NONLINEAR MODEL PREDICTIVE CONTROL

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Glossary

Control horizon: Time-interval over which the open-loop optimal input determined.
Cost function: Integral functional that is minimized during the open-loop optimization. It usually depends on the system states and inputs and spans over the prediction horizon in the future.
Guaranteed stability: Denotes control strategies that lead to stability of the closed loop independent of the design and plant parameters.
NMPC: Acronym for nonlinear model predictive control.
MHE: Acronym for moving horizon (state) estimation.
MPC: Acronym for model predictive control.
Prediction horizon: Time-interval over which the system is predicted in the future.
Receding horizon control: Often used as a synonym for model predictive control.
Stage cost: Integrand appearing in the integral of the cost function.

Terminal penalty term: Penalty term in the cost function that penalizes the final predicted state.

Terminal region constraint: Constraint that is added to the open-loop optimal control problem to enforce feasibility and stability.

Value function: Optimal cost function.

QP: Acronym for quadratic program. Optimization of a quadratic function over a polyhedron, defined by linear equations and/or inequalities.

SQP: Acronym for sequential quadratic programming. Solving a nonlinear program by a sequence of quadratic approximations and using quadratic programming to solve each one. The approximations are usually done by using the second-order Taylor expansion.

Summary

While linear model predictive control is popular since the 70s of the past century, the 90s have witnessed an steadily increasing attention from control theoretists as well as control practitioners in the area of nonlinear model predictive control (NMPC). The practical interest is driven by the fact that today's processes need to be operated under tighter performance specifications. At the same time more and more constraints, stemming for example from environmental and safety considerations, need to be satisfied. Often these demands can only be met when process nonlinearities and constraints are explicitly considered in the controller. Nonlinear predictive control, the extension of well established linear predictive control to the nonlinear world, appears to be a well suited approach for this kind of problems. In this note the basic principle of NMPC is reviewed, the key advantages/disadvantages of NMPC are outlined and some of the theoretical, computational, and implementation aspects of NMPC are discussed. Furthermore, some of the currently open questions in the area of NMPC are outlined.

1. Introduction

Model predictive control (MPC), also referred to as moving horizon control or receding horizon control, is a control strategy in which the applied input is determined on-line at each sampling instance by the solution of an open-loop optimal control problem using the current (estimated) state as initial state. The solution of the optimization leads an optimal input signal from which only the first part is implemented until the next measurement becomes available. The online calculation of the optimal input is the main difference of MPC to conventional control, in which normally a pre-computed control law is used. As is shown in the later sections, on one side the on-line calculation leads to a set of problems like the question of closed-loop stability and the necessity to solve an open-loop optimal control problem. On the other side it also allows together with the often used time-domain formulation, the consideration of otherwise difficult to include requirements, like hard constraints on inputs and states, optimal handling of multivariable control problems and the possibility to take cost functions that must be minimized into account.

In general linear MPC and nonlinear MPC (NMPC) are distinguished (see also Model Based Predictive Control for Linear Systems and Model Based Predictive Control). Linear MPC refers to a family of MPC schemes in which linear models are used to predict the system dynamics, even though the dynamics of the closed-loop system is nonlinear due to the presence of constraints. Linear MPC approaches have found successful applications, especially in the process industry. The success of linear MPC is mainly based on the fact, that model predictive control is able to handle constraints on inputs and states and has the ability to cope with multivariable systems in an optimal way. Several thousand applications in a very wide range from chemical to aerospace industry are reported so far. By now, linear MPC theory is quite mature. Important issues such as online computation, the interplay between modeling/identification and control and system theoretic issues like stability are well addressed (see Model Based Predictive Control for Linear Systems).

Many systems are, however, in general inherently nonlinear. This, together with higher product quality specifications and increasing productivity demands, tighter environmental regulations and demanding economical considerations in the process industry require to operate systems closer to the boundary of the admissible operating region. In these cases, linear models are often inadequate to describe the process dynamics and nonlinear models have to be used. This motivates the use of nonlinear model predictive control.

This paper reviews the main principles underlying NMPC and outlines the key advantages/disadvantages of NMPC and some of the theoretical, computational, and implementation aspects. In Section 1.1 and Section 1.2 the basic underlying concept of NMPC is introduced. In Section 2 some of the system theoretical aspects of NMPC are presented. Besides the basic question of the stability of the closed-loop, which stems from the fact that the predicted open-loop and the actual closed-loop behavior of the system is in general different, questions such as robust formulations of NMPC and some remarks on the performance of the closed-loop and the need to estimate the system states are examined. Section 3 contains some remarks and descriptions concerning the numerical solution of the open-loop optimal control problem. Finally some of the open issues in the area of NMPC are outlined and some conclusions are drawn.

Notice that in the text no references are given to make the text as self contained as possible. The corresponding references are however collected at the end of the note to allow the interested reader an easy access to the existing literature.

1.1. The Basic Principle of Model Predictive Control

In general, the model predictive control problem is formulated as solving on-line a finite horizon open-loop optimal control problem subject to system dynamics and constraints involving states and controls. Figure 1 shows the general principle of model predictive control. Based on measurements obtained at time t, the controller predicts the future dynamic behavior of the system over a prediction horizon T_p and determines (over a control horizon $T_c \leq T_p$) the input such that a predetermined open-loop performance objective functional is optimized. *If* there were no disturbances and no model-plant mismatch, and *if* the optimization problem could be solved for infinite horizons, then one could apply the input function found at time $t = 0$ to the system for all times $t \geq 0$. However, this is not possible in general. Due to disturbances and model-plant mismatch, the true system behavior is different from the predicted behavior. In order to incorporate some feedback mechanism, the open-loop manipulated input function obtained will be implemented only until the next measurement becomes available. The time difference between the recalculation/measurements can vary, however often it is assumed to be fixed, i.e the measurement will take place every δ sampling time-units. Using the new measurement at time $t + \delta$, the whole procedure – prediction and optimization – is repeated to

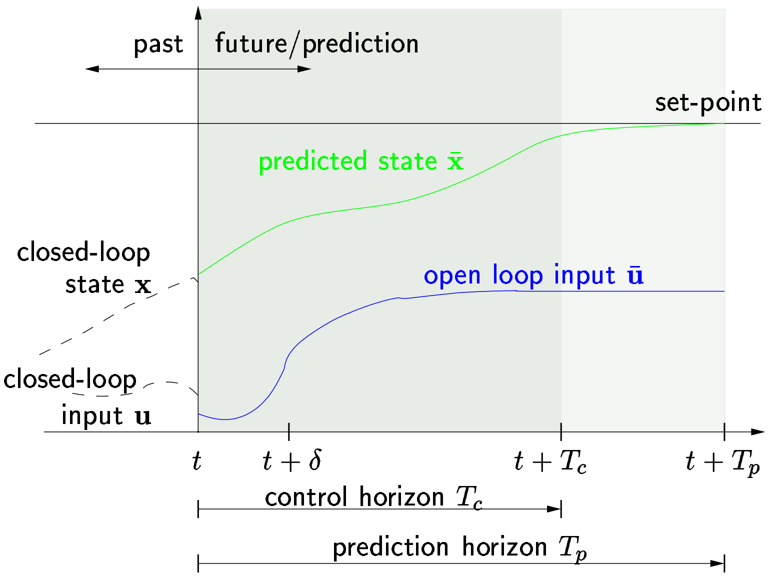

Figure 1: Principle of model predictive control.

find a new input function with the control and prediction horizons moving forward.

Notice, that in Fig. 1 the input is depicted as arbitrary function of time. As shown in Section 3 for numerical solutions of the open-loop optimal control problem it is often necessary to parameterize the input in an appropriate way. This is normally done by using a finite number of basis functions, e.g. the input could be approximated as piecewise constant over the sampling time δ.

As will be shown, the calculation of the applied input based on the predicted system behavior allows the inclusion of constraints on states and inputs as well as the optimization of a given cost function. However, since in general the predicted system behavior will differ from the closed-loop one, precaution must be taken to achieve closed-loop stability.

1.2. Mathematical Formulation of NMPC

Consider a class of continuous time systems described by the following nonlinear differential equation[1]

$$\dot{\mathbf{x}}(t) \quad = \quad \mathbf{f}\left(\mathbf{x}(t), \mathbf{u}(t)\right), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{1}$$

subject to input and state constraints of the form:

$$\mathbf{u}(t) \in \mathcal{U},\ \forall t \geq 0 \tag{2a}$$
$$\mathbf{x}(t) \in \mathcal{X},\ \forall t \geq 0, \tag{2b}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ and $\mathbf{u}(t) \in \mathbb{R}^m$ denote the vector of states and inputs, respectively. In general the input and state constraint sets $\mathcal{X}$ and $\mathcal{U}$ are compact and contain (in the case of the

[1]In this note only the continuous time formulation of NMPC is considered. However, notice that most of the presented topics have dual counterparts in the discrete time setting.

stabilization problem) the origin. In its simplest form, $\mathcal{U}$ and $\mathcal{X}$ are given by box constraints of the form:

$$\mathcal{U} := \{\mathbf{u} \in \mathbb{R}^m | \mathbf{u}_{min} \leq \mathbf{u} \leq \mathbf{u}_{max}\} \tag{3a}$$
$$\mathcal{X} := \{\mathbf{x} \in \mathbb{R}^n | \mathbf{x}_{min} \leq \mathbf{x} \leq \mathbf{x}_{max}\}. \tag{3b}$$

Here $\mathbf{u}_{min}$, $\mathbf{u}_{max}$ and $\mathbf{x}_{min}$, $\mathbf{x}_{max}$ are given constant vectors.

Usually, the finite horizon open-loop optimal control problem, as described above, is mathematically formulated as follows.
Problem 1 *Find* $\min_{\bar{\mathbf{u}}(\cdot)} J(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot); T_c, T_p)$

with $$J(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot); T_c, T_p) := \int_t^{t+T_p} F(\bar{\mathbf{x}}(\tau), \bar{\mathbf{u}}(\tau)) d\tau \tag{4}$$

subject to:
$$\dot{\bar{\mathbf{x}}}(\tau) = \mathbf{f}(\bar{\mathbf{x}}(\tau), \bar{\mathbf{u}}(\tau)), \quad \bar{\mathbf{x}}(t) = \mathbf{x}(t) \tag{5a}$$
$$\bar{\mathbf{u}}(\tau) \in \mathcal{U}, \quad \forall \tau \in [t, t+T_c] \tag{5b}$$
$$\bar{\mathbf{u}}(\tau) = \bar{\mathbf{u}}(\tau + T_c), \quad \forall \tau \in [t+T_c, t+T_p] \tag{5c}$$
$$\bar{\mathbf{x}}(\tau) \in \mathcal{X}, \quad \forall \tau \in [t, t+T_p] \tag{5d}$$

where T_p *and* T_c *are the prediction and the control horizon with* $T_c \leq T_p$ The bar denotes internal controller variables and $\bar{\mathbf{x}}(\cdot)$ is the solution of (5a) driven by the input $\bar{\mathbf{u}}(\cdot) : [t, t+T_p] \rightarrow \mathcal{U}$ with initial condition $\mathbf{x}(t)$. The distinction between the real system and the variables in the controller is necessary, since the predicted values, even in the nominal undisturbed case, need not, and in generally will not, be the same as the actual closed-loop values, since the optimal input is recalculated (over a moving finite horizon T_c) at every sampling instance.

The function F specifies the desired control performance that can arise, for example from economical and ecological considerations. The standard quadratic form is the simplest and most often used one:

$$F(\mathbf{x}, \mathbf{u}) = (\mathbf{x} - \mathbf{x}_s)^T Q (\mathbf{x} - \mathbf{x}_s) + (\mathbf{u} - \mathbf{u}_s)^T R (\mathbf{u} - \mathbf{u}_s), \tag{6}$$

where $\mathbf{x}_s$ and $\mathbf{u}_s$ denote a given reference trajectory, that can be constant or time-varying; Q and R denote positive definite, symmetric weighting matrices.

In order to have a feasible solution of Problem 1, $(\mathbf{x}_s, \mathbf{u}_s)$ should be contained in the interior of $\mathcal{X} \times \mathcal{U}$. In the case of a stabilization problem (no tracking), it can be assumed, without loss of generality, that $(\mathbf{x}_s, \mathbf{u}_s) = (\mathbf{0}, \mathbf{0})$ is the steady state point that should be stabilized. Note the initial condition in (5a): The system model used to predict the future in the controller is initialized by the actual system state; thus they are assumed to be measured or must be estimated. Equation (5c) is not a constraint but implies that beyond the control horizon the predicted control takes a constant value equal to that at the last step of the control horizon.

In the following an optimal solution to the optimization problem (existence assumed) is denoted by $\bar{\mathbf{u}}^*(\cdot; \mathbf{x}(t), T_p) : [t, t+T_p] \rightarrow \mathcal{U}$. The open-loop optimal control problem will be solved repeatedly at the sampling instances $t = j\delta$, $j = 0, 1, \cdots$, once new measurements are available. The *closed-loop* control is defined by the sequence of repeated optimal solutions of Problem 1:

$$\mathbf{u}^*(\tau) := \bar{\mathbf{u}}^*(\tau; \mathbf{x}(t)) \; , \; \tau \in [t, \delta] \, . \tag{7}$$

The *nominal closed-loop system* using the input (7) is given by:

$$\dot{\mathbf{x}}(t) \;\; = \;\; \mathbf{f}\left(\mathbf{x}(t), \mathbf{u}^*(t)\right) . \tag{8}$$

The optimal value of the NMPC open-loop optimal control problem as a function of the state will be denoted in the following as value function:

$$V(\mathbf{x}; T_p, T_c) = J(\mathbf{x}, \bar{\mathbf{u}}^\star(\cdot; \mathbf{x}(t)); T_p, T_c). \tag{9}$$

The value function plays an important role in the proof of the stability of various NMPC schemes, as it serves as a Lyapunov function candidate.

1.3. Properties, Advantages, and Disadvantages of NMPC

In general one would like to use an infinite prediction and control horizon, i.e. T_p and T_c in Problem 1 are set to ∞. This would allow (at least in the nominal case) to minimize the performance objective determined by the cost functional *over the infinite horizon*, and by this also the closed loop system performance as explained in the following. However, the open-loop optimal control Problem 1 that must be solved on-line, is often formulated over a finite horizon in order to allow a (real-time) numerical solution of the nonlinear open-loop optimal control problem. When a finite prediction horizon is used, the actual closed-loop input and state trajectories will differ from the predicted open-loop trajectories, even if no model plant mismatch and no disturbances are present (compare Model Based Predictive Control). This fact can be easiest explained considering somebody hiking in the mountains with no height map at hand. Lets assume he wants to take the shortest route to a certain place. Since he is walking in a hilly area, the only thing he can do is to climb the nearest mountain and plan the shortest route based on how far he can see. Then he follows the planned way until he comes to another hill, where he obtains new information. Then he normally re-optimizes the predicted path, since he can see further. In principle this is the same phenomena encountered using a finite horizon NMPC strategy. At a certain sampling instance the future is only predicted for a certain time interval in the future. At the next sampling instance one can obtain more information and re-plan. This fact is depicted in Figure 2 where the system can only move inside the shaded area as state constraints of the form $\mathbf{x}(\tau) \in \mathcal{X}$ are assumed. This makes the key difference between standard control strategies, where the feedback law is obtained a priori and NMPC where the feedback law is obtained on-line and has two immediate consequences. Firstly, the actual goal to compute a feedback such that the performance objective over the *infinite horizon* of the closed-loop is minimized is not achieved. In general it is by no means true that a repeated minimization over a *finite horizon objective* in a receding horizon manner leads to an optimal solution for the infinite horizon problem (with the same stage cost F). In fact, the two solutions will in general differ significantly if a finite horizon is chosen. Secondly, if the predicted and the actual trajectory differ, there is no guarantee that the closed-loop system will be stable. It is indeed easy to construct examples for which the closed-loop becomes unstable

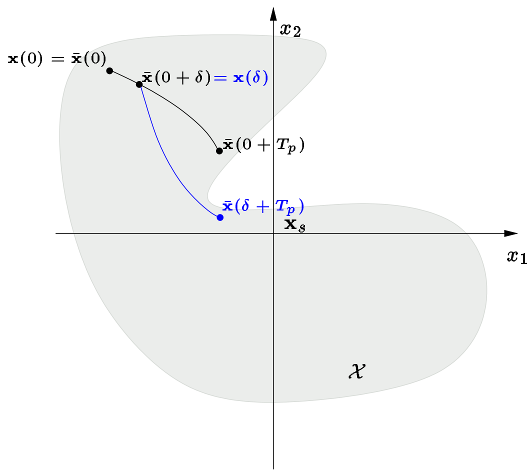

Figure 2: The difference between open-loop prediction and closed-loop behavior.

if a (small) finite horizon is chosen. Hence, when using finite horizons in standard NMPC, the stage cost cannot be simply chosen based on the desired physical objectives.

The overall basic structure of a NMPC control loop is depicted in Figure 3. As can be seen, it is

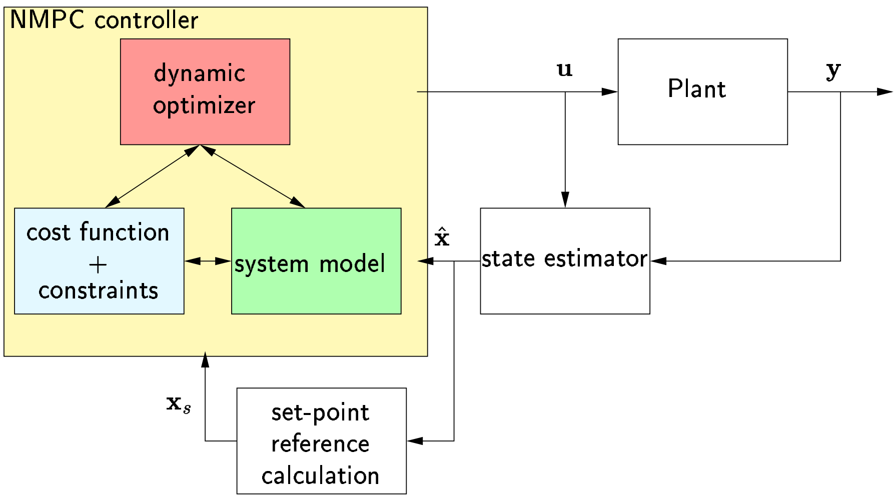

Figure 3: Basic NMPC control loop.

necessary to estimate the system states from the output measurements. Additionally, as shown, it is possible to add a reference/set-point or target calculation to the overall loop. However this will not be covered in this note.

Summarizing the basic NMPC scheme works as follows:

1. obtain measurements/estimates of the states of the system
2. obtain an optimal input signal by **minimizing** a given **cost function** over a certain **prediction horizon** in the future using a **model of the system**
3. **implement the first part of the optimal input signal** until new measurements/estimates of the state are available
4. continue with 2.

From the remarks given so far and from the basic NMPC setup, one can extract the following key characteristics of NMPC:

- NMPC allows the use of a nonlinear model for prediction.
- NMPC allows the explicit consideration of state and input constraints.
- In NMPC a specified performance criteria can be minimized on-line.
- In NMPC the predicted behavior is in general different from the closed loop behavior.
- The on-line solution of an open-loop optimal control problem is necessary for the application of NMPC.
- To perform the prediction the system states must be measured or estimated.
- A (nonlinear) model is explicitly needed for the predictions.

In the remaining sections various aspects of these properties will be discussed. The next Section focuses on system theoretical aspects of NMPC. Especially the questions on closed-loop stability, robustness and the output feedback problem are considered.

2. Systen Theoretical Aspects of NMPC

In this Section different system theoretical aspects of NMPC are considered. Besides the question of nominal stability of the closed-loop, which can be considered as somehow mature today, remarks on robust NMPC strategies as well as the output-feedback problem are given.

2.1. Stability

One of the key questions in NMPC is certainly, whether a finite horizon NMPC strategy does lead to stability of the closed-loop. As pointed out, the key problem with a finite prediction and control horizon stems from the fact that the predicted open and the resulting closed-loop behavior is in general different. Ideally one would seek for a NMPC strategy achieves closed-loop stability independent of the choice of the performance parameters in the cost functional and, if possible, approximates the infinite horizon NMPC scheme as good as possible. A NMPC strategy that achieves closed-loop stability independent of the choice of the performance parameters is usually referred to NMPC approach *with guaranteed stability*. Different possibilities to achieve closed-loop stability for NMPC using finite horizon length have been proposed. Here only the key ideas are reviewed and no detailed proofs nor a detailed review is given. Also most of the technical details and conditions are left out for reasons simplicity. For the stabilization problem, without loss of generality, it is assumed that the origin ($\mathbf{x} = \mathbf{0}$ and $\mathbf{u} = \mathbf{0}$) is the steady state that should be stabilized. Also the prediction horizon is set equal to the control horizon, $T_p = T_c$.

2.1.1. Infinite Horizon NMPC

The most intuitive way to achieve stability is the use of an infinite horizon cost, i.e. T_p in Problem 1 is set to ∞. In the nominal case feasibility at one sampling instant also implies feasibility and optimality at the next sampling interval. This follows from Bellman's Principle of Optimality (see Optimal Control), i.e. the input and state trajectories computed as the solution of the NMPC optimization Problem 1 at a specific instance in time, are in fact equal to the closed-loop trajectories of the nonlinear system, i.e. the remaining parts of the trajectories after one sampling instance are the optimal solution at the next sampling instance. This fact also implies closed-loop stability.

Key ideas of the stability proof: Since nearly all stability proofs for NMPC are similar to the infinite horizon proof, the key ideas are shortly outlined. In principle the proof is based on the use of the optimal cost function, i.e. the so called value function, as a Lyapunov function (see Stability/Lyapunov Functions). First it is shown, that feasibility at one sampling instance does imply feasibility at the next sampling instance for the nominal case. In a second step it is established that the value function is strictly decreasing and by this the state and input converge to the origin. Utilizing the continuity of the value function at the origin and the monotonicity property, asymptotic stability is established in the third step. As feasibility thus implies asymptotic stability, the set of all states, for which the open-loop optimal control problem has a solution does belong to the region of attraction of the origin.

2.1.2. Finite Horizon NMPC Schemes with Guaranteed Stability

Different possibilities to achieve closed-loop stability for NMPC using a finite horizon length have been proposed. Most of these approaches modify the NMPC setup such that stability of the closed-loop can be guaranteed independently of the plant and performance specifications. This is usually achieved by adding suitable equality or inequality constraints and suitable additional penalty terms to the cost functional to the standard setup. These additional constraints are usually not motivated by physical restrictions or desired performance requirements but have the sole purpose to enforce stability of the closed-loop. Therefore, they are usually termed *stability constraints*.

The key drawback of an infinite horizon NMPC formulation is that at every sampling instance an infinite dimensional optimization problem must be solved. The simplest possibility to enforce stability with a finite prediction horizon is to add a so called *zero terminal equality constraint* at the end of the prediction horizon, i.e. to add the equality constraint

$$\bar{\mathbf{x}}(t+T_p;\mathbf{x}(t),t,\bar{\mathbf{u}}) = \mathbf{0} \tag{10}$$

to Problem 1. This leads to stability of the closed-loop, if the optimal control problem posses a solution at $t = 0$, since the feasibility at one time instance does also lead to feasibility at the following time instances and a decrease in the value function. One disadvantage of a zero terminal constraint is that the system must be brought to the origin in finite time. This leads in general to feasibility problems for short prediction/control horizon lengths, i.e. a small region of attraction. Additionally, from a computational point of view, an exact satisfaction of a zero terminal equality constraint does require an infinite number of iterations in the nonlinear programming problem. On the other hand, the main advantages are the straightforward application and the conceptual simplicity.

Many schemes have been proposed, that try to overcome the use of a zero terminal constraint of the form (10). Most of them either use a so called *terminal region constraint*

$$\bar{\mathbf{x}}(t+T_p) \in \Omega \tag{11}$$

and/or a *terminal penalty term* $E(\bar{\mathbf{x}}(t+T_p))$ that are added to the basic NMPC setup. Note that the terminal penalty term is not a performance specification that can be chosen freely. Rather E and the terminal region Ω in (11) are determined off-line such that the cost functional *with* terminal penalty term

$$J\left(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot); T_p\right) \;=\; \int_t^{t+T_p} F\left(\bar{\mathbf{x}}(\tau), \bar{\mathbf{u}}(\tau)\right)\, d\tau + E(\bar{\mathbf{x}}(t+T_p)) \tag{12}$$

gives an upper bound of the infinite horizon cost functional with stage cost F. The resulting open-loop optimization problem is formulated as follows:

Problem 2: *Find* $\min_{\bar{\mathbf{u}}(\cdot)} J(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot); T_p)$

$$\textit{with} \quad J(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot); T_p) := \int_t^{t+T_p} F(\bar{\mathbf{x}}(\tau), \bar{\mathbf{u}}(\tau)) d\tau + E(\bar{\mathbf{x}}(t+T_p)) \tag{13}$$

$$\begin{aligned}
\textit{subject to:} \quad & \dot{\bar{\mathbf{x}}}(\tau) = \mathbf{f}\left(\bar{\mathbf{x}}(\tau), \bar{\mathbf{u}}(\tau)\right), \quad \bar{\mathbf{x}}(t) = \mathbf{x}(t) && \text{(14a)}\\
& \bar{\mathbf{u}}(\tau) \in \mathcal{U}, \quad \forall \tau \in [t, t+T_p] && \text{(14b)}\\
& \bar{\mathbf{x}}(\tau) \in \mathcal{X}, \quad \forall \tau \in [t, t+T_p] && \text{(14c)}\\
& \bar{\mathbf{x}}(t+T_p) \in \Omega. && \text{(14d)}
\end{aligned}$$

If the terminal penalty term E and the terminal region Ω are chosen suitably, stability of the closed-loop can be guaranteed. The following (simplified) stability theorem can be given (relaxed and more general versions exist):

Theorem 2.1 *Suppose*

(a) *$\mathcal{U} \subset \mathbb{R}^m$ is compact, $\mathcal{X} \subseteq \mathbb{R}^n$ is connected and the origin is contained in the interior of $\mathcal{U} \times \mathcal{X}$.*

(b) *The vector field $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and satisfies $\mathbf{f}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$. In addition, it is locally Lipschitz continuous in $\mathbf{x}$.*

(c) *The system has an unique continuous solution for any initial condition in the region of interest and any piecewise continuous and right continuous input function $\mathbf{u}(\cdot) : [0, T_p] \to \mathcal{U}$.*

(d) *$F : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ is continuous in all arguments with $F(\mathbf{0}, \mathbf{0}) = 0$ and $F(\mathbf{x}, \mathbf{u}) > 0 \;\; \forall (\mathbf{x}, \mathbf{u}) \in \mathbb{R}^n \times \mathcal{U} \setminus \{\mathbf{0}, \mathbf{0}\}$.*

(e) *E is C^1 with $E(\mathbf{0}, \mathbf{0}) = 0$, $\Omega \subseteq \mathcal{X}$ is closed and connected with the origin contained in Ω and for any $\mathbf{x} \in \Omega$ there exists a $\mathbf{u} \in \mathcal{U}$, s.t.*

$$\frac{\partial E}{\partial \mathbf{x}} f(\mathbf{x}, \mathbf{u}) + F(\mathbf{x}, \mathbf{u}) \leq 0, \quad \forall \mathbf{x} \in \Omega \tag{15}$$

(f) the NMPC open-loop optimal control problem has a feasible solution for $t = 0$.

Then for any sampling time $0 < \delta < T_p$ *the nominal closed-loop system is asymptotically stable with the region of attraction* $\mathcal{R}$ *being the set of states for which the open-loop optimal control problem has a feasible solution.*

Loosely speaking E is a local control Lyapunov function (see Lyapunov Design) in Ω. As is shown in Appendix A, equation (15) allows to upper bound the optimal infinite horizon cost inside Ω by the cost resulting from a local feedback $\mathbf{k}(\mathbf{x})$, namely by E. This together with the fact that the terminal region constraint enforces feasibility at the next sampling instances allows, similar to the infinite horizon case, to show that the optimal cost (the value function) is strictly decreasing. Thus stability can be established.

Notice, that this result is nonlocal in nature, i.e. there exists a region of attraction $\mathcal{R}$ of at least the size of Ω. The region of attraction is given by all states for which the open-loop optimal control problem has a feasible solution.

Many NMPC schemes that guarantee stability and fit into this setup exist. They vary in how the terminal region and terminal penalty terms are obtained and/or if they appear at all. For example if a global control Lyapunov function for the system is known no terminal region constraint is needed and the terminal penalty term is set to the control Lyapunov function. However, in this case the weighting function F must be chosen such that condition (e) of Theorem 2.1 is satisfied. Another possibility is to use a (known) local control law that stabilizes the system and satisfies the constraints and to calculate a corresponding terminal penalty and terminal region as is for example the case in the quasi-infinite horizon NMPC approach.

The use of the terminal inequality constraint gives the described NMPC scheme computational advantages. The satisfaction of a zero terminal constraint in finite time is not necessary. The solution time necessary for solving the open-loop optimal control problem is decreased, since no "boundary-value" problem stemming from the zero terminal constraint must be solved. Note also that it is not necessary to find optimal solutions of Problem 1 in order to guarantee stability. Only a decrease in the value function (which is used as Lyapunov function/decreasing function) is necessary, thus feasibility implies stability.

Summarizing, the nominal stability question of NMPC is understood quite well and various NMPC approaches with guaranteed stability exist. However, there are various system theoretical aspects of NMPC, like robustness and output-feedback, that must be further examined (see Section 4).

2.2. Performance of Finite Horizon NMPC Formulations

Ideally one would like to use an infinite horizon NMPC formulation, since in the nominal case, the closed-loop trajectories do coincide with the open-loop predicted ones (principle of optimality). The main problem is, that infinite horizon schemes can often not be applied in practice, since the open-loop optimal control problem cannot be solved sufficiently fast. Using finite horizons, however, it is by no means true that a repeated minimization over a *finite horizon objective* in a receding horizon manner leads to an optimal solution for the infinite horizon problem (with the same stage cost F). In fact, the two solutions will in general differ significantly if a short horizon is chosen.

From this discussion it is clear that short horizons are desirable from a computational point of view, but long horizons are required for closed-loop stability and in order to achieve the desired performance.

The NMPC strategy outlined in the previous section allows in principle to recover the performance of the infinite horizon scheme without jeopardizing the closed-loop stability. As shown in Appendix A the value function resulting from Problem 2 can be seen as an upper bound of the infinite horizon cost. To be more precise, if the terminal penalty function E is chosen such that a corresponding local control law is a good approximation of the control resulting from the infinite horizon control law in a neighborhood of the origin, the performance corresponding to Problem 2 can recover the performance of the infinite horizon cost even for short horizons (assuming the terminal region constraint can be satisfied).

2.3. Robust Stability

So far only the nominal control problem was considered. The NMPC schemes discussed before do require that the actual system is identical to the model used for prediction, i.e. that no model/plant mismatch or unknown disturbances are present. Clearly this is a very unrealistic assumption for practical applications and the development of a NMPC framework to address robustness issues is of paramount importance. In this note the nonlinear uncertain system is assumed to be given by:

$$\dot{\mathbf{x}}(t) \quad = \quad \mathbf{f}\left(\mathbf{x}(t), \mathbf{u}(t), \mathbf{d}(t)\right) \tag{16}$$

where the uncertainty $\mathbf{d}(\cdot)$ satisfies $\mathbf{d}(\tau) \in \mathcal{D}(x, u)$ and $\mathcal{D}$ is assumed to be compact. Like in the nominal stability and performance case, the resulting difference between the predicted open-loop and actual closed-loop trajectory is the main obstacle. As additional problem now the uncertainty $\mathbf{d}$ hitting the system leads not only to one single future trajectory in the prediction, instead a whole tree of possible solutions must be analyzed.

Even though the analysis of robustness properties in *nonlinear* NMPC must still be considered as an unsolved problem in general, some preliminary results are available.

In principle one must distinguish between two approaches to consider the robustness question. Firstly one can examine the robustness properties of the NMPC schemes designed for nominal stability and by this take the uncertainty/disturbances indirectly into account. Secondly one can consider to design NMPC schemes that directly take into account the uncertainty/disturbances.

2.3.1. Inherent Robustness of NMPC

As mentioned above, inherent robustness corresponds to the fact, that nominal NMPC can cope with input model uncertainties without taking them directly into account. This fact stems from the close relation of NMPC to optimal control. Assuming that the system under consideration is of the following (input affine) form:

$$\dot{\mathbf{x}}(t) \;=\; \mathbf{f}\left(\mathbf{x}(t)\right) + \mathbf{g}(\mathbf{x}(t))\mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{17}$$

and the cost function takes the form:

$$J(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot); T_p) := \int_t^{t+T_p} \frac{1}{2}\|\mathbf{u}\|^2 + \mathbf{q}(\mathbf{x})d\tau + E(\bar{\mathbf{x}}(t+T_p)) \tag{18}$$

where $\mathbf{q}$ is positive definite, there are no constraints on the state and the input and the resulting control law and the value function satisfies further technical assumptions ($\mathbf{u}^\star$ being continuously differentiable and the value function being twice continuously differentiable) one can show that the NMPC control law is inverse optimal, i.e. it is also optimal for a modified optimal control problem spanning over an infinite horizon. Due to this inverse optimality, the NMPC control law inherits the same robustness properties as infinite horizon optimal control (see Optimal Control) assuming that the sampling time δ goes to zero. In particular, the closed-loop is robust with respect to sector bounded input uncertainties; the nominal NMPC controller also stabilizes systems of the form:

$$\dot{\mathbf{x}}(t) \;=\; \mathbf{f}\left(\mathbf{x}(t)\right) + \mathbf{g}(\mathbf{x}(t))\phi(\mathbf{u}(t)), \tag{19}$$

where $\phi(\cdot)$ is a nonlinearity in the sector $(1/2, \infty)$.

2.3.2. Robust NMPC Schemes

Most of the NMPC schemes that directly take the uncertainty into account are based on a game-theoretic (see H_∞ Control, Game-Theory) setting of the robust stabilization problem. The disturbance/uncertainty in the system is considered as a player working against the input, thus the problem is formulated as a classical min-max problem over a finite horizon. At least three different formulations exist:

- **Robust NMPC solving an open-loop min-max problem**:
  In this formulation the standard NMPC setup is kept, however now the cost function optimized is given by the worst case disturbance "sequence" occurring, i.e.

  $$J(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot); T_p) := \max_{\bar{\mathbf{d}}(\cdot)} \int_t^{t+T_p} F(\bar{\mathbf{x}}(\tau), \bar{\mathbf{u}}(\tau))d\tau + E(\bar{\mathbf{x}}(t+T_p)). \tag{20}$$

  subject to

  $$\dot{\bar{\mathbf{x}}}(t) \;=\; \mathbf{f}\left(\bar{\mathbf{x}}(t), \bar{\mathbf{u}}(t), \bar{\mathbf{d}}(t)\right), \quad \bar{\mathbf{x}}(t) = \mathbf{x}(t). \tag{21}$$

  The resulting open-loop optimization is a min-max problem. The key problem is, that adding stability constraints like in the nominal case, might lead to the fact that no feasible solution can be found at all. This mainly stems from the fact, that one input signal must "reject" all possible disturbances and guarantee the satisfaction of the stability constraints.

- **$\mathbf{H_\infty}$-NMPC:** Another possibility is to consider the standard H_∞ problem (see H_∞ control) in a receding horizon framework. Nonlinear H_∞ achieves robust stability, as its linear counterpart, due to the choice of a special cost function. The key obstacle is, that an infinite horizon min-max problem must be solved (solution of the nonlinear Hamilton-Jacobi-Isaacs equation). Modifying the NMPC cost functions similar to the H_∞ problem and optimizing over a sequence of control laws robustly stabilizing finite horizon H_∞–NMPC formulations can be achieved. The main obstacle is the prohibitive computational time necessary. This approach is in close connection to the first approach.

- **Robust NMPC optimizing a feedback controller used during the sampling times:**
  The open-loop formulation of the robust stabilization problem can be seen as very conservative, since only open-loop control is used during the sampling times, i.e. the disturbances are not directly rejected in between the sampling instances. To overcome this problem it has been proposed not to optimize over the input signal. Instead of optimizing the open-loop input signal directly, a feedback controller is optimized, i.e. the decision variable $\bar{\mathbf{u}}$ is not considered as optimization variable instead a "sequence" of control laws $\mathbf{u}_i = \mathbf{k}_i(\mathbf{x})$ applied during the sampling times is optimized. Now the optimization problem has as optimization variables the parameterizations of the feedback controllers $\{\mathbf{k}_1, \ldots, \mathbf{k}_N\}$. While this formulation is very attractive since the conservatism is removed, the solution is often prohibitively complex.

Summarizing, a deficiency of the standard NMPC setup is the open-loop nature of the control scheme, i.e. no feedback is used during the sampling instances. Thus the input signal in the open-loop min-max formulation has to be chosen such, that it can cope with all disturbances over the whole prediction horizon. Due to this the solution is in general very conservative or does not exist. To overcome this problem, the use of a feedback during the sampling instances in combination with a min-max formulation is very promising. However no really implementable robust NMPC formulation exists so far.

2.4. Output Feedback

So far it was assumed, that the system state necessary for prediction is (perfectly) accessible through measurements. In general this is not the case and a state observer, as already shown in Figure 3 must be implicitly or explicitly included in the control loop. Two main questions arise from the use of a state observer. Firstly the question occurs, if the closed-loop including the state observer possesses the same stability properties as the state feedback contribution alone. Secondly the question arises, what kind of observer should be used to obtain a good state estimate and good closed loop performance. The second point is not considered in detail here, see for example State Reconstruction by Observers. It is only noted, that a dual of the NMPC approach for control does exist for the state estimation problem. It is formulated as an on-line optimization similar to NMPC and is named moving horizon estimation (MHE). It is dual in the context, that a moving window of old measurement data is used to obtain an optimization based estimate of the system state. More can be found in State Reconstruction by Observers.

2.4.1. Stability of Output-Feedback NMPC

The most often used approach for output-feedback NMPC is based on the "certainty equivalence principle". The estimate is measured via a state observer and the estimated state $\hat{\mathbf{x}}$ is, without any further precaution, used in the model predictive controller. Even assuming, that the observer error is exponentially stable, often only local stability of the closed-loop can be shown, i.e. the observer error must be small to guarantee stability of the closed-loop and in general nothing can be said about the necessary degree of smallness. This is a consequence of the fact that no general valid separation principle for nonlinear systems exists. Nevertheless this approach is applied successfully in many applications.

To achieve non-local stability results of the observer based output-feedback NMPC controller, at least two possibilities to attack the problem exist:

- **Consideration of the observer error in the NMPC controller**: One could in principle consider the observer error as disturbance in the controller and design a NMPC controller that can reject this disturbance. The hurdle of this approach is the fact, that so far an applicable robust NMPC scheme is not available.
- **Utilization of special system structures to "decouple" the observer and the controller error:** In this approach use of the system structures is made. Using special separation principles based on high-gain observers (see State Reconstruction by Observers) semi-regional stability results for the closed-loop can be established. The key component is, that the speed of the observer can be made as fast as necessary.

Summarizing it can be noticed, that only limited results with respect to the output-feedback NMPC problem are available at present.

3. Computational Aspects of NMPC

NMPC requires repeated on-line solution of a *nonlinear* optimal control problem. In the case of linear MPC the solution of the optimal control problem can be cast as the solution of a (convex) quadratic program and can be solved efficiently even on-line. This can be seen as one of the reasons why linear MPC is widely used in industry. For the NMPC problem the solution in general involves the solution of a *nonlinear* program, as is shown in the preceding sections. In general the solution of a nonlinear (non-convex) optimization problem can be computational expensive. However in the case of NMPC the nonlinear program shows special structure that can be exploited to still achieve a real-time feasible solution to the NMPC optimization problem.

For the purpose of this Section the open-loop optimal control Problem 2 of Section 2.1.2 will be considered in a more optimization focused setting. Especially it is considered, that the state and input constraints $\mathbf{x} \in \mathcal{X}$, $\mathbf{u} \in \mathcal{U}$ can be recasted as a nonlinear inequality constraint of the form $\mathbf{l}(\mathbf{x}, \mathbf{u}) \leq 0$. Furthermore for simplicity of exposition it is assumed that the control and prediction horizon coincide and that no final region constraint is present, i.e. we consider the following deterministic optimal control problem in Bolza form that must be solved at every sampling instance:

Problem 3: *Find*

$$\min_{\bar{\mathbf{u}}(\cdot)} J(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot); T_p) \tag{22}$$

$$\textit{with} \quad J(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot); T_p) := \int_t^{t+T_p} F(\bar{\mathbf{x}}(\tau), \bar{\mathbf{u}}(\tau)) d\tau + E(\bar{\mathbf{x}}(t+T_p)) \tag{23}$$

$$\textit{subject to:} \quad \dot{\bar{\mathbf{x}}}(\tau) = \mathbf{f}\left(\bar{\mathbf{x}}(\tau), \bar{\mathbf{u}}(\tau)\right), \quad \bar{\mathbf{x}}(t) = \mathbf{x}(t) \tag{24a}$$

$$\mathbf{l}(\bar{\mathbf{u}}(\tau), \bar{\mathbf{x}}(\tau)) \leq \mathbf{0}, \quad \forall \tau \in [t, t+T_p]. \tag{24b}$$

3.1. Solution Methods for the Open-Loop Optimal Control Problem

In principle three different approaches to solve the optimal control Problem 3 exist:

- **Hamilton-Jacobi-Bellmann partial differential equations/dynamic programming:** This approach is based on the direct solution of the so called Hamilton-Jacobi-Bellmann partial differential equations (see Optimal Control). Rather than just seeking for the optimal $\mathbf{u}(\tau)$ trajectory the problem is approach as finding a solution for all $\mathbf{x}(t)$. The solution derived is a state feedback law of the form $\mathbf{u}^\star = \mathbf{k}(\mathbf{x})$ and is valid for every initial condition. The key obstacle of this approach is, that since the "complete" solution is considered at once, it is in general computationally intractable and suffers from the so called curse of dimensionality, i.e. can be only solved for small systems. Ideally one would like to obtain such a closed loop state feedback law. In principle the intractability of the solution can be seen as the key motivation of receding horizon control.

- **Euler-Lagrange differential equations/calculus of variations/maximum principle:** This method employs classical calculus of variations to obtain an explicit solution of the input as a function of time $\mathbf{u}(\tau)$ and not as feedback law. Thus it is only valid for the specified initial condition $\mathbf{x}(t)$. The approach can be thought of as the application of the necessary conditions for constrained optimization with the twist, that the optimization is infinite dimensional. The solution of the optimal control problem is cast as a boundary value problem. Since an infinite dimensional problem must be solved, this approach can normally not be applied for on-line implementation.

- **Direct solution using a finite parameterization of the controls and/or constraints:** In this approach the input and/or the constraints are parametrized finitely, thus an approximation of the original open-loop optimal control problem is seeked. The resulting finite dimensional dynamic optimization problem is solved with "standard" static optimization techniques.

For an on-line solution of the NMPC problem only the last approach is normally used. Since no feedback is obtained, the optimization problem must be solved at every sampling instance with the new state information. In the following only the last solution method is considered in detail.

3.2. Solution of the NMPC Problem Using a Finite Parameterization of the Controls

As mentioned the basic idea behind the direct solution using a finite parameterization of the controls is to approximate/transcribe the original infinite dimensional problem into a finite dimensional nonlinear programming problem. In this note the presentation is limited to a parameterization of the input signal as piecewise constant over the sampling times. The controls are piecewise constant on each of the $N = \frac{T_p}{\delta}$ predicted sampling intervals: $\bar{\mathbf{u}}(\tau) = \bar{\mathbf{u}}_i$ for $\tau \in [\tau_i, \tau_{i+1})$, $\tau_i = t + i\delta$, compare also Figure 4. Thus in the optimal control problem the "input

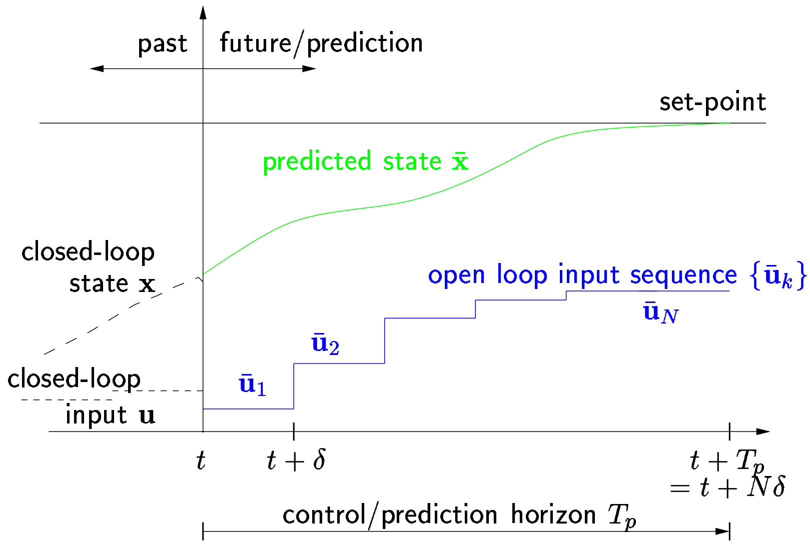

Figure 4: Piecewise constant input signal for the direct solution of the optimal control problem.

vector" $\{\bar{\mathbf{u}}_1, \ldots \bar{\mathbf{u}}_N\}$ is optimized, i.e. the optimization problem takes the form:

$$\min_{\{\bar{\mathbf{u}}_1,\ldots\bar{\mathbf{u}}_N\}} J(\mathbf{x}(t), \{\bar{\mathbf{u}}_1, \ldots \bar{\mathbf{u}}_N\}; T_p) \tag{25}$$

subject to the state and input constraints and the system dynamics. Basically two different solution strategies to this optimization problem exist:

- **Sequential approach:** In this method in every iteration step of the optimization strategy the differential equations (or in the discrete time case the difference equation) are solved exactly by a numerical integration, i.e. the solution of the system dynamics is implicitly done during the integration of the cost function and only the input vector $\{\bar{\mathbf{u}}_1, \ldots \bar{\mathbf{u}}_N\}$ appears directly in the optimization problem.
- **Simultaneous approach:** In this approach the system dynamics (24a) at the sampling points enter as nonlinear constraints to the optimization problems, i.e. at every sampling point the following equality constraint must be satisfied:

  $$\bar{\mathbf{s}}_{i+1} = \bar{\mathbf{x}}(t_{i+1}; \bar{\mathbf{s}}_i, \bar{\mathbf{u}}_i). \tag{26}$$

  Here $\bar{\mathbf{s}}_i$ is introduced as additional degree in the optimization problem and describes the "initial" condition for the sampling interval i, compare also Figure 5. This constraint requires, once the optimization has converged, that the state trajectory pieces fit together.

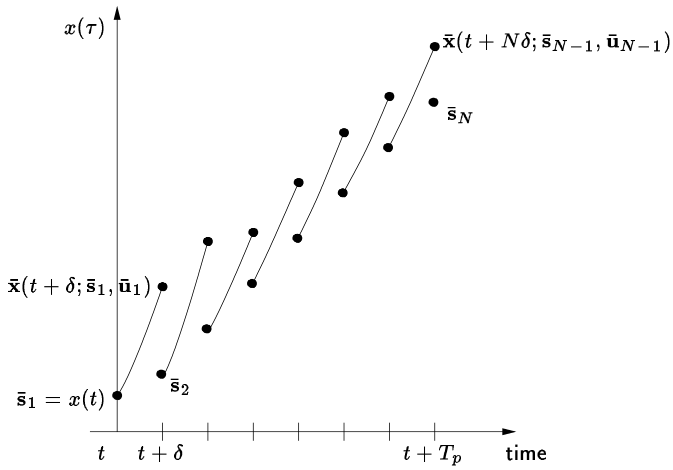

Figure 5: Simultaneous approach.

Thus additionally to the input vector $\{\bar{\mathbf{u}}_1, \ldots \bar{\mathbf{u}}_N\}$ also the vector of the $\bar{\mathbf{s}}_i$ appears as optimization variables.

For both approaches the resulting optimization problem is often solved using sequential quadratic programming techniques (SQP). Both approaches have different advantages and disadvantages. For example the introduction of the "initial" states $\bar{\mathbf{s}}_i$ as optimization variables does lead to a special banded-sparse structure of the underlying QP-problem. This structure can be taken into account to lead to a fast solution strategy. In comparison the matrices for the sequential approach are often dense and thus the solution is expensive to obtain. A drawback of the simultaneous approach is, that only at the end of the iteration a valid state trajectory for the system is available. Thus if the optimization cannot be finished in time, nothing can be said about the feasibility of the trajectory at all.

3.2.1. Remarks on State and Input Equality Constraints

In the description given above, the state and input constraints were not taken into account. The key problem is, that they should be satisfied for the whole state and input vector. While for a suitable parametrized input signal (e.g. parametrized as piecewise constant) it is not a problem to satisfy the constraints since only a finite number of points must be checked, the satisfaction of the state constraints must in general be enforced over the whole state trajectory. Different possibilities exist to consider them during the optimization:

- **Satisfaction of the constraints at the sampling instances:** An approximated satisfaction of the constraints can be achieved by requiring, that they are at least satisfied at the sampling instances, i.e. at the sampling times it is required:

  $$\mathbf{l}(\bar{\mathbf{u}}(t_i), \bar{\mathbf{x}}(t_i)) \leq \mathbf{0}. \tag{27}$$

  Notice, that this does not guarantee that the constraints are satisfied for the predicted trajectories in between the sampling instances. However, since this approach is easy to implement it is often used in practice.

- **Adding a penalty in the cost function:** An approach to enforce the constraint satisfaction exactly for the whole input/state trajectory is to add an additional penalty term to the cost function. This term is zero as long as the constraints are satisfied. Once the constraints are not satisfied the value of this term increases significantly, thus enforcing the satisfaction of the constraints. The resulting cost function may look as following:

$$J(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot); T_p) := \int_t^{t+T_p} \left(F(\bar{\mathbf{x}}(\tau), \bar{\mathbf{u}}(\tau)) + \mathbf{p}(\mathbf{l}(\bar{\mathbf{x}}(\tau)), \bar{\mathbf{u}}(\tau)) \right) d\tau + E(\bar{\mathbf{x}}(t+T_p)) \tag{28}$$

where $\mathbf{p}$ in the case that only one nonlinear constraint is present might look like shown in Figure 6. A drawback of this formulation is, that the resulting optimization problem

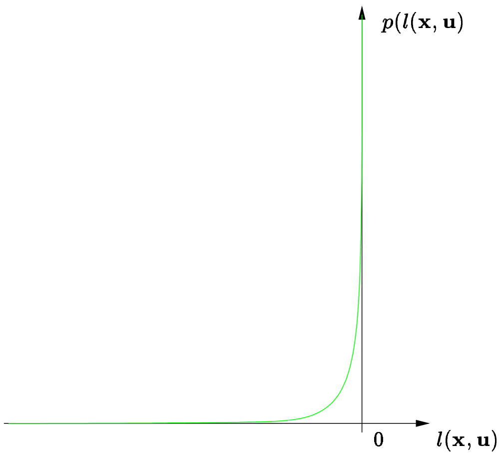

Figure 6: Constraint penalty function for one nonlinear constraint.

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is in general difficult to solve for example due to the resulting non-differentiability of the cost function outside the feasible region of attraction.

3.2.2. Efficient Solution of the Open-Loop Optimal Control Problem

In the previous section the general solution strategies for the open-loop optimal control problem have been outlined. To achieve a computationally feasible solution of the NMPC problem one should take the special structure of the problem into account. Here only two elementary strategies, that can decrease the solution time significantly, are outlined:

- **Utilization of the occurring sparsity:** One of the key observations is, that if the simultaneous approach for optimization is used and if a SQP optimization strategy is employed the sparse structure of the underlying QP matrices can be exploited. The sparsity results from the fact that a dynamical system is optimized, in which the state at one prediction time instance does only depend on the state and input of the previous sampling instance. Using special sparse solving strategies can thus lead to significant reductions in the necessary computational time.

- **Re-use as much information as possible from the previous sampling interval:** In the nominal case the optimal input sequence and the derivative as well as the second derivative information from subsequent sampling instances are quite similar. If an infinite horizon would be considered, the remaining input trajectory would even be a solution at the next sampling instance. Thus the remaining input from the previous sampling time plus the derivative information can be seen as a good initial guess for the current sampling instance. The use of this information is often called hot-starting and can lead to significant computational improvements in subsequent optimization steps. Only if the disturbance during the sampling instances is to big, a "completely new" solution must be found.

3.2.3. Efficient NMPC Formulations

One should notice, that besides an efficient solution strategy of the occurring open-loop optimal control problem the NMPC problem should be also formulated efficiently. Different possibilities for an efficient NMPC formulation exist:

- **Use of short horizon length without loss of performance and stability:** As was outlined in Section 2 short horizons are desirable from a computational point of view, but long horizons are required for closed-loop stability and in order to achieve the desired performance in closed-loop. The general NMPC scheme outlined in Section 2.1.2 offers a way out of this dilemma. It uses a terminal region constraint in combination with a terminal penalty term. The terminal penalty term can be used to given a good approximation of the infinite horizon cost utilizing a local control law. Additionally the terminal region constraint is in general not very restrictive, i.e. does not complicate the dynamic optimization problem in an unnecessary manner, as for example in the zero terminal constraint approach. In some cases, e.g. stable systems, feedback linearizable systems or systems for which a globally valid control Lyapunov function is known it can even be removed. Thus such an approach offers the possibility to formulate a computationally efficient NMPC scheme with a short horizon while not sacrificing stability and performance.
- **Use of suboptimal NMPC strategies, feasibility implies stability:** In general no global minima of the open-loop optimization must be found. It is sufficient to achieve a decrease in the value function at every time to guarantee stability. Thus stability can be seen as being implied by feasibility. If one uses an optimization strategy that delivers feasible solutions at every sub-iteration and a decrease in the cost function, the optimization can be stopped if no more time is available and still stability can be guaranteed. The key obstacle is that optimization strategies that guarantee a feasible and decreasing solution at every iteration are normally computationally expensive.
- **Taking the system structure into account, flatness based NMPC:** It is also noticeable, that the system structure should be taken into account. For example for systems for which a flat output is known (see Flatness Based Control) the dynamic optimization problem can be directly reduced to a static optimization problem. This results from the fact that for flat systems the input and the system state can be given in terms of the output and its derivatives as well as the system initial conditions. The drawback however is, that the algebraic relation between the output and the derivatives to the states and inputs must be known, which is not always possible.

Combining the presented approaches for an efficient formulation of the NMPC problem and the efficient solution strategies of the optimal control problem, the application of NMPC to realistically sized applications is possible even with nowadays computational power.

4. Conclusions and Further Outlook

Model predictive control for linear constrained systems has been shown to provide an excellent control solution both theoretically and practically. The incorporation of nonlinear models poses a much more challenging problem mainly because of computational and control theoretical difficulties, but also holds much promise for practical applications.

In this note an overview over the theoretical and computational aspects of NMPC is given. As outlined some of the challenges occurring in NMPC are already solvable. Nevertheless many unsolved questions remain. Here only a few are noticed as a guide for future research:

- **Output feedback NMPC:** While some first results in the area of output feedback NMPC exist, none of them seem to be applicable to real processes. Especially the incorporation of suitable state estimation strategies in the NMPC formulation must be further considered.
- **Robust NMPC Formulations:** By now a few robust NMPC formulations exist. While the existing schemes increase the general understanding they are computationally intractable to be applied in practice. Further research is required to develop implementable robust NMPC strategies.
- **Industrial Applications of NMPC:** The state of industrial application of NMPC is growing rapidly and seems to follow academically available results more closely than linear MPC. However, none of the NMPC algorithms provided by vendors include stability constraints as required by control theory for nominal stability; instead they rely implicitly upon setting the prediction horizon long enough to effectively approximate an infinite horizon. Future developments in NMPC control theory will hopefully contribute to making the gap between academic and industrial developments even smaller.

A. Outline of the Stability Proof for NMPC with Terminal Cost and Terminal Penalty

In Section 2.1.2 a general NMPC formulation that guarantees stability if the terminal penalty and terminal region constraint are chosen suitably. In this section it is shortly outlined, why the conditions of Theorem 2.1 do enforce stability. The key focus is lead on the decrease in the optimal cost function, i.e. the value function.

If the terminal penalty term and the terminal region are determined according to Theorem 2.1, the open-loop optimal trajectories found at each time instant approximate the optimal solution for the infinite horizon problem.

The following reasoning make this plausible: Consider an infinite horizon cost functional defined by

$$J^{\infty}\left(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot)\right) := \int_{t}^{\infty} F\left(\bar{\mathbf{x}}(\tau), \bar{\mathbf{u}}(\tau)\right)\, d\tau \tag{29}$$

with $\bar{\mathbf{u}}(\cdot)$ on $[t, \infty)$. This cost functional can be split up into two parts

$$\min_{\bar{\mathbf{u}}(\cdot)} J^{\infty}(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot)) = \min_{\bar{\mathbf{u}}(\cdot)} \left(\int_{t}^{t+T_p} F\left(\bar{\mathbf{x}}(\tau), \bar{\mathbf{u}}(\tau)\right) d\tau + \int_{t+T_p}^{\infty} F\left(\bar{\mathbf{x}}(\tau), \bar{\mathbf{u}}(\tau)\right) d\tau \right). \tag{30}$$

The goal is to upper approximate the second term by a terminal penalty term $E(\bar{\mathbf{x}}(t+T_p))$. Without further restrictions, this is not possible for general nonlinear systems. However, if we ensure that the trajectories of the closed-loop system remain within some neighborhood of the origin (terminal region) for the time interval $[t+T_p, \infty)$, then an upper bound on the second term can be found. One possibility is to determine the terminal region Ω such that a local state feedback law $\mathbf{u} = \mathbf{k}(\mathbf{x})$ asymptotically stabilizes the nonlinear system and renders Ω positively invariant for the closed-loop. If an additional terminal inequality constraint $\mathbf{x}(t+T_p) \in \Omega$ (see (14d)) is added to Problem 1, then the second term of equation (30) can be upper bounded by the cost resulting from the application of this local controller $\mathbf{u} = \mathbf{k}(\mathbf{x})$. Note that the predicted state will not leave Ω after $t+T_p$ since $\mathbf{u} = \mathbf{k}(\mathbf{x})$ renders Ω positively invariant. Furthermore the feasibility at the next sampling instance is guaranteed dismissing the first part of $\bar{\mathbf{u}}$ and replacing it by the nominal open-loop input resulting from the local controller. Requiring that $\mathbf{x}(t+T_p) \in \Omega$ and using the local controller for $\tau \in [t+T_p, \infty)$ leads to:

$$\min_{\bar{\mathbf{u}}(\cdot)} J^{\infty}(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot)) \leq \min_{\bar{\mathbf{u}}(\cdot)} \left(\int_{t}^{t+T_p} F\left(\bar{\mathbf{x}}(\tau), \bar{\mathbf{u}}(\tau)\right) d\tau + \int_{t+T_p}^{\infty} F\left(\bar{\mathbf{x}}(\tau), \mathbf{k}\left(\bar{\mathbf{x}}(\tau)\right)\right) d\tau \right). \tag{31}$$

If, furthermore, the terminal region Ω and the terminal penalty term are chosen according to condition b), integrating (15) leads to

$$\int_{t+T_p}^{\infty} F\left(\bar{\mathbf{x}}(\tau), \mathbf{k}\left(\bar{\mathbf{x}}(\tau)\right)\right) d\tau \leq E\left(\bar{\mathbf{x}}(t+T_p)\right). \tag{32}$$

Substituting (32) into (31) one obtains:

$$\min_{\bar{\mathbf{u}}(\cdot)} J^{\infty}\left(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot)\right) \leq \min_{\bar{\mathbf{u}}(\cdot)} J(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot); t+T_p). \tag{33}$$

This implies that the optimal value of the finite horizon problem bounds that of the corresponding infinite horizon problem. Thus, the prediction horizon can be thought of as extending to infinity. Equation (33) can be exploited to prove the decrease of the value function and thus Theorem 2.1.

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