Polynomial Chaos on Discontious Models

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Simula Research Laboratory AS

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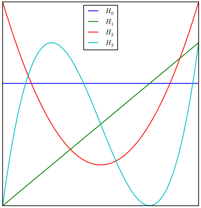
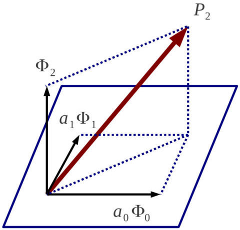
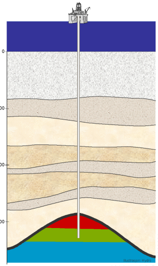
Overview



Polynomial Chaos



New Method

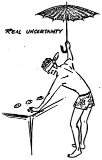


Case Study

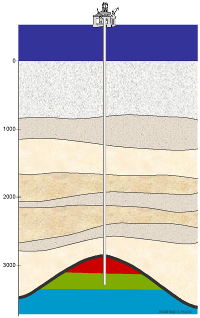
Defining the general model and it's variables

Time t

Coordinate x



Uncertain parameters q
w/density $p(q)$

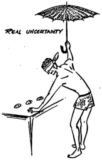


Solver $u(x, t; q)$

Defining the general model and it's variables

Time t

Coordinate x



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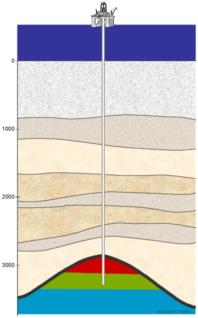

Solver $u(x, t; q)$

Uncertainty analysis
$\mathbb{E}[u]$, $\mathrm{Var}(u)$

$u(q)$-model is approximated non-intrusively from polynomial chaos expansion and stochastic collocation

Model solver		Orthogonal poly.	Fourier coef.
$u(x,t;q) \approx \hat{u}_M(x,t;q) =$	$\sum_{n=0}^{N}$	$\Phi_i(q)$	$\hat{c}_i(x,t)$

$u(q)$-model is approximated non-intrusively from polynomial chaos expansion and stochastic collocation

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$$u(x,t;q) \approx \hat{u}_M(x,t;q) = \sum_{n=0}^{N} \Phi_i(q) \, \hat{c}_i(x,t)$$

Askey-scheme

Discretized Stieltjes

$u(q)$-model is approximated non-intrusively from polynomial chaos expansion and stochastic collocation

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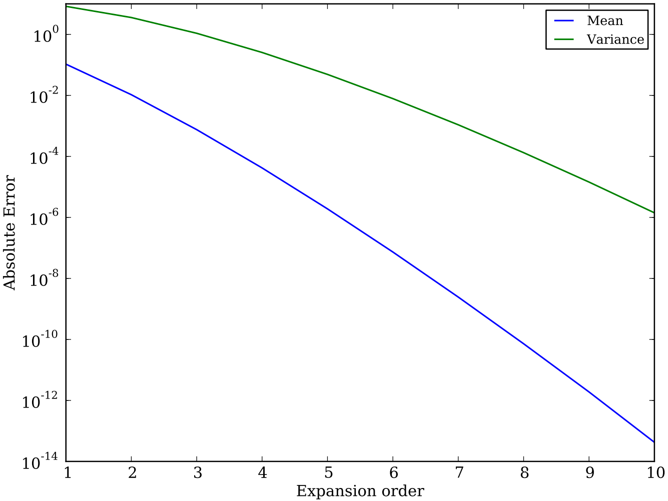
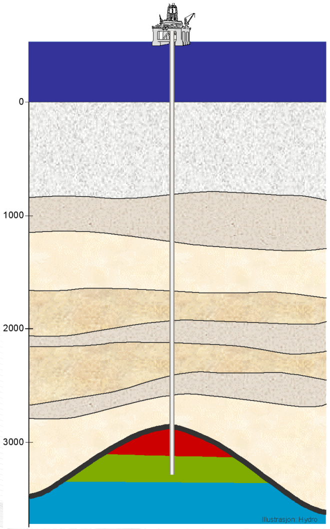
		⇧	⇧
		Askey-scheme	Least squares
		Discretized Stieltjes	Quadrature

The statistics of the approximation converges with exponential rate

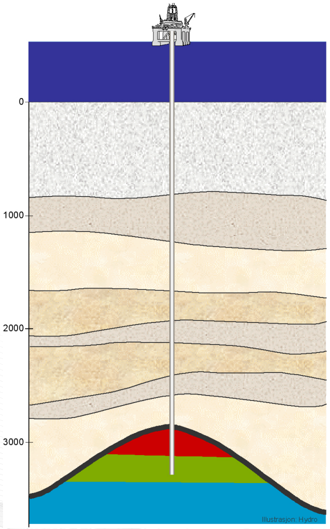


Case study: 1-dimensional diffusion equation



$$\frac{d}{dz}\left(a\frac{du}{dz}\right) = 0$$

Case study: 1-dimensional diffusion equation

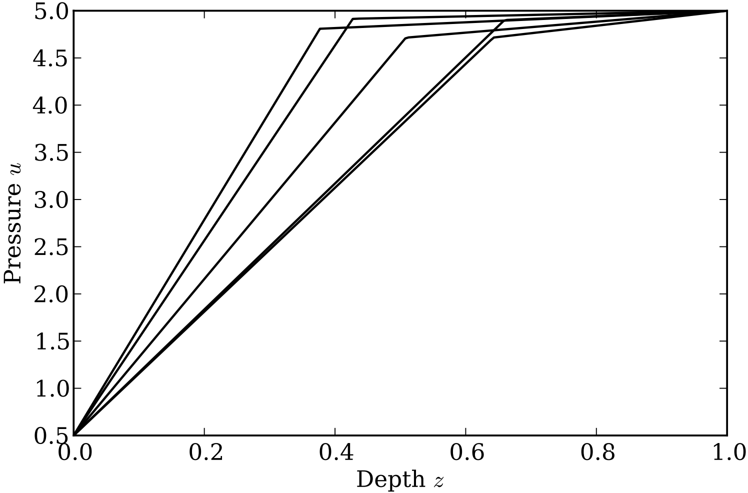


$$\frac{\mathrm{d}}{\mathrm{d}z}\left(a\frac{\mathrm{d}u}{\mathrm{d}z}\right) = 0$$

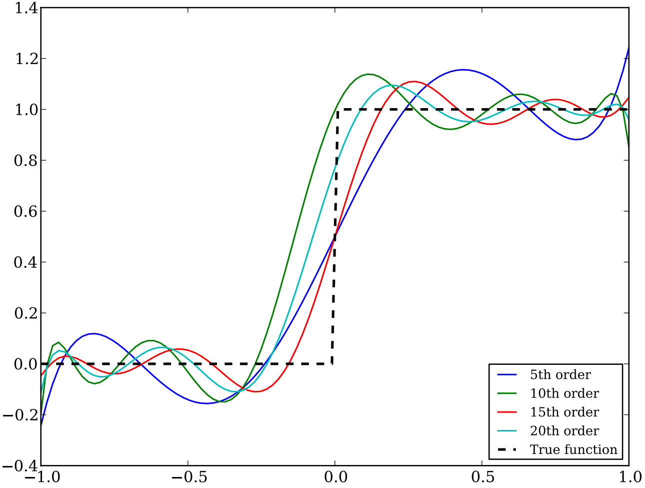
$$a(z, q) = \begin{cases} q_1 & \text{if } q3 < t \\ q_2 & \text{if } q3 \geqslant t \end{cases}$$

$$q_1 \sim Uni(0, 1)$$
$$q_2 \sim Uni(8, 10)$$
$$q_3 \sim Uni(0.3, 0.7)$$

Solution of equation is not smooth because of the discontinuity in layered medium



Polynomials are bad at approximating discontinuities



Remedy: transform the variables to deal with discontinueties

$$u(x, t; q) = u^*(x, t; T(x, t; q)) \approx \sum_{i=0}^{N} \Psi_i(T(x, t; q))\hat{c}_i^*(x, t)$$

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Problems...

- Selection of $r = T(x,t;q)$ non-trivial
- Distribution of $r = T(x,t;q)$ is unknown/difficult to calculate
- Polynomials spatio-temporal dependent
- Random variables not statistically independent

Construct orthogonal polynomials from covariance

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$$\{\Phi_i\}_i \text{ is orthogonal} \qquad \Longleftrightarrow \qquad \mathrm{Cov}\left([\Phi_i]_i\right) \text{ is uncorrelated}$$

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$\{\Phi_i\}_i$ is orthogonal $\iff$ Cov $([\Phi_i]_i)$ is uncorrelated

(Except for $\Phi_0 = 1$)

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$$\{\Phi_i\}_i \text{ is orthogonal} \quad \Longleftrightarrow \quad \text{Cov}\left([\Phi_i]_i\right) \text{ is uncorrelated}$$

(Except for $\Phi_0 = 1$)

Polynomial basis:

$$\begin{array}{llll} \text{ex. } v_r = r_1, & r_1^2, & r_1^3, & \ldots \\ & r_2, \; r_1 r_2, & r_1^2 r_2, & \\ & r_2^2, & r_1 r_2^2, & \\ & & r_2^3, & \end{array}$$

Construct orthogonal polynomials from covariance

$$\{\Phi_i\}_i \text{ is orthogonal} \quad \Longleftrightarrow \quad \mathrm{Cov}\left([\Phi_i]_i\right) \text{ is uncorrelated}$$

(Except for $\Phi_0 = 1$)

Polynomial basis:

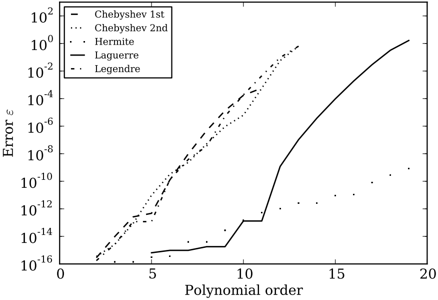
$$\begin{aligned}\text{ex. } v_r = r_1,\quad & r_1^2,\quad r_1^3,\ \ldots\\ r_2,\ r_1 r_2,\ & r_1^2 r_2,\\ & r_2^2,\ r_1 r_2^2,\\ & r_2^3,\end{aligned}$$

$$[\Phi_i]_i \setminus \Phi_0 = L^{-1}\left(v_r - \mathbb{E}\left[v_r\right]\right)$$

$$\Sigma_{v_r} = L^T L$$

$$\mathbb{E}_r\left[r_1^{k_1} \cdots r_C^{k_C}\right] = \mathbb{E}_q\left[T_1(q,x,t)^{k_1} \cdots T_C(q,x,t)^{k_C}\right]$$

Generating orthogonal polynomials is ill-posed: Solution is unstable



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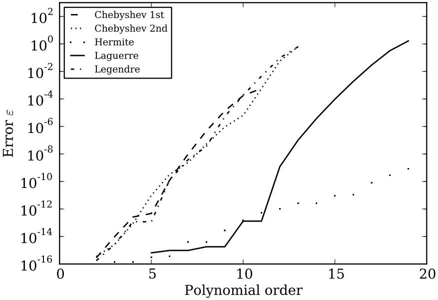
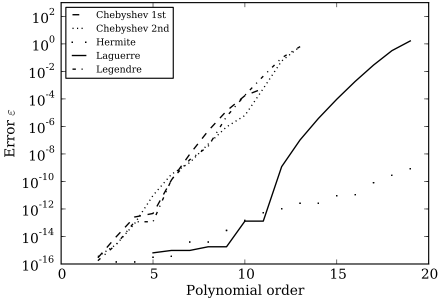


Error is small
for low order

Generating orthogonal polynomials is ill-posed: Solution is unstable

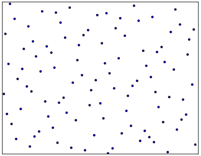


Error is small for low order

Pivoted Cholesky Decomposition for numerical stability

Estimate Fourier coefficients using collocation and linear least squares

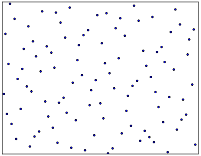
Select collocation nodes in q



$Q_1, \ldots, Q_K$

Estimate Fourier coefficients using collocation and linear least squares

Select collocation nodes in q



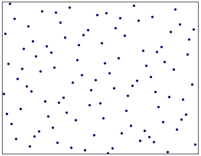
$Q_1, \ldots, Q_K$

$$L = \begin{bmatrix} \Psi_0(T(Q_1)) & \cdots & \Psi_N(T(Q_1)) \\ \vdots & & \vdots \\ \Psi_0(T(Q_K)) & \cdots & \Psi_N(T(Q_K)) \end{bmatrix} \qquad U = \begin{bmatrix} u(x,t;Q_1) \\ \vdots \\ u(x,t;Q_K) \end{bmatrix}$$

$$c = (L^T L)^{-1} L^T U$$

Estimate Fourier coefficients using collocation and linear least squares

Select collocation nodes in q



$Q_1, \ldots, Q_K$

Independent of T

$$L = \begin{bmatrix} \Psi_0(T(Q_1)) & \cdots & \Psi_N(T(Q_1)) \\ \vdots & & \vdots \\ \Psi_0(T(Q_K)) & \cdots & \Psi_N(T(Q_K)) \end{bmatrix} \qquad U = \begin{bmatrix} u(x,t;Q_1) \\ \vdots \\ u(x,t;Q_K) \end{bmatrix}$$

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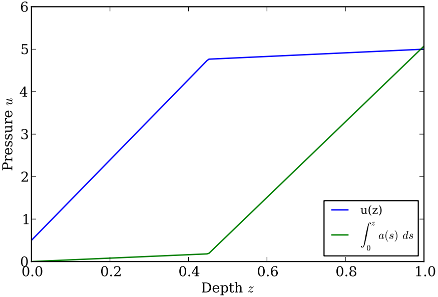
Case study: Select a transformation that has discontinuity in the same location



$$r_1 = q_1$$
$$r_2 = q_2$$
$$r_3 = q_3$$
$$r_4 = \int_0^z a(s)\, ds$$

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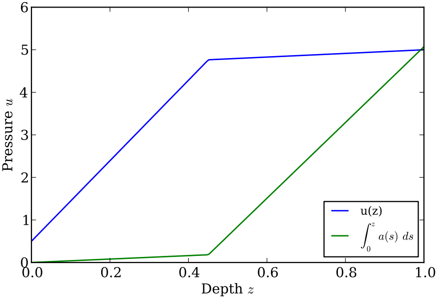
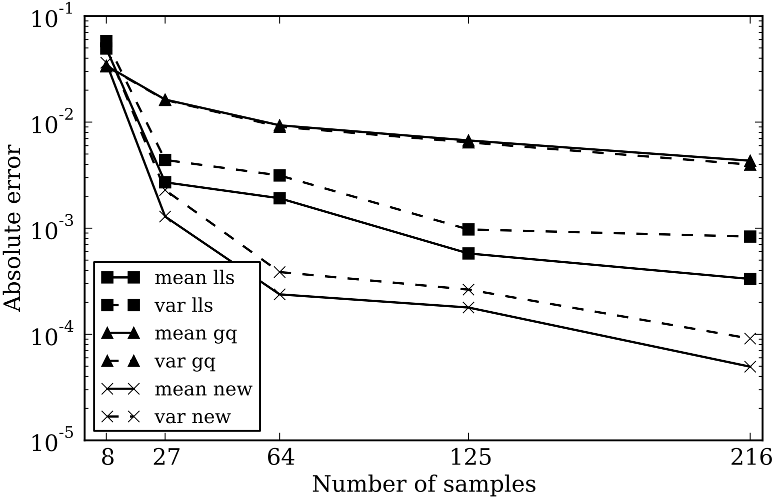


$$r_1 = q_1$$
$$r_2 = q_2$$
$$r_3 = q_3$$
$$r_4 = \int_0^z a(s)\, ds$$

$$\mathbb{E}_r\left[r_1^{k_1} r_2^{k_2} r_3^{k_3} r_4^{k_4}\right] = \mathbb{E}_q\left[q_1^{k_1} q_2^{k_2} q_3^{k_3} \left(\int_0^z a(s)\, ds\right)^{k_4}\right]$$

New method converges to true values fast



Concluding marks

- Discontinuous media: slowdown of poly chaos convergence
- Remedy: transform stochastic variables
- Problem: stochastic variables become *dependent*
- New method for constructing orthogonal polynomials in the dependent case
- Major computational enhancement in the model problem