Combinatorial algorithms

computing subset rank and unrank, Gray codes,
k-element subset rank and unrank,
computing permutation rank and unrank

Jiří Vyskočil, Radek Mařík

2012

Combinatorial Generation

- **definition:**

Suppose that S is a finite set. A *ranking function* will be a bijection

$$\mathrm{rank}\colon S \rightarrow \{0, \ldots, |S| - 1\}$$

and unrank function is an inverse function to rank function.

- **definition:**

Given a ranking function rank , defined on S, the $\mathrm{successor}$ function satisfies the following rule:

$$\mathrm{successor}(s) = t \quad \Leftrightarrow \quad \mathrm{rank}(t) = \mathrm{rank}(s) + 1$$

- **potential uses:**
  - storing combinatorial objects in the computer instead of storing a combinatorial structure which could be quite complicated
  - generation of random objects from S ensuring equal probability $1/|S|$

Subsets

- Suppose that n is a positive integer and $S = \{1, \dots, n\}$.
- Define $\mathcal{S}$ to consist of the 2^n subsets of S.
- Given a subset $T \subseteq S$, let us define the *characteristic vector* of T to be the one-dimensional binary array

$$\chi(T) = [x_{n-1}, x_{n-2}, \dots, x_0]$$

where

$$x_i = \begin{cases} 1 & \text{if } (n-i) \in T \\ 0 & \text{if } (n-i) \notin T \end{cases}$$

Subsets

- Example of the lexicographic ordering on subsets of $S = \{1,2,3\}$:

T	$\chi(T) = [x_2, x_1, x_0]$	$rank(T)$
$\emptyset$	$[0,0,0]$	0
$\{3\}$	$[0,0,1]$	1
$\{2\}$	$[0,1,0]$	2
$\{2,3\}$	$[0,1,1]$	3
$\{1\}$	$[1,0,0]$	4
$\{1,3\}$	$[1,0,1]$	5
$\{1,2\}$	$[1,1,0]$	6
$\{1,2,3\}$	$[1,1,1]$	7

Subsets

- **computing the subset rank over lexicographical ordering**

Function SubsetLexRank(size n; set T) : rank
$r = 0$;
for $i = 1$ **to** n **do** {
    if $i \in T$ **then** $r = r + 2^{n-i}$;
}
return r ;

Function SubsetLexUnrank(size n; rank r) : set
$T = \emptyset$;
for $i = n$ **downto** 1 **do** {
    if $r \bmod 2 = 1$ **then** $T = T \cup \{i\}$;
    $r = \left\lfloor \frac{r}{2} \right\rfloor$;
}
return T ;

Gray Code

- **definition:**

The *reflected binary code*, also known as *Gray code*, is a binary numeral system where two successive values differ in only one bit.

G^n will denote the reflected binary code for 2^n binary n-tuples, and it will be written as a list of 2^n vectors, as follows:

$$G^n = [G_0^n, G_1^n, \dots, G_{2^n-1}^n]$$

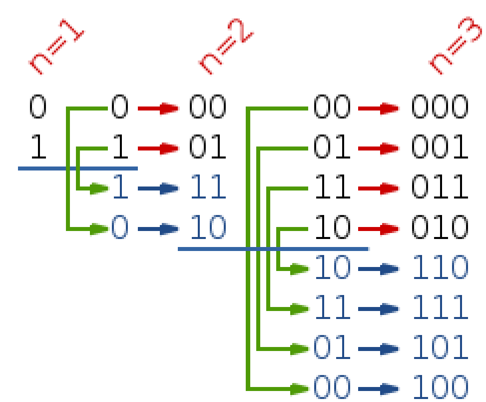
The codes G^n are defined recursively:

$$G^1 = [0,1]$$

$$G^n = [0G_0^{n-1}, 0G_1^{n-1}, \dots, 0G_{2^n-1}^{n-1}, 1G_{2^n-1}^{n-1}, \dots, 1G_1^{n-1}, 1G_0^{n-1}]$$

- **example:**

$G^3 = [000, 001, 011, 010, 110, 111, 101, 100]$



Gray Code

- Example:

G_r^3	binary representation of r	r
000	000	0
001	001	1
011	010	2
010	011	3
110	100	4
111	101	5
101	110	6
100	111	7

Gray Code

- **lemma 1**

Suppose

- $0 \leq r \leq 2^n - 1$
- $B = b_n, \ldots, b_0$ is a binary code of r
- $G = g_n, \ldots, g_0$ is a Gray code of r

Then for every $j \in \{0, 1, \ldots, n-1\}$

$$g_j = (b_j + b_{j+1}) \bmod 2$$

- **proof**

By induction on n.

- **lemma 2**

Suppose

- $0 \le r \le 2^n - 1$
- $B = b_n, \dots, b_0$ is a binary code of r
- $G = g_n, \dots, g_0$ is a Gray code of r

Then for every $j \in \{0,1, \dots, n-1\}$

$$b_j = (g_j + b_{j+1}) \bmod 2$$

- **proof**

$$g_j = (b_j + b_{j+1}) \bmod 2 \Rightarrow g_j \equiv (b_j + b_{j+1}) \pmod 2 \Rightarrow$$
$$b_j \equiv (g_j + b_{j+1}) \pmod 2 \Rightarrow b_j = (g_j + b_{j+1}) \bmod 2$$

lemma 3

Suppose

- $0 \le r \le 2^n - 1$
- $B = b_n, \ldots, b_0$ is a binary code of r
- $G = g_n, \ldots, g_0$ is a Gray code of r

Then for every $j \in \{0, 1, \ldots, n-1\}$

$$b_j = \left(\sum_{i=j}^{n-1} g_i \right) \bmod 2$$

proof

$$\left(\sum_{i=j}^{n-1} g_i \right) \bmod 2 = \left(\sum_{i=j}^{n-1} (b_i + b_{i+1}) \right) \bmod 2 = \left(b_j + b_n + 2 \sum_{i=j+1}^{n-1} b_i \right) \bmod 2 = (b_j + b_n) \bmod 2 = b_j$$

By lemma 1.

By the sum reordering.

By the property of modulo.

By the maximum range of r and the range of b_j.

- **converting to and from minimal change ordering (Gray code)**

```
Function BinaryToGray( binary code rank B ) : gray code rank
return  B xor (B >> 1) ;
```

```
Function GrayToBinary(gray code rank G ) : binary code rank
B = 0;
n = (number of bits in G) – 1;
 for i=0 to n do {
  B = B << 1;
  B = B or (1 and ( (B >> 1) xor (G >> n) ) );
  G = G << 1;
 }
 return B ;
```

Subsets – Gray Code

- **computing the subset rank over minimal change ordering**

```
Function GRAYCODERANK( size n; set T ) : rank
r = 0 ;
b = 0 ;
for i = n – 1 downto 0 do {
    if n – i ∈ T then b = 1 – b ;
    if b = 1 then r = r + 2^i ;
}
return r ;
```

- **computing the subset unrank over minimal change ordering**

```
Function GRAYCODEUNRANK( size n; rank r ) : set
T = ∅ ;
c = 0 ;
for i = n – 1 downto 0 do {
    b = ⌊ r / 2^i ⌋ ;
    if b ≠ c then T = T ∪ {n – i} ;
    c = b ;
    r = r – b·2^i ;
}
return T ;
```

k - Element subsets

- Suppose that n is a positive integer and $S = \{1, \dots, n\}$.
- $\binom{S}{k}$ consists of all k*-element subsets* of S.
- A k-element subset $T \subseteq S$ can be represented in a natural way as a sorted one-dimensional array $\vec{T} = [t_1, t_2, \dots, t_k]$ where $t_1 < t_2 < \cdots < t_k$.

k - Element subsets

- Example of the lexicographic ordering on k-element subsets:

T	$\vec{T}$	$rank(T)$
{1,2,3}	[1,2,3]	0
{1,2,4}	[1,2,4]	1
{1,2,5}	[1,2,5]	2
{1,3,4}	[1,3,4]	3
{1,3,5}	[1,3,5]	4
{1,4,5}	[1,4,5]	5
{2,3,4}	[2,3,4]	6
{2,3,5}	[2,3,5]	7
{2,4,5}	[2,4,5]	8
{3,4,5}	[3,4,5]	9

k - Element subsets

- **computing the k-element subset successor with lexicographic ordering**

```
Function KSubsetLexSuccesor(k-element subset as array T;
                              number n, k ) : k-element subset as array ;
U = T ;
i = k ;
while ( i ≥ 1 ) and ( T[i] = n – k + i )  do  i = i – 1 ;
if ( i = 0 ) then
    return "undefined" ;
else {
    for j = i  to  k  do  U[j] = T[i] + 1 + j – i ;
    return  U ;
}
```

- **computing the k-element subset rank with lexicographic ordering**

Function KSUBSETLEXRANK(k-element subset as array T;
number n, k) : rank ;
$r = 0$;
$T[0] = 0$;
for $i = 1$ **to** k **do** {
  if ($T[i\text{-}1]+1 \leq T[i]-1$) **then** {
    for $j = T[i\text{-}1]+1$ **to** $T[i]-1$ **do** $r = r + \binom{n-j}{k-i}$;
  }
}
return r ;

k - Element subsets

- **computing the k-element subset unrank with lexicographic ordering**

Function KSUBSETLEXUNRANK(rank r;
number n, k) : k-element subset as array ;
$x = 1$;
for $i = 1$ **to** k **do** {
    while ($\binom{n-x}{k-i} \le r$) **do** {
        $r = r - \binom{n-x}{k-i}$;
        $x = x + 1$;
    }
    $T[i] = x$;
    $x = x + 1$;
}
return T ;

Permutations

- A *permutation* is a bijection from a set to itself.
- one possible representation of a permutation

$$\pi: \{1, \dots, n\} \to \{1, \dots, n\}$$

is by storing its values in a one-dimensional array as follows:

index	1	2	…	n
value	$\boldsymbol{\pi[1]}$	$\boldsymbol{\pi[2]}$	**…**	$\boldsymbol{\pi[n]}$

Permutations

- **computing the permutation rank over lexicographical ordering**

```
Function PermLexRank( size n; permutation π ) : rank
r = 0 ;
ρ = π ;
for j = 1 to n do {
    r = r + (ρ[ j ] – 1)·(n – j)! ;
    for i = j +1 to n do if ρ[ i ] > ρ[ j ] then ρ[ i ] = ρ[ i ] – 1 ;
}
return r ;
```

- **computing the permutation unrank over lexicographical ordering**

Function PermLexUnRank(size n; rank r) : permutation

$\pi[\, n \,] = 1$;

for $j = 1$ **to** $n - 1$ **do** {

  $d = \frac{r \textbf{ mod } (j+1)!}{j!}$;

  $r = r - d \cdot j!$;

  $\pi[\, n - j \,] = d + 1$;

  for $i = n - j + 1$ **to** n **do if** $\pi[\, i \,] > d$ **then** $\pi[\, i \,] = \pi[\, i \,] + 1$;

}

return π ;

References

- D.L. Kreher and D.R. Stinson , *Combinatorial Algorithms: Generation, Enumeration and Search* , CRC press LTC , Boca Raton, Florida, 1998.