The non-linear shallow water equations

Mathematical formulation

The scaled non-linear shallow water equations is a set of partial differential equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \eta \tag{1}$$

$$\frac{\partial \eta}{\partial t} = -\nabla \cdot [\mathbf{u}\,(\eta + h)] \tag{2}$$

where $\mathbf{u}$ is the wave velocity, η is the surface elevation and h is the water depth. Eq.1 is the equation of motion derived from Newtons second law of motion and Eq.2 is the continuity equation.

Finite element formulation

By implementing the Galerkin method, i.e. multiplying each equation with a test function and performing integration by parts (IBP), we can obtain a finite element formulation; Find $u_h \in H^1$ and $\eta_h \in L^2$ such that

$$a(u_h, v) + b(\eta_h, v) = 0 \qquad \forall v \in H^1 \tag{3}$$

$$c(u_h, q) + d(\eta_h, q) = 0 \qquad \forall q \in L^2 \tag{4}$$

where

$$\begin{aligned} a(u_h, v) &= \left(\frac{\partial u_h}{\partial t}, v\right)_\Omega + (u_h \cdot \nabla u_h, v)_\Omega \\ b(\eta_h, v) &= (\eta_h, v) \\ c(u_h, q) &= \\ d(\eta_h, q) &= \end{aligned} \tag{5}$$

Temporal discretization