Finite Difference Method for PDE

Directed Reading Program (DRP)

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Presentation Outline

About PDE

Finite Difference Method (FDM)

Elliptic Equation and FDM

Programming and Results

Wrap Up

Partial Differential Equation

Example PDE Equation

$$a(t,x)U_t + b(t,x)U_x + c(t,x)U_{xt} = f(t,x)$$

- t,x are independent variables (in time and space)
- a,b,c are functions
- U denotes the partial derivatives

Solving a PDE

- Finding the unknown function U which satisfies the PDE
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- Most PDEs of interest cannot be solved analytically
- Numerical procedure is used to find the approximate solution

Finite Difference Method(FDM)

What is the Finite Difference Method?

- Fundamentally based on the Taylor's theorem
- FDM replaces region in which independent variables are defined by a finite grid of points
- Replaces all partial derivatives and other terms in the PDE by approximations.
- The partial derivatives at each grid point are approximated from its neighbors by using Taylors theorem.

Finite Difference Method Application

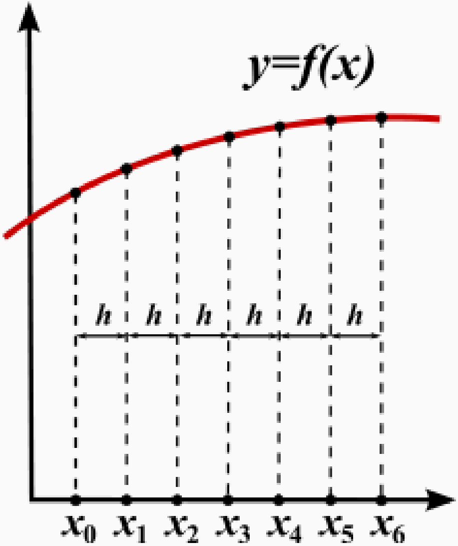

Figure 1: Finite Discrete Mesh Grid

Finite Difference Method(FDM)

Taylor's Theorem

U(x) have n continuous derivatives on $[a, b]$

$$U(x_0 + h) = U(x_0) + hU_x(x_0) + h^2 \frac{U_{xx}(x_0)}{2!} + \dots$$

Finite Difference Method(FDM)

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Applied to FDM

In FDM, both x_0 and $x_0 + h$ are grid points and $U(x_0)$ and $U(x_0 + h)$ are known.

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In FDM, both x_0 and $x_0 + h$ are grid points and $U(x_0)$ and $U(x_0 + h)$ are known.

$$U_x(x_0) = \frac{U(x_0 + h) - U(x_0)}{h} + O(h)$$

known as finite difference approximation to first order derivative

Finite Difference Method

Taylor's theorem for 3 grid points x_0, $x_0 + h$, $x_0 - h$ gives the second order of second derivative

$$U_{xx}(x_0) = \frac{U(x_0 - h) - 2U(x_0) + U(x_0 - h)}{h^2} + O(h^2)$$

is called second order FD approximation to $U_{xx}(x_0)$ since the approximation error $= O(h^2)$

Elliptic Equation's Formula

$$a_1 U_{xx} + a_2 U_{xy} + a_3 U_{yy} = 0$$

$$(x, y) \in \Omega = (a, b)x(c, d)$$

where the coefficients a_1, a_2, a_3 satisfy

$$a_2^2 - 4a_1 a_3 < 0$$

Laplace's and Poisson's Equation

In 2D, Laplace's Equation is

$$U_{xx} + U_{yy} = 0$$

Used to model steady state groundwater flow, temperature distribution over a region, etc.

In 2D, Poisson's equation is

$$U_{xx} + U_{yy} = f(x, y)$$

Used to model the gravitational fields, stress patterns, etc.

Poisson's Equation and FDM

Poisson's equation and Dirichlet Boundary Condition

$$U_{xx} + U_{yy} = f(x, y)$$

$$(x, y)|_{\partial\Omega} = u_0(x, y)$$

$$(x, y) \in \Omega = (a, b)x(c, d)$$

Poisson's equation and Dirichlet Boundary Condition

$$U_{xx} + U_{yy} = f(x, y)$$

$$(x, y)|_{\partial\Omega} = u_0(x, y)$$

$$(x, y) \in \Omega = (a, b)x(c, d)$$

Our goal: seeking for an approximate solution u_{ij} at the grid points (x_i, y_j), where $U(x, y)$ are unknown

Poisson's Equation and FDM

FDM procedure is as follows

- **Step 1:** Generating a grid.

$$x_i = a + i(\Delta x)$$

$$y_j = c + j(\Delta y)$$

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FDM procedure is as follows

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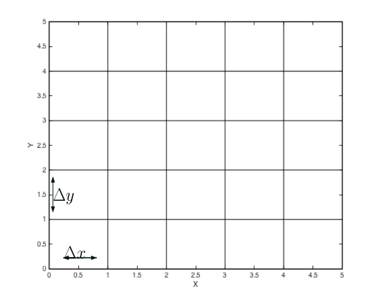

Figure 2: Finite Mesh Grid

- **Step 2:** Replacing partial derivatives with FD formulas involving the function values at the grid points

$$U_{xx} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{(\Delta x)^2} + O(\Delta x^2)$$

$$U_{yy} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{(\Delta y)^2} + O(\Delta y^2)$$

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- **Step 3:** Solving the linear system of algebraic equations of

$$U_{xx} + U_{yy} = \frac{u_{i-1,j} + u_{i+1,j} - 4u_{i,j} + u_{i,j-1} + u_{i,j+1}}{h^2} + O(h^2) = f_{i,j}$$

$$i = 1, 2, 3, \ldots n, \;\; j = 1, 2, 3, \ldots, n$$

Poisson's Equation and FDM

- **Step 3:** Solving the linear system of algebraic equations of

$$U_{xx} + U_{yy} = \frac{u_{i-1,j} + u_{i+1,j} - 4u_{i,j} + u_{i,j-1} + u_{i,j+1}}{h^2} + O(h^2) = f_{i,j}$$

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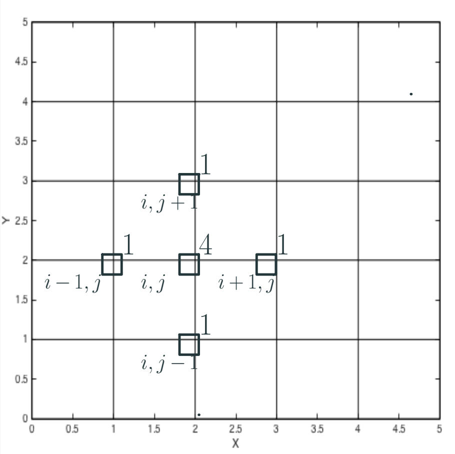
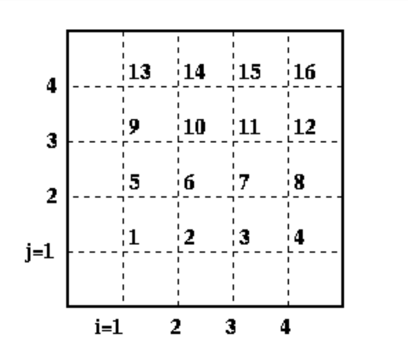


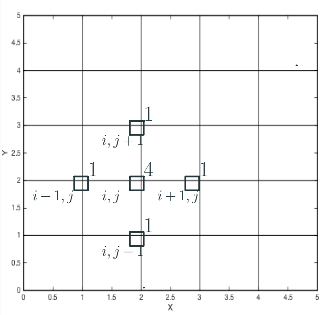
Poisson's Equation and FDM

How to solve the this linear system of equation?

1. Ordering the unknowns



(a) Natural Ordering



(b) Mesh Grid

How to solve the this linear system of equation?

2. Putting in Matrix

Discrete Poisson Problem on 4–by–4 Grid

$$
\left[\begin{array}{cccc|cccc|cccc|cccc}
4 & -1 & & & -1 & & & & & & & & & & & \\
-1 & 4 & -1 & & & -1 & & & & & & & & & & \\
& -1 & 4 & -1 & & & -1 & & & & & & & & & \\
& & -1 & 4 & & & & -1 & & & & & & & & \\
\hline
-1 & & & & 4 & -1 & & & -1 & & & & & & & \\
& -1 & & & -1 & 4 & -1 & & & -1 & & & & & & \\
& & -1 & & & -1 & 4 & -1 & & & -1 & & & & & \\
& & & -1 & & & -1 & 4 & & & & -1 & & & & \\
\hline
& & & & -1 & & & & 4 & -1 & & & -1 & & & \\
& & & & & -1 & & & -1 & 4 & -1 & & & -1 & & \\
& & & & & & -1 & & & -1 & 4 & -1 & & & -1 & \\
& & & & & & & -1 & & & -1 & 4 & & & & -1 \\
\hline
& & & & & & & & -1 & & & & 4 & -1 & & \\
& & & & & & & & & -1 & & & -1 & 4 & -1 & \\
& & & & & & & & & & -1 & & & -1 & 4 & -1 \\
& & & & & & & & & & & -1 & & & -1 & 4
\end{array}\right]
*
\begin{bmatrix} U(1,1) \\ U(2,1) \\ U(3,1) \\ U(4,1) \\ U(1,2) \\ U(2,2) \\ U(3,2) \\ U(4,2) \\ U(1,3) \\ U(2,3) \\ U(3,3) \\ U(4,3) \\ U(1,4) \\ U(2,4) \\ U(3,4) \\ U(4,4) \end{bmatrix}
=
\begin{bmatrix} b(1,1) \\ b(2,1) \\ b(3,1) \\ b(4,1) \\ b(1,2) \\ b(2,2) \\ b(3,2) \\ b(4,2) \\ b(1,3) \\ b(2,3) \\ b(3,3) \\ b(4,3) \\ b(1,4) \\ b(2,4) \\ b(3,4) \\ b(4,4) \end{bmatrix}
$$

Using FDM to Solve For

$$\Delta u = -2\pi^2 sin(\pi x) cos(\pi y)$$

$$(x, y)|_{\partial\Omega} = U_0(x, y) = sin(\pi x) cos(\pi y)$$

$$(x, y) \in \Omega = (0, 1)x(0, 1)$$

Verification

Is the program computing what it is supposed to?

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Is the program computing what it is supposed to?

- Find an exact solution of a simpler PDE then compare numerical and exact results
- Consider the error as you refine the grid:

$$N = \{2^i N_o\}_{i=0}^{4}$$

Summary

- Finite Difference Method is is used to find an approximate solution to PDE.
- It works by replacing all partial derivatives by an approximations
- Most widely used numerical method to solve PDE

Thank you! Questions?