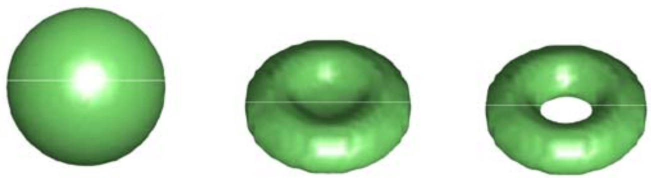 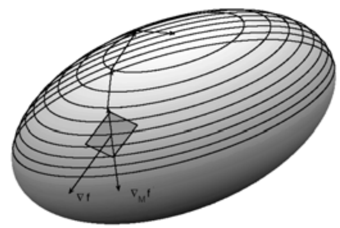 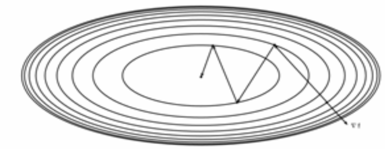

Gradient Descent for Variational Problems with Moving Curves and Surfaces

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The Problem

- Define gradient descent procedures to find extremals to variational problems of the form

$$E(\Gamma) = \int_\Gamma g(\mathbf{x}, \mathbf{n}(\mathbf{x}))\, d\sigma \quad ,$$

over the space of sufficiently regular m-surfaces in $\mathbf{R}^{m+1}$, e.g.

  - curves in images (1D in 2D)
  - surfaces in space (2D in 3D)

Example 1 - Area

- Find surface with minimal area (Plateau's Problem)

$$E(\Gamma) = \int_\Gamma 1 \, d\sigma$$

$$g = 1$$

- Without boundary conditions:

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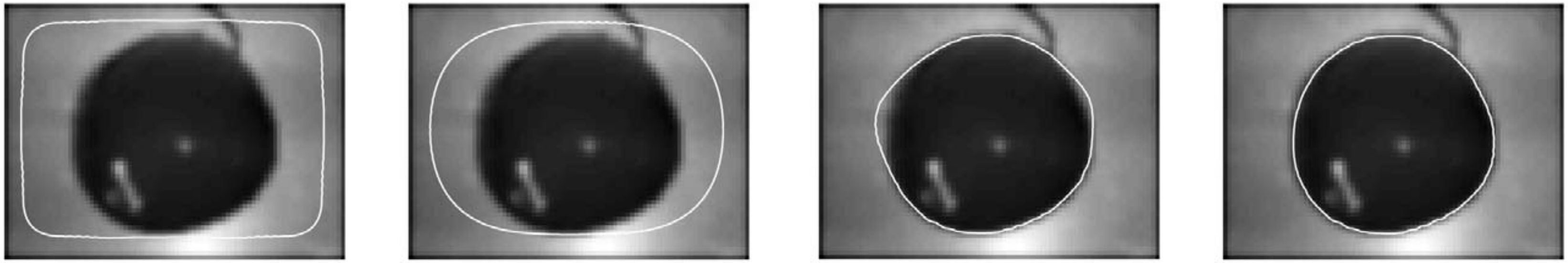
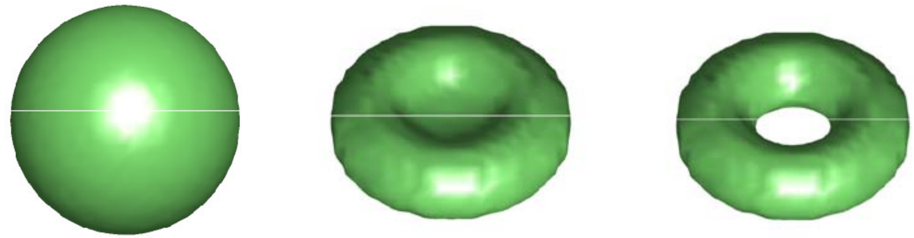
Example 2 – Curve fitting



- Find a curve aligned to the image edges using Geodesic Active Contours

$$E(\Gamma) = \int_\Gamma g(|\nabla I|)\, d\sigma$$

Example 3 – Surface reconstruction



- Surface fitting to 3D data using a distance potential

$$E(\Gamma) = \int_{\Gamma} d(\mathbf{x})\, d\sigma$$

[Zhao et al 2000]

And many, many more examples…

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The Level Set Method

- Numerical methods for interface evolution

  [Osher & Sethian 1988]
  [Dervieux & Thomasset 1979]
- Represent a dynamic surface Γ implicitly
- Properties such as surface normal and mean curvature can easily be computed
- The surface is moved/deformed by solving a PDE on a fixed grid.

$$\Gamma(t) = \{\mathbf{x} \in \Omega \ ; \ \phi(\mathbf{x}, t) = 0\}$$

$$\phi(\mathbf{x}, t) \begin{cases} < 0 & \text{inside } \Gamma \\ = 0 & \text{on } \Gamma \\ > 0 & \text{outside } \Gamma \end{cases}$$

$$\mathbf{n} = \frac{\nabla \phi}{|\nabla \phi|} \qquad \kappa = \nabla \cdot \frac{\nabla \phi}{|\nabla \phi|}$$

$$\frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla \phi = 0 \ \Leftrightarrow \ \frac{\partial \phi}{\partial t} + v_n |\nabla \phi| = 0$$

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Variational Level Set Method

- **Example:** (Plateau's Problem) Minimal surfaces
- "Gradient descent" of functional gives surface motion (mean curvature flow)

$$E(\Gamma) = \int_{\Gamma} 1 dS \quad \Longleftrightarrow \quad \frac{\partial \phi}{\partial t} = \kappa |\nabla \phi|$$

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Deriving Gradient Descent

- A ”common procedure”

$$E(\Gamma) = \int_{\Gamma} g(\mathbf{x}, \mathbf{n}(\mathbf{x}))\, d\sigma = \int_{\mathbf{R}^{m+1}} g(\mathbf{x}, \mathbf{n}(\mathbf{x}))|\nabla\phi|\delta(\phi)d\mathbf{x}$$

- Euler-Lagrange equation $G(\phi, \mathbf{x})\delta(\phi) = 0$
- Solve PDE until steady state

$$\frac{\partial\phi}{\partial t} = \pm G\delta(\phi) \Rightarrow \frac{\partial\phi}{\partial t} = \pm G|\nabla\phi|$$

- This is called the gradient descent evolution.

- Where is the gradient?
- What does the replacement of $\delta(\phi)$ mean?
- Can we give a geometric interpretation for this gradient descent procedure?

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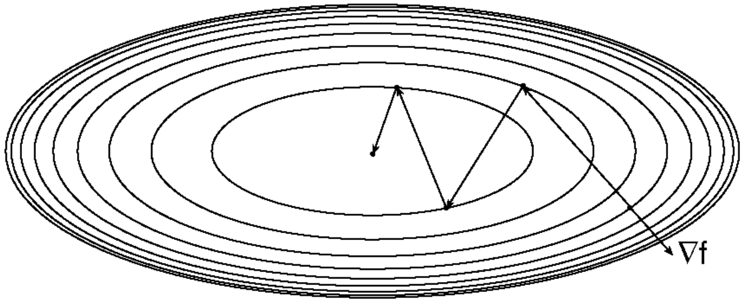
Gradient Descent in R^m



- Solve $$\dot{\mathbf{x}}(t) = -\nabla f(\mathbf{x}(t)), \qquad \mathbf{x}(0) = \mathbf{x}_0$$
  until a stationary point is reached. Here the gradient is

$$\nabla f = (\partial f/\partial x_1, \ldots, \partial f/\partial x_m)$$

and

$$f'_{\mathbf{v}}(\mathbf{x}) = \langle \nabla f(\mathbf{x}), \mathbf{v} \rangle$$

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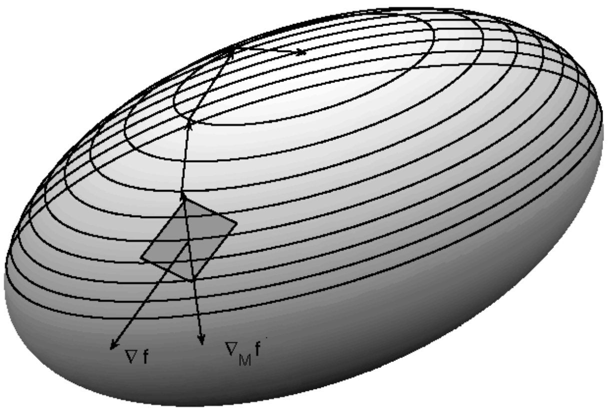
Gradient Descent on Manifolds



Example:

$M = S^m \quad \mathbf{n}(\mathbf{x}) = \mathbf{x}$

$\nabla_{S^m} f(\mathbf{x}) = \nabla \tilde{f} - \langle \mathbf{x}, \nabla \tilde{f} \rangle_{\mathbf{x}} \mathbf{x}$

- Solve $\dot{\mathbf{x}}(t) = -\nabla_M f(\mathbf{x}(t)), \qquad \mathbf{x}(0) = \mathbf{x}_0$
  until a stationary point on the manifold is reached.
- The gradient is defined as the unique vector $\nabla_M f(\mathbf{x}) \in T_{\mathbf{x}} M$ such that

$$df(\mathbf{x})\mathbf{v} = \langle \nabla_M f(\mathbf{x}), \mathbf{v} \rangle_{\mathbf{x}}$$

where

$$df(\mathbf{x})\mathbf{v} = \frac{d}{dt} f(\alpha(t)) \Big|_{t=0}$$

$$\langle \mathbf{v}, \mathbf{w} \rangle_{\mathbf{x}} = \mathbf{v} \cdot \mathbf{w} \quad \text{for} \quad \mathbf{v}, \mathbf{w} \in T_{\mathbf{x}} M$$

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The Ingredients

1. Tangent space at the point x: $T_{\mathbf{x}}M$
2. Scalar product on the tangent space
3. Differential on M $df(\mathbf{x})\mathbf{v}$

These are the necessary ingredients to define a gradient descent procedure. Can we do this for variational problems like

$$\min_{\Gamma} \; E(\Gamma) = \int_{\Gamma} g(\mathbf{x}, \mathbf{n}(\mathbf{x}))\, d\sigma \quad ,$$

???

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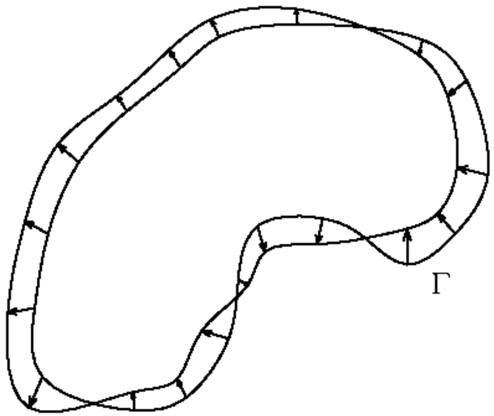
The Manifold of Admissible m-Surfaces



- Let Γ_0 be an m-dimensional surface in R^{m+1}
- The manifold M is defined as the set of surfaces which can be obtained by a regular evolution $t \mapsto \Gamma(t)$ from Γ_0
- Each surface is a "point" on M
- Need to find the "ingredients": tangent space at the point Γ_0 , and a scalar product.

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Normal Velocity

- For a differentiable curve $\alpha : (-\epsilon, \epsilon) \to \mathbf{R}^{m+1}$ such that α belongs to Γ for all t, we have the normal velocity for the evolution $\Gamma(t)$

$$\dot{\Gamma}(t, \mathbf{x}) = \dot{\alpha}(0) \cdot \mathbf{n}(\mathbf{x}_0) = -\frac{\partial \phi(\mathbf{x}_0, 0)/\partial t}{|\nabla \phi(\mathbf{x}_0, 0)|}$$

- This is a continuous function on Γ_0 which can be interpreted as a tangent vector to M.
- The surface evolution is then determined by normal velocities $v \in T_\Gamma M$
- In particular: $C^2(\Gamma_0) \subset T_{\Gamma_0} M \subset C(\Gamma_0)$

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The Gradient

- The differential at Γ_0, $dE(\Gamma_0) : T_{\Gamma_0}M \to \mathbf{R}$, is defined as

$$dE(\Gamma_0)v = \frac{d}{dt}E(\Gamma(t))\Big|_{t=0} \qquad \dot{\Gamma}(0) = v$$

- Introduce scalar product on $T_\Gamma M \subset L^2(\Gamma)$

$$\langle v, w \rangle_\Gamma = \int_\Gamma v(\mathbf{x})w(\mathbf{x})\, d\sigma$$

- By Riesz' lemma $dE(\Gamma)v = \langle w, v \rangle_\Gamma \qquad w = \nabla_M E(\Gamma)$
- Compare:

$$f'_{\mathbf{v}}(\mathbf{x}) = \langle \nabla f(\mathbf{x}), \mathbf{v} \rangle \qquad df(\mathbf{x})\mathbf{v} = \langle \nabla_M f(\mathbf{x}), \mathbf{v} \rangle_{\mathbf{x}}$$

$$dE(\Gamma)v = \langle \nabla_M E(\Gamma), v \rangle_\Gamma$$

- Gradient descent

$$\dot{\Gamma}(t) = -\nabla_M E(\Gamma(t))$$

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Back to the Problem

- Functional: $$E(\Gamma) = \int_\Gamma g(\mathbf{x}, \mathbf{n}(\mathbf{x}))\, d\sigma$$
- **Theorem:** The functional has differential

$$dE(\Gamma)v = \langle \nabla \cdot [g_\mathbf{n} + g\,\mathbf{n}]\,, v\,\rangle_\Gamma$$

$$\nabla_M E = \nabla \cdot [g_\mathbf{n} + g\,\mathbf{n}] \qquad g_\mathbf{n} = \nabla_{S^m} g$$

- Gradient descent is then

$$\dot{\Gamma} = -\nabla \cdot [g_\mathbf{n} + g\,\mathbf{n}]$$

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Examples Again

Area

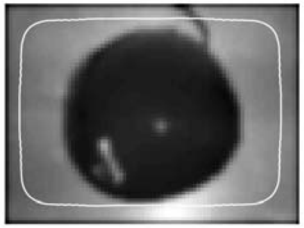
$$E(\Gamma) = \int_\Gamma 1 \, d\sigma$$

$$\nabla_M E = \kappa$$

Geodesic Active Contours

$$E(\Gamma) = \int_\Gamma f(|\nabla I|) \, d\sigma$$

$$\nabla_M E = f(|\nabla I|) \, \kappa + \nabla f(|\nabla I|) \cdot \mathbf{n}$$



Surface potential

$$E(\Gamma) = \int_\Gamma d(\mathbf{x}) \, d\sigma$$

$$\nabla_M E = g(\mathbf{x})\kappa + \nabla g(\mathbf{x}) \cdot \mathbf{n}$$

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Volume functionals

- Same analysis for volume (area) functionals

$$E(\Gamma) = \int_{\Omega^-} g(\mathbf{x})\, d\mathbf{x} \qquad \Omega^- = int(\Gamma)$$

- The functional has differential

$$dE(\Gamma)v = \langle\, g(\mathbf{x})\, , v\, \rangle_\Gamma$$

$$\nabla_M E = g(\mathbf{x})$$

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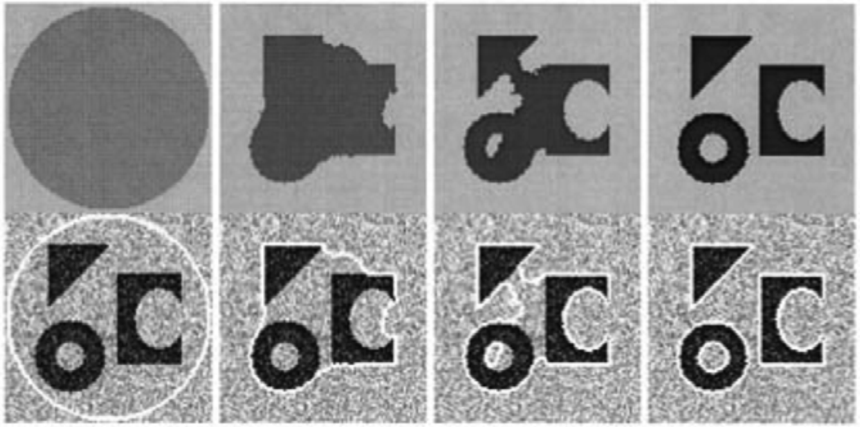
Example - Segmentation

- Chan-Vese model:

$$E(\Gamma) = \int_{\Omega^-} (I - I_0)^2 \, d\mathbf{x} + \int_{\Omega^+} (I - I_1)^2 \, d\mathbf{x}$$

$$\nabla_M E = (I - I_0)^2 - (I - I_1)^2$$



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Connection to Euler-Lagrange

$$E(\Gamma) = \int_\Gamma 1\, d\sigma$$

E-L approach

- E-L equation

$$\kappa\delta(\phi) = 0$$

$$\frac{\partial \phi}{\partial t} = \pm\kappa\delta(\phi)$$

- Find sign & extend

Gradient interpretation

- Differential $dE(\Gamma)v = \langle \kappa, v\rangle_\Gamma$
- Gradient $\nabla_M E = \kappa$
- Gradient descent

$$\dot{\Gamma} = -\nabla_M E \qquad \dot{\Gamma} = -\frac{\partial\phi/\partial t}{|\nabla\phi|}$$

$$\frac{\partial \phi}{\partial t} = \kappa|\nabla\phi|$$

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Gradient Projection for Variational Surface Problems with Constraints

- Functional

$$E(\Gamma) = \int_{\Gamma} g(\mathbf{x}, \mathbf{n}(\mathbf{x}))\, d\sigma$$

with constraints in "infinite" form, defined by a functional

$$G(\Gamma) = 0$$

- Constraint (sub-manifold)

$$N = \{\Gamma \in M : G(\Gamma) = 0\} \quad ,$$

- This gives a gradient on the constraint manifold as

$$\nabla_N E(\Gamma) := \nabla E(\Gamma) - \frac{\langle \nabla E(\Gamma), \nabla G(\Gamma) \rangle_{\Gamma}}{\|\nabla G(\Gamma)\|_{\Gamma}^2} \nabla G(\Gamma)$$

Note: this extends naturally to multiple constraints!

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Constrained Evolution - Lagrange Functional

- Find extremals (Γ,λ) to the Lagrange functional $E(\Gamma) - \lambda G(\Gamma)$ using gradient descent. This gives

$$\frac{\partial \phi}{\partial t} = (\nabla E - \lambda \nabla G)|\nabla \phi|$$

where the Lagrange multiplier λ is given by

$$\lambda = \frac{\int_\Gamma \nabla E \nabla G \, d\sigma}{\int_\Gamma (\nabla G)^2 \, d\sigma}$$

Note: this is a orthogonal projection on the constraint manifold.

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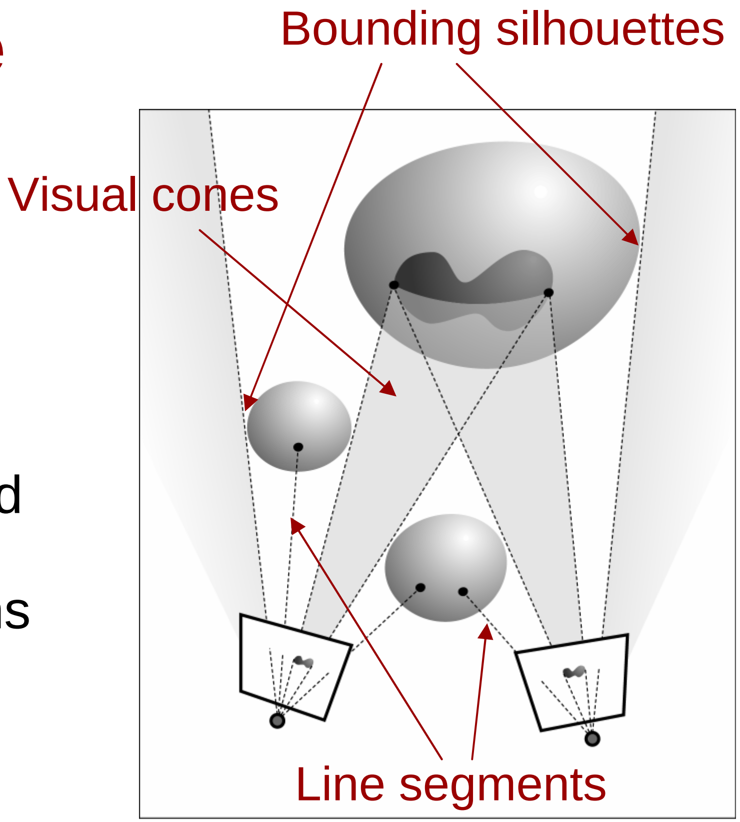
Example



Problem:

- Incorporate visibility constraints in variational surface fitting procedures.

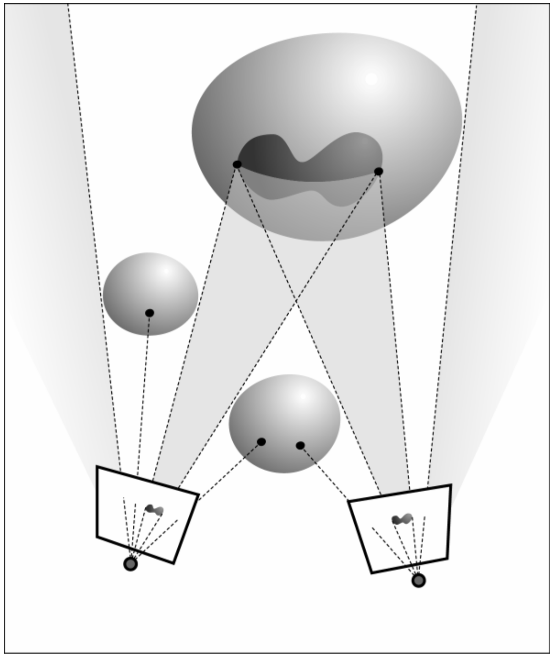
Solution:

- Define forbidden regions using signed distance functions
- Define constraints using these regions
- Use constrained gradient descent to evolve the surface

Why?

- Visibility is an important cue
- Faster convergence, correct topology, avoid oscillatory evolution, always obey visibility during evolution.

Visibility Constraints



- Group all forbidden regions in a set $W \subset \mathbf{R}^3$ defined using a signed distance function

$$W = \{\mathbf{x} \;;\; w(\mathbf{x}) \geq 0\} \quad \partial W = \{w = 0\}$$

- Define the constraint using a functional G

$$G(\Gamma, W) = \int_{\Omega^- \cap W} d\mathbf{x} = 0$$

- The gradient of the constraint is

$$\nabla G = H(w)$$

where $H(w)$ is the Heaviside function, and the constrained surface evolution is

$$\boxed{\frac{\partial \phi}{\partial t} = (\nabla E - \lambda H(w))|\nabla \phi| \qquad \lambda = \frac{\int_\Gamma \nabla E H(w)\, d\sigma}{\int_\Gamma (H(w))^2\, d\sigma}}$$

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Initialization

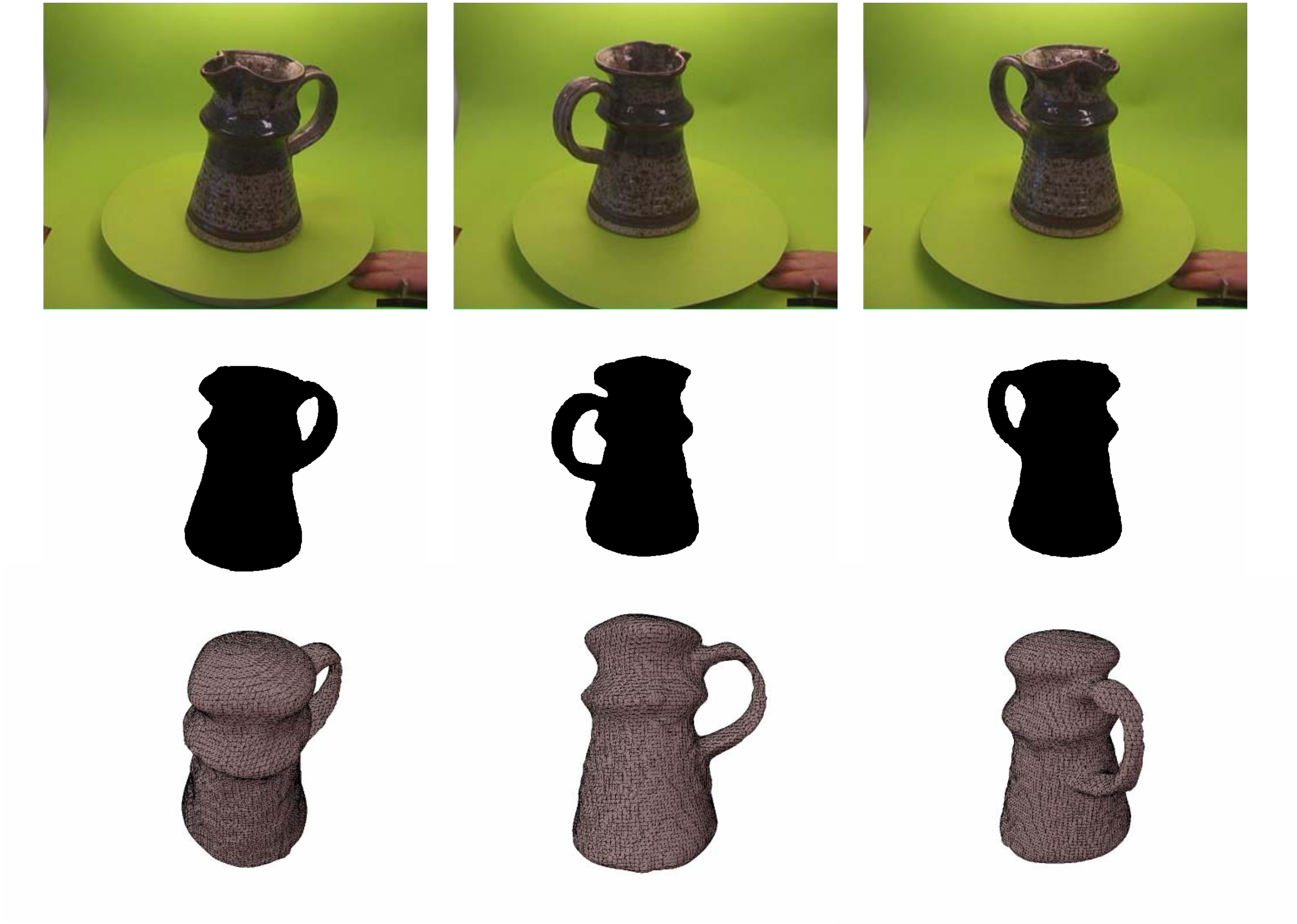
- Need initial surface that satisfies $G(\Gamma, W) = 0$
- This is easy to obtain using Boolean operations on the functions since $\boxed{\Omega^{-} \setminus W = \Omega^{-} \cap W^{c} = \{\mathbf{x}\ ;\ \max(\phi(\mathbf{x}), w(\mathbf{x})) \leq 0\}}$
- Given any initial function, this can be modified to satisfy the constraint by setting $\phi(\mathbf{x}) = \max(\phi(\mathbf{x}), w(\mathbf{x}))$

Example – pitcher:

convex hull

lines

silhouettes



Summary

- We introduced:
  - manifold of admissible m-surfaces
  - tangent space
  - scalar product
  - gradient
- Based on geometric quantities such as *normal velocity* and gradient.
- This means that the theory is valid
  - in any number of dimensions
  - for any surface representation
- References
  - Solem & Overgaard, Scale Space 2005
  - Solem & Overgaard, VLSM 2005

Thank you!

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