# Progress Report

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# 1 Solution of Navier-Stokes equations using Finite Volume Method for nonuniform grid

## 1.1 Discretization of the convective and diffusive terms

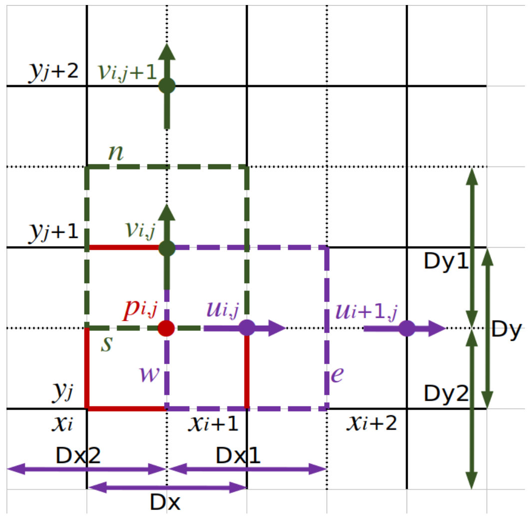


Figure 1: Visual representation of the staggered grid used for discretization in Finite Volume Method

The convective and diffusive terms of Navier-Stokes equations can be discretized using the individual components. Using Chorin's projection method, the $u$-component equation can be written as

$$\frac{\partial u}{\partial t} = -\frac{\partial uu}{\partial x} - \frac{\partial uv}{\partial y} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] \quad . \tag{1}$$

For nonuniform grids, using second-order central scheme for diffusion terms and the notation described in figure 1, the equation can be discretized as

$$\begin{aligned}\frac{\partial u}{\partial t} = &- \frac{u_e^2 - u_w^2}{Dx1_i} - \frac{u_n v_{n,u} - u_s v_{s,u}}{Dy_j} + \nu\left[\left\{\frac{u_{i+1,j} - u_{i,j}}{Dx_{i+1}} - \frac{u_{i,j} - u_{i-1,j}}{Dx_i}\right\}\frac{1}{Dx1_i}\right.\\ &\left.+\left\{\frac{u_{i,j+1} - u_{i,j}}{Dy1_j} - \frac{u_{i,j} - u_{i,j-1}}{Dy2_j}\right\}\frac{1}{Dy_j}\right] \quad .\end{aligned} \tag{2}$$

Similarly for the $v$-component,

$$\frac{\partial v}{\partial t} = -\frac{\partial uv}{\partial x} - \frac{\partial vv}{\partial y} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] \quad , \tag{3}$$

which can be and discretized as

$$
\begin{aligned}
\frac{\partial v}{\partial t} = & -\frac{u_{e,v} v_e - u_{w,v} v_w}{Dx_i} - \frac{v_n^2 - v_s^2}{Dy1_j} \\
& + \nu \left[ \left\{ \frac{v_{i+1,j} - v_{i,j}}{Dx1_i} - \frac{v_{i,j} - v_{i-1,j}}{Dx2_i} \right\} \frac{1}{Dx_i} \right. \\
& \left. + \left\{ \frac{v_{i,j+1} - v_{i,j}}{Dy_{j+1}} - \frac{v_{i,j} - v_{i,j-1}}{Dy_j} \right\} \frac{1}{Dy1_j} \right] \quad .
\end{aligned}
\tag{4}
$$

### 1.1.1 Upwind scheme for velocities in the convective terms

Linear interpolation is used for the velocities $u_n$, $u_s$, $v_e$ and $v_w$,

$$
\begin{aligned}
u_n &= u_{i,j} + \frac{u_{i,j+1} - u_{i,j}}{Dy1_j} & \qquad v_e &= v_{i,j} + \frac{v_{i+1,j} - v_{i,j}}{Dx1_i} \\
u_s &= u_{i,j} - \frac{u_{i,j} - u_{i,j-1}}{Dy2_j} & \qquad v_w &= v_{i,j} - \frac{v_{i,j} - v_{i-1,j}}{Dx2_i}
\end{aligned}
$$

For rest of the velocities in convective terms, the upwind scheme was used. For positive velocities,

$$
\begin{aligned}
u_e &= u_{i,j} & \qquad u_{n,v} &= u_{i-1,j+1} & \qquad v_n &= v_{i,j} & \qquad v_{e,u} &= v_{i+1,j-1} \\
u_w &= u_{i-1,j} & \qquad u_{s,v} &= u_{i-1,j} & \qquad v_s &= v_{i,j-1} & \qquad v_{w,u} &= v_{i,j-1}
\end{aligned}
$$

For negative velocities,

$$
\begin{aligned}
u_e &= u_{i+1,j} & \qquad u_{n,v} &= u_{i,j+1} & \qquad v_n &= v_{i,j+1} & \qquad v_{e,u} &= v_{i,j+1} \\
u_w &= u_{i,j} & \qquad u_{s,v} &= u_{i,j} & \qquad v_s &= v_{i,j} & \qquad v_{w,u} &= v_{i,j}
\end{aligned}
$$

For the first time step, Euler scheme is used and for subsequent time steps, Adams-Bashforth scheme is utilized.

## 1.2 Poisson equation of pressure

The Poisson equation for pressure can be written as

$$
\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \frac{1}{\Delta t} \left( \frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} \right) \quad . \tag{5}
$$

Integrating it twice, discretizing and rearranging leads to

$$
\begin{aligned}
p_{i,j}^{n+1} =& \frac{1}{\left[-\frac{Dy_j}{Dx1_i} - \frac{Dy_j}{Dx2_i} - \frac{Dx_i}{Dy1_j} - \frac{Dx_i}{Dy2_j}\right]} \\
& \times \left[-\frac{Dy_j}{Dx1_i} p_{i+1,j} - \frac{Dy_j}{Dx2_i} p_{i-1,j} - \frac{Dx_i}{Dy1_j} p_{i,j+1} - \frac{Dx_i}{Dy2_j} p_{i,j-1}\right. \\
& \left. + \frac{1}{\Delta t} \left\{ \left( u_{i,j}^{*} - u_{i-1,j}^{*} \right) Dy_j + \left( v_{i,j}^{*} - v_{i,j-1}^{*} \right) Dx_i \right\} \right] \quad . \qquad (6)
\end{aligned}
$$

This equation is employed in successive over-relaxation method (SOR).

## 1.3 Corrected velocity

The correct velocity can be found for $u$- and $v$-components using

$$
u_{i,j}^{n+1} = u_{i,j}^{*} - \frac{\Delta t}{\rho} \cdot \frac{p_{i+1,j}^{n+1} - p_{i,j}^{n+1}}{Dx1_i} \qquad (7)
$$

and

$$
v_{i,j}^{n+1} = v_{i,j}^{*} - \frac{\Delta t}{\rho} \cdot \frac{p_{i,j+1}^{n+1} - p_{i,j}^{n+1}}{Dy1_j} \qquad (8)
$$

## 1.4 Future work

- Remove the errors for 2D case
- Modify the 3D code accordingly