Progress Report

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Analysis of 2D Burgers' equation and solution of system of equations

2D form of Burgers' equation was considered, which is given mathematically as:

$$\frac{\partial u}{\partial t} = \frac{1}{Re}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] - u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} \tag{1}$$

The exact solution was carried out first and then the numerical solution was executed using code in C++.

$$\frac{\partial v}{\partial t} = \frac{1}{Re}\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] - u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} \tag{2}$$

Exact solution of 2D Burgers' equation

The velocities for the exact solution are expressed as:

$$u = \frac{-2[a_2 + a_4 y + \lambda a_5[e^{\lambda(x-x_0)} - e^{-\lambda(x-x_0)}]\cos(\lambda y)]}{Re[a_1 + a_2 x + a_3 y + a_4 xy + a_5[e^{\lambda(x-x_0)} + e^{-\lambda(x-x_0)}]\cos(\lambda y)]} \tag{3}$$

$$v = \frac{-2[a_3 + a_4 x - \lambda a_5[e^{\lambda(x-x_0)} + e^{-\lambda(x-x_0)}]\sin(\lambda y)]}{Re[a_1 + a_2 x + a_3 y + a_4 xy + a_5[e^{\lambda(x-x_0)} + e^{-\lambda(x-x_0)}]\cos(\lambda y)]} \tag{4}$$

where a_1 to a_5, λ and x_0 provide appropriate features to the exact solution. This equation was solved for $-1 \leq x \leq 1$ and $0 \leq y \leq 2$. The other parameters in the steady solution have the values $a_1 = a_2 = 1.3 \times 10^{13}, a_3 = a_4 = 0, a_5 = 1.0, \lambda = 25, x_0 = 1$ and $Re = 500$. The vector plot of the exact solution is given below.

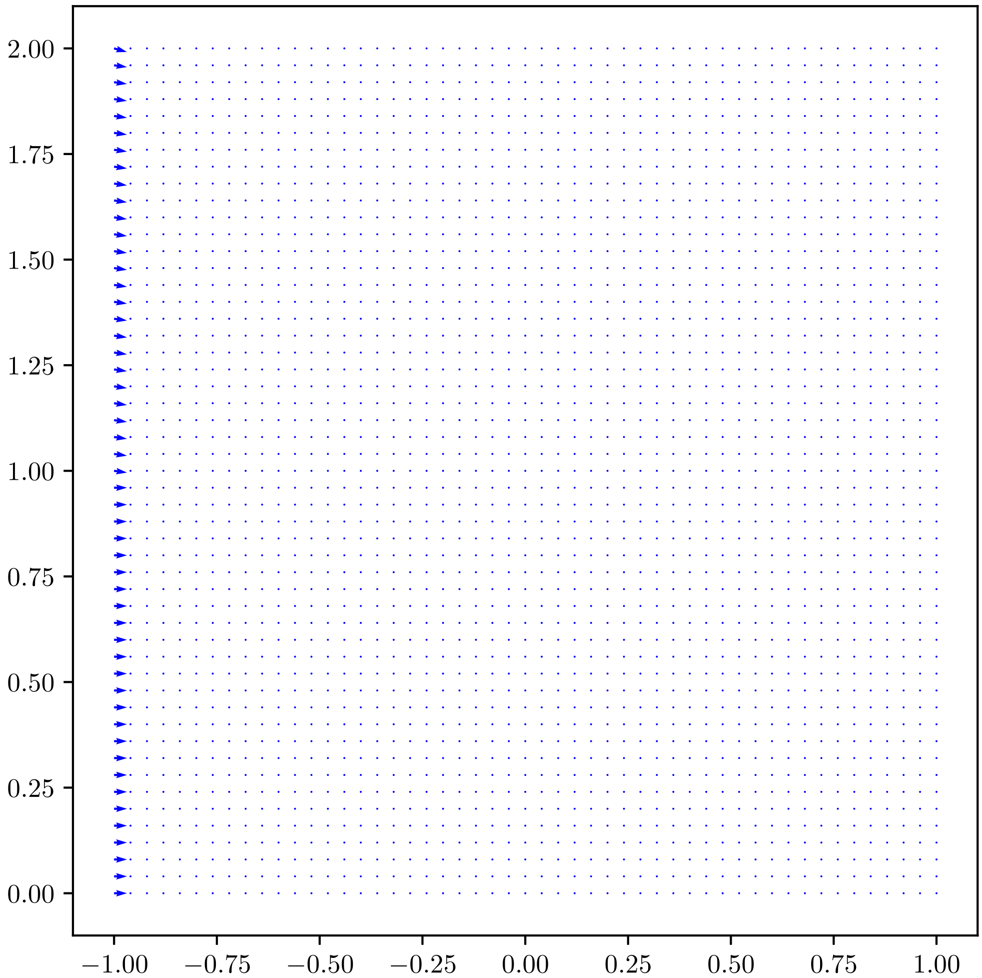

Figure 1: Vector plot of exact solution.

Numerical solution of Burgers' equation

An equally spaced 50×50 grid was used to solve for the numerical solution of the 2D Burgers' equation using code written in C++. A time step of 0.001 was selected for the solution. Dirichlet boundary conditions using the exact solution were chosen and initial conditions of $u_i = 0.1$ and $v_i = 0.1$ were selected for testing purposes (regardless of the physicality of the flow).

Central difference scheme was used for the solution, as given by the following formula:

$$
\begin{aligned}
u_{i,j}^{n+1} = \quad & u_{i,j}^{n} + \Delta t\Bigg[\frac{1}{Re}\left(\frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{(\Delta x)^2} + \frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{(\Delta y)^2}\right) \\
& - u_{i}^{n}\left(\frac{u_{i,j+1}^{n} - u_{i,j-1}^{n}}{2\Delta x}\right) - v_{i}^{n}\left(\frac{u_{i+1,j}^{n} - u_{i-1,j}^{n}}{2\Delta y}\right)\Bigg]
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
v_{i,j}^{n+1} = \quad & v_{i,j}^{n} + \Delta t\Bigg\{\frac{1}{Re}\left(\frac{v_{i,j+1}^{n} - 2v_{i,j}^{n} + v_{i,j-1}^{n}}{(\Delta x)^2} + \frac{v_{i+1,j}^{n} - 2v_{i,j}^{n} + v_{i-1,j}^{n}}{(\Delta y)^2}\right) \\
& - v_{i}^{n}\left(\frac{v_{i,j+1}^{n} - v_{i,j-1}^{n}}{2\Delta x}\right) - v_{i}^{n}\left(\frac{v_{i+1,j}^{n} - v_{i-1,j}^{n}}{2\Delta y}\right)\Bigg\}
\end{aligned} \tag{6}
$$

The vector plots at four different time instances are shown in the figure.

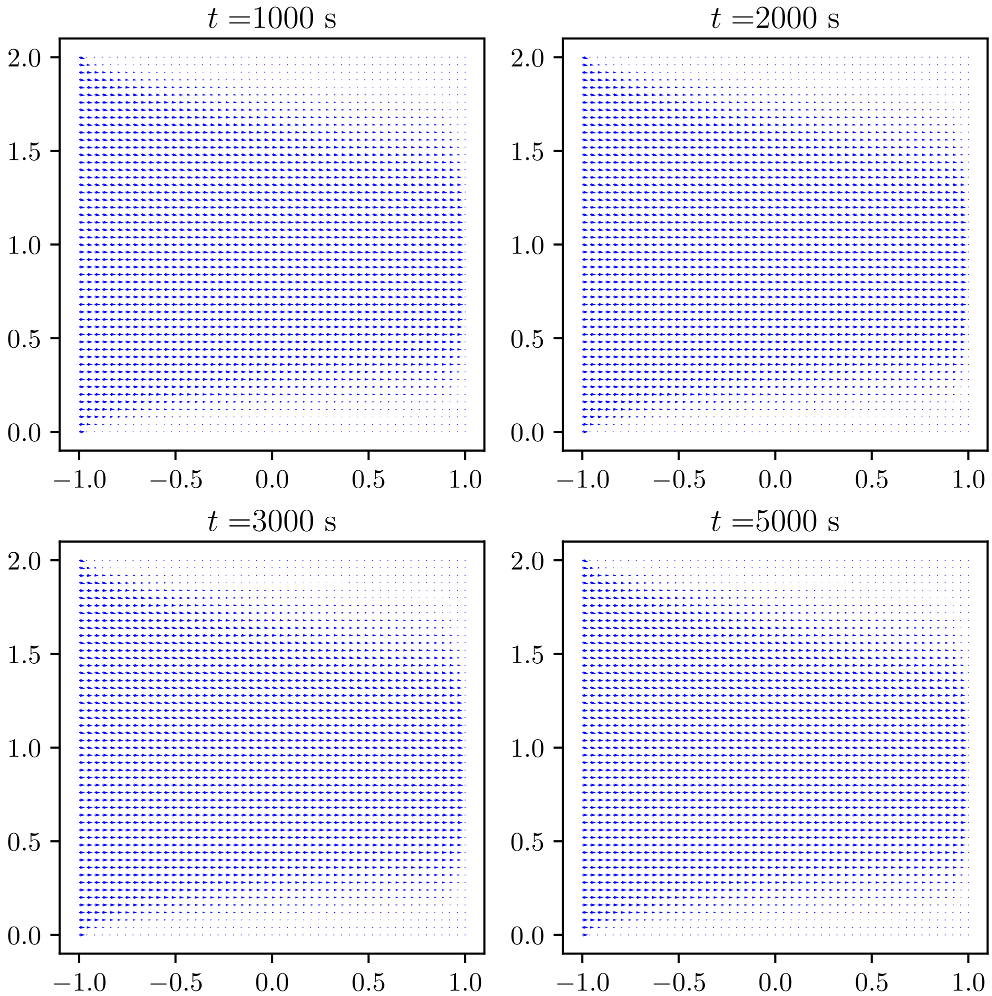

Figure 2: Vector plots of the numerical solution at different instances of time.

Solving a system of equations using SOR method

Successive overrelaxation variant of Gauss-Siedel method is used to solve the following set of equations in C++:

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \tag{7}$$

$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3 \tag{8}$$

$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4 \tag{9}$$

The value of relaxation parameter was taken as 1.8. The true solution is $x_1 = 3$, $x_2 = -2.5$ and $x_3 = 7$, which was achieved after 14 iterations with an error of less than 0.01%.