5. Systems of Conservation Law Equations

In the previous sections we have examined a number of issues relating to scalar valued quasilinear partial differential equations of the conservation law form in one space variable and one time variable. The most general problems of this type involve systems of equations in m unknown function of n space variables and a time variable. Problems of this generality are still poorly understood. However, we can generalize to problems involving m unknown functions in one space variable and one time variable. We list now several examples of such systems.

Example 5.1

1. ***The p-system***- Consider the system of two equations for two unknown functions $u = u(x,t),\ v = v(x,t)$

$$\partial_t u(x,t) - \partial_x v(x,t) = 0$$
$$\partial_t v(x,t) - \partial_x p(u) = 0$$

This can be written in vector notation as,

.

$$\partial_t \vec{u} = \partial_x \vec{F}(\vec{u}) \quad where \quad \vec{u} = [u,v], \quad \vec{F}(\vec{u}) = [-v,-p(u)]$$

.

Note that $\partial_{tt} u = \partial_{tx} v = \partial_{xt} v = \partial_{xx} p(u),$ hence this system is equivalent to a single nonlinear wave equation of order two for the unknown function u.

.

2. ***Shallow Water Waves*** The following system for unknown functions $v = v(x,t)\ and\ h = h(x,t)$ can be used to describe waves propagating with horizontal velocity v in a shallow pool of depth h.

$$\partial_t h + \partial_x (vh) = 0$$
$$\partial_t v + \partial_x (v^2/2 + h) = 0$$

.

The first equation expresses the conservation of mass and the second expresses the conservation of momentum in the fluid.

.

3. ***Euler's Formulation of the Equations of Gas Dynamics*** The equations for the 1-dimensional flow of a compressible gas are

.

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho v) = 0 && (conservation\ of\ mass) \\ &\partial_t(\rho v) + \partial_x(\rho v^2 + p) = 0 && (conservation\ of\ momentum) \\ &\partial_t s + v\,\partial_x s = 0 && (conservation\ of\ energy) \end{aligned}$$

.

where the unknowns $p, \rho, v\ and\ s$ are pressure, density, velocity and entropy. If we add a ***state equation*** $p = p(\rho, s)$, then the second equation can be replaced by

.

$$\partial_t v + v\,\partial_x v + \frac{c^2}{\rho}\partial_x \rho + \sigma\,\partial_x s = 0$$

.

where $c^2 = \partial_\rho p)_s$ and $\sigma = \frac{1}{\rho}\,\partial_s p)_\rho$. Then c has the interpretation of the local speed of propagation.

Problem 5.1 Show how the state equation can be used to eliminate the pressure from the momentum equation.

Problem 5.2 Suppose that the entropy is a constant, (i.e., isentropic flow) and that the equation of state reduces to $p = k\rho^\gamma$ (a polytropic gas). What does the momentum equation reduce to in this case?

Weak Solutions

As we did in the case of the scalar conservation law equations, we will define a weak solution of the system,

$$\partial_t \vec{u}(x,t) + \partial_x \vec{F}(\vec{u}(x,t)) = \vec{0}, \qquad \vec{u}(x,0) = \vec{u}_0(x) \qquad (5.1)$$

to be a function which is locally integrable in R^2_+ with values in R^n such that

$$\iint_{R^2_+} \vec{u} \bullet \partial_t \vec{\phi} + \vec{F}(\vec{u}) \bullet \partial_x \vec{\phi}\; dxdt + \int_R \vec{u}_0 \bullet \vec{\phi}(x,0)\,dx = 0$$

for all vector valued test functions $\vec{\phi}$.

Rankine-Hugoniot Condition

If there is a curve $x = x(t)$ in the x-t plane across which $\vec{u} = \vec{u}(x,t)$ experiences a jump discontinuity, then we can use essentially the same argument that was employed in the scalar case to show that

$$x'(t)\,[\vec{u}_L - \vec{u}_R] = [\vec{F}(\vec{u}_L) - \vec{F}(\vec{u}_R)] \qquad (5.2)$$

Note that unlike the scalar case, which was a scalar equation that could be solved for the unknown function $x(t)$, this is a system of n equations relating the $2n+1$ quantities, $\vec{u}_L,\ \vec{u}_R \text{ and } \sigma = x'(t)$. We will consider some shock problems later.

Travelling Waves and Strictly Hyperbolic Systems

Consider the quasilinear system of n equations in n unknown functions,

$$\partial_t \vec{u}(x,t) + A(\vec{u}(x,t))\,\partial_x \vec{u}(x,t) = \vec{0}. \qquad (5.3)$$

This is an equation of conservation law form if the n by n matrix A is equal to

$$grad\vec{F}(\vec{u}) = \begin{bmatrix} \partial_{u_1}F_1 & \partial_{u_2}F_1 & \cdots & \partial_{u_n}F_1 \\ \partial_{u_1}F_2 & \cdots & & \partial_{u_n}F_2 \\ \vdots & & \ddots & \vdots \\ \partial_{u_1}F_n & & & \partial_{u_n}F_n \end{bmatrix}$$

A ***travelling wave solution*** to the equation (5.3) is a solution of the form $\vec{u} = \vec{f}(x - \sigma t)$ where the wave form, $\vec{f},\ and\ \sigma,$ the wave speed, are to be found. We see that

$$\partial_t\vec{u}(x,t) = \vec{f}'(x - \sigma t)(-\sigma) \qquad and \qquad \partial_x\vec{u}(x,t) = \vec{f}'(x - \sigma t)(1)$$

hence (5.1) becomes $\quad \vec{f}'(\theta)(-\sigma) + A(\vec{u})\vec{f}'(\theta) = 0, \quad or$

$$A(\vec{u})\vec{x} = \sigma\vec{x} \qquad where \quad \vec{x} = \vec{f}'(\theta), \quad \theta = x - \sigma t.$$

Evidently, the eigenvalues of A are the wave speeds and the associated eigenvector determines the wave form that travels at that speed. Then the existence of travelling wave solutions to (5.1) is dependent on the existence of real eigenvalues for the matrix A. Therefore, we define the system (5.3) to be ***strictly hyperbolic*** if, for each $\vec{z} \in R^n,$ the n by n matrix $A(\vec{z})$ has n real, distinct eigenvalues. Although problems in which the matrix does not have real distinct eigenvalues are of interest, they are more diificult to analyze and we will therefore suppose throughout this section that our problems are strictly hyperbolic.

Example 5.2

1. ***The p-system***- Consider the p-system for the unknown functions $u = u(x,t),\ v = v(x,t)$
.

$$\partial_t\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ p'(u) & 0 \end{bmatrix}\partial_x\begin{Bmatrix} u \\ v \end{Bmatrix}$$

Then

$$\begin{array}{lll} matrix\ eigenvalue & \lambda_1(u) = -\sqrt{p'(u)} & \lambda_2(u) = \sqrt{p'(u)} \\ eigenvector & \vec{x}_1 = \left[1,\ \sqrt{p'(u)}\ \right] & \vec{x}_2 = \left[1,\ -\sqrt{p'(u)}\ \right] \end{array}$$

It is evident that this system is strictly hyperbolic provided that $p'(u) > 0$ and in that case, travelling wave solutions are of the form

$$\vec{x} = \vec{f}'(\theta) = \begin{Bmatrix} u'(\theta) \\ v'(\theta) \end{Bmatrix} = \begin{Bmatrix} 1 \\ \pm\sqrt{p'(u(\theta))} \end{Bmatrix}.$$

We find then

$$u(\theta) = \theta - \theta_0, \qquad v(\theta) = \int_{\theta_0}^{\theta} \sqrt{p'(s)}\, ds$$

.

2. ***Shallow Water Waves*** Consider the equations

.

$$\begin{aligned} \partial_t h + v\,\partial_x h + h\,\partial_x v &= 0 \\ \partial_t v + v\,\partial_x v + \partial_x h &= 0; \end{aligned}$$

i.e.,

$$\partial_t \begin{Bmatrix} h \\ v \end{Bmatrix} = \begin{bmatrix} v & h \\ 1 & v \end{bmatrix} \partial_x \begin{Bmatrix} h \\ v \end{Bmatrix}.$$

Then

$$\begin{array}{lll} \textit{matrix eigenvalue} & \lambda_1 = v - \sqrt{h} & \lambda_2 = v + \sqrt{h} \\ \textit{eigenvector} & \vec{x}_1 = \left[\sqrt{h}\,, -1\right] & \vec{x}_2 = \left[\sqrt{h}\,, 1\right] \end{array}$$

This example is also strictly hyperbolic, for $h > 0$, and simple wave solutions have the form

$$\vec{x} = \vec{f}'(\theta) = \begin{Bmatrix} h'(\theta) \\ v'(\theta) \end{Bmatrix} = \begin{Bmatrix} \sqrt{h(\theta)} \\ \pm 1 \end{Bmatrix}.$$

i.e.,

$$h(\theta) = \tfrac{1}{4}(\theta - \theta_0)^2 + h_0, \qquad v(\theta) = \pm(\theta - v_0)$$

.

3. ***Euler's Formulation of the Equations of Gas Dynamics***

The gas dynamics equations

$$\begin{aligned} \partial_t \rho + \partial_x(\rho v) &= 0 \\ \partial_t v + v\,\partial_x v + \frac{c^2}{\rho}\partial_x \rho + \sigma\,\partial_x s &= 0 \\ \partial_t s + v\,\partial_x s &= 0 \end{aligned}$$

can be written in matrix notation as follows,

$$\partial_t \begin{bmatrix} \rho \\ v \\ s \end{bmatrix} + \begin{bmatrix} v & \rho & 0 \\ \frac{c^2}{\rho} & v & \sigma \\ 0 & 0 & v \end{bmatrix} \partial_x \begin{bmatrix} \rho \\ v \\ s \end{bmatrix} = \vec{0}$$

Since this matrix has eigenvalues $\lambda = v, v \pm c$, it is clear that this system is also strictly hyperbolic if $c^2 = \partial_\rho p)_s > 0$. The eigens are,

$$matrix\ eigenvalue \quad \lambda_1 = v \qquad \begin{array}{l} \lambda_2 = v + c \\ \lambda_3 = v - c \end{array}$$

$$eigenvector \qquad \vec{x}_1 = [-\sigma, 0, c^2/\rho] \quad \begin{array}{l} \vec{x}_2 = [\rho, c, 0] \\ \vec{x}_3 = [-\rho, c, 0] \end{array}$$

Riemann Invariants

Suppose (5.3) is strictly hyperbolic and let the eigenvalues of $A = A(\vec{u})$ be arranged in increasing order,

$$\lambda_1(\vec{u}) < \lambda_2(\vec{u}) < \dots < \lambda_n(\vec{u}).$$

For $1 \leq m \leq n$, let $\vec{r}_m(\vec{u})$ denote the associated right eigenvector; i.e., $A(\vec{u})\vec{r}_m = \lambda_m \vec{r}_m$. Since the eigenvalues of A are distinct, the eigenvectors $\vec{r}_m$ span R^n for every $\vec{u}$. Since $A \ and\ A^\top$ share the same eigenvalues, we can denote by $\vec{q}_m(\vec{u})$ the eigenvector for $A^\top(\vec{u})$ corresponding to the eigenvalue $\lambda_m(\vec{u})$; *i.e.*,

$$A^\top(\vec{u})\vec{q}_m = \lambda_m \vec{q}_m \qquad 1 \leq m \leq n$$

or

$$\vec{q}_m^\top A(\vec{u}) = \lambda_m \vec{q}_m^\top \qquad \left(\vec{q}_m^\top \text{ is a left eigenvector for } A(\vec{u})\right)$$

Note that $\quad \vec{q}_j^\top \vec{r}_k = 0 \quad for \quad k \neq j;$

i.e.,

$$\lambda_k(\vec{q}_j^\top \vec{r}_k) = \vec{q}_j^\top \lambda_k \vec{r}_k = \vec{q}_j^\top A(\vec{u})\vec{r}_k = \lambda_j \vec{q}_j^\top \vec{r}_k$$

Then

$$(\lambda_j - \lambda_k)\,\vec{q}_j^\top \vec{r}_k = 0 \quad \text{and since } \lambda_j - \lambda_k \neq 0 \quad for \quad k \neq j,$$

the result follows.

Consider now a scalar valued function $Q_j = Q_j(\vec{u})$ and note that

$$\partial_t Q_j(\vec{u}) + \lambda_j \partial_x Q_j(\vec{u}) = \nabla Q_j(\vec{u}) \bullet \partial_t \vec{u} + \lambda_j \nabla Q_j(\vec{u}) \bullet \partial_x \vec{u}$$

$$= \nabla Q_j(\vec{u}) \bullet (\partial_t \vec{u} + \lambda_j \partial_x \vec{u}) = \nabla Q_j(\vec{u}) \bullet (-A(\vec{u})\partial_x \vec{u} + \lambda_j \partial_x \vec{u}).$$

Now if $\nabla Q_j(\vec{u}) = \vec{q}_j(\vec{u})$, it follows that

$$\partial_t Q_j(\vec{u}) + \lambda_j \partial_x Q_j(\vec{u}) = 0$$

which, in turn implies that

$Q_j(\vec{u})$ is constant along solution curves of $x'(t) = \lambda_j(\vec{u})$.

Evidently, the scalar valued functions $Q_j = Q_j(\vec{u}),\quad 1 \le j \le n,$ can serve as alternative dependent variables in the equation (5.3). The advantage of this choice of dependent variable is that the system becomes diagonal when written in terms of the $Q_j's$ instead of the u_j's. This notion can be exploited in various ways.

Example 5.3

1. Consider the following linear system

$$\partial_t \begin{Bmatrix} u \\ v \end{Bmatrix} + \begin{bmatrix} 0 & -1 \\ -c^2 & 0 \end{bmatrix} \partial_x \begin{Bmatrix} u \\ v \end{Bmatrix} = \vec{0}.$$

The distinct real eigenvalues and the associated left eigenvectors are

$$\lambda_1 = -c, \qquad \vec{q}_1 = [c, 1]$$
$$\lambda_2 = c\ , \qquad \vec{q}_2 = [-c, 1]$$

Then

$$\vec{q}_1 \bullet \partial_t \begin{Bmatrix} u \\ v \end{Bmatrix} + \vec{q}_1 \bullet \begin{bmatrix} 0 & -1 \\ -c^2 & 0 \end{bmatrix} \partial_x \begin{Bmatrix} u \\ v \end{Bmatrix}. =$$

.

$$c\partial_t u + \partial_t v + \lambda_1 (c\partial_x u + \partial_x v) = 0$$

and

$$\vec{q}_2 \bullet \partial_t \begin{Bmatrix} u \\ v \end{Bmatrix} + \vec{q}_2 \bullet \begin{bmatrix} 0 & -1 \\ -c^2 & 0 \end{bmatrix} \partial_x \begin{Bmatrix} u \\ v \end{Bmatrix}. =$$

.

$$-c\partial_t u + \partial_t v + \lambda_2 (-c\partial_x u + \partial_x v) = 0,$$

or

$$c(\partial_t u + \lambda_1 \partial_x u) + \partial_t v + \lambda_1 \partial_x v = 0$$
$$-c(\partial_t u + \lambda_2 \partial_x u) + \partial_t v + \lambda_2 \partial_x v = 0$$

Let C_1, C_2 denote solution curves for the systems

$$C_1 : t'(\alpha) = 1, \qquad x'(\alpha) = \lambda_1 = -c \qquad \text{i.e., } x(t) = -ct + x_0$$
$$C_2 : t'(\beta) = 1, \qquad x'(\beta) = \lambda_2 = c \qquad \text{i.e., } x(t) = ct + x_0$$

These straight line solution curves are the characteristics for this constant coefficient system of linear conservation law equations. Along characteristics, the conservation laws reduce to odes as follows,

$$c\frac{d}{d\alpha}u + \frac{d}{d\alpha}v = 0 \qquad on\ C_1$$

and

$$-c\frac{d}{d\beta}u + \frac{d}{d\beta}v = 0 \qquad on\ C_2$$

This implies that $R_\alpha(x,t) = v(x,t) + cu(x,t)$ is constant along C_1 characteristics and $R_\beta(x,t) = v(x,t) - cu(x,t)$ is constant along C_2 characteristics. This leads to,

$$R_\alpha(x,t) = v(x,t) + cu(x,t) = F(x+ct)$$
$$R_\beta(x,t) = v(x,t) - cu(x,t) = G(x-ct)$$

If we are given initial values for u and v, then these expressions permit the computation of u and v at every point in the upper half plane.

2. Recall that for the system

.

$$\partial_t \begin{Bmatrix} h \\ v \end{Bmatrix} + \begin{bmatrix} v & h \\ 1 & v \end{bmatrix} \partial_x \begin{Bmatrix} h \\ v \end{Bmatrix} = \vec{0}.$$

.

we have distinct real eigenvalues $\lambda(u) = v \pm \sqrt{h}$ and the associated left eigenvectors are

$$\lambda_1(u) = v - \sqrt{h} \text{ corresponds to left eigenvector } \vec{q}_1 = \left[-1, \sqrt{h}\,, \right]$$
$$\lambda_2(u) = v + \sqrt{h} \text{ corresponds to left eigenvector } \vec{q}_2 = \left[1, \sqrt{h}\,\right].$$

Then

$$\vec{q}_1 \bullet \partial_t \begin{Bmatrix} h \\ v \end{Bmatrix} + \vec{q}_1 \bullet \begin{bmatrix} v & h \\ 1 & v \end{bmatrix} \partial_x \begin{Bmatrix} h \\ v \end{Bmatrix}. =$$

.

$$-\partial_t h + \sqrt{h}\,\partial_t v + \lambda_1 \left(-\partial_x h + \sqrt{h}\,\partial_x v\right) = 0$$

.

$$\vec{q}_2 \bullet \partial_t \begin{Bmatrix} h \\ v \end{Bmatrix} + \vec{q}_2 \bullet \begin{bmatrix} v & h \\ 1 & v \end{bmatrix} \partial_x \begin{Bmatrix} h \\ v \end{Bmatrix}.$$

.

$$= \partial_t h + \sqrt{h}\,\partial_t v + \lambda_2 \left(\partial_x h + \sqrt{h}\,\partial_x v\right) = 0$$

or

$$\partial_t h + \lambda_1\,\partial_x h - \sqrt{h}\,(\partial_t v + \lambda_1\,\partial_t v) = 0$$
$$\partial_t h + \lambda_2\,\partial_x h + \sqrt{h}\,(\partial_t v + \lambda_2\,\partial_t v) = 0$$

.

Let C_1, C_2 denote solution curves for the systems

$$C_1 : t'(\alpha) = 1, \qquad x'(\alpha) = \lambda_1$$
$$C_2 : t'(\beta) = 1, \qquad x'(\beta) = \lambda_2$$

These solutions curves are the characteristics for this system of conservation law equations. Along characteristics, the conservation laws reduce to odes as follows,

.

$$\frac{d}{d\alpha} h - \sqrt{h}\,\frac{d}{d\alpha} v = 0 \qquad on\ C_1$$

and

$$\frac{d}{d\beta}h + \sqrt{h}\,\frac{d}{d\beta}v = 0 \qquad on\ C_2$$

.

If we let $h = c^2$ then $\frac{d}{d\alpha}h = 2c\frac{d}{d\alpha}c$ etc and these equations reduce further to

.

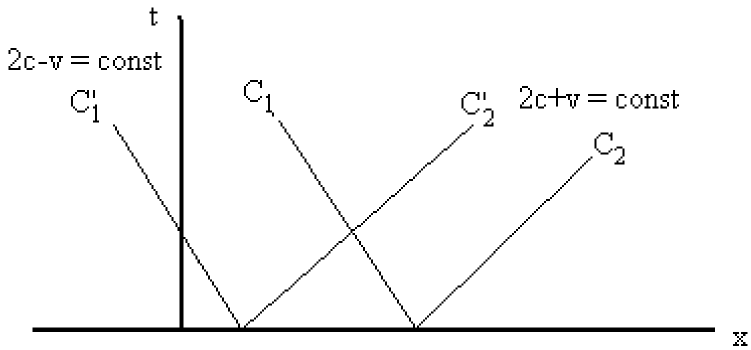
$$2c\frac{d}{d\alpha}c \ - c\frac{d}{d\alpha}v = 0 \qquad and \qquad 2c\frac{d}{d\beta}c \ + c\frac{d}{d\beta}v = 0;$$

i.e.,

$$\frac{d}{d\alpha}(2c \ - v) = 0 \qquad and \qquad \frac{d}{d\beta}(2c \ + v) = 0$$

.

Evidently, $2c \pm v$ are constant along characteristics (which are curves, in general).



2c-v, 2c+v constant on characteristics

.

3. The eigenvalues and left eigenvectors for the system

.

$$\partial_t \begin{bmatrix} \rho \\ v \\ s \end{bmatrix} + \begin{bmatrix} v & \rho & 0 \\ \frac{c^2}{\rho} & v & \sigma \\ 0 & 0 & v \end{bmatrix} \partial_x \begin{bmatrix} \rho \\ v \\ s \end{bmatrix} = \vec{0}$$

are

$$\begin{array}{lll} matrix\ eigenvalue & \lambda_1 = v & \begin{array}{l} \lambda_2 = v + c \\ \lambda_3 = v - c \end{array} \\ eigenvector & \vec{q}_1 = [0,0,1] & \begin{array}{l} \vec{q}_2 = [c^2, \rho c, \sigma] \\ \vec{q}_3 = [c^2, -\rho c, \sigma] \end{array} \end{array}$$

Then

$$\vec{q}_1 \bullet (\partial_t \vec{u} + A\,\partial_x \vec{u}) = \partial_t s + \lambda_1\,\partial_x s = 0$$
$$\vec{q}_2 \bullet (\partial_t \vec{u} + A\,\partial_x \vec{u}) = \partial_t \rho + \frac{\rho}{c}\partial_t v + \lambda_2(\,\partial_x \rho + \frac{\rho}{c}\partial_x v) = 0$$

$$\vec{q}_3 \bullet (\partial_t \vec{u} + A\,\partial_x \vec{u}) = \partial_t \rho - \frac{\rho}{c}\partial_t v + \lambda_3(\partial_x \rho - \frac{\rho}{c}\partial_x v) = 0.$$

.

If we now suppose that the gas is isentropic (i.e., s is constant) then $\partial_x p = c^2 \partial_x \rho$. In addition, if the gas is a so called polytropic gas, then $p = k\rho^\gamma$ and $c^2 = k\gamma\rho^{\gamma-1}$. The equations now reduce to

.

$$\frac{c}{\rho}(\partial_t \rho + \lambda_2\,\partial_x \rho) + \partial_t v + \lambda_2\,\partial_x v = 0 \qquad or \qquad \frac{c}{\rho}\frac{d\rho}{d\alpha} + \frac{dv}{d\alpha} = 0$$

$$\frac{c}{\rho}(\partial_t \rho + \lambda_3\,\partial_x \rho) - (\partial_t v + \lambda_3\,\partial_x v) = 0 \qquad or \qquad \frac{c}{\rho}\frac{d\rho}{d\beta} - \frac{dv}{d\beta} = 0,$$

.

where $d/d\alpha,\ d/d\beta$ denote differentiation along the characteristics associated with $x'(t) = \lambda_2,\ x'(t) = \lambda_3,$ respectively. Since

.

$$\int \frac{c}{\rho} d\rho = \frac{2c(\rho)}{\gamma - 1}$$

.

It follows that if $r_1 = \dfrac{2c(\rho)}{\gamma-1} + v \ and \ r_2 = \dfrac{2c(\rho)}{\gamma-1} - v,\ then$

$r_1 = const$ along the characteristics associated with $x'(t) = c + v$

$r_2 = const$ along the characteristics associated with $x'(t) = c - v$

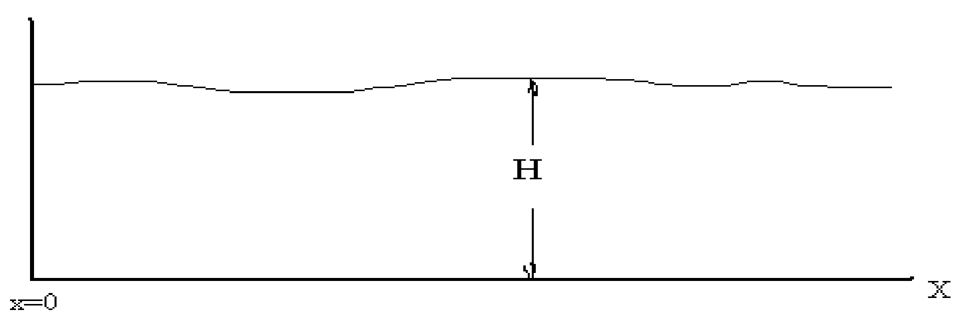
i.e., $r_1 \ and \ r_2$ are the Riemann invariants for this system.

We will now show how the Riemann invariants can be exploited in solving some problems.

6. Applications of Riemann Invariants

We will now show how Riemann invariants can be used to solve various problems for systems of quasilinear pde's.

Example 6.1 A Ripple Tank



A Ripple Tank of depth H

We consider the propagation through a tank of shallow water of a wave introduced at the boundary of the tank. If h and v denote, respectively, the depth of the water and the fluid velocity, then the governing equations are

$$\partial_t h + v\,\partial_x h + h\,\partial_x v = 0 \qquad (6.1)$$
$$\partial_t v + v\,\partial_x v + \partial_x h = 0.$$

Now we assume $h(x,t) = H + w(x,t) \quad where \quad w << 1,$ and $v(x,t) << 1,$

and we suppose $w(x,0) = v(x,0) = 0, \qquad and \qquad v(0,t) = \varepsilon \sin \Omega t.$

These conditions could be induced by moving the end of an initially quiet water tank back and forth in a periodic fashion but with small amplitude. Then if we neglect all products of small terms, the equations simplify to

$$\partial_t \begin{bmatrix} w \\ v \end{bmatrix} + \begin{bmatrix} 0 & H \\ 1 & 0 \end{bmatrix} \partial_x \begin{bmatrix} w \\ v \end{bmatrix} = \vec{0}. \qquad (6.2)$$

These are the linearized shallow water equations. The eigenvalues and left eigenvectors of the coefficient matrix are,

$$\begin{array}{ll} \lambda_1 = \sqrt{H} & \vec{q}_1 = \left(\sqrt{H}, H\right) \\ \lambda_2 = -\sqrt{H} & \vec{q}_2 = \left(\sqrt{H}, -H\right) \end{array}.$$

Then $\quad \vec{q}_j \bullet (\partial_t \vec{u} + A\,\partial_x \vec{u}) = 0, \quad j = 1,2 \quad$ leads to,

$$\sqrt{H}\,(\partial_t w + \lambda_1\,\partial_x w) + H(\partial_t v + \lambda_1\,\partial_x v) = 0$$
$$(6.3)$$
$$\sqrt{H}\,(\partial_t w + \lambda_2\,\partial_x w) - H(\partial_t v + \lambda_2\,\partial_x v) = 0.$$

The characteristics for this system are solution curves C_k for $x_k'(t) = \lambda_k = \pm\sqrt{H},\ k = 1,2;$ i.e.,

$$C_1 : \quad x_1(t) = t\sqrt{H} + x_0$$
$$C_2 : \quad x_2(t) = -t\sqrt{H} + x_0.$$

Note that since the system is linear with constant coefficients, the characteristics are two families of parallel straight lines. The equations (6.3) can be written as

$$\sqrt{H}\,\frac{dw}{d\alpha} + H\frac{dv}{d\alpha} = 0$$

$$\sqrt{H}\,\frac{dw}{d\beta} - H\frac{dv}{d\beta} = 0$$

where $d/d\alpha \ and \ d/d\beta$ denote differentiation along characteristics $C_1 \ and \ C_2$ respectively. It follows that the Riemann invariants are

$$r_1(x,t) = \sqrt{H}\, w(x,t) + Hv(x,t) \quad on \text{ characteristics } C_1$$

$$r_2(x,t) = \sqrt{H}\, w(x,t) - Hv(x,t) \quad on \text{ characteristics } C_2$$

For C_1 characteristics originating at points $(x_0, 0)$ with $x_0 \geq 0$, we have

$$r_1(x,t) = r_1(x_0,0) \qquad on \quad x - t\sqrt{H} = x_0.$$

But $\quad r_1(x_0,0) = \sqrt{H}\, w(x_0,0) + Hv(x_0,0) = 0$

hence

$$r_1(x,t) = 0 \qquad on \quad x - t\sqrt{H} = x_0 \geq 0, \quad t > 0.$$

Similarly, $\quad r_2(x,t) = r_2(x_0,0) \qquad on \quad x + t\sqrt{H} = x_0 \geq 0,$

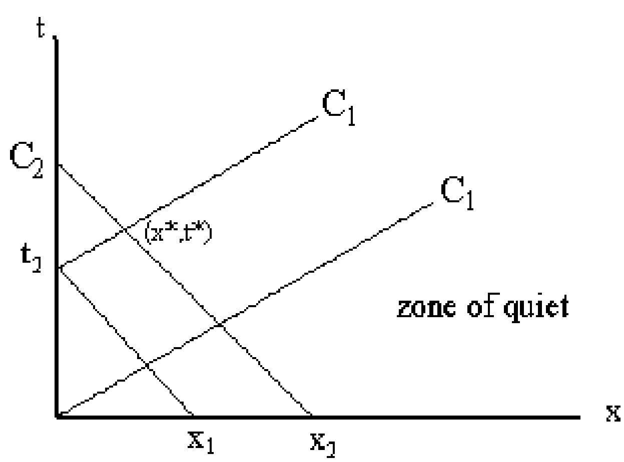
and

$$r_2(x_0,0) = \sqrt{H}\, w(x_0,0) - Hv(x_0,0) = 0,$$

hence

$$r_2(x,t) = 0 \qquad on \quad x + t\sqrt{H} = x_0 \geq 0, \quad t > 0$$

Then $r_1(x,t) = r_2(x,t) = 0 \; for \; t > 0, \; x > t\sqrt{H}$ and this leads to $w(x,t) = v(x,t) = 0$ in $\{t > 0, \; x > t\sqrt{H}\}$. This region is referred to as the "zone of quiet". Now consider the region $\{t > 0, \; 0 < x < t\sqrt{H}\}$.



Since $r_1(x,t)$ is constant on C_1 characteristics and $r_2(x,t)$ is constant on C_2, we have, for any $(x^*, t^*) \in \{t > 0, \; 0 < x < t\sqrt{H}\}$ there exist $x_2 > x_1 > 0$, and $t_2 > 0$ such that

$$i) \;\; r_1(0,t_2) = r_1(x^*,t^*),$$

$$ii) \;\; r_2(x_2,0) = r_2(x^*,t^*),$$

and, $\quad iii) \;\; r_2(x_1,0) = r_2(0,t_2).$

Then iii) implies

$$\sqrt{H}\, w(0,t_2) - Hv(0,t_2),= \sqrt{H}\, w(x_1,0) - Hv(x_1,0) = 0$$

i.e.,

$$w(0,t_2) = \sqrt{H}\ v(0,t_2) = \varepsilon\sqrt{H}\ \sin \Omega t_2.$$

Also, the C_1 characteristic originating at $(0,t_2)$ is given by $x_1(t) = (t-t_2)\sqrt{H}$ and since (x^*,t^*) lies on this line, it follows that $t_2 = t^* - \dfrac{x^*}{\sqrt{H}}$.

Now ii) implies $r_2(x^*,t^*) = 0$ so $w(x^*,t^*) = \sqrt{H}\ v(x^*,t^*)$, and finally, $i)$ implies

$$w(x^*,t^*) + \sqrt{H}\ v(x^*,t^*) = w(0,t_2) + \sqrt{H}\ v(0,t_2).$$

and we can combine all these equalities to conclude that at any point $(x^*,t^*) \in \left\{t > 0,\ 0 < x < t\sqrt{H}\right\}$

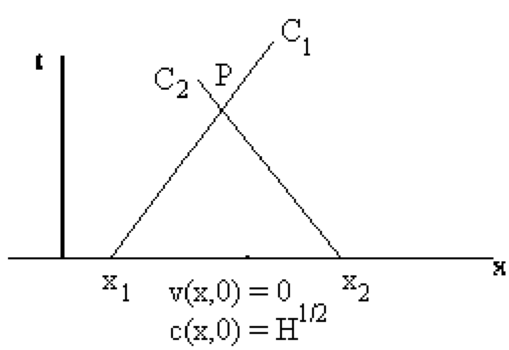
$$\begin{aligned}
v(x^*,t^*) &= v(0,t_2) = \varepsilon \sin \Omega\left(t^* - \frac{x^*}{\sqrt{H}}\right)\\
w(x^*,t^*) &= \sqrt{H}\ v(x^*,t^*) = \varepsilon\sqrt{H}\ \sin \Omega\left(t^* - \frac{x^*}{\sqrt{H}}\right)\\
h(x^*,t^*) &= H + \sqrt{H}\,\varepsilon \sin \Omega\left(t^* - \frac{x^*}{\sqrt{H}}\right),
\end{aligned}$$

from which we see that the response to the moving boundary is a travelling wave of amplitude $\sqrt{H}\,\varepsilon$, having the same frequency as the input but with a delay due to the finite speed of propagation.

Now compare this result to what we find from solving the quasilinear system (6.1). We know from example 5.3.1 that

$r_1(x,t) = 2c(x,t) + v(x,t)$ is constant along solution curves of $x_1'(t) = v + \sqrt{h}$
$r_2(x,t) = 2c(x,t) - v(x,t)$ is constant along solution curves of $x_2'(t) = v - \sqrt{h}$,

where $c = \sqrt{h}$.



Then for characteristics $C_1.C_2$ originating at arbitrary points $x_2 > x_1 > 0$ we have

$$r_1(x,t) = r_1(x_1,0) = 2c(x_1,0) + v(x_1,0) = 2\sqrt{H}, \quad on \quad C_1 : \ x_1'(t) = v + \sqrt{h}, \quad x_1(0) = x_1$$

$$r_2(x,t) = r_2(x_2,0) = 2c(x_2,0) - v(x_2,0) = 2\sqrt{H}, \quad on \quad C_2 : \ x_2'(t) = v - \sqrt{h}, \quad x_2(0) = x_2.$$

It is not hard to see that since r_1 *and* r_2 are **both** constant along characteristics

$$\begin{aligned} 2c_P + v_P &= 2\sqrt{H} + 0 = 2\sqrt{H} \\ 2c_P - v_P &= 2\sqrt{H} - 0 = 2\sqrt{H} \end{aligned}$$

hence

$$c_P = \sqrt{H} \qquad \text{and} \qquad v_P = 0;$$

i.e., $c(x,t)$ and $v(x,t)$ are separately constant along C_1 characteristics so that $c(x,t) = \sqrt{H}$, and $v(x,t) = 0$ at all points in the zone of quiet, $\{t > 0,\ x > t\sqrt{H}\}$. Both families of characteristics throughout that region are families of straight lines

$$x_1(t) = t\sqrt{H} + x_0 \qquad and \qquad x_2(t) = -t\sqrt{H} + x_0 \qquad \text{for } x_0 \geq 0.$$

This part of the solution agrees with what we found for the linearized problem.

Now consider the region $\{t > 0,\ 0 < x < t\sqrt{H}\}$. We can show that the C_1 characteristics in this region are straight lines although the C_2 characteristics are, in general, not straight lines in this region. Note first in the following figure that we have $c(P) = c(Q)$ and $v(P) = v(Q)$ since v and c are separately constant along C_1 characteristics in the zone of quiet. In addition, we have

$$r_1 = 2c(R) + v(R) = 2c(S) + v(S),$$

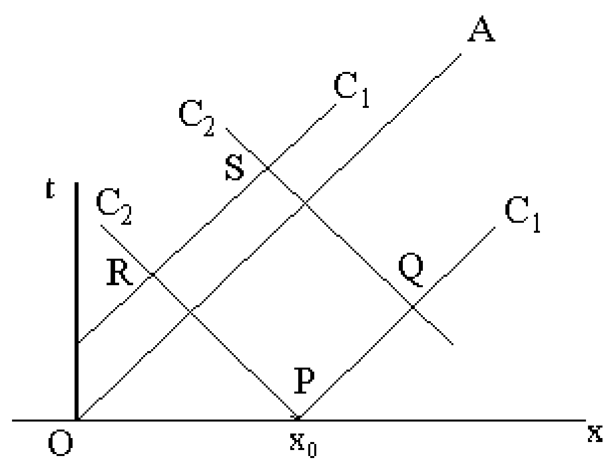
and

$$\begin{aligned} r_2 &= 2c(P) - v(P) = 2c(R) - v(R), \\ r_2 &= 2c(Q) - v(Q) = 2c(S) - v(S). \end{aligned}$$

Since $c(P) = c(Q)$ and $v(P) = v(Q)$, the left sides of these last two equations are identical, leading to

$$2c(R) - v(R) = 2c(S) - v(S).$$

In combination with the first equation for r_1 this implies $c(R) = c(S)$ and $v(R) = v(S)$.



Then the equation for C_1 characteristics in this region is

$$on \quad C_1 : \; x_1'(R) = v(R) + c(R) = v(S) + c(S) = \sqrt{H}, \quad \text{constant}$$

Then the C_1 characteristics in this region are straight lines. Note that this same argument does not extend to the C_2 characteristics.

Now note that for arbitrary $(x^*, t^*) \in \{t > 0,\ 0 < x < t\sqrt{H}\}$ there exist $x_1 > x_2 > 0$ and $t_2 > 0$ such that

$$r_2(0, t_2) = r_2(x_2, 0)$$
$$r_1(x^*, t^*) = r_1(0, t_2).$$
$$r_2(x^*, t^*) = r_2(x_1, 0).$$

Then

$$r_2(x_2, 0) = 2\sqrt{H}, \quad \text{and} \quad r_2(0, t_2) = 2c(0, t_2) - v(0, t_2) = 2\sqrt{H} - \varepsilon \sin \Omega t_2,$$

leads to $$c(0, t_2) = \sqrt{H} + \frac{\varepsilon}{2} \sin \Omega t_2.$$

In addition, $r_2(x^*, t^*) = r_2(x_1, 0) = 2\sqrt{H},\ \textit{and}\ \ r_1(0, t_2) = r_1(x^*, t^*)$;
i.e.,

$$2\sqrt{H} = 2c(x^*, t^*) - v(x^*, t^*)$$
$$2\sqrt{H} + \varepsilon \sin \Omega t_2 + \varepsilon \sin \Omega t_2 = 2c(x^*, t^*) + v(x^*, t^*).$$

This leads to

$$c(x^*, t^*) = \sqrt{H} + \frac{\varepsilon}{2} \sin \Omega \left(t^* - \frac{x^*}{\sqrt{H}} \right)$$
$$v(x^*, t^*) = \varepsilon \sin \Omega \left(t^* - \frac{x^*}{\sqrt{H}} \right),$$

and, since $h = c^2$

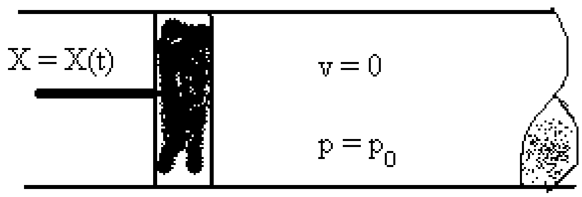
$$h(x^*, t^*) = H + \sqrt{H}\, \varepsilon \sin \Omega \left(t^* - \frac{x^*}{\sqrt{H}} \right) + \frac{\varepsilon^2}{4} \sin^2 \Omega \left(t^* - \frac{x^*}{\sqrt{H}} \right)$$
$$= H + \sqrt{H}\, \varepsilon \sin \Omega \left(t^* - \frac{x^*}{\sqrt{H}} \right) + \frac{\varepsilon^2}{8} \left(1 - \cos 2\Omega \left(t^* - \frac{x^*}{\sqrt{H}} \right) \right)$$

Evidently, the nonlinear solution consists of the linearized solution plus an additional term which, although of small amplitude, contains a travelling wave with double the frequency of the input.

Example 6.2 A Wave in a Polytropic Gas

Consider a polytropic gas in a long tube containing a piston. Suppose the gas is initially at rest in a uniform state (i.e., constant pressure and density) and at $t = 0$ the piston begins to move. If $X = X(t)$ denotes the position of the piston at time t, then we suppose

$$X(0) = X'(0) = 0, \qquad \textit{and} \qquad X'(t) < 0, \qquad X''(t) < 0,\ \textit{for}\ t > 0. \tag{1}$$



A long tube with a moveable piston

In example 5.3.2 we showed that the polytropic gas equations have the following Riemann invariants,

$$r_1 = \frac{2c(\rho)}{\gamma - 1} + v \quad and \quad r_2 = \frac{2c(\rho)}{\gamma - 1} - v, \qquad (2)$$

constant along solution curves

$$\begin{aligned} & C_1(x_0) : \ x_1'(t) = v + c, \quad x_1(0) = x_0 \\ and \quad & C_2(x_0) : \ x_2'(t) = v - c, \quad x_2(0) = x_0, \end{aligned}$$

respectively. Since we have the initial conditions

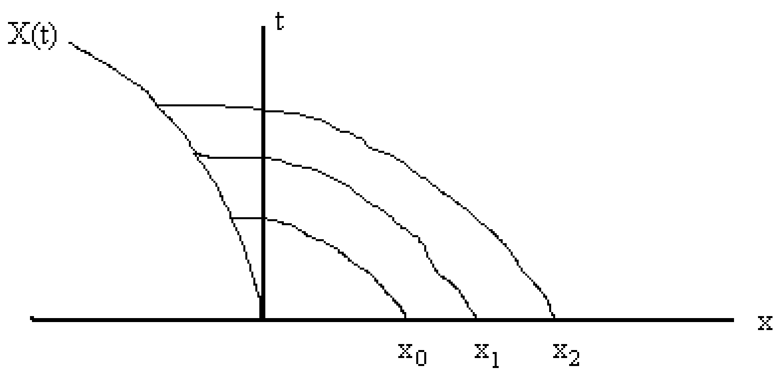
$$v(x_0, 0) = 0, \quad \rho(x_0, 0) = \rho_0, \quad c(x_0, 0) = c_0, \quad p(x_0, 0) = p_0, \ \ for \ \ x_0 \geq 0, \qquad (3)$$

It is clear that $x_2'(0) = v(x_0, 0) - c(x_0, 0) = 0 - c_0 < 0$ which means that this characteristic curve will go initially to the left.
Now right at the surface of the piston, the fluid velocity is equal to the speed of the piston, so we have

$$v(X(t), t) = X'(t) < 0. \qquad (4)$$

But this implies that the $C_2(x_0)$ characteristic will meet the piston curve at some time $t_0 > 0$ since at this point we will have

$$x_2'(t) = X'(t_0) - c(X(t_0), t_0) < X'(t_0).$$



Now $r_2(x_0, 0) = \dfrac{2c_0}{\gamma - 1} - 0 = r_2(x_2(t; x_0), t) =$ constant along $C_2(x_0)$.

Similarly,

$$r_2(x_1,0) = \frac{2c_0}{\gamma-1} - 0 = r_2(x_2(t;x_1),t) = \text{constant along } C_2(x_1)$$

$$r_2(x_2,0) = \frac{2c_0}{\gamma-1} - 0 = r_2(x_2(t;x_2),t) = \text{constant along } C_2(x_2).$$

Evidently, not only is $r_2(x,t)$ constant along every $C_2(x_p)$, it equals the same constant on all the curves in this family that originate at a point $(x_p,0),\ x_p \ge 0$. That is,

$$r_2(x,t) = \frac{2c_0}{\gamma-1} \quad \textit{for all} \quad (x,t) \in S = \{t > 0,\ x \ge X(t)\}. \qquad (5)$$

Now consider the solution curves, C_1, for $x_1'(t) = v + c$. If the curve originates at a point $(x_p,0)$ for $x_p \ge 0$, then in just the same way as with $r_2(x,t)$, we can show that not only is $r_1(x,t)$ constant along every $C_1(x_p)$ originating at $x_p \ge 0$, it is the case that $r_1(x,t)$ equals the same constant on every $C_1(x_p)$ originating at $x_p \ge 0$. In particular,

$$r_1(x,t) = \frac{2c_0}{\gamma-1} \qquad \text{along every } C_1(x_p) \text{ originating at } x_p \ge 0 \qquad (6)$$

Using (5) and (6) together, we conclude that $v = 0,\ \textit{and}\ c = c_0$ throughout the "zone of quiet" $\{x \ge c_0 t,\ t > 0\}$. Then all the C_1 characteristics in the region are straight lines given by $x_1(t;x_p) = c_0 t + x_p,\ x_p \ge 0$. Now consider the C_1 characteristics which originate at a point on the piston curve. Although $v\ \textit{and}\ c$ are not constants in the region $\{X(t) \le x \le c_0 t,\ t > 0\}$, these curves also turn out to be straight lines. To see this, write

$$r_1(x,t) = \frac{2c}{\gamma-1} + v = K_p \text{ (constant) on } C_1(x_p) \text{ originating at } (x_p,t_p) = (X(t_p),t_p),$$

$$r_2(x,t) = \frac{2c}{\gamma-1} - v = \frac{2c_0}{\gamma-1}\ , \quad \textit{for all} \quad (x,t) \in S = \{t > 0,\ x \ge X(t)\}.$$

Then, in particular, $r_2(x,t)$ is constant on all the $C_1(x_p) - \textit{characteristics}$ so we can add and subtract these two equations to conclude that along the curve $C_1(x_p)$,

$$c = \frac{\gamma-1}{4}\left(K_p + \frac{2c_0}{\gamma-1}\right) = \frac{1}{2}\left(c_0 + \frac{\gamma-1}{2}K_p\right) := c_p$$

$$v = \frac{1}{2}\left(K_p - \frac{2c_0}{\gamma-1}\right) := v_p.$$

Then c and v are constant along $C_1(x_p)$ characteristics, but they are equal to different constants on distinct characteristics. Since $C_1(x_p)$ is the solution curve for

$$x_1'(t) = v + c = v_p + c_p, \quad t > t_p, \qquad x_1(t_p) = x_p,$$

it follows that the C_1 characteristic that originates at (x_p,t_p) is the straight line whose equation is, $x_1(t) = (v_p + c_p)(t - t_p) + x_p$.

Now consider the characteristic $C_1(t_p)$ that originates at a point $(X(t_p),t_p),\ t_p \ge 0,$ on the piston curve. This line has the equation

$$x_1(t) = (v + c)(t - t_p) + X(t_p) \qquad \textit{for} \quad t \ge t_p.$$

To determine the values for $v\ \textit{and}\ c$ on this line, $C_1(t_p)$, note that $v(X(t_p),t_p) = X'(t_p)$. But then for some $x_p > 0$, there is a curve $C_2(x_p)$ which meets the piston curve at $(X(t_p),t_p)$ and then

$$r_2 = \frac{2c(x,t)}{\gamma - 1} - X'(t_p) = \frac{2c_0}{\gamma - 1}.$$

This allows us to solve for $c(x,t) = c_0 + \frac{\gamma - 1}{2}X'(t_p)$. which leads to

$$v + c = X'(t_p) + c_0 + \frac{\gamma - 1}{2}X'(t_p) = c_0 + \frac{\gamma + 1}{2}X'(t_p)$$

and,

$$x_1(t) = \left(c_0 + \frac{\gamma + 1}{2}X'(t_p)\right)(t - t_p) + X(t_p) \qquad for \quad t \geq t_p.$$

Then the region $S\backslash Q$ is covered by C_1 characteristics. In order to determine v at some point $(\hat{x},\hat{t}) \in S\backslash Q$ we find the C_1 –characteristic passing through the point, $(\hat{x},\hat{t})$, and find the point $(X(t_0), t_0)$ where this characteristic meets the piston curve. Then $v(\hat{x},\hat{t}) = X'(t_0)$ and $c(\hat{x},\hat{t}) = c_0 + \frac{\gamma - 1}{2}v(\hat{x},\hat{t})$. Then ρ, p are obtained from the equation of state and the definition $c^2 = p'(\rho)$.

Note that the C_1 characteristics have slopes $m_p = c_0 + \frac{\gamma + 1}{2}X'(t_p)$. Then for $t_q > t_p$, it follows from $X''(t) < 0$ that $m_q < m_p$ which implies that the lines $x_1(t;t_p)$ *and* $x_1(t;t_q)$ don't cross. Instead, we have an expansion wave in the region $S\backslash Q$. As long as $X''(t) < 0$, the expansion wave persists and the solution is a classical solution. On the other hand, suppose $X''(t) > 0$ *for* $t_1 < t < t_2$. Then $x_1(t;t_1)$ has a steeper slope than $x_1(t;t_2)$ and these lines will meet, leading to the formation of a shock.

In a situation where, instead of a gradual withdrawl, the piston is suddenly withdrawn, we would have

$$X(0) = 0, \qquad X'(0) = -V < 0.$$

Then $v = 0$ *and* $c = c_0$ *in* $Q = \{t > 0,\ x > c_0 t\}$. In addition, by the same reasoning used previously, we see that the C_1 characteristics are straight lines and at each point of S,

$$r_2(x,t) = \frac{2c}{\gamma - 1} - v = \frac{2c_0}{\gamma - 1}.$$

Now it follows that there is a wedge shaped region, bounded on the left by the piston curve, where $v = -V$ *and* $c =$ constant. In this region, we have

$$\frac{2c}{\gamma - 1} - (-V) = \frac{2c_0}{\gamma - 1}$$

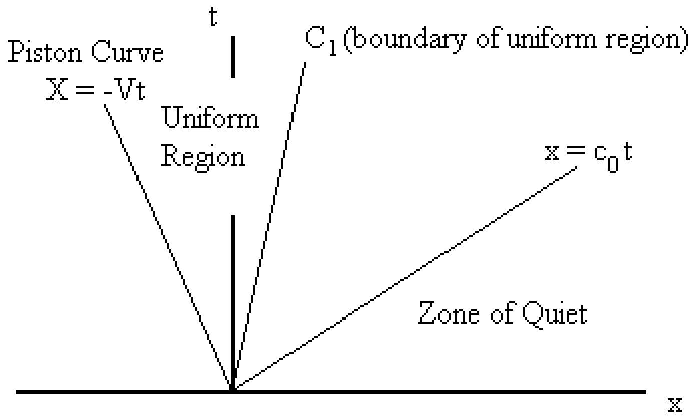
i.e.,

$$c = c_0 - \left(\frac{\gamma - 1}{2}\right)V.$$

Then the C_1 characteristic that meets the piston curve at the origin forms the right boundary of this wedge. This C_1 characteristic is the solution curve for

$$x_1'(t) = c + v = c_0 - \left(\frac{\gamma - 1}{2}\right)V - V = c_0 - \left(\frac{\gamma + 1}{2}\right)V, \qquad x_1(0) = 0$$

i.e., $$x_1(t) = [c_0 - \left(\frac{\gamma + 1}{2}\right)V]t.$$



Evidently, the zone of quiet, where $v = 0$ *and* $c = c_0$, is a wedge whose left boundary is the C_1 characteristic, $x_1(t) = c_0 t$. Also there is the uniform region where $v = -V$ *and* $c =$ constant, and this wedge has as its right boundary, the C_1 characteristic we just computed. In between these two C_1 characteristics, there is a wedge shaped region containing no characteristics. This is the region

$$[c_0 - \left(\frac{\gamma+1}{2}\right)V]\,t < x < c_0 t, \qquad t > 0.$$

i.e., $$1 - \left(\frac{\gamma+1}{2}\right)\frac{V}{c_0} < \frac{x}{c_0 t} < 1.$$

If we let

$$v(x,t) = \frac{2c_0}{\gamma+1}\left(\frac{x}{c_0 t} - 1\right) \quad \text{in this wedge shaped region}$$

then, since

$$r_2(x,t) = \frac{2c(x,t)}{\gamma-1} - v(x,t) = \frac{2c_0}{\gamma-1}$$

at all (x,t), it then follows that in the characteristic-free wedge,

$$c(x,t) = \frac{\gamma-1}{2}v(x,t) + c_0 = c_0\left[\frac{\gamma-1}{\gamma+1}\frac{x}{c_0 t} + \frac{2}{\gamma+1}\right].$$

This solution joins the two uniform regions with an expansion fan. This, is consistent with the entropy condition discussed previously since

$$[c_0 - \left(\frac{\gamma+1}{2}\right)V]\,t < c_0 t$$

i.e., the velocity of the wave on the left is less than the velocity of the wave on the right, leading to the formation of an expansion fan.