Progress Report

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1 Solution of Navier-Stokes equations using Finite Volume Method

Continuity equation is

$$\nabla \cdot \mathbf{u} = 0 \quad , \tag{1}$$

and Navier-Stokes equation is

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f} \quad . \tag{2}$$

where $\mathbf{u}$ is the velocity vector, p is the pressure, ρ is density of the fluid, and $\mathbf{f}$ is body force per unit mass.

Ignoring the body force $\mathbf{f}$, we are left with

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho}\nabla p - \nabla \cdot (\mathbf{u}\mathbf{u}) + \nu \nabla^2 \mathbf{u} \quad . \tag{3}$$

1.1 Projection method

Using a new vector $\mathbf{u}^*$ for intermediate velocity and n as the index for time step, projection method is used to decompose Navier-Stokes equation in (3) into two parts,

$$\frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} = -\nabla \cdot (\mathbf{u}\mathbf{u}) + \nu \nabla^2 \mathbf{u} \quad , \tag{4}$$

which accounts for the convective and diffusive terms, and

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^*}{\Delta t} = -\frac{1}{\rho}\nabla p \quad , \tag{5}$$

which accounts for the pressure term.

1.2 Discretization of the convective and diffusive terms

The convective and diffusive terms from equation (4) can be discretized using the individual components. For the u-component, equation (4) can be written as

$$\frac{\partial u}{\partial t} = -\frac{\partial uu}{\partial x} - \frac{\partial uv}{\partial y} + \nu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] \quad , \tag{6}$$

which can be discretized as

$$\frac{\partial u}{\partial t} = -\frac{u_e^2 - u_w^2}{\Delta x} - \frac{u_n v_n - u_s v_s}{\Delta y} + \nu \left[\left\{ \frac{u_{i+1,j} - u_{i,j}}{x_{i+2} - x_{i+1}} - \frac{u_{i,j} - u_{i-1,j}}{x_{i+1} - x_i} \right\} \frac{1}{\Delta x} + \left\{ \frac{u_{i,j+1} - u_{i,j}}{(y_{j+2} - y_j)/2} - \frac{u_{i,j} - u_{i,j-1}}{(y_{j+1} - y_{j-1})/2} \right\} \frac{1}{\Delta y} \right] \quad , \tag{7}$$

where, for the *diffusion terms*, second-order central scheme has been used. Similarly for the v-component, equation (4) will be

$$\frac{\partial v}{\partial t} = -\frac{\partial uv}{\partial x} - \frac{\partial vv}{\partial y} + \nu \left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right] \quad , \tag{8}$$

which can be and discretized as

$$\frac{\partial v}{\partial t} = -\frac{u_e v_e - u_w v_w}{\Delta x} - \frac{v_n^2 - v_s^2}{\Delta y} + \nu \left[\left\{ \frac{v_{i+1,j} - v_{i,j}}{(x_{i+2} - x_i)/2} - \frac{v_{i,j} - v_{i-1,j}}{(x_{i+1} - x_{i-1})/2} \right\} \frac{1}{\Delta x} + \left\{ \frac{v_{i,j+1} - v_{i,j}}{y_{j+2} - y_{j+1}} - \frac{v_{i,j} - v_{i,j-1}}{y_{j+1} - y_j} \right\} \frac{1}{\Delta y} \right] \quad . \tag{9}$$

1.2.1 Substitutions for velocities in the convective terms

A simple mean is used for the velocities u_n, u_s, v_e and v_w,

$$u_n = u_{i,j} + \frac{u_{i,j+1} - u_{i,j}}{(y_{j+2} - y_j)/2} \qquad v_e = v_{i,j} + \frac{v_{i+1,j} - v_{i,j}}{(x_{i+2} - x_i)/2}$$

$$u_s = u_{i,j} - \frac{u_{i,j} - u_{i,j-1}}{(y_{j+1} - y_{j-1})/2} \qquad v_w = v_{i,j} - \frac{v_{i,j} - v_{i-1,j}}{(x_{i+1} - x_{i-1})/2}$$

For rest of the velocities in the convective terms, namely u_e, u_w, v_n and v_s, the following three schemes were used:

1. Upwind scheme
2. Central difference scheme
3. Quadratic Upwind Interpolation for Convective Kinematics (QUICK)

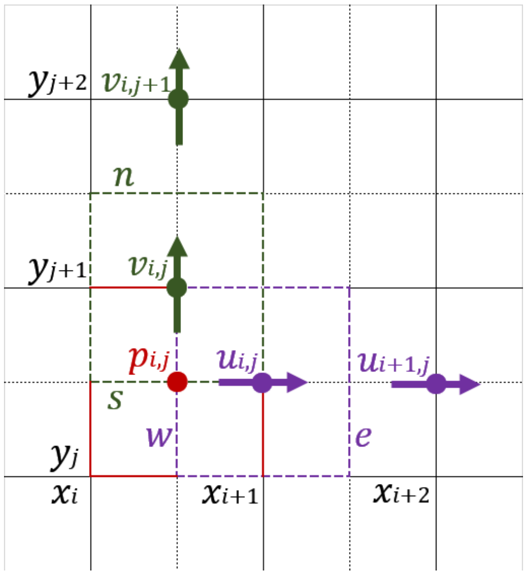

Figure 1: Visual representation of the staggered grid used for discretization in Finite Volume Method

Upwind scheme
For positive velocities,

$$u_e = u_{i,j} \qquad v_n = v_{i,j}$$
$$u_w = u_{i-1,j} \qquad v_s = v_{i,j-1}$$

For negative velocities,

$$u_e = u_{i+1,j} \qquad v_n = v_{i,j+1}$$
$$u_w = u_{i,j} \qquad v_s = v_{i,j}$$

Central scheme

$$u_e = \frac{u_{i,j} + u_{i+1,j}}{2} \qquad v_n = \frac{v_{i,j} + v_{i,j+1}}{2}$$
$$u_w = \frac{u_{i-1,j} + u_{i+1,j}}{2} \qquad v_s = \frac{v_{i,j-1} + v_{i,j}}{2}$$

QUICK scheme
For positive velocities,

$$u_e = \tfrac{6}{8}u_{i,j} + \tfrac{3}{8}u_{i+1,j} - \tfrac{1}{8}u_{i-1,j} \qquad v_n = \tfrac{6}{8}v_{i,j} + \tfrac{3}{8}v_{i,j+1} - \tfrac{1}{8}v_{i,j-1}$$
$$u_w = \tfrac{6}{8}u_{i-1,j} + \tfrac{3}{8}u_{i,j} - \tfrac{1}{8}u_{i-2,j} \qquad v_s = \tfrac{6}{8}v_{i,j-1} + \tfrac{3}{8}v_{i,j} - \tfrac{1}{8}v_{i,j-2}$$

For negative velocities,

$$u_e = \tfrac{6}{8}u_{i+1,j} + \tfrac{3}{8}u_{i,j} - \tfrac{1}{8}u_{i+2,j} \qquad v_n = \tfrac{6}{8}v_{i,j+1} + \tfrac{3}{8}v_{i,j} - \tfrac{1}{8}v_{i,j+2}$$
$$u_w = \tfrac{6}{8}u_{i,j} + \tfrac{3}{8}u_{i-1,j} - \tfrac{1}{8}u_{i+1,j} \qquad v_s = \tfrac{6}{8}v_{i,j} + \tfrac{3}{8}v_{i,j-1} - \tfrac{1}{8}v_{i,j+1}$$

1.2.2 Euler scheme for first time step

For the first time step, the Euler scheme is adopted for the intermediate velocity using

$$\begin{aligned}\frac{u^*_{i,j} - u_{i,j}}{\Delta t} = & -\frac{u_e^2 - u_w^2}{\Delta x} - \frac{u_n v_n - u_s v_s}{\Delta y} \\ & + \nu\left[\left\{\frac{u_{i+1,j} - u_{i,j}}{x_{i+2} - x_{i+1}} - \frac{u_{i,j} - u_{i-1,j}}{x_{i+1} - x_i}\right\}\frac{1}{\Delta x}\right. \\ & \left. + \left\{\frac{u_{i,j+1} - u_{i,j}}{(y_{j+2} - y_j)/2} - \frac{u_{i,j} - u_{i,j-1}}{(y_{j+1} - y_{j-1})/2}\right\}\frac{1}{\Delta y}\right] \quad , \end{aligned} \tag{10}$$

and

$$\begin{aligned}\frac{v^*_{i,j} - v_{i,j}}{\Delta t} = & -\frac{u_e v_e - u_w v_w}{\Delta x} - \frac{v_n^2 - v_s^2}{\Delta y} \\ & + \nu\left[\left\{\frac{v_{i+1,j} - v_{i,j}}{(x_{i+2} - x_i)/2} - \frac{v_{i,j} - v_{i-1,j}}{(x_{i+1} - x_{i-1})/2}\right\}\frac{1}{\Delta x}\right. \\ & \left. + \left\{\frac{v_{i,j+1} - v_{i,j}}{y_{j+2} - y_{j+1}} - \frac{v_{i,j} - v_{i,j-1}}{y_{j+1} - y_j}\right\}\frac{1}{\Delta y}\right] \quad . \end{aligned} \tag{11}$$

1.2.3 Adams-Bashforth scheme

Denoting the terms on the right-hand side of equation (4) with $\mathcal{F}(\mathbf{u})$, the second order Adams-Bashforth scheme can be applied using

$$\frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} = \frac{3}{2}\mathcal{F}(\mathbf{u}^n) - \frac{1}{2}\mathcal{F}(\mathbf{u}^{n-1}) \quad . \tag{12}$$

For a 2D problem, the discretized equations can be written as

$$\begin{aligned}\mathcal{F}(u^n_{i,j}) = & -\frac{u_e^2 - u_w^2}{\Delta x} - \frac{u_n v_n - u_s v_s}{\Delta y} + \nu\left[\left\{\frac{u_{i+1,j} - u_{i,j}}{x_{i+2} - x_{i+1}} - \frac{u_{i,j} - u_{i-1,j}}{x_{i+1} - x_i}\right\}\frac{1}{\Delta x}\right. \\ & \left. + \left\{\frac{u_{i,j+1} - u_{i,j}}{(y_{j+2} - y_j)/2} - \frac{u_{i,j} - u_{i,j-1}}{(y_{j+1} - y_{j-1})/2}\right\}\frac{1}{\Delta y}\right] \quad , \end{aligned} \tag{13}$$

and

$$
\begin{aligned}
\mathcal{F}(v^n_{i,j}) = & -\frac{u_e v_e - u_w v_w}{\Delta x} - \frac{v_n^2 - v_s^2}{\Delta y} \\
& + \nu \left[\left\{ \frac{v_{i+1,j} - v_{i,j}}{(x_{i+2} - x_i)/2} - \frac{v_{i,j} - v_{i-1,j}}{(x_{i+1} - x_{i-1})/2} \right\} \frac{1}{\Delta x} \right. \\
& \left. + \left\{ \frac{v_{i,j+1} - v_{i,j}}{y_{j+2} - y_{j+1}} - \frac{v_{i,j} - v_{i,j-1}}{y_{j+1} - y_j} \right\} \frac{1}{\Delta y} \right] \quad . \qquad (14)
\end{aligned}
$$

Then, the intermediate velocities will be

$$
\frac{u^*_{i,j} - u_{i,j}}{\Delta t} = \frac{3}{2}\mathcal{F}(u^{n+1}_{i,j}) - \frac{1}{2}\mathcal{F}(u^n_{i,j}) \qquad (15)
$$

$$
\frac{v^*_{i,j} - v_{i,j}}{\Delta t} = \frac{3}{2}\mathcal{F}(v^{n+1}_{i,j}) - \frac{1}{2}\mathcal{F}(v^n_{i,j}) \qquad (16)
$$

1.3 Poisson equation of pressure

Using the conservation of mass principle for the $n + 1^{\text{st}}$ time step,

$$
\nabla \cdot \mathbf{u}^{n+1} = 0 \quad , \qquad (17)
$$

and substituting equation (5), we get the Poisson equation for pressure

$$
\nabla^2 p^{n+1} = \frac{\rho}{\Delta t} \nabla \cdot \mathbf{u}^* \quad , \qquad (18)
$$

which can be written as

$$
\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \frac{1}{\Delta t}\left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} \right) \quad . \qquad (19)
$$

Integrating it twice, discretizing and rearranging leads to

$$
\begin{aligned}
p^{n+1}_{i,j} = & \frac{1}{\left[-\frac{Dy}{Dx1} - \frac{Dy}{Dx2} - \frac{Dx}{Dy1} - \frac{Dx}{Dy2} \right]} \\
& \times \left[-\frac{Dy}{Dx1} p_{i+1,j} - \frac{Dy}{Dx2} p_{i-1,j} - \frac{Dx}{Dy1} p_{i,j+1} - \frac{Dx}{Dy2} p_{i,j-1} \right. \\
& \left. + \frac{1}{\Delta t} \left\{ \left(u^*_{i,j} - u^*_{i-1,j} \right) \Delta y + \left(v^*_{i,j} - v^*_{i,j-1} \right) \Delta x \right\} \right] \quad . \qquad (20)
\end{aligned}
$$

Using successive over-relaxation method (SOR),

$$
\begin{aligned}
p_{i,j}^{n+1} = (1-\omega)\, p_{i,j} + \omega \Bigg[& \frac{1}{\left(-\frac{Dy}{Dx1} - \frac{Dy}{Dx2} - \frac{Dx}{Dy1} - \frac{Dx}{Dy2}\right)} \\
& \times \Bigg\{ -\frac{Dy}{Dx1} p_{i+1,j} - \frac{Dy}{Dx2} p_{i-1,j} - \frac{Dx}{Dy1} p_{i,j+1} - \frac{Dx}{Dy2} p_{i,j-1} \\
& + \frac{1}{\Delta t} \left(\left(u_{i,j}^{*} - u_{i-1,j}^{*}\right) \Delta y + \left(v_{i,j}^{*} - v_{i,j-1}^{*}\right) \Delta x \right) \Bigg\} \Bigg] \quad ,
\end{aligned}
\tag{21}
$$

where the relaxation factor, $\omega = 1.8$. Finally, the correct velocity can be found using

$$
\mathbf{u}^{n+1} = \mathbf{u}^{*} - \frac{\Delta t}{\rho} \cdot \nabla p^{n+1} \quad , \tag{22}
$$

which is, for u- and v-components,

$$
u_{i,j}^{n+1} = u_{i,j}^{*} - \frac{\Delta t}{\rho} \cdot \frac{p_{i+1,j}^{n+1} - p_{i,j}^{n+1}}{\Delta x} \tag{23}
$$

and

$$
v_{i,j}^{n+1} = v_{i,j}^{*} - \frac{\Delta t}{\rho} \cdot \frac{p_{i,j+1}^{n+1} - p_{i,j}^{n+1}}{\Delta y} \tag{24}
$$

1.4 Cases

The following cases of 2D laminar duct flow were studied:

1. *Plane Poiseuille flow*: both the upper and lower surfaces are stationary with fluid entering from the left and exiting at the right

2. *Couette-Poiseuille flow*: the upper surface is moving right and the bottom surface is stationary, with flow moving from left to right

3. *Couette-Poiseuille flow*: the upper surface is moving right and the bottom surface is moving left, with flow moving from left to right

1.4.1 Boundary conditions

In order to model 2D flow in a cylindrical pipe, appropriate boundary condtions were selected as described in figure 2.

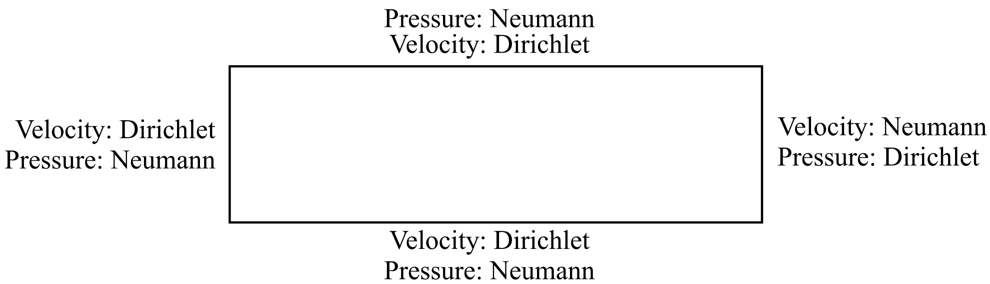

Figure 2: Velocity and pressure boundary conditions at each boundary of the domain are labeled.

1.4.2 Initial conditions

The following initial conditions were utilized:

1. $u_{i,j} = u_{\text{inlet}}$
2. $v_{i,j} = 0$
3. $p_{i,j} = \text{constant}$
4. The values of constants have been selected such that Re = 10.

1.5 Results

1.5.1 Case 1: Plane Poiseuille flow

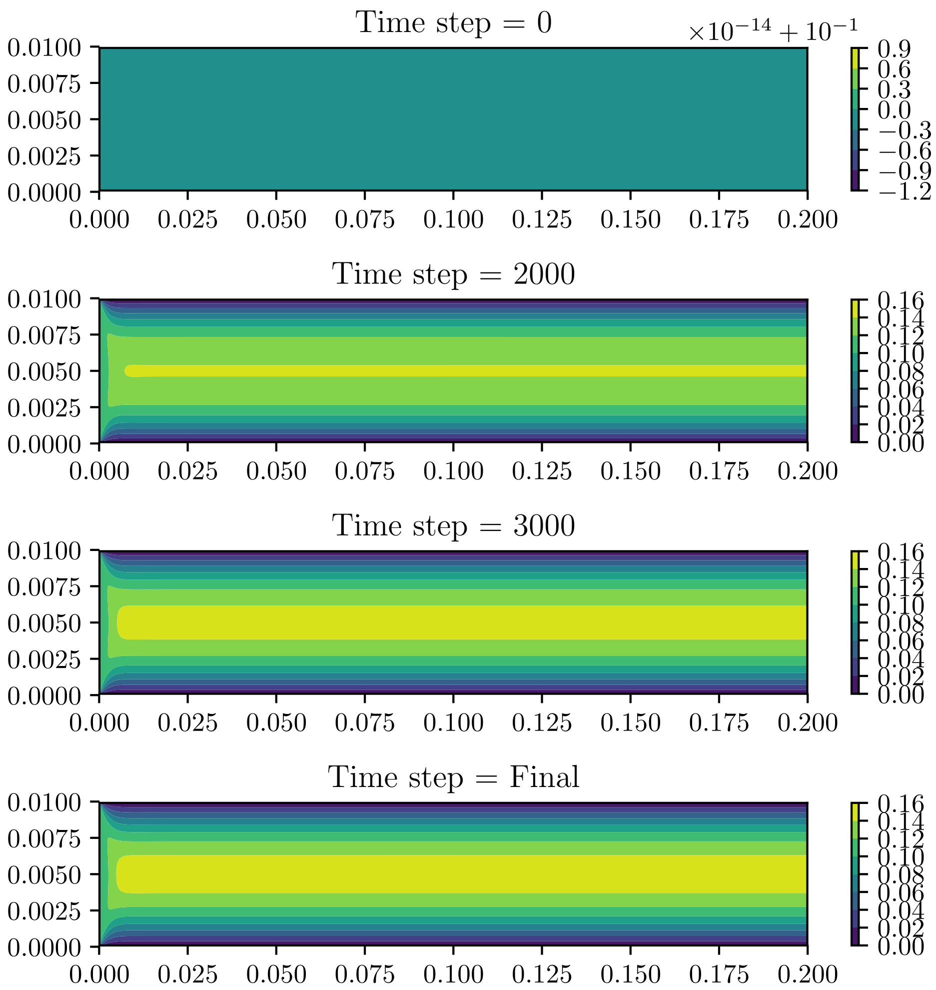

Figure 3: u-velocity contours in the computational domain at various time steps during the numerical solution for Plane Poiseuille flow using QUICK scheme and n_y = 110

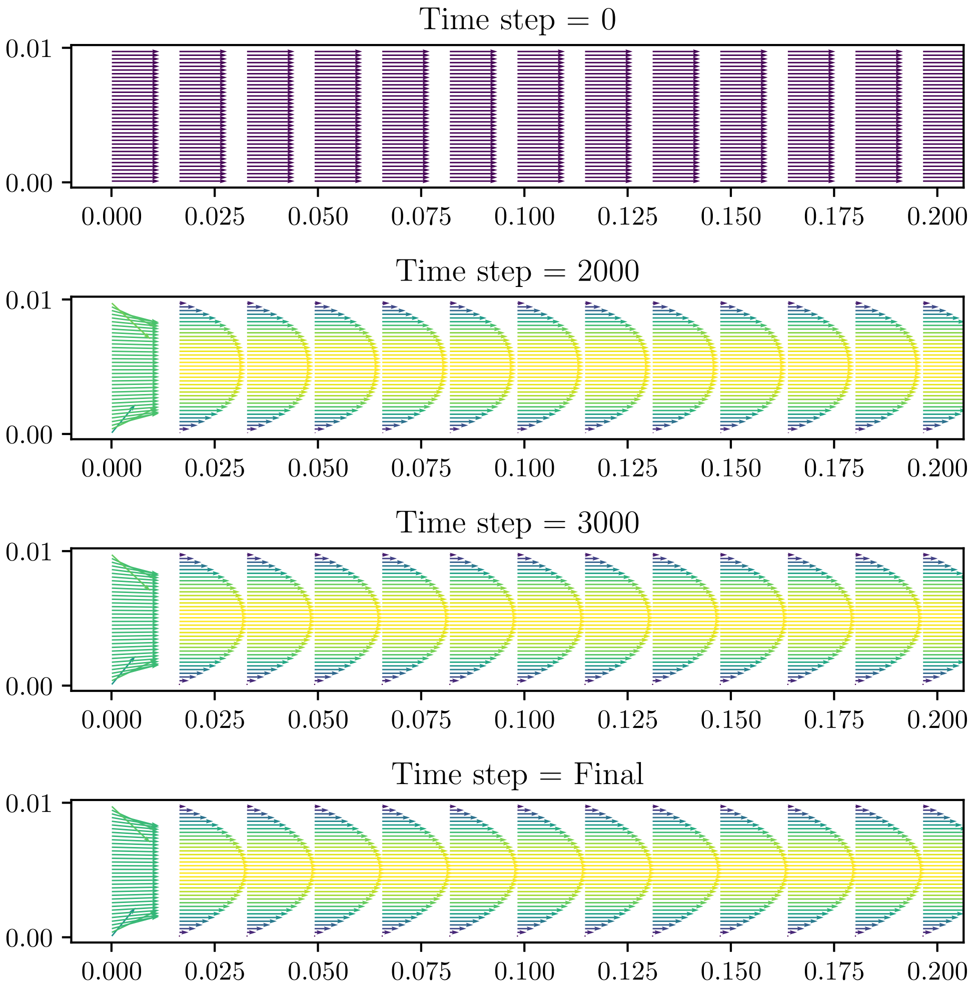

Figure 4: Velocity vectors at various time steps during the numerical solution for Plane Poiseuille flow using QUICK scheme and $n_y = 110$

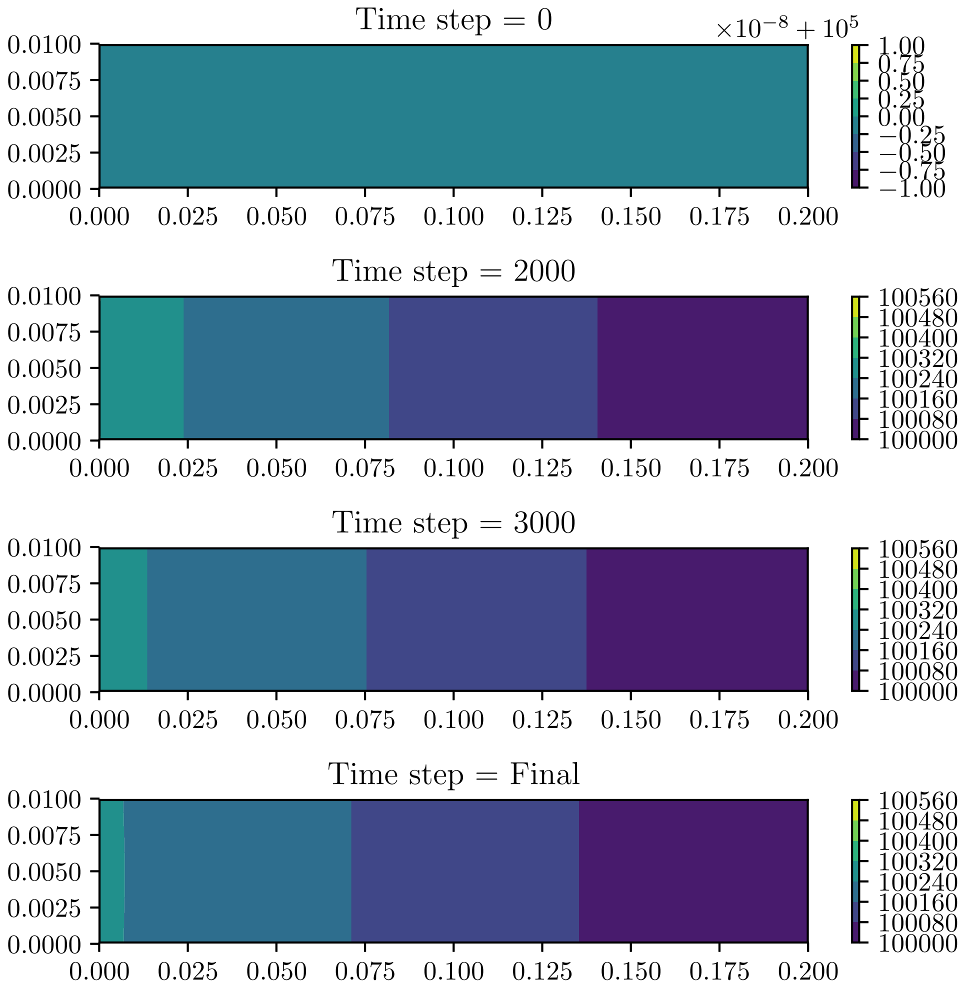

Figure 5: Pressure contours at various time steps during the numerical solution for Plane Poiseuille flow using QUICK scheme and $n_y = 110$

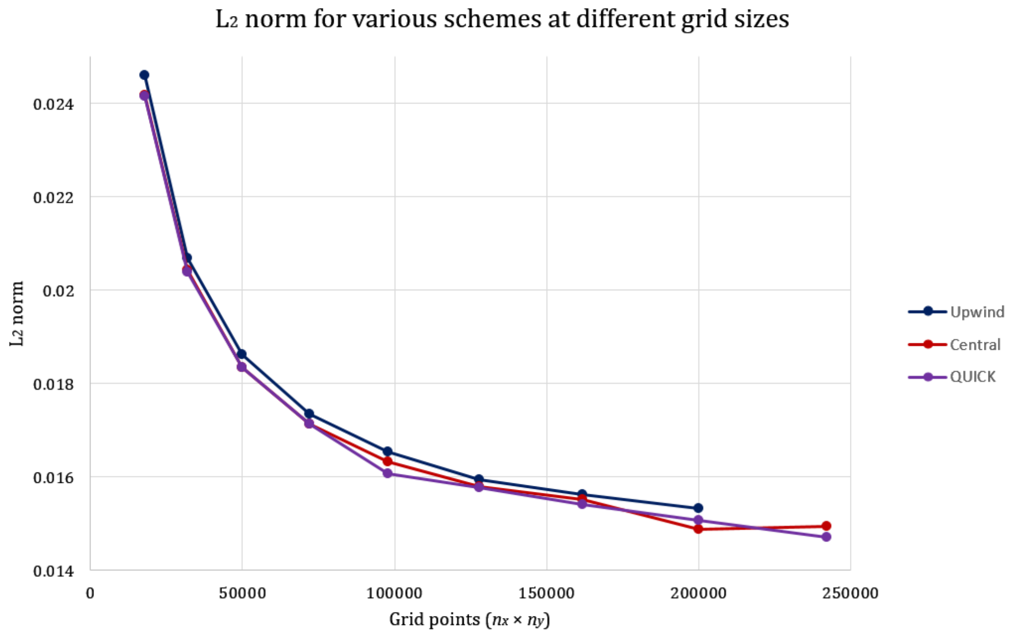

Figure 6: Pressure contours at various time steps during the numerical solution for Plane Poiseuille flow using QUICK scheme and $n_y = 110$

1.5.2 Case 2: Couette-Poiseuille flow with one surface moving

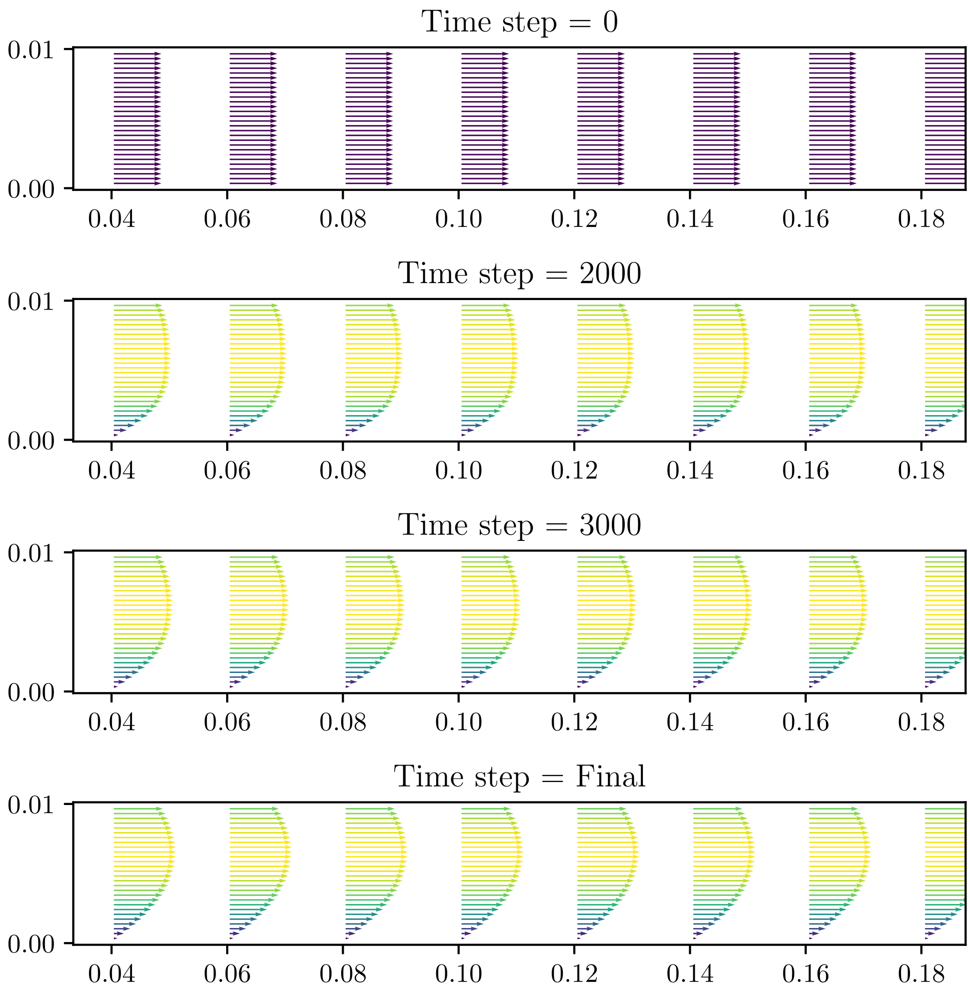

Figure 7: Velocity vectors at various time steps during the numerical solution for Couette-Poiseuille flow with one surface moving, using QUICK scheme and $n_y = 110$

1.5.3 Case 3: Couette-Poiseuille flow with both surfaces moving in opposite directions

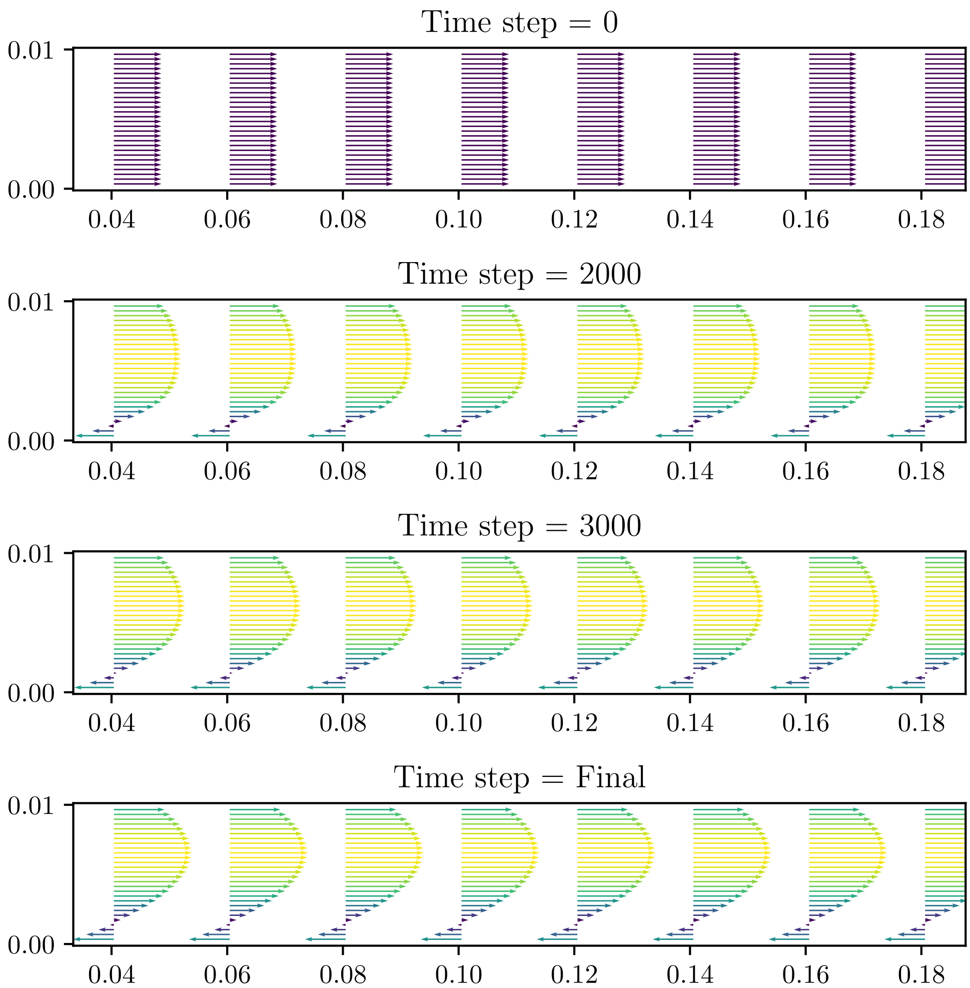

Figure 8: Velocity vectors at various time steps during the numerical solution for Couette-Poiseuille flow with both surfaces moving, using QUICK scheme and $n_y = 110$

1.6 Pending cases

The following cases for larger grid sizes are still running. The table below lists the cases that have failed to converge and the currently running cases.

Table 1: List of cases for $n_y = 500$ that have been tested so far

Case	Time step	Max. pressure iterations	Status
1	1×10^{-5}	2000	Failed
2	1×10^{-5}	10	Failed
3	1×10^{-6}	10	Failed
4	1×10^{-7}	10	Failed
5	1×10^{-7}	100	Failed
6	1×10^{-8}	10	Running

1.7 Future work

- Solve energy equation for heat transfer
- Solve for 3D domain