# Progress Report

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# 1 Weighted essentially non-oscillatory scheme (WENO) with finite volume method for 3D Navier-Stokes equations on nonuniform grid

The three-dimensional solver for Navier-Stokes equations is being updated to utilize a fifth order WENO scheme. WENO schemes are an extension of essentially non-oscillatory schemes (ENO) [1].

## 1.1 Calculation of constants for WENO reconstruction

Let us analyze the procedure for calculating reconstructed values at a cell boundary for a $k$-th order WENO approximation.

At a location $I_i$ and the order of accuracy $k$, we select a stencil,

$$S(i) \equiv \{I_{i-r}, \ldots, I_{i+s}\} \quad , \tag{1}$$

based on $r$ cells to the left, $s$ cells to the right and $I_i$ itself, where $r, s \geq 0$ and $r + s + 1 = k$.

If we have the $k$ cell averages,

$$\overline{v}_{i-r}, \ldots, \overline{v}_{i-r+k-1} \quad , \tag{2}$$

the reconstructed value at the cell boundary $x_{i+1/2}$ can be found using constants $c_{rj}$ such that

$$v_{i+1/2} = \sum_{j=0}^{k-1} c_{rj} \overline{v}_{i-r+j} \tag{3}$$

is $k$-th order accurate with

$$v_{i+1/2} = v(x_{i+1/2}) + O(\Delta x^k) \quad . \tag{4}$$

### 1.1.1 Uniform grid

For a uniform grid, $\Delta x_i = \Delta x$ and $c_{rj}$ can be calculated as:

$$c_{rj} = \sum_{m=j+1}^{k} \frac{\sum_{\substack{l=0\\l\neq m}}^{k} \prod_{\substack{q=0\\q\neq m,l}}^{k} (r-q+1)}{\prod_{\substack{l=0\\l\neq m}}^{k} (m-l)} \tag{5}$$

For $k = 3$ at $j = 1, r = 1$ and $m = j + 1 = 2$,

$$c_{11} = \sum_{m=2}^{3} \frac{\sum_{\substack{l=0\\l\neq m}}^{3} \prod_{\substack{q=0\\q\neq m,l}}^{3} (2-q)}{\prod_{\substack{l=0\\l\neq m}}^{k} (m-l)}$$

$$c_{11} = \left.\frac{\overbrace{(2-1)(2-3)}^{l=0} + \overbrace{(2-0)(2-3)}^{l=1} + \overbrace{(2-0)(2-1)}^{l=3}}{(2-0)(2-1)(2-3)}\right\} m = 2$$

$$+ \left.\frac{\overbrace{(2-1)(2-2)}^{l=0} + \overbrace{(2-0)(2-2)}^{l=1} + \overbrace{(2-0)(2-1)}^{l=3}}{(3-0)(3-1)(3-2)}\right\} m = 3$$

$$c_{11} = \frac{5}{6}$$

### 1.1.2 Nonuniform grid

In case of nonuniform grid, $c_{rj}$ has to be calculated using the formula:

$$c_{rj} = \left( \sum_{m=j+1}^{k} \frac{\sum_{\substack{l=0\\ l\neq m}}^{k} \prod_{\substack{q=0\\ q\neq m,l}}^{k} \left(x_{i+1/2} - x_{i-r+q-1/2}\right)}{\prod_{\substack{l=0\\ l\neq m}}^{k} \left(x_{i-r+m-1/2} - x_{i-r+l-1/2}\right)} \right) \Delta x_{i-r+j} \tag{6}$$

For the same case with $k = 3$ at $j = 1, r = 1$ and $m = j + 1 = 2$,

$$c_{11} = \left( \sum_{m=2}^{3} \frac{\sum_{\substack{l=0\\ l\neq m}}^{3} \prod_{\substack{q=0\\ q\neq m,l}}^{3} \left(x_{i+1/2} - x_{i-1+q-1/2}\right)}{\prod_{\substack{l=0\\ l\neq m}}^{3} \left(x_{i-1+m-1/2} - x_{i-1+l-1/2}\right)} \right) \Delta x_{i-1+j}$$

$$\begin{aligned} c_{11} = \Bigg( & \frac{\begin{array}{c}(x_{i+1/2} - x_{i-1+1-1/2})(x_{i+1/2} - x_{i-1+3-1/2}) \\ + (x_{i+1/2} - x_{i-1+0-1/2})(x_{i+1/2} - x_{i-1+3-1/2}) \\ + (x_{i+1/2} - x_{i-1+0-1/2})(x_{i+1/2} - x_{i-1+1-1/2})\end{array}}{(x_{i-1+2-1/2} - x_{i-1+0-1/2})(x_{i-1+2-1/2} - x_{i-1+1-1/2})(x_{i-1+2-1/2} - x_{i-1+3-1/2})} \\ & + \frac{\begin{array}{c}(x_{i+1/2} - x_{i-1+1-1/2})(x_{i+1/2} - x_{i-1+2-1/2}) \\ + (x_{i+1/2} - x_{i-1+0-1/2})(x_{i+1/2} - x_{i-1+2-1/2}) \\ + (x_{i+1/2} - x_{i-1+0-1/2})(x_{i+1/2} - x_{i-1+1-1/2})\end{array}}{(x_{i-1+3-1/2} - x_{i-1+0-1/2})(x_{i-1+3-1/2} - x_{i-1+1-1/2})(x_{i-1+3-1/2} - x_{i-1+2-1/2})} \Bigg) \\ & \times \quad \Delta x_{i-1+1} \end{aligned}$$

$$\begin{aligned} c_{11} = \Bigg( & \frac{(x_{i+1/2} - x_{i-1/2})(x_{i+1/2} - x_{i+3/2}) + (x_{i+1/2} - x_{i-3/2})(x_{i+1/2} - x_{i+3/2}) + (x_{i+1/2} - x_{i-3/2})(x_{i+1/2} - x_{i-1/2})}{(x_{i+1/2} - x_{i-3/2})(x_{i+1/2} - x_{i-1/2})(x_{i+1/2} - x_{i+3/2})} \\ & + \frac{(x_{i+1/2} - x_{i-1/2})(x_{i+1/2} - x_{i+1/2}) + (x_{i+1/2} - x_{i-3/2})(x_{i+1/2} - x_{i+1/2}) + (x_{i+1/2} - x_{i-3/2})(x_{i+1/2} - x_{i-1/2})}{(x_{i+3/2} - x_{i-3/2})(x_{i+3/2} - x_{i-1/2})(x_{i+3/2} - x_{i+1/2})} \Bigg) \Delta x_i \end{aligned}$$

Similarly, constants $c_{rj}$ have to be calculated for all cells at all locations.

# References

[1] B. Cockburn, Chi-Wang Shu, Claes Johnson, Eitan Tadmor, and Alfio Quarteroni. *Advanced Numerical Approximation of Nonlinear Hyperbolic Equations Lectures given at the 2nd Session of the Centro Internazionale Matematico Estivo (C.I.M.E.) held in Cetraro, Italy, June 23-28, 1997*. Springer Berlin Heidelberg, 1998.