Exam MEK4250
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1 Weak formulation and finite element error estimate

1.1 Finite element formulation for Poisson equation

Strong form:

$$
\begin{aligned}
-\nabla \cdot (\kappa \nabla u) &= f \quad in\ \Omega \\
u &= u_0 \quad on\ \partial\Omega_D \\
-\kappa \frac{\partial u}{\partial n} &= g \quad on\ \partial\Omega_N
\end{aligned}
\tag{1}
$$

To obtain a FEM formulation we can use Galerkins method. Multiply with a test function $v \in V_{0,D}$, meaning that they are chosen such that they are zero on the Dirichlet boundaries, and perform integration by parts (IBP). First we obtain the weak formulation

$$
\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx + \int_{\partial\Omega_N} g v dx \tag{2}
$$

The finite element formulation can now be formulated as:
find $u_h \in V_h$ such that:

$$
a(u_h, v) = L(v) \quad \forall v \in V_{0,D} \tag{3}
$$

where

$$
\begin{aligned}
a(u_h, v) &= \int_\Omega \nabla u_h \cdot \nabla v dx \\
L(v) &= \int_\Omega f v dx + \int_{\partial\Omega_N} g v dS
\end{aligned}
\tag{4}
$$

We see from the equation that the least criteria we can have for V is H^1.

1.2 Show that Lax-Milgram is satisfied

Lax-Milgram is a set of three conditions that must be satisfied in order for the problem to be well-posed. The theorem states:

Lax-Milgram theorem
Let V be a Hilbert Space, $a(\cdot,\cdot)$ a bilinear form, $L(\cdot)$ a linear form and let the following three conditions be satisfied:

$$a(u,u) \geq \alpha||u||_V^2 \quad \forall u \in V \tag{5}$$

$$a(u,v) \leq C||u||_V||v||_V \quad \forall u,v \in V \tag{6}$$

$$L(v) \leq D||v||_V \quad \forall v \in V \tag{7}$$

Then the problem
find $u \in V$ such that $a(u,v) = L(v)$ is well posed, meaning there exists an unique solution, and have the following stability condition

$$||u|| \leq \frac{C}{\alpha}||L||_{V^*} \tag{8}$$

In order to prove that the Lax-Milgram theorem is satisfied I will use some mathematical inequalities. The Cauchy-Schwarz (C-S) inequality;

$$|(u,v)| \leq ||u|| \cdot ||v|| \tag{9}$$

The Poincare inequality (PC):

$$||u||_0 \leq C|u|_1 \tag{10}$$

I also use the fact that an H^1 norm returns a greater value than a L^2 norm.

Starting with the first criteria which is called coersivity or positivity. The left hand side reads

$$\begin{aligned}\alpha||u||_{H^1}^2 &= \alpha\left(||u||_{L^2}^2 + ||\nabla u||_{L^2}^2\right)\\ &\overset{\text{PC}}{\leq} \alpha\left(C_1^2||\nabla u||_{L^2}^2 + ||\nabla u||_{L^2}^2\right)\\ &= \alpha\left(C_1^2+1\right)||\nabla u||_{L^2}^2\end{aligned} \tag{11}$$

This leaves us with

$$a(u,u) = \int \kappa\nabla u^2 \geq \alpha\left(C_1^2+1\right)\int \nabla u^2 \tag{12}$$

For this to be true, κ must be a positive definite matrix and all $eig(\kappa) \geq \alpha(1 + C_1^2)$.

The second condition is called boundedness of a, and the left hand side looks like

$$\begin{aligned}
a(u, v) &\leq |(\kappa\nabla u, \nabla v)| \\
&\overset{\text{C-S}}{\leq} ||\kappa\nabla u||_{L^2}||\nabla v||_{L^2} \\
&\leq ||\kappa||_{op}||\nabla u||_{L^2}||\nabla v||_{L^2} \\
&= |\nabla u|_{H^1}|\nabla v|_{H^1} \\
&\leq C||u||_{H^1}||v||_{H^1}
\end{aligned} \tag{13}$$

The second condition is then satisfied for all constants $C \geq 1$. The third condition:

$$\begin{aligned}
L(v) &\leq |(f, v)|_\Omega + |(g, v)|_{\partial\Omega_N} \\
&\overset{\text{C-S}}{\leq} ||f||_{L^2(\Omega)}||v||_{L^2(\Omega)} + ||g||_{L^2(\partial\Omega_N)}||v||_{L^2(\partial\Omega_N)} \\
&\overset{\text{trace}}{\leq} ||f||_{L^2(\Omega)}||v||_{L^2(\Omega)} + C_2||g||_{L^2(\partial\Omega_N)}||v||_{L^2(\Omega)} \\
&= \left(||f||_{L^2(\Omega)} + C_2||g||_{L^2(\partial\Omega_N)}\right)||v||_{L^2} \\
&\leq \left(||f||_{L^2(\Omega)} + C_2||g||_{L^2(\partial\Omega_N)}\right)||v||_{H^1} \\
&\leq D||v||_{H^1}
\end{aligned} \tag{14}$$

The third condition is satisfied for all $D \geq ||f||_{L^2(\Omega)} + C_2||g||_{L^2(\partial\Omega_N)}$.

1.3 Extensions

If we consider the linear elasticity equation

$$2\mu\nabla \cdot \epsilon(u) + \lambda\nabla(\nabla \cdot u) = f \tag{15}$$

Since $\epsilon(u)$ is Cauchy's stress tensor we know that the first term will be a Poisson equation. For the second term we must consider what u can be. We know that any function $u \in L^2$ can be written as a composition of a gradient and a curl. We also know that

$$\begin{aligned}
\nabla \cdot curl &= 0 \\
rot\nabla &= 0
\end{aligned} \tag{16}$$

If we rewrite the Laplacian operator as

$$\Delta u = \nabla(\nabla \cdot u) - curl(rot(u)) \tag{17}$$

This gives us that

$$\begin{aligned} \nabla(\nabla \cdot u) &= \Delta u, \quad if\ u = gradient \\ \nabla(\nabla \cdot u) &= 0 \quad if\ u = curl \end{aligned} \tag{18}$$

This basically gives us that the linear elasticity equation can be written as something like

$$(\mu + \lambda)\Delta u = f \tag{19}$$

which is the Poisson equation. We have already proved that Lax-Milgram is satisfied for this.

1.4 A priori error estimate for FEM in energy norm

Energy norm is defined as

$$||e||_E = \sqrt{a(e,e)} \tag{20}$$

where $e = u - u_h$.

Inserted we get

$$\begin{aligned} ||e||_E^2 &= a(e, u - u_h) \\ &= a(e, u - v) - a(e, u_h - v) \\ &= a(e, u - v) \\ &\overset{\text{C-S}}{\leq} \sqrt{a(e,e)}\sqrt{a(u-v, u-v)} \\ &= ||e||_E ||u - v||_E \end{aligned} \tag{21}$$

Hence we have obtained an error estimate

$$||e||_E \leq ||u - v||_E \tag{22}$$

This tells us that the error is less or equal to all linear combinations in V_h and thus makes the FEM solution the optimal solution in the energy norm.

By using a interpolation result and setting $v = \pi_h u$

$$\begin{aligned} ||u - u_h|| &\leq ||u - \pi_h u|| \\ &\leq C(p,q)||h^{q+1-p} D^{q+1} u|| \end{aligned} \tag{23}$$

For piecewise linear functions we then obtain a priori error estimate

$$||e||_E \leq Ch||D^2 u|| \tag{24}$$

1.5 Convergence rate

Compute for different mesh sizes, various h, and then find the convergence rate by the formula

$$r = \frac{log\left(error_i/error_{i-1}\right)}{log\left(h_i/h_{i-1}\right)} \tag{25}$$

2 Discretization of Converction-Diffusion

Strong form:

$$
\begin{aligned}
-\mu\Delta u + w\cdot\nabla u &= f \quad in\ \Omega \\
u &= u_0 \quad on\ \partial\Omega_D
\end{aligned} \tag{26}
$$

2.1 Derive proper variational formulation

We multiply with a test function $v \in V$ and perform IBP. The problem then becomes
Find $u \in V$ such that

$$
a(u,v) = L(v) \quad \forall v \in V_{0,D} \tag{27}
$$

where

$$
\begin{aligned}
a(u,v) &= \mu\int_\Omega \nabla u\cdot\nabla v dx + \int_\Omega w v\cdot\nabla u dx \\
L(v) &= \int_\Omega f v dx
\end{aligned} \tag{28}
$$

2.2 Prove that Lax-Milgram is satisfied

The problem is well posed, there exists an unique solution, if the three Lax-Milgram conditions are satisfied.
For the first one, coersivity, we start with the right hand side we can use the result from the previous exercise

$$
\alpha||u||_{H^1}^2 \leq \alpha\left(1 + C_1^2\right)||\nabla u||_{L^2}^2 \tag{29}
$$

When we tackle the left hand side $a(u,u)$ we divide it into two terms. We call the first one $b(u,u)$ and the second $c(u,u)$. Starting with b:

$$
b(u,u) = \mu\int \nabla u^2 = \mu||\nabla u||_{L^2}^2 \tag{30}
$$

For c, we assume incompressible fluid, $\nabla\cdot w = 0$ and that $v \in V_{0,D}$.

$$
\begin{aligned}
c(u,u) &= \int wu\cdot\nabla u \\
&= \int w\nabla u^2 - \int wu\cdot\nabla u \\
&= wu^2\big|_{\partial\Omega} - \int u^2\nabla\cdot w - \int wu\cdot\nabla u \\
&= -c(u,u)
\end{aligned} \tag{31}
$$

The last jump is due to incompressibility of w and Dirichlet boundary conditions in v. The only value $c(u,u)$ can have to fulfil this is zero. Thus we end up with $a(u,u) = b(u,u)$ and the condition is satisfied when

$$\alpha \leq \frac{\mu}{1 + C_1^2} \tag{32}$$

The second condition, boundedness, starting the left hand side

$$\begin{aligned} a(u,v) &\leq |(\mu \nabla u, \nabla v)| + |(wv, \nabla u)| \\ &\overset{\text{C-S}}{\leq} \mu ||\nabla u||_{L^2} ||\nabla v||_{L^2} + ||w||_\infty ||\nabla u||_{L^2} ||v||_{L^2} \\ &\leq \mu ||\nabla u||_{L^2} ||\nabla v||_{L^2} + C_2 ||w||_\infty ||\nabla u||_{L^2} ||\nabla v||_{L^2} \\ &= (\mu + C_2 ||w||_\infty) ||\nabla u||_{L^2} ||\nabla v||_{L^2} \\ &\leq (\mu + C_2 ||w||_\infty) ||u||_{H^1} ||v||_{H^1} \\ &\leq C ||u||_{H^1} ||v||_{H^1} \end{aligned} \tag{33}$$

The condition is satisfied for $C \geq \mu + C_2 ||w||_\infty$.
The last condition is satisfied as long as f is bounded in V

$$L(v) \overset{\text{C-S}}{\leq} ||f||_{L^2} ||v||_{L^2} \leq D ||v||_{H^1} \tag{34}$$

and as long as $D \geq ||f||_{L^2}$.
Lax-Milgrams conditions are satisfied and the problem is well posed.

2.3 Oscillations and SUPG

To see why and how oscillations can appear we use an easy example. The 1D convection diffusion equation with $f = 0$. For this problem, the discretization, corresponds to the finite difference CD discretization.

$$-\mu \frac{u^{n+1} - 2u^n + u^{n-1}}{h^2} + w \frac{u^{n+1} - u^{n-1}}{2h} = 0 \tag{35}$$

for convection dominated problem, $\mu << w$, we have that $u^{n+1} = u^{n-1}$. This will give rise to oscillations. An easy fix for this is to use a forward euler or backward euler instead of CD. Using backward Euler we get

$$\frac{u^n - u^{n-1}}{h} = \frac{u^{n+1} - u^{n-1}}{2h} - \frac{h}{2} \frac{u^{n+1} - 2u^n + u^{n-1}}{h^2} \tag{36}$$

This gives us back our second derivative term that we lost due to low viscosity multiplied with a coefficient $h/2$. A solution to the problem would then be to add his to the equation. The problem then becomes

$$-(\mu + h/2) \Delta u + w \cdot \nabla u = f \tag{37}$$

This is what is called adding artificial diffusion. This takes care of the problem with oscillations, but it is not consistent. We no longer have our the same equation since what we have done is to add another term. Updating the weak formulation with added diffusion we get
Find $u \in V$ such that

$$a(u_h, v) + \frac{h}{2}(\nabla u, \nabla v) = (f, v) \tag{38}$$

If we look at the truncation error

$$\tau = a(u, v) - (f, v) \tag{39}$$

We see that the error will be of $\mathcal{O}(h)$. It is consistent in the sense that the error goes towards zero as h goes towards zero, but the error will be big for big h.
Solving this is done by a method called streamwise upwind Petrov Galerkin. Instead of adding diffusion ad-hoc, we include it in our test function. We now use discrete spaces $V_{u_0,D}$ and $W_{0,D}$. If we write the right hand side we get from standard Galerkin on matrix form we get

$$A_{ij} = a(N_i, N_j) = -\mu \int \nabla N_i \cdot \nabla N_j + \int w \cdot \nabla N_i N_j \tag{40}$$

For SUPG we get

$$A_{ij} = a(N_i, L_j) = -\mu \int \nabla N_i \cdot \nabla L_j + \int w \cdot \nabla N_i L_j \tag{41}$$

where our new test function L_j is defined as

$$L_j = N_j + \beta w \cdot \nabla N_j \tag{42}$$

Inserted in the equation we get a right hand side

$$\begin{aligned} A_{ij} = &- \mu \int \nabla N_i \cdot \nabla N_j + \int w \cdot \nabla N_i N_j \\ &- \mu \int \nabla N_i \cdot \nabla(\beta w \cdot \nabla N_j) + \int w \cdot \nabla N_i (\beta w \cdot \nabla N_j) \end{aligned} \tag{43}$$

We see that the two first terms are what we got with standard Galerkin method, the third term will vanish for 1'st order functions, and the fourth term is artificial diffusion in the direction of w. The right hand side looks like

$$b(L_j) = \int f N_j + \int f(\beta w \cdot \nabla N_j) \tag{44}$$

We see that we now have included artificial diffusion in a consistent way.

2.4 Cea's lemma

By using the results from Lax-Milgrams theorem we get that

$$\begin{aligned} ||e||_1^2 &\leq \frac{1}{\alpha}a(e,e) \\ &= \frac{1}{\alpha}a(e,u-v) \\ &\leq \frac{C}{\alpha}||e||_1||u-v||_1 \end{aligned} \tag{45}$$

Hence, we end up with

$$||u-u_h||_V \leq \frac{C}{\alpha}||u-v||_V \tag{46}$$

Using the interpolation results and $v = \pi_h u$ we obtain

$$||u-u_h||_1 \leq \frac{BC}{\alpha}h^t||u||_{t+1} \tag{47}$$

We see now that if we introduce the coefficients from the Lax-Milgram theorem we can write this as something like

$$||u-u_h||_1 \leq B\frac{\mu + C_2||w||_\infty}{\mu}h^t||u||_{t+1} \tag{48}$$

Here we see that if the velocity if much greater than the viscosity, the error will increase also.

This is an error estimate for the standard Galerkin method. For SUPG we introduce an alternative norm

$$||u||_{sd} = \left(h||v\cdot\nabla u||^2 + \mu||\nabla u||^2\right)^{1/2} \tag{49}$$

Assuming that the Lax-Milgram conditions are still satisfied we can obtain an error estimate for SUPG

$$||u-u_h||_{sd} \leq Ch^{3/2}||u||_2 \tag{50}$$

3 Discretization of Stokes

Stokes problem

$$
\begin{aligned}
-\mu \Delta u + \nabla p &= f \quad in\ \Omega \\
\nabla \cdot u &= 0 \quad in\ \Omega \\
u &= u_0 \quad on\ \partial\Omega_D \\
\frac{\partial u}{\partial n} - p\vec{n} &= g \quad on\ \partial\Omega_N
\end{aligned} \tag{51}
$$

3.1 Derive proper variational formulation

We use Galerkins method. This means that we multiply the main equation with a test function $v \in H_0^1$ and the continuity equation with a test function $q \in L^2$ and then perform IBP. This gives us a variational problem:
Find $u \in H^1_{u_0,D}$ and $p \in L^2$ such that

$$
\begin{aligned}
a(u,v) + b(p,v) &= L(v) \quad \forall v \in H^1_{0,D} \\
b(u,q) &= 0 \quad \forall q \in L^2
\end{aligned} \tag{52}
$$

where

$$
\begin{aligned}
a(u,v) &= \mu\,(\nabla u, \nabla v) \\
b(p,v) &= (p, \nabla \cdot v) \\
L(v) &= (f,v)_\Omega + (g,v)_{\partial\Omega_N} \\
b(u,q) &= (\nabla \cdot u, q)
\end{aligned} \tag{53}
$$

Note: we have let the $b(u,v)$ operator "swallow" the negative sign in order to achieve symmetry in our system.

3.2 Brezzi conditions

There are four Brezzi conditions. The three first is positivity of $a(u,u)$ and boundedness of $a(u,v)$ and $b(p,v)$. The last is the famous Babuska-Brezzi condition which is more complicated. We start with listing them

$$
\begin{aligned}
&1) \quad a(u,u) \geq \alpha ||u||_1 \\
&2) \quad a(u,v) \leq C_1 ||u||_1 ||v||_1 \\
&3) \quad b(p,v) \leq C_2 ||p||_0 ||v||_1 \\
&4) \quad sup_v \frac{(q, \nabla \cdot v)}{||v||} \leq \beta ||p||_0 < 0
\end{aligned} \tag{54}
$$

We start with number 1, positivity of a. We have that

$$
\begin{aligned}
\alpha||u||_1^2 &= \alpha(||u||_0^2 + ||\nabla u||_0^2) \\
&\overset{\text{PC}}{\leq} \alpha(c^2+1)||\nabla u||_0^2 \\
&\leq \mu||\nabla u||_0^2
\end{aligned}
\tag{55}
$$

This condition is then satisfied as long as $\alpha \leq \frac{\mu}{1+c^2}$. The second condition, boundedness of a, goes like

$$
\begin{aligned}
a(u,v) &\overset{\text{C-S}}{\leq} \mu||\nabla u||_0||\nabla v||_0 \\
&\leq \mu||u||_1||v||_1 \\
&\leq C_1||u||_1||v||_1
\end{aligned}
\tag{56}
$$

This is then satisfied as long as $C_1 \geq \mu$. The third condition, boundedness of b;

$$
\begin{aligned}
b(p,v) &\overset{\text{C-S}}{\leq} ||p||_0||\nabla \cdot v||_0 \\
&\leq ||p||_0||\nabla u||_0 \\
&\leq ||p||_0||u||_1
\end{aligned}
\tag{57}
$$

This is satisfied as long as $C_2 \geq 1$. The fourth condition is not really clear what it means, but it has something to do with the eigenvalues of b transposed being positive. This has also something to do with oscillations in the pressure.

3.3 Pressure oscillations

To see what this can mean we write out the equation in a matrix form

$$
\begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} L \\ 0 \end{bmatrix}
\tag{58}
$$

This linear system gives us two equations

$$
\begin{aligned}
Au + Bp &= L \\
B^T u &= 0
\end{aligned}
\tag{59}
$$

To obtain an equation for the pressure we write the first equation as

$$
u = A^{-1}L - A^{-1}Bp \tag{60}
$$

and insert it in the second equation and get

$$B^T A^{-1} B p = B^T A^{-1} L \tag{61}$$

This equation is called the schur compliment. This equation is solvable if $B^T A^{-1} B$ is invertible. We already know from the first three Brezzi conditions that A is an invertible matrix. This means that we need to find out if $B^T B$ is invertible. This is equivalent to kernel(B^T)=0. This means that the only vector that will take B^T to the zero room is the zero vector. This means that we have no zero columns of rows in B^T. And by the fourth Brezzi condition we should have something like the eigenvalues being positive. This would mean that $B^T B$ is invertible. But further inspection can be done. We introduce $u = \sum_{i=1}^n u_i N_i$, $v = N_j$, $p = \sum_{i+1}^m p_i L_i$ and $q = L_j$. Here, n is the degrees of freedom for velocity and m is the degrees of freedom for the pressure. From our matrix system we have that the A matrix is a square matrix, nxn. The B matrix is mxn, B^T is nxm and the zero matrix is mxm. In order to ensure that the zero vector is small we need that n is much larger than m. This gives us that we need more dofs for velocity than for pressure. The factor between n and m is given by the fourth Brezzi condition. When it is satisfied, $B^T B$ is invertible.

3.4 Finite elements

We have a optimal convergence approximation that yields

$$||u - u_h||_1 + ||p - p_h||_0 \leq Ch^k ||u||_{k+1} + Dh^{l+1} ||p||_{l+1} \tag{62}$$

A very used element that is used is the Taylor-Hood element which is a P2-P1 element giving second order functions for velocity and first order for pressure. We see from our error estimate that this will have second order convergence. The Crouzeix-Raviart is linear both in velocity and pressure. It satisfies the fourth condition but is only first order convergent. The mini element is also linear in velocity and pressure. Usually these elements wont satisfy the fourth condition but with the extra bubble in the middle it does.

3.5 Circumvent the sup-inf

A way to circumvent the sup-inf condition is to eliminate the zero matrix. A way to do this is to fill it up with something. We do this by replacing our equation set by

$$\begin{aligned} Au + Bp &= L \\ B^T u - \epsilon D p &= \epsilon d \end{aligned} \tag{63}$$

where ϵ is a small number. We now need to figure out what D is. We do the same as earlier and end up with an pressure equation

$$(-\epsilon D - B^T A^{-1} B)p = B^T A^{-1} L + \epsilon d \tag{64}$$

If D is non singular then $(B^T A^{-1} B + \epsilon D)$ will be non singular since both D and $B^T A^{-1} B$ is positive (only D is definite). We can factorize for p and get an equation for u as well. Here we will also obtain a non singular matrix in front of u. Three techniques is listed in the compendium to stabilize:
Pressure stabilization:

$$\nabla \cdot u = \epsilon \Delta p \tag{65}$$

This gives us $D = A$ where A is the stiffness matrix (discrete laplace operator).
Penalty method:

$$\nabla \cdot u = -\epsilon p \tag{66}$$

This gives us $D = M$ where M is the mass matrix.
Last one is pressure correction:

$$\nabla \cdot = \epsilon \frac{\partial p}{\partial n} \tag{67}$$

This gives $D = \frac{1}{\Delta t} M$.

4 Discretization of Navier-Stokes

Navier-Stokes equation can be written as

$$
\begin{aligned}
\frac{\partial u}{\partial t} + u \cdot \nabla u &= -\frac{1}{\rho}\nabla p + \nu \nabla^2 u + g \\
\nabla \cdot u &= 0
\end{aligned}
\tag{68}
$$

4.1 Operator splitting vs Algebraic splitting

For our purposes, the main difference is that for operator splitting, we discretize in time prior to space where as for algebraic splitting we discretize in space prior to time. For algebraic splitting we can achieve a mixed formulation scheme.

4.2 Operator splitting and boundary conditions

We start of with a simple forward Euler scheme

$$
\begin{aligned}
u^{n+1} &= u^n + dt\left(-u^n \cdot \nabla u^n - \frac{1}{\rho}\nabla p^n + \nu \nabla^2 u^n + g^n\right) \\
\nabla \cdot u^{n+1} &= 0
\end{aligned}
\tag{69}
$$

There are several problems with this scheme. First of all, continuity of u^{n+1} is not satisfied, and we have no way of determining p^{n+1}.
One way of dealing with this problem is to introduce two new velocities. First, we guess an initial tentative velocity

$$
u^* = u^n + dt\left(-u^n \cdot \nabla u^n - \frac{\beta}{\rho}\nabla p^n + \nu \nabla^2 u^n + g^n\right) \tag{70}
$$

where β is a constant that needs to be determined. We also update u^{n+1} to include updated pressure

$$
u^{n+1} = u^n + dt\left(-u^n \cdot \nabla u^n - \frac{1}{\rho}\nabla p^{n+1} + \nu \nabla^2 u^n + g^n\right) \tag{71}
$$

The second velocity we introduce is a correction velocity in order to ensure that we have continuity for the updated velocity.

$$
u^c = u^{n+1} - u^* = -\frac{dt}{\rho}\left(\nabla p^{n+1} - \beta \nabla p^n\right) \tag{72}
$$

and we define $\phi = p^{n+1} - \beta p^n$. By utilizing the continuity of u^{n+1} we can take the divergence of the correction velocity to obtain

$$\nabla \cdot u^c = -\nabla \cdot u^* = -\frac{dt}{\rho}\Delta\phi \tag{73}$$

We now have a Poisson equation for the pressure

$$\Delta\phi = \frac{\rho}{dt}\nabla \cdot u^* \tag{74}$$

We then continue with

$$p^{n+1} = \phi + \beta p^n \tag{75}$$

and

$$u^{n+1} = u^* - \frac{dt}{\rho}\nabla\phi \tag{76}$$

We now have a scheme that can be solved in four steps. The problem is the Poisson equation for the pressure. It demands one boundary condition for each boundary where as Navier-Stokes equation only demands one. We are in other words short of boundary conditions. We can create some new boundary conditions but these will cause problems in the boundary layers. Two ways to do this is :
First we can dot the full Navier-Stokes equation with a normal vector. This will give us a Neuman boundary condition for the pressure. Additional problem with this is that N-S is not valid on the boundaries but we now force it to be.
The second way to do this is to create some Dirichlet conditions by setting

$$\nabla\phi\big|_{\partial\Omega} = \frac{\rho}{dt}\left(u^* - u^{n+1}\right)\big|_{\partial\Omega} = 0 \tag{77}$$

This must equal zero since u^* and u^{n+1} have the same boundary conditions. This leaves the derivative of ϕ is zero and thus ϕ must be constant on the boundary.
Now we have created some boundary conditions for the Poisson equation but this leaves the N-S equation overdetermined. This will cause problems in the boundary layer. This scheme is fairly unstable meaning *Deltat* must be small compared to Δx for high velocities.

4.3 Algebraic splitting

We can use standard Galerkin method, multiply N-S with test function $v = N_j \in H_0^1$ and the continuity equation with test function $q = L_j \in L^2$. We then perform IBP to obtain an equation set on the form;
Find $u = \sum u_i N_i \in H_0^1$ and $p = \sum p_i L_i \in L^2$ such that

$$\begin{aligned} M\dot{u} + K(u)u &= -Qp + Au \\ Q^T u &= 0 \end{aligned} \tag{78}$$

Here, $M = \int N_i Nj$ is the mass matrix, $A = \int \nabla N_i \nabla N_j$ is the stiffness matrix. Q is a gradient operator and Q^T is a divergence operator. $K(u)$ is something like $u \cdot \nabla$.
Using a classical scheme called the theta-scheme, where we have discretized in time, we can obtain a mixed formulation of linear equations on the form

$$\begin{aligned} Nu^{n+1} + \Delta t Q p^{n+1} &= q \\ Q^T u^{n+1} &= 0 \end{aligned} \tag{79}$$

where

$$\begin{aligned} N &= M + \theta \Delta t R(u^n) \\ R(u^n) &= K(u^n) - A \\ q &= (M - (1-\theta)\Delta t R(u^n))u^n + \Delta t f^{n+1} \end{aligned} \tag{80}$$

This system is non symmetric and indefinite and is the worst kind to solve. This can be solved with a projection scheme like we did with the operator splitting but here we go for a fully implicit scheme with pre conditioners. Solving the first equation for u^{n+1} (and assuming N is invertible), we end up with

$$Q^T N^{-1} Q p^{n+1} = \frac{1}{\Delta t} Q^T N^{-1} q \tag{81}$$

This is called the Schur complement pressure equation. Here we can use Richardson iterations which becomes

$$p^{n+1,k+1} = p^{n+1,k} - C^{-1}\left(Q^T N^{-1} Q p^{n+1,k} - \frac{1}{\Delta t} Q^T N^{-1} q\right) \tag{82}$$

Here C^{-1} is a pre conditioner. We can do the same with the velocity and we end up with

$$\begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} N & Q \\ Q^T & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} q \\ 0 \end{bmatrix} \tag{83}$$

here, we see that C_1 should be something like N^{-1} and C_2 should be something like $(Q^T N^{-1} Q)^{-1}$. We have no optimal spectral equivalent pre conditioner for this problem.

5 Iterative methods

We want to solve a system of linear equations on the form

$$Au = b \tag{84}$$

We have that A is an nxn matrix and that it is invertible. This means that this system can be solved directly by a finite number of operations. But however this make take a long time. Even though A is sparse, A^{-1} might be dense and the operations is somewhere around $\mathcal{O}(N^2)$ $\mathcal{O}(N^3)$. In order to avoid this, we might use a preconditioner. This is simply multiplying our system with a matrix that resembles A^{-1}. But first we look at an iterative method called Richardson iteration.

5.1 Richardson iteration

The Richardson iteration is

$$u^{n+1} = u^n - \tau(Au^n - b) \tag{85}$$

where τ is a relaxation factor. We see that this is convergent in that sense that if u^n is the exact solution then the residual will be zero and thus $u^{n+1} = u^n$. No further iterations will improve the solution and we have convergence. The system now requires $\mathcal{O}(N)$ operations and memory for each iteration. This is much better than Gaussian elimination. To figure out how well this works we must look at the error. We define $e^{n+1} = u^{n+1} - u$. We simply subtract u from each side of the original equation and end up with

$$e^{n+1} = e^n - \tau Ae^n \tag{86}$$

We put this into an L^2 norm

$$||e^{n+1}||_0 = ||e^n - \tau Ae^n||_0 \leq ||I - \tau A||_0||e^n||_0 \tag{87}$$

We see that if the convergence factor $||I - \tau A|| < 1$ this will be convergent. Assuming we know what A is and that $eig(A) = [\lambda_0, ..., \lambda_N]$ are organized in rising order then the norm of $||I - \tau A||$,

$$||I - \tau A|| = max_x \frac{||(I - \tau A)x||}{||x||} \tag{88}$$

will be obtained either by the smallest or the biggest eigenvalue $1 - \tau\lambda_0$ or $-(1 - \tau\lambda_N)$. If we have these equal each other we will have a optimal τ as the eigenvalues will have the same value on the edges. This gives us

$$\tau_{opt} = \frac{2}{\lambda_N + \lambda_0} \tag{89}$$

The convergence factor with the optimal relation will then be

$$||I - \tau A|| = max_i(1 - \tau\lambda_i) = 1 - \tau_{opt}\lambda_0 = \frac{\lambda_N - \lambda_0}{\lambda_N + \lambda_0} \tag{90}$$

We introduce now the condition number κ which here equals $\kappa = \frac{\lambda_N}{\lambda_0}$. We now have

$$\rho = ||I - \tau A|| = \frac{\kappa - 1}{\kappa + 1} \tag{91}$$

If we insert this back into our error estimate we now have

$$||e^{n+1}|| \leq \frac{\kappa - 1}{\kappa + 1}||e^n|| = \left(\frac{\kappa - 1}{\kappa + 1}\right)^n ||e^0|| \tag{92}$$

For an iterative method we never iterate until true convergence, but until we have reached a set convergence criteria. We define

$$\frac{||e^{n+1}||}{||e^0||} \leq \epsilon \leq \left(\frac{\kappa - 1}{\kappa + 1}\right)^n \tag{93}$$

Taking the logarithm on both sides we have convergence in

$$n = \frac{log(\epsilon)}{log\left(\frac{\kappa-1}{\kappa+1}\right)} \tag{94}$$

iterations. But what is this dependent on? If we take an example, the Poisson equation. Then the A matrix will give us eigenvalues $\lambda = \frac{4}{h^2} sin^2\left(\frac{k\pi h}{2}\right)$. This gives us a $\lambda_{min} = \pi^2$ and a $\lambda_{max} = \frac{4}{h^2}$. A's condition number $\kappa(A) = \frac{4}{\pi^2 h^2}$ We see now that the condition number is mesh dependent, and that if we refine the mesh we will need more iterations until we reach convergence. This is now NOT order optimal. This might be solved by preconditioning.

5.2 Preconditioning

The idea is to replace our system

$$Au = b \tag{95}$$

with

$$BAu = Bb \tag{96}$$

Both systems will have the same solution if B is non singular but B should be chosen such that BA has a smaller condition number that A. If we choose $B = A^{-1}$ we would have convergence in one iteration but this preconditioner is hard to calculate. We will therefore seek a preconditioner that is $\mathcal{O}(N)$ in both memory and evaluation. The three conditions we state for out preconditioner is
1) B should be spectral equivalent with A^1.
2) The evaluation of B on a vector, Bu, should be $\mathcal{O}(N)$.
3) The storage of B should be $\mathcal{O}(N)$.

5.3 Spectral equivalence

Definition, *Two linear operators or matrices A^{-1} and B that are symmetric and positive definite are spectral equivalent if*

$$c_1(A^{-1}v, v) \leq (Bv, v) \leq c_2(A^{-1}v, v) \quad \forall v \tag{97}$$

If A^{-1} and B are spectral equivalent then the condition number of the matrix BA is $\kappa(BA) \leq \frac{c_2}{c_1}$. If they are equal then we would have convergence in on iteration as the condition number would be one. But if they are spectral equivalent then we have a mesh independent condition number. This would give the same number of iterations for any refinement in the mesh.

5.4 Poisson equation

See exercise 8.6 in chapter 8.

6 Linear elasticity

We have from Newton's second law

$$
\begin{aligned}
-\nabla \cdot \sigma &= f \\
\sigma &= 2\mu\epsilon(u) + \lambda tr(\epsilon(u))\delta \\
\epsilon(u) &= \frac{1}{2}\left(\nabla u + \nabla u^T\right)
\end{aligned}
\tag{98}
$$

Where σ is the stress tensor. This gives us an equation for linear elasticity

$$
-2\mu\nabla \cdot \epsilon(u) - \lambda\nabla\left(\nabla \cdot u\right) = f \tag{99}
$$

There is possible to write this equation as a displacement equation as well. It comes on the form

$$
-\mu\Delta u - (\mu + \lambda)\nabla(\nabla \cdot u) = f \tag{100}
$$

6.1 Variational form

Using standard Galerkin we multiply with test function $v \in V$ and perform IBP

$$
\begin{aligned}
-\left(2\mu\nabla \cdot \epsilon(u), v\right) - \left(\lambda\nabla(\nabla \cdot u), v\right) &= (f, v) \\
(2\mu\epsilon(u), \nabla v) - (2\mu\epsilon(u) \cdot \vec{n}, v) + (\lambda(\nabla \cdot u), \nabla \cdot v) - (\lambda(\nabla \cdot u), v) &= (f, v) \\
(2\mu\epsilon(u), \nabla v) + (\lambda(\nabla \cdot u), \nabla \cdot v) &= (f, v) + (\sigma \cdot \vec{n}, v)
\end{aligned}
\tag{101}
$$

We now have a variational form for linear elasticity. We can set $\sigma \cdot \vec{n} = t$ as a Neuman condition (t for traction). All Dirichlet conditions will give a unique solution. A combination of Dirichlet and Neuman is also ok, but a pure Neuman problem will keep us from obtaining a unique solution. For the displacement equation we have a variational form of

$$
(2\mu\nabla u, \nabla v) + (\mu + \lambda)(\nabla \cdot u, \nabla \cdot v) = (f, v) \tag{102}
$$

6.2 Rigid motion

To demonstrate the problem with rigid motion we have in 2D

$$
u = a + c\begin{pmatrix} y \\ -x \end{pmatrix} \tag{103}
$$

It is easy to see that the divergence of u now is zero. Also, if we insert it into $\epsilon(u)$ we get that

$$\epsilon(u) = \frac{1}{2}\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) \tag{104}$$

This is also zero. Then we have our equation on the form

$$\begin{aligned} -2\mu\nabla\cdot\epsilon(u) - \lambda\nabla(\nabla\cdot u) &= f \\ 0 - 0 &= f \end{aligned} \tag{105}$$

Hence f must be zero and we have no equation. This obviously is a solution but we need Dirichlet conditions in order to ensure a unique solution.

6.3 Korn's lemma

Korn's lemma states that our problem is well-posed if we have Dirichlet conditions. It thereby states that Lax-Milgram's theorem is satisfied if

$$||\epsilon(u)||_0 \geq C||u||_1 \tag{106}$$

We see if this helps us. Condition number one

$$a(u,u) \geq \alpha||u||_1^2 \tag{107}$$

We start with writing

$$\begin{aligned} a(u,u) &= 2\mu||\epsilon(u)||_0^2 + \lambda||\nabla\cdot u||_0^2 \\ &\geq 2C^2\mu||u||_1^2 + \lambda||\nabla\cdot u||_0^2 \end{aligned} \tag{108}$$

Since the L^2 norm of the divergence is positive this is clearly satisfied when

$$\begin{aligned} &\alpha \leq 2C^2\mu \\ &\lambda > 0 \end{aligned} \tag{109}$$

For the second condition we use the displacement formulation of the equation. The condition is

$$a(u,v) \leq D||u||_1||v||_1 \tag{110}$$

This is easy with the displacement form and if we use the fact that $||\nabla u||_0 \geq ||\nabla\cdot u||_0$. This can be proven by the identity of the laplace operator.

$$\Delta = \nabla\nabla\cdot - \nabla x\nabla x \tag{111}$$

We now use IBP with homogenous Dirichlet conditions to obtain

$$(\nabla u, \nabla u) = (\nabla \cdot u, \nabla \cdot u) + \nabla x u, \nabla x u) \tag{112}$$

Now we can show

$$\begin{aligned} a(u,v) &\overset{\text{C-S}}{\leq} \mu||\nabla u||_0||\nabla v||_0 + (\mu+\lambda)||\nabla \cdot u||_0||\nabla \cdot v||_0 \\ &\leq \mu||\nabla u||_0||\nabla v||_0 + (\mu+\lambda)||\nabla u||_0||\nabla v||_0 \\ &\leq (2\mu+\lambda)||u||_1||v||_1 \end{aligned} \tag{113}$$

Hence this is satisfied when $D \geq 2\mu + \lambda$.

6.4 Locking

To figure out what can cause locking, which is a numerical artefact, which causes the compression of a substance to be smaller than what we would get in reality, we first revisit Cea's lemma. Since Lax-Milgram's theorem is satisfied we can use this. We have that Cea's lemma provides an error estimate on the form of something like

$$||e||_1 \leq B\frac{D}{\alpha}h^t||u||_0 \tag{114}$$

if we insert D and α from Lax-Milgram we get

$$||e||_1 \leq B\frac{2\mu+\lambda}{\mu}h^t||u||_0 \tag{115}$$

We see that for a given mesh size, large values of λ, meaning $\lambda >> \mu$ will give a big maximum of the error. We can also see this in the variational form. Since we haven't used any elements that model divergence well we have that the error they produce is enhanced by the big value of λ.

6.5 Mixed Formulation

A nice way to avoid locking is to redefine the divergence as

$$\nabla \cdot u = \frac{p}{\lambda} \tag{116}$$

Now we force the divergence to get smaller as λ gets larger. This also provides us with a mixed formulation which now can be written as

$$\begin{gathered} 2\mu(\epsilon(u), \epsilon(v)) + (p, \nabla v) = (f, v) \\ (\nabla \cdot u - \frac{p}{\nabla}, q) = 0 \end{gathered} \tag{117}$$

We recognize this form as Stokes problem for compressible fluids. This is solvable and gives good results for any λ. For linear elements it should give a convergence of second order.