PhD Thesis

Investigation of complex liquid-gas turbulent interfacial flows

A Numerical Study

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June 15, 2020

Institut Jean le Rond ∂'Alembert
Sorbonne Université

PhD Thesis

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The harmony of the world is made manifest in Form and Number, and the heart and soul and all the poetry of Natural Philosophy are embodied in the concept of mathematical beauty.

– D'Arcy Wentworth Thompson

Contents

List of Figures

List of Tables

Introduction | 1

Multiphase Flows

Brief description of multiphase flows in nature Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Surface tension dominated flows Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Fragmentation

Brief description of atomization Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Importance of drop size distributions Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Numerical Platforms

PARIS Simulator Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Basilisk Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Numerical Development

Methodology 2

In this chapter, we describe the basic numerical methodology behind our models concerning the dynamics of immiscible liquid-gas interfacial flows, at the incompressible and isothermal limits. These implementations are developed on the platforms 'PARIS Simulator' [1] and 'Basilisk' [2], with considerable overlap between the two platforms in terms of the treatment of the interface capturing schemes, transport of conserved quantities and surface tension models. [1] The numerical implementations are based on finite volume discretizations on uniform or dynamically refined Cartesian grids , utilizing state of the art methods in interfacial reconstruction coupled with geometric transport of the corresponding fluxes, curvature computation and surface tension modeling. For more detailed descriptions of the general capabilities of 'PARIS Simulator' and 'Basilisk', we refer the reader to the previously cited references.

1: The principle difference between 'PARIS Simulator' and 'Basilisk' is the ability to resolve the conservation laws on dynamically adaptive meshes in the case of 'Basilisk', whereas 'PARIS Simulator' only deals with regular Cartesian meshes.

2.1 Governing Equations

We use the one-fluid formulation for our system of governing equations, thus solving the incompressible Navier-Stokes equations throughout the whole domain including regions of variable density and viscosity, which itself depend on the explicit location of the interface separating the two fluids. In the absence of mass tranfer, the velocity field is continuous across the interface at the incompressible limit, with the interface evolving according to the local velocity vector.

Conservative Formulation

Generally, we have a choice regarding how to model the convective operator of the incompressible Navier-Stokes equations. There is a well established corpus of numerical methods tailored specifically to deal with non-conservative [2] form of the convective operator that appear in transport equations of mass and momentum [3] , which perform quite well in the context of single phase flows. However, in interfacial flows we often deal with discontinuities that arise as a consequence of the contrast in material properties between the two fluids. Therefore, even though the velocity field remains continuous throughout the domain, the otherwise smooth density (mass) and momentum fields contain sharp jumps (discontinuities) localized at the interfacial position. Therefore, we choose to formulate our governing equations in a conservative form i.e involving divergence of fluxes instead of gradients of the primitive variables when it comes to the convective operator. More detailed discussions and analyses about the comparative advantages of the conservative formulation in the context of flows involving large density-ratios is the focus of the subsequent chapters. Thus, the equations are as follows :

2: also referred to as the strong form, necessitating certain orders of smoothness of the primitive variable

3: These methods are descendants of the class of numerical schemes used to solve hyperbolic partial differential equations.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \boldsymbol{u}\right) = 0 \tag{2.1}$$

$$\frac{\partial}{\partial t}\left(\rho \boldsymbol{u}\right) + \nabla\left(\rho \boldsymbol{u} \otimes \boldsymbol{u}\right) = -\nabla p + \nabla \cdot \left(2\mu \boldsymbol{D}\right) + \sigma\kappa\delta_s \boldsymbol{n} + \rho \boldsymbol{g} \tag{2.2}$$

with ρ and μ being the density and dynamical viscosity respectively, and therefore are the physical quantities which are discontinuous across the interface. The volumetric sources are modeled by the acceleration g, and the deformation rate tensor $\boldsymbol{D}$ used to model the viscous stresses is defined as:

$$\boldsymbol{D} = \frac{1}{2}\left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right] \tag{2.3}$$

The term $\sigma\kappa\delta_s\boldsymbol{n}$ models the surface tension forces in the framework of the continuum surface-force (CSF) method. The normal vector to the interface is $\boldsymbol{n}$, with σ being the coefficient of surface tension and κ the local interfacial curvature. The operator δ_s is the Dirac delta function, the numerical approximation of which allows us to map the singular surface force distribution along the interface onto their volumetric equivalents for our Cartesian control volumes. At the incompressible limit, the advection of mass given by equation 2.1 can be treated as equivalent to that of the advection of volume.

Material Properties

Within the framework of interface capturing schemes, the temporal evolution of the interface separating the two fluids can be modeled by the following advection equation :

$$\frac{\partial \chi}{\partial t} + \boldsymbol{u}\nabla\chi = 0 \tag{2.4}$$

where χ is the phase-characteristic function, that has different values in each phase [4] . Mathematically, the function χ is equivalent to a Heaviside function in space and time. At the macroscopic length scales under consideration, the interface evolution as described by equation 2.4 is modeled as having infinitesimal thickness under the continuum hypothesis. The coupling of the interfacial evolution with the equations of fluid motion as described in 2.1 and 2.2 is provided by :

4: Generally, χ is assigned values of 0 in one phase and 1 in the other.

$$\rho = \rho_1\chi + (1-\chi)\,\rho_2 \tag{2.5}$$

$$\mu = \mu_1\chi + (1-\chi)\,\mu_2 \tag{2.6}$$

where ρ_1, ρ_2 are the densities of fluids 1 and 2 respectively, likewise for viscosities μ_1 and μ_2. Under certain circumstances, it is beneficial to opt for a weighted harmonic mean description of the variable dynamic viscosity, instead of the weighted arithmetic mean as in equation 2.6.

The two main (and the most popular) approaches in the context of interface capturing schemes are the are the volume-of-fluid (VOF) method first developed by Hirt and Nichols [3], and the level set class of methods pioneered by Osher and Sethian [4]. The principal difference between the two approaches lies in the manner in which the Heaviside function χ is modeled in the discrete sense, either as a smooth differentiable field in the case of level sets, or as a sharp discontinuous field in the volume-of-fluid (VOF) context. Each class of methods has its own set of merits (and demerits) relative to each other. Generally speaking, volume-of-fluid based methods display superior mass conservation [5] whereas in terms of interface curvature computation, level set based methods hold an advantage [6] . A detailed exposition into the different classes of interfacial transport methods can be found in the seminal monograph by Tryggvason, Scardovelli and Zaleski [5]

5: VOF based methods implicitly track the evolution of the discontinuous density field, which is not the case in level set based methods.

6: The differentiable nature of the level set function lends itself to straightforward curvature computation routines.

2.2 Interfacial Transport : VOF

Our numerical studies are based on the Volume-of-Fluid methodology. We refer to the discontinuous approximation to the Heaviside function [7] χ as the volume fraction field or colour function interchageably, which is defined below in the context of finite volume discretization :

7: A comprehensive discussion about the different types of approximations to the interface Heaviside function can be found in Popinet [6]

$$C_{ijk}(t) = \frac{1}{\Delta V} \int_{\Delta V} \chi(\boldsymbol{x}, t)\, d\boldsymbol{x} \tag{2.7}$$

where C is the colour function with its values lying between 0 and 1, with i,j and k being the indices to the corresponding discretized control volume of volume ΔV. There are two steps involved in the VOF method, the reconstruction of the interface and its subsequent propagation (advection). We present a brief overview of the two steps in the following sections, as going into detailed descriptions of the reconstruction and propagation procedures are not the focus of the present body of work [8] .

8: In-depth explanations into these numerical techniques can be found in [5, 7–9].

PLIC representation

We employ means of geometric reconstructions to explicitly define the interface location using the discrete colour function information. The interface is represented by disjointed line segments under the PLIC (piecewise linear interface construction) framework as illustrated in figure 2.1, with the images reproduced from the the review by Scardovelli and Zaleski [7] . Such reconstructions involve the determination of interface normals using the Mixed Youngs Centered method, the detailed description of which can be found in [5].

[7]: Scardovelli et al. (1999), 'Direct numerical simulation of free-surface and interfacial flow'

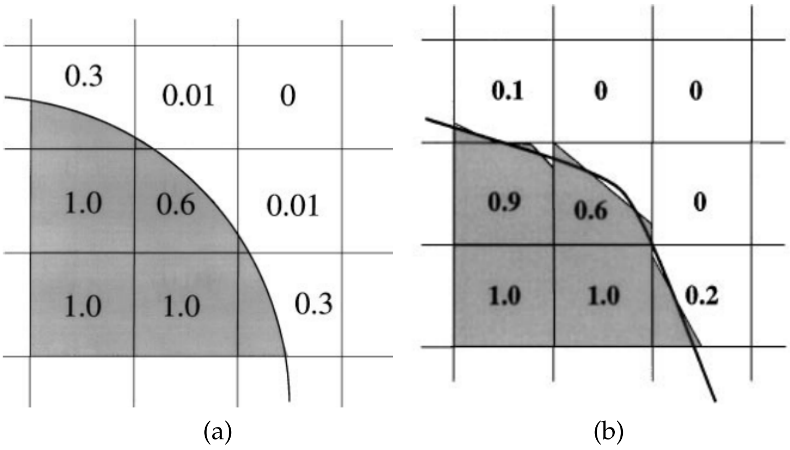

Figure 2.1: Exlicit definition of the interface location using the volume-of-fluid approach. These images are reproduced with permission from Scardovelli and Zaleski [7]. (a) The exact discrete representation of a circular arc on a regular Cartesian grid using the colour function field (volume fraction). (b) The piecewise linear (PLIC) approximation to the smooth circular arc shown in (a), which entails second-order spatial accuracy.

Flux Computation

Once the geometric PLIC reconstructions have been carried out, the interface segments are advected using the the velocity field. This entails computation of fluxes of the colour function, which can be computed via algebraic transport schemes (generally less accurate), or by using geometric reconstructions in either Eulerian, Lagrangian or hybrid frameworks. In the context of our numerical platforms ('PARIS' and 'Basilisk'), state-of-the-art [9] geometrical flux reconstruction procedues are utilised. The temporal integration of the fluxes could be carried out either as a series of one dimensional propagations along each of the spatial directions, termed as direction-split, or carried out in one single sweep, termed as multidimensional or unsplit.

9: The reader can refer to [1, 2] for further details

Direction-split methods are more intuitive and easier to develop (extension to 3D in particular), but suffer from lack of conservation (to the order of machine precision) when it comes to 3D [10] . Multidimensional (unsplit) methods have an advantage in that respect due to the fact that they are conservative by nature of their design, but are inherently more complicated to develop and implement, with no straightforward extension from 2D to 3D. For a more detailed and nuanced evaluation of the comparitive advantages of interfacial transport methods, we refer the reader to the recent review by Mirjalili et al. [11] on the given subject. The propagation of the interface can be described by the evolution of the colour function (volume fraction field) as -

10: A detailed exposition of this problem along with a noteworthy solution can be found in the work by Weymouth and Yue [10]

$$\frac{\partial C}{\partial t} + \boldsymbol{u} \cdot \nabla C = 0 \tag{2.8}$$

We can express the same in the conservative form as -

$$\frac{\partial C}{\partial t} + \nabla \cdot (C\boldsymbol{u}) = C\,(\nabla \cdot \boldsymbol{u}) \tag{2.9}$$

As one can observe, the "compression" term on the right hand side of equation 2.9 equals to zero in the context of incompressible flows without mass transfer, but it is important to keep this term in our numerical formulation within the direction-split framework. Discretization of the above equation results in :

$$C^{n,d+1}_{i,j,k} = C^{n,d}_{i,j,k} - F^{n,d}_{+}(C) - F^{n,d}_{-}(C) + \overline{C}^{n,d}_{i,j,k}\left(\frac{\Delta u_q}{\Delta x}\right)^{n,d}_{i,j,k} \tag{2.10}$$

The above equation represents an advection substep, the superscripts n and d refer to the timestep and direction of integration respectively. The notation $d = 0$ refers to the field at the n^{th} timestep, before any integration is performed along any direction. The fluxes $F^{n,d}_{\pm}(C)$ in equation 2.10 are derived through geometrical reconstructions [11] . The $+$ and $-$ subcripts refer to the orientation with respect to the central cell (i, j, k). The subscript q refers to the direction corresponding to that given advection substep i.e either X, Y or Z. In our case, our numerical platforms are based on cubic (regular Cartesian) grids, consequently there is no requirement for a subscript with Δx. After each substep, the interface is reconstructed once again with the updated volume fraction field in order to compute the fluxes for the next advection substep. Finally, the volume fraction field for the next timestep is given by -

11: For details regarding geometric flux reconstruction, refer to [5, 7]

$$C^{n+1}_{i,j,k} = C^{n,3}_{i,j,k} \tag{2.11}$$

The interpretation and numerical approximation of the prefactor $\overline{C}^{n,d}_{i,j,k}$ to the directional divergence, as well as the fluxes $F^{n,d}_{\pm}(C)$, depend on the exact nature of the geometrical advection scheme in question, which in our context is either CIAM (Lagrangian explicit) or Weymouth-Yue (Eulerian implicit) [12] . A brief outline of these two methods is presented in the subsequent sections.

12: The classification of Lagrangian explicit and Eulerian implicit are in accordance with the paper by Aulisa et al. [12]

Lagrangian Explicit This scheme was originally described in the work of Li [13], 'CIAM' being an abbreviation for the French title '*Calcul d'interface affine par morceaux*', which can be thought of as a straightforward Lagrangian transport of the interface Heaviside function. After the interface segments are reconstructed from the discrete colour function at the start of the time-step, the interfacial points are transported by the component of the velocity field corresponding to the direction of transport. A geometrical interpretation of the scheme is illustrated in figure 2.2, reproduced from the seminal work of Gueyffier et al. [14].

As one can infer from the geometrical representation in figure 2.2, the fluxes $F^{n,d}_{\pm}(C)$ correspond to the volumes Φ^- and Φ^+. Thus, the updated field $C^{n,d+1}_{i,j,k}$ is the sum of the three contributions Φ^- , Φ^0 and Φ^+. We can rewrite equation 2.10 specifically for the CIAM scheme as -

$$C^{n,d+1}_{i,j,k} = C^{n,d}_{i,j,k}\left[1 + \left(\frac{\Delta u_q}{\Delta x}\right)^{n,d}_{i,j,k}\right] - F^{n,d}_{+}(C) - F^{n,d}_{-}(C) \tag{2.12}$$

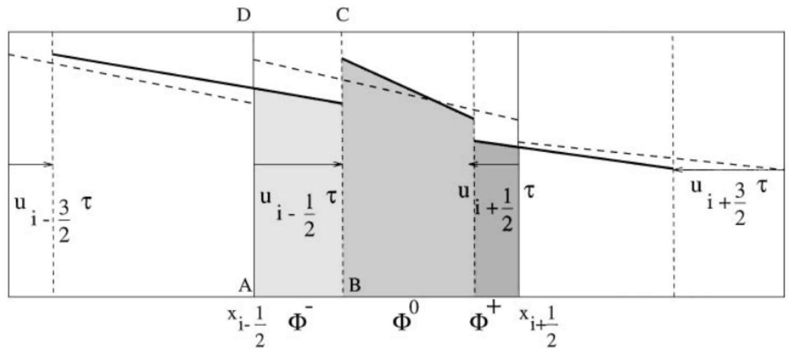

Figure 2.2: Lagrangian transport of the interface segments using the CIAM scheme, the image is reproduced with permission from Gueyffier et al. [14]. A 2D schemetic of the geometric calculation of the fluxes of the volume fraction field is shown, for the advection substep along the horizontal direction. The central cell (i, j, k) undergoes a net compression during this substep. The fluxes Φ^- , Φ^0 and Φ^+ are the volumes under the advected interface segments advected by the interpolated velocity field, intersected by the (i, j, k) cell boundaries.

where the compression coefficient $\overline{C}^{n,d}_{i,j,k}$ is simply equal to the value of the colour function $C^{n,d}_{i,j,k}$ at the start of the corresponding advection substep. Although the flux terms cancel upon integration throughout the whole domain, one can clearly see that the compression terms do not sum up to zero due to the changing prefactor in front of the directional divergences. This precise issue brings us to the next advection scheme.

Eulerian Implicit This advection scheme was developed by Weymouth and Yue [10] in order to specifically tackle the problem of discrete conservation when it comes to direction-split geometrical advection schemes [13]. The scheme fundamentally employs a forward Eulerian method in order to carry out temporal integration of the fluxes, with the fluxes themselves computed as the quantity of the substance entering or exiting a given control volume through its fixed surfaces, as shown in figure 2.3. This is in contrast with the flux computation method in the case of CIAM, where the interface segments are propagated forward in time in a Lagrangian fashion.

13: By 'discrete conservation' we mean that the sum of the directional divergences sum upto zero, to the accuracy of machine precision.

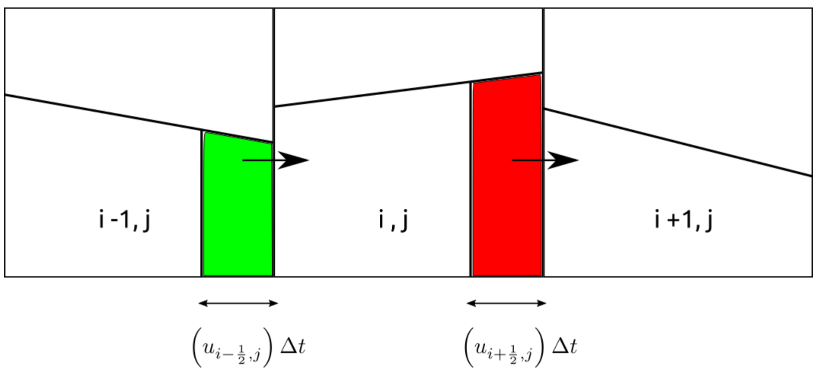

Figure 2.3: A 2D schemetic of the Eulerian (geometric) flux calculation using the Weymouth-Yue [10] scheme for the advection substep along the horizontal direction, with the interface reconstructed using the volume fraction field at the start of the substep. The colour fraction of the central cell (i, j) is updated during this substep through the addition of the fluxes (coloured regions), with the green polygon corresponding to the volume entering the cell i, j from the $i-1, j$ and the red one corresponding to that exiting i, j into $i+1, j$. The geometric flux calculations are made on the basis of the interfacial positions at the start of the substep, and the face centered velocities of the cell in question.

The subtle but important tactic used in this scheme lies in the manner in which the prefactor to the compression term [14] is treated, with its definition being :

14: Compression coefficient is used as a short-hand version of 'prefactor to the compression term'.

$$\overline{C}^{n,d}_{i,j,k} = H\left(C^{n,0}_{i,j,k} - 1/2\right) \tag{2.13}$$

where H is a one-dimensional Heaviside function. This renders the compression coefficient independent of the direction of the advection substep, consequently enabling the three discrete directional divergences to sum up to zero [15] Therefore, the scheme is able to demonstrate volume conservation, subject to local CFL restrictions [16] . To summarise, we can rewrite equation 2.10 for the Weymouth-Yue scheme as -

$$C_{i,j,k}^{n,d+1} = C\left[1 + \left(\frac{\Delta u_q}{\Delta x}\right)_{i,j,k}^{n,d}\right] - F_{+}^{n,d}(C) - F_{-}^{n,d}(C) \tag{2.14}$$

where C is a constant with a value of either 0 or 1, determined by the value of $C_{i,j,k}^{n,0}$ according to equation 2.13.

15: In numerical terms, we can only ensure that they sum up to the accuracy of the Poisson solver $\sim 10^{-3} - 10^{-6}$, with the limiting factor being the level of machine accuracy ($\sim 10^{-14} - 10^{-17}$).

16: For a proof of discrete volume conservation subject to certain CFL criteria, refer to the appendix of [10].

2.3 Time Marching

In order to describe the overall numerical algorithm for the one-fluid Navier-Stokes equations with variable density and viscosity, we choose to the reframe our equations in a more convenient operator form, as presented below :

$$\frac{\partial}{\partial t}(\rho \boldsymbol{u}) = L(\rho, \boldsymbol{u}) - \nabla p \tag{2.15}$$

The operator L in the above expression can be decomposed in an explicit fashion as :

$$L = L_{adv} + L_{\mu} + L_{\sigma} + L_{g} \tag{2.16}$$

where the L_{adv} represents the conservative advection, L_{μ} represents the diffusive forces generated by viscous stresses, L_{σ} represents the capillary forces arising from the surface tension model and finally L_g represents the volumetric (body forces) source term.

Spatio-temporal Discretization

We apply the spatially discretized versions of these operators (denoted by the superscript h) onto the primary variables ($C, \boldsymbol{u}$), and march forward in time using a small, possibly variable time-step τ such that $t_{n+1} = t_n + \tau$. We shall be dropping the subscript i, j, k from this point onwards, with the understanding that the operators in equation 2.16 apply uniformly to all control volumes. In the first part of the algorithm, the volume fraction field C^n is updated to the next timestep , with the superscript n signifying discretization in time. The operation can be written as follows -

$$C^{n+1} = C^n + \tau L^h_{vof}\left(C^n, \boldsymbol{u}^n\right) \tag{2.17}$$

The temporal evolution of the volume fraction field represented above by the operator L^h_{vof} is in accordance with the Lagrangian explicit or Eulerian implicit advection schemes, as described in the previous sections. Once we have obtained the updated field C^{n+1}, we can move on to the temporal update of our momentum field given by -

$$\begin{aligned} \rho^{n+1} \cdot \boldsymbol{u}^* = \rho^n \cdot \boldsymbol{u}^n + \tau L^h_{adv}\left(C^n, \boldsymbol{u}^n\right) + \\ \tau\left[L^h_\mu\left(C^{n+1}, \boldsymbol{u}^n\right) + L^h_\sigma\left(C^{n+1}\right) + L^h_g\left(C^{n+1}\right)\right] \end{aligned} \tag{2.18}$$

The advection operator L^h_{adv} is implemented as high-order spatial schemes coupled with a choice of non-linear flux limiters such as QUICK, ENO, WENO, Superbee, Verstappen, BCG, for regions of constant density [17] . For control volumes in the vicinity of the interface location, we revert to lower order schemes due to the sharp jumps in the material properties across the interface. The functionality of the operator L^h_{adv} near the interface is tighly coupled to that of L^h_{vof} from equation 2.17, so as to ensure consistency in the discrete advection of mass and momentum The details regarding this coupling shall be the focus of subsequent chapters, where it is covered in more depth.

17: These high-order spatial schemes are based on well established methods developed to deal with hyperbolic conservation laws, for more details refer to the studies of Leveque [15] and Sweby [16].

Pressure-Poisson Projection

The velocity field is evolved using a classical time-splitting projection method as described in the seminal work of Chorin [17], which involves predicting an 'intermediate' velocity field $\boldsymbol{u}^*$ as given by equation 2.18, followed by a correction step as follows -

$$\boldsymbol{u}^{n+1} = \boldsymbol{u}^* - \frac{\tau}{\rho^{n+1}}\nabla^h p^{n+1} \tag{2.19}$$

The discrete pressure field required to correct the intermediate velocity is determined by imposing the conservation of mass, which in our incompressible framework reduces to necessitating the resulting velocity field to be divergence-free (solenoidal) -

$$\nabla^h \cdot \boldsymbol{u}^{n+1} = 0 \tag{2.20}$$

Thus, combining equations 2.19 and 2.20, we are left with a variable coefficient Poisson equation for the pressure :

$$\nabla^h \cdot \left(\frac{\tau}{\rho^{n+1}}\nabla^h p^{n+1}\right) = \nabla^h \cdot \boldsymbol{u}^* \tag{2.21}$$

The default Poisson solver used in 'PARIS Simulator' to invert the elliptic operator appearing in eqn. 2.21 is a red-black Gauss-Seidel (GS) solver with overrelaxation [18] . There is also has an in-house implementation of a multigrid solver for structured grids with 2^n number of points per direction, utilizing a fully parallelized V-Cycle scheme [18]. Relaxation operations are applied starting from the finest to the coarsest first, and then from the coarsest to the finest, with the number of relaxation operations being a user-adjustable parameter. Having a native multigrid solver allows for an efficient solution of the Poisson equation without the necessity of having external libraries/pre-conditioners (e.g. HYPRE) installed on the system. When it comes to 'Basilisk', an atypical multigrid solver is implemented using a "half" V-cycle in order to deal with the spatial inhomogeity of the grid size arising due to adaptive mesh refinement. For more details regarding the differences between the multigrid solver of 'Basilisk' and the classical implementation of a multigrid, one can refer to [19] .

[18]: Briggs (1987), 'A Multigrid Tutorial, SIAM'

[19]: Popinet (2003), 'Gerris: a tree-based adaptive solver for the incompressible Euler equations in complex geometries'

The whole set of operations described up to this point, constitutes a temporal integration scheme of the first-order, which can be expressed as -

$$\left(C^{n+1}, \boldsymbol{u}^{n+1}\right) = L_1\left(C^n, \boldsymbol{u}^n\right) \tag{2.22}$$

where L_1 is the operator consisting of all the steps described so far, applied to the primary fields C and $\boldsymbol{u}$. Therefore, a second-order time integration can easily be computed by using L_1 to get a first prediction -

$$\left(C^{**}, \boldsymbol{u}^{**}\right) = L_1\left(C^n, \boldsymbol{u}^n\right) \tag{2.23}$$

The superscript $**$ refers to our first order prediction of the primary variables. Therefore, the second-order estimate can be obtained via averaging -

$$\left(C^{n+1}, \boldsymbol{u}^{n+1}\right) = \frac{1}{2}\left[\left(C^{**}, \boldsymbol{u}^{**}\right) + L_1\left(C^{**}, \boldsymbol{u}^{**}\right)\right] \tag{2.24}$$

2.4 Source Terms

The detailed descriptions of the methods used in our numerical platforms to deal with surface tension, viscosity and body forces have already been carried out in [1, 2, 20], therefore we briefly touch upon certain aspects of the operators in question, in particular, their interaction with the volume fraction field.

Surface Tension

We use the Continuum Surface Force method (CSF) as our model for surface tension, coupled with height functions for curvature computation. The height functions used in our implementation were first introduced in [20] , subsequently tested, revised and improved in [21, 22]. In general, the height functions are used to compute the curvature field based on second-order finite differences applied to the heights. Although, in regions of poor interfacial resolution [18] , the method reverts to certain fallbacks, one of which is curve fitting instead of height functions. The resulting curvature field is coupled with a well-balanced discretization with respect to the discrete pressure gradient, with the same discretization stencil applied to the volumetric (body) force term as well.

18: These are regions where the local radius of curvature is comparable to the grid size

Viscous Diffusion

We use second-order spatial discretizations of the viscous stresses, using centered differences [19] . The (variable) dynamic viscosity is computed based on the volume fraction field via weighted arithmetic or harmonic averaging (equations 2.6). The temporal treatment of the viscous term can be either in explicit or semi-implicit fashion, , but in the context of the present study we will be sticking exclusively with the explicit version.

19: The exact implementation differs slightly between 'Basilisk' and 'PARIS Simulator'

3 Artificial Atomization : The Falling Raindrop

3.1 Computational Setup

Parameterization : Reynolds, Weber, Bond Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

3.2 Exploration of Blowups

Combinations of Advection Scheme & Flux Limiters Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

3.3 Origin of Numerical Instabilities

Un-physical Stagnation Pressures Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

3.4 Effect of Mass-Momentum Consistent Transport

Stabilization of Numerical Instabilities Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Consistent Mass-Momentum Transport 4

4.1 Principles of Momentum Consistent Schemes

Major Iterations in Literature Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Overview of Methods Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Our Strategies Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

4.2 Consistent Flux Computation

Schematic Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Numerical Stencils Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

4.3 Reconstruction on Staggered Cells

Half-Fractions Method Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Sub-Grid Method Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

4.4 Sub-Grid Strategy

Consistency and Conservation Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Restriction and Prolongation Operators Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

4.5 Summary of Methods

Flowchart : Half-Fractions Method Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Flowchart : Sub-Grid Method Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Numerical Benchmarks | 5

In the upcoming sections, we demonstrate the robustness and accuracy of our class of mass-momentum consistent numerical methods when applied to challenging high density-ratio flow configurations, primarily in comparison to the version of our method which does not maintain consistency between the mass and momentum advection. Most of the standard tests that exist in the current literature concerning numerical methods to tackle liquid-gas flows such as the decay of spurious currents in static and moving droplets, viscous damping of capillary waves etc., are carried out in the absence of any density jump (or viscosity jump) across the interface separating the fluids. In this chapter, we shall take a closer look in detail at the behavior of our methods when dealing with difficulties that arise due to the non-linear coupling between interfacial deformation/propagation, capillary and viscous forces, especially in the regime where the material properties across the interface are separated by orders of magnitude, particularly in which the flow features in question are poorly resolved.

In order to assess the performance of the different methods, we shall use an easier nomenclature to describe the different methods, which are as follows :

- **M1** Method with non-consistent momentum-mass transport.
- **M2** Method with consistent momentum-mass transport, but not conservative. Uses half-fractions strategy.
- **M3** Method with consistent and conservative momentum-mass transport. Uses sub-grid strategy.

5.1 Static Droplet

A popular numerical benchmark in the existing literature relevant to surface tension dominated flows is the case of a spherical droplet of the denser fluid immersed in a quiescent surrounding medium of the lighter fluid. In the hydrostatic limit of the Navier-Stokes equations, the droplet should stay in equilibrium, with a curvature induced pressure jump across the interface corresponding to Laplace's equilibrium. In practice however, numerically reproducing such a trivial equilibrium condition is not as straighforward, as there exists a slight difference between the initial numerical interface and the exact analytical shape of the sphere, thereby resulting in the generation of the well documented '*spurious*' or '*parasitic*' currents of varying intensity in the velocity field [23–25] . A lot or progress has been made since in the context of *well-balanced* surface tension formulations, that ensure consistency between the numerical stencils used for the discretization of the pressure gradient and the Heaviside approximation ($n\delta_s$) that projects the the surface force distribution onto the control volumes [20, 26]. A significant contribution to the interpretation of these parasitic currents within the well-balanced

[23]: Lafaurie et al. (1994), 'Modelling merging and fragmentation in multiphase flows with SURFER'
[24]: Harvie et al. (2006), 'An analysis of parasitic current generation in volume of fluid simulations'
[25]: Popinet et al. (1999), 'A front-tracking algorithm for accurate representation of surface tension'

framework was made by Popinet [20] which demonstrated that given sufficient time (of the order of viscous dissipation time-scales), a well-balanced method will relax to the '*numerical*' equilibrium shape through the damping of the 'physically consistent' numerical capillary waves, therefore allowing us to recover the exact (to machine precision) Laplace equilibrium condition.

Setup

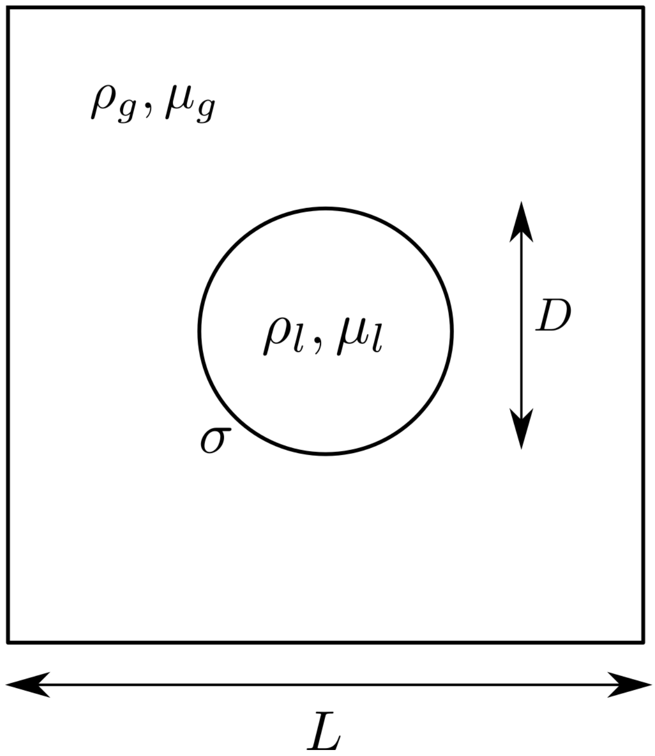

Figure 5.1: Schematic of the static droplet of dense fluid surrounded by a quiescent medium of lighter fluid. A 40×40 grid is employed to spatially discretize the domain.

The key difference in our implementation of this classic test case from that of Popinet [20] is that we consider the effect of density contrast across the interface separating the fluids. As we have previously discussed, a sharp density jump across the interface may have an amplification effect on the numerical errors incurred as a result of interfacial reconstructions, curvature estimation and various other truncations, thereby rendering the method unstable. We demonstrate that in our framework of mass consistent momentum transport coupled with a well-balanced surface tension discretization, density-ratios as large as 1000 : 1 can be simulated without loss of numerical stability, in conjuction with the ability to recover the exact numerical equilibrium through the dissipation of spurious currents within relevant time-scales [1] .

[20]: Popinet (2009), 'An accurate adaptive solver for surface-tension-driven interfacial flows'

1: The viscous time-scale corresponding to the droplet length-scale is the most commonly used in literature.

We consider a circular droplet of size D placed at the centre of a square domain of side L. The densities of the heavier and lighter phases are ρ_l and ρ_g respectively, likewise for the viscosities μ_l and μ_g, and σ being the surface tension coefficient (fig. 5.1). The ratio of the droplet size to the box is chosen as $D/L = 0.4$, coupled with a numerical resolution of $D/\Delta x = 16$ (where Δx is the grid size). As for boundary conditions, we use symmetry conditions on all sides of the square domain.

The problem incorporates two natural time-scales, the capillary oscillation scale and the viscous dissipation scale, which are defined below :

$$T_\sigma = \left(\frac{\rho_l D^3}{\sigma}\right)^{1/2} \quad , \quad T_\mu = \frac{\rho_l D^2}{\mu_l} \tag{5.1}$$

The ratio of these time-scales give us -

$$\frac{T_\mu}{T_\sigma} = \sqrt{\rho_l \sigma D}/\mu_l = \sqrt{La} \tag{5.2}$$

where La is the Laplace number based upon the heavier fluid. In the present study, we introduce the density-ratio ρ_l/ρ_g as another important parameter. In order to rescale our 'parasitic' velocity field, we define a velocity scale based on capillary oscillations as -

$$U_\sigma = \sqrt{\sigma/\rho_l D} \tag{5.3}$$

Additionally, the time-step in our numerical simulation must be smaller than the oscillation period corresponding to the grid wavenumber (fastest capillary wave with a time period $\sim \left(\rho_l \Delta x^3/\sigma\right)^{1/2}$) as a stability criterion [2] , as our surface tension model is explicit in time. For the scope of the present study, we shall not consider any viscosity contrast between the two fluids while varying the density-ratio, therefore $\mu_l/\mu_g = 1$ for all the cases under study.

2: Similar criteria are defined on the basis of the viscous and advection operators as well, with the smallest amongst the three selecting the numerical time-step

Decay of Spurious Currents

In figures 5.2 to 5.4, we illustrate the decay of the root-mean-square of the spurious currents as a function of time, in the case of four different density-ratios, with three different Laplace numbers for each ratio. The first figure (5.2) refers to simulations carried out without consistency between the momentum-mass transport (**M1**), the second (5.3) corresponds to that of the consistent but not conservative method (**M2**), and final one (5.4) refers to that of the consistent and conservative method (**M3**). The time is rescaled by the viscous dissipation scale, and the spurious currents by the capillary velocity scale. We have two main observations, the rapid decay of the rescaled spurious currents for all combinations of density-ratios and Laplace numbers within approximately $0.2T_\mu$, and the slower re-growth of the currents in question for combinations of non-unity density-ratios and large Laplace numbers, in all simulations except those carried out with **M3**. With method **M3**, the decayed currents keep hovering around levels of machine precision for remainder of time. Although there is a re-growth of the currents using the consistent method (**M2**) after $0.2T_\mu$, the behavior is not quite alarming as the rate of this re-growth is quite low. Therefore, out of all the methods tested, the consistent and conservative method (**M3**) does seem to demonstrate the desired performance, especially when it comes to combinations of large density contrasts coupled with large Laplace numbers.

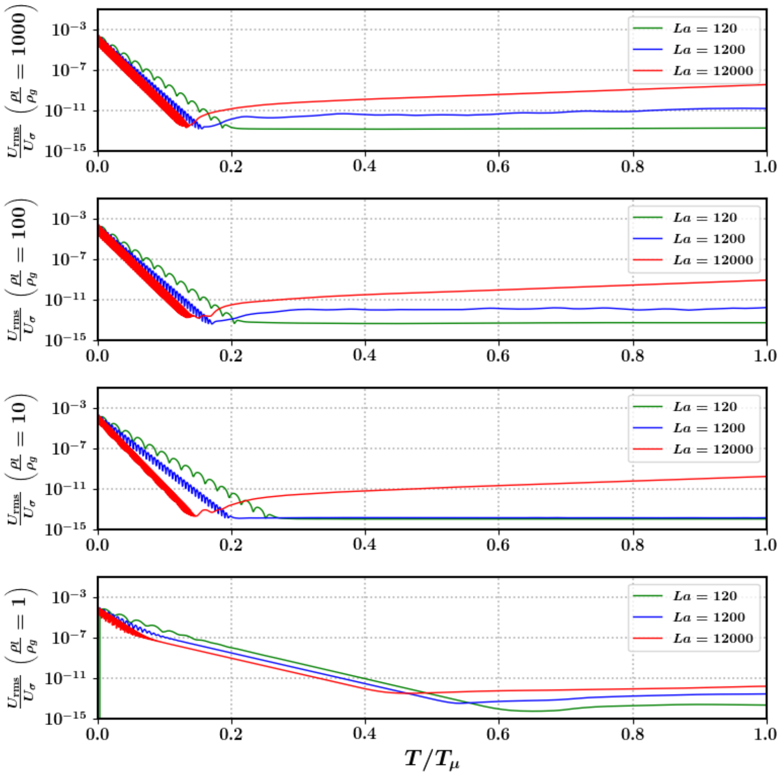

Figure 5.2: M1 Decay of normalized spurious currents as a function of viscous dissipation time-scales for different density-ratios and Laplace numbers. The currents seem to initially decay quickly for all higher density-ratios, and relax to the numerical equilibrium curvature even within $0.2 \cdot T_\mu$. For combinations of large ρ_l/ρ_g and large La, the spurious currents seem to grow back to an order of magnitude (10^{-8}) which is quite far from that of machine precision (10^{-14}).

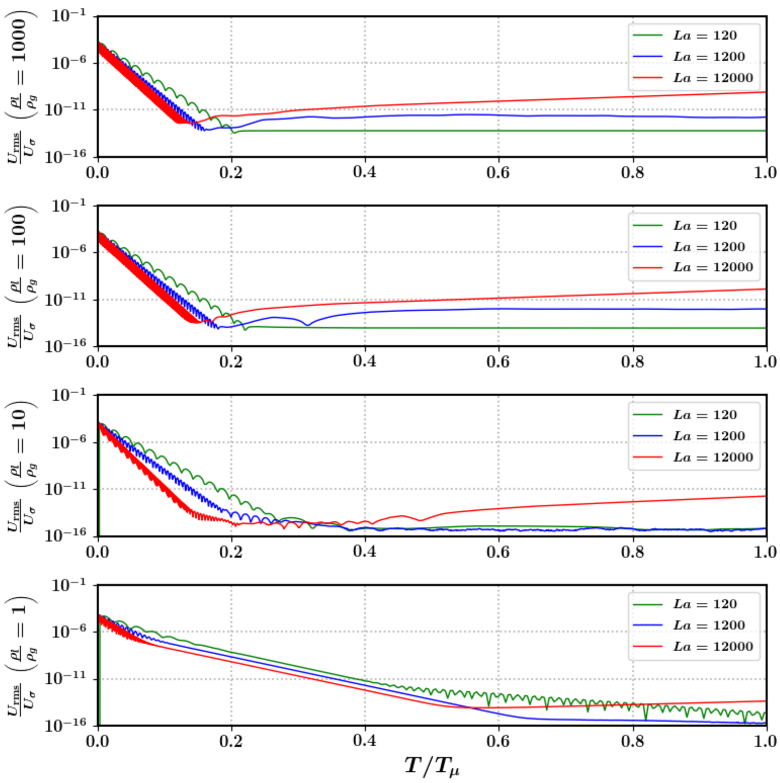

Figure 5.3: M2 Decay of normalized spurious currents as a function of viscous dissipation time-scales for different density-ratios and Laplace numbers. The currents seem to initially decay quickly for all higher density-ratios, and relax to the numerical equilibrium curvature even within $0.2 \cdot T_\mu$. For combinations of large ρ_l/ρ_g and large La, the spurious currents seem to grow back to an order of magnitude (10^{-8}) which is quite far from that of machine precision (10^{-14}). No considerable improvement is observed with respect to **M1**.

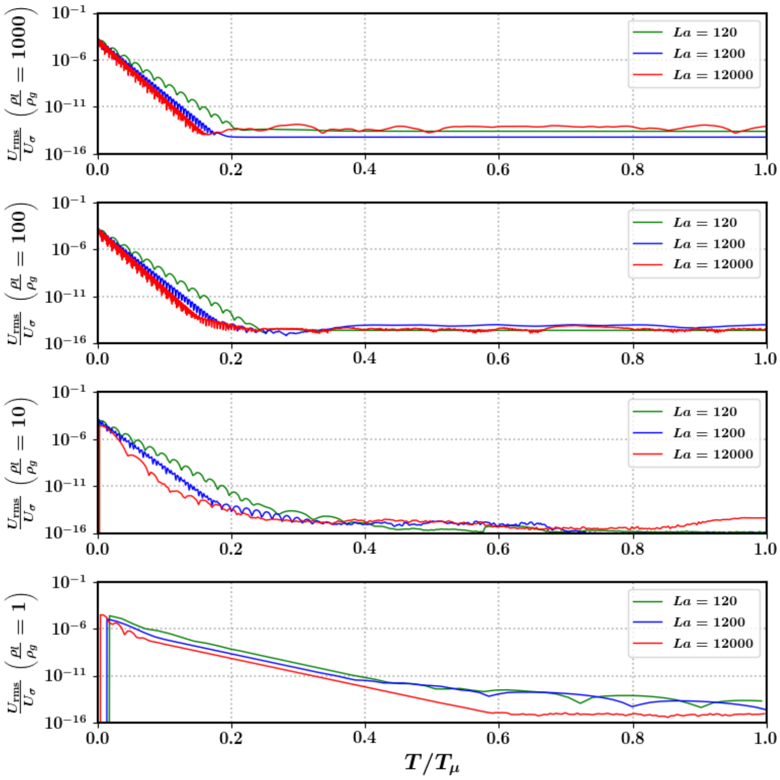

Figure 5.4: M3 Decay of normalized spurious currents as a function of viscous dissipation time-scales for different density-ratios and Laplace numbers. The currents seem to decay very quickly in the case of higher density-ratios, and relax to the numerical equilibrium curvature even within $0.2 \cdot T_\mu$. For all combinations of ρ_l/ρ_g and La numbers, the decayed spurious currents are not observed to grow back as in the cases of **M1** and **M2**, and hover around values close to machine precision (10^{-14}).

Spatial Convergence

Once the solution relaxes to a numerical equilibrium curvature (spurious currents are approximately at the order of machine precision), there still exists a difference between the numerical curvature and the exact analytical curvature corresponding to the spherical (circular) shape. We use the definitions of the shape errors as introduced in the seminal work of Popinet [20] to assess the convergence of our class of methods to the exact (analytical) curvature as we increase spatial resolution. The norms are defined as follows :

$$L_2 = \sqrt{\frac{\sum_i \left(C_i - C_i^{\text{exact}}\right)^2}{\sum_i}} \quad , \quad L_\infty = \max_i \left(|C_i - C_i^{\text{exact}}|\right) \tag{5.4}$$

where C_i is the volume fraction of a cell after the solution has relaxed to the numerical equilibrium curvature, and C_i^{exact} is the volume fraction corresponding to the exact circular shape which was initialized at the start of the simulation.

Fig. 5.5 demonstrates the behavior of the shape errors defined in eqn. 5.4 for the case of the most stringent parameter combination ($\rho_l/\rho_g = 1000$, $La = 12000$) as a function of the droplet resolution. As one can clearly observe, all the methods tested display a roughly second-order convergence in space for both the error norms. In terms of the L_2 norm, the consistent and conservative method (**M3**) does indeed achieve smaller errors as compared to both **M1** and **M2** for all spatial resolutions. As a minor remark, there is not much to discern in terms of shape error when it comes to comparing the performances of the consistent (**M2**) method with the non-consistent one (**M1**).

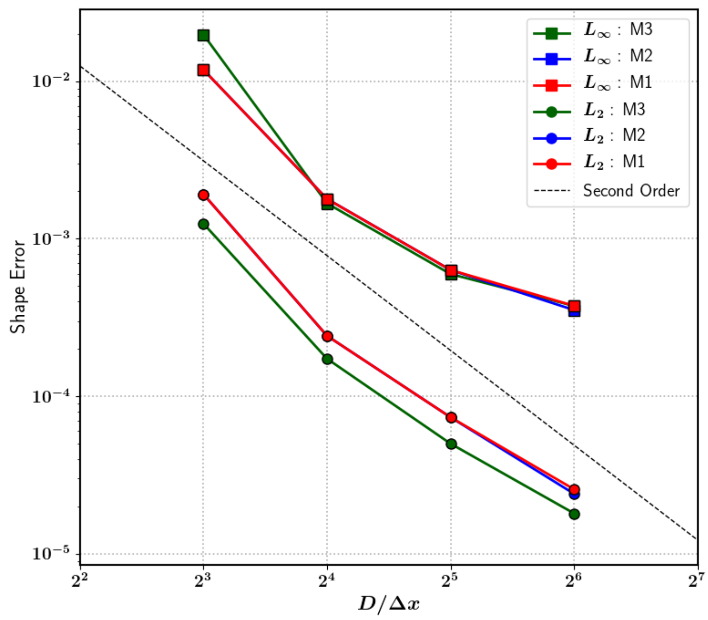

Figure 5.5: Second-order spatial convergence for the spurious current error norms corresponding to the most stringent parameter combination ($\rho_l/\rho_g = 1000$, $La = 12000$) . Both of the norms (L_∞ and L_2) seem to demonstrate a roughly second order rate of spatial convergence with each of the methods tested. However, **M3** has a marginally lower L_2 error compared to both **M1** and **M2** for all resolutions tested. There is negligible difference observed in the shape errors between **M1** and **M2** in both of the norm definitions.

5.2 Moving Droplet

An incisive numerical setup that enables us to evaluate the accuracy of the coupling between interfacial propagation and surface tension discretization was first proposed by Popinet [20], and subsequently employed in the comparative study of Abadie et al. [27] . The manner in which this test differs from that of the static droplet is the addition of a uniform background velocity field, therefore serving as a better representation of droplets in complex surface tension dominated flows where they might be advected by the mean flow. In terms of the Laplace equilibrium, the hydrostatic solution is still valid in the frame of reference of the moving droplet. The point at which the solution in the moving reference frame diverges from that of the static droplet (5.1) is through the continuous injection of noise at the scale of the grid size. This 'numerical' noise emanates from the perturbations to the curvature estimates, which are in turn induced by the interfacial reconstructions carried out to propagate the interface (temporal integration) . These fluctuating errors act as source terms for the momentum, thereby transforming the problem into that of viscous dissipation in the presence of continuous forcing (in the reference frame of the moving drop).

[27]: Abadie et al. (2015), 'On the combined effects of surface tension force calculation and interface advection on spurious currents within Volume of Fluid and Level Set frameworks'

Setup

In the present study, we evaluate our class of methods using the advection of a droplet in a spatially periodic domain using an identical setup as [20], but with the important difference of including sharp density jumps across the interface as well as using lower spatial resolutions.

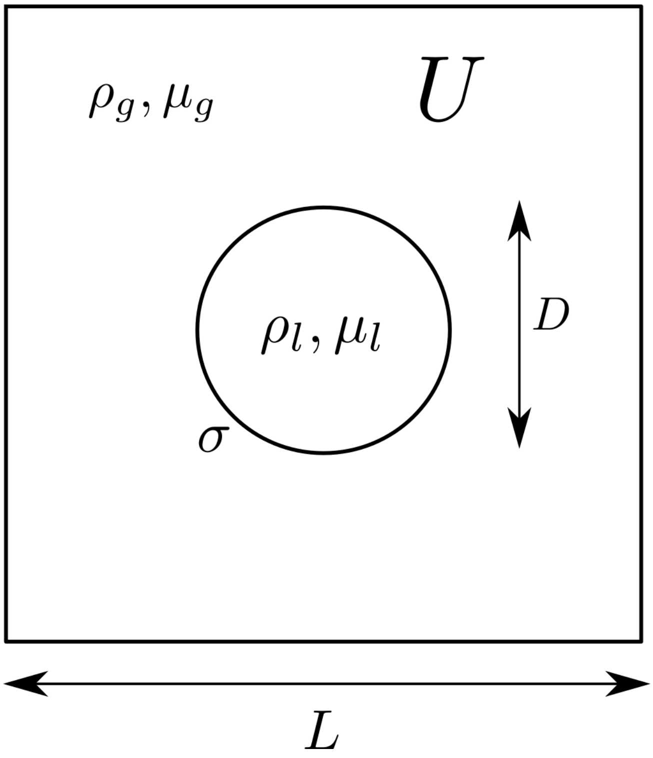

Figure 5.6: Schematic of the droplet of dense fluid advected in a surrounding medium of lighter fluid. A 50×50 grid is employed to spatially discretize the domain, which is spatially periodic in the direction of droplet advection.

As previously discussed (5.1), high density-ratios tend to amplify the fluctuations induced by the myriad numerical approximations (interface reconstruction, curvature estimation etc) involved in the algorithm.

We consider a circular droplet of diameter D placed at the centre of a square domain of side L. The densities of the heavier and lighter phases are ρ_l and ρ_g respectively, likewise for the viscosities μ_l and μ_g, and σ being the surface tension coefficient (fig. 5.6). A uniform velocity field $\boldsymbol{U}$ is initialized on the entire domain (only a horizontal component). The ratio of the droplet size to the box is $D/L = 0.4$, with $D/\Delta x = 20$ (Δx being the grid size. [3]). As for boundary conditions, we use symmetry conditions on the top and bottom sides, and periodic boundary conditions on the horizontal direction (along which advection by U takes place). We characterize by problem by introducing the following adimensional parameters (based on the heavier fluid) :

3: In Popinet [20], a resolution of $D/\Delta x = 25.6$ corresponding to a grid of 64×64 is used

$$La = \frac{\rho_l \sigma D}{\mu_l^2} \quad , \quad We = \frac{\rho_l U^2 D}{\sigma} \tag{5.5}$$

In addition to the capillary and viscous time-scales for the static case (eqns. 5.1), we have an additional scale defined as :

$$T_u = D/U \tag{5.6}$$

which is the time-scale of advection. In our subsequent analysis, we shall use T_u and U as the time and velocity scales, repectively.

Evolution of Spurious Currents

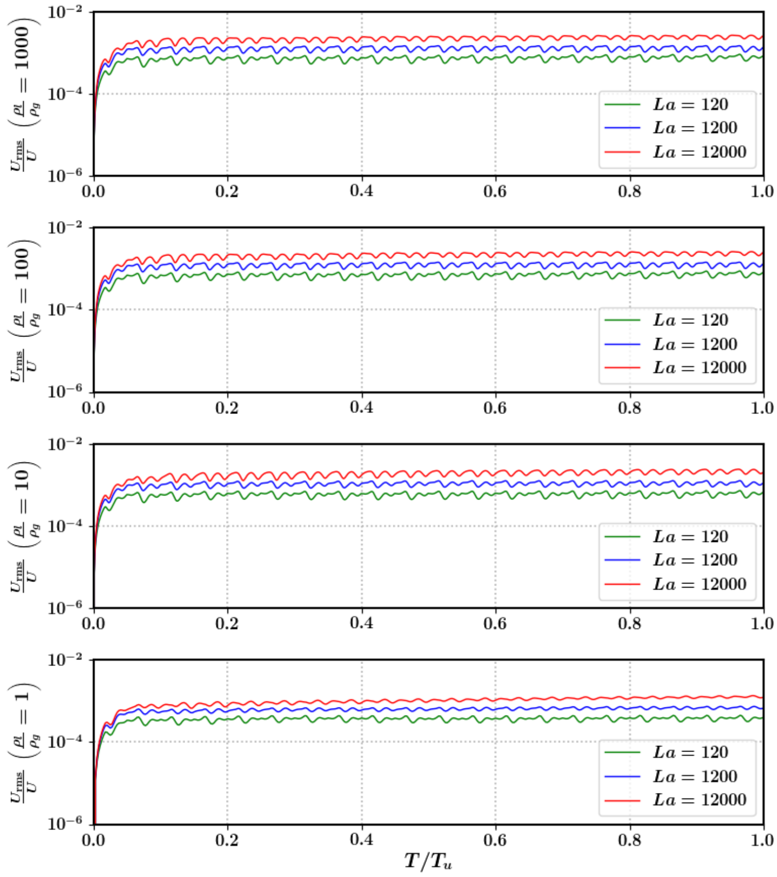

Figure 5.7: **M1** Time evolution of normalized spurious currents as a function of advection time-scales (T_u) for different combinations of density-ratio and Laplace numbers. The currents seem to hover around 10^{-3}, with a larger Laplace number corresponding to a higher error for all density-ratios. $We = 0.4$ for all the cases presented.

Figures 5.7 to 5.9 depict the evolution of the root-mean-square (RMS) error of the velocity field in the moving frame of reference, as a function of different Laplace numbers, spanning over density-ratios separated by orders of magnitude. The first figure (5.7) refers to simulations carried out without consistency between the momentum-mass transport (**M1**), the second (5.8) corresponds to that of the consistent but not conservative method (**M2**), and final one (5.9) refers to that of the consistent and conservative method (**M3**). We again have a couple of important observations, the first being that spurious currents do not decay to machine precision as in static droplet case for all of the combinations and methods tested, instead they oscillate around a mean value of the order of $0.1 - 0.01\%$ of the constant field U. The second observation is regarding the significantly smaller error (almost by one order of magnitude) in the case of the consistent and conservative method (**M3**) when compared to that of **M1** and **M2**. As a minor remark, in case of large Laplace numbers, the **M3** method displays a slight upward trend in the error evolution, which is not the case in either **M1** or **M2**. This is not too worrisome as the growth is over a time-scale much larger than T_u, with the oscillations corresponding to a time-scale of the order $U/\Delta x$. All of the plots in figures 5.7 to 5.9 correspond to $We = 0.4$, alongside an additional simplification of equal viscosities across the interface i.e $\mu_l/\mu_g = 1$.

As evindenced by the persistence of these spurious currents due to the

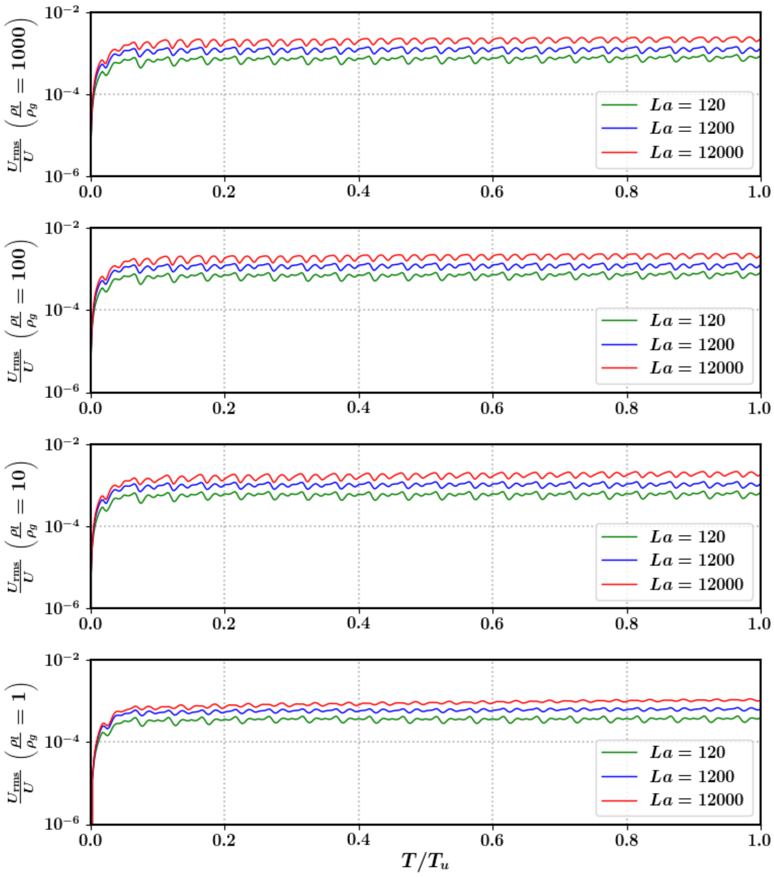

Figure 5.8: M2 Time evolution of normalized spurious currents as a function of advection time-scales (T_u) for different combinations of density-ratio and Laplace numbers. There seems to be no appreciable difference from the evolution seen in the case of **M1** (fig. 5.7). The currents seem to hover around 10^{-3}, with a larger Laplace number corresponding to a higher error for all density-ratios. $We = 0.4$ for all the cases presented.

addition of grid-level noise emanating from interfacial reconstructions, further advancements should be made with respect to the combined performace of the interfacial transport, curvature computation and the surface tension model. Nonetheless, all the methods tested do seem to be quite numerically stable when dealing with the high density-ratios, and are not subject to rapid uncontrollable amplifications of the interfacial perturbations even for high Laplace numbers.

Spatial Convergence

In order to evaluate the performance of our class of methods at different resolutions, we define the errors as the maximum values of the norms L_∞ and L_2 of the rescaled field U_{rms}/U over time (5 times T_u). In fig. 5.10, we show the scaling of the error as a function of spatial resolution for the most stringent case of $\rho_l/\rho_g = 1000$, $La = 12000$, for each of our different methods. As similarly observed in section 5.1, in terms of both L_∞ and L_2 norms, there is no appreciable difference in the behaviors of **M1** and **M2**. For **M3**, we do observe significantly lower maximum errors compared to other two methods, but at a cost of slightly less than first-order convergence. The overall convergence behavior of the class of methods we have tested seem to be consistent with earlier studies of Popinet [20] and others [4] .

4: In existing literature, convergence rates have only been studied in case of equal density fluids across the interface

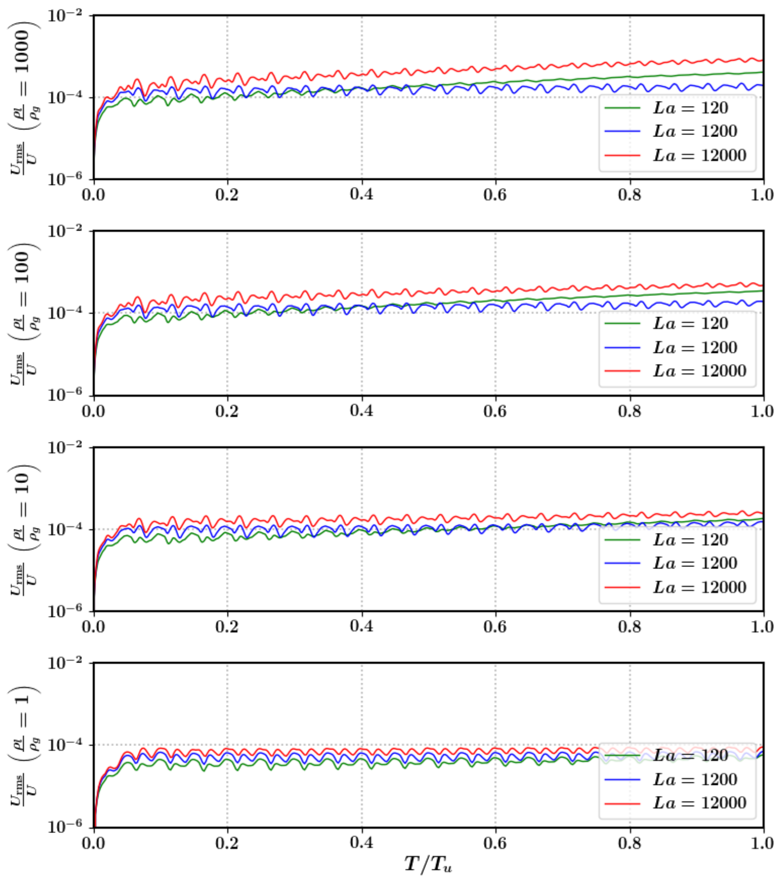

Figure 5.9: M3 Time evolution of normalized spurious currents as a function of advection time-scales (T_u) for different combinations of density-ratio and Laplace numbers. In terms of the errors observed in **M1** and **M2**, we observe a decrease of roughly one order of magnitude. Although an upward trend is observed for large Laplace numbers, the growth rate is quite low. The currents seem to hover slightly above 10^{-4}, with larger Laplace numbers corresponding to larger errors for all density-ratios. $We = 0.4$ for all the cases presented.

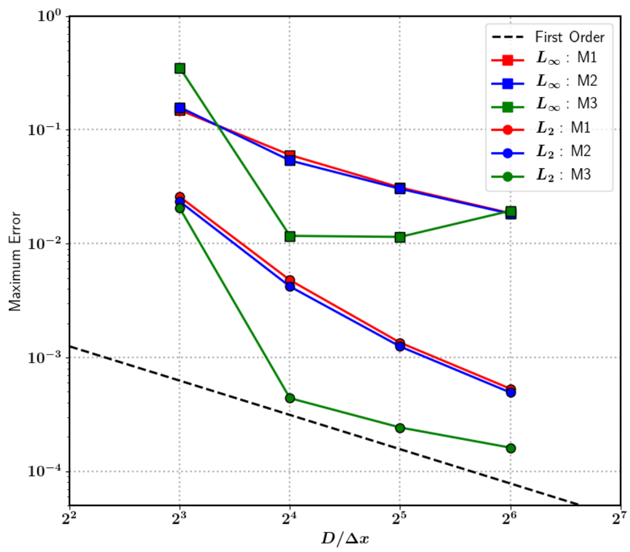

Figure 5.10: First-order (approximately) spatial convergence of the maximum of the spurious current error norms in the frame of reference of the moving droplet, for the most stringent parameter combination ($\rho_l/\rho_g = 1000$, $La = 12000$, $We = 0.4$). Methods **M1** and **M2** display similar convergence properties, whereas **M3** leads to significantly lower errors even though it doesn't quite follow the first-order convergence rate.

Error Dependence : Laplace & Weber numbers

As the final point of inquiry into the performance of our class of methods, figures 5.11 and 5.12 demonstrate the influence of the Laplace and Weber numbers on the behavior of the maximum error norm, carried out for the largest density-ratio ($\rho_l/\rho_g = 1000$). We only present the results obtained using the consistent and conservative method (**M3**), for a resolution corresponding to $D/\Delta x = 25.6$. As we can observe, the error (both L_∞

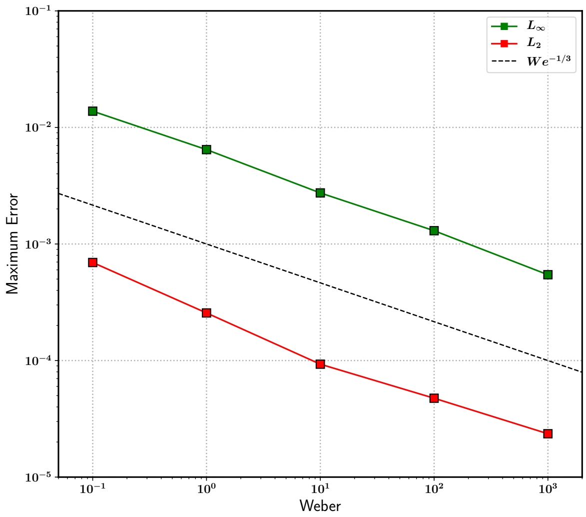

Figure 5.11: Scaling of the maximum error norm as a function of Weber (La = 12000, ρ_l/ρ_g = 1000).

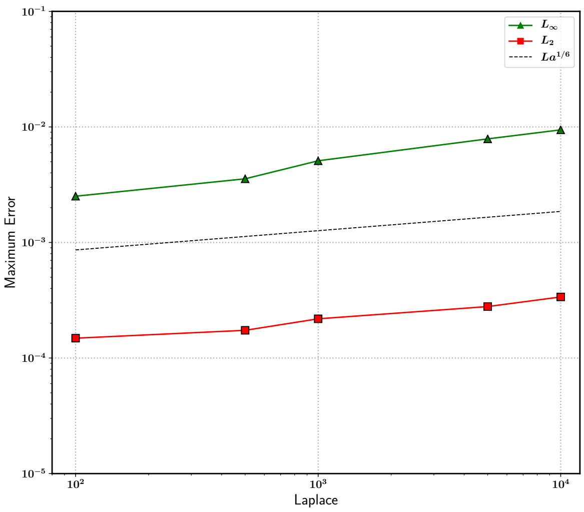

Figure 5.12: Scaling of the maximum error norm as a function of Laplace (We = 0.4, ρ_l/ρ_g = 1000).

and L_2) scales as $We^{-1/3}$ over 4 orders of magnitude, which is different from the $We^{-1/2}$ scaling observed by Popinet [20] [5] . In terms of Laplace numbers, the errors scale as $La^{1/6}$ over two orders of magnitude, which is the same as that observed in [20] (for equal densities).

5: Although Popinet [20] had equal densities (ρ_l/ρ_g = 1)

5.3 Capillary Wave

One of fundamental features of immiscible multiphase flows involving interfaces are the presense and propagation of capillary waves. Therefore, a robust and accurate numerical method should not only be able to adequately resolve, but also accurately emulate the spatio-temporal evolution of such surface tension induced oscillations. A brief outline on the state-of-the-art numerical implementations of capillary waves (and surface tension models in general) existing in current literature is provided by Popinet in the comprehensive review [6] .

[6]: Popinet (2018), 'Numerical models of surface tension'

Setup

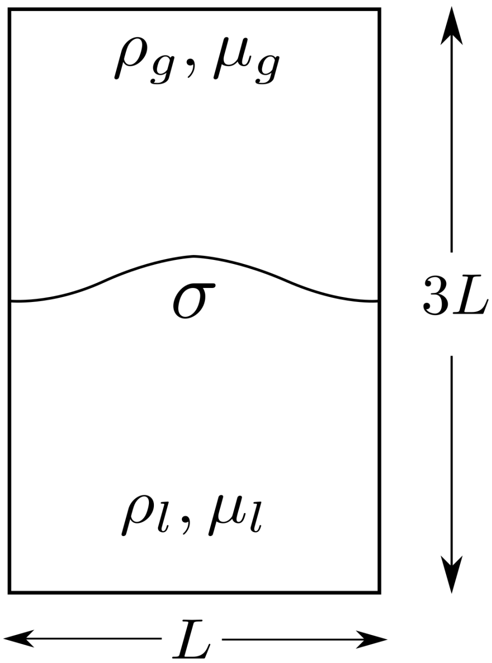

Figure 5.13: Schematic of the initially perturbed planar interface separating two immiscible fluids of different densities and viscosities. A spatial resolution of 32×96 is used for spatial discretization (compared to 64×192 in Popinet [20]), with the width of the box corresponding to the size of the perturbed wavelength.

In the present study, we evaluate the accuracy of our class of methods by comparing with an analytical solution of damped capillary oscillations. Generally, analytical solutions exist only for cases corresponding to extremely small initial perturbations, that too either in the inviscid limit (Lamb [28]) or the asymptotic limit of vanishing viscosity (Prosperetti [29, 30]). For our purposes, we use the configuration of the viscosity-damped capillary oscillations of a planar interface, as was first implemented and popularized by Popinet & Zaleski [25].

[28]: Lamb (1993), *Hydrodynamics*

[29]: Prosperetti (1980), 'Free oscillations of drops and bubbles: the initial-value problem'

[30]: Prosperetti (1981), 'Motion of two superposed viscous fluids'

We consider a rectangular domain of dimensions $L \times 3L$, where L corresponds to the wavelength of our initial perturbation. The densities of the heavier and lighter phases are ρ_l and ρ_g respectively, likewise for the viscosities μ_l and μ_g, and σ being the surface tension coefficient (fig. 5.13). An intial perturbation amplitude of $L/100$ is used, coupled with a numerical resolution given by $L/\Delta x = 32$ (Δx being the grid size). Symmetry conditions are applied on the top and bottom sides, with periodic conditions along the horizontal direction. We use the following adimensional parameters to characterize our problem :

$$T_0 = T\omega_0 \quad , \quad La = \frac{\rho_l \sigma L}{\mu_l^2} \tag{5.7}$$

where La is the Laplace number based on the heavier fluid, and ω_0 is defined using the dispersion relation used in Popinet [20] given as :

$$\omega_0^2 = \frac{\sigma k^3}{2\rho_l} \quad , \qquad \text{where} \quad k = \frac{2\pi}{L} \tag{5.8}$$

The dispersion relation is obtained via linear stability analysis at the inviscid limit [28] . In order to evaluate the influence of density-ratio on the performance of our class of methods, we use three different numerical setups keeping the same Laplace number ($La = 3000$) as follows :

[28]: Lamb (1993), *Hydrodynamics*

- $\rho_l/\rho_g = 1$, $\mu_l/\mu_g = 1$ (Popinet [20])
- $\rho_l/\rho_g = 10$, $\mu_l/\mu_g = 1$
- $\rho_l/\rho_g = 1000.0/1.2$, $\mu_l/\mu_g = 1.003 \cdot 10^{-3}/1.8 \cdot 10^{-5}$ (Air-Water)

The final setup corresponds to that of an air-water interface (physical properties corresponding to 20^o Celsuis), which is the most stringent due to the significant density and viscosity jumps.

Comparison with Prosperetti Solution

The theoretical solution to this configuration corresponds to the closed-form expressions of the planar interface shape evolution established by Prosperetti [29, 30], which takes into account the finite time-scales at which the vorticity (generated due to interface oscillations) diffuses into the bulk medium. These closed-form expressions are subsequently integrated using a fourth-order Runge-Kutta time integrator (details of which not described here), and used to assess the accuracy of the results obtained by our class of numerical methods.

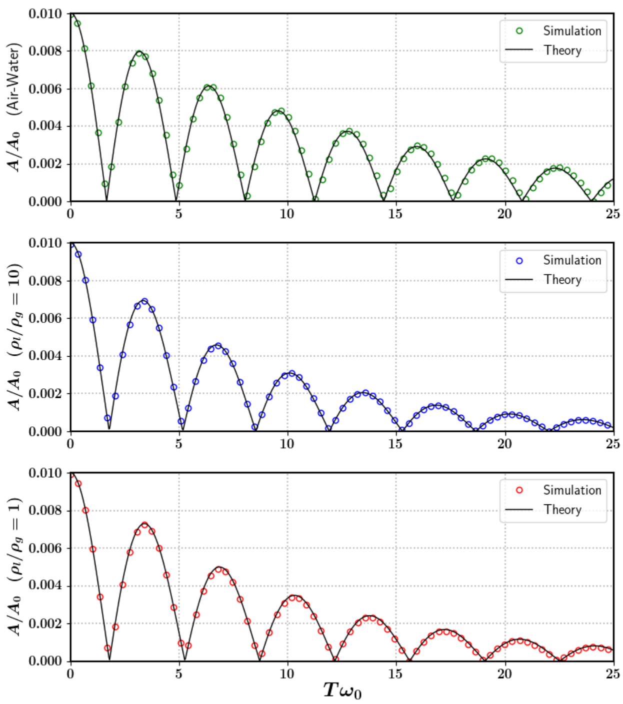

Figure 5.14: M1 Time evolution of the amplitude of the planar interface undergoing damped capillary oscillations, comparing the solution obtained by our numerical method with the closed-from Prosperetti solution. More or less good agreement with theory is observed for all the density-ratios tested.

As we can in figures 5.14 to 5.16, solutions from our class of numerical methods (circles) are compared to that of the theoretical (Prosperetti) solution (black curves), where the amplitude is normalized by the initial value (A_0) and the time rescaled by T_0. The first figure (5.14) refers to simulations carried out by the non-consistent method, the second (5.15) corresponds to that of the consistent method, and the final one (5.16) refers to that of the consistent and conservative method. We observe that there is hardly any appreciable qualitative difference between the results obtained via the different methods **M1**, **M2** and **M3**, although **M3** does seem to perforn marginally better when it comes to the most stringent case (air-water configuration). Surprisingly, even the non-consistent method (**M1**) does not seem to show any un-physical interfacial deformations for all the density-ratios tested, and that it is difficult to distinguish between the different methods for the lower density-ratios.

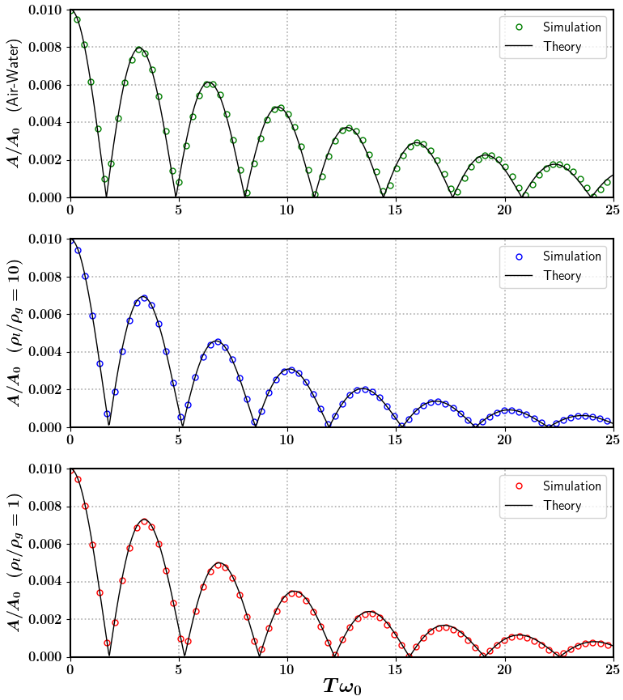

Figure 5.15: M2 Time evolution of the amplitude of the planar interface undergoing damped capillary oscillations, comparing the solution obtained by our numerical method with the closed-from Prosperetti solution. Behavior is quite similar to **M1**, with good agreement with theory for all the density-ratios tested.

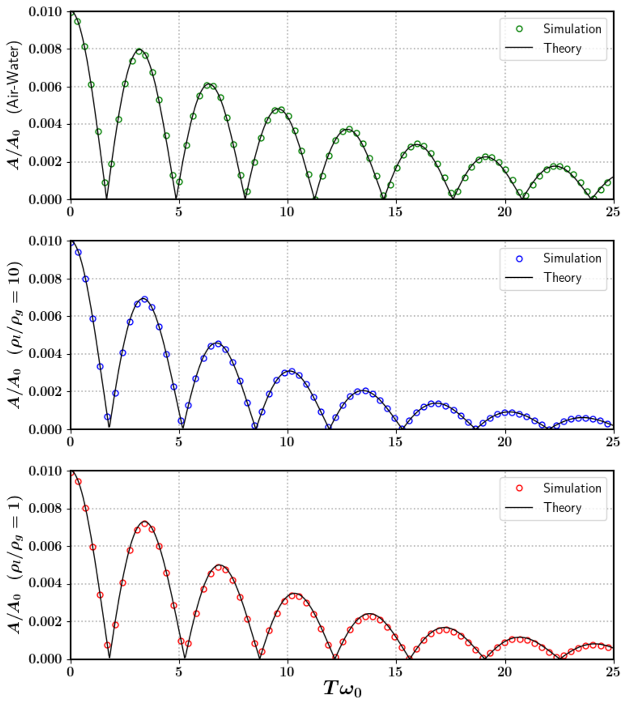

Figure 5.16: M3 Time evolution of the amplitude of the planar interface undergoing damped capillary oscillations, comparing the solution obtained by our numerical method with the closed-from Prosperetti solution. Slightly better agreement with theory when comparing to **M1** and **M2**, for all density-ratios tested.

Spatial Convergence

The next step in our evaluation would be to quantify the accuracy of our numerical results to the Prosperetti solution using an integral (in time) error norm, the same as defined in [20] :

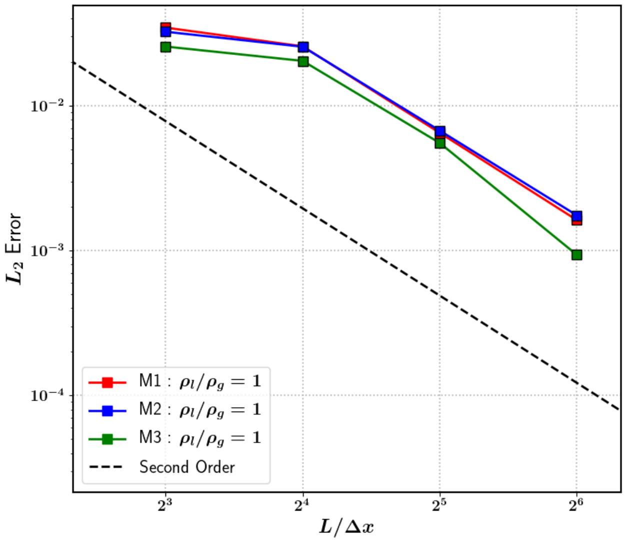

Figure 5.17: Comparison of spatial convergence for the case of $\rho_l/\rho_g = 1, La = 3000$, for our class of methods. There is no viscosity jump across the interface. All methods seem to demonstrate approximately second-order convergence. There seems to be no appreciable difference in the behavior of **M1** and **M2**, with **M3** displaying marginally lower errors compared to the others.

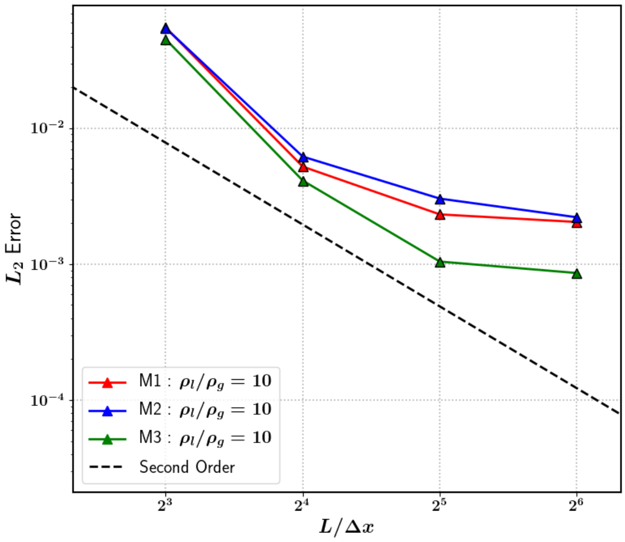

Figure 5.18: Comparison of spatial convergence for the case of $\rho_l/\rho_g = 10, La = 3000$, for our class of methods. Again, there is no viscosity jump across the interface. All methods seem to demonstrate approximately second-order convergence upto $/\Delta x = 32$, beyond which there is a slight saturation in the rate of convergence. Qualitatively, **M1** and **M2** demonstrate similar behavior, with **M3** delivering slightly lower errors. In case of **M3**, the errors are marginally lower compared to **M1** and **M2** for higher resolutions.

$$L_2 = \frac{1}{L}\sqrt{\frac{\omega_0}{25}\int_{t=0}^{T}(h - h_{exact})^2} \tag{5.9}$$

where h is the maximum inteface height obtained using our numerical simulations, and h_{exact} being the maximum height obtained via time integration of the Prosperetti solution. In figures 5.17 to 5.19 we demonstrate the rate of spatial convergence of the L_2 error norms for different density-ratios, simultaneously comparing the behavior of the different methods **M1**, **M2** and **M3** at each density-ratio. In all the results presented, we maintain $La = 3000$ for all density-ratios, spatial resolutions

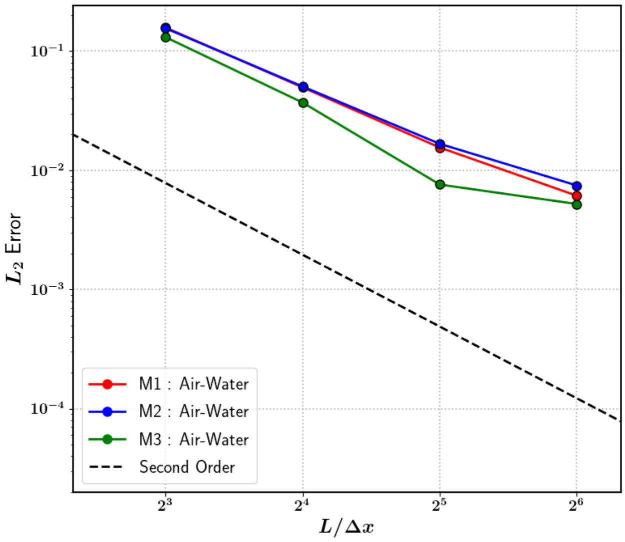

Figure 5.19: Comparison of spatial convergence for the Air-Water case corresponding to $\rho_l/\rho_g = 1000.0/1.2$, $\mu_l/\mu_g = 1.003 \cdot 10^{-3}/1.8 \cdot 10^{-5}$, $La = 3000$, for our class of methods. All methods seem to demonstrate approximately second-order convergence. No appreciable difference is observed between **M1** and **M2**, with **M3** delivering slightly lower errors although there is some saturation in the convergence rate at higher resolutions.

and methods tested.

In figure 5.17 we observe roughly second-order spatial convergence when it comes to equal densities across the interface, with **M1** and **M2** displaying nearly identical behavior, whereas **M3** does marginally better with lower errors for all resolutions. When it comes to $\rho_l/\rho_g = 10$ in figure 5.18, we observe a saturation in the initial second-order convergence rate irrespective of whichever method is used, however **M3** performs slightly better in terms of error when compared **M1** and **M2**. Finally, figure 5.19 demonstrates the roughly second-order convergence of all three methods when it comes to the air-water configuration, again, with **M3** performing merginally better with lower errors. Not surprisingly, the largest errors arise for the air-water configuration errors across all methods.

Physics of Fragmentation

Ligament Mediated Paradigm 6

6.1 Mechanism of Drop Formation

Disintegration of Jets & Shear Layers Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Expansion of Sheets Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Effervescent Atomization Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Drop Impacts Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

6.2 Theories of Fragmentation

Cascade Mechanism : Log-Normal Distribution Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Corrugation-Coalescence Mechanism : Gamma Distribution Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Droplet Generation in Corrugated Ligaments

7

7.1 Numerical Setup

Platform : Basilisk Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Computational Schematic Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Random Surface Generation Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Parameterization Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

7.2 Ligament Breakup

3D vs. 2D Simulations Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Effect of Spatial Resolution Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Effect of Droplet Removal Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Effect of Corrugation Amplitude Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Effect of Ohnesorge Number Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Effect of Cut-Off Wavenumber Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Effect of Aspect Ratio Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Quantization of Waves Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Statistics of Drop Sizes 8

8.1 Monte Carlo Approach to DNS

Characterization of Ligament Ensembles Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

8.2 Millimeter Scale Ensembles

Diameter Distributions Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Mass Distributions Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Equivalent Diameters Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Local Distribution of Large Drop Sizes Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

8.3 Exploration of Parameter Space Φ

Bifurcation Parameter : Corrugation Amplitude Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Scaling of D/W : Function of Parameter Space Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

To be added Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Conclusions & Perspectives

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

This is the second paragraph. Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

And after the second paragraph follows the third paragraph. Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

After this fourth paragraph, we start a new paragraph sequence. Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

APPENDIX

A

Heading on Level 0 (chapter)

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

A.1 Heading on Level 1 (section)

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Heading on Level 2 (subsection)

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Heading on Level 3 (subsubsection)

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift –

not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Heading on Level 4 (paragraph) Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

A.2 Lists

Example for list (itemize)

- First item in a list
- Second item in a list
- Third item in a list
- Fourth item in a list
- Fifth item in a list

Example for list (4*itemize)

- First item in a list
  - First item in a list
    - First item in a list
      - First item in a list
      - Second item in a list
    - Second item in a list
  - Second item in a list
- Second item in a list

Example for list (enumerate)

1. First item in a list
2. Second item in a list
3. Third item in a list
4. Fourth item in a list
5. Fifth item in a list

Example for list (4*enumerate)

1. First item in a list
    a) First item in a list
        i. First item in a list
            A. First item in a list
            B. Second item in a list
        ii. Second item in a list
    b) Second item in a list
2. Second item in a list

Example for list (description)

First item in a list
Second item in a list
Third item in a list
Fourth item in a list
Fifth item in a list

Example for list (4*description)

First item in a list
    First item in a list
        First item in a list
            First item in a list
            Second item in a list
        Second item in a list
    Second item in a list
Second item in a list

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Notation

The next list describes several symbols that will be later used within the body of the document.

c Speed of light in a vacuum inertial frame

h Planck constant

Greek Letters with Pronounciation

Character	Name	Character	Name
α	alpha *AL-fuh*	ν	nu *NEW*
β	beta *BAY-tuh*	ξ, Ξ	xi *KSIGH*
γ, Γ	gamma *GAM-muh*	o	omicron *OM-uh-CRON*
δ, Δ	delta *DEL-tuh*	π, Π	pi *PIE*
ϵ	epsilon *EP-suh-lon*	ρ	rho *ROW*
ζ	zeta *ZAY-tuh*	σ, Σ	sigma *SIG-muh*
η	eta *AY-tuh*	τ	tau *TOW (as in cow)*
θ, Θ	theta *THAY-tuh*	υ, Υ	upsilon *OOP-suh-LON*
ι	iota *eye-OH-tuh*	ϕ, Φ	phi *FEE, or FI (as in hi)*
κ	kappa *KAP-uh*	χ	chi *KI (as in hi)*
λ, Λ	lambda *LAM-duh*	ψ, Ψ	psi *SIGH, or PSIGH*
μ	mu *MEW*	ω, Ω	omega *oh-MAY-guh*

Capitals shown are the ones that differ from Roman capitals.