Adaptive Model Predictive Control based on Gaussian Process

Statistical Learning and Stochastic Control

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Motivation

Gaussian Process

Definition (Gaussian Process (GP))

Let $X = [x_1, \dots, x_N] \in \mathbb{R}^{n \times N}$ be a set of points, $f : \mathbb{X} \to \mathbb{R}$, $\mathbb{X} \subset \mathbb{R}^n$ a random function, $m : \mathbb{X} \to \mathbb{R}$ be any function and $k : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ be a valid covariance function.

Then a Gaussian Process $f(x) \sim \mathcal{GP}(m(x), k(x, x'))$ is defined as a probability distribution over the function space f, such that the joint distribution of any finite collection of function evaluations Y:

$$Y = [y_1, \dots, y_N]^\mathsf{T} = [f(x_1), \dots, f(x_N)]^\mathsf{T} = f(X) \tag{1}$$

is Gaussian, i.e.:

$$Y \sim \mathcal{N}(\underbrace{\mathbb{E}(f(X))}_{m(X)}, \underbrace{\mathbb{V}ar(f(X))}_{K(X,X)}) \tag{2}$$

where $m(X) = [m(x_1), \dots, m(x_N)]^T \in \mathbb{R}^n$ is any function that evaluates the joint mean and $K(X, X) \in \mathbb{R}^{n \times n}$ is a joint covariance matrix defined element-wise by any valid covariance function $K(X, X)_{i,j} = k(x_i, x_j)$.

Gaussian Process Regression

Assume now an unknown function $f(x) : \mathbb{R}^n \to \mathbb{R}$ to be learned that defines a Gaussian Process $f(x) \sim \mathcal{GP}(m(x), k(x,x))$.

Further, assume that the observation outputs y are corrupted by an additive i.i.d. Gaussian noise $\epsilon \sim \mathcal{N}(0, \sigma_n^2)$. such that:

$$y = f(x) + \epsilon \tag{3}$$

Consider now a training Dataset $\mathcal{D} = \{(x_i, y_i) | i = 1, \dots, N\}$ with N observations pairs (x_i, y_i), such that $X = [x_1, \dots, x_N]$ and $Y = [y_1, \dots, y_N]$.

Then, clearly, the Likelihood $p(Y|X)$ is given by:

$$Y|X \sim \mathcal{N}\left(m(X), K(X,X) + \sigma_n^2 I\right) \tag{4}$$

Gaussian Process Regression

Given a new set of test points X^*, one may now wish to calculate the posterior distribution $p(f^*|X^*, Y, X)$, where we denote $f^* = f(X^*)$.

Since we know that the prior is distributed as $p(f^*|X^*) = \mathcal{N}(m(X^*), K(X^*))$, we can calculate the posterior distribution $p(f^*|X^*, Y, X)$ by means of Bayesian inference:

$$\underbrace{p(f^*|X^*, Y, X)}_{posterior} \propto \underbrace{p(f^*|X^*)}_{prior} \underbrace{p(Y|X)}_{likelihood} = \underbrace{p(f^*, Y|X^*, X)}_{joint\ probability} \tag{5}$$

where

$$\begin{bmatrix} f^* \\ Y \end{bmatrix} = \begin{bmatrix} f(X^*) \\ f(X) \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} \Xi, \qquad \Xi \sim (0, \sigma_n^2 I) \tag{6}$$

leads to the joint distribution

$$\left[\begin{matrix} f^* \\ Y \end{matrix} \Big/ X^*, X \right] \sim \mathcal{N}\left(\begin{bmatrix} m(X^*) \\ m(X) \end{bmatrix}, \begin{bmatrix} K(X^*, X^*) & K(X, X^*) \\ K(X^*, X) & K(X, X) + \sigma_n^2 I \end{bmatrix} \right), \tag{7}$$

and, as before, $m(X) = \mathbb{E}[f(X)]$ and $K(X, X^*) = \mathbb{V}ar(f(X), f(X^*))$ are functions that define the mean and covariance of the function f at different locations.

Gaussian Process Regression

Finally, using the Update Lemma

Lemma (Update Lemma (von Misses))

$$\begin{bmatrix} x \\ y \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix} , \begin{bmatrix} P_{xx} & P_{xy} \\ P_{yx} & P_{yy} \end{bmatrix} \right)$$
$$\Rightarrow \quad x|y \quad \sim \mathcal{N}\left(\mu_x + P_{xy}\, P_{yy}^{-1}(y-\mu_y) \ , \ P_{xx} - P_{xy}\, P_{yy}^{-1}\, P_{yx}\right) \tag{8}$$

with equation (7) leads to the posterior (predictive distribution of the function values at X^*):

Posterior / Predictive Distribution

$$f^*|X^*, Y, X \sim \mathcal{N}\left(\begin{array}{l} m(X^*) + K(X^*, X)\,(K(X,X) + I\sigma_n^2)^{-1}\,(Y - m(X)) \ , \\ K(X^*, X^*) - K(X^*, X)\,(K(X,X) + I\sigma_n^2)^{-1}\,K(X, X^*) \end{array} \right) \tag{9}$$

which is also a Gaussian Process.

Kernel Trick and Function-Space View

Explain very briefly Kernel trick and show Squared Exponential Kernel (SE)

$$k(x, x') = \sigma_f^2 \, exp \left(0.5 \, \|x - x'\|^2_{M^{-1}}\right) \tag{10}$$

mean function $m(x) = \mathbf{0}$

Show 1D GP plots of the GP prior and posterior

Efficient Implementation

Each output dimension is treated as one GP

Show briefly the trick of the Cholesky decomposition

Matrices alpha and L

Hyper-parameter Optimization

The GP Likelihood is Gaussian and in the case of the zero mean function, is given by:

$$Y|X,\theta \sim \mathcal{N}(0, \underbrace{K(X,X) + \sigma_n^2 I}_{K_y}) \quad (11)$$

where $\theta = [\{M\}, \sigma_f^2, \sigma_n^2]$ is a vector containing all hyper-parameters.

Among many possible choices, we chose to parameterize the length-scale covariance matrix M as diagonal positive-semidefinite:

$$M = \begin{bmatrix} l_1 & & 0 \\ & \ddots & \\ 0 & & l_n \end{bmatrix} \quad (12)$$

with $l_i \geq 0, \forall i \in 0, \ldots, n$

such that the hyperparameter vector becomes $\theta = [l_1, \ldots, l_n, \sigma_f^2, \sigma_n^2]$

Hyper-parameter Optimization

One may optimize the GP hyper-parameters by maximizing the Log Likehood (LL):

$$log\, p(Y|X,\theta) = -\frac{1}{2}y^T K_y^{-1} y - \frac{1}{2}\log|K_y| - \frac{n}{2}\log(2\pi)$$

$$\theta = \arg\max_{\theta} \; log\, p(Y|X,\theta) \tag{13}$$

which allows local optimization of the hyper-parameters, since (13) is nonconvex (one optimization problem for each output dimension).

Obs: Even gradient-free tools like `fmincon` from Matlab, showed to be efficient. Nevertheless, the gradient of (13) can be easily derived, as shown in [RW06]

Sparse Gaussian Process Regression

The computational complexity of a GP regression strongly depends on the number of data points N.

For larger number of N, the evaluation of $\mathcal{GP}$ becomes impractible for real-time applications.

Explain briefly what do we do if the dictionary is full: we select the so called, **inducing points**

MPC Formulation

MPC is a control strategy which predicts the future dynamic behaviour within a finite prediction horizon using a dynamics model and chooses the control input such that a performance functional is minimized.

NMPC

$$
\begin{aligned}
x^*_{0:N}, u^*_{0:N-1} = \underset{x_{0:N}, u_{0:N-1}}{\arg\min} \quad & \sum_{k=0}^{N-1} f_o(x_k, u_k) + \phi(X_N) \\
s.t. \quad & x_{k+1} = f_d(x_k, u_k) \\
& x_0 = \bar{x} \\
& x_k \in \mathcal{X}(x_k) \\
& x_N \in \mathcal{X}_f \\
& u_k \in \mathcal{U}(x_k)
\end{aligned}
\tag{14}
$$

NMPC Algorithm:

1) Measure (estimate) current state $\bar{x}$
2) Solve the open-loop discrete-time infinite-horizon optimal control problem (14)
3) Implement the first portion of the optimal input $u_{NMPC}(\bar{x}) = u^*_0$
4) Go to 1)

Adaptive Process Model

One of the challenges of MPC is that unmodeled dynamics might deteriorate its performance. To this end, an adaptive process model [KVR+19] is proposed as follows:

Adaptive Process Model

$$x_{k+1} = f_d(x_k, u_k) + B_d \, (d(z_k) + w) \tag{15}$$

where

$z_k = [Bz_x.x_k; Bz_u.u_k] \in \mathbb{R}^{n_z}$	collection of relevant features
$w \sim \mathcal{N}(0, \sigma_n^2)$	process noise
$f_d(z_k) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$	nominal discrete-time process model
$d(x_k, u_k) \sim \mathcal{GP}(m(z_k), K(z_k))$	Gaussian Process function

The matrix $B_d \in \mathbb{R}^{n \times n_d}$ is used to select a subspace of states that are affected by both GP process and noise.

In a similar way, the matrices Bz_x and Bz_u select a subset of states that are relevant for regression.

Collecting Data Points to the GP

Given the adaptive process model

$$x_{k+1} = f_d(x_k, u_k) + B_d \, (d(z_k) + w) \tag{16}$$

Assume that the state observation $\hat{x}_{k+1}$ at time k+1 can be perfectly measured.

This means that the training data (X, Y)

$$X = z_k \tag{17}$$

$$Y = B_d^{\dagger} \, (x_{k+1} - f_d(x_k, u_k)) = d_{true} + w \tag{18}$$

Propagation of uncertainty

The adaptive process model allows us to

Efficient MPC Formulation

The equality constraints (usually corresponding to the state space variables $x_{0:N}$) can be removed. and introduce relaxed barrier function to remove the states and inputs constraints while ensuring feasibility

$$x_k \in \mathcal{X} = \{x : g(x) \leq \lambda\} \tag{19}$$

$$\beta(x) = \frac{q_\beta}{2}\left(\sqrt{\frac{(4+\gamma(\lambda-x)^2)}{\gamma}} - (\lambda - x)\right) \approx q_\beta \, \frac{(|\lambda - x| - (\lambda - x))}{2} \tag{20}$$

Show equation and plot of the Relaxed-Barrier function (draw.io) - lambda and then slope q_β

Efficient MPC Formulation

Second, the equality constraints, usually corresponding only to the state space variables $x_{0:N}$, can be easily removed from the set of variables to be optimized.

This is usually possible because the state variables at any time step are a function of the given initial state and the sequence of inputs to be optimized.

To this end, we define the unconstrained nonlinear MPC as follows:

Efficient Unconstrained NMPC

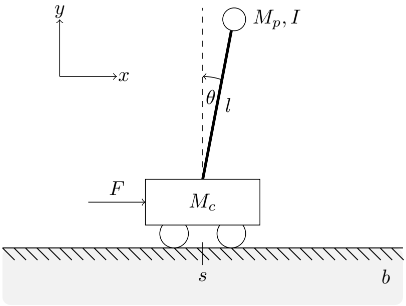
$$\min_{u_{0:N-1}} \quad \sum_{k=0}^{N-1} f_o(x_k, u_k) + B_x(x_k) + B_u(u_k) + \phi(X_N)$$

$$\text{where} \quad x_k(\bar{x}, u_{0:k-1}) = f(f(\cdots f(f(\bar{x}, u_0), u_1) \dots), u_{k-1}) \tag{21}$$

$$\beta(x) = \frac{q_\beta}{2}\left(\sqrt{\frac{(4+\gamma(\lambda - x)^2)}{\gamma}} - (\lambda - x)\right)$$

This unconstrained optimization problem, can be solved very efficiently with nonlinear optimization solvers and is always feasible.

The Inverted Pendulum Problem



Continuous time dynamics

$$(M_c + M_p)\ddot{x} + b\dot{x} + \frac{1}{2}M_p l\ddot{\theta}\cos\theta - \frac{1}{2}M_p l\dot{\theta}^2\sin\theta = F \tag{22}$$

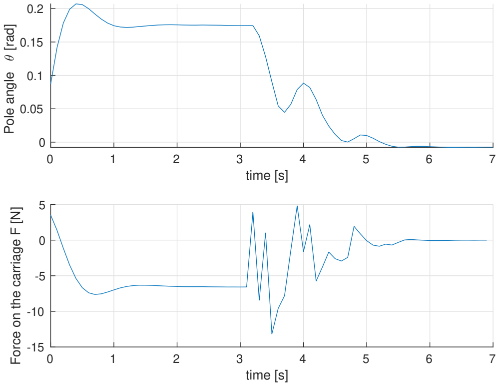
$$(I + M_p\left(\frac{l}{2}\right)^2)\ddot{\theta} - \frac{1}{2}M_p gl\sin\theta + M_p l\ddot{x}\cos\theta = 0. \tag{23}$$

Nominal and True Dynamics

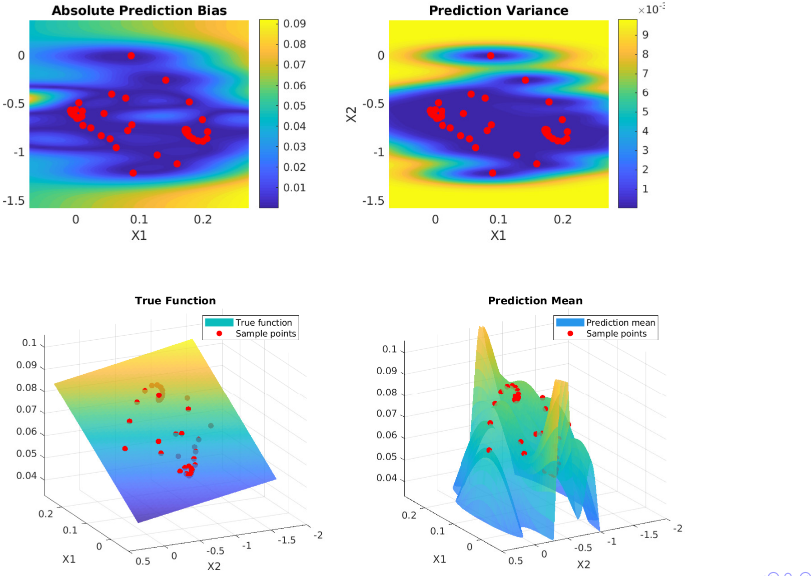
Cost function and Constraints

Simulation results

Starting positon at XXX degress. GP accumulates data to the dictionary. At t=3s, we activate predictions with GP



Learning analysis



Training

Discuss the posterior function before and after Hyper-parameter optimization

Show image of d before and after hyper-parameter optimization

Racing Car Problem

Vehicle Dynamics

Show basic vehicle dynamics equations

True and Nominal Dynamics

Show difference between true and nominal model

Efficient MPC Formulation

Show how the inequality constraints can be removed

Show equation and plot of the Relaxed-Barrier function

Results: unconstrained optimization problem, which can be solved very efficiently with nonlinear optimization solvers

Cost function and constraints

Inspired by [KHLZ19]

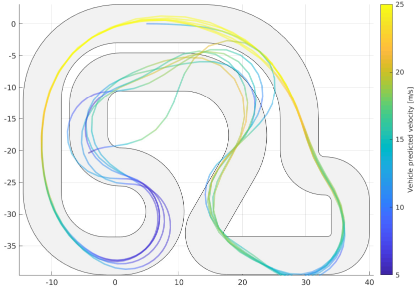
Try to formulate the problem as a $\min\max$ problem

min distance from predictions (colored points) - which depend on the inputs: steering angle, and gas pedal

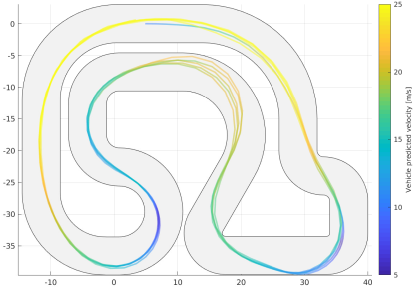
maximize centerline projections (grey points) - which depend on the input: centerline velocity

Results

MPC with unkown dynamics



Adaptive GP MPC



Outlook and Conclusion

- GP can introduce high nonlinearities in the prediction, making more difficult for the optimizer to a find a good local optimum.
- Replacing the inequality constraints by relaxed barrier functions increase significantly the computational performance, while always ensuring feasibility.
- Hyper-parameter optimization (GP training) plays an important role in the final controller performance

Bibliography I

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