LECTURE NOTES ON FINITE VOLUME METHOD

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September 10, 2015

Governing Equations

Fluid flow should conserve mass and momentum, so the continuity equation and Navier-Stokes equation are used and denoted as

$$\nabla \cdot \mathbf{u} = 0 \tag{1}$$

and

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f} \tag{2}$$

where $\mathbf{u}$ is a velocity vector, p is pressure, ρ is density of fluid, and $\mathbf{f}$ is body force per unit mass. Proper boundary conditions for velocity or pressure should be given for solving those governing equations.

Projection Method

Chorin(1967) proposed the projection method to solve those governing equations. First of all, the Navier-Stokes equation is decomposed into

$$\frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} = -\nabla \cdot (\mathbf{u}\mathbf{u}) + \nu \nabla^2 \mathbf{u} \tag{3}$$

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^*}{\Delta t} = -\frac{1}{\rho} \nabla p \ . \tag{4}$$

where n is the time level index. The new vector $\mathbf{u}^*$ is the intermediate velocity.

Ref: Chorin, A. J. (1967), The numerical solution of the Navier-Stokes equations for an incompressible fluid, *Bull. Am. Math. Soc.* **73**: 928931

Poisson Equation of Pressure

The velocity at the $n+1^{th}$ time level should also obey the mass conservation law, i.e.

$$\nabla \cdot \mathbf{u}^{n+1} = 0 \quad . \tag{5}$$

Substituting Eq. 4 to Eq. 5 gives

$$\nabla \cdot \left(\mathbf{u}^* - \frac{\Delta t}{\rho} \nabla p \right) = 0 \tag{6}$$

or

$$\nabla^2 p = \frac{\rho}{\Delta t} \nabla \cdot \mathbf{u}^* \quad . \tag{7}$$

Eq. 7 can be used for calculation of pressure.

Temporal Scheme

In order to solve Eq. 3, a temporal integral scheme should be applied. Given the terms in the right hand side are denoted as $\mathcal{F}(\mathbf{u})$, the second order Adams-Bashforth scheme can be one of choices for the temporal scheme. It is undertaken by the formula

$$\frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} = \frac{3}{2}\mathcal{F}(\mathbf{u}^n) - \frac{1}{2}\mathcal{F}(\mathbf{u}^{n-1}) \ . \tag{8}$$

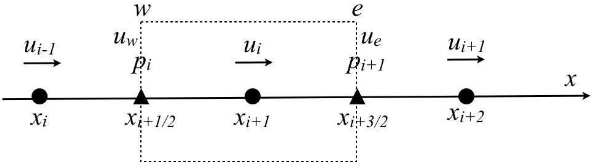
The Adams-Bashforth scheme is an explicit scheme, so the terms in the right hand sides are known or calculated at the n^{th} and the $n{+}1^{th}$ time levels.

1-D example

The term $\mathcal{F}(\mathbf{u})$ is $-\nabla \cdot (\mathbf{uu}) + \nu\nabla^2\mathbf{u}$. Given that the 1-D Cartesian coordinate system is used, it becomes

$$\mathcal{F}(\mathbf{u}) = -\frac{\partial(uu)}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2} \quad . \tag{9}$$

Consider a computational cell in the x-axis as shown in the following figure. The cell is surrounded by the dash line. Its east and west sides are denoted by e and w, respectively. Since it is an one-dimensional cell, its volume is Δx.



Taking the integral of $\mathcal{F}(\mathbf{u})$ over the computational cell gives

$$\int_w^e -\frac{\partial(uu)}{\partial x} + \nu \frac{\partial^2 u}{\partial x^2} \mathrm{d}x = \int_w^e \mathcal{F}(\mathbf{u}) \mathrm{d}x$$
$$= -\left(uu\big|_e - uu\big|_w \right) + \nu \left(\frac{\partial u}{\partial x}\bigg|_e - \frac{\partial u}{\partial x}\bigg|_w \right) . \tag{10}$$

Eq. 3 becomes

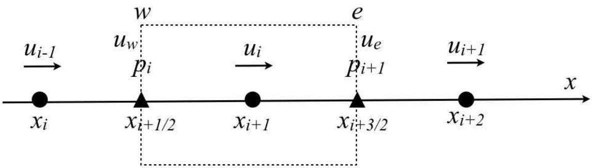
$$\int_w^e \left(\frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} \right) \mathrm{d}x = \int_w^e \mathcal{F}(\mathbf{u}) \mathrm{d}x \tag{11}$$

or

$$\frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} = -\frac{uu\big|_e - uu\big|_w}{\Delta x} + \frac{\nu}{\Delta x} \left(\frac{\partial u}{\partial x}\bigg|_e - \frac{\partial u}{\partial x}\bigg|_w \right) \tag{12}$$

Staggered Grids

Velocity u and pressure p are allocated at different points in the finite volume method as shown in the figure below. It is called a stagger grid arrangement.



The purpose of a stagger grid is to prevent pressure oscillation in numerical solutions. The other point is that $u_e, u_w, \left.\frac{\partial u}{\partial x}\right|_e, \left.\frac{\partial u}{\partial x}\right|_w$ are unknown.

Central Difference Scheme

The central difference scheme is applied to calculation of diffusion terms, i.e.

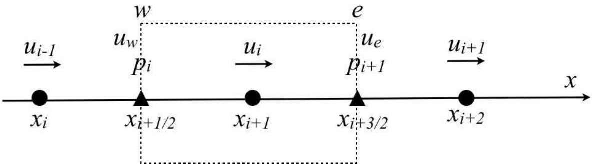
$$\left.\frac{\partial u}{\partial x}\right|_e = \frac{u_{i+1} - u_i}{x_{i+2} - x_{i+1}} \tag{13}$$

$$\left.\frac{\partial u}{\partial x}\right|_w = \frac{u_i - u_{i-1}}{x_{i+1} - x_i} \quad . \tag{14}$$

Essentially, its accuracy is second-order.

Upwind scheme

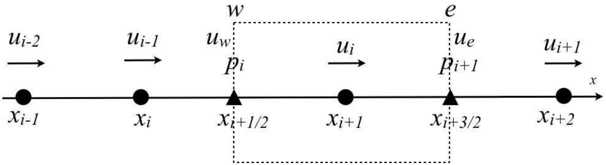
Velocity components at control faces e and w should be determined. It depends on the velocity direction as the upwind scheme is utilized.



If u_e is positive, then u_e will become u_i. It means that the momentum at the cell centre is conveyed to the control face e. On the other hand, if u_e is negative, then u_e will become u_{i+1}. This is the so-called upwind scheme. It is only first-order accurate. The same principle can be applied to u_w.

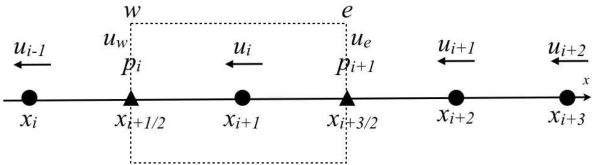
QUICK scheme 1/2

Since the upwind scheme is only first order accurate, the numerical solution will be also first order accurate. To improve the order of accuracy, the QUICK scheme is proposed. It involves more velocity components to determine a face velocity. The Taylor expansion is used. Given that u_e and u_w are positive, u_{i-1}, u_i, and u_{i+1} are used to determine u_e. For a uniform grid,
$u_e = \frac{6}{8}u_i + \frac{3}{8}u_{i+1} - \frac{1}{8}u_{i-1}$ and $u_w = \frac{6}{8}u_{i-1} + \frac{3}{8}u_i - \frac{1}{8}u_{i-2}$.



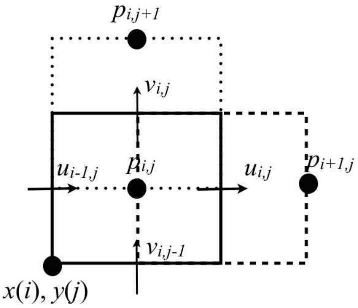
QUICK scheme 2/2

Given that u_e and u_w are negative, u_i, u_{i+1}, and u_{i+2} are used to determine u_e. For a uniform grid, $u_e = \frac{6}{8}u_{i+1} + \frac{3}{8}u_i - \frac{1}{8}u_{i+2}$ and $u_w = \frac{6}{8}u_i + \frac{3}{8}u_{i-1} - \frac{1}{8}u_{i+1}$.



Projection Method 1/6

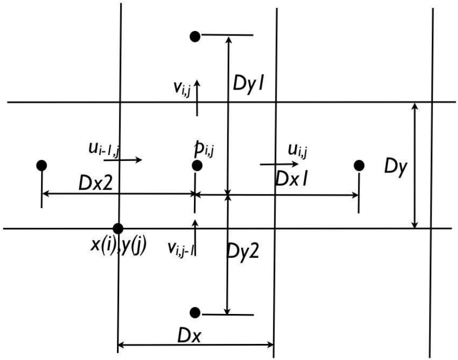
Once $\mathbf{u}^*$ is calculated by Eq. (8), the Poisson equation of pressure, Eq. (7), can be solved by an iterative method. For a 2-D flow, a computational cell for pressure is denoted by the solid lines in the figure below.



Projection Method 2/6

Eq. (7) is denoted as

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \frac{1}{\Delta t}\left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y}\right) \tag{15}$$



Projection Method 3/6

$$\int \left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} \right) \mathrm{d}x = \int \frac{1}{\Delta t} \left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} \right) \mathrm{d}x \tag{16}$$

It gives

$$\left[\left.\frac{\partial p}{\partial x}\right|_e - \left.\frac{\partial p}{\partial x}\right|_w \right] + \left(\frac{\partial^2 p}{\partial y^2} \right) \Delta x = \frac{1}{\Delta t} \left[\left(u^*_{i,j} - u^*_{i-1,j} \right) + \frac{\partial v^*}{\partial y} \Delta x \right]. \tag{17}$$

Again, $\int$(Eq. (17))dy gives

$$\left[\left.\frac{\partial p}{\partial x}\right|_e - \left.\frac{\partial p}{\partial x}\right|_w \right] \Delta y + \left[\left.\frac{\partial p}{\partial y}\right|_n - \left.\frac{\partial p}{\partial y}\right|_s \right] \Delta x =$$
$$\frac{1}{\Delta t} \left[\left(u^*_{i,j} - u^*_{i-1,j} \right) \Delta y + \left(v^*_{i,j} - v^*_{i,j-1} \right) \Delta x \right]. \tag{18}$$

Projection Method 4/6

Furthermore,

$$\left[\frac{p_{i+1,j}-p_{i,j}}{Dx1}-\frac{p_{i,j}-p_{i-1,j}}{Dx2}\right]Dy+\left[\frac{p_{i,j+1}-p_{i,j}}{Dy1}-\frac{p_{i,j}-p_{i,j-1}}{Dy2}\right]Dx = \frac{1}{\Delta t}\left[\left(u^*_{i,j}-u^*_{i-1,j}\right)\Delta y+\left(v^*_{i,j}-v^*_{i,j-1}\right)\Delta x\right]. \tag{19}$$

As a result,

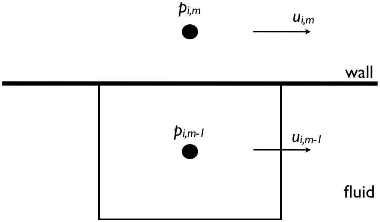
$$\frac{Dy}{Dx1}p_{i+1,j}+\left[-\frac{Dy}{Dx1}-\frac{Dy}{Dx2}-\frac{Dx}{Dy1}-\frac{Dx}{Dy2}\right]p_{i,j} +\frac{Dy}{Dx2}p_{i-1,j}+\frac{Dx}{Dy1}p_{i,j+1}+\frac{Dx}{Dy2}p_{i,j-1} = \frac{1}{\Delta t}\left[\left(u^*_{i,j}-u^*_{i-1,j}\right)\Delta y+\left(v^*_{i,j}-v^*_{i,j-1}\right)\Delta x\right]. \tag{20}$$

Projection Method 5/6

The Neumann boundary condition can used for Eq. (7), i.e.

$$\frac{\partial p}{\partial n} = 0. \tag{21}$$

For example, $p_{i,m-1}$ is inside the fluid domain, but $p_{i,m}$ is in the ghost cell. Since the Neumann boundary condition is applied, $p_{i,j} = p_{i,m-1}$.



Projection Method 6/6

Eq. (20) and Eq. (21) should solved together by an matrix solver, such as SOR method or conjugate gradient method. As a result,

$$\mathbf{u}^{n+1} = \mathbf{u}^* - \frac{\Delta t}{\rho} \cdot \nabla p \tag{22}$$

For 1D case,

$$u_i^{n+1} = u_i^* - \frac{\Delta t}{\rho} \cdot \frac{p_{i+1} - p_i}{x_{i+3/2} - x_{i+1/2}}. \tag{23}$$