Progress Report

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1 Solution of Navier-Stokes equations using Finite Volume Method for nonuniform grid

1.1 Discretization of the convective and diffusive terms

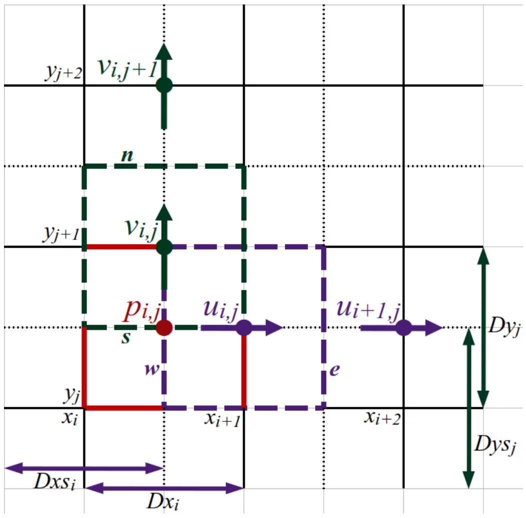

Figure 1: Visual representation of the staggered grid used for discretization in Finite Volume Method

The convective and diffusive terms of Navier-Stokes equations can be discretized using the individual components. Using Chorin's projection method, the u-component equation can be written as

$$\frac{\partial u}{\partial t} = -\frac{\partial uu}{\partial x} - \frac{\partial uv}{\partial y} + \nu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] \quad . \tag{1}$$

For nonuniform grids, using second-order central scheme for diffusion terms and the notation described in figure 1, the equation can be discretized as

$$\begin{aligned} \frac{\partial u}{\partial t} = & - \frac{u_e^2 - u_w^2}{Dxs_{i+1}} - \frac{u_n v_{n,u} - u_s v_{s,u}}{Dy_j} \\ & + \nu \left[\left\{ \frac{u_{i+1,j} - u_{i,j}}{Dx_{i+1}} - \frac{u_{i,j} - u_{i-1,j}}{Dx_i} \right\} \frac{1}{Dxs_{i+1}} \right. \\ & \left. + \left\{ \frac{u_{i,j+1} - u_{i,j}}{Dys_{j+1}} - \frac{u_{i,j} - u_{i,j-1}}{Dys_j} \right\} \frac{1}{Dy_j} \right] \quad . \end{aligned} \tag{2}$$

Similarly for the v-component,

$$\frac{\partial v}{\partial t} = -\frac{\partial uv}{\partial x} - \frac{\partial vv}{\partial y} + \nu \left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right] \quad , \tag{3}$$

which can be and discretized as

$$\begin{aligned}
\frac{\partial v}{\partial t} = &-\frac{u_{e,v}v_e - u_{w,v}v_w}{Dx_i} - \frac{v_n^2 - v_s^2}{Dys_{j+1}} \\
&+ \nu \left[\left\{ \frac{v_{i+1,j} - v_{i,j}}{Dxs_{i+1}} - \frac{v_{i,j} - v_{i-1,j}}{Dxs_i} \right\} \frac{1}{Dx_i} \right. \\
&+ \left. \left\{ \frac{v_{i,j+1} - v_{i,j}}{Dy_{j+1}} - \frac{v_{i,j} - v_{i,j-1}}{Dy_j} \right\} \frac{1}{Dys_{j+1}} \right] \quad . \qquad (4)
\end{aligned}$$

1.1.1 Upwind scheme for velocities in the convective terms

Linear interpolation is used for the velocities u_n, u_s, $u_{e,v}$, $u_{w,v}$, v_e, v_w, $v_{n,u}$ and $v_{s,u}$,

$$\begin{aligned}
u_n &= u_{i,j} + \frac{u_{i,j+1} - u_{i,j}}{2} & v_e &= v_{i,j} + \frac{v_{i+1,j} - v_{i,j}}{2} \\
u_s &= u_{i,j-1} + \frac{u_{i,j} - u_{i,j-1}}{2} & v_w &= v_{i-1,j} + \frac{v_{i,j} - v_{i-1,j}}{2} \\
u_{e,v} &= u_{i,j} + \frac{u_{i,j+1} - u_{i,j}}{2} & v_{n,u} &= v_{i,j} + \frac{v_{i+1,j} - v_{i,j}}{2} \\
u_{w,v} &= u_{i-1,j} + \frac{u_{i-1,j+1} - u_{i-1,j}}{2} & v_{s,u} &= v_{i,j-1} + \frac{v_{i+1,j-1} - v_{i,j-1}}{2}
\end{aligned}$$

For rest of the velocities in convective terms, the upwind scheme was used. For positive velocities,

$$\begin{aligned}
u_e &= u_{i,j} & v_n &= v_{i,j} \\
u_w &= u_{i-1,j} & v_s &= v_{i,j-1}
\end{aligned}$$

For negative velocities,

$$\begin{aligned}
u_e &= u_{i+1,j} & v_n &= v_{i,j+1} \\
u_w &= u_{i,j} & v_s &= v_{i,j}
\end{aligned}$$

For the first time step, Euler scheme is used and for subsequent time steps, Adams-Bashforth scheme is utilized.

1.2 Poisson equation of pressure

The Poisson equation for pressure can be written as

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \frac{1}{\Delta t}\left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y}\right) \quad . \qquad (5)$$

Integrating it twice, discretizing and rearranging leads to

$$
\begin{aligned}
p_{i,j}^{n+1} = & \frac{1}{\left[-\frac{Dy_j}{Dxs_{i+1}} - \frac{Dy_j}{Dxs_i} - \frac{Dx_i}{Dys_{j+1}} - \frac{Dx_i}{Dys_j}\right]} \\
& \times \left[-\frac{Dy_j}{Dxs_{i+1}} p_{i+1,j} - \frac{Dy_j}{Dxs_i} p_{i-1,j} - \frac{Dx_i}{Dys_{j+1}} p_{i,j+1} - \frac{Dx_i}{Dys_j} p_{i,j-1} \right. \\
& \left. + \frac{1}{\Delta t} \left\{ \left(u_{i,j}^* - u_{i-1,j}^* \right) Dy_j + \left(v_{i,j}^* - v_{i,j-1}^* \right) Dx_i \right\} \right] \quad .
\end{aligned} \tag{6}
$$

This equation is employed using successive over-relaxation method (SOR).

1.3 Corrected velocity

The correct velocity for u- and v-components can be found using

$$
u_{i,j}^{n+1} = u_{i,j}^* - \frac{\Delta t}{\rho} \cdot \frac{p_{i+1,j}^{n+1} - p_{i,j}^{n+1}}{Dxs_{i+1}} \tag{7}
$$

and

$$
v_{i,j}^{n+1} = v_{i,j}^* - \frac{\Delta t}{\rho} \cdot \frac{p_{i,j+1}^{n+1} - p_{i,j}^{n+1}}{Dys_{j+1}} \tag{8}
$$

1.4 Results using code for nonuniform grid

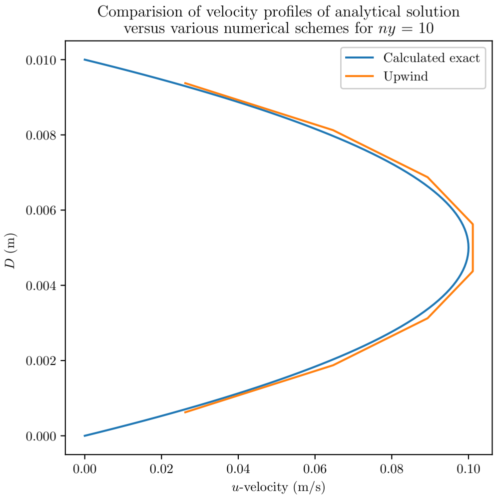

Figure 2: Results.

1.5 Future work

- Remove the errors for 2D case
- Modify the 3D code accordingly