(\theta + \pi) = \text{constant}$$

for constant-width shapes and interpret in Fourier terms.

4.2 Radial vs support

Inquiry 4.2 (What does the support function forget?)**.** Compare a nonconvex star-shaped example $r(\theta) = 1 + \varepsilon \cos(k\theta)$ with its convex hull and explain which features vanish in $h(\theta)$.

Chapter 5

Toward Packing: Collision, Minkowski Sums, and Configuration Space

Inquiry 5.1 (Why support functions are good for packing)**.** Formulate collision/clearance in terms of support functions (e.g., via Minkowski difference / separating hyperplanes). Identify what becomes linear and what remains nonlinear.

Task 5.1 (Minkowski sum identity)**.** Prove $h_{A+B}(\theta) = h_A(\theta) + h_B(\theta)$ for convex compact sets A, B.

Task 5.2 (Computational check)**.** Numerically verify Minkowski-sum additivity for polygons by sampling h.

Chapter 6

Application: Christmas Tree Shape

6.1 Tree geometry

Inquiry 6.1 (Tree in support-function space)**.** Compute $h(\theta)$ for the Christmas tree polygon and its convex hull. Compare spectra and discuss what geometric information is lost under convexification.

6.2 Band-limited approximations

Inquiry 6.2 (How many Fourier modes are enough?)**.** Define an accuracy metric (Hausdorff proxy, area error, max support error) and study how approximation quality scales with truncation level K.

Appendix A

Reproducibility Ledger

A.1 Figure checklist

For each figure, record:

- Script path:
- Data / parameters:
- Random seed:
- Output file path:

A.2 Code snippets

Listing A.1: Support function sampling (placeholder)

```
# TODO: implement in your repo; include path and commit hash in ledger.
def support_function_polygon(poly, thetas):
    # return max_{v in vertices} <v, u(theta)> for each theta
    ...
```

Change Log