Development of Navier-Stokes code by projection method

Problem equations

- Conservative form of NS equations

$$
\begin{cases}
\dfrac{\partial \mathbf{u}}{\partial \mathbf{x}} + \dfrac{\partial \mathbf{v}}{\partial \mathbf{y}} = 0 \\[2ex]
\dfrac{\partial \mathbf{u}}{\partial \mathbf{t}} + \dfrac{\partial \mathbf{u}^2}{\partial \mathbf{x}} + \dfrac{\partial \mathbf{uv}}{\partial \mathbf{y}} = -\dfrac{\partial \mathbf{p}}{\partial \mathbf{x}} + \boldsymbol{\nu}\left(\dfrac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \dfrac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2}\right) \\[2ex]
\dfrac{\partial \mathbf{v}}{\partial \mathbf{t}} + \dfrac{\partial \mathbf{uv}}{\partial \mathbf{x}} + \dfrac{\partial \mathbf{v}^2}{\partial \mathbf{y}} = -\dfrac{\partial \mathbf{p}}{\partial \mathbf{y}} + \boldsymbol{\nu}\left(\dfrac{\partial^2 \mathbf{v}}{\partial \mathbf{x}^2} + \dfrac{\partial^2 \mathbf{v}}{\partial \mathbf{y}^2}\right)
\end{cases}
$$

Projection method

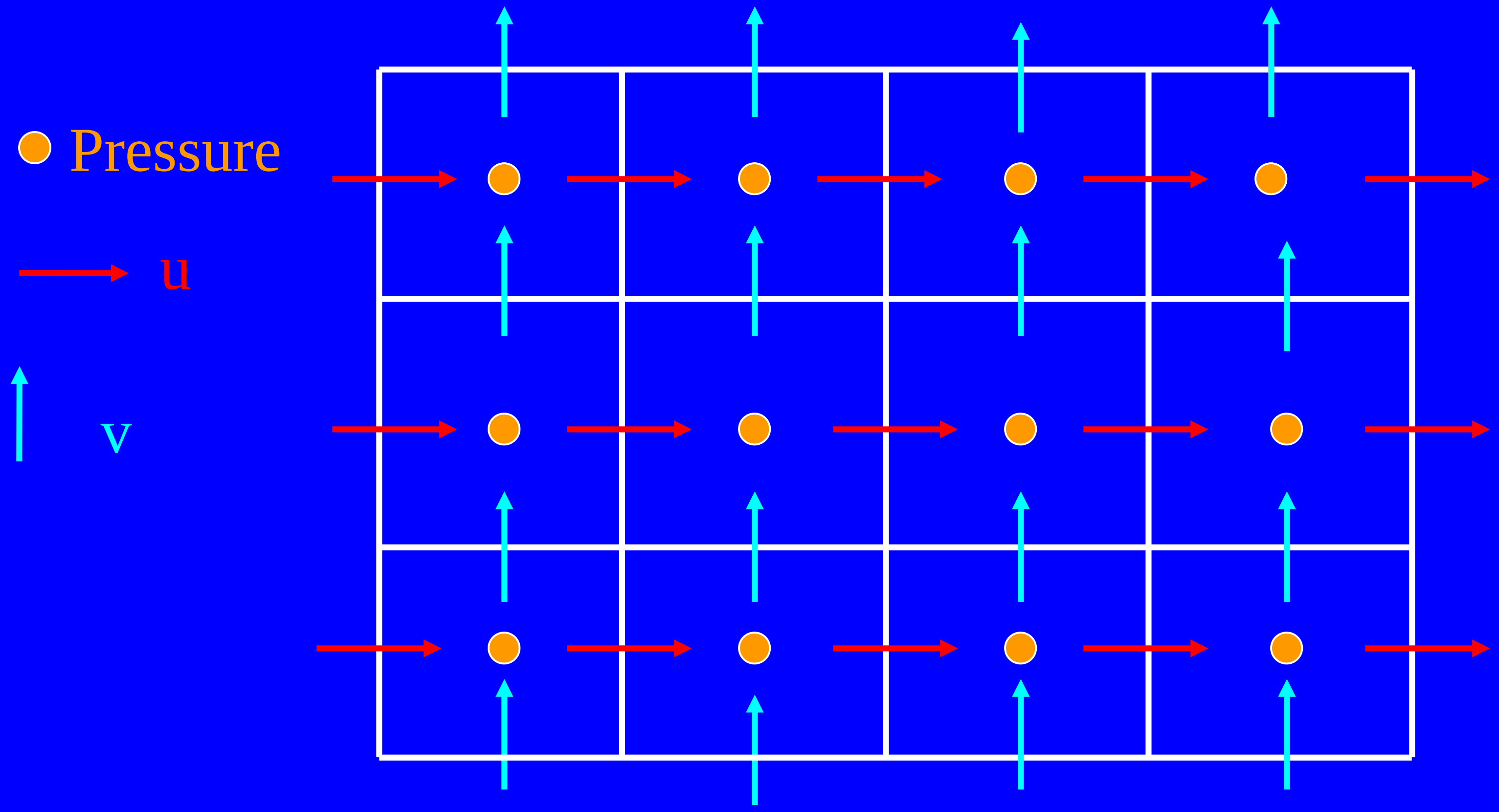
Forecast step

$$\begin{cases} \dfrac{\mathbf{u}^{*} - \mathbf{u}^{n}}{\Delta t} + \nabla(\mathbf{u} \otimes \mathbf{u})^{n} = \nu \nabla \cdot \mathbf{D}^{n} \\ \dfrac{\mathbf{u}^{n+1} - \mathbf{u}^{*}}{\Delta t} = -\nabla p^{n+1} \end{cases}$$

Correction step to ensure no divergence

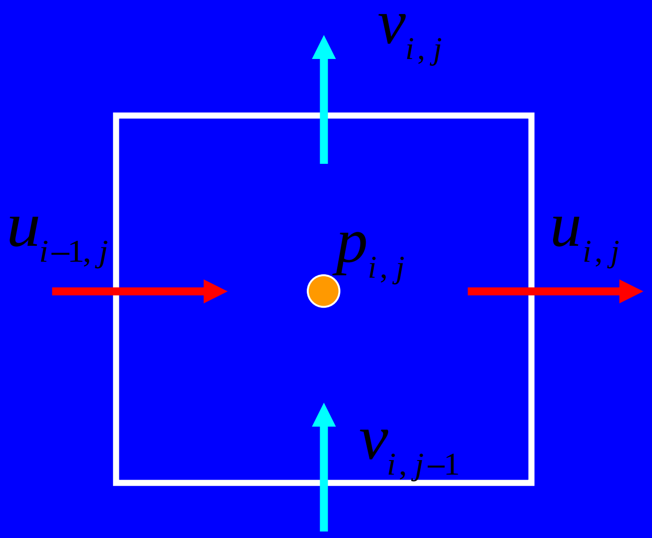
Staggered grid

- We use a velocity-pressure staggered grid

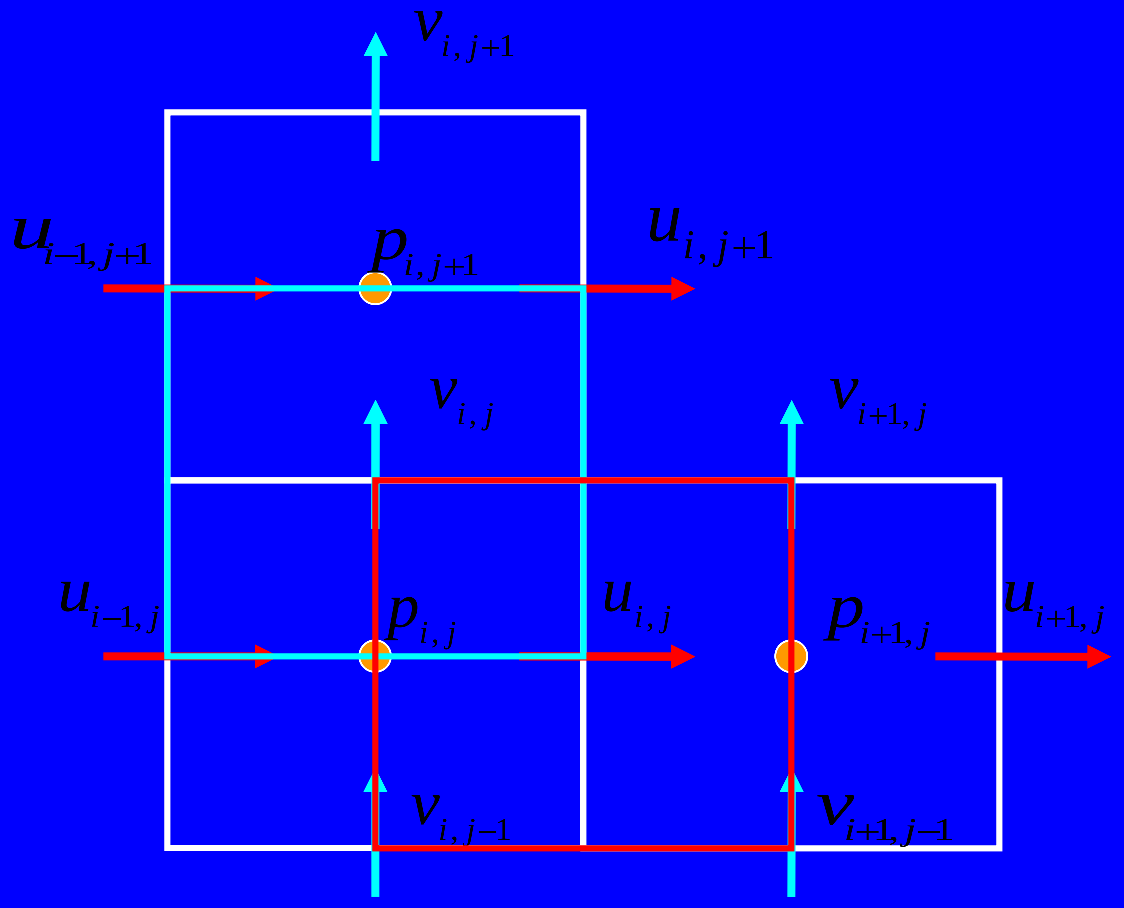


Notations for the unknowns

- Convention for the indices of the different variables



Finite volume approach



Convection scheme

- Conservative form

$$\frac{\partial \Phi}{\partial t} + \frac{\partial}{\partial x}\left(u\Phi\right) + \frac{\partial}{\partial y}\left(v\Phi\right) = SM \text{ avec } \begin{cases} \Phi = u \\ \Phi = v \end{cases}$$

Discrete form with Euler explicit time integration

$$\frac{\Phi_{i,j}^{n+1}-\Phi_{i,j}^{n}}{\Delta t}+\frac{1}{\Delta x}\left(\tilde{u}_{i+1/2,j}\Phi_{i+1/2,j}-\tilde{u}_{i-1/2,j}\Phi_{i-1/2,j}\right)$$

$$+\frac{1}{\Delta y}\left(\tilde{v}_{i,j+1/2}\Phi_{i,j+1/2}-\tilde{v}_{i,j-1/2}\Phi_{i,j-1/2}\right)=SM$$

With $\tilde{u}_{i-1/2,j}$, convection speed at interface i-1/2,j

With $\tilde{v}_{i,j-1/2}$, convection speed at interface i,j-1/2

With $\tilde{u}_{i+1/2,j}$, convection speed at interface i+1/2,j

With $\tilde{v}_{i,j+1/2}$, convection speed at interface i,j+1/2

Equation for velocity u

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} + \frac{1}{\Delta x}\left(\tilde{u}_{i+1/2,j} u_{i+1/2,j} - \tilde{u}_{i-1/2,j} u_{i-1/2,j}\right)$$

$$+ \frac{1}{\Delta y}\left(\tilde{v}_{i,j+1/2} u_{i,j+1/2} - \tilde{v}_{i,j-1/2} u_{i,j-1/2}\right) = SM$$

Assessment of convection velocities for the 'u' equation

- For the convection velocities $\tilde{u}_{i+1/2,j}$ and $\tilde{u}_{i-1/2,j}$, we simply take

$$\tilde{u}_{i+1/2,j} = \frac{u^n_{i,j} + u^n_{i+1,j}}{2} \qquad \tilde{u}_{i-1/2,j} = \frac{u^n_{i-1,j} + u^n_{i,j}}{2}$$

- For the convection velocities $\tilde{v}_{i,j+1/2}$ and $\tilde{v}_{i,j-1/2}$, we consider

$$\tilde{v}_{i,j+1/2} = \frac{v^n_{i,j} + v^n_{i+1,j}}{2} \qquad \tilde{v}_{i,j-1/2} = \frac{v^n_{i,j-1} + v^n_{i+1,j-1}}{2}$$

Equation for velocity v

$$\frac{v_{i,j}^{n+1} - v_{i,j}^{n}}{\Delta t} + \frac{1}{\Delta x}\left(\tilde{u}_{i+1/2,j} v_{i+1/2,j} - \tilde{u}_{i-1/2,j} v_{i-1/2,j}\right)$$

$$+ \frac{1}{\Delta y}\left(\tilde{v}_{i,j+1/2} v_{i,j+1/2} - \tilde{v}_{i,j-1/2} v_{i,j-1/2}\right) = SM$$

Assessment of convection velocities for the 'v' equation

- For the convection velocities $\tilde{v}_{i,j+1/2}$ and $\tilde{v}_{i,j-1/2}$, we simply take

$$\tilde{\mathbf{v}}_{\mathbf{i,j+1/2}} = \frac{\mathbf{v}^{\mathbf{n}}_{\mathbf{i,j}} + \mathbf{v}^{\mathbf{n}}_{\mathbf{i,j+1}}}{2} \qquad \tilde{\mathbf{v}}_{\mathbf{i,j-1/2}} = \frac{\mathbf{v}^{\mathbf{n}}_{\mathbf{i,j-1}} + \mathbf{v}^{\mathbf{n}}_{\mathbf{i,j}}}{2}$$

- For the convection velocities $\tilde{u}_{i+1/2,j}$ and $\tilde{u}_{i-1/2,j}$, we consider

$$\tilde{\mathbf{u}}_{\mathbf{i+1/2,j}} = \frac{\mathbf{u}^{\mathbf{n}}_{\mathbf{i,j}} + \mathbf{u}^{\mathbf{n}}_{\mathbf{i,j+1}}}{2} \qquad \tilde{\mathbf{u}}_{\mathbf{i-1/2,j}} = \frac{\mathbf{u}^{\mathbf{n}}_{\mathbf{i-1,j}} + \mathbf{u}^{\mathbf{n}}_{\mathbf{i-1,j+1}}}{2}$$

Upwind numerical scheme for u

- On interface i+1/2,j : If $\tilde{u}_{i+1/2,j} \geq 0$ then $u_{i+1/2,j} = u^n_{i,j}$
  otherwise $u_{i+1/2,j} = u^n_{i+1,j}$
- On interface i-1/2,j : If $\tilde{u}_{i-1/2,j} \geq 0$ then $u_{i-1/2,j} = u^n_{i-1,j}$
  otherwise $u_{i-1/2,j} = u^n_{i,j}$
- On interface i,j+1/2 : If $\tilde{v}_{i,j+1/2} \geq 0$ then $u_{i,j+1/2} = u^n_{i,j}$
  otherwise $u_{i,j+1/2} = u^n_{i,j+1}$
- On interface i,j-1/2 : If $\tilde{v}_{i,j-1/2} \geq 0$ then $u_{i,j-1/2} = u^n_{i,j-1}$
  otherwise $u_{i,j-1/2} = u^n_{i,j}$

Upwind numerical scheme for v

- On interface i+1/2,j : If $\tilde{u}_{i+1/2,j} \geq 0$ then $v_{i+1/2,j} = v_{i,j}^n$ otherwise $v_{i+1/2,j} = v_{i+1,j}^n$
- On interface i-1/2,j : If $\tilde{u}_{i-1/2,j} \geq 0$ then $v_{i-1/2,j} = v_{i-1,j}^n$ otherwise $v_{i-1/2,j} = v_{i,j}^n$
- On interface i,j+1/2 : If $\tilde{v}_{i,j+1/2} \geq 0$ then $v_{i,j+1/2} = v_{i,j}^n$ otherwise $v_{i,j+1/2} = v_{i,j+1}^n$
- On interface i,j-1/2 : If $\tilde{v}_{i,j-1/2} \geq 0$ then $v_{i,j-1/2} = v_{i,j-1}^n$ otherwise $v_{i,j-1/2} = v_{i,j}^n$

Central differencing scheme for u

- On interface i+1/2,j : $$\mathbf{u}_{i+1/2,j} = \frac{\mathbf{u}^n_{i,j} + \mathbf{u}^n_{i+1,j}}{2}$$
- On interface i-1/2,j : $$\mathbf{u}_{i-1/2,j} = \frac{\mathbf{u}^n_{i-1,j} + \mathbf{u}^n_{i,j}}{2}$$
- On interface i,j+1/2 : $$\mathbf{u}_{i,j+1/2} = \frac{\mathbf{u}^n_{i,j} + \mathbf{u}^n_{i,j+1}}{2}$$
- On interface i,j-1/2 : $$\mathbf{u}_{i,j-1/2} = \frac{\mathbf{u}^n_{i,j-1} + \mathbf{u}^n_{i,j}}{2}$$

Central differencing scheme for v

- On interface i+1/2,j : $$\mathbf{v}_{\mathbf{i}+1/2,\mathbf{j}} = \frac{\mathbf{v}^{\mathbf{n}}_{\mathbf{i},\mathbf{j}} + \mathbf{v}^{\mathbf{n}}_{\mathbf{i}+1,\mathbf{j}}}{2}$$
- On interface i-1/2,j : $$\mathbf{v}_{\mathbf{i}-1/2,\mathbf{j}} = \frac{\mathbf{v}^{\mathbf{n}}_{\mathbf{i}-1,\mathbf{j}} + \mathbf{v}^{\mathbf{n}}_{\mathbf{i},\mathbf{j}}}{2}$$
- On interface i,j+1/2 : $$\mathbf{v}_{\mathbf{i},\mathbf{j}+1/2} = \frac{\mathbf{v}^{\mathbf{n}}_{\mathbf{i},\mathbf{j}} + \mathbf{v}^{\mathbf{n}}_{\mathbf{i},\mathbf{j}+1}}{2}$$
- On interface i,j-1/2 : $$\mathbf{v}_{\mathbf{i},\mathbf{j}-1/2} = \frac{\mathbf{v}^{\mathbf{n}}_{\mathbf{i},\mathbf{j}-1} + \mathbf{v}^{\mathbf{n}}_{\mathbf{i},\mathbf{j}}}{2}$$

Viscous terms treatment

- We use the classic 2nd order centered scheme:

- For u : $$\nu\left(\frac{u^n_{i+1,j}-2u^n_{i,j}+u^n_{i-1,j}}{\Delta x^2}\right)+\nu\left(\frac{u^n_{i,j+1}-2u^n_{i,j}+u^n_{i,j-1}}{\Delta y^2}\right)$$

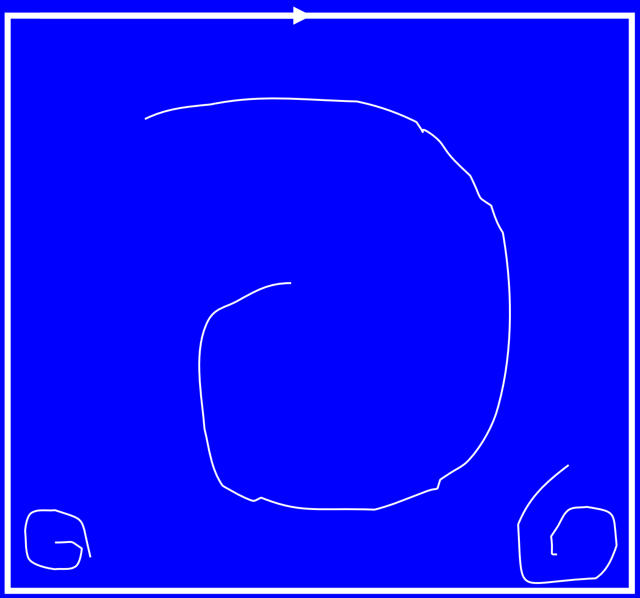
- For v : $$\nu\left(\frac{v^n_{i+1,j}-2v^n_{i,j}+v^n_{i-1,j}}{\Delta x^2}\right)+\nu\left(\frac{v^n_{i,j+1}-2v^n_{i,j}+v^n_{i,j-1}}{\Delta y^2}\right)$$

Boundary conditions for velocity fields

- Here, we are interested in a 2D square-shaped lid-driven cavity with walls of 1m length.
- Wall adhesion

$u = 1m/s, v = 0$

$u = v = 0$

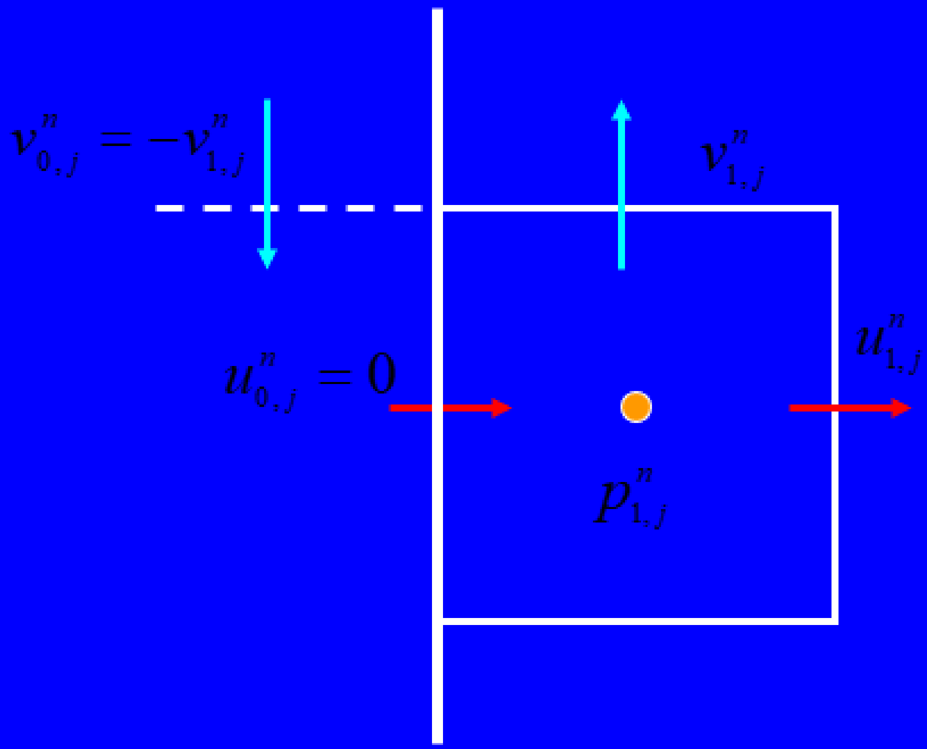


$u = v = 0$

$u = v = 0$

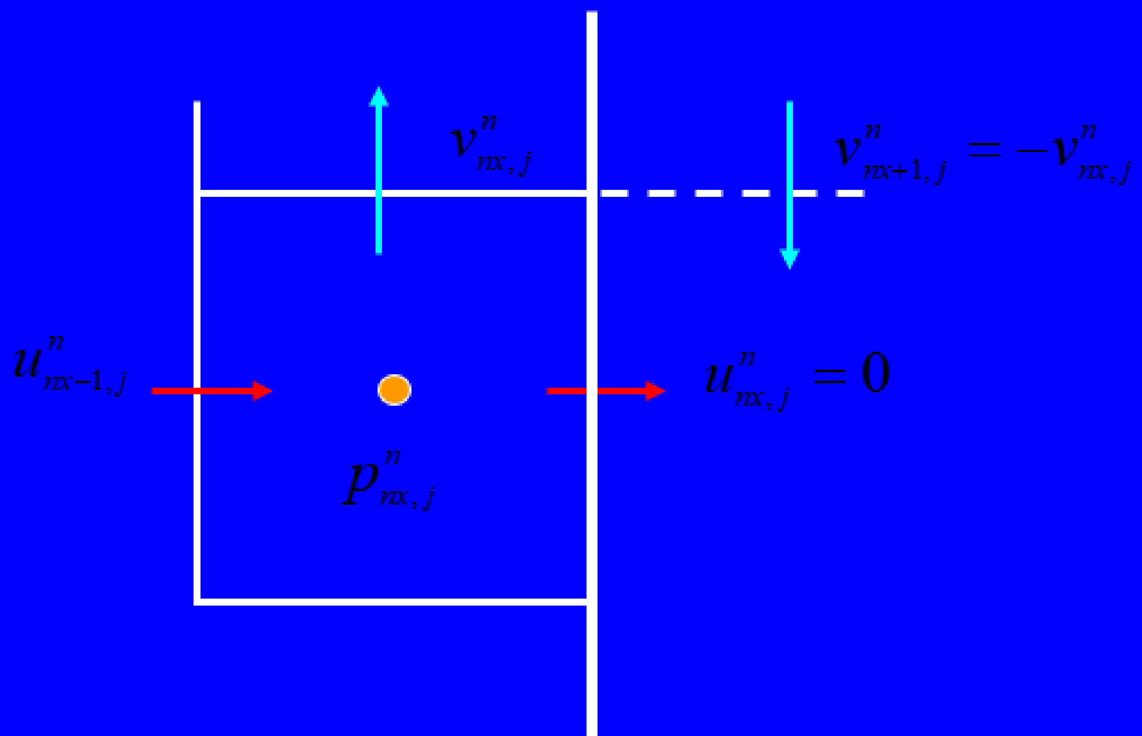
Implementation of boundary conditions for staggered grid

- Left wall boundary:



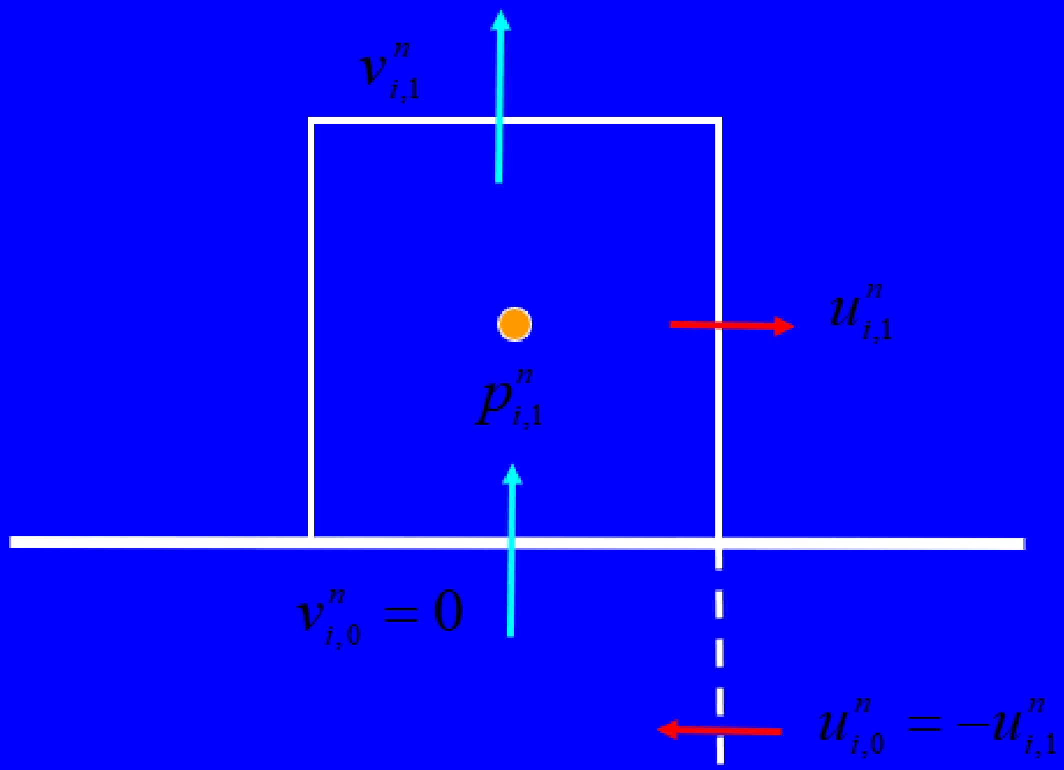
Implementation of boundary conditions for staggered grid

- Right wall boundary:



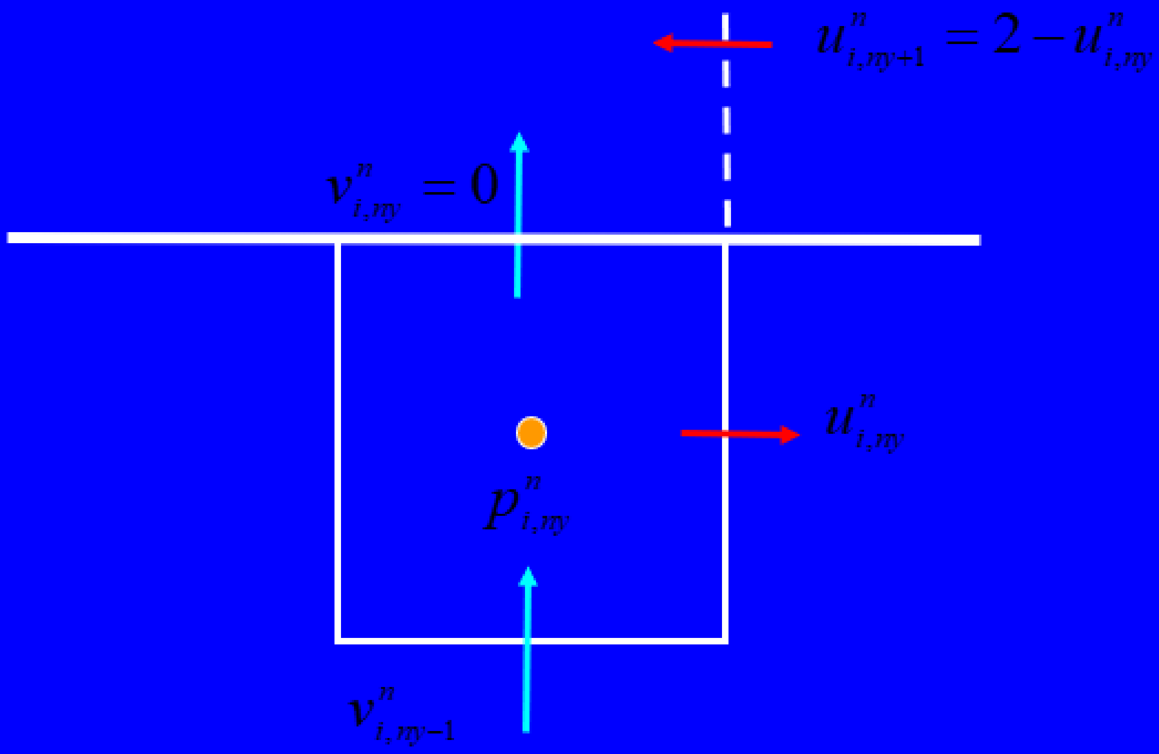
Implementation of boundary conditions for staggered grid

- Bottom wall boundary:



Implementation of boundary conditions for staggered grid

- Top wall boundary:



Remarks

- These boundary conditions are applied for both the velocities at current time, and also to the starry ones during the forecast step of the projection method.

Discretisation of pressure equation 1

- The correction step with the projection method is given by :

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^{*}}{\Delta t} = -\nabla p^{n+1}$$

- Taking the divergence of the above equation :

$$\frac{\nabla \cdot \mathbf{u}^{*}}{\Delta t} = \Delta p^{n+1}$$

Discretisation of pressure equation 2

- Under discrete form, we obtain:

$$\left(\frac{\mathbf{p}_{i+1,j}^{n+1}-2\mathbf{p}_{i,j}^{n+1}+\mathbf{p}_{i-1,j}^{n+1}}{\Delta\mathbf{x}^2}\right)+\left(\frac{\mathbf{p}_{i,j+1}^{n+1}-2\mathbf{p}_{i,j}^{n+1}+\mathbf{p}_{i,j-1}^{n+1}}{\Delta\mathbf{y}^2}\right)=\frac{1}{\Delta\mathbf{t}}\left(\frac{\mathbf{u}_{i,j}^{*}-\mathbf{u}_{i-1,j}^{*}}{\Delta\mathbf{x}}+\frac{\mathbf{v}_{i,j}^{*}-\mathbf{v}_{i,j-1}^{*}}{\Delta\mathbf{y}}\right)$$

- Pressure boundary conditions are Neumann homogen types:

$$\frac{\partial p}{\partial n} \text{ on the 4 walls of the box}$$

Pressure boundary conditions implementation

Left vertical wall: $p_{0,j}^{n+1} = p_{1,j}^{n+1}$

Right vertical wall: $p_{nx+1,j}^{n+1} = p_{nx,j}^{n+1}$

Bottom horizontal wall: $p_{i,0}^{n+1} = p_{i,1}^{n+1}$

Top horizontal wall: $p_{i,ny+1}^{n+1} = p_{i,ny}^{n+1}$

Resolution of pressure equation

- We notice that with the given boundary conditions, the matrix is singular, with its determinant = 0.
- The solution is then known to a constant.
- We use an iterative method to calculate the numerical solution, in to order to solve:

$$AX = B$$

Resolution of pressure equation

- If we use a mesh NXxNY, the matrix of the system to solve is hollow (5 bands) and of dimensions (NXxNY) x (NXxNY).
- The unknown vector **X** has dimension NXxNY
- The second membre **B** also has dimension NXxNY

Resolution of pressure equation

- Generating matrix for pressure equation: use the subroutine matgen_cavity.f

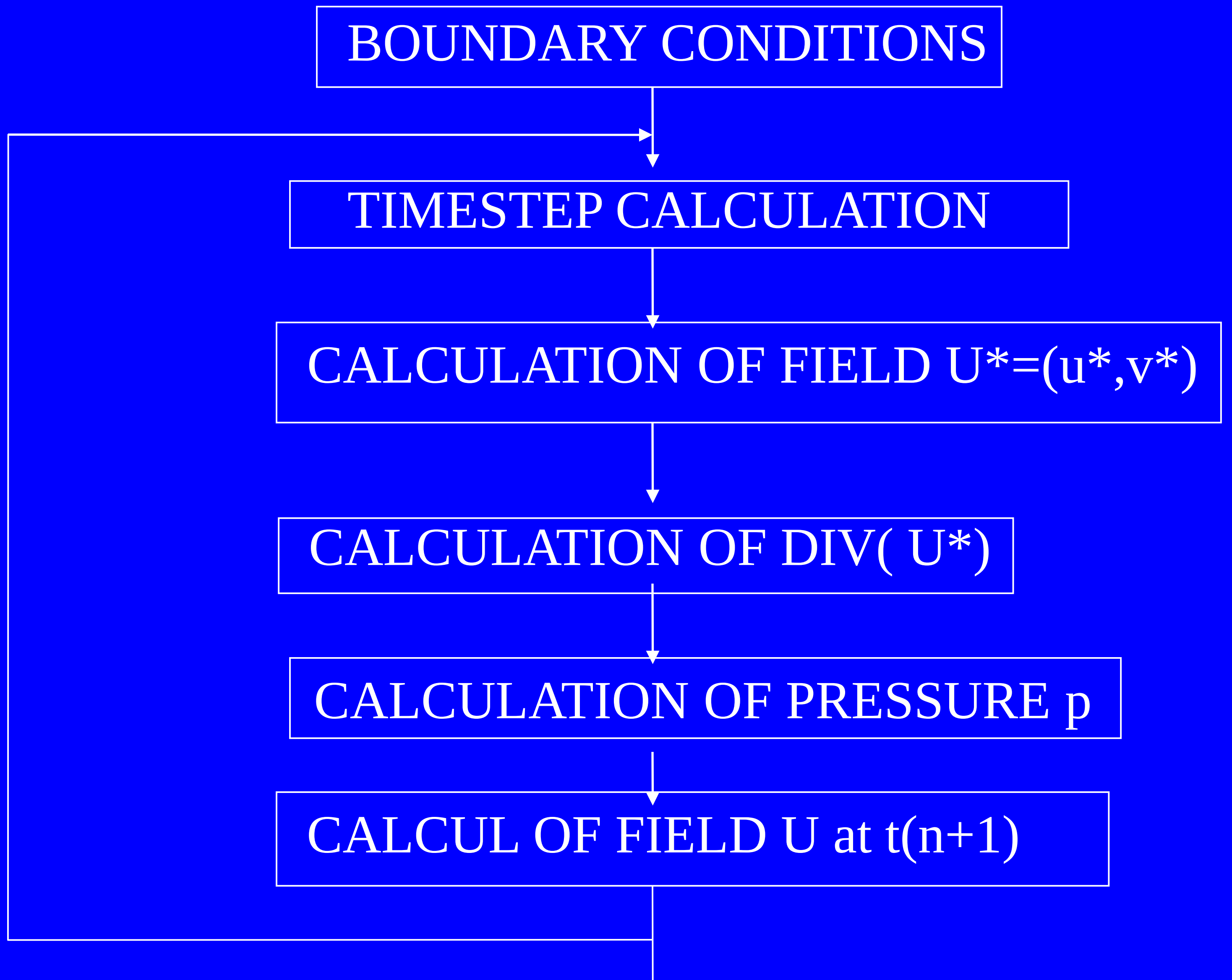
- Solving pressure equation: use the subroutine ICCG2 in the file Solver.f90

Resolution of pressure equation

- To obtain unicity for the solution of the system $AX = B$, we modify the second member by:

```
sum = 0
do j = 1, ny
    do i = 1, nx
        sum = sum + B(i, j)
    enddo
enddo
sum = sum/(nx × ny)
do j = 1, ny
    do i = 1, nx
        B(i, j) = B(i, j) - sum
    enddo
enddo
```

General algorithm



Conditions of stability

- **Upwind** scheme for convection; centered scheme for diffusion:

$$\Delta t \le \min\left(\min\left(\frac{\Delta x}{\max_{i,j}(|u(i,j)|)}, \frac{\Delta x^2}{2\nu}\right), \min\left(\frac{y}{\max_{i,j}(|v(i,j)|)}, \frac{\Delta y^2}{2\nu}\right) \right)$$

Conditions of stability

- **Centered** scheme for convection; centered scheme for diffusion:

$$\Delta t \le \min\left(\min\left(\frac{\nu}{\max\limits_{i,j}\left(\left|u(i,j)\right|\right)^2}, \frac{\Delta x^2}{2\nu}\right), \min\left(\frac{\nu}{\max\limits_{i,j}\left(\left|v(i,j)\right|\right)^2}, \frac{\Delta y^2}{2\nu}\right) \right)$$