A Numerical Approach to the Shallow Water Wave Equations

Christian Pedersen

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The nondimensional shallow water wave equations (NLSW) are a set of partial differential equations on the form

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla \eta \tag{1}$$

$$\frac{\partial \eta}{\partial t} = -\nabla \cdot (\mathbf{u}\,(\eta + h)) \tag{2}$$

where $\eta(\mathbf{x}, t)$ is the surface elevation, $\mathbf{u}(\mathbf{x}, \mathbf{t})$ is the horizontal velocity and $h(\mathbf{x})$ is the unperturbed water depth. Eqn.(1) is the equation of motion and eqn.(2) is the equation of continuity. These equations are derived from the famous Navier-Stokes equation where the following assumptions and simplifications have been made; viscocity is neglected, the velocity is depth averaged, hydrostatic pressure, the wavelength is larger than the water depth and we have no dispersion.

1 Linear shallow water equations

By assuming that the surface elevation is sufficiently small we can linearize the NLSW to obtain the linear shallow water equations (LSW)

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla \eta \tag{3}$$

$$\frac{\partial \eta}{\partial t} = -\nabla \cdot (uh) \tag{4}$$

1.1 A finite difference approach

We start with a simple 1HD[1] example. We use a staggered grid discretization in both space and time to obtain the discrete LSW equations

$$\frac{u_j^{n+\frac{1}{2}} - u_j^{n-\frac{1}{2}}}{\Delta t} = -\frac{\eta_{j+\frac{1}{2}}^n - \eta_{j-\frac{1}{2}}^n}{\Delta x} \tag{5}$$

$$\frac{\eta_{j+\frac{1}{2}}^{n+1} - \eta_{j+\frac{1}{2}}^n}{\Delta t} = -\frac{u_{j+1}^{n+\frac{1}{2}} - u_j^{n+\frac{1}{2}}}{\Delta x} \tag{6}$$

[1] One horizontal dimension