# Progress Report

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## Numerical solution of Laplace equation using unequally spaced intervals

### Derivation of central difference formula for first derivative with unequal intervals

Using the Taylor series expansion for a function $f(x)$ while neglecting second and higher order terms, we can write:

$$f_{i+1} = f_i + f_i'(x_{i+1} - x_i) + \dots \tag{1}$$

$$f_{i-1} = f_i - f_i'(x_i - x_{i-1}) + \dots \tag{2}$$

Subtracting equation (2) from (1) gives:

$$f_{i+1} - f_{i-1} \cong f_i'(x_{i+1} - x_{i-1})$$

$$f_i' = \frac{f_{i+1} - f_{i-1}}{x_{i+1} - x_{i-1}} \tag{3}$$

### Derivation of central difference formula for second derivative with unequal intervals

Now using the Taylor series expansion for the function $f(x)$ with second order terms but neglecting higher order ones, we have:

$$f_{i+1} = f_i + f_i'(x_{i+1} - x_i) + f_i''\frac{(x_{i+1} - x_i)^2}{2!} + \dots \tag{4}$$

$$f_{i-1} = f_i - f_i'(x_i - x_{i-1}) + f_i''\frac{(x_i - x_{i-1})^2}{2!} - \dots \tag{5}$$

Adding the above equations and rearranging gives:

$$f_i''[(x_{i+1}-x_i)^2+(x_i-x_{i-1})^2] \cong 2f_{i+1}+2f_{i-1}-4f_i-2f_i'(x_{i+1}-2x_i+x_{i-1})$$

$$f_i'' = \frac{2(f_{i+1}-2f_i+f_{i-1})}{(x_{i+1}-x_i)^2+(x_i-x_{i-1})^2} - \frac{2f_i'(x_{i+1}-2x_i+x_{i-1})}{(x_{i+1}-x_i)^2+(x_i-x_{i-1})^2}$$

Substituting (3) in the above equation gives:

$$f_i'' = \frac{2(f_{i+1}-2f_i+f_{i-1})}{(x_{i+1}-x_i)^2+(x_i-x_{i-1})^2} - \frac{2(f_{i+1}-f_{i-1})(x_{i+1}-2x_i+x_{i-1})}{(x_{i+1}-x_{i-1})[(x_{i+1}-x_i)^2+(x_i-x_{i-1})^2]} \quad (6)$$

## Discretization of Laplace equation for unequally spaced intervals

The following Laplace equation was considered:

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \quad (7)$$

For equally spaced intervals, this equation can be discretized as follows:

$$\frac{\phi_{i+1,j}-2\phi_{i,j}+\phi_{i-1,j}}{(\Delta x)^2}+\frac{\phi_{i,j+1}-2\phi_{i,j}+\phi_{i,j-1}}{(\Delta y)^2}=0 \quad (8)$$

For use in numerical solutions, equation (8) can be rearranged as follows:

$$\phi_{i,j} = \left(\frac{\phi_{i+1,j}+\phi_{i-1,j}}{(\Delta x)^2}+\frac{\phi_{i,j+1}+\phi_{i,j-1}}{(\Delta y)^2}\right)\left(\frac{(\Delta x)^2(\Delta y)^2}{2[(\Delta x)^2+(\Delta y)^2]}\right) \quad (9)$$

However, for unequally spaced intervals, equation (6) must be used to discretize each of the two terms in (7) as follows:

$$\begin{aligned}&\frac{2(\phi_{i+1,j}-2\phi_{i,j}+\phi_{i-1,j})}{(x_{i+1}-x_i)^2+(x_i-x_{i-1})^2}-\frac{2(\phi_{i+1,j}-\phi_{i-1,j})(x_{i+1}-2x_i+x_{i-1})}{(x_{i+1}-x_{i-1})[(x_{i+1}-x_i)^2+(x_i-x_{i-1})^2]}+\\&\frac{2(\phi_{i,j+1}-2\phi_{i,j}+\phi_{i,j-1})}{(y_{j+1}-y_j)^2+(y_j-y_{j-1})^2}-\frac{2(\phi_{j+1}-\phi_{j-1})(y_{j+1}-2y_j+y_{j-1})}{(y_{j+1}-y_{j-1})[(y_{j+1}-y_j)^2+(y_j-y_{j-1})^2]}\\&=0\end{aligned} \quad (10)$$

Rearranging the above equation for use in numerical solutions:

$$\begin{aligned}&\frac{2(\phi_{i+1,j}+\phi_{i-1,j})}{(x_{i+1}-x_i)^2+(x_i-x_{i-1})^2}-\frac{2(\phi_{i+1,j}-\phi_{i-1,j})(x_{i+1}-2x_i+x_{i-1})}{(x_{i+1}-x_{i-1})[(x_{i+1}-x_i)^2+(x_i-x_{i-1})^2]}\\&+\frac{2(\phi_{i,j+1}+\phi_{i,j-1})}{(y_{j+1}-y_j)^2+(y_j-y_{j-1})^2}-\frac{2(\phi_{j+1}-\phi_{j-1})(y_{j+1}-2y_j+y_{j-1})}{(y_{j+1}-y_{j-1})[(y_{j+1}-y_j)^2+(y_j-y_{j-1})^2]}\\&=\frac{4\phi_{i,j}}{(x_{i+1}-x_i)^2+(x_i-x_{i-1})^2}+\frac{4\phi_{i,j}}{(y_{j+1}-y_j)^2+(y_j-y_{j-1})^2}\end{aligned}$$

$$\phi_{i,j} = \left[\frac{2(\phi_{i+1,j}+\phi_{i-1,j})}{(x_{i+1}-x_i)^2+(x_i-x_{i-1})^2} - \frac{2(\phi_{i+1,j}-\phi_{i-1,j})(x_{i+1}-2x_i+x_{i-1})}{(x_{i+1}-x_{i-1})[(x_{i+1}-x_i)^2+(x_i-x_{i-1})^2]} + \frac{2(\phi_{i,j+1}+\phi_{i,j-1})}{(y_{j+1}-y_j)^2+(y_j-y_{j-1})^2} - \frac{2(\phi_{j+1}-\phi_{j-1})(y_{j+1}-2y_j+y_{j-1})}{(y_{j+1}-y_{j-1})[(y_{j+1}-y_j)^2+(y_j-y_{j-1})^2]}\right] \times \frac{1}{4}\left[\frac{1}{(x_{i+1}-x_i)^2+(x_i-x_{i-1})^2} + \frac{1}{(y_{j+1}-y_j)^2+(y_j-y_{j-1})^2}\right]^{-1}$$

$$\phi_{i,j} = \left[\frac{2(\phi_{i+1,j}+\phi_{i-1,j})}{(x_{i+1}-x_i)^2+(x_i-x_{i-1})^2} - \frac{2(\phi_{i+1,j}-\phi_{i-1,j})(x_{i+1}-2x_i+x_{i-1})}{(x_{i+1}-x_{i-1})[(x_{i+1}-x_i)^2+(x_i-x_{i-1})^2]} + \frac{2(\phi_{i,j+1}+\phi_{i,j-1})}{(y_{j+1}-y_j)^2+(y_j-y_{j-1})^2} - \frac{2(\phi_{j+1}-\phi_{j-1})(y_{j+1}-2y_j+y_{j-1})}{(y_{j+1}-y_{j-1})[(y_{j+1}-y_j)^2+(y_j-y_{j-1})^2]}\right] \times \frac{1}{4}\left[\frac{[(x_{i+1}-x_i)^2+(x_i-x_{i-1})^2][(y_{j+1}-y_j)^2+(y_j-y_{j-1})^2]}{(x_{i+1}-x_i)^2+(x_i-x_{i-1})^2+(y_{j+1}-y_j)^2+(y_j-y_{j-1})^2}\right] \quad (11)$$

For the case of equally spaced intervals when

$$(x_{i+1}-x_i) = (x_i-x_{i-1}) = 0\,,$$
$$(y_{j+1}-y_j) = (y_j-y_{j-1}) = 0\,,$$
$$(x_{i+1}-2x_i+x_{i-1}) = 0 \text{ and}$$
$$(y_{j+1}-2y_j+y_{j-1}) = 0\,,$$

then equation (11) reduces to equation (9).

## Unequal grid coordinates

A $100 \times 100$ grid was selected with the following functions for the x and y axes:

$$f(x) = \frac{L}{2}\left[1+\sin\left(\pi\left(\frac{i}{n_x}-\frac{1}{2}\right)\right)\right] \quad (12)$$

$$f(y) = W\sin\left(\frac{\pi}{2}\times\frac{j}{n_y}\right) \quad (13)$$

The resulting grid and the graphs of its functions are shown in the figure below.

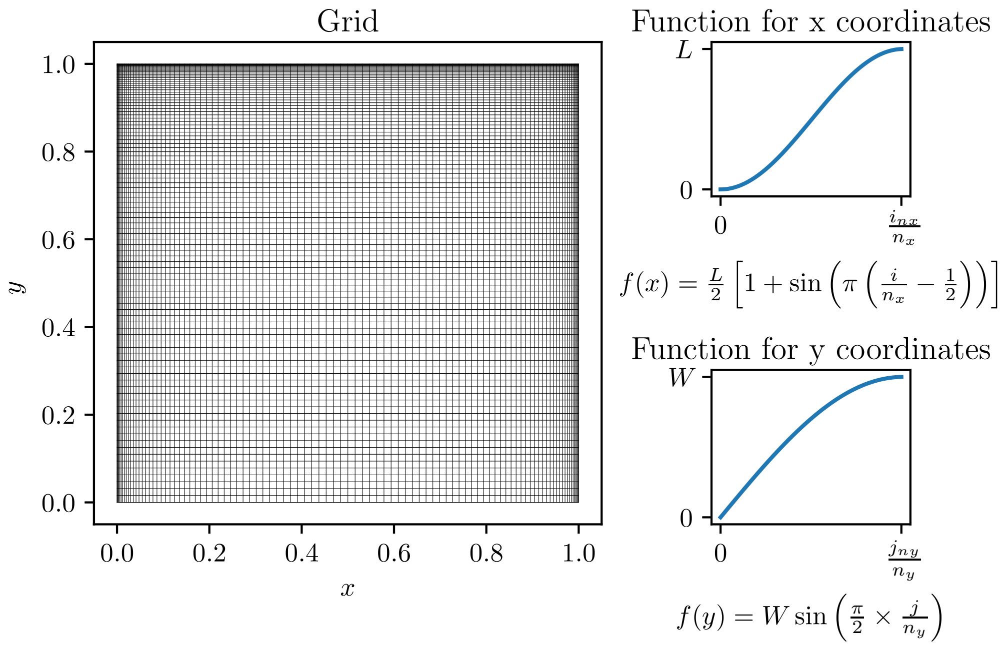


Figure 1: The selected grid with the chosen $x$ and $y$ functions and their graphs

## SOR Method

For equal grid intervals, Successive Over-Relaxation (SOR) method applied to Laplace equation was as follows:

$$\phi_{i,j}^{m+1} = (1-\omega)\phi_{i,j}^{m} + \omega\left(\frac{\phi_{i+1,j}^{m} + \phi_{i-1,j}^{m+1}}{(\Delta x)^2} + \frac{\phi_{i,j+1}^{m} + \phi_{i,j-1}^{m+1}}{(\Delta y)^2}\right)\left(\frac{(\Delta x)^2(\Delta y)^2}{2(\Delta x^2)(\Delta y^2)}\right) \tag{14}$$

For unequal grid intervals, SOR method can be applied to Laplace equation using equation (11) as follows:

$$\begin{aligned}
&\phi_{i,j}^{m+1} = (1-\omega)\phi_{i,j}^{m} + \omega \times \\
&\left[\frac{2(\phi_{i+1,j}^{m} + \phi_{i-1,j}^{m+1})}{(x_{i+1}-x_i)^2 + (x_i - x_{i-1})^2} - \frac{2(\phi_{i+1,j}^{m} - \phi_{i-1,j}^{m+1})(x_{i+1} - 2x_i + x_{i-1})}{(x_{i+1}-x_{i-1})[(x_{i+1}-x_i)^2 + (x_i - x_{i-1})^2]} + \right. \\
&\left. \frac{2(\phi_{i,j+1}^{m} + \phi_{i,j-1}^{m+1})}{(y_{j+1}-y_j)^2 + (y_j - y_{j-1})^2} - \frac{2(\phi_{j+1}^{m} - \phi_{j-1}^{m+1})(y_{j+1} - 2y_j + y_{j-1})}{(y_{j+1}-y_{j-1})[(y_{j+1}-y_j)^2 + (y_j - y_{j-1})^2]}\right] \\
&\times \frac{1}{4}\left[\frac{[(x_{i+1}-x_i)^2 + (x_i - x_{i-1})^2][(y_{j+1}-y_j)^2 + (y_j - y_{j-1})^2]}{(x_{i+1}-x_i)^2 + (x_i - x_{i-1})^2 + (y_{j+1}-y_j)^2 + (y_j - y_{j-1})^2}\right]
\end{aligned} \tag{15}$$

This equation was solved using $\omega = 1.8$ for a $100 \times 100$ grid. After $1,114$ iterations, the numerical solution stopped changing, according to the following formula:

$$\max\left(|\phi_{i,j}^{m+1} - \phi_{i,j}^{m}|\right) \leq 1 \times 10^{-9} \tag{16}$$

The error between the numerical solution and the exact solution is shown below in the figure.

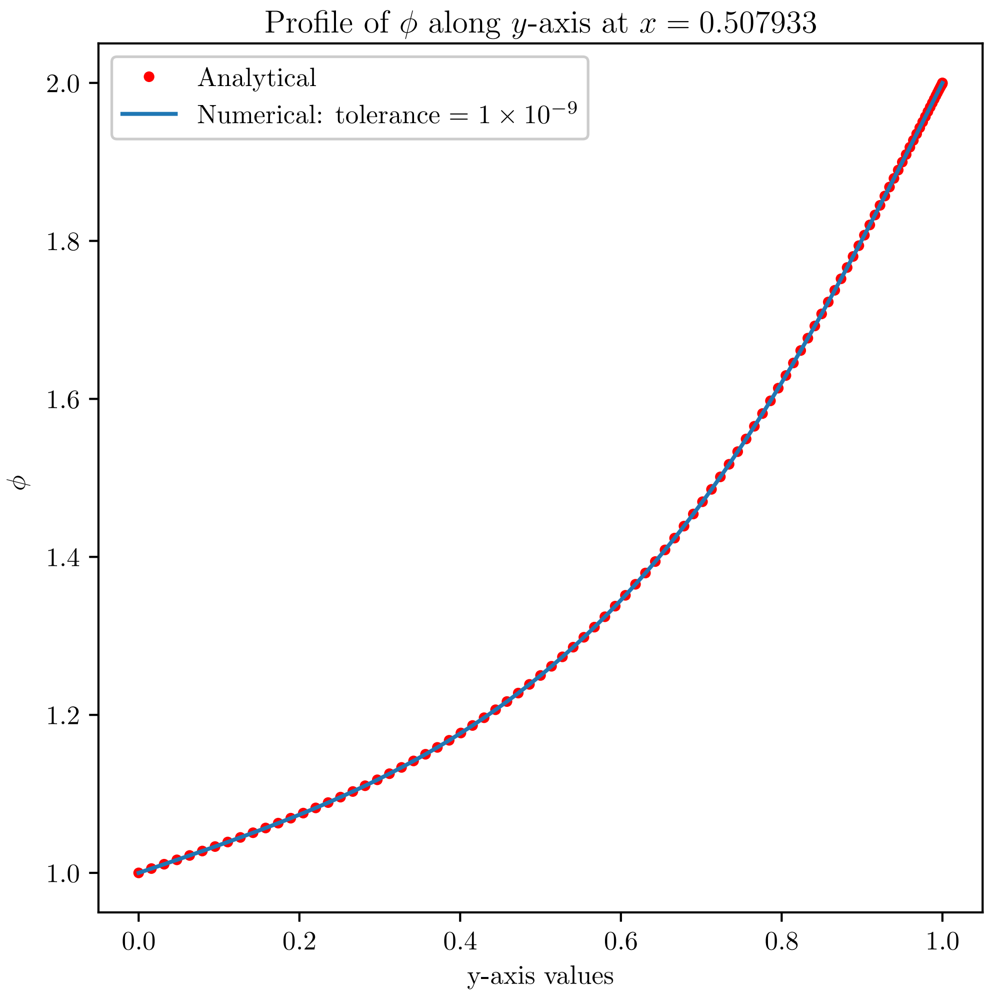


Figure 2: Comparison of numerical solution and exact solution of Laplace equation with unequal grid intervals

# Further work in progress

## Numerical solution of 2D Navier-Stokes equation

Following is the Navier-Stokes equation:

$$\frac{\partial u}{\partial t} + \frac{\partial (uu)}{\partial x} + \frac{\partial (uv)}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{17}$$

Navier-Stokes equation has to be solved numerically for a 2D problem using SOR method for pressure term and upwind scheme for the convective term.