Progress Report

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Contents

1 Solution of Navier-Stokes equations using Finite Volume Method for unequal grid intervals

Continuity equation is

$$\nabla \cdot \mathbf{u} = 0 \quad , \tag{1}$$

and Navier-Stokes equation is

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f} \quad . \tag{2}$$

where $\mathbf{u}$ is the velocity vector, p is the pressure, ρ is density of the fluid, and $\mathbf{f}$ is body force per unit mass.

Ignoring the body force $\mathbf{f}$, we are left with

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho}\nabla p - \nabla \cdot (\mathbf{u}\mathbf{u}) + \nu \nabla^2 \mathbf{u} \quad . \tag{3}$$

1.1 Projection method

Using a new vector $\mathbf{u}^*$ for intermediate velocity and n as the index for time step, projection method is used to decompose Navier-Stokes equation in (3) into two parts,

$$\frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} = -\nabla \cdot (\mathbf{u}\mathbf{u}) + \nu \nabla^2 \mathbf{u} \quad , \tag{4}$$

which accounts for the convective and diffusive terms, and

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^*}{\Delta t} = -\frac{1}{\rho}\nabla p \quad , \tag{5}$$

which accounts for the pressure term.

1.2 Discretization of the convective and diffusive terms

The convective and diffusive terms from equation (4) can be discretized using the individual components. For the u-component, equation (4) can be written as

$$\frac{\partial u}{\partial t} = -\frac{\partial uu}{\partial x} - \frac{\partial uv}{\partial y} + \nu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] \quad , \tag{6}$$

which can be discretized as

$$\frac{\partial u}{\partial t} = -\frac{u_e^2 - u_w^2}{Dx} - \frac{u_n v_n - u_s v_s}{Dy} + \nu \left[\left\{ \frac{u_{i+1,j} - u_{i,j}}{Dx1} - \frac{u_{i,j} - u_{i-1,j}}{Dx2} \right\} \frac{1}{Dx} + \left\{ \frac{u_{i,j+1} - u_{i,j}}{Dy} - \frac{u_{i,j} - u_{i,j-1}}{Dy3} \right\} \frac{1}{Dy2} \right] \quad , \tag{7}$$

where, for the *diffusion terms*, second-order central scheme has been used. Similarly for the v-component, equation (4) will be

$$\frac{\partial v}{\partial t} = -\frac{\partial uv}{\partial x} - \frac{\partial vv}{\partial y} + \nu \left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right] \quad , \tag{8}$$

which can be and discretized as

$$\frac{\partial v}{\partial t} = -\frac{u_e v_e - u_w v_w}{Dx} - \frac{v_n^2 - v_s^2}{Dy} + \nu \left[\left\{ \frac{v_{i+1,j} - v_{i,j}}{Dx} - \frac{v_{i,j} - v_{i-1,j}}{Dx3} \right\} \frac{1}{Dx2} + \left\{ \frac{v_{i,j+1} - v_{i,j}}{Dy1} - \frac{v_{i,j} - v_{i,j-1}}{Dy2} \right\} \frac{1}{Dy} \right] \quad . \tag{9}$$

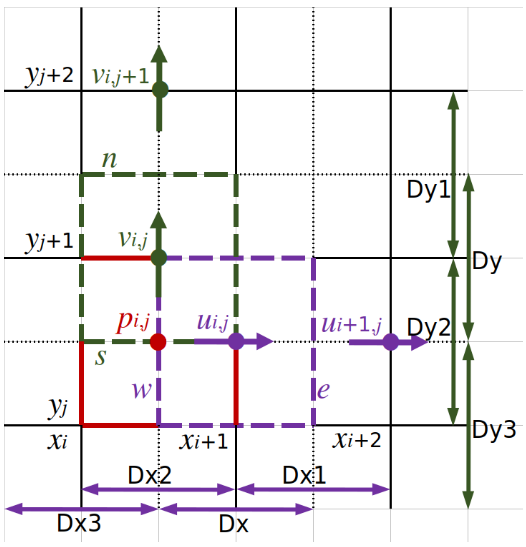

Figure 1: Visual representation of the staggered grid used for discretization in Finite Volume Method

1.2.1 Substitutions for velocities in the convective terms

Following formulas are used for the velocities u_n, u_s, v_e and v_w,

$$\begin{aligned} u_n &= u_{i,j} + \frac{u_{i,j+1} - u_{i,j}}{Dy} & \qquad v_e &= v_{i,j} + \frac{v_{i+1,j} - v_{i,j}}{Dx} \\ u_s &= u_{i,j} - \frac{u_{i,j} - u_{i,j-1}}{Dy3} & \qquad v_w &= v_{i,j} - \frac{v_{i,j} - v_{i-1,j}}{Dx3} \end{aligned}$$

For rest of the velocities in the convective terms, namely u_e, u_w, v_n and v_s, the upwind scheme was used.

Upwind scheme
For positive velocities,

$$\begin{aligned} u_e &= u_{i,j} & \qquad v_n &= v_{i,j} \\ u_w &= u_{i-1,j} & \qquad v_s &= v_{i,j-1} \end{aligned}$$

For negative velocities,

$$\begin{aligned} u_e &= u_{i+1,j} & \qquad v_n &= v_{i,j+1} \\ u_w &= u_{i,j} & \qquad v_s &= v_{i,j} \end{aligned}$$

1.2.2 Adams-Bashforth scheme

Euler scheme was utilized for the first time step and Adam-Bashforth scheme was used for the remaining. Denoting the terms on the right-hand side of equation (4) with $\mathcal{F}(\mathbf{u})$, the second order Adams-Bashforth scheme can be applied using

$$\frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} = \frac{3}{2}\mathcal{F}(\mathbf{u}^n) - \frac{1}{2}\mathcal{F}(\mathbf{u}^{n-1}) \quad . \tag{10}$$

1.3 Poisson equation of pressure

Using the conservation of mass principle for the $n + 1^{\text{st}}$ time step,

$$\nabla \cdot \mathbf{u}^{n+1} = 0 \quad , \tag{11}$$

and substituting equation (5), we get the Poisson equation for pressure

$$\nabla^2 p^{n+1} = \frac{\rho}{\Delta t} \nabla \cdot \mathbf{u}^* \quad , \tag{12}$$

which can be written as

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \frac{1}{\Delta t}\left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y}\right) \quad . \tag{13}$$

Integrating it twice, discretizing and rearranging leads to

$$
\begin{aligned}
p_{i,j}^{n+1} = & \frac{1}{\left[-\frac{Dy}{Dx1} - \frac{Dy}{Dx2} - \frac{Dx}{Dy1} - \frac{Dx}{Dy2}\right]} \\
& \times \left[-\frac{Dy}{Dx1} p_{i+1,j} - \frac{Dy}{Dx2} p_{i-1,j} - \frac{Dx}{Dy1} p_{i,j+1} - \frac{Dx}{Dy2} p_{i,j-1}\right. \\
& \left. + \frac{1}{\Delta t}\left\{\left(u_{i,j}^{*} - u_{i-1,j}^{*}\right)\Delta y + \left(v_{i,j}^{*} - v_{i,j-1}^{*}\right)\Delta x\right\}\right] \quad . \qquad (14)
\end{aligned}
$$

Using successive over-relaxation method (SOR),

$$
\begin{aligned}
p_{i,j}^{n+1} = & (1-\omega)\, p_{i,j} + \omega \left[\frac{1}{\left(-\frac{Dy}{Dx1} - \frac{Dy}{Dx2} - \frac{Dx}{Dy1} - \frac{Dx}{Dy2}\right)}\right. \\
& \times \left\{-\frac{Dy}{Dx1} p_{i+1,j} - \frac{Dy}{Dx2} p_{i-1,j} - \frac{Dx}{Dy1} p_{i,j+1} - \frac{Dx}{Dy2} p_{i,j-1}\right. \\
& \left.\left. + \frac{1}{\Delta t}\left(\left(u_{i,j}^{*} - u_{i-1,j}^{*}\right)\Delta y + \left(v_{i,j}^{*} - v_{i,j-1}^{*}\right)\Delta x\right)\right\}\right] \quad , \qquad (15)
\end{aligned}
$$

where the relaxation factor, $\omega = 1.8$. Finally, the correct velocity can be found using

$$
\mathbf{u}^{n+1} = \mathbf{u}^{*} - \frac{\Delta t}{\rho} \cdot \nabla p^{n+1} \quad , \qquad (16)
$$

which is, for u- and v-components,

$$
u_{i,j}^{n+1} = u_{i,j}^{*} - \frac{\Delta t}{\rho} \cdot \frac{p_{i+1,j}^{n+1} - p_{i,j}^{n+1}}{\Delta x} \qquad (17)
$$

and

$$
v_{i,j}^{n+1} = v_{i,j}^{*} - \frac{\Delta t}{\rho} \cdot \frac{p_{i,j+1}^{n+1} - p_{i,j}^{n+1}}{\Delta y} \qquad (18)
$$

1.4 Future work

- Solve energy equation for heat transfer
- Solve for 3D domain