Applications in Scientific Computing
Assignment 5: Numerical integration

530.390.13

Due: Friday 15 January 2016

Submit all code by committing it to the directory `assignments/assignment5` in your `530.390.13` GitHub repository. For a reminder of how to use Git, refer to the repository file `notes/using-git`.

1. We saw in class the definition of the *scalar product*:

$$(f, g) \equiv \int_a^b f(x)\, g(x)\; dx.$$

Using the code that we constructed in class, numerically compute the following scalar products in the interval $0 \le x \le 2\pi$:

$$(\sin(x), \sin(x))$$
$$(\sin(x), \sin(2x))$$
$$(\sin(2x), \sin(8x))$$
$$(\sin(8x), \sin(8x))$$

What do you find? As we discussed in class, the what you have seen here is an example of the *orthogonality* of the sine functions, where

$$(\sin(mx), \sin(nx)) = \begin{cases} \frac{1}{2}L; & m = n \\ 0; & m \neq n \end{cases}.$$

- See the code in `integrate.py`, which generates the following output using `N=100`:
```
(sin(1*x),sin(1*x) = 3.14159265359
(sin(1*x),sin(2*x) = -1.59640445774e-17
(sin(2*x),sin(8*x) = -2.7083827352e-16
(sin(8*x),sin(8*x) = 3.14159265359
```
- Note the typo in the orthogonality rule in the original assignment. The integral should be equal to $L/2$, not 1.
- We find, to near machine precision, the numerical equivalent of the orthogonality rule.

2. In the interval $0 \le x \le 1$, integrate the following functions:

$$y = x$$
$$y = x^2$$
$$y = x^{11}$$
$$y = x^{12}$$
$$y = e^x$$

Integrate by using the trapezoidal rule with $N = \{10, 100, 1000\}$ and using Gauss-Legendre quadrature for the orders of $n = \{1, 2, 6\}$. Complete the following table and comment on the results. Do you expect the exponential to integrate exactly? Why or why not?

		Trapezoidal rule			Gauss-Legendre quadrature		
Function	Exact result	$N = 10$	$N = 100$	$N = 1000$	$n = 1$	$n = 2$	$n = 6$
$y = x$	$\frac{1}{2} = 0.5$	0.5000+	0.5000-	0.5000-	0.5	0.5	0.5
$y = x^2$	$\frac{1}{3} \approx 0.3333333333333333$	0.3354	0.3334	0.3333+	0.25	0.3333...	0.3333...
$y = x^{11}$	$\frac{1}{12} \approx 0.08333333333333333$	0.09444	0.0834	0.08333+	4.8828e-4	0.03672	0.08333...
$y = x^{12}$	$\frac{1}{13} \approx 0.07692307692307693$	0.08900	0.07703	0.07692-	2.4414e-4	0.02896	0.07692-
$y = e^x$	$e - 1 \approx 1.7182818284590451$	1.7200	1.7183-	1.7183+	1.6487	1.7179	1.718

- See the code in `integrate.py`.
- The table shows that the trapezoidal rule never integrates any of the functions precisely except for $y = x$ (the + and - at the end of a number in the table indicates that it is either slightly high or slightly low in the significant digits beyond those listed). Generally, as N grows larger for the trapezoidal rule, the number of matching significant digits increases. For the Gauss-Legendre quadrature, we see that polynomials of order $(2n - 1)$ are integrated exactly.
- We do not expect either integration scheme to integrate the exponential exactly because they are both designed to integrate polynomials, but we see that the trapezoidal rule is extremely accurate at $N = 1000$ and the Gauss-Legendre quadrature integrates the exponential to five significant digits for $n = 6$.