Elasticity
an introduction

Christian Pedersen

March 12, 2017

Linear elastostatics

Linear elastostatics can be described by the governing equations

$$f_b + \nabla \cdot \sigma = 0 \tag{1a}$$

$$\sigma = 2\mu\epsilon(u) + \lambda tr(\epsilon(u))\delta \tag{1b}$$

$$\epsilon(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right) \tag{1c}$$

The first equation is Newton's second law of mechanical equilibrium where f_b is the body force, i.e. gravity, and $\nabla \cdot \sigma$ is the contact force. The stress tensor σ is derived from Hooke's law where we have a linear relation between stress and strain u and $\epsilon(u)$ is Cauchy's strain tensor. By combining the equation set (1) we obtain

$$f_b + \mu\nabla^2 u + (\lambda + \mu)\nabla\nabla \cdot u = 0 \tag{2}$$

which is called the *Navier-Cauchy equilibrium equation.*

Finite element formulation

u is now expressed as a sum of piecewise linear basis functions $u = \sum_j c_j \psi_j$. After we multiply equation (2) with a test function v and perform integration by parts, a FEM formulation is;

Find $u \in H^1$ such that

$$a(u, v) = L(v)\ , \quad \forall v \in H^1 \tag{3}$$

where

$$\begin{aligned} a(u, v) &= \mu(\nabla u, \nabla v) + (\mu + \lambda)(\nabla \cdot u, \nabla \cdot v) \\ L(v) &= (f, v) \end{aligned} \tag{4}$$

Linear elastodynamics

Linear elastodynamics can be described as linear elastostatics with an additional inertia term

$$f_b + \mu\nabla^2 u + (\lambda + \mu)\nabla\nabla \cdot u = \rho\frac{\partial^2 u}{\partial t^2} \tag{5}$$