A Gradient Descent Procedure for Variational Dynamic Surface Problems with Constraints

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Abstract. Many problems in image analysis and computer vision involving boundaries and regions can be cast in a variational formulation. This means that m-surfaces, e.g. curves and surfaces, are determined as minimizers of functionals using e.g. the variational level set method. In this paper we consider such variational problems with constraints given by functionals. We use the geometric interpretation of gradients for functionals to construct gradient descent evolutions for these constrained problems. The result is a generalization of the standard gradient projection method to an infinite-dimensional level set framework. The method is illustrated with examples and the results are valid for surfaces of any dimension.

1 Introduction

Variational formulations have been successfully used by many researchers to solve a wide variety of problems within computer vision and image analysis. Benefits of using variational formulations are among others; a solid mathematical framework, well developed numerical techniques, and the fact that a variational formulation clearly and unambiguously shows the type of solutions that are sought.

From a variational formulation one can determine what the solutions look like. If they cannot be computed directly from the Euler-Lagrange equations a gradient descent procedure can be used, cf. [1]. This paper deals with gradient descent for variational problems involving dynamic m-dimensional surfaces and interfaces with side conditions. The side conditions appear in the form of constraints on the solutions. Constraints can appear in "finite" form such as boundary conditions for curves and surfaces and in "infinite" form, defined by functionals. Here the latter type is considered.

Variational problems with such constraints appear naturally in many vision applications such as overlapping of regions in multi-phase segmentation [2,3] and as obstacles for surface fitting problems [4].

This paper introduces mathematical techniques that give a geometric interpretation and introduces an infinite-dimensional gradient projection method as an extension of the finite-dimensional theory, cf. [5]. A projected gradient is introduced as the orthogonal projection on a manifold determined by the constraint functional and we derive the corresponding gradient descent procedure. In doing so, we show that a gradient descent Lagrange method is in fact a projection. Furthermore, we give some illustrative examples and treat issues related with practical implementations.

The paper is organized as follows; background material on finite-dimensional constrained optimization, the level set method and infinite-dimensional gradient descent is covered in Section 2. Our contributions are described in Sections 3 to 5. In Section 3 we derive the projected gradient on the constraint manifold and in Section 4 we give some illustrative examples. Finally, we comment on the practical implementation in Section 5.

2 Background

As a courtesy to the reader, the necessary background on finite dimensional constrained optimization, the level set method, and the geometric gradient interpretation for variational m-surface problems is briefly recalled here.

2.1 Gradient Projection for Finite Dimensional Problems

Suppose we are asked to find the minimum of $f(\mathbf{x})$ subject to the side condition $g(\mathbf{x}) = 0$, where $f, g : \mathbf{R}^{m+1} \to \mathbf{R}$ are a pair of differentiable functions. Set $N = \{\mathbf{x} \in \mathbf{R}^{m+1} : g(\mathbf{x}) = 0\}$ and assume for simplicity that $\nabla g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in N$. Then N is a differentiable m-dimensional surface, by the implicit function theorem, and the above minimization problem becomes that of finding $\mathbf{x}^* \in N$ such that

$$f(\mathbf{x}^*) = \min_{\mathbf{x}\in N} f(\mathbf{x}). \tag{1}$$

The classical method for solving such a problem is to use Lagrange multipliers: If $\mathbf{x}^*$ minimizes f on N, then there exists a constant $\lambda^* \in \mathbf{R}$ such that

$$\nabla f(\mathbf{x}^*) - \lambda^* \nabla g(\mathbf{x}^*) = 0.$$

Thus, $(\mathbf{x}^*, \lambda^*)$ is to be found among the stationary points of the Lagrange function $L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda g(\mathbf{x})$. However, computing the stationary points of $L(\mathbf{x}, \lambda)$ is generally a highly non-trivial matter, so in practice one tries to find $\mathbf{x}^*$ directly by using some kind of descent method.

Let us describe the construction of a gradient descent procedure for the minimization problem (1). First, let $\mathbf{x} \in N$ and define the *gradient* $\nabla_N f(\mathbf{x})$ *of* f *on* N *at* $\mathbf{x} \in N$ by

$$\nabla_N f(\mathbf{x}) = \nabla f(\mathbf{x}) - \frac{\nabla g(\mathbf{x}) \cdot \nabla f(\mathbf{x})}{\|\nabla g(\mathbf{x})\|^2} \nabla g(\mathbf{x}), \tag{2}$$

where $\mathbf{v} \cdot \mathbf{w}$ denotes the usual scalar product between vectors $\mathbf{v}, \mathbf{w} \in \mathbf{R}^{m+1}$, and $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$ is the corresponding norm. Notice that the N-*gradient* $\nabla_N f(\mathbf{x})$ is simply the orthogonal projection of the usual gradient $\nabla f(\mathbf{x})$ onto the tangent space of N at $\mathbf{x}$: $T_{\mathbf{x}}N = \{\mathbf{v} \in \mathbf{R}^{m+1} : \nabla g(\mathbf{x}) \cdot \mathbf{v} = 0\}$, hence the term: *the gradient projection method*, cf. Figure 1.

Next, given a point $\mathbf{x}_0 \in N$ then the gradient descent motion for (1) is the solution of the initial value problem:

$$\dot{\mathbf{x}}(t) = -\nabla_N f(\mathbf{x}(t)), \qquad \mathbf{x}(0) = \mathbf{x}_0 . \tag{3}$$

If the limit $\mathbf{x}^\circ = \lim_{t\to\infty} \mathbf{x}(t)$ exists then $\nabla_N f(\mathbf{x}^\circ) = 0$, hence $\mathbf{x}^\circ$ is a stationary point of f's restriction to the manifold N. Hopefully the gradient descent motion solves the minimization problem (1), i.e. $\mathbf{x}^\circ = \mathbf{x}^*$, but $\mathbf{x}^\circ$ may of course turn out to be some other stationary point.

The purpose of this work is to generalize this procedure to variational problems for curves and surfaces, where both the objective function and the side conditions are given by functionals.

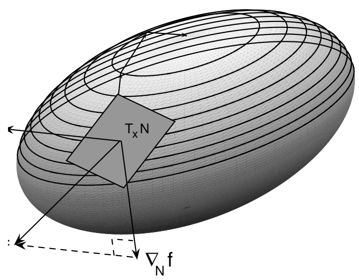

Fig. 1. Projected gradient descent on an ellipsoid-shaped manifold defined as the set $N = \{\mathbf{x} \in \mathbf{R}^{m+1} : g(\mathbf{x}) = 0\}$. The gradient is projected on the tangent space using the gradient of the constraint, ∇g.

2.2 The Kinematics of Dynamic Surfaces

A regular m-surface in $\mathbf{R}^{m+1}$ has codimension equal to one and can be represented implicitly as the zero set of a differentiable function $\phi : \mathbf{R}^{m+1} \to \mathbf{R}$, the *level set function*, as

$$\Gamma = \{\mathbf{x} \ : \ \phi(\mathbf{x}) = 0\} \ . \tag{4}$$

The sets $\Omega = \{\mathbf{x} : \phi(\mathbf{x}) < 0\}$ and $\{\mathbf{x} : \phi(\mathbf{x}) > 0\}$ are called the *inside* and the *outside* of Γ, respectively. Using this convention, the outward unit normal $\mathbf{n}$ and the mean curvature κ of Γ are given by (c.f. [6])

$$\mathbf{n} = \frac{\nabla\phi}{|\nabla\phi|} \quad \text{and} \quad \kappa = \nabla \cdot \frac{\nabla\phi}{|\nabla\phi|} \ . \tag{5}$$

The implicit representation introduced above can be used to define a *dynamic surface* (or *surface evolution*), $t \to \Gamma(t)$, by adding a time dependence, $\phi = \phi(\mathbf{x}, t)$, to the level set function, where $\phi : \mathbf{R}^{m+1} \times I \to \mathbf{R}$ is a sufficiently differentiable function. The dynamic surface is then given by

$$t \mapsto \Gamma(t) := \{\mathbf{x} \;:\; \phi(\mathbf{x}, t) = 0\} \;. \tag{6}$$

Let us recall from [1] the notion of the *normal velocity* of the surface evolution (6). Suppose a particle moves along with the dynamic surface $\Gamma(t)$. If the motion of the particle is given by the parameterized curve $\boldsymbol{\alpha} : I \to \mathbf{R}^{m+1}$ with $\boldsymbol{\alpha}(0) = \mathbf{x}_0$, then the equality $\phi(\boldsymbol{\alpha}(t), t) = 0$ holds identically at all times t. Differentiation of this identity yields

$$\dot{\boldsymbol{\alpha}}(0) \cdot \mathbf{n} = -\frac{\partial\phi(\mathbf{x}_0, 0)/\partial t}{|\nabla\phi(\mathbf{x}_0, 0)|} \;, \tag{7}$$

where the left-hand side is the normal component of the velocity $\dot{\boldsymbol{\alpha}}(0)$ of the particle at $t = 0$. This normal component is an intrinsic property of the evolution since it does not depend on the choice of $\boldsymbol{\alpha}$ or $\phi(\mathbf{x}, t)$, cf. [1]. We can then define the *normal velocity* of the evolution $\Gamma(t)$ as the function

$$\dot{\Gamma}(t) = -\frac{\partial\phi(\mathbf{x}, t)/\partial t}{|\nabla\phi(\mathbf{x}, t)|} \qquad (\mathbf{x} \in \Gamma(t)) \;. \tag{8}$$

Using the notation $v = v(\Gamma) = -\dot{\Gamma}(t)$ we can rewrite this equation as

$$\frac{\partial\phi}{\partial t} = v|\nabla\phi| \;, \tag{9}$$

where we have dropped the dependence on $\mathbf{x}$ and t to simplify the notation. This is the well-known *level set equation* which is the basis for the level set method, introduced independently by [7] and [8] as a tool for evolving implicit surfaces.

2.3 Geometric Gradient Descent for Dynamic Surfaces

In this section we recall from [1] the construction of gradient descent evolutions for the minimization of functionals $E(\Gamma)$ defined on manifolds of admissible m-surfaces Γ. Here we are primarily concerned with functionals of the following types

$$E_\circ(\Gamma) = \int_\Gamma g(\mathbf{x})\, d\sigma \quad \text{or} \quad E_\bullet(\Gamma) = \int_\Omega g(\mathbf{x})\, d\mathbf{x} \;, \tag{10}$$

where $\Gamma = \partial\Omega$ is a closed m-surface, $d\sigma$ the Euclidean surface measure, and $g : \mathbf{R}^{m+1} \to \mathbf{R}$ is a given weight function.

As in [1], let M denote the (pre-)manifold of admissible m-surfaces Γ. If $\Gamma \in M$ then the *tangent space of* M *at* Γ is the set $T_\Gamma M$ of all functions $v : \Gamma \to \mathbf{R}$ such that v correspond to the normal velocity of some (regular) surface evolution through Γ. Each tangent space $T_\Gamma M$ of M is endowed with a scalar product $\langle\cdot,\cdot\rangle_\Gamma$ defined as the integral

$$\langle v, w\rangle_\Gamma = \int_\Gamma v(\mathbf{x})w(\mathbf{x})\, d\sigma \qquad (v, w \in T_\Gamma M) \;. \tag{11}$$

If the norm of $v \in T_\Gamma M$ is defined by by $\|v\|_\Gamma = \sqrt{\langle v, v\rangle_\Gamma}$, then we have Schwarz' inequality:

$$|\langle v, w\rangle_\Gamma| \leq \|v\|_\Gamma \|w\|_\Gamma \qquad (v, w \in T_\Gamma M) \ . \tag{12}$$

Now, consider a functional $E : M \to \mathbf{R}$ and let $\Gamma \in M$ be fixed. The functional E is said to be Gâteaux-differentiable at Γ, if the derivative

$$dE(\Gamma)v = \frac{d}{dt} E(\Gamma(t))\big|_{t=0} \tag{13}$$

exists for every $v \in T_\Gamma M$. Here $\Gamma(t)$ is a surface evolution which satisfies $\Gamma(0) = \Gamma$ and $\dot{\Gamma}(0) = v$. The linear functional on the left hand side of (13) is called the Gâteaux derivative (or the functional derivative) of E at Γ. There sometimes exists a vector $\nabla E(\Gamma) \in T_\Gamma M$ such that the following identity holds for all normal velocities $v \in T_\Gamma M$:

$$dE(\Gamma)v = \langle \nabla E(\Gamma), v\rangle_\Gamma \qquad \text{(Riesz)} \ . \tag{14}$$

If this is the case, then $\nabla E(\Gamma)$ is called the *gradient of* E *at* Γ, and it is uniquely determined by the property (14)[1].

The gradient descent for the variational problem $E(\Gamma^*) = \min_\Gamma E(\Gamma)$ is, analogous to (3), given by the solution of the following initial value problem:

$$\dot{\Gamma}(t) = -\nabla E(\Gamma(t)); \qquad \Gamma(0) = \Gamma_0 \ , \tag{15}$$

where Γ_0 is the initial m-surface.

As an example we apply this procedure to the two functionals in (10), and derive the corresponding gradient descent evolutions in the level set framework. First we notice that the Gâteaux derivatives of these functionals are $dE_\circ(\Gamma)v = \int_\Gamma (\nabla g \cdot \mathbf{n} + g\kappa)v \, d\sigma$ and $dE_\bullet(\Gamma)v = \int_\Gamma gv \, d\sigma$, respectively. The first of these derivatives, which is the classical *geodesic active contours*, is derived in [1,9,10], and the second derivative can be found in e.g. [1,11]. Using (14) we see that

$$\nabla E_\circ(\Gamma) = \nabla g \cdot \mathbf{n} + g\kappa \quad \text{and} \quad \nabla E_\bullet(\Gamma) = g \ . \tag{16}$$

Using the formula (8) for the normal velocity it follows from (15) that the gradient descent evolutions for the minimization of $E_\circ$ and $E_\bullet$ are

$$\frac{\partial \phi}{\partial t} = (\nabla g \cdot \mathbf{n} + g\kappa)|\nabla \phi| \quad \text{and} \quad \frac{\partial \phi}{\partial t} = g|\nabla \phi| \ , \tag{17}$$

respectively, where the initial level set function $\phi_0(\mathbf{x}) = \phi(\mathbf{x}, 0)$ must be specified.

[1] It would be more correct to use the notation $\nabla_M E$ for the gradient of E, as it is actually the gradient of E on the manifold M of admissible m-surfaces. However, always insisting on correct names ultimately leads to cumbersome notation, and since functionals on M are always denoted by upper case letters, we trust the reader understands that ∇E means the (functional) gradient in the infinite-dimensional setting of surfaces.

3 Gradient Projection for Variational Surface Problems

In this section we show how the notion of a functional gradient, defined in Section 2.3, can be used to give a geometric interpretation of descent evolutions for variational level set problems with constraints. Let $F, G : M \to \mathbf{R}$ be two Gâteaux-differentiable functionals of either of the forms in (10). Define N as the sub-manifold of admissible m-surfaces Γ given by

$$N = \{\Gamma \in M : G(\Gamma) = 0\} \ ,$$

and consider the constrained variational problem of finding $\Gamma^* \in N$ such that

$$F(\Gamma^*) = \min_{\Gamma \in N} \ F(\Gamma) \ . \tag{18}$$

Assume that $\nabla G \neq 0$ on N. If Γ^* solves (18) then, according to the Lagrange multiplier method, there exists a number $\lambda^* \in \mathbf{R}$ such that the pair (Γ^*, λ^*) is a stationary point of the Lagrange function $L(\Gamma, \lambda) = F(\Gamma) - \lambda G(\Gamma)$, that is, (Γ^*, λ^*) solves the following system of equations:

$$\begin{cases} \nabla L(\Gamma, \lambda) = \nabla F(\Gamma) - \lambda \nabla G(\Gamma) = 0 \\ \dfrac{\partial L}{\partial \lambda} L(\Gamma, \lambda) = G(\Gamma) = 0 \ . \end{cases} \tag{19}$$

In order to find (Γ^*, λ^*) we construct a gradient descent motion $t \mapsto (\Gamma(t), \lambda(t))$ for the Lagrange function L in such a way that the constraint $\partial L / \partial \lambda = 0$ is enforced at all times. This means that $t \to \Gamma(t)$ solves the initial value problem:

$$\dot{\Gamma}(t) = -\nabla L(\Gamma(t), \lambda(t)), \qquad \Gamma(0) = \Gamma_0 \ , \tag{20}$$

where it remains to determine the value of $\lambda = \lambda(t)$ in (20). To do this, we differentiate the identity $G(\Gamma(t)) = 0$, in the second equation of the system (19),

$$\begin{aligned} 0 &= \frac{d}{dt} G(\Gamma) = dG(\Gamma)\dot{\Gamma} = -\langle \nabla G(\Gamma), \nabla L(\Gamma, \lambda) \rangle_\Gamma \\ &= -\langle \nabla G(\Gamma), \nabla F(\Gamma) - \lambda \nabla G(\Gamma) \rangle_\Gamma \end{aligned}$$

then we see that

$$\lambda = \frac{\langle \nabla F(\Gamma), \nabla G(\Gamma) \rangle_\Gamma}{\|\nabla G(\Gamma)\|_\Gamma^2} \ .$$

It follows from this calculation that the right-hand side of (20) is $\nabla L(\Gamma, \lambda) = \nabla_N F(\Gamma)$, where

$$\nabla_N F(\Gamma) := \nabla F(\Gamma) - \frac{\langle \nabla F(\Gamma), \nabla G(\Gamma) \rangle_\Gamma}{\|\nabla G(\Gamma)\|_\Gamma^2} \nabla G(\Gamma) \tag{21}$$

is the *N-gradient of F at $\Gamma \in N$*, which is defined as the orthogonal projection (in $T_\Gamma M$) of ∇F onto the tangent space $T_\Gamma N = \{v \in T_\Gamma M : \langle \nabla G(\Gamma), v \rangle_\Gamma = 0\}$, precisely as in the finite dimensional case in Section 2.1. In other words, the gradient for the Lagrange function is a projection. In the level set formulation, the N-gradient descent motion for (18) becomes

$$\frac{\partial \phi}{\partial t} = \nabla_N F \, |\nabla \phi| \ , \tag{22}$$

where $\nabla_N F$ is given by (21). This result is easily generalized to variational problems with several constraints.

Proposition 1. *Let $F, G_1, \dots, G_n : M \to \mathbf{R}$ be Gâteaux differentiable functionals, and assume that the functional gradients $\nabla G_1(\Gamma), \dots, \nabla G_n(\Gamma)$ are linearly independent on $N = \{\Gamma \in M : G_1(\Gamma) = \cdots = G_n(\Gamma) = 0\}$. Then the gradient descent motion for the minimization problem (18) is the solution of the initial value problem:*

$$\dot{\Gamma}(t) = -\nabla_N F(\Gamma(t)), \qquad \Gamma(0) = \Gamma_0 \ . \tag{23}$$

Here the N-gradient $\nabla_N F$ is given by

$$\nabla_N F(\Gamma) = \nabla F(\Gamma) - \sum_{i=1}^{n} \lambda_i \nabla G_i(\Gamma) \ ,$$

where $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ solves the following system of equations:

$$\sum_{j=1}^{n} \lambda_j \langle \nabla G_j, \nabla G_i \rangle_\Gamma = \langle \nabla F, \nabla G_i \rangle_\Gamma \qquad (i = 1, \dots, n) \ . \tag{24}$$

That is, $\nabla_N F(\Gamma)$ is the orthogonal projection of F onto the tangent space $T_\Gamma N$.

Proof. The dynamic surface defined by (23) is the gradient descent motion for the Lagrange function $L(\Gamma, \boldsymbol{\lambda}) = F(\Gamma) - \sum_{i=1}^{n} \lambda_i G_i(\Gamma)$ which satisfies the conditions $\partial L / \partial \lambda_i(\Gamma(t)) = 0$ for $i = 1, \dots, n$. The latter implies that the equalities $G_i(\Gamma(t)) = 0$, $i = 1, \dots, n$ hold for all t. Differentiation of these identities gives the system (24) for the Lagrange multipliers, which is solvable, by the assumption on the functional gradients $\nabla G_1, \dots, \nabla G_n$. The details are left to the reader.

4 Examples

In this section we will show some examples of how to apply the geometric analysis above. First we use a classical result from differential geometry as a pedagogical example to illustrate the theory in Section 3, then we give a practical case where such analysis can be used for surface fitting to 3D data.

4.1 Illustrative Example: The Isoperimetric Problem

Let us define the length and area functionals for a closed planar curve as

$$L(\Gamma) = \int_\Gamma d\sigma \quad \text{and} \quad A(\Gamma) = \int_\Omega d\mathbf{x} \ , \tag{25}$$

where, as above, Ω is the interior of Γ. Since in both cases these expressions correspond to $g(\mathbf{x}) = 1$ in (10), the gradients are simply

$$\nabla L = \kappa \quad \text{and} \quad \nabla A = 1 \ , \tag{26}$$

from (16).

The isoperimetric problem (IP) is to *find the curve with a fixed length that encloses the largest area*, or equivalently, *find the shortest curve which encloses a fixed area*. If we decide to use the second formulation we can define the manifold satisfying the constraint as

$$N = \{\Gamma \ : \ A(\Gamma) = A_0\} \ ,$$

for some constant A_0. The problem is then formulated as

$$IP: \quad \min_{\Gamma \in N} L(\Gamma) \ , \tag{27}$$

and the projected gradient is simply

$$\nabla_N L = \nabla L - \frac{\langle \nabla L, \nabla A\rangle_\Gamma}{||\nabla A||_\Gamma^2} \nabla A = \kappa - \frac{\langle \kappa, 1\rangle_\Gamma}{||1||_\Gamma^2} 1 \ .$$

Remark: For a simple closed curve in the plane it is well known that $\langle \kappa, 1\rangle_\Gamma = 2\pi$, cf. [12, p.37]. Also, the norm in the denominator is equal to the curve length, $||1||_\Gamma^2 = \int_\Gamma d\sigma = L(\Gamma)$. Using Schwarz' inequality (12) it is clear that the constraints $L(\Gamma) = L_0$ and $A(\Gamma) = A_0$ always have nonzero gradient because $2\pi = \langle \kappa, 1\rangle_\Gamma \leq ||\kappa||_\Gamma ||1||_\Gamma = ||\nabla L||_\Gamma ||\nabla A||_\Gamma$.

Using the projected gradient above, we can now state and prove the following classical result, which tells us what the extremals are.

Proposition 2. *For a simple closed curve in the plane*

$$\nabla_N L(\Gamma) = 0 \quad \Leftrightarrow \quad \Gamma = \text{circle} \ .$$

Proof. Recall that a circle with radius r has positive constant curvature $\kappa = 1/r$. If Γ is a circle with radius r, then

$$\nabla_N L = \kappa - \frac{\langle \kappa, 1\rangle_\Gamma}{\langle 1, 1\rangle_\Gamma} = \frac{1}{r} - \frac{1}{r}\frac{\langle 1, 1\rangle_\Gamma}{\langle 1, 1\rangle_\Gamma} = 0 \ .$$

Conversely, if $\nabla_N L = 0$, then

$$\nabla_N L = \kappa - \frac{2\pi}{L(\Gamma)} = 0 \ ,$$

which implies $\kappa = 2\pi/L(\Gamma) = 1/r > 0$, so Γ is a circle.

Figure 2 illustrates two examples of the projected gradient descent

$$\frac{\partial \phi}{\partial t} = (\kappa - \frac{2\pi}{L(\Gamma)})|\nabla \phi| \ , \tag{28}$$

for the problem (27).

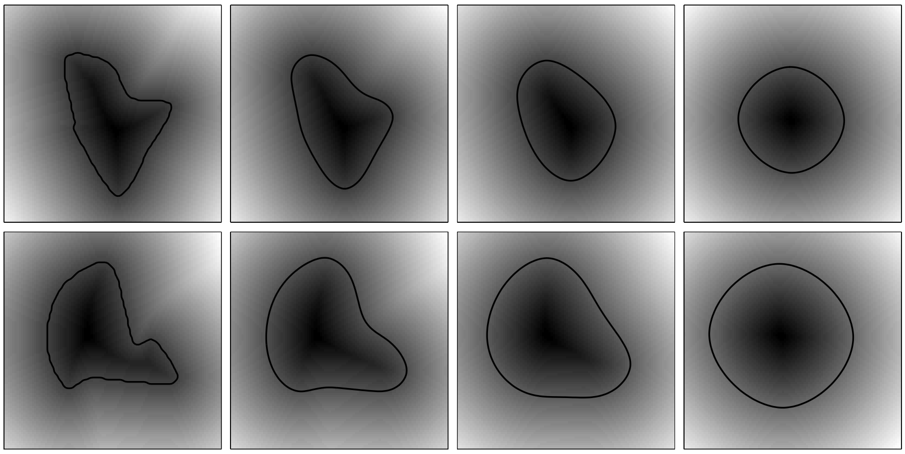

Fig. 2. Curve evolution for projected gradient descent of the isoperimetric problem. From left to right: initial shape, after 100 iterations, after 400 iterations and final shape for two randomly created planar curves. The value of the function ϕ is indicated in grayscale.

4.2 Practical Example: Visibility

This example concerns evolving surfaces in a surface fitting scheme such that they never violate visibility constraints cf. [4]. Some form of 3D data, e.g. unorganized points, is recovered from a sequence of images e.g. using structure from motion [13]. The goal is then to fit a surface to this data as a part of the scene reconstruction procedure. To do this the method from [14] can be used. In this case the functional is

$$F(\mathit{\Gamma}) = \int_{\mathit{\Gamma}} d(\mathbf{x})\, d\sigma \enspace , \tag{29}$$

where $d(\mathbf{x}) : \mathbf{R}^{m+1} \to \mathbf{R}$ is the distance potential to the data set. The gradient is $\nabla F = \nabla d \cdot \mathbf{n} + d\kappa$ from (16).

Let $W \subset \mathbf{R}^3$ denote the set corresponding to the forbidden regions in space determined from observations in the images. The following functional,

$$G(\mathit{\Gamma}) = \int_{\mathit{\Omega}} \chi_W \, d\mathbf{x} \enspace , \tag{30}$$

where $\mathit{\Omega}$ is the interior of $\mathit{\Gamma}$ and χ_W is a characteristic function for the set W, was used in [4] to detect if the visibility condition is violated. The problem of evolving $\mathit{\Gamma}$ such that no seen parts of the 3D data are occluded by the surface during evolution, leads to a variational problem of minimizing (29) under the constraint $G(\mathit{\Gamma}) = 0$.

The gradient of the constraint functional is $\nabla G = \chi_W$, by (16). This means that the projected gradient is

$$\nabla_N F = \nabla d \cdot \mathbf{n} + d\kappa - \frac{\langle \nabla d \cdot \mathbf{n} + d\kappa, \chi_W \rangle_\Gamma}{||\chi_W||_\Gamma^2} \chi_W \ ,$$

and the gradient descent evolution is

$$\frac{\partial \phi}{\partial t} = (\nabla d \cdot \mathbf{n} + d\kappa - \frac{\langle \nabla d \cdot \mathbf{n} + d\kappa, \chi_W \rangle_\Gamma}{||\chi_W||_\Gamma^2} \chi_W) |\nabla \phi| \, .$$

An example[2] reconstruction is shown in Figure 3 where the forbidden set W is the complement of the visual hull.

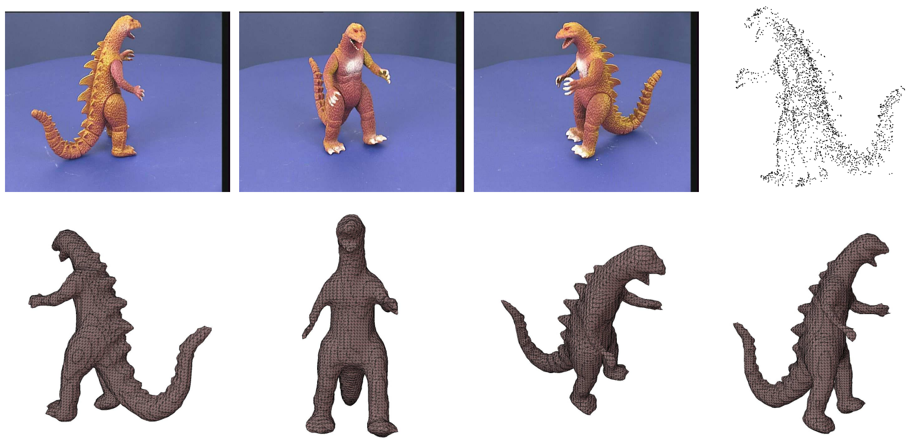

Fig. 3. Reconstruction from the Oxford dinosaur sequence. (top) Sample images recovered 3D points. (bottom) Four views of the reconstructed surface, using the visibility constraint.

5 Implementation Issues

This section deals with some practical considerations for implementing a projected gradient descent evolution, as described in Sections 3 and 4. There are some issues related to the fact that at the implementation stage we use finite resolution and numerical approximations.

When computing the projected gradient, one needs to compute surface (and volume) integrals such as the curve length $L(\Gamma)$ for the isoperimetric problem.

[2] The example is from [4].

This is not trivial to do in the implicit level set representation. Sometimes it is enough to compute these values with approximations using the Dirac and Heaviside functions as in e.g. [3,15]. If more accurate values are needed, methods like the marching cubes algorithm [16] can be used. Whatever the choice, there will be small errors in these computed values.

Another issue is that of satisfying the constraints. Only the gradient of the constraint functional appear in the evolution, not the value. In the isoperimetric example, the value of $A(\Gamma)$ never appear in the evolution equation (28). During the decent, finite step lengths and numerical errors in the calculations of the curve length $L(\Gamma)$ may introduce a "drift" in the value of $A(\Gamma)$, so that the constraint $G(\Gamma) = A(\Gamma) - A_0 = 0$ fails to hold after a while. One way to counter this drift is to add the gradient of a second order term, $G(\Gamma)^2/2$, to the projected gradient $\nabla_N F$, such that the evolution becomes $\dot{\Gamma}(t) = -v(\Gamma)$ with

$$v(\Gamma) = \nabla_N F + G\,\nabla G \ . \tag{31}$$

For the isoperimetric problem (27) this is

$$v(\Gamma) = \nabla_N L + (A(\Gamma) - A_0)1 \ . \tag{32}$$

The extra term in $v(\Gamma)$ is zero if the constraint $G(\Gamma) = 0$ holds, otherwise this term will try to restore the constraint.

Adding this term to the N-gradient of L does not change the extremals. In fact, the stationary points of the new augmented evolution are the same as for the one with $v(\Gamma) = \nabla_N L$.

Proposition 3. *$v(\Gamma) = 0$ in (32) if and only if $G(\Gamma) = A(\Gamma) - A_0 = 0$ and $\nabla_N L(\Gamma) = 0$.*

Proof. If $v(\Gamma) = 0$ then pairing with $\nabla G = 1$ yields

$$\begin{aligned} 0 = \langle v(\Gamma), 1\rangle_\Gamma &= \langle \kappa - \frac{2\pi}{L(\Gamma)} + (A(\Gamma) - A_0)1, 1\rangle_\Gamma \\ &= \langle \kappa - \frac{2\pi}{L(\Gamma)}, 1\rangle_\Gamma + (A(\Gamma) - A_0)\langle 1, 1\rangle_\Gamma = 0 + (A(\Gamma) - A_0)L(\Gamma) \ . \end{aligned}$$

Since $L(\Gamma) \neq 0$ this implies that $G(\Gamma) = A(\Gamma) - A_0 = 0$, and consequently $\nabla_N L(\Gamma) = 0$ by (32). In particular, Γ is a circle (by Proposition 2). The other direction is trivial.

This result is in fact true for the general case (31). The proof is a straight-forward adaption of the proof of Proposition 3 and is left to the reader.

6 Conclusions

In this paper we introduced a geometric infinite-dimensional gradient projection method for variational problems with constraints, as an extension of the finite-dimensional theory. Using a scalar product on the manifold of admissible m-surfaces we showed that gradient descent for the Lagrange method is equivalent to an orthogonal projection on the tangent space of the constraint manifold. We gave examples of how to use this theory in practice together with some useful ideas for implementation. This includes a way of stabilizing the evolution by augmenting the normal velocity. We also prove that this modified evolution still solves the original problem.

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