Progress Report

Syed Ahmad Raza

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Numerical solution for 2D heat transfer using unequally spaced intervals and its comparison with analytical solution

Heat transfer in a 2D block of pure silver (99.9%) is considered. The equation for unsteady heat transfer is as follows:

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \tag{1}$$

Two different sets of boundary conditions were analyzed. For the first case, the analytical and numerical solutions were computed and compared. For the second set of boundary conditions, only the numerical solution was computed. The numerical solutions for both cases were computed until steady state was achieved.

All the codes were written in C++ and all the plots were generated through Matplotlib. 50×50 grids were used with unequal intervals selected for their suitability to the solution.

Grid generation for unequally spaced intervals

To practice grid generation with unequally spaced intervals, the following function was used to generate the x-coordinates:

$$f(x) = L \sin\left(\frac{\pi}{L} \times \frac{i}{n_x} \right) \tag{2}$$

where L is the length of the block, i is the index of the coordinate and n_x is the total number of intervals on the axis. The same function was also used for the y-coordinates, with the necessary replacements of variables. The resulting grid is shown in the figure below.

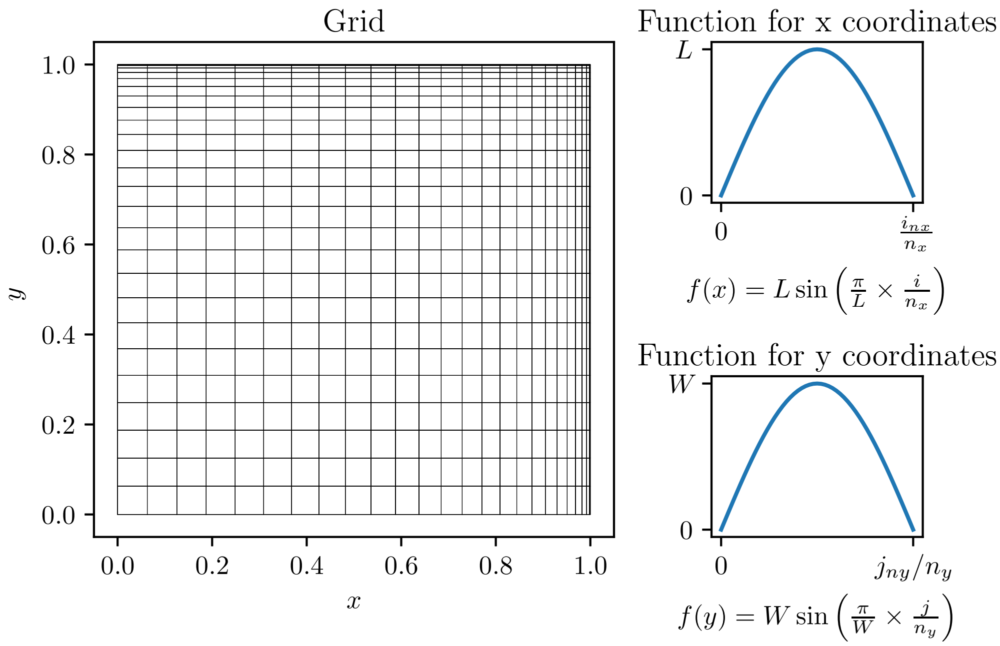

Figure 1: Practice grid with x and y functions and their graphs.

First case

The boundary conditions for this case have been selected as follows:

1. $T_{xi} = 25°C$ at $x = 0$
2. $T_{xf} = 25°C$ at $x = L$
3. $T_{yi} = 25°C$ at $y = 0$
4. $T_{yf} = 50°C$ at $y = L$

Analytical solution for first case

The simplified governing equation for the steady state reduces to:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{3}$$

The analytical solution to this problem is given by:

$$T(x, y) = \frac{2T_3}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1} + 1}{n}\right] \frac{\sin \lambda_n x \sinh \lambda_n y}{\sinh \lambda_n W} + T_1 \tag{4}$$

where

$$\lambda_n = \frac{n\pi}{L} \quad \text{for} \quad n = 0, 1, 2, \ldots \tag{5}$$

The temperatures in this equation are defined as:

$$T_1 = T_{xi} = T_{xf} = T_{yi} \quad \text{and} \quad T_3 = T_{yf} - T_1 \tag{6}$$

The analytical solution was computed for $n = 1$ to $n = 200$. The results are shown in the figure below.

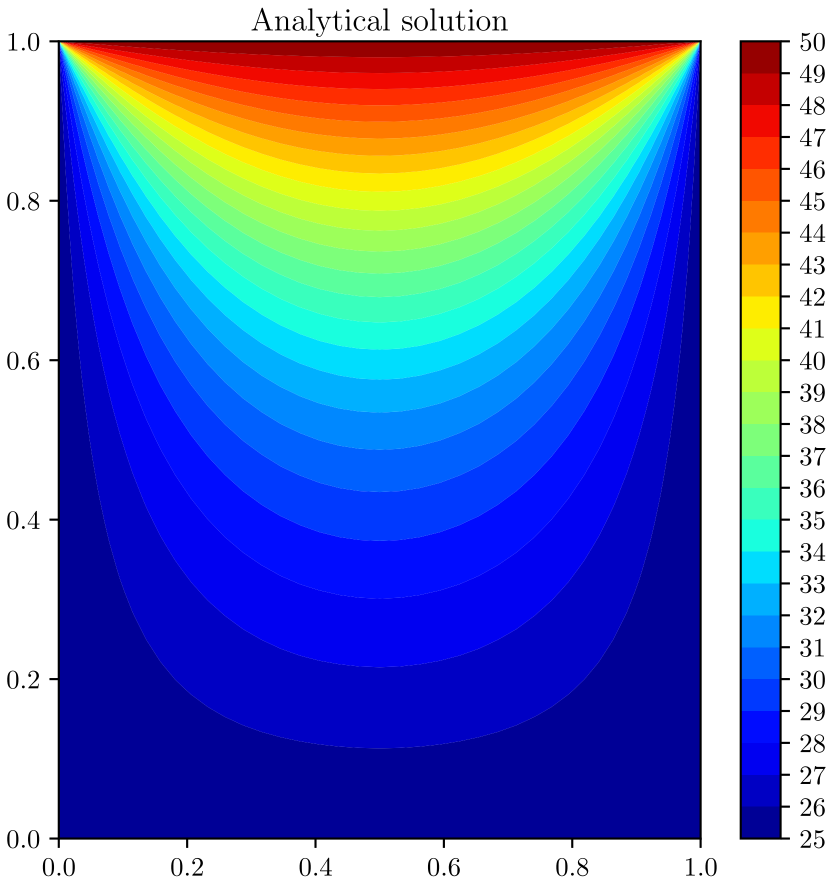

Figure 2: Plot for the analytical solution of case 1.

Numerical solution for first case

A C++ code was written for numerical solution of this unsteady 2D heat transfer problem. Initial condition was $T_i = 25°C$ throughout the block.

A time step of 0.001 was utilized. A 50×50 grid was selected with the following functions for the x and y axes:

$$f(x) = \frac{L}{2}\left[1 + \sin\left(\pi\left(\frac{i}{n_x} - \frac{1}{2}\right)\right)\right] \tag{7}$$

$$f(y) = W \sin\left(\frac{\pi}{2} \times \frac{j}{n_y}\right) \tag{8}$$

These functions were selected so that a denser mesh could be obtained in areas of the block where a larger temperature gradient is expected, from analysis of the analytical solution. The resulting grid is shown in the figure below.

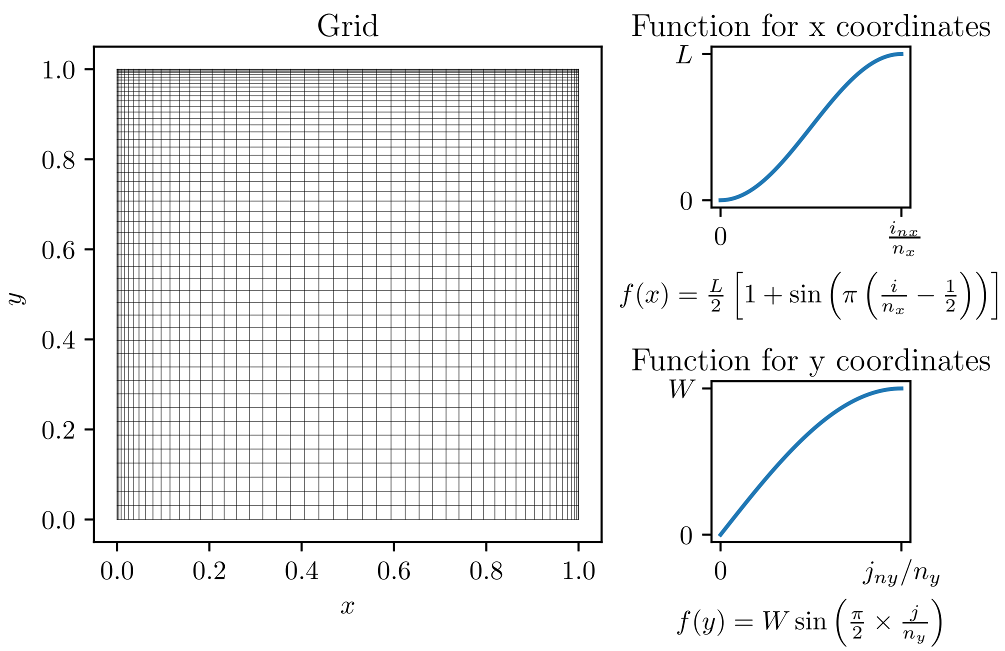

Figure 3: Grid for first case with x and y functions and their graphs

The following formula for forward in time, center in space method (FTCS) was derived and used:

$$\left.\frac{d^2 f}{dx^2}\right|_i = \frac{2(f_{i+1} + f_{i-1}) - 4f_i}{(x_{i+1} - x_i)^2 + (x_{i-1} - x_i)^2} - \frac{2(f_{i+1} - f_{i-1})(x_{i+1} - 2x_i + x_{i-1})}{(x_{i+1} - x_{i-1})[(x_{i+1} - x_i)^2 + (x_{i-1} - x_i)^2]} \tag{9}$$

The root mean square difference was calculated at $x = L/2$ for all y values between successive iterations of time to determine the steady state, using the following formula:

$$\text{RMS Difference} = \sqrt{\frac{\sum_{y=0}^{y_{n+1}} (T_{i-1} - T_i)^2}{n+1}} \tag{10}$$

Steady state, assumed when RMS difference $\leq 1 \times 10^{-10}$, was achieved at $t = 3741.219$ seconds. The numerical solution was also compared with the analytical solution. The root mean square difference (determined using the above formula) between the analytical and numerical solution was computed at $x = L/2$ for all y values. The result was found to be $0.00900°C$.

The results for four time conditions, including the steady state, are shown below.

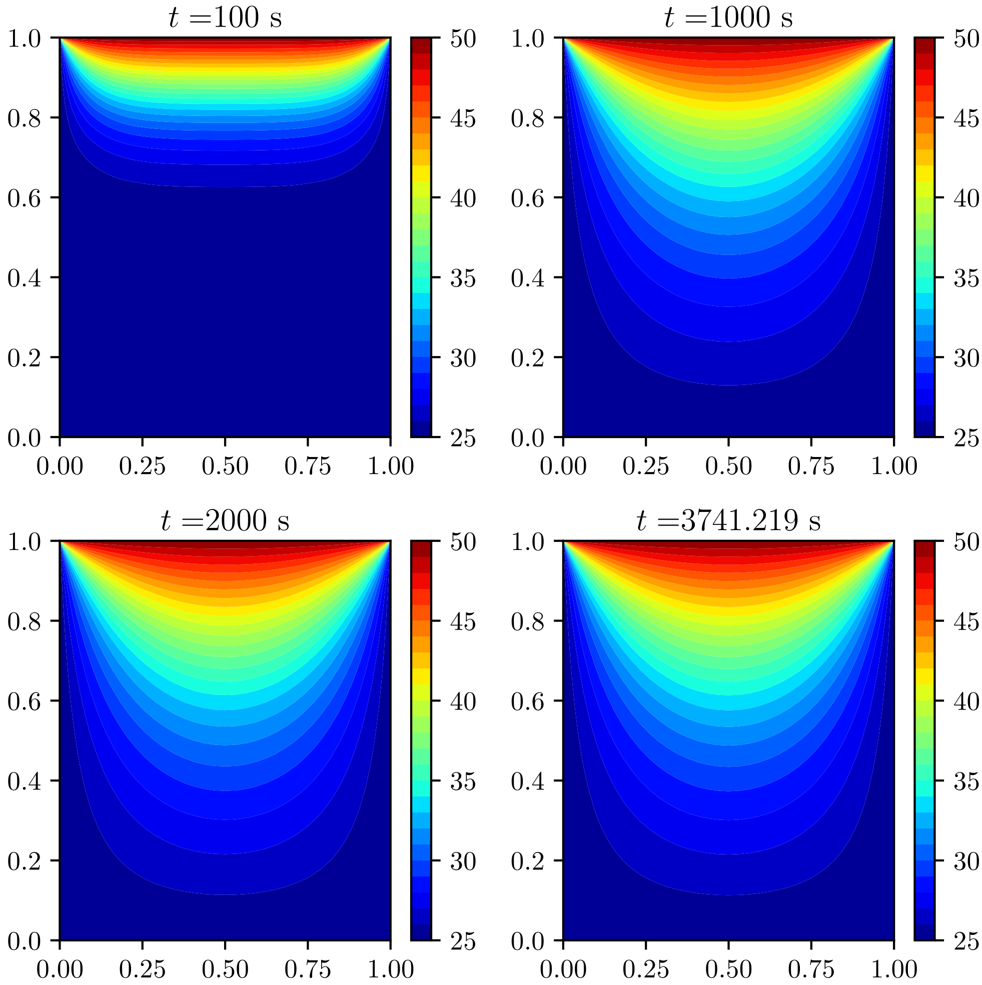

Figure 4: Plots for first case at various values of time (time step of 0.001).

Second case

The boundary conditions for the second case were selected as follows:

1. $T_{xi} = 25°C$ at $x = 0$
2. $T_{xf} = 50°C$ at $x = L$
3. $T_{yi} = 25°C$ at $y = 0$
4. $T_{yf} = 50°C$ at $y = L$

Numerical solution for second case

Again C++ code was written for numerical solution of this unsteady 2D heat transfer problem. Initial condition was $T_i = 25°C$ throughout the block.

A time step of 0.001 was utilized. A 50×50 grid was selected with the following functions for the x and y axes:

$$f(x) = L \sin\left(\frac{\pi}{2} \times \frac{i}{n_x}\right) \tag{11}$$

$$f(y) = W \sin\left(\frac{\pi}{2} \times \frac{j}{n_y}\right) \tag{12}$$

Again, the reason for selecting these functions was to achieve a denser mesh in areas of the block having a larger temperature gradient, during the intial time steps. The resulting grid is shown in the figure below.

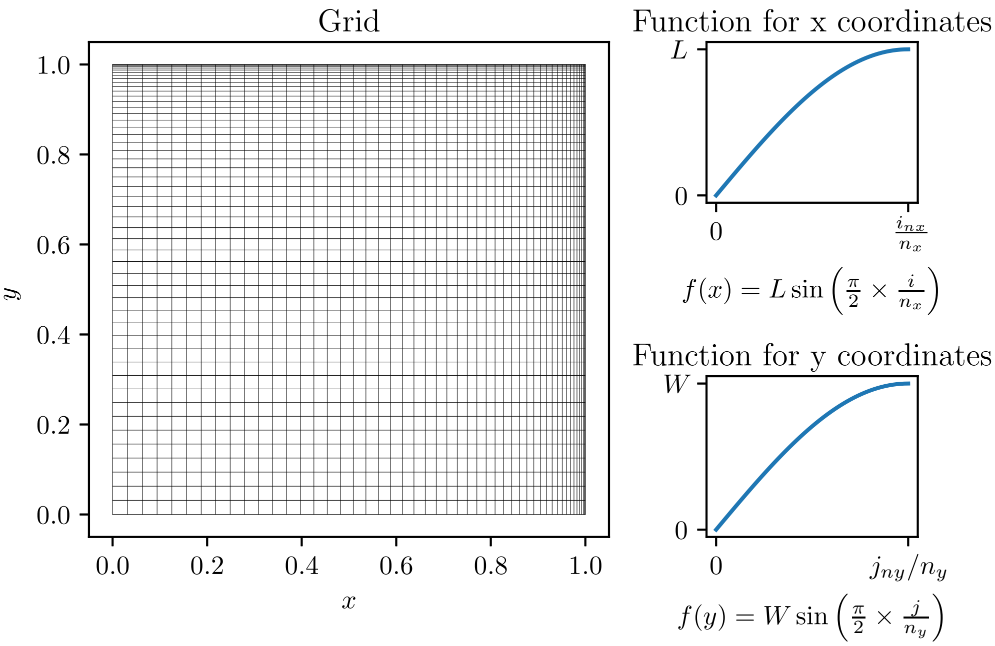

Figure 5: Grid for second case with x and y functions and their graphs

The same formula used earlier for forward in time, center in space method (FTCS) was utilized. The root mean square difference was calculated at $x = L/2$ for all y values between successive iterations of time to determine the steady state. Steady state, assumed when RMS difference $\leq 1 \times 10^{-10}$, was achieved at $t = 3883.242$ seconds.

The results for four time conditions, including the steady state, are shown below.

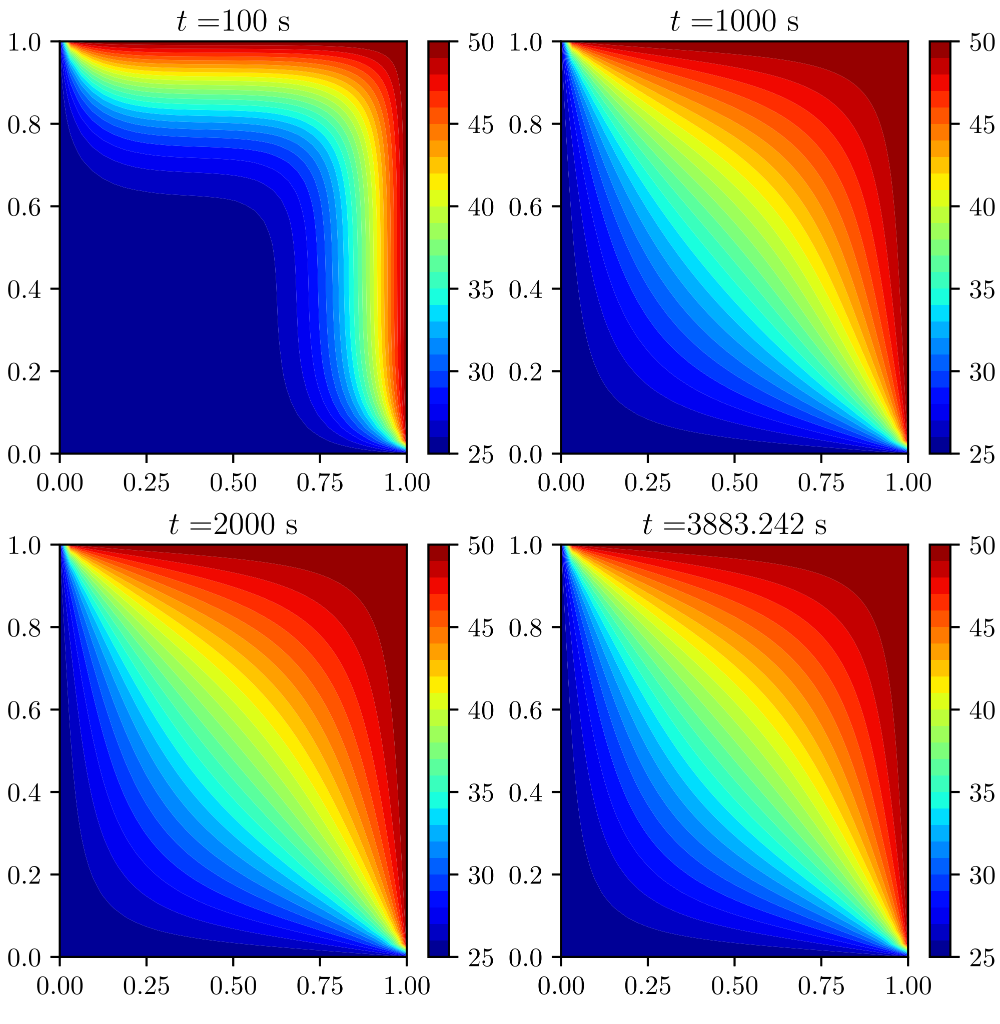

Figure 6: Plots for second case at various values of time (time step of 0.001).