Progress Report

Syed Ahmad Raza

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Numerical solution of Navier-Stokes equations using Finite Volume Method

Continuity equation and Navier-Stokes equation are used and denoted as

$$\nabla \cdot \mathbf{u} = 0 \tag{1}$$

and

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f} \tag{2}$$

where $\mathbf{u}$ is a velocity vector, p is pressure, ρ is density of fluid, and $\mathbf{f}$ is body force per unit mass.

Projection method

Projection method was proposed by Chorin in 1967 to solve the governing equations for fluid flow. Using this method, Navier-Stokes equation is decomposed into

$$\frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} = -\nabla \cdot (\mathbf{u}\mathbf{u}) + \nu \nabla^2 \mathbf{u} \tag{3}$$

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^*}{\Delta t} = -\frac{1}{\rho}\nabla p \tag{4}$$

where n is the time level index. For the current problem, the body force is ignored.

The vector $\mathbf{u}^*$ is an intermediate velocity that allows us to break the solution into two convenient steps. This technique permits the solution of convective and diffusive terms separately from the pressure term. Later, the pressure term can be solved using

$$\nabla^2 p = \frac{\rho}{\Delta t}\nabla \cdot \mathbf{u}^* \quad . \tag{5}$$

Discretization of the convective and diffusive terms

The convective and diffusive terms from equation (3) can be discretized as explained below. Starting from the u-component of (3)

$$\frac{\partial u}{\partial t} = -\frac{\partial uu}{\partial x} - \frac{\partial uv}{\partial y} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] \tag{6}$$

and integrating throughout

$$\begin{aligned}\iint \frac{\partial u}{\partial t}\,dx\,dy = -\iint \frac{\partial (uu)}{\partial x}\,dx\,dy \;-\; \iint \frac{\partial (uv)}{\partial y}\,dx\,dy \\ + \nu\left[\iint \frac{\partial^2 u}{\partial x^2}\,dx\,dy + \iint \frac{\partial^2 u}{\partial y^2}\,dx\,dy\right]\end{aligned} \tag{7}$$

$$\begin{aligned}\frac{\partial u}{\partial t}\Delta x\Delta y = {} & -\left(u_e^2 - u_w^2\right)\Delta y - \left(u_n v_n - u_s v_s\right)\Delta x \\ & + \nu\left[\left\{\left(\frac{\partial u}{\partial x}\right)_e - \left(\frac{\partial u}{\partial x}\right)_w\right\}\Delta y + \left\{\left(\frac{\partial u}{\partial y}\right)_n - \left(\frac{\partial u}{\partial y}\right)_s\right\}\Delta x\right]\end{aligned} \tag{8}$$

$$\begin{aligned}\frac{\partial u}{\partial t} = {} & -\frac{u_e^2 - u_w^2}{\Delta x} - \frac{u_n v_n - u_s v_s}{\Delta y} \\ & + \nu\left[\left\{\left(\frac{\partial u}{\partial x}\right)_e - \left(\frac{\partial u}{\partial x}\right)_w\right\}\frac{1}{\Delta x} + \left\{\left(\frac{\partial u}{\partial y}\right)_n - \left(\frac{\partial u}{\partial y}\right)_s\right\}\frac{1}{\Delta y}\right]\end{aligned} \tag{9}$$

$$\begin{aligned}\frac{\partial u}{\partial t} = {} & -\frac{u_e^2 - u_w^2}{\Delta x} - \frac{u_n v_n - u_s v_s}{\Delta y} + \nu\left[\left\{\frac{u_{i+1,j} - u_{i,j}}{x_{i+2} - x_{i+1}} - \frac{u_{i,j} - u_{i-1,j}}{x_{i+1} - x_i}\right\}\frac{1}{\Delta x}\right. \\ & \left. + \left\{\frac{u_{i,j+1} - u_{i,j}}{(y_{j+2} - y_j)/2} - \frac{u_{i,j} - u_{i,j-1}}{(y_{j+1} - y_{j-1})/2}\right\}\frac{1}{\Delta y}\right] \quad ,\end{aligned} \tag{10}$$

where

$$\begin{aligned} u_e &= u_{i,j} \\ u_w &= u_{i-1,j} \\ u_n &= u_{i,j} + \frac{u_{i,j+1} - u_{i,j}}{(y_{j+2} - y_j)/2} \\ u_s &= u_{i,j} - \frac{u_{i,j} - u_{i,j-1}}{(y_{j+1} - y_{j-1})/2} \end{aligned} \qquad \begin{aligned} v_n &= v_{i,j} \\ v_s &= v_{i,j-1} \quad . \end{aligned}$$

Similarly, for the v-component of (3), starting from

$$\frac{\partial v}{\partial t} = -\frac{\partial uv}{\partial x} - \frac{\partial vv}{\partial y} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] \quad , \tag{11}$$

and finally leading to

$$\begin{aligned}\frac{\partial v}{\partial t} &= -\frac{u_e v_e - u_w v_w}{\Delta x} - \frac{v_n^2 - v_s^2}{\Delta y} + \nu \left[\left\{ \frac{v_{i+1,j} - v_{i,j}}{(x_{i+2} - x_i)/2} - \frac{v_{i,j} - v_{i-1,j}}{(x_{i+1} - x_{i-1})/2} \right\} \frac{1}{\Delta x} \right. \\ &\quad \left. + \left\{ \frac{v_{i,j+1} - v_{i,j}}{y_{j+2} - y_{j+1}} - \frac{v_{i,j} - v_{i,j-1}}{y_{j+1} - y_{j-1}} \right\} \frac{1}{\Delta y} \right] \quad , \end{aligned} \tag{12}$$

where

$$\begin{aligned} v_n &= v_{i,j} & u_e &= u_{i,j} \\ v_s &= v_{i,j-1} & u_w &= u_{i-1,j} \quad . \\ v_e &= v_{i,j} + \frac{v_{i+1,j} - v_{i,j}}{(x_{i+2} - x_i)/2} \\ v_w &= v_{i,j} - \frac{v_{i,j} - v_{i-1,j}}{(x_{i+1} - x_{i-1})/2} \end{aligned}$$

First time step

For the first time step, the Euler scheme is adopted for the intermediate velocity using

$$\begin{aligned}\frac{u_{i,j}^* - u_{i,j}}{\Delta t} &= -\frac{u_e^2 - u_w^2}{\Delta x} - \frac{u_n v_n - u_s v_s}{\Delta y} \\ &\quad + \nu \left[\left\{ \frac{u_{i+1,j} - u_{i,j}}{x_{i+2} - x_{i+1}} - \frac{u_{i,j} - u_{i-1,j}}{x_{i+1} - x_i} \right\} \frac{1}{\Delta x} \right. \\ &\quad \left. + \left\{ \frac{u_{i,j+1} - u_{i,j}}{(y_{j+2} - y_j)/2} - \frac{u_{i,j} - u_{i,j-1}}{(y_{j+1} - y_{j-1})/2} \right\} \frac{1}{\Delta y} \right] \quad , \end{aligned} \tag{13}$$

and

$$\begin{aligned}\frac{v_{i,j}^* - v_{i,j}}{\Delta t} &= -\frac{u_e v_e - u_w v_w}{\Delta x} - \frac{v_n^2 - v_s^2}{\Delta y} \\ &\quad + \nu \left[\left\{ \frac{v_{i+1,j} - v_{i,j}}{x_{i+2} - x_{i+1}} - \frac{v_{i,j} - v_{i-1,j}}{x_{i+1} - x_i} \right\} \frac{1}{\Delta x} \right. \\ &\quad \left. + \left\{ \frac{v_{i,j+1} - v_{i,j}}{(y_{j+2} - y_{j+1})/2} - \frac{v_{i,j} - v_{i,j-1}}{(y_{j+1} - y_{j-1})/2} \right\} \frac{1}{\Delta y} \right] \quad . \end{aligned} \tag{14}$$

Adams-Bashforth scheme

Denoting the terms on the right-hand side of equation (3) with $\mathcal{F}(\mathbf{u})$, the second order Adams-Bashforth scheme can be applied using

$$\frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} = \frac{3}{2}\mathcal{F}(\mathbf{u}^n) - \frac{1}{2}\mathcal{F}(\mathbf{u}^{n-1}) \quad . \tag{15}$$

Applying this formula to a 2D problem, the discretized equations can be written as

$$\mathcal{F}(u_{i,j}^n) = -\frac{u_e^2 - u_w^2}{\Delta x} - \frac{u_n v_n - u_s v_s}{\Delta y} + \nu\left[\left\{\frac{u_{i+1,j} - u_{i,j}}{x_{i+2} - x_{i+1}} - \frac{u_{i,j} - u_{i-1,j}}{x_{i+1} - x_i}\right\}\frac{1}{\Delta x} + \left\{\frac{u_{i,j+1} - u_{i,j}}{(y_{j+2} - y_j)/2} - \frac{u_{i,j} - u_{i,j-1}}{(y_{j+1} - y_{j-1})/2}\right\}\frac{1}{\Delta y}\right] \quad , \tag{16}$$

and

$$\mathcal{F}(v_{i,j}^n) = -\frac{u_e v_e - u_w v_w}{\Delta x} - \frac{v_n^2 - v_s^2}{\Delta y} + \nu\left[\left\{\frac{v_{i+1,j} - v_{i,j}}{x_{i+2} - x_{i+1}} - \frac{v_{i,j} - v_{i-1,j}}{x_{i+1} - x_i}\right\}\frac{1}{\Delta x} + \left\{\frac{v_{i,j+1} - v_{i,j}}{(y_{j+2} - y_{j+1})/2} - \frac{v_{i,j} - v_{i,j-1}}{(y_{j+1} - y_{j-1})/2}\right\}\frac{1}{\Delta y}\right] \quad . \tag{17}$$

Then, the intermediate velocity will be

$$\frac{u_{i,j}^* - u_{i,j}}{\Delta t} = \frac{3}{2}\mathcal{F}(u_{i,j}^{n+1}) - \frac{1}{2}\mathcal{F}(u_{i,j}^n) \tag{18}$$

$$\frac{v_{i,j}^* - v_{i,j}}{\Delta t} = \frac{3}{2}\mathcal{F}(v_{i,j}^{n+1}) - \frac{1}{2}\mathcal{F}(v_{i,j}^n) \tag{19}$$

Boundary conditions

Velocity boundary conditions are described below.

1. Top boundary (Dirichlet):

$$u_{i,n_y-1} = -u_{i,n_y-2} \qquad \text{for all } i$$
$$v_{i,n_y-1} = -v_{i,n_y-2} \qquad \text{for all } i$$

2. Bottom boundary (Dirichlet):

$$u_{i,0} = -u_{i,1} \qquad \text{for all } i$$
$$v_{i,0} = -v_{i,1} \qquad \text{for all } i$$

3. Left boundary (Dirichlet):

$$u_{0,j} = u_{\text{in}} \qquad \text{for all } j$$
$$v_{0,j} = v_{\text{in}} \qquad \text{for all } j$$

4. Right boundary (Neumann):

$$u_{n_x-1,j} = u_{n_x-2,j} \qquad \text{for all } j$$
$$v_{n_x-1,j} = v_{n_x-2,j} \qquad \text{for all } j$$