# 3D Non-linear Wave Propagation by Spectral Element Method

**Filippo Gatti**[1,2]

[1]Laboratoire MSSMat - CentraleSupélec

[2]DICA - Politecnico di Milano

14 décembre 2016

# Set-up and Nomenclature

## PRELIMINARIES

$$\underline{\boldsymbol{I}} = \sum_{n=1}^{3} e_n \otimes e_n$$ 2$^{nd}$order Identity Tensor

$$\underline{\underline{\boldsymbol{A}}} = \underline{\underline{\boldsymbol{A}}}_D + \frac{1}{3}\mathcal{I}_1^A \underline{\underline{\boldsymbol{I}}}_2$$ Tensor decomposition : deviatoric and spherical part

$$\underline{\underline{\boldsymbol{A}}} : \underline{\underline{\boldsymbol{A}}} = Tr\left(\underline{\underline{\boldsymbol{A}}}\underline{\underline{\boldsymbol{A}}}^T\right)$$ 2$^{nd}$order Tensor Double Contraction

$$\mathcal{I}_1^A = Tr\left(\underline{\underline{\boldsymbol{A}}}\right) = \underline{\boldsymbol{I}} : \underline{\underline{\boldsymbol{A}}}$$ Tensor 1$^{st}$Invariant

$$\mathcal{J}_1^A = Tr\left(\underline{\underline{\boldsymbol{A}}}_D\right) = 0$$ Deviatoric Tensor 1$^{st}$Invariant

$$\mathcal{J}_2^A\left(\underline{\underline{\boldsymbol{A}}}\right) = \frac{1}{2}\underline{\underline{\boldsymbol{A}}}_D : \underline{\underline{\boldsymbol{A}}}_D$$ Deviatoric Tensor 2$^{nd}$Invariant

# Set-up and Nomenclature

## PRELIMINARIES

$$f\left(\underline{\underline{\boldsymbol{A}}}\right)$$ Tensor Valued Scalar Function

$$\underline{\underline{\boldsymbol{\nabla}}}_A f = \sum_{m,n=1}^{3} \frac{\partial f}{\partial A_{mn}} e_m \otimes e_n$$ Gradient of Tensor Valued Scalar Function

$$f\left(\underline{\underline{\boldsymbol{A}}};\underline{\underline{\boldsymbol{B}}}\right)$$ Tensor Bi-Valued Scalar Function

$$\mathcal{K}_1^{A-B} = \underline{\underline{\boldsymbol{A}}} : \underline{\underline{\boldsymbol{B}}}$$ Tensor $1^{st}$ Joint Invariant

$$\mathcal{K}_2^{A-B} = \underline{\underline{\boldsymbol{A}}} : \underline{\underline{\boldsymbol{B}}}^2$$ Tensor $2^{nd}$ Joint Invariant

$$\mathcal{K}_3^{A-B} = \underline{\underline{\boldsymbol{A}}}^2 : \underline{\underline{\boldsymbol{B}}}$$ Tensor $3^{rd}$ Joint Invariant

$$\mathcal{K}_4^{A-B} = \underline{\underline{\boldsymbol{A}}}^2 : \underline{\underline{\boldsymbol{B}}}^2$$ Tensor $4^{th}$ Joint Invariant

**Objectivity :** $f\left(\underline{\underline{\boldsymbol{Q}}}^T \underline{\underline{\boldsymbol{A}}}\,\underline{\underline{\boldsymbol{Q}}};\underline{\underline{\boldsymbol{Q}}}^T \underline{\underline{\boldsymbol{B}}}\,\underline{\underline{\boldsymbol{Q}}}\right) = f\left(\underline{\underline{\boldsymbol{A}}};\underline{\underline{\boldsymbol{B}}}\right) \forall \underline{\underline{\boldsymbol{Q}}} \in SO(3)$

$\rightarrow f\left(\underline{\underline{\boldsymbol{A}}};\underline{\underline{\boldsymbol{B}}}\right) = f\left(\mathcal{I}_m^A;\mathcal{I}_m^B;\mathcal{K}_m^{A-B}\right)$

# Set-up and Nomenclature

## PRELIMINARIES

$$f\left(\underline{\underline{\boldsymbol{A}}};\underline{\underline{\boldsymbol{B}}}\right)=f\left(\underline{\underline{\boldsymbol{A}}}-\underline{\underline{\boldsymbol{B}}}\right)$$ Tensor Bi-Valued Scalar Function

$$\mathcal{L}_1^{A-B}=\underline{\boldsymbol{I}}:\left(\underline{\underline{\boldsymbol{A}}}-\underline{\underline{\boldsymbol{B}}}\right)$$ Tensor $1^{st}$Invariant

$$\mathcal{L}_2^{A-B}=\underline{\boldsymbol{I}}:\left(\underline{\underline{\boldsymbol{A}}}-\underline{\underline{\boldsymbol{B}}}\right)^2$$ Tensor $2^{nd}$Invariant

$$\mathcal{L}_3^{A-B}=\underline{\boldsymbol{I}}:\left(\underline{\underline{\boldsymbol{A}}}-\underline{\underline{\boldsymbol{B}}}\right)^3$$ Tensor $3^{st}$Invariant

**Objectivity :** $f\left(\underline{\underline{\boldsymbol{Q}}}^T\underline{\underline{\boldsymbol{A}}}\underline{\underline{\boldsymbol{Q}}};\underline{\underline{\boldsymbol{Q}}}^T\underline{\underline{\boldsymbol{B}}}\underline{\underline{\boldsymbol{Q}}}\right)=f\left(\underline{\underline{\boldsymbol{A}}};\underline{\underline{\boldsymbol{B}}}\right)\forall\underline{\underline{\boldsymbol{Q}}}\in SO(3)$

$\rightarrow f\left(\underline{\underline{\boldsymbol{A}}};\underline{\underline{\boldsymbol{B}}}\right)=f\left(\mathcal{L}_m^{A-B}\right)$

# Set-up and Nomenclature

## PRELIMINARIES

$$f\left(\underline{\underline{\boldsymbol{A}}};\underset{\sim}{\boldsymbol{\xi}}\right)$$

$\underset{\sim}{\boldsymbol{\xi}}$ Structural Hidden Variables

$$\underset{\sim}{\boldsymbol{\xi}}=\left[\underline{\underline{\boldsymbol{A}}};\underline{\boldsymbol{a}};a\right]$$

Hidden Variables : Tensors,Vectors and Scalars

**Objectivity :** $f\left(\underline{\underline{\boldsymbol{Q}}}^{T}\underline{\underline{\boldsymbol{A}}}\underline{\underline{\boldsymbol{Q}}};\underline{\underline{\boldsymbol{Q}}}\underset{\sim}{\boldsymbol{\xi}}\right)=f\left(\underline{\underline{\boldsymbol{A}}};\underset{\sim}{\boldsymbol{\xi}}\right)\forall\underline{\underline{\boldsymbol{Q}}}\in SO(3)$

$\rightarrow\underline{\underline{\boldsymbol{Q}}}\underset{\sim}{\boldsymbol{\xi}}=\left[\underline{\underline{\boldsymbol{Q}}}^{T}\underline{\underline{\boldsymbol{A}}}\underline{\underline{\boldsymbol{Q}}};\underline{\underline{\boldsymbol{Q}}}\underline{\boldsymbol{a}};a\right]$

# Set-up and Nomenclature

## STRAIN

$\underline{\underline{\boldsymbol{\varepsilon}}} = \underline{\underline{\boldsymbol{e}}} + \frac{1}{3}\mathcal{I}_1^{\varepsilon}\underline{\underline{\boldsymbol{I}}}$ Deviatoric and Spherical decomposition

$\mathcal{I}_1^{\varepsilon} = Tr\left(\underline{\underline{\boldsymbol{\varepsilon}}}\right) = \varepsilon_{vol}$ $1^{st}$ Strain Tensor Invariant = Volumetric Strain

$\mathcal{J}_2^{\varepsilon}\left(\underline{\underline{\boldsymbol{\varepsilon}}}\right) = \frac{1}{2}\underline{\underline{\boldsymbol{e}}} : \underline{\underline{\boldsymbol{e}}}$ $2^{nd}$ Strain Tensor Invariant

$\underline{\underline{\dot{\boldsymbol{\varepsilon}}}} = \underline{\underline{\dot{\boldsymbol{\varepsilon}}}}^{el} + \underline{\underline{\dot{\boldsymbol{\varepsilon}}}}^{pl}$ Small Strain Additivity

**Assumption :** $\mathcal{I}_1^{\varepsilon^{pl}} = 0 \rightarrow \underline{\underline{\dot{\boldsymbol{\varepsilon}}}}^{pl} = \underline{\underline{\dot{\boldsymbol{e}}}}^{pl}$

# Set-up and Nomenclature

## HARDENING VARIABLES

$$\underset{\sim}{\boldsymbol{\eta}} = \left[\underline{\underline{\boldsymbol{\alpha}}}; r\right]$$ Kinematic Internal Variables

$$\underset{\sim}{\boldsymbol{\chi}}\left(\underline{\underline{\boldsymbol{\eta}}}\right) = \left[\underline{\underline{\boldsymbol{X}}}\left(\underline{\underline{\boldsymbol{\alpha}}}\right); R\left(r\right)\right]$$ Static Internal Variables

$\underline{\underline{\boldsymbol{X}}}\left(\underline{\underline{\boldsymbol{\alpha}}}\right)$ Back Stress (Kinematic Hardening)

$R\left(r\right)$ Yield Limit (Isotropic Hardening)

**Assumption :** $\underset{\sim}{\boldsymbol{\chi}} = -\dot{\lambda}\boldsymbol{\nabla}_{\underset{\sim}{\boldsymbol{\eta}}}\boldsymbol{g}$

# NL model constitutive equations

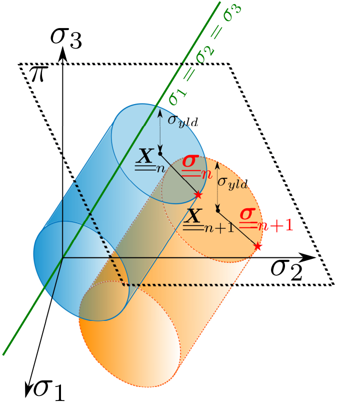



FIGURE 1 – Von Mises yield locus in principal stress 3D space

- Von Mises yield locus :

$$f\left(\underline{\underline{\boldsymbol{\sigma}}};\underset{\sim}{\chi}\right) \leq 0$$

$$f = \sqrt{3\mathcal{J}_2\left(\underline{\underline{\boldsymbol{\sigma}}}^D - \underline{\underline{\boldsymbol{X}}}\right)} - R(r)$$

$$\underline{\underline{\boldsymbol{D_\sigma f}}} = \frac{3}{2}\frac{\underline{\underline{\boldsymbol{\sigma}}}^D - \underline{\underline{\boldsymbol{X}}}}{\sqrt{3\mathcal{J}_2\left(\underline{\underline{\boldsymbol{\sigma}}}^D - \underline{\underline{\boldsymbol{X}}}\right)}}$$

$$\dot{\lambda} = \sqrt{\frac{2}{3}}\|\underline{\underline{\dot{\boldsymbol{\varepsilon}}}}^{pl}\| = \sqrt{\frac{2}{3}}\|\underline{\underline{\dot{\boldsymbol{e}}}}^{pl}\|$$

- Non-associative flow-rule

$$g\left(\underline{\underline{\boldsymbol{\sigma}}};\underline{\underline{\boldsymbol{\chi}}}\right) = f\left(\underline{\underline{\boldsymbol{\sigma}}};\underline{\underline{\boldsymbol{\chi}}}\right) + \underbrace{\frac{3}{4}\frac{\kappa}{C}Tr\left(\underline{\underline{\boldsymbol{X}}}.\underline{\underline{\boldsymbol{X}}}^T\right)}_{\text{fading memory}}$$

$$\underline{\underline{\dot{\boldsymbol{\varepsilon}}}}^{pl} = \underline{\underline{\dot{\boldsymbol{e}}}}^{pl} = \dot{\lambda}\underline{\underline{\boldsymbol{D_\sigma g}}}$$

# Nonlinear Hardening

- Prager kinematic hardening :

$$\underline{\underline{\boldsymbol{X}}} = \frac{2}{3} C \underline{\underline{\boldsymbol{\alpha}}} \rightarrow \underline{\underline{\dot{\boldsymbol{X}}}} = \frac{2}{3} C \underline{\underline{\dot{\boldsymbol{\alpha}}}}$$

$$\underline{\underline{\dot{\boldsymbol{\alpha}}}} = -\dot{\lambda} \underline{\underline{\boldsymbol{D_X g}}} = \dot{\lambda} \left( \underline{\underline{\boldsymbol{D_\sigma g}}} - \underbrace{\frac{3}{2} \frac{\kappa}{C} \underline{\underline{\boldsymbol{X}}}}_{\textbf{recall term}} \right)$$

- Isotropic hardening :

$$R = R(r) = R \left( \int_0^t \| \underline{\underline{\dot{\boldsymbol{\varepsilon}}}}^{pl} \| dt \right) = \sigma_{yld} + R_\infty \left( 1 - e^{-br} \right)$$

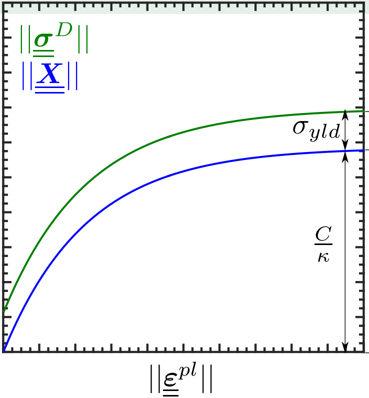
$$\dot{R} = b \left( R_\infty + \sigma_{yld} - R \right) \dot{r}$$

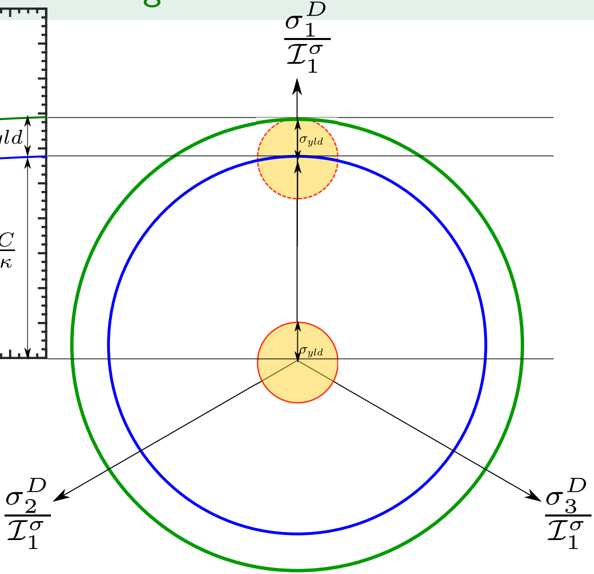
$$\dot{r} = -\dot{\lambda} \frac{\partial g}{\partial R} = \dot{\lambda} = \sqrt{\frac{2}{3}} \| \underline{\underline{\dot{\boldsymbol{\varepsilon}}}}^{pl} \|$$

**Deviatoric stress plane :**

- $\underline{\underline{\boldsymbol{X}}}$ : back-stress **centre** of moving yield locus
- $R$ : evolving **radius** of yield locus
- $\sigma_{yld}$ : first yielding limit

# Nonlinear Hardening



# Nonlinear Hardening

Hardening saturation

- Isotropic hardening
  $\dot{R} = 0 \rightarrow R = R_{\infty} + \sigma_{yld}$
- Kinematic hardening
  $\|\dot{\underline{\underline{\boldsymbol{X}}}}\| = 0 = \|\dot{\underline{\underline{\boldsymbol{\alpha}}}}\|$
  $\rightarrow \|\underline{\underline{\boldsymbol{X}}}\| = \frac{C}{\kappa}$

# Elastic-Plastic solution

- **Hardening modulus h**

$$h = C - \frac{3}{2} \frac{\left(\underline{\underline{\boldsymbol{\sigma}}}^D - \underline{\underline{\boldsymbol{X}}}\right) : \underline{\underline{\boldsymbol{X}}}}{\sqrt{3\mathcal{J}_2\left(\underline{\underline{\boldsymbol{\sigma}}}^D - \underline{\underline{\boldsymbol{X}}}\right)}} + b\left(R_\infty + \sigma_{yld} - R\right)$$

- **Plastic multiplier $\dot{\lambda}$**

$$\dot{\lambda} = \frac{\langle \underline{\underline{\boldsymbol{\nabla}}}_\sigma f : \mathbb{D}^{el} : \underline{\underline{\dot{\boldsymbol{\varepsilon}}}} \rangle}{h + \underline{\underline{\boldsymbol{\nabla}}}_\sigma f : \mathbb{D}^{el} : \underline{\underline{\boldsymbol{\nabla}}}_\sigma g}$$

- **Elastic-Plastic stiffness matrix $\mathbb{D}^{ep}$**

$$\underline{\underline{\dot{\boldsymbol{\sigma}}}} = \mathbb{D}^{ep} : \underline{\underline{\dot{\boldsymbol{\varepsilon}}}} \Longrightarrow \mathbb{D}^{ep} = \mathbb{D}^{el} - \frac{\mathbb{D}^{el} : \underline{\underline{\boldsymbol{\nabla}}}_\sigma g \otimes \mathbb{D}^{el} : \underline{\underline{\boldsymbol{\nabla}}}_\sigma f}{h + \underline{\underline{\boldsymbol{\nabla}}}_\sigma f : \mathbb{D}^{el} : \underline{\underline{\boldsymbol{\nabla}}}_\sigma g}$$

# Numerical implementation : key points

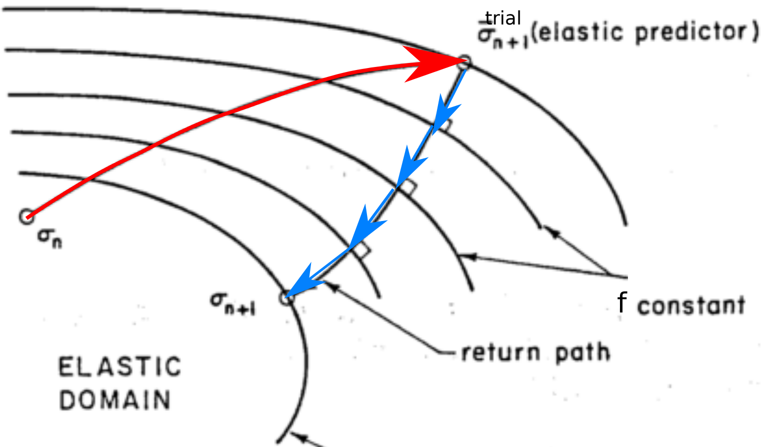
## Sub-stepping explicit algorithm ($\Delta t_n \ll$)

1. Updating current state : $t_n$
2. Strain increment : $\underline{\underline{\boldsymbol{\Delta\varepsilon}}}_n = \int_{t_n}^{t_{n+1}} \underline{\underline{\dot{\varepsilon}}}\left(s\right) ds = \frac{1}{\Delta t_n} \int_0^1 \left[\underline{\underline{\dot{\varepsilon}}}^{el}\left(s\right) + \underline{\underline{\dot{\varepsilon}}}^{pl}\left(s\right)\right] ds$
3. Elastic trial stress state prediction : $\underline{\underline{\boldsymbol{\Delta\sigma}}}_n^{trial} = \mathbb{D}^{el} : \underline{\underline{\boldsymbol{\Delta\varepsilon}}}_n$
4. Plasticity-Check : $f\left(\underline{\underline{\boldsymbol{\sigma}}}_n + \underline{\underline{\boldsymbol{\Delta\sigma}}}_n^{trial}; \underset{\sim}{\chi}_n\right)$
   - $f < 0$ : elastic step $\rightarrow$ next strain increment
   - $f \geq 0$ : elastic-plastic step $\rightarrow$ **PLASTIC CORRECTION !**

Plastic Correction :

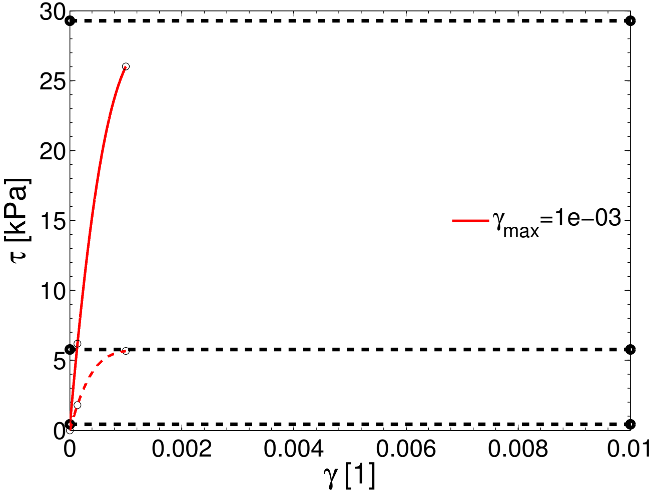
- sub-stepping method (radial return) **Sloan et al. 2001** :
  $f\left(\underline{\underline{\boldsymbol{\sigma}}}_n + \underline{\underline{\boldsymbol{\Delta\sigma}}}_n^{trial-k}; \underline{\underline{\chi}}_n\right) = 0$
- hardening update : $\underset{\sim}{\chi}_n \rightarrow \underset{\sim}{\chi}_{n+1}$
- drift correction : $f\left(\underline{\underline{\boldsymbol{\sigma}}}_n + \underline{\underline{\boldsymbol{\Delta\sigma}}}_n^{trial-k} + \underline{\underline{\boldsymbol{\Delta\sigma}}}_n^{drift}; \underset{\sim}{\chi}_{n+1}\right) = 0$

# Numerical implementation : key points



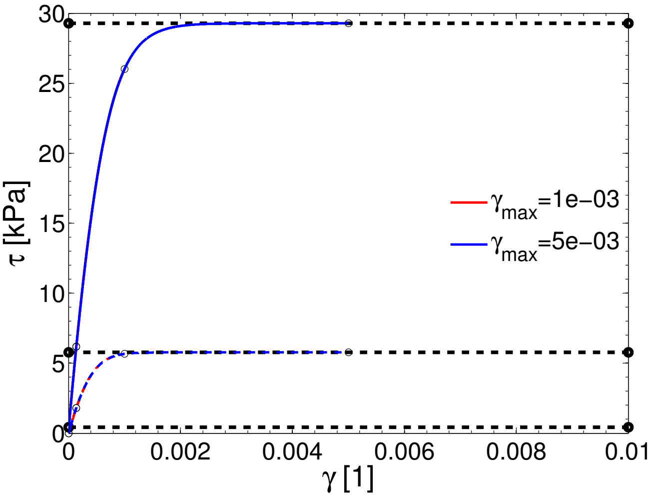
# Numerical example

## LINEAR MONOTONIC strain path



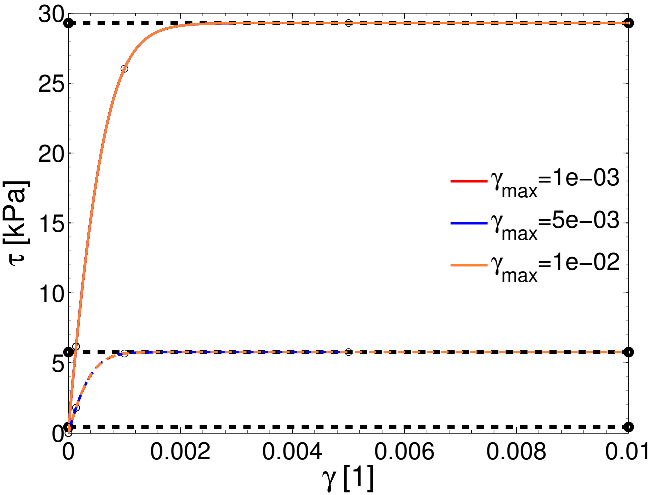
# Numerical example

## LINEAR MONOTONIC strain path



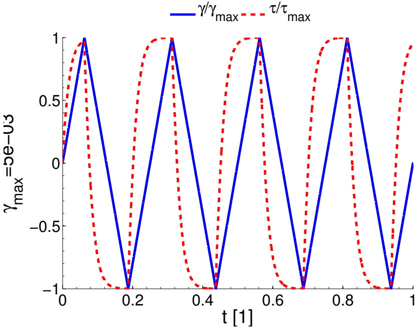
# Numerical example

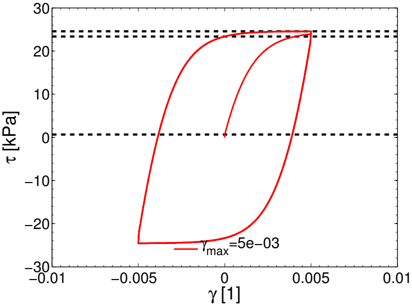
## LINEAR MONOTONIC strain path



# Numerical example

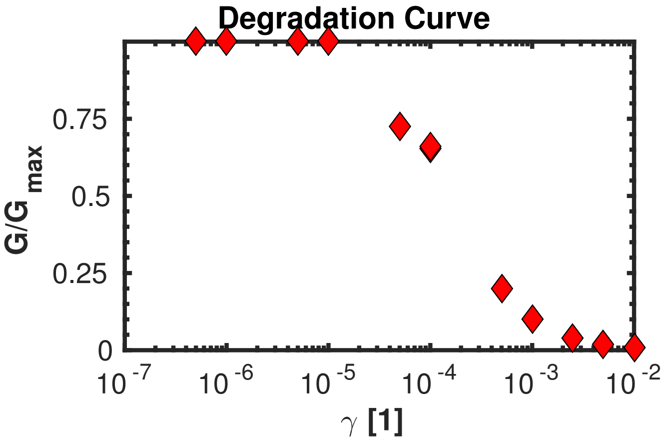
## SAW-TOOTH strain path



# Numerical example

## SAW-TOOTH strain path



# SEM implementation

## FOLLOW UP

- **3D implementation in SEM3D**
- **2D implementation in SPEED** (to be coupled with DRM ?)
- **Model calibration** for clays and sands
- **Validation & Verification** :
  - Soil column test
  - Non-linear random soil material

# GRAZIE DELL'ATTENZIONE
# MERCI DE VOTRE ATTENTION
# THANKS FOR THE ATTENTION

**Filippo Gatti**[3, 7]

**filippo.gatti@centralesupelec.fr**
**filippo.gatti@polimi.it**

GitHub

`https://github.com/FilLTP89`

# REFERENCES & ACKNOWLEDGEMENTS

JP Bardet, K Ichii, and CH Lin.
*EERA - A computer program for Equivalent-linear Earthquake site Response Analyses of layered soil deposits.*
University of Southern California, August 2000.

I.A. Beresnev and Kuo-liang Wen.
Nonlinear Soil Response-A Reality ?
*Bulletin of the Seismological Society of America*, 86(6) :1964–1978, 1996.

Filippo Gatti, Fernando Lopez-Caballero, and Didier Clouteau.
One-Dimensional Seismic Soil Response at the Nuclear Power Plant of Kashiwazaki-Kariwa during the 2007 Niigata-Chuetsu-Oki Earthquake.
In J. Kruis, Y. Tsompanakis, and B.H.V. Topping, editors, *Proceedings of the Fifteenth International Conference on Civil, Structural and Environmental Engineering Computing*, number 157, pages 1–16, Stirlingshire, UK., 2015. Civil-Comp Press.

I. M. Idriss.
Procedures for Selecting Earthquake Ground Motions at Rock Sites.
Technical report, 1991.

G Masing.
On Heyn's hardening theory of metals due to inner elastic stresses.
*Wiss. Veröff. Siemens-Konzern*, 3 :231–239, 1923.

Alain Pecker.
*Dynamique des Sols.*
Presse de l'École Nationale des Ponts et Chausses, 1984.

C. Smerzini, F. Gatti, and R. Paolucci.
An artificial neural network approach to support physics-based generation of broadband earthquake ground motions, Septembre 2016.

The work carried out under the SINAPS@ project has benefited from French funding managed by the National Research Agency under the program "Future Investments" - SINAPS@ reference No. ANR-11-RSNR-0022.