A simple implementation of polynomial chaos expansions

Jonathan Feinberg

Simula Research Laboratory AS

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Overview of the three parts of the talk.

1. Problem Description
2. Theory
3. Implementation

Description of a general model with uncertain components.

Space x, time t and unknown model parameters q with density p_q

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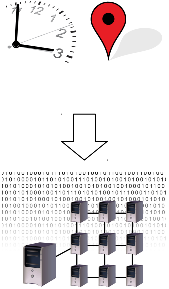

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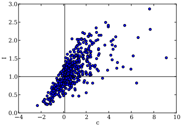

Model analysis $V(U)$

Model approximation using polynomial chaos expansion.

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Choose polynomials to be orthogonal in custom inner product

$$\langle \Phi_i, \Phi_j \rangle = \mathrm{E}[\Phi_i \Phi_j] = 0 \qquad\qquad i \neq j$$

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- Cheap model analysis
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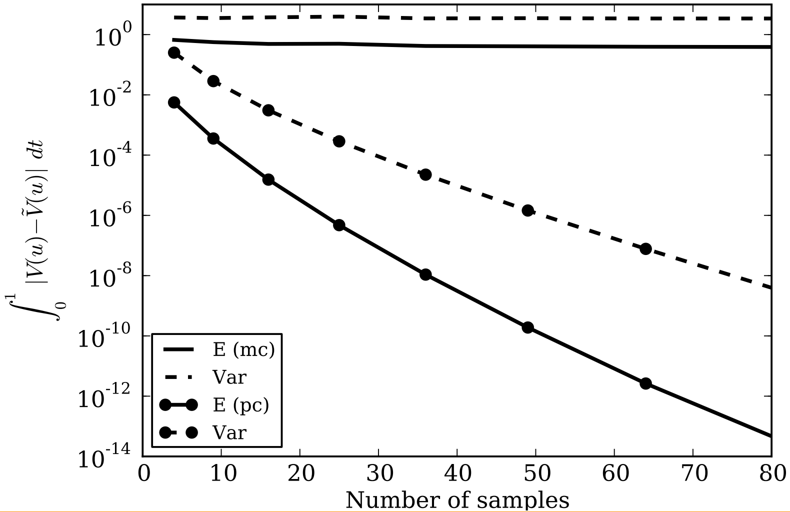
- Coefficients c_i are Fourier

Polynomial chaos often outperforms other methods with several orders of magnitude.



Example

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import chaospy as cp

# Uncertain model parameters
dist_c = cp.Uniform(2, 10)
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# Analysis
print cp.E(Q, Z)
# [ 2.          1.88290007  1.77362073   ...
print cp.Var(Q, Z)
# [ 4.          3.54780499  3.15456747   ...
```

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- Easy setup
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Thank you!

Calculating Fourier coefficients c_i using numerical integration.

$$
\begin{aligned}
c_i = \tfrac{\langle u, \Phi_i \rangle}{\|\Phi_i\|^2} &= \frac{\mathrm{E}[u\Phi_i]}{\mathrm{E}\left[\Phi_i^2\right]} \\
&= \frac{1}{\mathrm{E}\left[\Phi_i^2\right]} \int u(x,t;q)\Phi_i(q)p_q(q)\,\mathrm{d}q \\
&\approx \frac{1}{\mathrm{E}\left[\Phi_i^2\right]} \sum_{k=0}^{K} u(x,t;q_k)\Phi_i(q_k)
\end{aligned}
$$

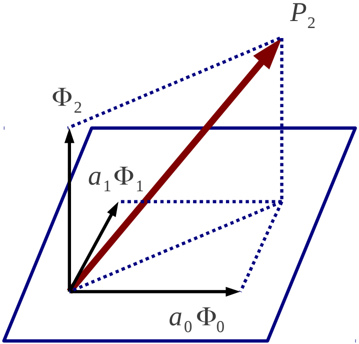
Constructing Orthogonal Polynomials Φ_i using orthogonalization methods.

Gram-Schmidt orthogonalization:

$$\Phi_0 = P_0$$

$$\Phi_n = P_n - \sum_{i=0}^{n-1} \frac{\langle P_n, \Phi_i \rangle}{\|\Phi_i\|^2} \Phi_i$$

$$= P_n - \sum_{i=0}^{n-1} \frac{\mathrm{E}[P_n \Phi_i]}{\mathrm{E}[P_n^2]} \Phi_i$$



P_i any linear independent basis.