Progress Report

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Contents

1 Analytical solution of plane Poiseuille flow and 3D discretization of Navier-Stokes equation

Ignoring the body force $\mathbf{f}$, the Navier-Stokes equation is represented by

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho}\nabla p - \nabla \cdot (\mathbf{u}\mathbf{u}) + \nu \nabla^2 \mathbf{u} \quad . \tag{1}$$

1.1 Analytical solution for 2D duct flow

$$\frac{U}{U_{\text{avg}}} = \frac{3}{2}\left[1 - \left(\frac{y}{\frac{D}{2}}\right)^2\right] \quad , \tag{2}$$

where

$$U_{\text{avg}} = \frac{D^2}{12\mu}\left(-\frac{dP}{dx}\right) \quad , \tag{3}$$

where the fraction $\frac{dP}{dx} \approx \frac{\Delta P}{\Delta x}$ across the whole domain. Values of pressure are constant on the right edge of the domain where a Dirichlet pressure boundary condition has been selected. But on the left edge of the domain, where a Neumann boundary condition exists, the values vary from top to bottom and they have been averaged.

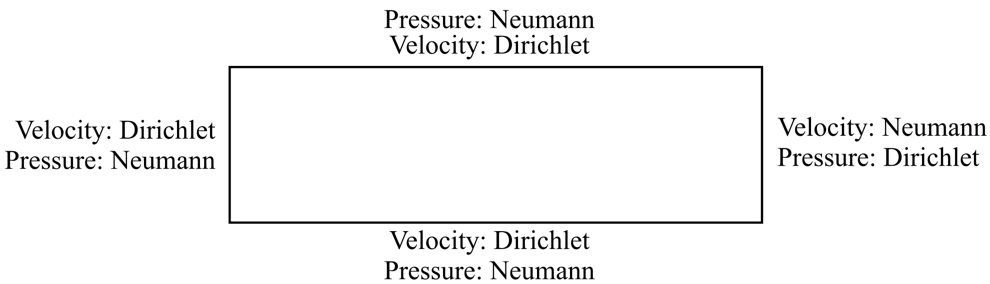

Figure 1: Velocity and pressure boundary conditions at each boundary of the domain are labeled.

1.2 Calculation of L_2 norm

$$L_2 \text{ norm} = \sqrt{\frac{\sum_{i=1}^{n_y/2} (U_a - U_n)^2}{n_y/2}} \quad , \tag{4}$$

where U_a is the value u-velocity value at the node from the analytical solution, whereas U_n is the velocity at the same node determined from the numerical solution, and n_y is the number of grid points on the y-axis.

1.3 Discretization of the convective and diffusive terms for 3D domain

The convective and diffusive terms can be discretized using the individual components. To apply the projection method, the equation (1) for the u-component can be written as

$$\frac{\partial u}{\partial t} = -\frac{\partial uu}{\partial x} - \frac{\partial uv}{\partial y} - \frac{\partial uw}{\partial z} + \nu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right] \quad , \tag{5}$$

which can be discretized as

$$\frac{\partial u}{\partial t} = -\frac{u_e^2 - u_w^2}{\Delta x} - \frac{u_n v_n - u_s v_s}{\Delta y} - \frac{u_o w_o - u_i w_i}{\Delta z} \tag{6}$$

$$\begin{aligned} &+ \nu \left[\left\{ \frac{u_{i+1,j,k} - u_{i,j,k}}{x_{i+2} - x_{i+1}} - \frac{u_{i,j,k} - u_{i-1,j,k}}{x_{i+1} - x_i} \right\} \frac{1}{\Delta x} \right. \\ &+ \left\{ \frac{u_{i,j+1,k} - u_{i,j,k}}{(y_{j+2} - y_j)/2} - \frac{u_{i,j,k} - u_{i,j-1,k}}{(y_{j+1} - y_{j-1})/2} \right\} \frac{1}{\Delta y} \\ &+ \left. \left\{ \frac{u_{i,j,k+1} - u_{i,j,k}}{(z_{k+2} - z_k)/2} - \frac{u_{i,j,k} - u_{i,j,k-1}}{(z_{k+1} - z_{k-1})/2} \right\} \frac{1}{\Delta z} \right] \quad , \end{aligned} \tag{7}$$

where, for the *diffusion terms*, second-order central scheme has been used. Similarly for the v- and w-components, equation (1) will be

$$\frac{\partial v}{\partial t} = -\frac{\partial vu}{\partial x} - \frac{\partial vv}{\partial y} - \frac{\partial vw}{\partial z} + \nu \left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right] \tag{8}$$

and

$$\frac{\partial w}{\partial t} = -\frac{\partial wu}{\partial x} - \frac{\partial wv}{\partial y} - \frac{\partial ww}{\partial z} + \nu \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right] \quad , \tag{9}$$

which can be and discretized as

$$\frac{\partial v}{\partial t} = -\frac{v_e u_e - v_w u_w}{\Delta x} - \frac{v_n^2 - v_s^2}{\Delta y} - \frac{v_o w_o - v_i w_i}{\Delta z} \tag{10}$$

$$+ \nu \left[\left\{ \frac{v_{i+1,j,k} - v_{i,j,k}}{(x_{i+2} - x_i)/2} - \frac{v_{i,j,k} - v_{i-1,j,k}}{(x_{i+1} - x_{i-1})/2} \right\} \frac{1}{\Delta x} \right.$$
$$+ \left\{ \frac{v_{i,j+1,k} - v_{i,j,k}}{y_{j+2} - y_{j+1}} - \frac{v_{i,j,k} - v_{i,j-1,k}}{y_{j+1} - y_j} \right\} \frac{1}{\Delta y}$$
$$\left. + \left\{ \frac{v_{i,j,k+1} - v_{i,j,k}}{(z_{k+2} - z_k)/2} - \frac{v_{i,j,k} - v_{i,j,k-1}}{(z_{k+1} - z_{k-1})/2} \right\} \frac{1}{\Delta z} \right] \tag{11}$$

and

$$\frac{\partial w}{\partial t} = -\frac{w_e u_e - w_w u_w}{\Delta x} - \frac{w_n v_n - w_s v_s}{\Delta y} - \frac{w_o^2 - w_i^2}{\Delta z} \tag{12}$$

$$+ \nu \left[\left\{ \frac{w_{i+1,j,k} - w_{i,j,k}}{(x_{i+2} - x_i)/2} - \frac{w_{i,j,k} - w_{i-1,j,k}}{(x_{i+1} - x_{i-1})/2} \right\} \frac{1}{\Delta x} \right.$$
$$+ \left\{ \frac{w_{i,j+1,k} - w_{i,j,k}}{(y_{j+2} - y_j)/2} - \frac{w_{i,j,k} - w_{i,j-1,k}}{(y_{j+1} - y_{j-1})/2} \right\} \frac{1}{\Delta y}$$
$$\left. + \left\{ \frac{w_{i,j,k+1} - w_{i,j,k}}{z_{k+2} - z_{k+1}} - \frac{w_{i,j,k} - w_{i,j,k-1}}{z_{k+1} - z_k} \right\} \frac{1}{\Delta z} \right] \quad . \tag{13}$$

1.4 Poisson equation of pressure

The Poisson equation of pressure in 3D may be written as

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = \frac{1}{\Delta t} \left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} + \frac{\partial w^*}{\partial z} \right) \quad . \tag{14}$$

Pressure may be found from the following discretized equation,

$$p_{i,j,k}^{n+1} = \frac{1}{\left[-\frac{Dy}{Dx1} - \frac{Dy}{Dx2} - \frac{Dy}{Dz1} - \frac{Dy}{Dz2} - \frac{Dx}{Dy1} - \frac{Dx}{Dy2} - \frac{Dx}{Dz1} - \frac{Dx}{Dz2} - \frac{Dz}{Dx1} - \frac{Dz}{Dx2} - \frac{Dz}{Dy1} - \frac{Dz}{Dy2} \right]}$$
$$\times \left[-\frac{Dz}{Dx1} p_{i+1,j,k} - \frac{Dz}{Dx2} p_{i-1,j,k} - \frac{Dz}{Dy1} p_{i,j+1,k} - \frac{Dz}{Dy2} p_{i,j-1,k} \right.$$
$$- \frac{Dy}{Dx1} p_{i+1,j,k} - \frac{Dy}{Dx2} p_{i-1,j,k} - \frac{Dy}{Dz1} p_{i,j,k+1} - \frac{Dy}{Dz2} p_{i,j,k-1}$$
$$- \frac{Dx}{Dy1} p_{i,j+1,k} - \frac{Dx}{Dy2} p_{i,j-1,k} - \frac{Dx}{Dz1} p_{i,j,k+1} - \frac{Dx}{Dz2} p_{i,j,k-1}$$
$$\left. + \frac{1}{\Delta t} \left\{ \left(u_{i,j,k}^* - u_{i-1,j,k}^* \right) \Delta y \Delta z + \left(v_{i,j,k}^* - v_{i,j-1,k}^* \right) \Delta x \Delta z + \left(w_{i,j,k}^* - w_{i,j,k-1}^* \right) \Delta x \Delta y \right\} \right] \quad . \tag{15}$$