Applications in Scientific Computing
Assignment 4: Fast Fourier Transforms

530.390.13

Due: Wednesday 13 January 2016

Submit all code by committing it to the directory `assignments/assignment4` in your `530.390.13` GitHub repository. For a reminder of how to use Git, refer to the repository file `notes/using-git`.

1. We saw in class that the convolution

$$g * h \equiv \int_{-\infty}^{\infty} g(\tau)\, h(t-\tau)\; d\tau$$

is equal to the inverse Fourier transform of the product of $G(f)$ and $H(f)$, which are the Fourier transforms of $g(t)$ and $h(t)$, respectively:

$$g * h = \int_{-\infty}^{\infty} G(f)\, H(f)\, e^{-2\pi i f t}\, df.$$

In a similar way, show the following relation for the *correlation* between g and h:

$$\operatorname{corr}(g,h) \equiv \int_{-\infty}^{\infty} g(\tau+t)\, h(\tau)\; d\tau = \int_{-\infty}^{\infty} G(f)\, H(-f)\, e^{-2\pi i f t}\, df.$$

Why, in the case of purely real-valued g and h, does $H(-f) = [H(f)]^*$, where $[z]^*$ indicates complex conjugation of $z = a + ib$?

- Correlation:

$$\begin{aligned}
\operatorname{corr}(g,h) &\equiv \int_{-\infty}^{\infty} g(\tau+t)\, h(\tau)\; d\tau \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} G(f)\, e^{-2\pi i f(\tau+t)}\, df\right] h(\tau)\; d\tau \\
&= \int_{-\infty}^{\infty} G(f) \left[\int_{-\infty}^{\infty} h(\tau)\, e^{2\pi i(-f)\tau}\, d\tau\right] e^{-2\pi i f t}\, df \\
&= \int_{-\infty}^{\infty} G(f)\, H(-f)\, e^{-2\pi i f t}\, df
\end{aligned}$$

- $H(-f) = [H(f)]^*$:

$$
\begin{aligned}
H(f) &= \int_{-\infty}^{\infty} h(t)\, e^{2\pi i f t}\, dt \\
&= \int_{-\infty}^{\infty} (g_r + i g_i)\, e^{2\pi i f t}\, dt \\
&(g_i = 0) \\
&= \int_{-\infty}^{\infty} g_r e^{2\pi i f t}\, dt \\
H(-f) &= \int_{-\infty}^{\infty} g_r e^{2\pi i (-f) t}\, dt \\
&= \int_{-\infty}^{\infty} g_r e^{-2\pi i f t}\, dt \\
&= \int_{-\infty}^{\infty} g_r \left[e^{2\pi i f t}\right]^*\, dt \\
&= \left[\int_{-\infty}^{\infty} g_r e^{2\pi i f t}\, dt\right]^* \\
H(-f) &= [H(f)]^*
\end{aligned}
$$

2. Beginning with the convolution code that we constructed in class, write a function to compute the correlation.

   - See `fft.py`.

3. Using two different discretizations N of a signal of length $L = 1$, take the correlation of the following two functions:

$$
g(t) = \begin{cases} 1; & 0 \leq x \leq 0.1 \\ 0; & 0.1 < x \leq 1 \end{cases} \qquad h(t) = \begin{cases} 0; & 0 \leq x < 0.4 \\ 1; & 0.4 \leq x \leq 0.6 \\ 0; & 0.6 < x \leq 1 \end{cases} .
$$

   How must you normalize g for the maximum value of the correlation function to be equal to one? Why?

   - Divide g by $N/10$.
   - More generally, divide g by the sum of all of the components of g so that the sum of all components is equal to one and it does not add or remove energy to or from h during the correlation.