Progress Report

Syed Ahmad Raza

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1 Solving the pressure term for solution of Navier-Stokes equations using Finite Volume Method

Continuity equation is

$$\nabla \cdot \mathbf{u} = 0 \tag{1}$$

and Navier-Stokes equation is

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f} \quad . \tag{2}$$

where $\mathbf{u}$ is the velocity vector, p is the pressure, ρ is density of the fluid, and $\mathbf{f}$ is body force per unit mass.

Ignoring the body force $\mathbf{f}$, we are left with

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} \quad . \tag{3}$$

1.1 Projection method

Using a new vector $\mathbf{u}^*$ for intermediate velocity and n as the index for time step, projection method is used to decompose Navier-Stokes equation in (3) into two parts,

$$\frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} = -\nabla \cdot (\mathbf{u}\mathbf{u}) + \nu \nabla^2 \mathbf{u} \quad , \tag{4}$$

which accounts for the convective and diffusive terms, and

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^*}{\Delta t} = -\frac{1}{\rho}\nabla p \quad , \tag{5}$$

which accounts for the pressure term.

1.2 Discretization of the convective and diffusive terms

The convective and diffusive terms from equation (4) can be discretized using the individual components. For the u-component, equation (4) can be written as

$$\frac{\partial u}{\partial t} = -\frac{\partial uu}{\partial x} - \frac{\partial uv}{\partial y} + \nu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] \quad , \tag{6}$$

which can be discretized as

$$\begin{aligned} \frac{\partial u}{\partial t} = & -\frac{u_e^2 - u_w^2}{\Delta x} - \frac{u_n v_n - u_s v_s}{\Delta y} + \nu \left[\left\{ \frac{u_{i+1,j} - u_{i,j}}{x_{i+2} - x_{i+1}} - \frac{u_{i,j} - u_{i-1,j}}{x_{i+1} - x_i} \right\} \frac{1}{\Delta x} \right. \\ & \left. + \left\{ \frac{u_{i,j+1} - u_{i,j}}{(y_{j+2} - y_j)/2} - \frac{u_{i,j} - u_{i,j-1}}{(y_{j+1} - y_{j-1})/2} \right\} \frac{1}{\Delta y} \right] \quad , \end{aligned} \tag{7}$$

For flow from left to right, the substitutions will be as follows

$$\begin{aligned} u_e &= u_{i,j} & v_n &= v_{i,j} \\ u_w &= u_{i-1,j} & v_s &= v_{i,j-1} \quad . \\ u_n &= u_{i,j} + \frac{u_{i,j+1} - u_{i,j}}{(y_{j+2} - y_j)/2} \\ u_s &= u_{i,j} - \frac{u_{i,j} - u_{i,j-1}}{(y_{j+1} - y_{j-1})/2} \end{aligned}$$

Similarly for the v-component, equation (4) will be

$$\frac{\partial v}{\partial t} = -\frac{\partial uv}{\partial x} - \frac{\partial vv}{\partial y} + \nu \left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right] \quad , \tag{8}$$

which can be and discretized as

$$\frac{\partial v}{\partial t} = -\frac{u_e v_e - u_w v_w}{\Delta x} - \frac{v_n^2 - v_s^2}{\Delta y} \tag{9}$$

$$\begin{aligned} & + \nu \left[\left\{ \frac{v_{i+1,j} - v_{i,j}}{(x_{i+2} - x_i)/2} - \frac{v_{i,j} - v_{i-1,j}}{(x_{i+1} - x_{i-1})/2} \right\} \frac{1}{\Delta x} \right. \\ & \left. + \left\{ \frac{v_{i,j+1} - v_{i,j}}{y_{j+2} - y_{j+1}} - \frac{v_{i,j} - v_{i,j-1}}{y_{j+1} - y_{j-1}} \right\} \frac{1}{\Delta y} \right] \quad , \end{aligned} \tag{10}$$

where, for flow from bottom to top, the substitutions will be

$$\begin{aligned} v_n &= v_{i,j} & u_e &= u_{i,j} \\ v_s &= v_{i,j-1} & u_w &= u_{i-1,j} \quad . \\ v_e &= v_{i,j} + \frac{v_{i+1,j} - v_{i,j}}{(x_{i+2} - x_i)/2} \\ v_w &= v_{i,j} - \frac{v_{i,j} - v_{i-1,j}}{(x_{i+1} - x_{i-1})/2} \end{aligned}$$

1.2.1 Euler scheme for first time step

For the first time step, the Euler scheme is adopted for the intermediate velocity using

$$
\begin{aligned}
\frac{u^*_{i,j} - u_{i,j}}{\Delta t} = & -\frac{u_e^2 - u_w^2}{\Delta x} - \frac{u_n v_n - u_s v_s}{\Delta y} \\
& + \nu \left[\left\{ \frac{u_{i+1,j} - u_{i,j}}{x_{i+2} - x_{i+1}} - \frac{u_{i,j} - u_{i-1,j}}{x_{i+1} - x_i} \right\} \frac{1}{\Delta x} \right. \\
& \left. + \left\{ \frac{u_{i,j+1} - u_{i,j}}{(y_{j+2} - y_j)/2} - \frac{u_{i,j} - u_{i,j-1}}{(y_{j+1} - y_{j-1})/2} \right\} \frac{1}{\Delta y} \right] \quad ,
\end{aligned}
\tag{11}
$$

and

$$
\begin{aligned}
\frac{v^*_{i,j} - v_{i,j}}{\Delta t} = & -\frac{u_e v_e - u_w v_w}{\Delta x} - \frac{v_n^2 - v_s^2}{\Delta y} \\
& + \nu \left[\left\{ \frac{v_{i+1,j} - v_{i,j}}{x_{i+2} - x_{i+1}} - \frac{v_{i,j} - v_{i-1,j}}{x_{i+1} - x_i} \right\} \frac{1}{\Delta x} \right. \\
& \left. + \left\{ \frac{v_{i,j+1} - v_{i,j}}{(y_{j+2} - y_{j+1})/2} - \frac{v_{i,j} - v_{i,j-1}}{(y_{j+1} - y_{j-1})/2} \right\} \frac{1}{\Delta y} \right] \quad .
\end{aligned}
\tag{12}
$$

1.2.2 Adams-Bashforth scheme

Denoting the terms on the right-hand side of equation (4) with $\mathcal{F}(\mathbf{u})$, the second order Adams-Bashforth scheme can be applied using

$$
\frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} = \frac{3}{2}\mathcal{F}(\mathbf{u}^n) - \frac{1}{2}\mathcal{F}(\mathbf{u}^{n-1}) \quad .
\tag{13}
$$

Applying this formula to a 2D problem, the discretized equations can be written as

$$
\begin{aligned}
\mathcal{F}(u^n_{i,j}) = & -\frac{u_e^2 - u_w^2}{\Delta x} - \frac{u_n v_n - u_s v_s}{\Delta y} + \nu \left[\left\{ \frac{u_{i+1,j} - u_{i,j}}{x_{i+2} - x_{i+1}} - \frac{u_{i,j} - u_{i-1,j}}{x_{i+1} - x_i} \right\} \frac{1}{\Delta x} \right. \\
& \left. + \left\{ \frac{u_{i,j+1} - u_{i,j}}{(y_{j+2} - y_j)/2} - \frac{u_{i,j} - u_{i,j-1}}{(y_{j+1} - y_{j-1})/2} \right\} \frac{1}{\Delta y} \right] \quad ,
\end{aligned}
\tag{14}
$$

and

$$
\begin{aligned}
\mathcal{F}(v^n_{i,j}) = & -\frac{u_e v_e - u_w v_w}{\Delta x} - \frac{v_n^2 - v_s^2}{\Delta y} + \nu \left[\left\{ \frac{v_{i+1,j} - v_{i,j}}{x_{i+2} - x_{i+1}} - \frac{v_{i,j} - v_{i-1,j}}{x_{i+1} - x_i} \right\} \frac{1}{\Delta x} \right. \\
& \left. + \left\{ \frac{v_{i,j+1} - v_{i,j}}{(y_{j+2} - y_{j+1})/2} - \frac{v_{i,j} - v_{i,j-1}}{(y_{j+1} - y_{j-1})/2} \right\} \frac{1}{\Delta y} \right] \quad .
\end{aligned}
\tag{15}
$$

Then, the intermediate velocity will be

$$\frac{u^*_{i,j} - u_{i,j}}{\Delta t} = \frac{3}{2}\mathcal{F}(u^{n+1}_{i,j}) - \frac{1}{2}\mathcal{F}(u^{n}_{i,j}) \tag{16}$$

$$\frac{v^*_{i,j} - v_{i,j}}{\Delta t} = \frac{3}{2}\mathcal{F}(v^{n+1}_{i,j}) - \frac{1}{2}\mathcal{F}(v^{n}_{i,j}) \tag{17}$$

1.2.3 Boundary conditions

Velocity boundary conditions are described below.

1. Top boundary (Dirichlet):

$$\begin{aligned} u_{i,n_y-1} &= -u_{i,n_y-2} \qquad \text{for all } i \\ v_{i,n_y-1} &= -v_{i,n_y-2} \qquad \text{for all } i \end{aligned}$$

2. Bottom boundary (Dirichlet):

$$\begin{aligned} u_{i,0} &= -u_{i,1} \qquad \text{for all } i \\ v_{i,0} &= -v_{i,1} \qquad \text{for all } i \end{aligned}$$

3. Left boundary (Dirichlet):

$$\begin{aligned} u_{0,j} &= u_{\text{in}} \qquad \text{for all } j \\ v_{0,j} &= v_{\text{in}} \qquad \text{for all } j \end{aligned}$$

4. Right boundary (Neumann):

$$\begin{aligned} u_{n_x-1,j} &= u_{n_x-2,j} \qquad \text{for all } j \\ v_{n_x-1,j} &= v_{n_x-2,j} \qquad \text{for all } j \end{aligned}$$

1.3 Poisson equation of pressure

Applying the conservation of mass principle on the $n+1^{\text{st}}$ time step and substituting in equation (4), The pressure term can be solved using

$$\nabla^2 p = \frac{\rho}{\Delta t}\nabla \cdot \mathbf{u}^* \quad , \tag{18}$$

which can be written as

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \frac{1}{\Delta t}\left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y}\right) \quad . \tag{19}$$

Integrating it twice, discretizing and rearranging leads to

$$
\begin{aligned}
p_{i,j} = & \frac{1}{\left[-\frac{Dy}{Dx1} - \frac{Dy}{Dx2} - \frac{Dx}{Dy1} - \frac{Dx}{Dy2}\right]} \\
& \times \left[-\frac{Dy}{Dx1} p_{i+1,j} - \frac{Dy}{Dx2} p_{i-1,j} - \frac{Dx}{Dy1} p_{i,j+1} - \frac{Dx}{Dy2} p_{i,j-1} \right. \\
& \left. + \frac{1}{\Delta t} \left\{ \left(u^*_{i,j} - u^*_{i-1,j} \right) \Delta y + \left(v^*_{i,j} - v^*_{i,j-1} \right) \Delta x \right\} \right] \quad . \qquad (20)
\end{aligned}
$$

Using successive over-relaxation method (SOR),

$$
\begin{aligned}
p_{i,j} = & (1-\omega) p_{i,j} + \omega \left[\frac{1}{\left(-\frac{Dy}{Dx1} - \frac{Dy}{Dx2} - \frac{Dx}{Dy1} - \frac{Dx}{Dy2}\right)} \right. \\
& \times \left\{ -\frac{Dy}{Dx1} p_{i+1,j} - \frac{Dy}{Dx2} p_{i-1,j} - \frac{Dx}{Dy1} p_{i,j+1} - \frac{Dx}{Dy2} p_{i,j-1} \right. \\
& \left. \left. + \frac{1}{\Delta t} \left((u^*_{i,j} - u^*_{i-1,j}) \Delta y + (v^*_{i,j} - v^*_{i,j-1}) \Delta x \right) \right\} \right] \quad , \qquad (21)
\end{aligned}
$$

where the relaxation factor, $\omega = 1.8$.

Finally, the correct velocity can be found using

$$
\mathbf{u}^{n+1} = \mathbf{u}^* - \frac{\Delta t}{\rho} \cdot \nabla p \quad , \qquad (22)
$$

which is, for u- and v-components,

$$
u^{n+1}_{i,j} = u^*_{i,j} - \frac{\Delta t}{\rho} \cdot \frac{p_{i+1,j} - p_{i,j}}{\Delta x} \qquad (23)
$$

and

$$
v^{n+1}_{i,j} = v^*_{i,j} - \frac{\Delta t}{\rho} \cdot \frac{p_{i,j+1} - p_{i,j}}{\Delta y} \qquad (24)
$$