

# Artificial Intelligence (CS111)

## Lecture 6: Constraint Satisfaction Problems

Credit: Ansaf Salleb-Aouissi, and “Artificial Intelligence: A Modern Approach” , Stuart Russell and Peter Norvig, and “The Elements of Statistical Learning” , Trevor Hastie, Robert Tibshirani, and Jerome Friedman, and “Machine Learning” , Tom Mitchell.

# Recall

- **Search problems:**

- Find the sequence of actions that leads to the goal.
- Sequence of actions means a path in the search space.
- Paths come with different costs and depths.
- We use heuristics to guide the search efficiently.

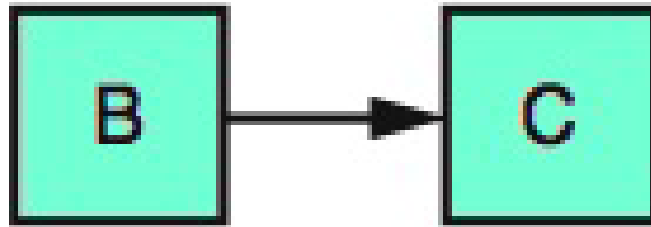
- **Constraint satisfaction problems:**

- A search problem too!
- We care about the **goal itself**.

# CSPs definition

- Search problems:

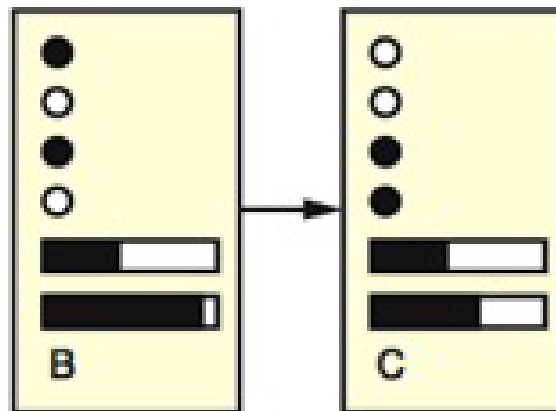
- A state is a **black box**, implemented as some data structure.  
Recall **atomic representation**.
- A goal test is a function over the states.



# CSPs definition

- CSPs problems:

- A state: defined by variables  $X_i$  with values from domain  $D_i$ . Recall **factored representation**.
- A goal test is a **set of constraints** specifying **allowable combinations** of values for subsets of variables.



# CSPs definition



# CSPs definition

- A constraint satisfaction problem consists of **three elements**:
  - A set of **variables**,  $X = \{X_1, X_2, \dots, X_n\}$
  - A set of **domains** for each variable:  $D = \{D_1, D_2, \dots, D_n\}$
  - A set of **constraints**  $C$  that specify allowable combinations of values.
- Solving the CSP: **finding the assignment(s)** that **satisfy all constraints**.
- Concepts: problem formalization, backtracking search, arc consistency, etc.
- We call a solution, a **consistent assignment**.

# Example: Map coloring



**Variables:**  $X = \{WA, NT, Q, NSW, V, SA, T\}$

**Domains:**  $D_i = \{\text{red, green, blue}\}$

**Constraints:** adjacent regions must have different colors;  
e.g.,  $WA \neq NT$  or  $(WA, NT) \in \{(\text{red, green}), (\text{red, blue}), \text{etc.}\}$

# Example: Map coloring



**Example:**

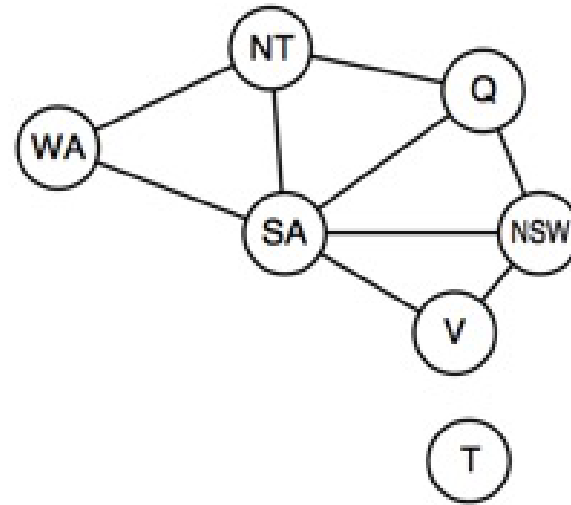
{WA=red, NT=green, Q=red, NSW=green, V=red, SA=blue, T=green}



# Real-world CSPs

- Assignment problems, e.g., who teaches what class?
- Timetabling problems, e.g., which class is offered when and where?
- Hardware configuration
- Transportation scheduling
- Factory scheduling
- Floor planning
- etc.
- Notice that many real-world problems involve real-valued variables

# Constraint graph



- **Binary CSP:** each constraint relates at most two variables  
Constraint graph: nodes are variables, arcs show constraints
- **CSP algorithms:** use the graph structure to speed up search. E.g., Tasmania is an independent sub-problem!

# Varieties of variables

- **Discrete variables:**

- Finite domains: assume  $n$  variables,  $d$  values, then the number of complete assignments is  $O(d^n)$ , e.g., map coloring, 8-queen problem
- Infinite domains (integers, strings, etc.): need to use a constraint language, e.g., job scheduling.  $T_1 + d \leq T_2$ .

- **Continuous variables:**

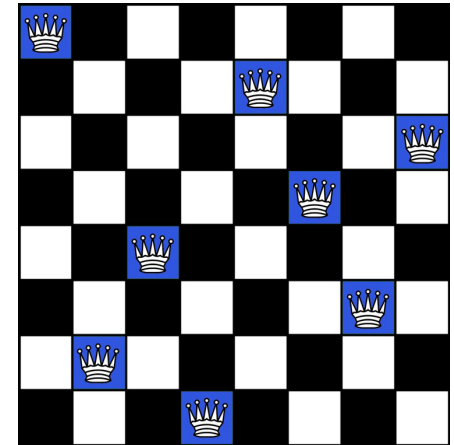
- Common in operation research
- Linear programming problems with linear or non linear equalities

# Varieties of constraints

- **Unary constraints:** involve a single variable e.g.,  $SA \neq \text{green}$
- **Binary constraints:** involve pairs of variables e.g.,  $SA \neq WA$
- **Global constraints:** involve 3 or more variables e.g., *Alldiff* that specifies that all variables must have different values (e.g., Cryptarithmic puzzles, Sudoku)
- **Preferences (soft constraints):**
  - Example: red is better than green
  - Often represented by a cost for each variable assignment
  - constrained optimization problems

# Example: 8-queen

**8-Queen:** Place 8 queens on an 8x8 chess board so no queen can attack another one.

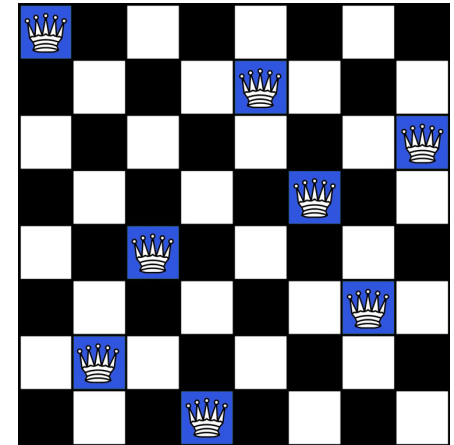


## Problem formalization 1:

- One variable per queen,  $Q_1, Q_2, \dots, Q_8$ .
- Each variable could have a value between 1 and 64.
- Solution:  $Q_1 = 1, Q_2 = 13, Q_3 = 24, \dots, Q_8 = 60$ .

# Example: 8-queen

**8-Queen:** Place 8 queens on an 8x8 chess board so no queen can attack another one.

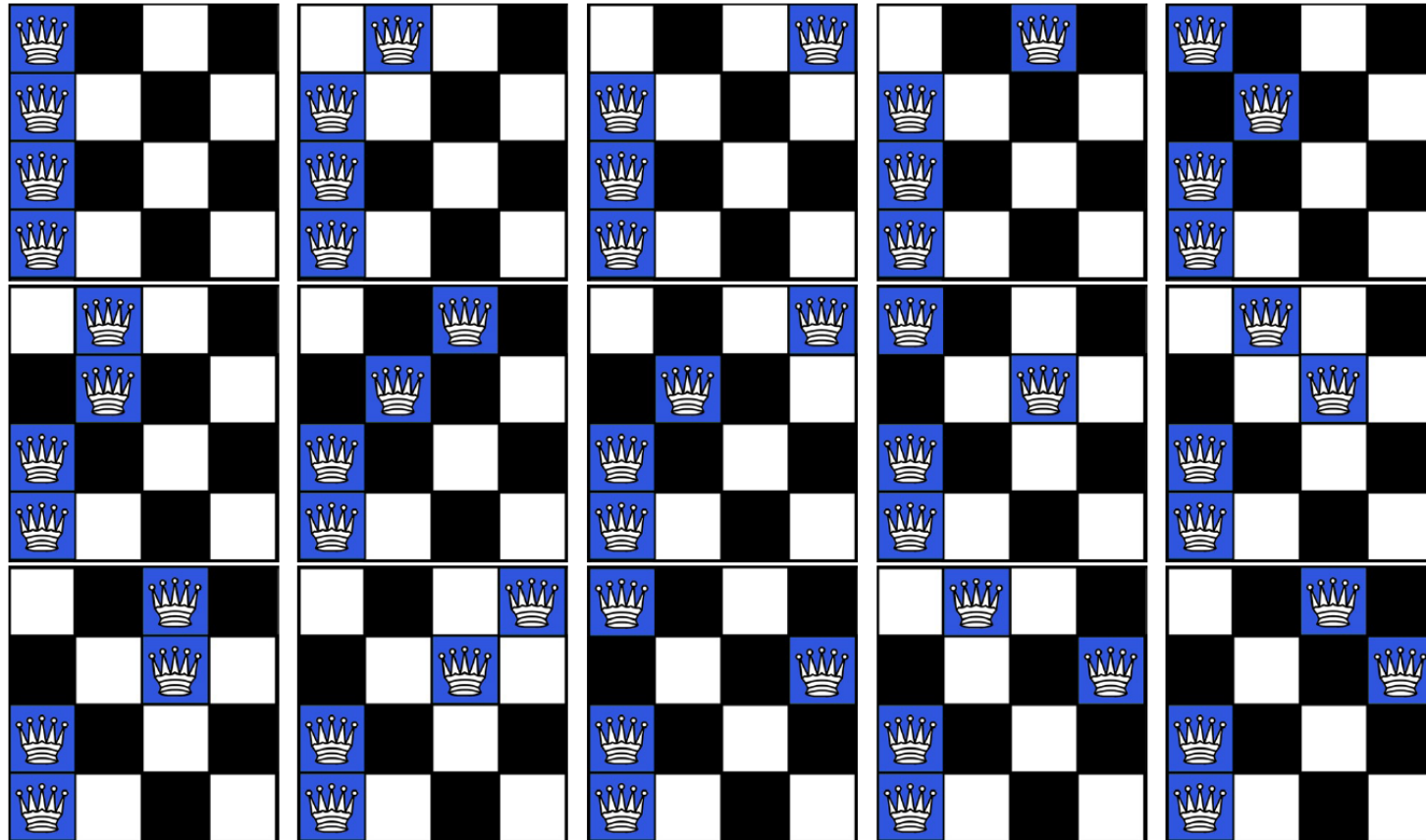


## Problem formalization 2:

- One variable per queen,  $Q_1, Q_2, \dots, Q_8$ .
- Each variable could have a value between 1 and 8 (rows).
- Solution:  $Q_1 = 1, Q_2 = 7, Q_3 = 5, \dots, Q_8 = 3$ .

# Brute force?

Should we simply generate and test all configurations?



# Example Cryptarithmic

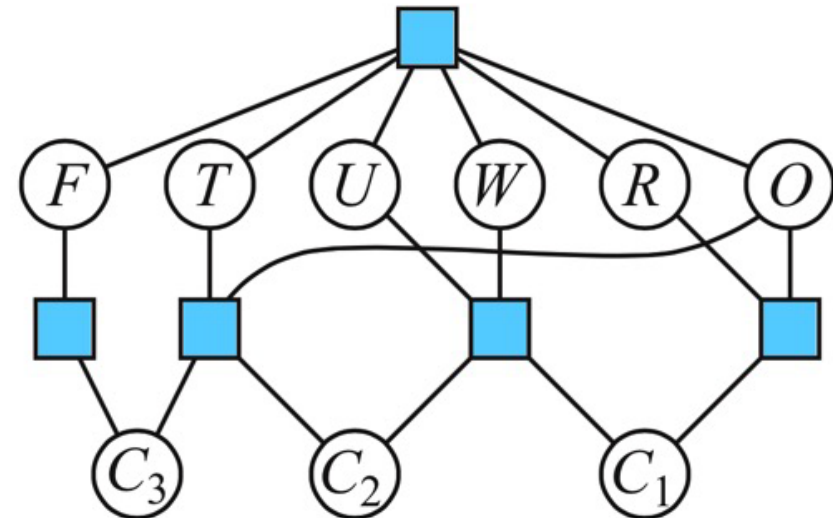
**Variables:**  $X = \{F, T, U, W, R, O, C_1, C_2, C_3\}$

**Domain:**  $D = \{0, 1, 2, \dots, 9\}$

**Constraints:**

- $\text{Alldiff}(F, T, U, W, R, O)$
- $T \neq 0, F \neq 0$
- $O + O = R + 10 * C_1$
- $C_1 + W + W = U + 10 * C_2$
- $C_2 + T + T = O + 10 * C_3$
- $C_3 = F$

$$\begin{array}{r} T \ W \ O \\ + \ T \ W \ O \\ \hline F \ O \ U \ R \end{array}$$





# Solving CSPs

**IMPORTANT**

- **State-space search algorithms:** search!
- **CSP Algorithms:** Algorithm can do two things:
  - **Search:** choose a new variable assignment from many possibilities
  - **Inference:** constraint propagation, use the constraints to spread the word: reduce the number of values for a variable which will reduce the legal values of other variables etc.
- As a preprocessing step, constraint propagation can sometimes solve the problem entirely without search.
- Constraint propagation can be intertwined with search.

# Solving CSPs

- **BFS:** Develop the complete tree
- **DFS:** Fine but time consuming
- **BTS: Backtracking search** is the basic uninformed search for CSPs. It's a DFS s.t.
  1. Assign one variable at a time (**expansion**): assignments are commutative, e.g., (WA = red, NT = green) is same as (NT = green, WA = red)
  2. Check constraints on the go: consider values that do not conflict with previous assignments.

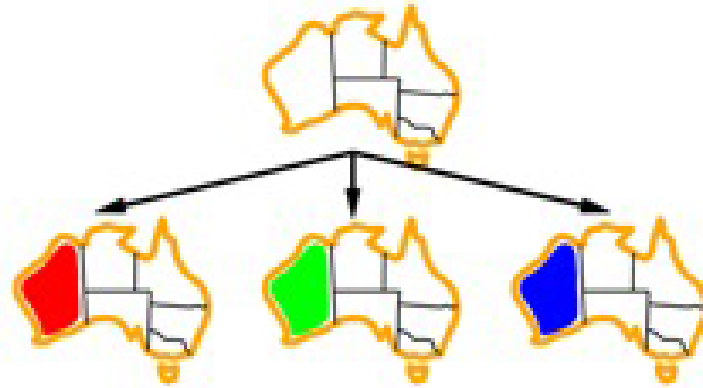
# Solving CSPs

- **Initial state:** empty assignment  $\{ \}$
- **States:** are partial assignments
- **Successor function:** assign a value to an unassigned variable
- **Goal test:** the current assignment is complete and satisfies all constraints

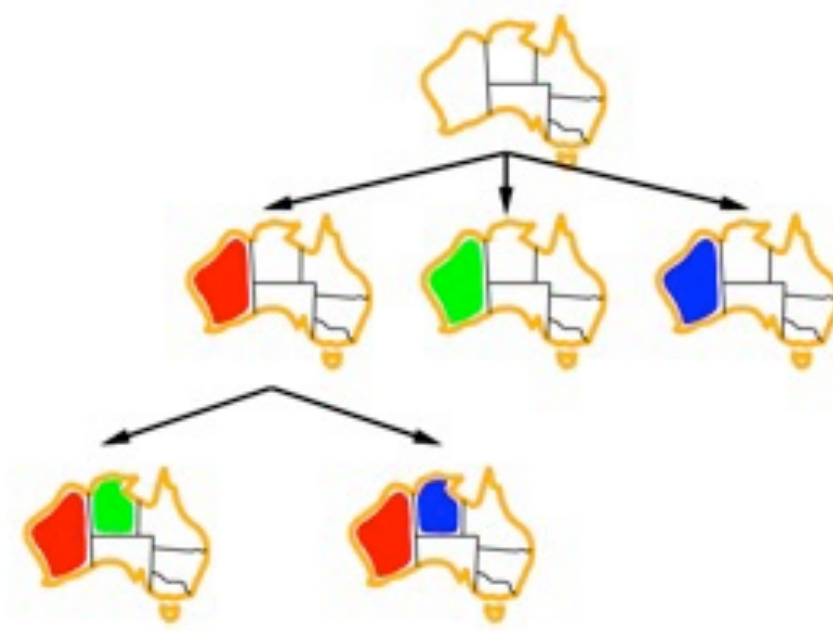
# Backtracking search



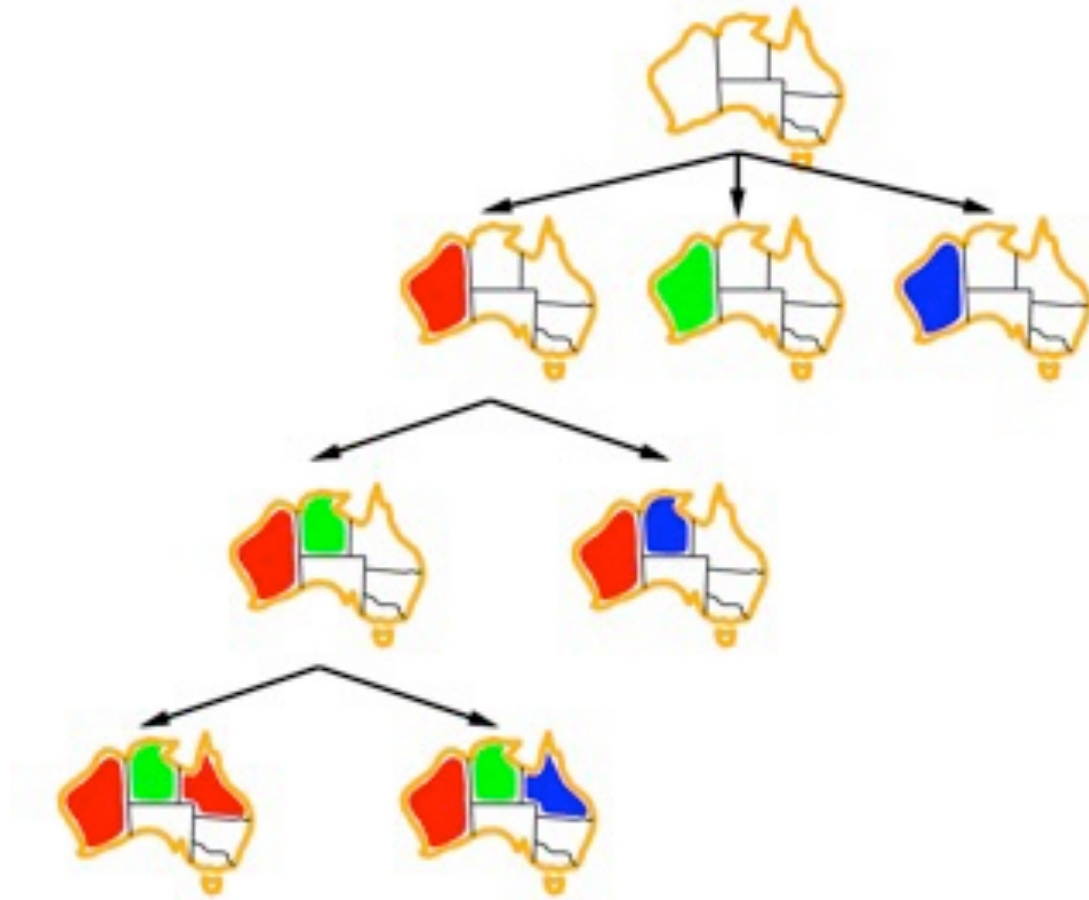
# Backtracking search



# Backtracking search



# Backtracking search



# Improving BTS

**Heuristics are back!**

1. Which variable should be assigned next?
2. In what order should its values be tried?
3. Can we detect inevitable failure early?

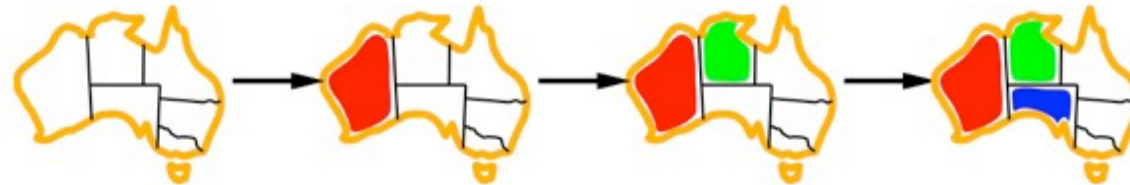


# Minimum Remaining Value

1. Which variable should be assigned next?



- **MRV:** Choose the variable with the fewest legal values in its domain



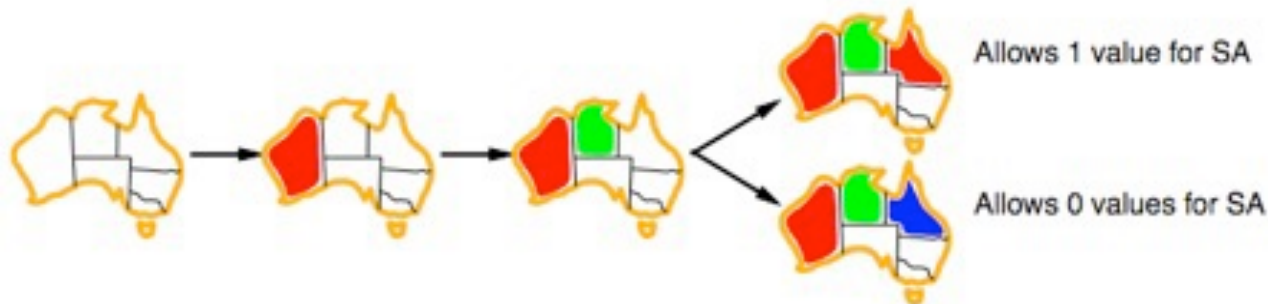
Pick the hardest!

# Least constraining value

2. In what order should its values be tried?



- **LCV:** Given a variable, choose the least constraining value: the one that rules out the fewest values in the remaining variables



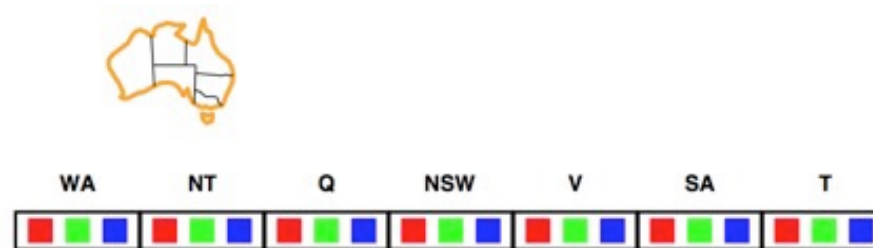
Pick the ones that are likely to work!

# Forward checking

## 3. Can we detect inevitable failure early?



- **FC:** Keep track of remaining legal values for the unassigned variables. Terminate when any variable has no legal values.



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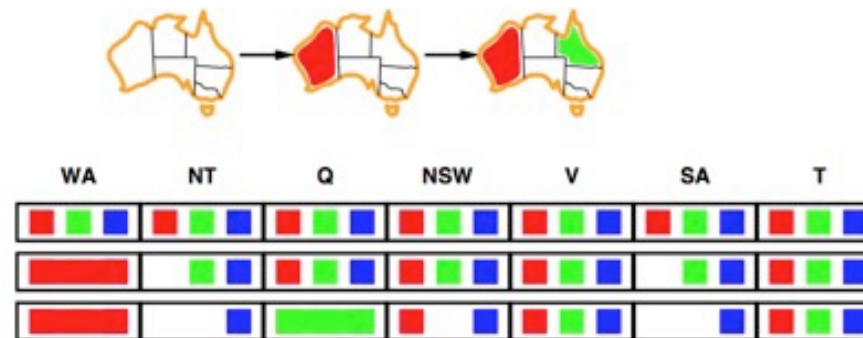


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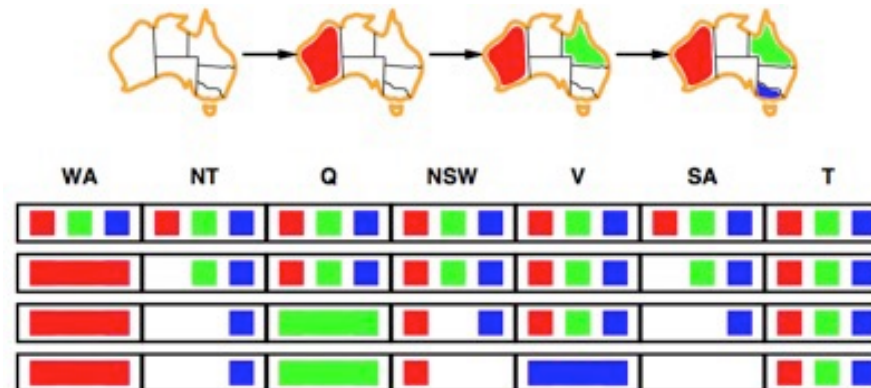


# Forward checking

## 3. Can we detect inevitable failure early?



- **FC:** Keep track of remaining legal values for the unassigned variables. Terminate when any variable has no legal values.



# Backtracking search

**function** BACKTRACKING\_SEARCH(csp) returns a solution, or failure  
    *return* BACKTRACK({}, csp)

**function** BACKTRACK(assignment, csp) returns a solution, or failure  
    **if** assignment is complete **then return** assignment  
    var = SELECT\_UNASSIGNED\_VARIABLE(csp)  
    **for each** value in ORDER\_DOMAIN\_VALUES (var, assignment, csp)  
        **if** value is consistent with assignment **then**  
            add {var = value} to assignment  
            result = BACKTRACK(assignment, csp)  
            **if** result  $\neq$  failure **then return** result  
            remove {var = value} from assignment  
    *return* failure

# Solving CSPs: Sudoku

All 3x3 boxes, rows, columns, **must contain all digits 1...9.**

**Variables:**  $V = \{A_1, \dots, A_9, B_1, \dots, B_9, \dots, I_1, \dots, I_9\}$ ,  $|V| = 81$ .

**Domain:**  $D = \{1, 2, \dots, 9\}$ , the filled squares have a single value.

**Constraints:** 27 constraints

- $\text{Alldiff}(A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9)$   
...
- $\text{Alldiff}(A_1, B_1, C_1, D_1, E_1, F_1, G_1, H_1, I_1)$   
...
- $\text{Alldiff}(A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3)$

8		9	5		1	7	3	6
2		7		6	3			
1	6							
				9		4		7
	9		3		7		2	
7		6		8				
							6	3
			9	3		5		2
5	3	2	6		4	8		9



# Solving CSPs: Sudoku

All 3x3 boxes, rows, columns, **must contain all digits 1...9.**

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2		7		6	3			
1	6							
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	9		3		7		2	
7		6		8				
							6	3
			9	3		5		2
5	3	2	6		4	8		9

- Naked doubles (triples): find two (three) cells in a 3x3 grid that have only the same candidates left, eliminate these two (three) values from all possible assignments in that box.
- Locked pair, Locked triples, etc.

# Solving CSPs: Sudoku

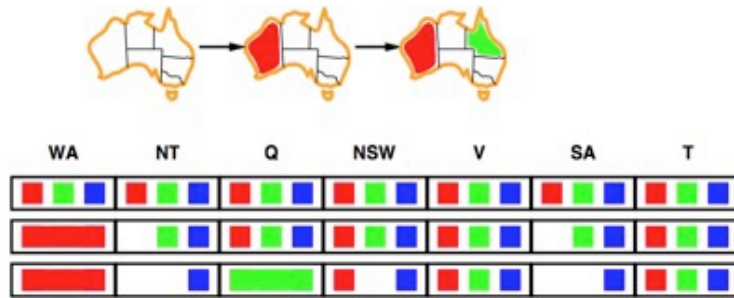
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2		7		6	3			
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8	4	9	5	2	1	7	3	6
2	5	7	8	6	3	9	1	4
1	6	3	7	4	9	2	5	8
3	2	5	1	9	6	4	8	7
4	9	8	3	5	7	6	2	1
7	1	6	4	8	2	3	9	5
9	8	4	2	7	5	1	6	3
6	7	1	9	3	8	5	4	2
5	3	2	6	1	4	8	7	9

# Constraint propagation

- Forward checking propagates information from assigned to unassigned variables.
- Observe:



Forward checking does not check interaction between unassigned variables! Here SA and NT! (They both must be blue but can't be blue!).

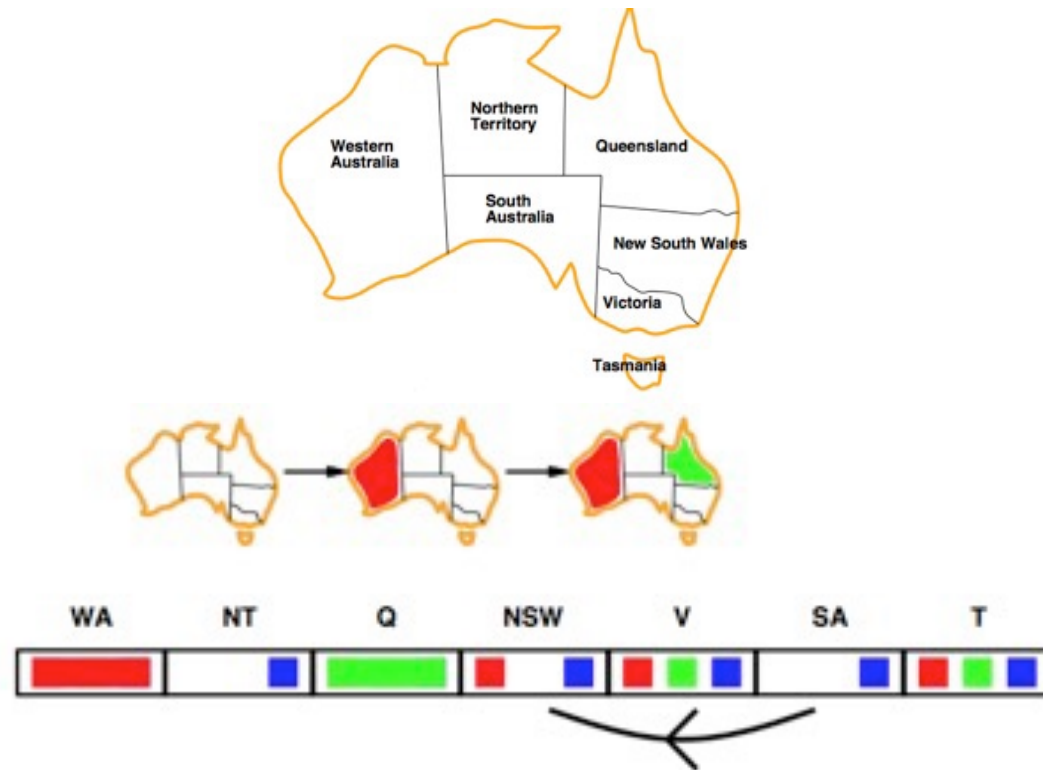
- Forward checking improves backtracking search but does not look very far in the future, hence does not detect all failures.
- We use constraint propagation, reasoning from constraint to constraint. e.g., arc consistency test.

# Types of Consistency

- **Node-consistency** (unary constraints): A variable  $X_i$  is node-consistent if all the values of  $Domain(X_i)$  satisfy all unary constraints.
- **Arc-consistency** (binary constraints):  $X \rightarrow Y$  is arc-consistent if and only if every value  $x$  of  $X$  is consistent with some value  $y$  of  $Y$ .
  - **E.g.**  $Domain(A)=\{3, 4, 5\}$ ,  $Domain(B)=\{3, 4, 5, 6\}$ ,  $A < B$ .
- **Path-consistency and  $k$ -consistency** (n-ary constraints): generalizes arc-consistency from binary to multiple constraints.
- **Note:** It is always possible to transform all n-ary constraints into binary constraints. Often, CSPs solvers are designed to work with binary constraints.

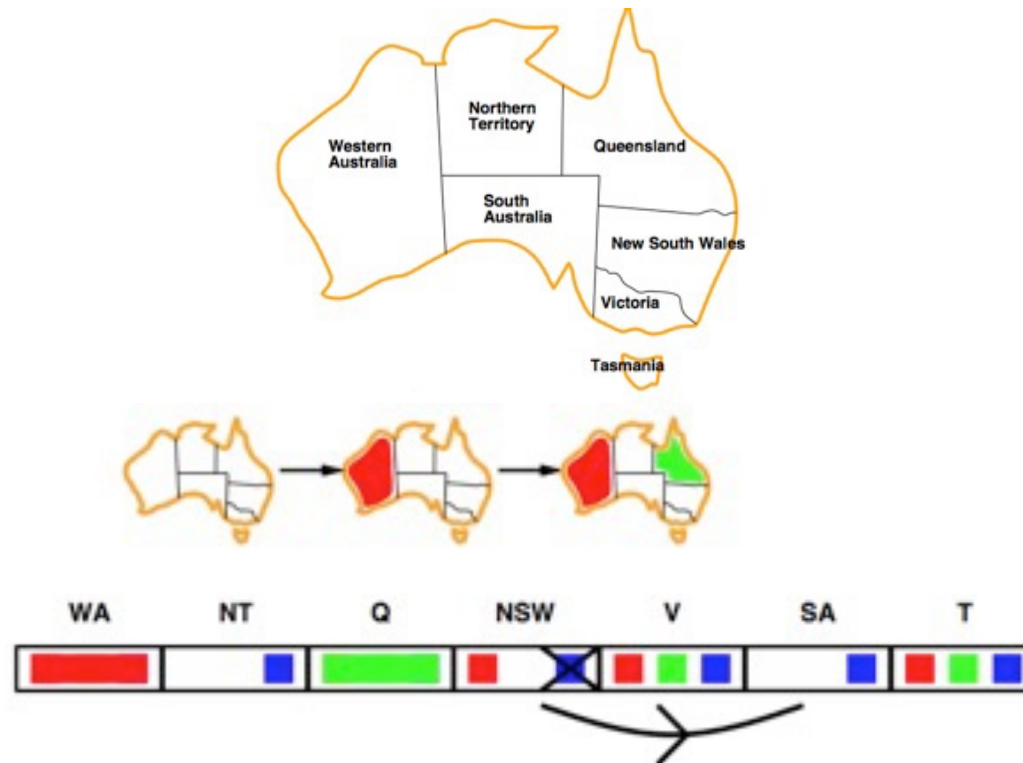
# Arc consistency

- **AC:** Simplest form of propagation makes each arc consistent.
- $X \rightarrow Y$  is consistent IFF for every value  $x$  of  $X$ , there is some allowed  $y$ .



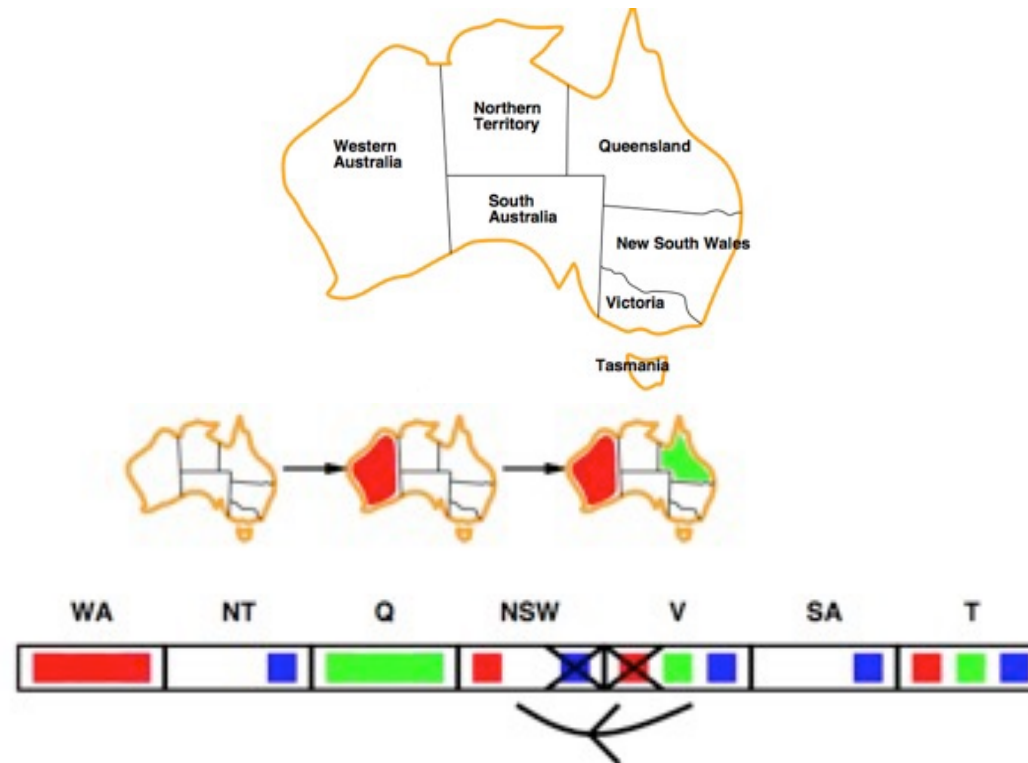
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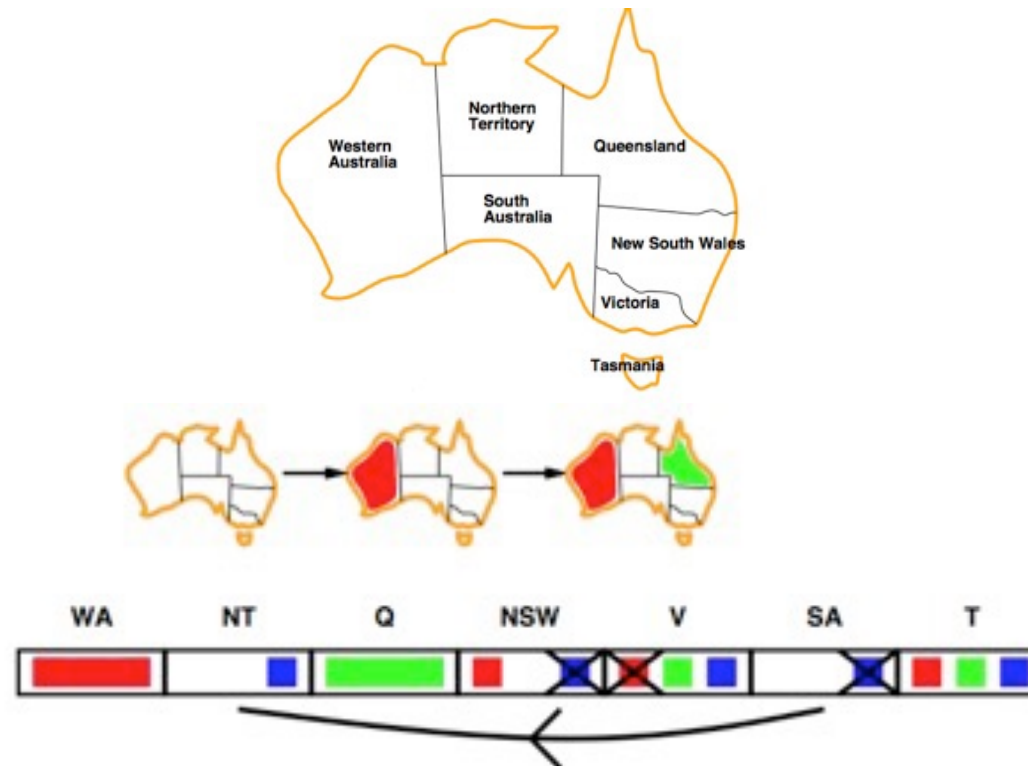
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# Arc consistency

## Algorithm that makes a CSP arc-consistent!

```
function AC-3( csp)
returns False if an inconsistency is found, True otherwise
inputs: csp, a binary CSP with components (X, D, C)
local variables: queue, a queue of arcs, initially all the arcs in csp
while queue is not empty do
     $(X_i, X_j) = \text{REMOVE-FIRST}(\text{queue})$ 
    if REVISE(csp,  $X_i, X_j$ ) then
        if size of  $D_i = 0$  then return False
        for each  $X_k$  in  $X_i.\text{NEIGHBORS} - \{X_j\}$  do
            add  $(X_k, X_i)$  to queue
return true

function REVISE( csp,  $X_i, X_j$ )
returns True iff we revise the domain of  $X_i$ 
revised = False
for each  $x$  in  $D_i$  do
    if no value  $y$  in  $D_j$  allows  $(x, y)$  to satisfy the constraint between  $X_i$  and  $X_j$  then
        delete  $x$  from  $D_i$ 
        revised = True
return revised
```

# Complexity of AC-3

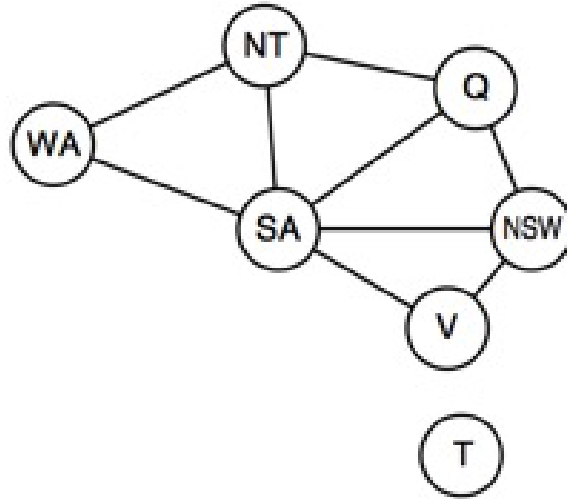
- Let  $n$  be the number of variables, and  $d$  be the domain size.
- If every node (variable) is connected to the rest of the variables, then we have  $n * (n - 1)$  arcs (constraints)  $\rightarrow O(n^2)$
- Each arc can be inserted in the queue  $d$  times  $\rightarrow O(d)$
- Checking the consistency of an arc costs  $\rightarrow O(d^2)$ .
- Overall complexity is  $O(n^2 d^3)$ .

# Backtracking w/inference

**function** BACKTRACKING-SEARCH(*csp*) **returns** a solution or *failure*  
    **return** BACKTRACK(*csp*, { })

**function** BACKTRACK(*csp*, *assignment*) **returns** a solution or *failure*  
    **if** *assignment* is complete **then return** *assignment*  
    *var*  $\leftarrow$  SELECT-UNASSIGNED-VARIABLE(*csp*, *assignment*)  
    **for each** *value* **in** ORDER-DOMAIN-VALUES(*csp*, *var*, *assignment*) **do**  
        **if** *value* is consistent with *assignment* **then**  
            add { *var* = *value* } to *assignment*  
            *inferences*  $\leftarrow$  INFERENCE(*csp*, *var*, *assignment*)  
            **if** *inferences*  $\neq$  *failure* **then**  
                add *inferences* to *csp*  
                *result*  $\leftarrow$  BACKTRACK(*csp*, *assignment*)  
                **if** *result*  $\neq$  *failure* **then return** *result*  
                remove *inferences* from *csp*  
            remove { *var* = *value* } from *assignment*  
    **return** *failure*

# Problem structure



- Idea: Leverage the problem structure to make the search more efficient.
- Example: Tasmania is an independent problem.
- Identify the connected component of a constraint graph.
- Work on independent sub-problems.

# Problem structure

## Complexity:

- Let  $d$  be the size of the domain and  $n$  be the number of variables.
- Time complexity for BTS is  $O(d^n)$ .
- Suppose we decompose into sub-problems, with  $c$  variables per sub-problem.
- Then we have  $n/c$  sub-problems.
- $c$  variables per sub-problem takes  $O(d^c)$ .
- The total for all sub-problems takes  $O(n/c \cdot d^c)$  in the worst case.

# Problem structure

## Example:

- Assume  $n = 80$ ,  $d = 2$ .
- Assume we can decompose into 4 sub-problems with  $c = 20$ .
- Assume processing at 10 million nodes per second.
- Without decomposition of the problem we need:

$$2^{80} = 1.2 \times 10^{24}$$

**3.83 million years!**

- With decomposition of the problem we need:

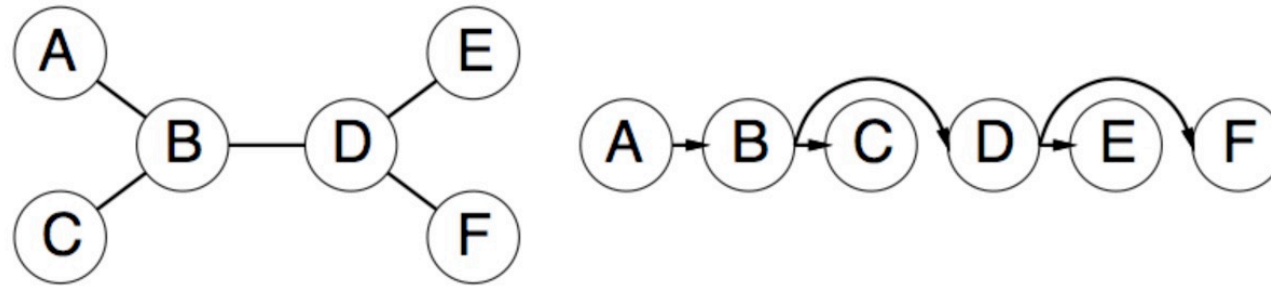
$$4 \times 2^{20} = 4.2 \times 10^6$$

**0.4 seconds!**

# Problem structure

- Turning a problem into independent sub-problems is not always possible.
- Can we leverage other graph structures?
- Yes, if the graph is tree-structured or nearly tree-structured.
- A graph is a **tree** if any two variables are connected by **only one path**.
- Idea: use DAC, Directed Arc Consistency
- A CSP is said to be **directed arc-consistent** under an ordering  $X_1, X_2, \dots, X_n$  IFF every  $X_i$  is arc-consistent with each  $X_j$  for  $j > i$ .

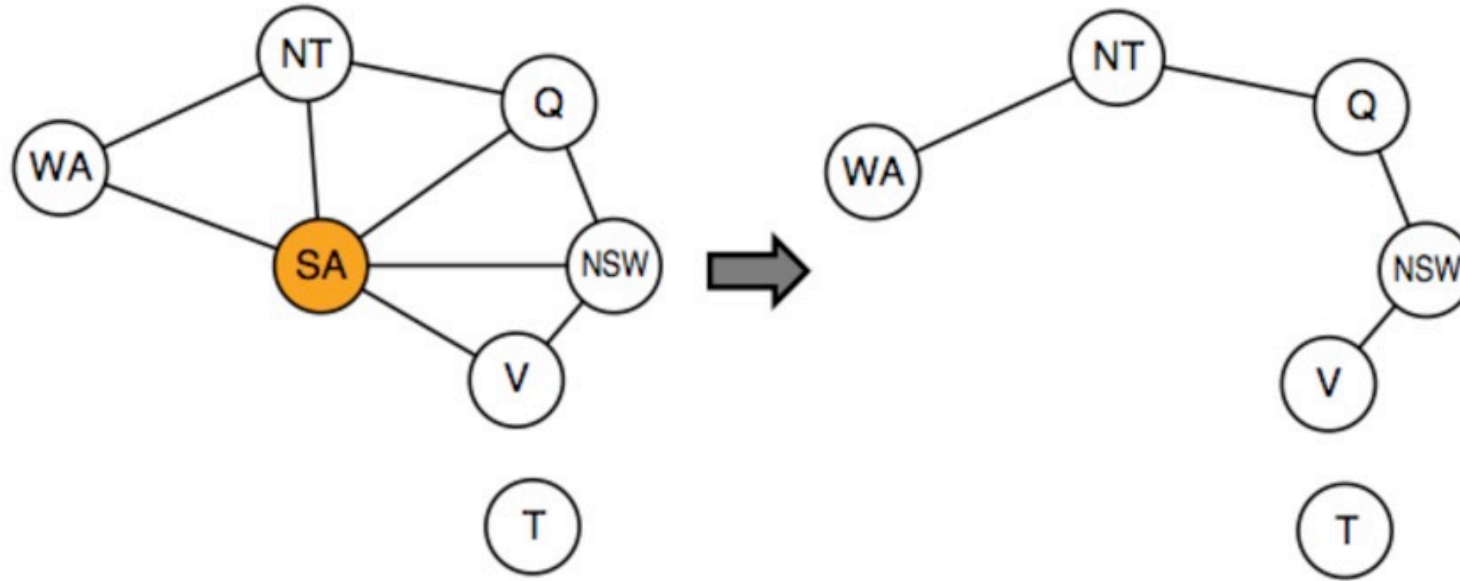
# Problem structure



- First pick a variable to be the root.
- Do a **topological sorting**: choose an ordering of the variables s.t. each variable appears after its parent in the tree.
- For  $n$  nodes, we have  $n - 1$  edges.
- Make the tree directed arc-consistent takes  $O(n)$
- Each consistency check takes up to  $O(d^2)$  (compare  $d$  possible values for 2 variables).
- The CSP can be solved in  $O(nd^2)$



# Nearly tree-structured CSPs



- Assign a variable or a set of variables and prune all the neighbors domains.
- This will turn the constraint graph into a tree :)
- There are other tricks to explore, have fun!

# Summary

- CSPs are a special kind of search problems:
  - states defined by values of a fixed set of variables
  - goal test defined by constraints on variable values
- Backtracking = depth-first search with one variable assigned per node
- Variable ordering and value selection heuristics help
- Forward checking prevents assignments that guarantee later failure

# Summary

- Constraint propagation (e.g., arc consistency) is an important mechanism in CSPs.
- It does additional work to constrain values and detect inconsistencies.
- Tree-structured CSPs can be solved in linear time
- Further exploration: How can local search be used for CSPs?
- **The power of CSPs: domain-independent, that is you only need to define the problem and then use a solver that implements CSPs mechanisms.**

To be continued