Efficient solutions of implicit discretizations of the immersed boundary method

Jordan Fisher

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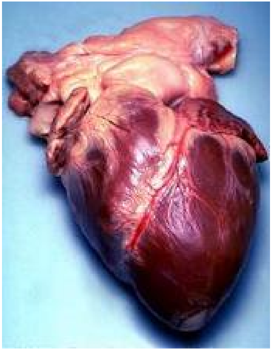

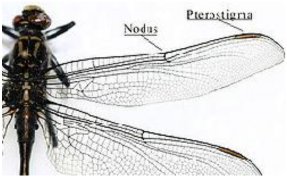

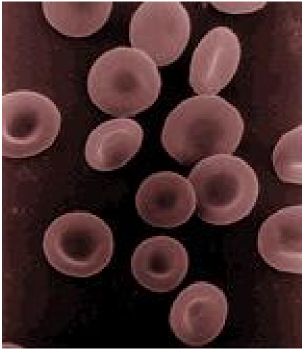

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$$\mathbf{f}(\mathbf{x}, t) = \int_B \mathbf{F}(\mathbf{X}(\cdot,\cdot), s, t)\delta(\mathbf{x} - \mathbf{X}(s,t))ds \tag{4}$$

Standard discretization: Forward Euler/Backward Euler

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We approximate δ with a discrete function

$$d_h(r) = \begin{cases} \frac{1}{4h}\left(1+\cos(\frac{\pi r}{2h})\right) & \text{if } |r| \leq 2h \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

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and approximate the spreading and smoothing via

$$(\mathcal{S}_n G)(\mathbf{x}) = \sum_{s \in \mathcal{G}_B} G(s)\delta_h(\mathbf{x} - \mathbf{X}^n(s))h_B, \tag{6}$$

$$(\mathcal{S}_n^* w)(s) = \sum_{\mathbf{x} \in \mathcal{G}_\Omega} w(\mathbf{x})\delta_h(\mathbf{x} - \mathbf{X}^n(s))h^2. \tag{7}$$

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$$\mathbf{D}_h\cdot\mathbf{u}^{n+1}=0, \quad (10)$$

$$\frac{\mathbf{X}^{n+1}-\mathbf{X}^{n}}{\Delta t}=\mathcal{S}_n^{*}\mathbf{u}^{n+1} \quad (11)$$

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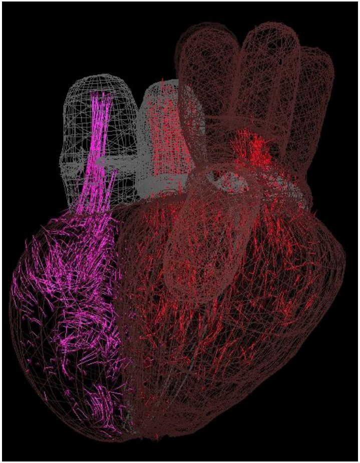

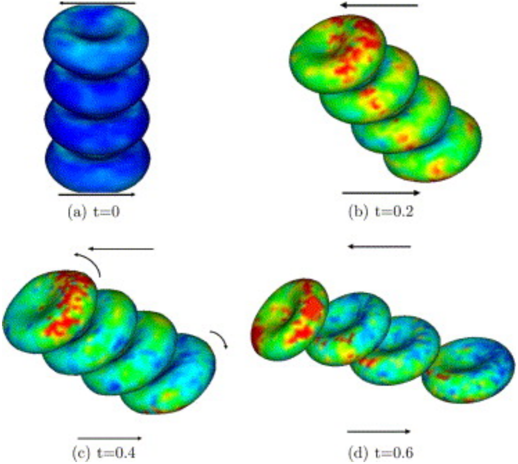

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Implicit discretizations!

What should we make implicit?

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Our work has focused on solving this system efficiently

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which we refer to as the fluid solver.

Now we may write

$$\mathbf{u}^{n+1} = \mathcal{L}_h \left[\frac{\Delta t}{\rho} \mathcal{S}_n \mathcal{A}_{h_B}(\mathbf{X}^{n+1}) + \mathbf{u}^n - \Delta t \ \mathbf{u}^n \cdot \nabla_h \mathbf{u}^n \right] \tag{15}$$

$$\mathbf{X}^{n+1} = \mathbf{X}^n + \Delta t \mathcal{S}_n^* \mathbf{u}^{n+1}. \tag{16}$$

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In fact, $I - \mathcal{M}_n \mathcal{A}_{h_B}$ is positive-definite if $\mathcal{A}_{h_B}$ is negative-semidefinite.

How can we efficiently solve $(I - \mathcal{M}_n \mathcal{A}_{h_B})\mathbf{X}^{n+1} = \mathbf{b}^n$?

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A matrix representation of $I - \mathcal{M}_n \mathcal{A}_{h_B}$ would greatly speed iterative methods

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The key is an approximation inspired by the continuous equations, as well as a reliance on lookup tables.

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- Slightly more complicated code

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- Faster evaluations of quantities of the form $\mathcal{M}_n\mathbf{F}$, by a factor of 20 to 100. This greatly streamlines iterative methods.

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- Multigrid!

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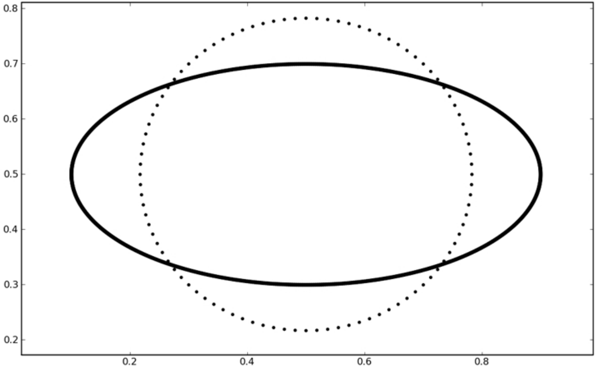

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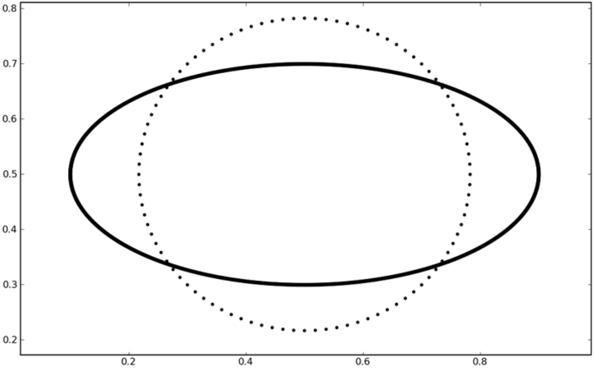

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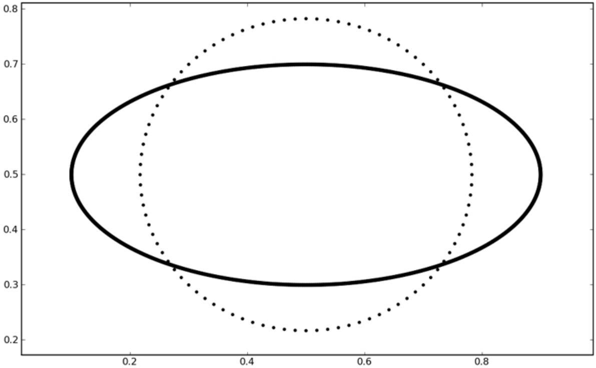

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$\mathcal{A}_{h_B}$ is linear, negative-semidefinite, hence we can apply our multigrid.

Comparison of total CPU between an explicit method and our implicit implementation

Table: Elliptical drop relaxation for Navier Stokes. The average CPU time per time-step and the total CPU time up to a simulation time of $T = 0.005$ is given. Δt is the time-step taken and is the maximum allowed while maintaining stability.

	Explicit			Implicit		
N	Δt	Average	Total	Δt	Average	Total
128	$1.95 \cdot 10^{-6}$	0.03	73.35	$1.17 \cdot 10^{-4}$	0.10	4.34
256	$9.76 \cdot 10^{-7}$	0.14	730.21	$7.81 \cdot 10^{-5}$	0.55	35.06
384	$6.51 \cdot 10^{-7}$	0.34	2613.19	$5.21 \cdot 10^{-5}$	1.24	118.98
512	$4.87 \cdot 10^{-7}$	0.63	6491.95	$3.91 \cdot 10^{-5}$	2.34	299.10

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What if $\mathcal{A}_{h_B}$ isn't linear?

- When the Jacobian of $\mathcal{A}_{h_B}$ is negative semidefinite we can solve Newton iterations via multigrid. This works well.
- Unfortunately, multigrid fails if the Jacobian isn't definite
- In certain important cases, we can fix the problem by opting for a fixed-point iteration

Fixed-point iteration

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The most immediate choice for a fixed point iteration is

$$\mathbf{X}^{n+1,k+1} = M_n \mathcal{A}_{h_B}(\mathbf{X}^{n+1,k}) + \mathbf{b}^n. \tag{21}$$

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Intuitively, the reverse of this iteration should be stable.

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We dub the following the splitting method

$$\begin{cases} M_n \mathbf{F}^{k+1} = \mathbf{X}^{n+1,k} - \mathbf{b}^n, \\ \mathcal{A}_{h_B}(\mathbf{X}^{n+1,k+1}) = \mathbf{F}^{k+1}. \end{cases} \tag{22}$$

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Typically the Jacobian of $\mathcal{A}_{h_B}$ is sparse, so this process is cheap.

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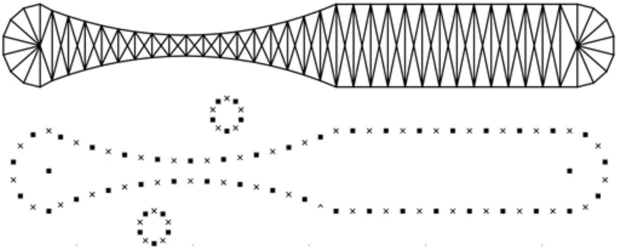
Provided the eigenvalues of the Jacobian of $\mathcal{A}_{h_B}$ are negative and large enough, (22) converges rapidly.

Application: blood flow past a heart valve



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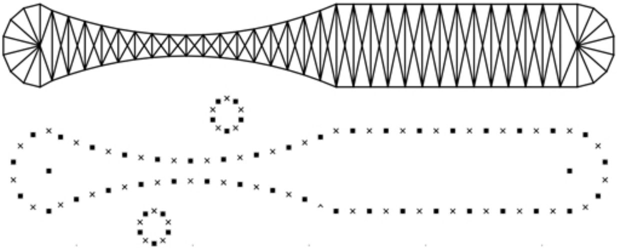

The force between two nodes is given by Hooke's law

$$\mathbf{f}_{i,j} = -k_{i,j} \frac{\mathbf{X}_i - \mathbf{X}_j}{|\mathbf{X}_i - \mathbf{X}_j|}(|\mathbf{X}_i - \mathbf{X}_j| - L_{i,j}) \tag{23}$$

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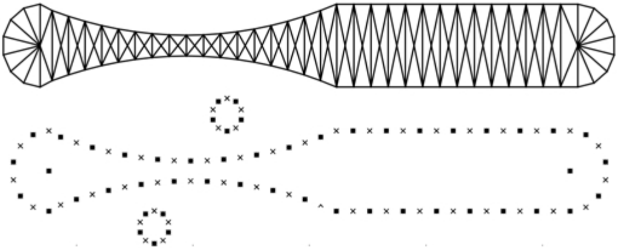

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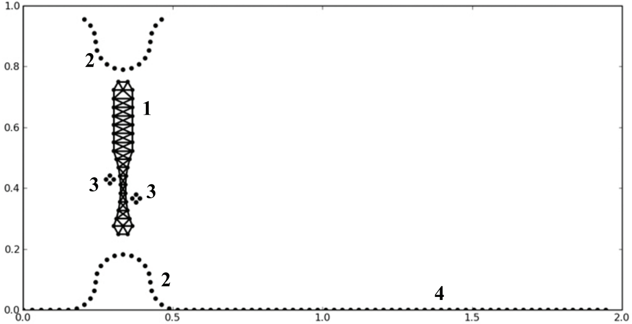
This is nonlinear when $L_{i,j} \neq 0$!

Application: blood flow past a heart valve



- 1. Valve
- 2. Cushions
- 3. Hinges
- 4. Artery wall

Application: blood flow past a heart valve

A horizontal flow is induced by adding a a time dependent component to the force field:

$$v_{flow}(t) = \begin{cases} 60000t(0.1 - t) & t < 0.1 \\ -60000(t - 0.1)(0.2 - t) & t \geq 0.1 \end{cases} . \tag{24}$$

Application: blood flow past a heart valve

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$\mathcal{A}_{h_B}$ can't produce translational or rotational forces.
$\mathcal{A}_{h_B}$ is insensitive to translation of its input

Factoring out components outside the image of $\mathcal{A}_{h_B}$

We define vectors

$$\mathbf{V}_I^1 = \begin{cases} (1,0) & \text{if } I \in \mathcal{G}_V, \\ (0,0) & \text{otherwise}, \end{cases} \tag{25}$$

$$\mathbf{V}_I^2 = \begin{cases} (0,1) & \text{if } I \in \mathcal{G}_V, \\ (0,0) & \text{otherwise}, \end{cases} \tag{26}$$

$$\mathbf{V}_I^3 = \begin{cases} (-Y_I + y_c, X_I - x_c) & \text{if } I \in \mathcal{G}_V, \\ (0,0) & \text{otherwise}, \end{cases} \tag{27}$$

where $\mathcal{G}_V$ is the collection of all indices I such that $\mathbf{X}_I$ is a fiber point belonging to the valve, and (x_c, y_c) is some fixed point.

Factoring out components outside the image of $\mathcal{A}_{h_B}$

The vectors $\mathbf{V}^1, \mathbf{V}^2, \mathbf{V}^3$ are points outside the image of $\mathcal{A}_{h_B}$

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$$\begin{cases} \mathbf{X}^{n+1,k+1,0} = \mathbf{X}^{n+1,k}, \\ J(\mathbf{X}^{n+1,k+1,l})(\mathbf{X}^{n+1,k+1,l+1} - \mathbf{X}^{n+1,k+1,l}) \\ \qquad = P(\mathbf{F}^{k+1} - \mathcal{A}_{h_B}(\mathbf{X}^{n+1,k+1,l})), \end{cases} \tag{28}$$

where

$$P(\mathbf{F}) = \mathbf{F} - \sum_{j=1}^{3} \mathbf{V}^j(\mathbf{X}^{n+1,k})(\mathbf{F}, \mathbf{V}^j(\mathbf{X}^{n+1,k}))_B \tag{29}$$

Reintroducing translation and rotation

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There are simple, fully explicit, and stable procedures for calculating the translation and rotation.

Comparison of total CPU time for the heart valve simulation

Table: Heart valve simulation. The average CPU time per time-step and the total CPU time up to a simulation time of $T = 0.5$ is given. For the explicit method Δt is the time-step taken and is the maximum allowed while maintaining stability. For the implicit method Δt is held constant, and is $O(h)$ or smaller.

	Explicit			Implicit		
N	Δt	Average	Total	Δt	Average	Total
128	$1.15 \cdot 10^{-7}$	0.024	105089*	0.0025	2.001	118
256	$2.89 \cdot 10^{-8}$	0.109	1884358*	0.0025	7.640	402
384	$1.29 \cdot 10^{-8}$	0.249	9658428*	0.0025	18.735	949
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Current research

Current research is focused on extending the methodology to 3D. The main problems are

- The matrix $\mathcal{M}_n$ has $O(N^4)$ entrees, so can't be used directly
- Without the matrix we can't make use of Gauss-Seidel
- Without a good smoother we can't implement a multigrid

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Current lines of attack include

- Looking at alternative smoothers, potentially involving banded components of $\mathcal{M}_n$
- Looking at good preconditioners for non-multigrid solvers, such as GMRES
- Modifying a Fast Multipole Method to implement Gauss-Seidel in $O(N_b)$ steps