Progress Report

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1 Solution of Navier-Stokes equations using Finite Volume Method for nonuniform grid

1.1 Discretization of the convective and diffusive terms for 2D nonuniform grid

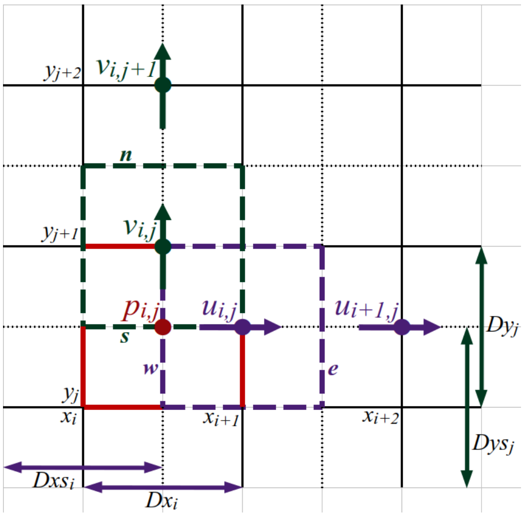

Figure 1: Visual representation of the staggered grid used for discretization in Finite Volume Method

Linear interpolation is used for the velocities u_n, u_s, v_e, and v_w. Let us analyze one of the velocities u_n.

$$\frac{u_n - u_{i,j}}{y_{j+1} - y_{j+1/2}} = \frac{u_{i,j+1} - u_{i,j}}{y_{j+3/2} - y_{j+1/2}}$$

$$\frac{u_n - u_{i,j}}{Dy_j/2} = \frac{u_{i,j+1} - u_{i,j}}{(Dy_{j+1}/2) + (Dy_j/2)}$$

$$\frac{u_n - u_{i,j}}{Dy_j/2} = \frac{u_{i,j+1} - u_{i,j}}{(Dy_{j+1}+Dy_j)/2}$$

Therefore, the terms reduce to the following:

$$u_n = u_{i,j} + \left(\frac{u_{i,j+1} - u_{i,j}}{Dys_{j+1}}\right)\left(\frac{Dy_j}{2}\right) \qquad v_e = v_{i,j} + \left(\frac{v_{i+1,j} - v_{i,j}}{Dxs_{i+1}}\right)\left(\frac{Dx_i}{2}\right)$$

$$u_s = u_{i,j-1} + \left(\frac{u_{i,j} - u_{i,j-1}}{Dys_j}\right)\left(\frac{Dy_{j-1}}{2}\right) \qquad v_w = v_{i-1,j} + \left(\frac{v_{i,j} - v_{i-1,j}}{Dxs_i}\right)\left(\frac{Dx_{i-1}}{2}\right)$$

1.1.1 Upwind scheme for velocities in the convective terms

For rest of the velocities in convective terms, the upwind scheme may be used. For positive velocities,

$$u_e = u_{i,j} \qquad u_w = u_{i-1,j} \qquad u_{e,v} = u_{i,j} \qquad u_{w,v} = u_{i-1,j}$$
$$v_n = v_{i,j} \qquad v_s = v_{i,j-1} \qquad v_{n,u} = v_{i,j} \qquad v_{s,u} = v_{i,j-1}$$

For negative velocities,

$$u_e = u_{i+1,j} \qquad u_w = u_{i,j} \qquad u_{e,v} = u_{i,j+1} \qquad u_{w,v} = u_{i-1,j+1}$$
$$v_n = v_{i,j+1} \qquad v_s = v_{i,j} \qquad v_{n,u} = v_{i+1,j} \qquad v_{s,u} = v_{i+1,j-1}$$

For the first time step, Euler scheme is used and for subsequent time steps, Adams-Bashforth scheme is utilized.

1.1.2 QUICK scheme for velocities in the convective terms

QUICK scheme is also incorporated in the code, with an option to switch between upwind (first order) or QUICK scheme (second order). For positive velocities,

$$u_e = \frac{u_i + u_{i+1}}{2} - \frac{Dx_{i+1}^2}{8Dxs_{i+1}}\left(\frac{u_{i+1} - u_i}{Dx_{i+1}} - \frac{u_i - u_{i-1}}{Dx_i}\right)$$

$$u_w = \frac{u_{i-1} + u_i}{2} - \frac{Dx_i^2}{8Dxs_i}\left(\frac{u_i - u_{i-1}}{Dx_i} - \frac{u_{i-1} - u_{i-2}}{Dx_{i-1}}\right)$$

$$v_n = \frac{v_j + v_{j+1}}{2} - \frac{Dy_{j+1}^2}{8Dys_{j+1}}\left(\frac{v_{j+1} - v_j}{Dy_{j+1}} - \frac{v_j - v_{j-1}}{Dy_j}\right)$$

$$v_s = \frac{v_{j-1} + v_j}{2} - \frac{Dy_j^2}{8Dys_j}\left(\frac{v_j - v_{j-1}}{Dy_j} - \frac{v_{j-1} - v_{j-2}}{Dy_{j-1}}\right)$$

For negative velocities,

$$u_e = \frac{u_i + u_{i+1}}{2} - \frac{Dx_{i+1}^2}{8Dxs_{i+2}}\left(\frac{u_{i+2} - u_{i+1}}{Dx_{i+2}} - \frac{u_{i+1} - u_i}{Dx_{i+1}}\right)$$

$$u_w = \frac{u_{i-1} + u_i}{2} - \frac{Dx_i^2}{8Dxs_{i+1}}\left(\frac{u_{i+1} - u_i}{Dx_{i+1}} - \frac{u_i - u_{i-1}}{Dx_i}\right)$$

$$v_e = \frac{v_j + v_{j+1}}{2} - \frac{Dy_{j+1}^2}{8Dys_{j+2}}\left(\frac{v_{j+2} - v_{j+1}}{Dy_{j+2}} - \frac{v_{j+1} - v_j}{Dy_{j+1}}\right)$$

$$v_w = \frac{v_{j-1} + v_j}{2} - \frac{Dy_j^2}{8Dys_{j+1}}\left(\frac{v_{j+1} - v_j}{Dy_{j+1}} - \frac{v_j - v_{j-1}}{Dy_j}\right)$$

1.2 2D results for nonuniform grid with higher order scheme

A nonuniform grid was selected for the y-direction to provide a greater number of cells in the middle of channel flow, as per the following equation:

$$y_j = \frac{L}{2}\left[\tan\left\{\pi\left(\frac{x_i}{2L} - \frac{1}{4}\right)\right\} + 1\right] \tag{1}$$

The code was run with first order upwind scheme and then using second order QUICK scheme. The results for two grid sizes are shown in the figures below.

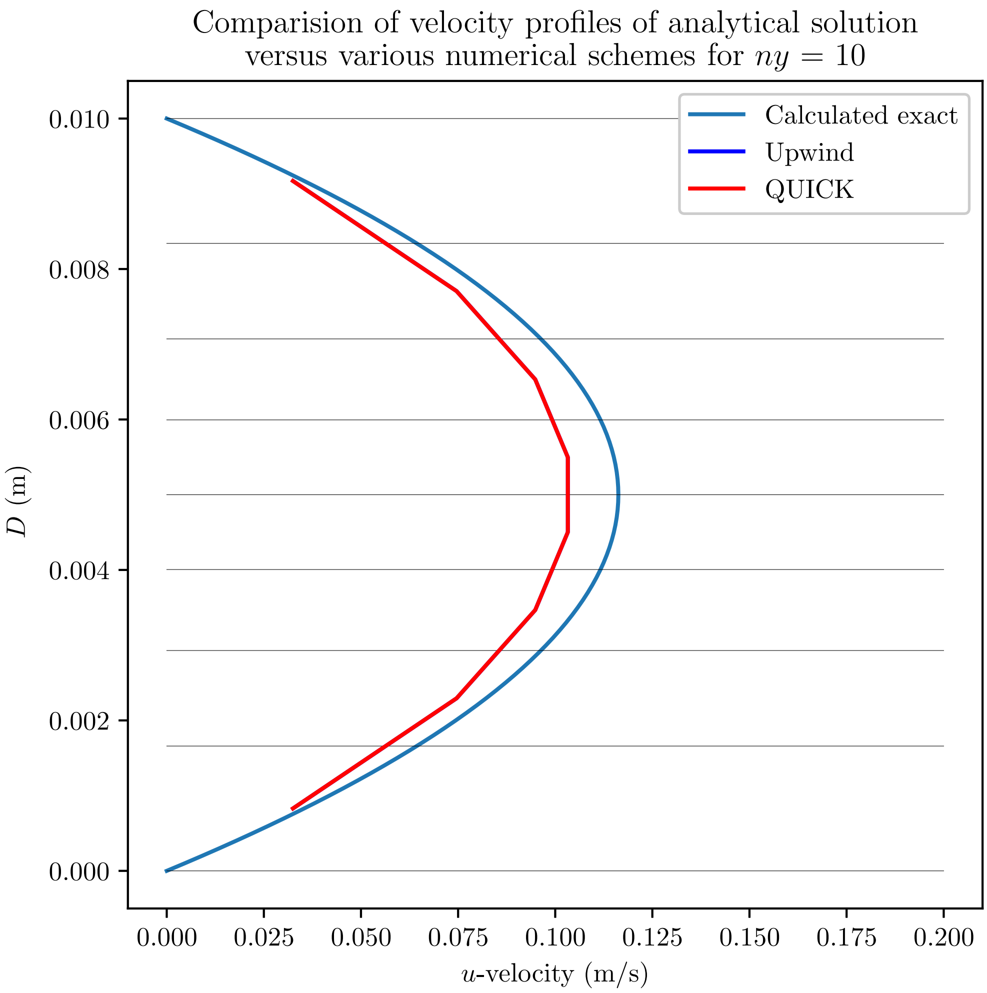

Figure 2: Comparison of numerical results with analytical solution for nonuniform grid with two different schemes using y-direction grid intervals $n_y = 10$.

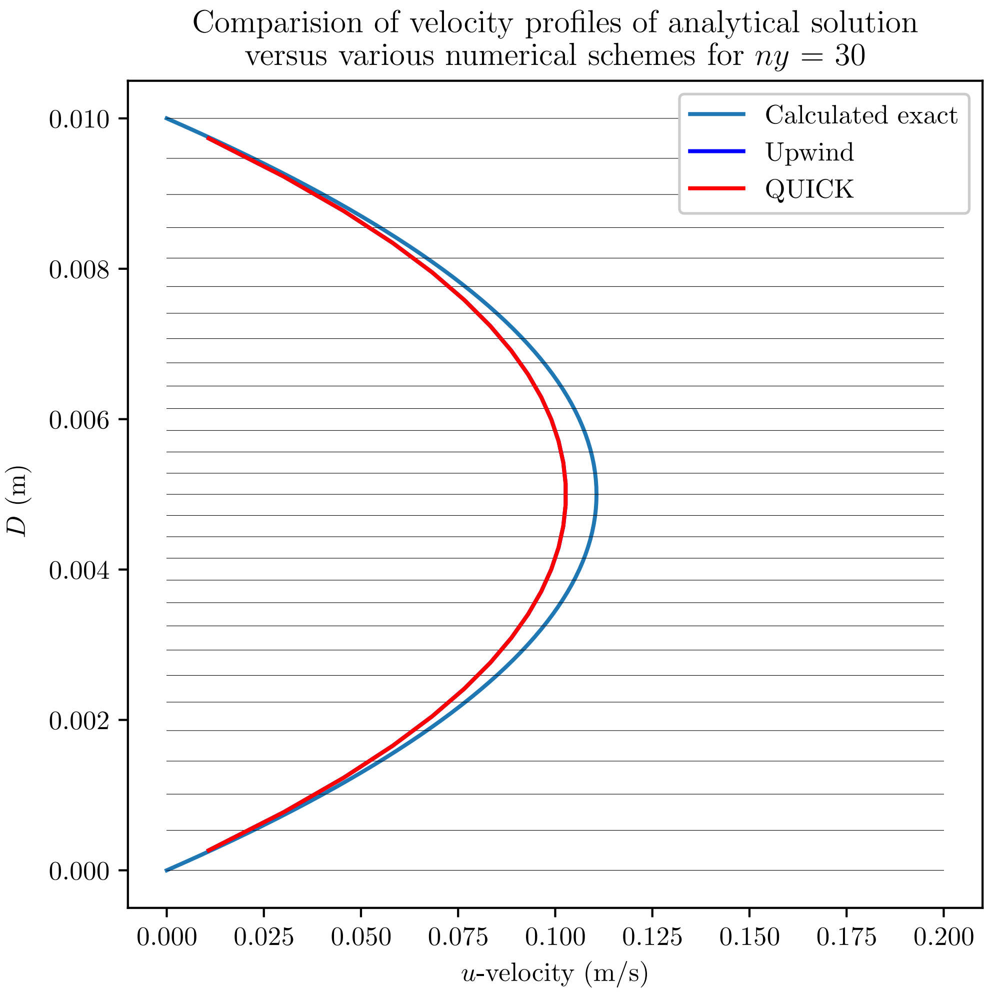

Figure 3: Comparison of numerical results with analytical solution for nonuniform grid with two different schemes using y-direction grid intervals $n_y = 30$.

1.3 Discretization of the convective and diffusive terms for 3D nonuniform grid

The convective and diffusive terms can be discretized using the individual components. To apply the projection method, the u-component equation can be written as

$$\frac{\partial u}{\partial t} = -\frac{\partial uu}{\partial x} - \frac{\partial uv}{\partial y} - \frac{\partial uw}{\partial z} + \nu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right] \quad , \tag{2}$$

which can be discretized as

$$\begin{aligned}\frac{\partial u}{\partial t} = &- \frac{u_e^2 - u_w^2}{Dxs_{i+1}} - \frac{u_n v_{n,u} - u_s v_{s,u}}{Dy_j} - \frac{u_t w_{t,u} - u_b w_{b,u}}{Dz_k} \\ &+\nu \left[\left\{ \frac{u_{i+1,j,k} - u_{i,j,k}}{Dx_{i+1}} - \frac{u_{i,j,k} - u_{i-1,j,k}}{Dx_i} \right\} \frac{1}{Dxs_{i+1}} \right. \\ &+ \left\{ \frac{u_{i,j+1,k} - u_{i,j,k}}{Dys_{j+1}} - \frac{u_{i,j,k} - u_{i,j-1,k}}{Dys_j} \right\} \frac{1}{Dy_j} \\ &\left. + \left\{ \frac{u_{i,j,k+1} - u_{i,j,k}}{Dzs_{k+1}} - \frac{u_{i,j,k} - u_{i,j,k-1}}{Dzs_k} \right\} \frac{1}{Dz_k} \right] \quad .\end{aligned} \tag{3}$$

where, for the *diffusion terms*, second-order central scheme has been used. Similarly for the v- and w-components, the respective equations will be

$$\frac{\partial v}{\partial t} = -\frac{\partial vu}{\partial x} - \frac{\partial vv}{\partial y} - \frac{\partial vw}{\partial z} + \nu \left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right] \tag{4}$$

and

$$\frac{\partial w}{\partial t} = -\frac{\partial wu}{\partial x} - \frac{\partial wv}{\partial y} - \frac{\partial ww}{\partial z} + \nu \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right] \quad , \tag{5}$$

which can be discretized as

$$\begin{aligned}\frac{\partial v}{\partial t} = &- \frac{v_e u_{e,v} - v_w u_{w,v}}{Dx_i} - \frac{v_n^2 - v_s^2}{Dys_{j+1}} - \frac{v_t w_{t,v} - v_b w_{b,v}}{Dz_k} \\ &+\nu \left[\left\{ \frac{v_{i+1,j,k} - v_{i,j,k}}{Dxs_{i+1}} - \frac{v_{i,j,k} - v_{i-1,j,k}}{Dxs_i} \right\} \frac{1}{Dx_i} \right. \\ &+ \left\{ \frac{v_{i,j+1,k} - v_{i,j,k}}{Dy_{j+1}} - \frac{v_{i,j,k} - v_{i,j-1,k}}{Dy_j} \right\} \frac{1}{Dys_{j+1}} \\ &\left. + \left\{ \frac{v_{i,j,k+1} - v_{i,j,k}}{Dzs_{k+1}} - \frac{v_{i,j,k} - v_{i,j,k-1}}{Dzs_k} \right\} \frac{1}{Dz_k} \right] \quad ,\end{aligned} \tag{6}$$

and

$$
\begin{aligned}
\frac{\partial w}{\partial t} = & -\frac{w_e u_{e,w} - w_w u_{w,w}}{Dx_i} - \frac{w_n v_{n,w} - w_s v_{s,w}}{Dy_j} - \frac{w_t^2 - w_b^2}{Dzs_{k+1}} \\
& +\nu \left[\left\{ \frac{w_{i+1,j,k} - w_{i,j,k}}{Dxs_{i+1}} - \frac{w_{i,j,k} - w_{i-1,j,k}}{Dxs_i} \right\} \frac{1}{Dx_i} \right. \\
& + \left\{ \frac{w_{i,j+1,k} - w_{i,j,k}}{Dys_{j+1}} - \frac{w_{i,j,k} - w_{i,j-1,k}}{Dys_j} \right\} \frac{1}{Dy_j} \\
& \left. + \left\{ \frac{w_{i,j,k+1} - w_{i,j,k}}{Dz_{k+1}} - \frac{w_{i,j,k} - w_{i,j,k-1}}{Dz_k} \right\} \frac{1}{Dzs_{k+1}} \right] \quad . \qquad (7)
\end{aligned}
$$

1.4 Poisson equation of pressure

The Poisson equation of pressure in 3D may be written as

$$
\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = \frac{1}{\Delta t} \left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} + \frac{\partial w^*}{\partial z} \right) \quad . \qquad (8)
$$

Pressure may be found from the following discretized equation,

$$
\begin{aligned}
p_{i,j,k}^{n+1} = & \frac{1}{\left[-\frac{Dy}{Dx1} - \frac{Dy}{Dx2} - \frac{Dy}{Dz1} - \frac{Dy}{Dz2} - \frac{Dx}{Dy1} - \frac{Dx}{Dy2} - \frac{Dx}{Dz1} - \frac{Dx}{Dz2} - \frac{Dz}{Dx1} - \frac{Dz}{Dx2} - \frac{Dz}{Dy1} - \frac{Dz}{Dy2} \right]} \\
& \times \left[-\frac{Dz}{Dx1} p_{i+1,j,k} - \frac{Dz}{Dx2} p_{i-1,j,k} - \frac{Dz}{Dy1} p_{i,j+1,k} - \frac{Dz}{Dy2} p_{i,j-1,k} \right. \\
& - \frac{Dy}{Dx1} p_{i+1,j,k} - \frac{Dy}{Dx2} p_{i-1,j,k} - \frac{Dy}{Dz1} p_{i,j,k+1} - \frac{Dy}{Dz2} p_{i,j,k-1} \\
& - \frac{Dx}{Dy1} p_{i,j+1,k} - \frac{Dx}{Dy2} p_{i,j-1,k} - \frac{Dx}{Dz1} p_{i,j,k+1} - \frac{Dx}{Dz2} p_{i,j,k-1} \\
& \left. + \frac{1}{\Delta t} \left\{ \left(u_{i,j,k}^* - u_{i-1,j,k}^* \right) \Delta y \Delta z + \left(v_{i,j,k}^* - v_{i,j-1,k}^* \right) \Delta x \Delta z + \left(w_{i,j,k}^* - w_{i,j,k-1}^* \right) \Delta x \Delta y \right\} \right] \quad . \\
& \qquad (9)
\end{aligned}
$$

1.5 Future work

- Generate figures for 3D grid
- Apply grid average and nodal value in QUICK scheme
- Use Smagorinsky-Lilly turbulent model for 2D case