Computing the Efficiency of a Wind Turbine

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Introduction

Renewable energy is a rapidly growing industry that harnesses many natural phenomena to generate electricity without the harmful effects of burning fossil fuels. Using large aerodynamic blades, wind turbines harness the kinetic energy of the wind to rotate a generator, and in turn, produce electricity. This paper will describe a computational method used to calculate the power coefficient of a wind turbine, which is the ratio of effective electric power produced by a wind turbine divided by the total wind power flowing into the turbine blades at a given wind speed. The precise question we will try to answer is: what is the electrical efficiency, measured in terms of the power coefficient, of a three-bladed horizontal axis wind turbine (HAWT) designed using a NACA 2412 airfoil?

Model

The computational method calculates the electrical efficiency of the turbine using blade element momentum (BEM) Theory. This theory equates the forces acting on the wake rotating in the air with speed ω (generated by the motion of the turbine) and the forces acting on the turbine blades as they rotate with speed Ω. The forces acting on the wake are determined from the changes in momentum of each section of the wake as it passes through the turbine. The forces acting on the turbine blades are determined by examining the lift and drag coefficients, respectively C_L and C_D, for different sections along the blade. Once the equations for the axial forces and torques have been derived and equated, they simplify into equations for a, the axial induction factor and a', the angular induction factor. As the air stream passes the turbine, the axial induction factor is the fractional decrease in wind velocity and the tangential induction factor is the fractional increase in wind rotational velocity. The values for a and a' are then found iteratively and used to calculate the torque acting on each section of the turbine. Once the sum of torques is known, it is possible to calculate the power coefficient of a given wind turbine operating in known wind conditions.

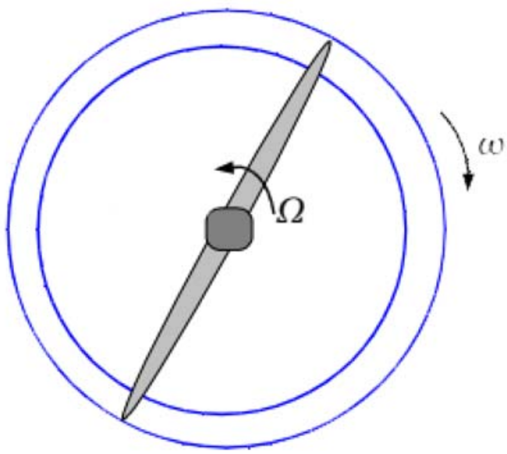

Figure 1: Notation for the rotating turbine blades and for a section of the rotating wake (Ingram, 2011)

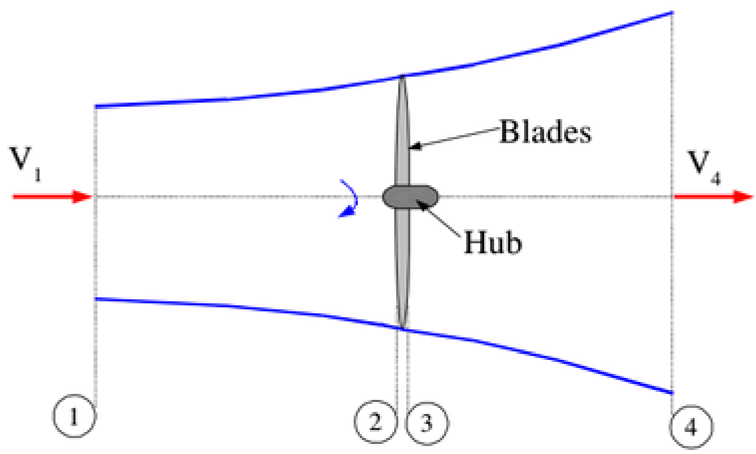

Figure 2: Axial stream tube around a wind turbine (Ingram, 2011)

Momentum theory describes a mathematical model for an ideal actuator disk, which creates a frictionless flow around the turbine, and connects power, blade radius, torque and induced velocity. Assuming that the air pressures far in front, p_1, and far behind the wind turbine, p_4, are the same, that the wind velocity immediately before, V_2, and immediately after the turbine, V_3, are equivalent, and that the flow is frictionless, one can use Bernoulli's equation and some algebra to arrive at the following equations:

$$p_2 - p_3 = \tfrac{1}{2}\rho(V_1^2 - V_4^2) \tag{1}$$

We define a, the axial induction as:

$$a = \frac{V_1 - V_2}{V_1} \tag{2}$$

$$V_2 = V_1(1 - a) \tag{3}$$

$$V_4 = V_1(1 - 2a) \tag{4}$$

Considering that force is pressure times area:

$$dFx = (p_2 - p_3)dA = \tfrac{1}{2}\rho(V_1^{\ 2} - V_4^{\ 2})dA \tag{5}$$

one arrives at the axial force on an annular element of fluid:

$$dFx = \tfrac{1}{2}\rho V_1^2 [4a(1 - a)]\, 2\pi r dr = \rho V_1^2 [4a(1 - a)]\, \pi r dr \tag{6}$$

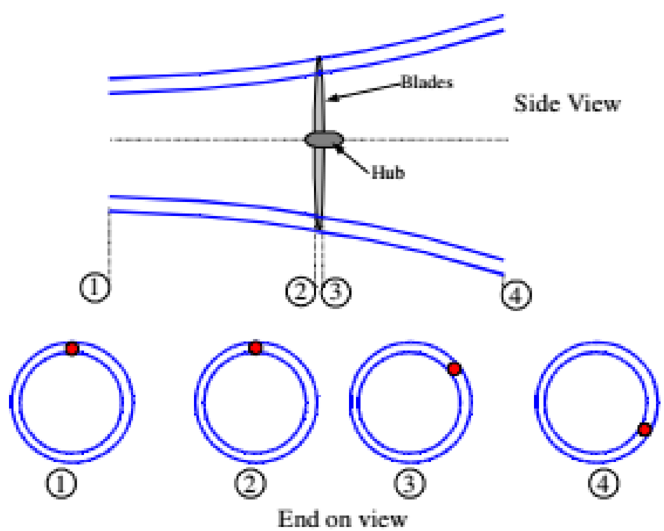

Figure 3: Rotating annular stream tube (Ingram, 2011)

The torque acting on the wake produced by the blades can be determined using conservation of momentum in a rotating annular stream tube. The wake of the blades rotates with an angular velocity ω and the blades rotate with an angular velocity Ω. From mechanics one can derive the torque for a small blade element:

$$dT = d\dot{m}\omega r^2 \tag{7}$$

Where the mass flow $d\dot{m}$ equals:

$$d\dot{m} = \rho A V_2 = \rho 2\pi r dr V_2 \tag{8}$$

$$\Rightarrow dT = \rho 2\pi r dr V_2 \omega r^2 = \rho V_2 \omega r^2 2\pi r dr \tag{9}$$

We then define the angular induction factor a':

$$a' = \frac{\omega}{2\Omega} \tag{10}$$

Recall that:

$$V_2 = V_1(1 - a) = V_0(1 - a) \tag{11}$$

Therefore the torque on an annular element of fluid is:

$$dT = 4a'(1 - a)\rho V_0 \Omega r^3 \pi dr \tag{12}$$

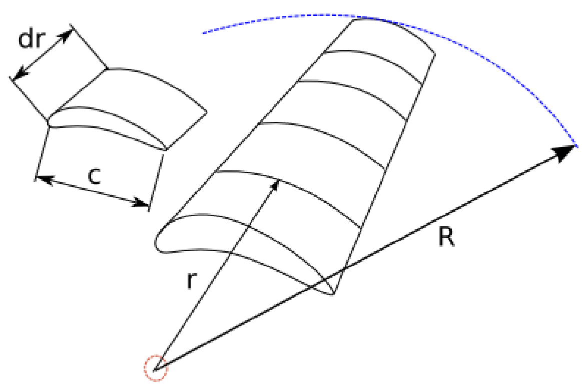

Figure 4: The blade element model (Ingram, 2011)

Blade element theory models the blades as a composite of N elements. For our model, 60 elements will be used. It then describes the axial forces and torques acting on the turbine blades, with two key assumptions. The first is that there are no aerodynamic effects between the different blade elements, and the second is that the forces acting on the blades are solely dependent on the lift and drag coefficients. The axial forces and torques depend on the flow conditions around each blade segment. An average of inlet and exit flow conditions are used because they vary for each segment, and change as the wind passes through the blades. The lift and drag coefficients for various airfoils are available from wind tunnel data, and they are used to calculate the axial force and torque acting on each blade element using the following equations:

$$dFx = \rho\sigma'\pi r C_t \left(\frac{V_0^2(1-a)^2}{sin^2\varphi}\right) dr \tag{13}$$

$$dT = \rho\sigma'\pi r^2 C_n \left(\frac{V_0^2(1-a)^2}{sin^2\varphi}\right) dr \tag{14}$$

Where the relative flow angle φ is defined as:

$$\varphi = arctan\left(\frac{1-a}{(1+a')\lambda_r}\right) \tag{15}$$

The relative flow angle φ is the relative angle of the wind as it hits the blades.

The local tip speed ratio λ_r is:

$$\lambda_r = \frac{\Omega r}{V_0} \tag{16}$$

The coefficients C_n and C_t are used to simplify the calculations and are defined as:

$$C_n = C_L cos\varphi + C_D sin\varphi \tag{17}$$

$$C_t = C_L sin\varphi - C_D cos\varphi \tag{18}$$

Finally the local solidity σ' is defined as:

$$\sigma' = \frac{Bc}{2\pi r} \tag{19}$$

where B is the number of blades and c is the length of the chord, which is the line connecting the leading and trailing edges of the blade element.

In our model, the chord length varies according to the linear relation:

$$c(r) = \frac{R}{15} - \frac{3r}{50} \tag{20}$$

where R is the radius of each blade and r is the current radial position on the blade. In our model, we are assuming $B = 3$ and that the blades are identical.

The pitch angle, or twist of the blade varies according to the logarithmic relation:

$$\theta_p(r) = 0.02\ \frac{3}{R} log_{0.3} r \tag{21}$$

We chose this distribution for θ_p as it accurately and succinctly represents the conventional blade design where the initial twist of the blade is large, and then tapers to a flat tip.

We then define the angle of attack, α, to be the angle between the position of the relative wind and the chord line as well as the local pitch angle, θ_p, to be the angle between the reference plane and the chord line. The geometry of the airfoil at a single cross section of the blade is illustrated in Figure 5.

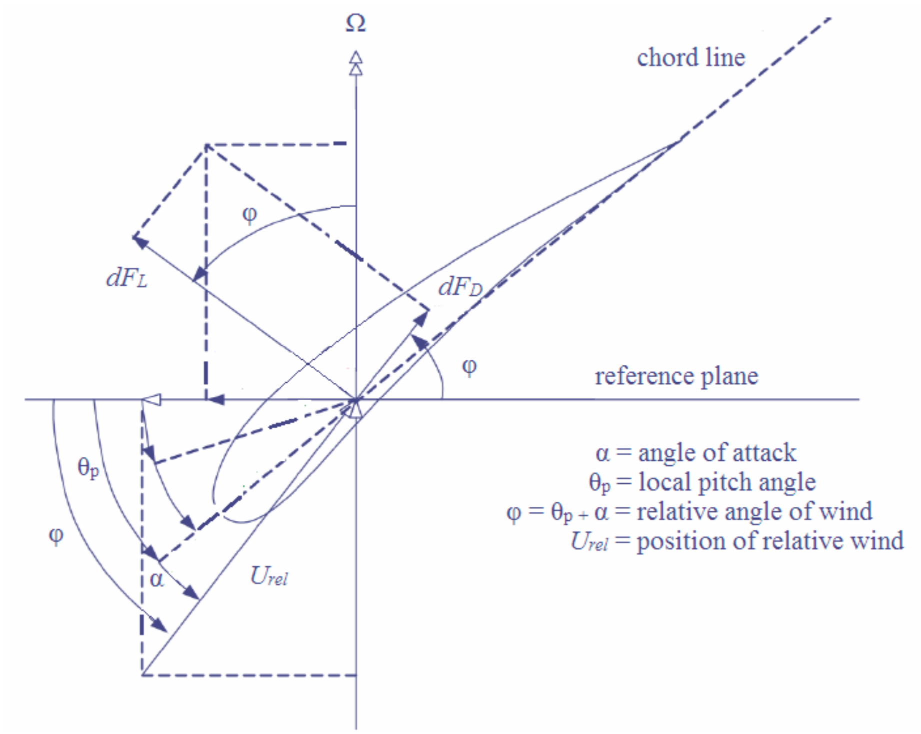

Figure 5: The angle definitions for each blade element (Masson, 2015)

Tip Loss Correction

Realistically, the flow of air past the tips of the turbine blades is reduced as the wind seeks the path of least resistance. It moves off the end of the blades and provides much less axial force and torque on the furthest segments. The adjustments for tip loss are determined by actuator models and accomplished by including the tip loss correction factor, F, to the equations for momentum theory. F describes the loss of forces along the length of the blade and will vary from 0 rad to 1 rad :

$$F = \frac{2}{\pi} arccos(e^{-f}), \qquad f = \frac{B(R-r)}{2r\, sin\varphi} \tag{21}$$

Momentum theory is also only an accurate representation of the wake forces for smaller axial induction factors. When a is larger than 0.4, the wake becomes significantly more complicated to model, and as a result, an additional empirical correction, $\tilde{F}$, must be included in the axial force and torque equations:

$$\tilde{F} = F \bullet max\{1, \frac{1 - \frac{a}{4}(5-3a)}{1-a} \} \tag{22}$$

The force and torque equations now become:

$$dFx = \tilde{F}\rho V_0^2 [4a(1 - a)]\pi r dr$$

$$dT = \tilde{F} 4a' (1 - a)\rho V \Omega r^3 \pi dr$$

Blade Element Momentum Equations

Momentum theory has given us the axial force and torque in terms of flow parameters, and blade element theory has given us the axial force and torque in terms of the lift and drag coefficients. When these two sets of equations are equated, they simplify into the two equations that allow for the iterative calculation of a and a'' at each blade element:

$$\frac{a}{1-a} = \frac{\sigma'[C_L sin\varphi - C_D cos\varphi]}{4\tilde{F} sin^2\varphi} \tag{23}$$

$$\frac{a'}{1-a} = \frac{\sigma'[C_L cos\varphi + C_D sin\varphi]}{4\tilde{F}\lambda_r sin^2\varphi} \tag{24}$$

Power Output and Power Coefficient

Once the values for a and a' have been determined for each set of blade segments, the contribution of each annulus, or ring, to the total power can be calculated:

Each annulus contributes:

$$dP = \Omega dT \tag{25}$$

Where r_h is the hub radius, the total power from the rotor is:

$$P = \int_{r_h}^{R} dP\,dr = \int_{r_h}^{R} \Omega dT\,dr \tag{26}$$

The power coefficient, C_P, is equal to:

$$C_P = \frac{P}{P_{wind}} = \frac{\int_{r_h}^{R} \Omega dT\,dr}{\frac{1}{2}\rho\pi R^2 V_0^3} \tag{27}$$

The C_P can also be expressed directly using the tip speed ratio:

$$C_P = \frac{8}{\lambda^2} \int_{\lambda_h}^{\lambda} \tilde{F}\lambda_r^3 a'(1-a)d\lambda_r \tag{28}$$

Computational Method and Testing

In order to calculate C_P for the given three-bladed HAWT with a radius $R = 3.0\ m/s$ and a maximal tip speed ratio $\lambda_R = 6$ at the tip of the blade, an iterative method will have to be used with the aforementioned equations. The algorithm for evaluating C_P is as follows:

1. Calculate the ratio $\frac{\Omega}{V_0}$ using $\frac{\Omega}{V_0} = \frac{\lambda_R}{R}$. Then set dr, the increment for the radial position, to $dr = 0.05m$. This will give 60 sections. Calculate $d\lambda_r$, the

increment in tip speed ratio, using $d\lambda_r = dr\frac{\Omega}{V_0}$ Set the current radial position to $r = dr$.

2. Calculate the local tip speed ratio λ_r, using $\lambda_r = r\frac{\Omega}{V_0}$. Calculate σ' for the current segment using equations (19) and (20). Set a to $a = \frac{1}{3}$ and a' to $a' = 0$.

3. Calculate φ using equation (15) and then calculate θ_p using equation (21). Calculate α using $\alpha = \varphi - \theta_p$. Look up $C_L(\alpha)$ and $C_D(\alpha)$ using a predefined method which will return the coefficients as piecewise functions of α. These piecewise functions are curve-fitted representations of the extended polars for the xFoil predicted data for the C_L and C_D of the NACA 2412 airfoil design with a Reynold's number of 200,000 (NACA 2412 (naca2412-il) Xfoil prediction polar, 2016). In order to obtain the extended polars for an alpha range of $-\pi$ to π radians, we applied the Viterna method.

4. Calculate C_n using equation (17) and C_t using equation (18). Then calculate the tip loss correction factor $\tilde{F}$ using equations (21) and (22). If the current segment is at the tip of the blade, then set $\tilde{F}$ and F to $\tilde{F} = F = 1$. Calculate a using $a = \frac{1}{\frac{4\tilde{F}sin^2\varphi}{C_n\sigma'}+1}$ and a' using $a' = \frac{1}{\frac{4F sin\varphi cos\varphi}{C_t\sigma'}-1}$.

5. Repeat steps 2 to 3 until a and a' converge to their ideal values. The criterion for convergence will be such that $|a_{n+1} - a_n| \leq 10^{-15}$ and $|a'_{n+1} - a'_n| \leq 10^{-15}$.
6. Calculate the current segment's contribution to the total power coefficient, C_P, using $dC_P = \frac{8}{\lambda^2} \tilde{F}\lambda_r^3 a'(1 - a)d\lambda_r$, and add this to the running sum for C_P.
7. Repeat steps 2 to 5, taking sections at each new increment of r by dr, with the last segment being at $r = R$.

The above algorithm was implemented in the Java programming language. The wind turbine program consists of a package containing three classes. The main class contains the method, `getPowerCoefficient`, for calculating the power coefficient using the above algorithm, as well as helper methods for calculating position dependent quantities, such as the chord length, c and the local pitch angle, θ_p. This class is used for creating `WindTurbine` objects. In the second class, `TurbineSimulation`,a `WindTurbine` object is instantiated and then has its `getPowerCoefficient` method called upon it. The returned value is then displayed, along with any intermediary values such as a and a'. Optionally, the `printLambdaVsCp` method may be called from the method `main`, which will print the λ vs. C_P curve for a given wind turbine radius, number of blades, tip speed ratio incrementor as well as lower and upper bounds for the tip speed ratio. Within this method, a loop continuously creates `WindTurbine` objects by looping through the tip speed ratio values provided by the lower and upper bounds and the incrementor. The `getPowerCoefficient` method is called upon the created objects, and then the returned power coefficients are stored in an array. A matching array is created to store the corresponding tip speed ratio value for each of these power coefficients. The graph of the power coefficient as a function of tip speed ratio is then generated using the given arrays and the `JMathPlot` library by Yann Richet (Richet, 2016). The third class, `Coefficients`, holds methods for calculating C_L and C_D as functions of α. These methods contain the curve-fitted piecewise functions generated using MATLAB for the Xfoil prediction polars as previously described.

In regards to testing, thorough debugging was carried out in the Eclipse IDE, which was used to develop the software. Specifically, we had set breakpoints at critical points in the code, such as at assignment statements for calculating a and a' in order to test the behaviour of the program. For example, we tested to see whether a and a' were converging properly. Initially we saw that the difference threshold could not be less than $\approx 2.7 \times 10^{-3}$ in order for the convergence loop to terminate. Then we realized that the curve-fitted representations we were using for C_L and C_D had discontinuities which was causing oscillations between boundary values of the different pieces of the function, thereby preventing successful convergence. We then re-fitted the curves to eliminate the discontinuities, and in turn, we were able to get the loop to converge within a difference threshold as small as 10^{-15}.

Results and Conclusion

For a three-bladed HAWT with a radius $R = 3.0\ m/s$ and a tip speed ratio $\lambda_R = 6$, we determined the power coefficient to be 34.04%. This computed value is reasonable, as a modern three-bladed wind turbine has a power coefficient of around 40% (Ingram, 2011). Moreover, the geometry of the blades we used to represent our blades is a highly simplified version of standard blade turbine design, which usually consists of sinusoidal distributions of chord length and twist as functions of angle of attack, radial position and local tip speed ratio (Ingram, 2011). This thereby explains why our turbine model produces a lower power coefficient than a typical modern wind turbine based on standard blade design. Figure 6 shows the power coefficient of the wind turbine as a function of tip speed ratio.

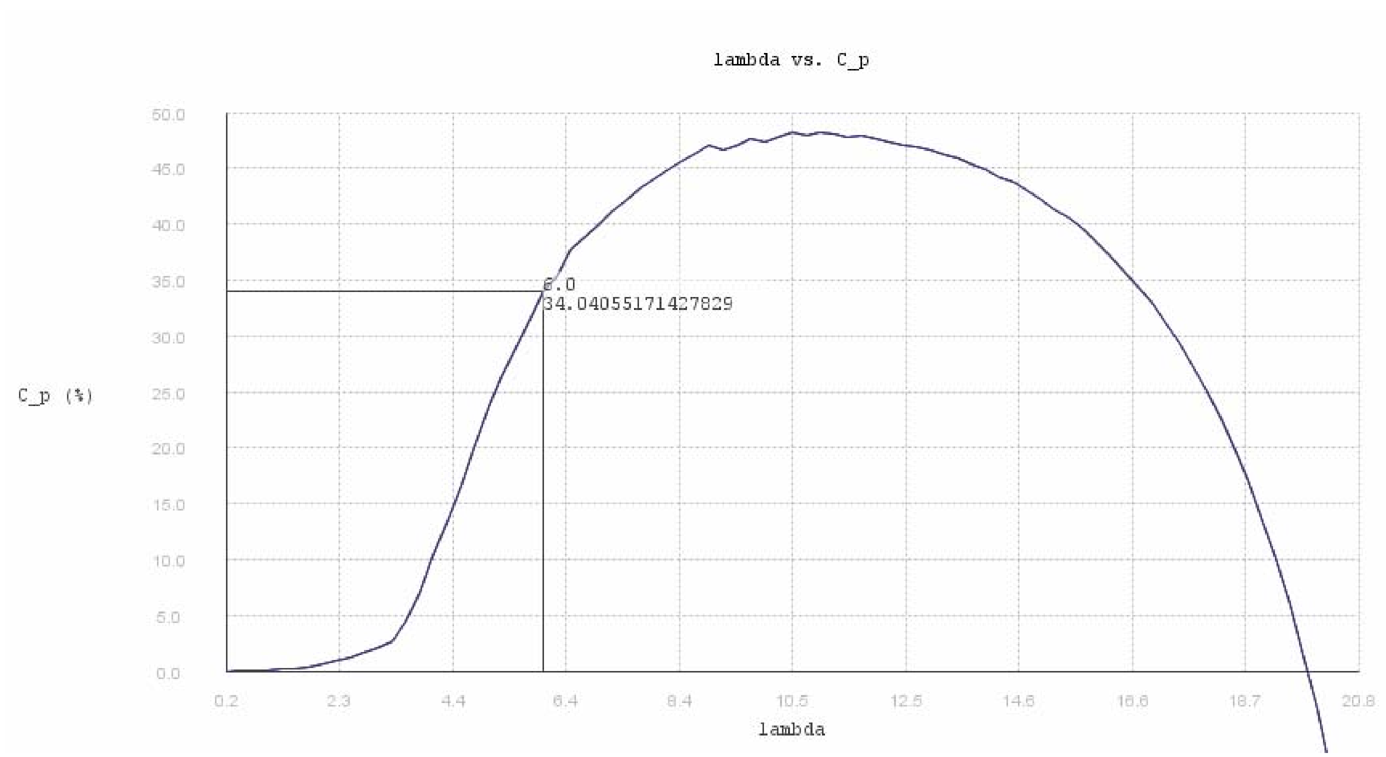

Figure 6: The power coefficient of the turbine as a function of tip speed ratio.

As it can be seen, the curve reaches a maximum power coefficient of about 48.25% at a tip speed ratio of 11.1 before falling to an efficiency of 0% at a tip speed ratio of about 19.8. These experimental results agree strongly with the results obtained from blade element momentum theory for conventional wind turbine blade design as can be seen from figure 7, representing the power coefficient vs. tip speed ratio curve for a three-bladed turbine.

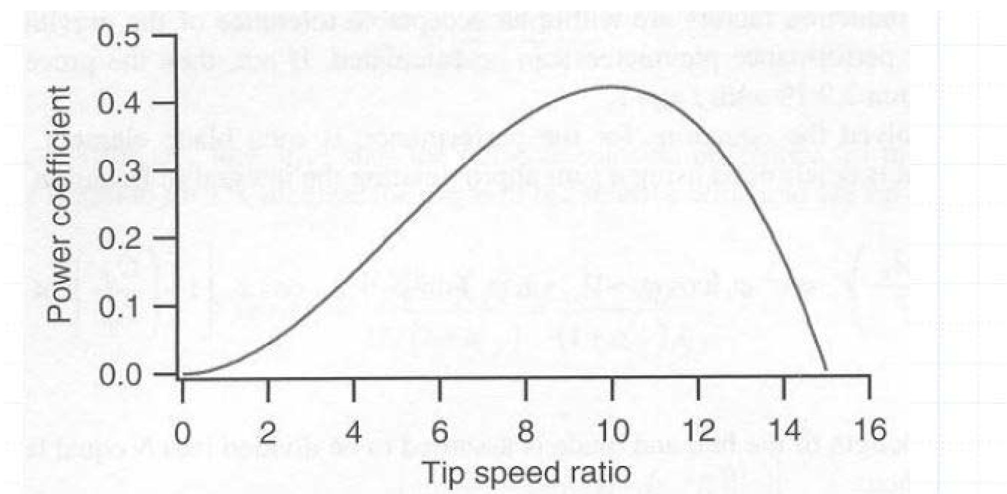

Figure 7: The power coefficient vs. tip speed ratio curve of a three-bladed wind turbine based on conventional blade design (Masson, 2015)

From the above figure obtained from the literature, it can be seen that the curve is initially sigmoidal and reaches a peak power coefficient of about 42% at a tip speed ratio of about 10, before falling down to an efficiency of 0% at a tip speed ratio of about 15. Although the maximum power coefficient and maximum tip speed ratio of the literature results differ by about 8% and 4.8 respectively from our obtained results, our power coefficient vs. tip speed ratio curve accurately reflects the shape and critical points of the literature curve. The difference in the maximum power coefficient and maximum tip speed ratio can be explained by a variety of factors such as the utilization of a different airfoil corresponding to different polars of coefficients of lift and drag as well as different chord and twist distributions. It can also be seen that the literature curve appears to be less steep initially than our power coefficient curve, which would be explained by the variation in blade geometry and more importantly, the C_D and C_L curves used in the model. Figure 8 shows the curve-fitted representations we used in our model, and Figure 9 shows the curves used in the literature model. The literature curves correspond to the curves in our model for an angle of attack between 0 rad and 0.7 rad.

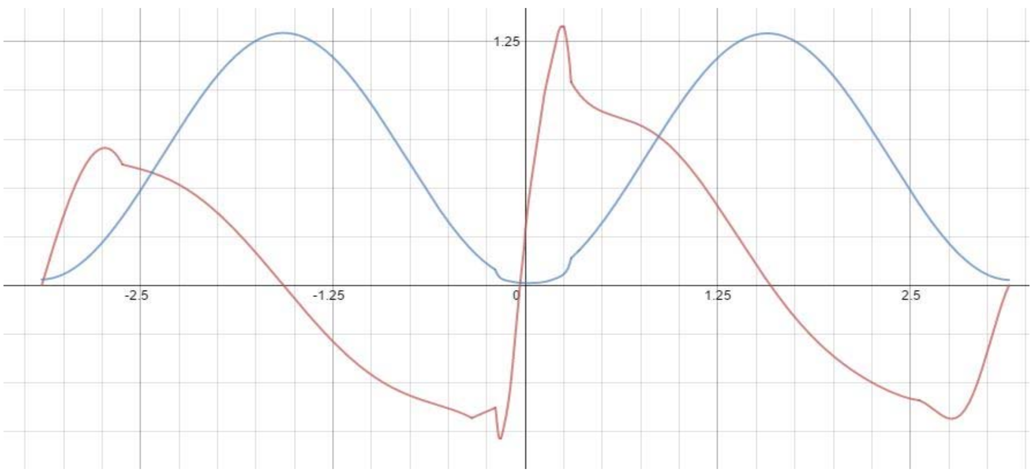

Figure 8: The curve-fitted representations of the C_L (red) and C_D (blue) Xfoil data as a function of α

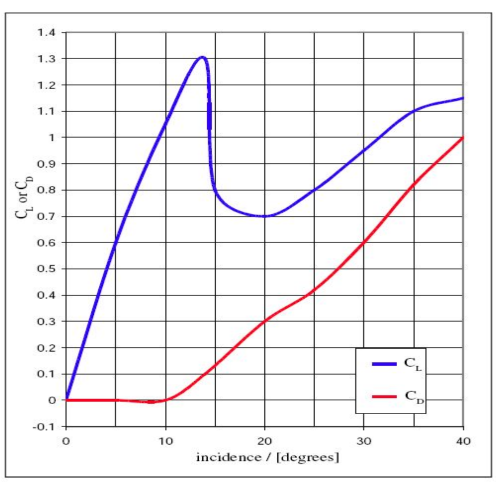

Figure 8: The C_L (blue) and C_D (red) curves as a function of α for the literature model (Ingram, 2011)

In comparing the two figures, it can be seen that our C_L curve is quite different from that of the literature model, thereby explaining how even a slight change in aerodynamic properties can greatly influence the electrical efficiency and output of a wind turbine. Ultimately, our results show that through blade element momentum theory, we were able to develop and implement a computational method in order to accurately compute the efficiency of a wind turbine of any given radius, number of blades and tip speed ratio.

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