Advanced Types

A4M36TPJ, 2013/2014

Basic Setup

- $\Gamma \vdash \diamond$ ("Γ is a well-formed environment")
- $\Gamma \vdash A$ ("A is a well-formed type in environment Γ")
- $\Gamma \vdash e : A$ ("e is a well-formed term of type A in environment Γ")

$$\frac{}{\emptyset \vdash \diamond}$$

$$\frac{\Gamma \vdash A \quad a \notin dom(\Gamma)}{\Gamma \cup \{(a, A)\} \vdash \diamond}$$

$$\frac{\Gamma \vdash \diamond}{\Gamma \vdash x : \Gamma(x)}$$

$$\frac{\Gamma : \diamond \quad A \in Basic}{\Gamma \vdash A}$$

Sets vs. Types

- What is the difference between the set Bool = {true, false} and the Boolean type?

- Answer: Boolean can be a type of an expression, e.g. true and false. This expression is not a member of the set Bool. On the other hand true is a Boolean and false also. Hence, the set of Boolean expressions is different from the set of Bool.

Type Constructor

- We can construct user-defined types in many languages (Java, C++, …).
- To construct new types from existing types we use type constructors.
- For example Int x Boolean is a type constructor of a new product type.

Function Type

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \to B}$$

$$\frac{\Gamma \cup \{(x, A)\} \vdash e : B}{\Gamma \vdash (\lambda x : A.e) : A \to B}$$

$$\frac{\Gamma \vdash e : A \to B \quad \Gamma \vdash p : A}{\Gamma \vdash e(p) : B}$$

Product Type

$$\frac{\Gamma \vdash A_1 \quad \Gamma \vdash A_2}{\Gamma \vdash A_1 \times A_2}$$

$$\frac{\Gamma \vdash e_1 : A_1 \quad \Gamma \vdash e_2 : A_2}{\Gamma \vdash (e_1, e_2) : A_1 \times A_2}$$

$$\frac{\Gamma \vdash e : A_1 \times A_2}{\Gamma \vdash \texttt{first} \;\; e : A_1}$$

$$\frac{\Gamma \vdash e : A_1 \times A_2}{\Gamma \vdash \texttt{second} \;\; e : A_2}$$

Tagged Union Type

$$\frac{\Gamma \vdash A_1 \quad \Gamma \vdash A_2}{\Gamma \vdash A_1 + A_2}$$

$$\frac{\Gamma \vdash e : A_1 \quad \Gamma \vdash A_2}{\Gamma \vdash \texttt{inLeft}_{A_2} e : A_1 + A_2}$$

$$\frac{\Gamma \vdash A_1 \quad \Gamma \vdash e : A_2}{\Gamma \vdash \texttt{inRight}_{A_1} e : A_1 + A_2}$$

Tagged Union Type

$$\frac{\Gamma \vdash e : A_1 + A_2}{\Gamma \vdash \texttt{isLeft}\ \ e : Boolean} \qquad \frac{\Gamma \vdash e : A_1 + A_2}{\Gamma \vdash \texttt{asLeft}\ \ e : A_1}$$

$$\frac{\Gamma \vdash e : A_1 + A_2}{\Gamma \vdash \texttt{isRight}\ \ e : Boolean} \qquad \frac{\Gamma \vdash e : A_1 + A_2}{\Gamma \vdash \texttt{asRight}\ \ e : A_2}$$

Record Type

$$\frac{\Gamma \vdash A_1 \quad \ldots \quad \Gamma \vdash A_n}{\Gamma \vdash \texttt{Record}(l_1 : A_1, \ldots, l_n : A_n)}$$

$$\frac{\Gamma \vdash e_1 : A_1 \quad \ldots \quad \Gamma \vdash e_n : A_n}{\Gamma \vdash \texttt{record}(l_1\texttt{=}e_1, \ldots, l_n\texttt{=}e_n) : \texttt{Record}(l_1 : A_1, \ldots, l_n : A_n)}$$

$$\frac{\Gamma \vdash e : \texttt{Record}(l_1 : A_1, \ldots, l_j : A_j, \ldots, l_n : A_n)}{e.l_j : A_j}$$

Reference Type

$$\frac{\Gamma \vdash A}{\Gamma \vdash \texttt{Ref}\ \ A}$$

$$\frac{\Gamma \vdash e : A}{\Gamma \vdash \texttt{ref}\ \ e : \texttt{Ref}\ \ A}$$

$$\frac{\Gamma \vdash e : \texttt{Ref}\ \ A}{\Gamma \vdash \texttt{deref}\ \ e : A}$$

$$\frac{\Gamma \vdash e : \texttt{Ref}\ \ A \quad \Gamma \vdash e' : A}{\Gamma \vdash e\ \texttt{=}\ e' : \diamond}$$

Type Variables

$$\frac{\Gamma \vdash \diamond \quad X \notin dom(\Gamma)}{\Gamma \cup \{X\} \vdash \diamond}$$

Recursive Type

$$\frac{\Gamma \cup \{X\} \vdash A}{\Gamma \vdash \mu X.A}$$

$$\frac{\Gamma \vdash e : \mu X.A}{\Gamma \vdash \texttt{unfold e} : A[X \mapsto \mu X.A]}$$

$$\frac{\Gamma \vdash e : A[X \mapsto \mu X.A]}{\Gamma \vdash \texttt{fold e} : \mu X.A}$$

Universal Type

$$\frac{\Gamma \cup \{X\} \vdash A}{\Gamma \vdash \forall X.A}$$

$$\frac{\Gamma \cup \{X\} \vdash e : A}{\Gamma \vdash \lambda X.e : \forall X.A}$$

$$\frac{\Gamma \vdash e : \forall X.A \quad \Gamma \vdash B}{\Gamma \vdash e(B) : A[X \mapsto B]}$$

Subtype Polymorphism

- We define a new binary relation <: on types and a new judgement: $\Gamma \vdash A <: B$ ("A is a subtype of B in environment Γ")

$$\frac{}{\Gamma \vdash A <: A}$$

$$\frac{\Gamma \vdash A <: B \quad \Gamma \vdash B <: C}{\Gamma \vdash A <: C}$$

Subsumption

$$\frac{\Gamma \vdash e : A \quad \Gamma \vdash A <: B}{\Gamma \vdash e : B}$$

Top Type

$$\frac{\Gamma \vdash \diamond}{\Gamma \vdash Top}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A <: Top}$$

Subtyping

$$\frac{\Gamma \vdash A' <: A \quad \Gamma \vdash B <: B'}{\Gamma \vdash A \to B <: A' \to B'}$$

$$\frac{\Gamma \vdash A' <: A \quad \Gamma \vdash B' <: B}{\Gamma \vdash A' \times B' <: A \times B}$$

$$\frac{\Gamma \vdash A' <: A \quad \Gamma \vdash B' <: B}{\Gamma \vdash A' + B' <: A + B}$$

$$\frac{\Gamma \vdash A'_1 <: A_1 \quad \ldots \quad \Gamma \vdash A'_n <: A_n \quad \Gamma \vdash A'_{n+1} \quad \ldots \quad \Gamma \vdash A'_{n+m}}{\Gamma \vdash \texttt{Record}(l_1 : A'_1, \ldots, l_{n+m} : A'_{n+m}) <: \texttt{Record}(l_1 : A_1, \ldots, l_n : A_n)}$$

Bounded Type Variables

$$\frac{\Gamma \vdash A \quad X \notin dom(\Gamma)}{\Gamma \cup \{X <: A\} \vdash \diamond}$$

$$\frac{\Gamma \cup \{X <: A\} \vdash \diamond}{\Gamma \cup \{X <: A\} \vdash X}$$

$$\frac{\Gamma \cup \{X <: A\} \vdash \diamond}{\Gamma \cup \{X <: A\} \vdash X <: A}$$

Subtyping of Recursive Types

$$\frac{\Gamma \cup \{X <: Top\} \vdash A}{\Gamma \vdash \mu X.A}$$

$$\frac{\Gamma \vdash \mu X.A \quad \Gamma \vdash \mu Y.B \quad \Gamma \cup \{Y <: Top, X <: Y\} \vdash A <: B}{\Gamma \vdash \mu X.A <: \mu Y.B}$$

Subtyping of Universal Types

$$\frac{\Gamma \cup \{X <: A\} \vdash B}{\Gamma \vdash \forall X <: A.B}$$

$$\frac{\Gamma \vdash A' <: A \quad \Gamma \cup \{X <: A'\} \vdash B <: B'}{\Gamma \vdash (\forall X <: A.B) <: (\forall X <: A'.B')}$$

$$\frac{\Gamma \cup \{X <: A\} \vdash e : B}{\Gamma \vdash \lambda X <: A.e : \forall X <: A.B}$$

$$\frac{\Gamma \vdash e : \forall X <: A.B \quad \Gamma \vdash A' <: A}{\Gamma \vdash e(A') : B[X \mapsto A']}$$