Weighted essentially non-oscillatory schemes

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1 Introduction

We consider a one-dimensional hyperbolic conservation law of the form

$$q_t + \big(f(q)\big)_x = 0. \tag{1.1}$$

For finite-volume schemes we do not solve (1.1) directly, but its integrated version instead. Integrating (1.1) over the interval $[a, b]$ we obtain

$$\frac{d}{dt}\overline{q}(t) + \frac{1}{b-a}\Big(f\big(q(b,t)\big) - f\big(q(a,t)\big)\Big) = 0$$

where

$$\overline{q}(t) \equiv \frac{1}{b-a}\int_a^b q(\xi, t)\, d\xi$$

is the average value of q over $[a, b]$. This leads us to one of the central problems in implementing a numerical scheme to solve (1.1): obtaining the values of q at the boundaries a and b based on the average $\overline{q}$ of q. This is the *reconsturction* problem.

It is our intention here to study the reconstruction problem.

2 Grid

We consider a grid over the interval $[a, b]$ with N cells. We denote the $N + 1$ cell boundaries by

$$x_{i-1/2} \quad \text{for} \quad i = 1, \dots, N + 1 \tag{2.1}$$

so that

$$a = x_{1/2} < x_{3/2} < \dots < x_{N-1/2} < x_{N+1/2} = b. \tag{2.2}$$

Subsequently, we denote the N cells by

$$C_i = [x_{i-1/2}, x_{i+1/2}] \quad \text{for} \quad i = 1, \dots, N; \tag{2.3}$$

the N cell centres by

$$x_i = \frac{x_{i-1/2} + x_{i+1/2}}{2} \quad \text{for} \quad i = 1, \dots, N; \tag{2.4}$$

the N cell sizes by

$$\Delta x_i = x_{i+1/2} - x_{i-1/2} \quad \text{for} \quad i = 1, \dots, N; \tag{2.5}$$

and the maximum cell size by

$$\Delta x = \max_{i=1,\dots,N} \Delta x_i. \tag{2.6}$$

We denote the contiguous stencil around the cell C_i, containing k cells shifted to the left by r cells, by

$$S_i^{r,k} = C_{i-r} \cup \dots \cup C_{i-r+k-1}. \tag{2.7}$$

Note that $S_i^{r,k}$ spans k cells and contains $k + 1$ cell boundaries.

3 One dimensional reconstruction for smooth functions

Given the cell averages $\overline{v}_j$ of a function v where

$$\overline{v}_j = \frac{1}{\Delta x_j} \int_{x_{j-1/2}}^{x_{j+1/2}} v(\xi)\, d\xi \tag{3.1}$$

we wish to find approximations $v_{i+1/2}$ to the function v at the cell boundaries $x_{i+1/2}$, based on k cell averages, that are k-order accruate. The remainder of this section will be devoted to finding these approximations and showing that they are k-order accuarate. As it turns out, we will show that there are constants c_{ij}^r (hereafter called *reconstruction coefficients*) such that the reconstructed values $v_{i+1/2}$ at the cell boundaries $x_{i+1/2}$ are given by

$$v_{i+1/2} = \sum_{j=0}^{k-1} c_{ij}^r\, \overline{v}_{i-r+j}.$$

That is, given a stencil $S_i^{r,k}$ that spans the k cells $C_{i-r}, \ldots, C_{i-r+k-1}$, the reconstructed value $v_{i+1/2}$ of the original function v at the cell boundary $x_{i+1/2}$ can be obtained using the cell averages $\overline{v}_j$ over the cells C_j in the stencil $S_i^{r,k}$. In general, the reconstruction coefficients depend on the order k, left shift r, and cell i.

We can also reconstruct the values $v_{i-1/2}$ using the reconstruction coefficients $\tilde{c}_{ij}^r$ so that

$$v_{i-1/2} = \sum_{j=0}^{k-1} \tilde{c}_{ij}^r\, \overline{v}_{i-r+j} \tag{3.2}$$

As it turns out, the reconstruction coefficients c_{ij}^r and $\tilde{c}_{ij}^r$ are related. This can be seen by considering the cell C_{i-1} and the stencil $S_{i-1}^{r-1,k}$. Then

$$v_{i-1/2} = v_{(i-1)+1/2} = \sum_{j=0}^{k-1} c_{(i-1)\,j}^{r-1}\, \overline{v}_{(i-1)-(r-1)+j} = \sum_{j=0}^{k-1} c_{(i-1)\,j}^{r-1}\, \overline{v}_{i-r+j} \tag{3.3}$$

Comparing (3.2) and (3.3), we see that

$$\tilde{c}_{ij}^r = c_{(i-1)\,j}^{r-1}.$$

In order to obtain the reconstruction coefficients c_{ij}^r and prove accuracy, we first consider the following approximation problem: Given the cell averages $\overline{v}_j$ of a function v, find the polynomials p_i^r of degree at most $k-1$ such that each p_i^r is a k-order accurate approximation to v inside C_i. That is, given the cell averages $\overline{v}_j$, find polynomials p_i^r such that

$$p_i^r(x) = v(x) + O(\Delta x^k) \quad \text{for} \quad x \in C_i.$$

In order to find these polynomials, we consider the function

$$V(x) = \int_a^x v(\xi)\, d\xi. \tag{3.4}$$

Using the cell averages $\overline{v}_j$ we can compute V at the cell boundaries $x_{i+1/2}$ through

$$\begin{aligned} V(x_{i+1/2}) &= \int_a^{x_{i+1/2}} v(\xi)\, d\xi \\ &= \sum_{j=1}^{i} \int_{x_{j-1/2}}^{x_{j+1/2}} v(\xi)\, d\xi \\ &= \sum_{j=1}^{i} \overline{v}_j \Delta x_j. \end{aligned} \tag{3.5}$$

Focusing on a particular cell C_i and stencil $S_i^{r,k}$, the unique polynomial P_i^r of order k which interpolates V at the $k+1$ points

$$x_{i-r-1/2}, \ldots, x_{i-r+k-1/2}$$

is given by

$$P_i^r(x) = \sum_{l=0}^{k} \left(V(x_{i-r+l-1/2}) \prod_{m=0, m\neq l}^{k} \frac{(x - x_{i-r+m-1/2})}{(x_{i-r+l-1/2} - x_{i-r+m-1/2})} \right). \tag{3.6}$$

(This is the interpolating polynomial of V in Lagrange form.) We denote the derivative of P_i^r by p_i^r, so that

$$p_i^r(x) = \frac{d}{dx} P_i^r(x).$$

It can be shown (see Appendix A) that

$$P_i^r(x) = V(x) + O(\Delta x^{k+1}) \quad \text{for} \quad x \in S_i^{r,k}.$$

Therefore

$$p_i^r(x) = v(x) + O(\Delta x^{k}) \quad \text{for} \quad x \in S_i^{r,k}$$

and $p_i^r(x)$ is of order $k-1$.

Furthermore, the cell averages of p_i^r over the cells C_j that comprise the stencil $S_i^{r,k}$ satisfy

$$\begin{aligned}
\frac{1}{\Delta x_j} \int_{x_{j-1/2}}^{x_{j+1/2}} p_i^r(\xi)\, d\xi &= \frac{1}{\Delta x_j} \int_{x_{j-1/2}}^{x_{j+1/2}} P_i'(\xi)\, d\xi \\
&= \frac{1}{\Delta x_j} \Big(P_i^r(x_{j+1/2}) - P_i^r(x_{j-1/2}) \Big) \\
&= \frac{1}{\Delta x_j} \Big(V(x_{j+1/2}) - V(x_{j-1/2}) \Big) \\
&= \frac{1}{\Delta x_j} \left(\int_a^{x_{j+1/2}} v(\xi)\, d\xi - \int_a^{x_{j-1/2}} v(\xi)\, d\xi \right) \\
&= \frac{1}{\Delta x_j} \left(\int_{x_{j-1/2}}^{x_{j+1/2}} v(\xi)\, d\xi \right) \\
&= \overline{v}_j \quad \text{for} \quad j = i-r, \ldots, i-r+k-1.
\end{aligned}$$

That is, the cell averages of the approximating polynomials p_i^r match the cell averages the original function v in each of the cells C_j which comprise the stencil $S_i^{r,k}$.

So far we have constructed polynomials p_i^r that approximate the original function v on the stencils $S_i^{r,k}$ to k-order using only the cell averages $\overline{v}_j$ for $j = i-r, \ldots, i-r+k-1$.

Now we consider the practical problem of finding the constants c_{ij}^r. Subtracting $V(x_{i-r-1/2})$ from $P_i^r(x)$ and using

$$\sum_{l=0}^{k} \prod_{m=0, m\neq l}^{k} \frac{(x - x_{i-r+m-1/2})}{(x_{i-r+l-1/2} - x_{i-r+m-1/2})} = 1 \quad \text{and} \quad V(x_{i-r+l-1/2}) - V(x_{i-r-1/2}) \equiv 0 \text{ for } l = 0$$

we obtain

$$P_i^r(x) - V(x_{i-r-1/2}) = \sum_{l=1}^{k} \left(\big(V(x_{i-r+l-1/2}) - V(x_{i-r-1/2})\big) \prod_{m=0, m\neq l}^{k} \frac{(x - x_{i-r+m-1/2})}{(x_{i-r+l-1/2} - x_{i-r+m-1/2})} \right).$$

Taking the derivative of the above, we obtain

$$\frac{d}{dx} P_i^r(x) = \frac{d}{dx} \left[\sum_{l=1}^{k} \left(\big(V(x_{i-r+l-1/2}) - V(x_{i-r-1/2})\big) \prod_{m=0, m\neq l}^{k} \frac{(x - x_{i-r+m-1/2})}{(x_{i-r+l-1/2} - x_{i-r+m-1/2})} \right) \right]$$

and hence

$$p_i^r(x) = \sum_{l=1}^{k} \left(\left(V(x_{i-r+l-1/2}) - V(x_{i-r-1/2}) \right) \frac{\sum_{m=0,m\neq l}^{k} \prod_{n=0,n\neq l,m}^{k} (x - x_{i-r+n-1/2})}{\prod_{m=0,m\neq l}^{k} (x_{i-r+l-1/2} - x_{i-r+m-1/2})} \right). \tag{3.7}$$

Evaluing p_i^r at the cell boundary $x_{i+1/2}$ and employing (3.5), we obtain

$$p_i^r(x_{i+1/2}) = \sum_{l=1}^{k} \left(\left(\sum_{j=0}^{l-1} \overline{v}_{i-r+j} \Delta x_{i-r+j} \right) \frac{\sum_{m=0,m\neq l}^{k} \prod_{n=0,n\neq l,m}^{k} (x_{i+1/2} - x_{i-r+n-1/2})}{\prod_{m=0,m\neq l}^{k} (x_{i-r+l-1/2} - x_{i-r+m-1/2})} \right). \tag{3.8}$$

Rearranging, we obtain

$$v_{i+1/2} = p_i^r(x_{i+1/2}) = \sum_{j=0}^{k-1} \sum_{l=j+1}^{k} \frac{\sum_{m=0,m\neq l}^{k} \prod_{n=0,n\neq l,m}^{k} (x_{i+1/2} - x_{i-r+n-1/2})}{\prod_{m=0,m\neq l}^{k} (x_{i-r+l-1/2} - x_{i-r+m-1/2})} \ \Delta x_{i-r+j} \overline{v}_{i-r+j}.$$

Therefore, the reconstruction coefficients c_{ij}^r are given by

$$c_{ij}^r = \sum_{l=j+1}^{k} \frac{\sum_{m=0,m\neq l}^{k} \prod_{n=0,n\neq l,m}^{k} (x_{i+1/2} - x_{i-r+n-1/2})}{\prod_{m=0,m\neq l}^{k} (x_{i-r+l-1/2} - x_{i-r+m-1/2})} \ \Delta x_{i-r+j} \tag{3.9}$$

and depend on r and k.

3.1 Summary

In summary, given a stencil $S_i^{r,k}$ and the cell averages $\overline{v}_j$ of a function v, the approximation $v_{i+1/2}$ to the function v at the cell boundary $x_{i+1/2}$ is given by

$$v_{i+1/2} = \sum_{j=0}^{k-1} c_{ij}^r \ \overline{v}_{i-r+j} \tag{3.10}$$

where

$$c_{ij}^r = \sum_{l=j+1}^{k} \frac{\sum_{m=0,m\neq l}^{k} \prod_{n=0,n\neq l,m}^{k} (x_{i+1/2} - x_{i-r+n-1/2})}{\prod_{m=0,m\neq l}^{k} (x_{i-r+l-1/2} - x_{i-r+m-1/2})} \ \Delta x_{i-r+j}. \tag{3.11}$$

Furthermore, the approximation $v_{i+1/2}$ is accurate to order k so that

$$v_{i+1/2} = v(x_{i+1/2}) + O(\Delta x^k)$$

where

$$\Delta x = \max_{j=i-r,\ldots,i-r+k-1} \Delta x_j.$$

Finally, the approximation $v_{i-1/2}$ to the function v at the cell boundary $x_{i-1/2}$ is given by

$$v_{i-1/2} = \sum_{j=0}^{k-1} c_{(i-1)\,j}^{r-1} \ \overline{v}_{i-r+j}.$$

The permissable values of left shift parameter r in (3.10) are $-1, \ldots, k-1$ so that the results of Appendix A hold.

4 One dimensional reconstruction for piece-wise smooth functions

The solutions of hyperbolic conservation laws may contain discontinuities, and therefore we are interested in reconstructing piecewise smooth functions. A piecewise smooth function v is smooth except at finitely many isolated points. At these points, v and its derivatives (at least up to the order of the scheme) are assumed to have finite left and right limits.

For such piecewise smooth functions, the order of accuracy herein referred to is formal- that is, it is defined as the accuracy determined by the local error in the smooth regions of the function.

The basic idea of WENO is to use a convex combination of several stencils to form the reconstruction of v at the cell boundaries, and, if a stencil contains a discontinuity, its weight should be close to zero. In smooth regions, using several stencils will also serve to increase the order of accuracy.

Consider the k stencils

$$S_i^{r,k} \quad \text{for} \quad r = 0, \dots, k-1$$

that can be used to reconstruct the value of v at the cell boundaries $x_{i-1/2}$ and $x_{i+1/2}$. These stencils span $2k-1$ cells. We denote the k different reconstructions by

$$v_{i+1/2}^r = \sum_{j=0}^{k-1} c_{ij}^r \bar{v}_{i-r+j} \quad \text{for} \quad r = 0, \dots, k-1 \tag{4.1}$$

where we have added the superscript r to $v_{i+1/2}$ to make the dependance upon the left shift r explicit.

A WENO reconstruction takes a convex combination of all $v_{i+1/2}^r$ defined in (4.1) as a new approximation to $v_{i+1/2}$ according to

$$v_{i+1/2} = \sum_{r=0}^{k-1} \omega_i^r v_{i+1/2}^r \tag{4.2}$$

where we require

$$\omega_i^r \geq 0 \quad \text{and} \quad \sum_{r=0}^{k-1} \omega_i^r = 1. \tag{4.3}$$

In smooth regions where all k stencils that can be used to reconstruct $v_{i+1/2}$ in (4.1) do not contain discontinuities, we could reconstruct $v_{i+1/2}$ to order $2k-1$ using the stencil $S_i^{k-1,2k-1}$ to obtain

$$v_{i+1/2} = \sum_{j=0}^{2k-2} c_{ij}^* \bar{v}_{i-(k-1)+j}. \tag{4.4}$$

Combining (4.1), (4.2), and (4.4), we obtain

$$\sum_{j=0}^{2k-2} c_{ij}^* \bar{v}_{i-(k-1)+j} = \sum_{r=0}^{k-1} \omega_i^r \left(\sum_{l=0}^{k-1} c_{il}^r \bar{v}_{i-r+l} \right). \tag{4.5}$$

Rearranging, we obtain

$$\sum_{j=0}^{2k-2} c_{ij}^* \bar{v}_{i-(k-1)+j} = \sum_{j=0}^{2k-2} \left(\sum_{l=\max(0,j-k+1)}^{\min(k-1,j)} \omega_i^{k-(j+1)+l} c_{il}^{k-(j+1)+l} \right) \bar{v}_{i-(k-1)+j}$$

and hence we obtain systems of the $2k-1$ equations

$$\sum_{l=\max(0,j-k+1)}^{\min(k-1,j)} \omega_i^{k-(j+1)+l} c_{il}^{k-(j+1)+l} = c_{ij}^* \quad \text{for} \quad j = 0, \dots, 2k-2 \tag{4.6}$$

at each i for the weights ω_i^r. For unstructured grids the systems (4.6) are over-determined, and thereeofore we must use some kind of optimisation algorithm in order to find the weights ω_i^r. For structured grids the systems (4.6) are no longer over-determined, and the weights ω_i^r can be found explicity (and are independent of i).

The weights ω_i^r defined by (4.5) and determined by (4.6) are called *optimal weights* since they can be used to reconstruct $v_{i+1/2}$ to order $2k-1$ in regions where v is smooth. We will henceforth denote the optimal weights by ϖ_i^r.

A similar procedure can be used to determine the optimal weights for reconstructing $v_{i-1/2}$ on the stencil $S_i^{r,k}$.

We now consider the practical problem of choosing the weights ω_i^r. If we choose the weights ω_i^r sufficiently close to the optimal weights ϖ_i^r in regions where v is smooth then we can achieve $2k-1$ order accuracy. In order to determine how close to the optimal weights ϖ_i^r the weights ω_i^r must be choosen we consider the reconstruction

$$v_{i+1/2} = \sum_{r=0}^{k-1} \omega_i^r v_{i+1/2}^r = \sum_{r=0}^{k-1} \varpi_i^r v_{i+1/2}^r + \sum_{r=0}^{k-1} \big(\omega_i^r - \varpi_i^r\big) v_{i+1/2}^r. \tag{4.7}$$

If we choose

$$\omega_i^r = \varpi_i^r + O(\Delta x^{k-1}) \tag{4.8}$$

then each term in the last summation of (4.8) becomes $O(\Delta x^{2k-1})$ hence $2k-1$ order accuracy is preserved by the reconstruction.

If we define

$$\omega_i^r = \frac{\alpha_i^r}{\alpha_i^0 + \cdots + \alpha_i^{k-1}} \tag{4.9}$$

where

$$\alpha_i^r = \frac{\varpi_i^r}{(\epsilon + \sigma_i^r)^p} \qquad \text{for} \qquad r = 0, \ldots, k-1; \tag{4.10}$$

and ϵ is a positive real number used to avoid dividing by zero (usually $\epsilon = 10^{-6}$), p is some power, and σ_i^r is a measure of the smoothness of the function v in the stencil $S_i^{r,k}$; with the smoothnesses σ_i^r chosen appropriately, then (4.8) is satisfied.

A Error of Lagrange interpolating polynomials

Let $f(x) \in C^n([a,b])$, and $p(x)$ be the interpolating polynomial of degree $n-1$ such that

$$p(x_i) = f(x_i) \quad \text{for} \quad i = 1, \dots, n \tag{A.1}$$

where

$$a = x_1 < x_2 < \cdots < x_{n-1} < x_n = b. \tag{A.2}$$

Then

$$p(x) = f(x) + O(\Delta x^n) \quad \text{for} \quad x \in [a,b] \tag{A.3}$$

where

$$\Delta x = \max_{i=2,\dots,n} x_i - x_{i-1}. \tag{A.4}$$

Proof. Let $x \in [a,b]$. If $x = x_i$ for some $i = 1, \dots, n$ then $f(x) - p(x) = 0$ since $p(x)$ is the interpolating polynomial. Otherwise, let

$$\Phi(x) = \frac{f(x) - p(x)}{\prod_{i=1}^{n}(x - x_i)} \tag{A.5}$$

and

$$g(x,\xi) = f(\xi) - p(\xi) - \Phi(x)\prod_{i=1}^{n}(\xi - x_i). \tag{A.6}$$

Then $g(x,\xi)$ is n times differentiable with respect to ξ, $g(x, x_i) = 0$ for $i = 1, \dots, n$, and $g(x,x) = 0$. Applying Rolle's theorem successively across all interpolation points x_i and x we obtain

$$\left.\frac{\partial^n}{\partial \xi^n} g(x,\xi)\right|_{\xi=\xi^*} = 0 \tag{A.7}$$

for some $\xi^* \in (a,b)$. Futhermore

$$\frac{\partial^n}{\partial \xi^n} g(x,\xi) = f^n(\xi) - n!\,\Phi(x) \tag{A.8}$$

so that, combininig (A.7) and (A.8), we obtain

$$\Phi(x) = \frac{f^n(\xi^*)}{n!} \tag{A.9}$$

and therefore

$$f(x) - p(x) = \frac{f^n(\xi^*)}{n!}\prod_{i=1}^{n}(x - x_i). \tag{A.10}$$

Finally, we conclude that

$$p(x) = f(x) + O(\Delta x^n). \tag{A.11}$$

That is, if $p(x)$ interpolates $f(x)$ at n points, then it is accurate to $O(\Delta x^n)$ where Δx is the maximum space between the interpolating points.