Question 1.

Now we need to prove the proposition 4.1. There are 4 statements, and we need to prove that they are equivalent. Actually we just need to justify that (i) is equivalent to (ii), (iii) and (iv). Then all this 4 statements will be equivalent to each other naturally. Since we can get (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) i.e (i) $\Leftrightarrow$ (iii) logically.

(a). (i) is equivalent to (ii)

Proof. Suppose that $\mathcal{T}$ is firmly nonexpansive, then we need to find a nonexpansive operator $\mathcal{N} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathcal{T} = (1-\alpha)\mathcal{I} + \alpha\mathcal{N}$. Firstly, because $\mathcal{T}$ is nonexpansive, we have

$$\|\mathcal{T}x - \mathcal{T}y\|^2 \leq \langle \mathcal{T}x - \mathcal{T}y, x - y\rangle.$$

And there is a nonexpansive operator $\mathcal{N}$ which satisfying

$$\|\mathcal{N}x - \mathcal{N}y\| \leq \|x - y\|.$$

Now we want to substitute $\mathcal{T}$ by $\frac{1}{2}\mathcal{I} + \frac{1}{2}\mathcal{N}$. We can get

$$\begin{aligned}\|(\frac{1}{2}\mathcal{I} + \frac{1}{2}\mathcal{N})x - (\frac{1}{2}\mathcal{I} + \frac{1}{2}\mathcal{N})y\|^2 &\leq \langle(\frac{1}{2}\mathcal{I} + \frac{1}{2}\mathcal{N})x - (\frac{1}{2}\mathcal{I} + \frac{1}{2}\mathcal{N})y, x - y\rangle \\ \|(\mathcal{I} + \mathcal{N})x - (\mathcal{I} + \mathcal{N})y\|^2 &\leq \langle(\mathcal{I} + \mathcal{N})x - (\mathcal{I} + \mathcal{N})y, 2(x - y)\rangle.\end{aligned}$$

If we calculate the left side of this inequation, we can easily get that

$$\begin{aligned}LHS &= (\mathcal{I} + \mathcal{N})^2\|x - y\|^2 \\ &= \|x - y\|^2 + 2\mathcal{N}\|x - y\|^2 + \|\mathcal{N}x - \mathcal{N}y\|^2.\end{aligned}$$

And the right side equals to

$$\begin{aligned}RHS &= \langle x - y + \mathcal{N}x - \mathcal{N}y, 2(x - y)\rangle \\ &= 2\|x - y\|^2 + 2\mathcal{N}\|x - y\|^2.\end{aligned}$$

Then we can get that

$$\begin{aligned}\|x - y\|^2 + 2\mathcal{N}\|x - y\|^2 + \|\mathcal{N}x - \mathcal{N}y\|^2 &\leq 2\|x - y\|^2 + 2\mathcal{N}\|x - y\|^2 \\ \|\mathcal{N}x - \mathcal{N}y\|^2 &\leq \|x - y\|^2.\end{aligned}$$

That is to say that, $\mathcal{N}$ is a nonexpansive operator. At the same time, $\mathcal{T}$ is a $\frac{1}{2}$-averaged nonexpansive operator.

Now let's proof that (ii) $\Rightarrow$ (i). That is to say that there is a nonexpansive operator $\mathcal{N}$ such that $\mathcal{T}$ satisfying $\mathcal{T} = \frac{1}{2}\mathcal{I} + \frac{1}{2}\mathcal{N}$. Then $\mathcal{T}$ is firmly nonexpansive. Since $\mathcal{N}$ is nonexpansive we have

$$\|\mathcal{N}x - \mathcal{N}y\| \leq \|x - y\|.$$

Substituting this inequaltion by $\mathcal{N} = 2\mathcal{T} - \mathcal{I}$, we can get

$$\begin{aligned}\|(2\mathcal{T} - \mathcal{I})(x - y)\| &\leq \|x - y\| \\ \|(2\mathcal{T} - \mathcal{I})(x - y)\|^2 &\leq \|x - y\|^2 \\ (2\mathcal{T} - \mathcal{I})^2\|x - y\|^2 &\leq \|x - y\|^2 \\ 4\|\mathcal{T}x - \mathcal{T}y\|^2 - 4\mathcal{T}\|x - y\|^2 + \|x - y\|^2 &\leq \|x - y\|^2 \\ \|\mathcal{T}x - \mathcal{T}y\|^2 &\leq \mathcal{T}\|x - y\|^2 \\ \|\mathcal{T}x - \mathcal{T}y\|^2 &\leq \langle\mathcal{T}x - \mathcal{T}y, x - y\rangle.\end{aligned}$$

So we can get that $\mathcal{T}$ is firmly nonexpansive. And we can claim that (i) is equivalent to (ii). $\square$

(b). (i) is equivalent to (iii)

Proof. Now let's prove (i) $\Leftrightarrow$ (iii). Actually from (a) we can get that $\mathcal{T}$ is firmly nonexpansive and then $\mathcal{T}$ is also $\frac{1}{2}$-averaged nonexpansive. From that we have $\mathcal{I} - \mathcal{T} = \mathcal{T} - \mathcal{N}$. If $\mathcal{T}$ is firmly nonexpansive, we can get that

$$
\begin{aligned}
&\|(\mathcal{I} - \mathcal{T})(x - y)\|^2 \\
=&\|x - y\|^2 + \|\mathcal{T}x - \mathcal{T}y\|^2 - 2\langle \mathcal{T}x - \mathcal{T}y, x - y\rangle.
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
&\langle (\mathcal{I} - \mathcal{T})x - (\mathcal{I} - \mathcal{T})y, x - y\rangle \\
=&\|x - y\|^2 - \langle \mathcal{T}x - \mathcal{T}y, x - y\rangle.
\end{aligned}
$$

Since $\mathcal{T}$ is firmly nonexpansive, if we put this 2 into left side and right side respectively, we have

$$
\|\mathcal{T}x - \mathcal{T}y\|^2 \leq \langle \mathcal{T}x - \mathcal{T}y, x - y\rangle.
$$

So $\mathcal{I} - \mathcal{T}$ is firmly nonexpansive. If $\mathcal{I} - \mathcal{T}$ is firmly nonexpansive, we can also get the same result through the same method. Then easily we have (i) $\Leftrightarrow$ (iii). □

(c). (i) is equivalent to (iv)

Proof. If $\mathcal{T}$ is firmly nonexpansive, so as it is $\frac{1}{2}$-averaged nonexpansive, and there is a nonexpansive operator $\mathcal{N}$ satisfying $\mathcal{T} = \frac{1}{2}\mathcal{I} + \frac{1}{2}\mathcal{N}$. That is to say that $\mathcal{N} = 2\mathcal{T} - \mathcal{I}$. And $\mathcal{N}$ is nonexpansive, so as $2\mathcal{T} - \mathcal{I}$.

If $2\mathcal{T} - \mathcal{I}$ is nonexpansive, we can define $\mathcal{N} = 2\mathcal{T} - \mathcal{I}$ as a new nonexpansive operator. Then it satisfying $\mathcal{T} = \frac{1}{2}\mathcal{I} + \frac{1}{2}\mathcal{N}$, it can show that $\mathcal{T}$ is firmly nonexpansive. So (iv) $\Rightarrow$ (i). Then (i) is equivalent to (iv). □

We have been proved that (i) $\Leftrightarrow$ (ii), (i) $\Leftrightarrow$ (iii) and (i) $\Leftrightarrow$ (iv). That is to say (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). Then this 4 statements are equivalent. So the proposition 4.1 is true.

Question 2.

Proof. In this question, we need to prove that $\text{prox}_{\|\cdot\|_1}$ is firmly nonexpansive. That is to say $\text{prox}_{\|\cdot\|_1}$ satisfying

$$\|\text{prox}_{\|\cdot\|_1}(\boldsymbol{x}) - \text{prox}_{\|\cdot\|_1}(\boldsymbol{y})\|^2 \leq \langle \text{prox}_{\|\cdot\|_1}(\boldsymbol{x}) - \text{prox}_{\|\cdot\|_1}(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y}\rangle. \tag{2.1}$$

From (3.7) we know that for $\boldsymbol{x} \in \mathbb{R}$,

$$\text{prox}_{|\cdot|}(x) = \max\{|x| - 1, 0\} \cdot \text{sgn}(x). \tag{2.2}$$

From (3.8) we know that for $\boldsymbol{x} \in \mathbb{R}^n$,

$$\text{prox}_{\|\cdot\|_1}(\boldsymbol{x}) = (\text{prox}_{|\cdot|}(x_1), \text{prox}_{|\cdot|}(x_2), \ldots, \text{prox}_{|\cdot|}(x_n))^T. \tag{2.3}$$

In this question, I denote $\text{prox}_{\|\cdot\|}$ as $\text{prox}_{\|\cdot\|_1}$, since there is only ℓ_1-norm. So I ignore the 1 index of $\|\cdot\|_1$. And I denote $\text{prox}(x)$ as $\text{prox}_{|\cdot|}(x)$.

For $x \in \mathbb{R}$, from the lecture 4 we can rewrite $\text{prox}(x)$ as

$$\text{prox}(x) = \begin{cases} x-1, & x \geq 1 \\ 0, & -1 \leq x \leq 1 \\ x+1, & x \leq -1 \end{cases}$$

So as $y \in \mathbb{R}$.

Then we can expand the inequaltion (2.1) using (2.3) and get

$$\|\text{prox}_{\|\cdot\|}(\boldsymbol{x}) - \text{prox}_{\|\cdot\|}(\boldsymbol{y})\|^2 \leq \langle \text{prox}_{\|\cdot\|}(\boldsymbol{x}) - \text{prox}_{\|\cdot\|}(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y}\rangle$$

$$\|(\text{prox}(x_1), \ldots, \text{prox}(x_n)) - (\text{prox}(y_1), \ldots, \text{prox}(y_n))\|^2 \leq \\ \langle(\text{prox}(x_1) - \text{prox}(y_1), \ldots, \text{prox}(x_n) - \text{prox}(y_n)), (x_1 - y_1, \ldots, x_n - y_n)\rangle.$$

Continuing we can get

$$\sum_{i=0}^{n} (\text{prox}(x_i) - \text{prox}(y_i))^2 \leq \sum_{i=0}^{n} (\text{prox}(x_i) - \text{prox}(y_i))(x_i - y_i).$$

That is what we need to prove. Now we want to prove for any $i \in \{1, 2, \ldots, n\}$, there exist

$$(\text{prox}(x_i) - \text{prox}(y_i))^2 = (\text{prox}(x_i) - \text{prox}(y_i))(x_i - y_i). \tag{2.4}$$

Firstly, if $\boldsymbol{x} = \boldsymbol{y}$ i.e $x_i = y_i$ for any $i \in \{1, 2, \ldots, n\}$, then $\boldsymbol{x}$ is a fixed point.
And $(\text{prox}(x_i) - \text{prox}(y_i))^2 = (\text{prox}(x_i) - \text{prox}(y_i))(x_i - y_i)$. This satisfies the definition of firmly nonexpansive.

Now let's focus on $x \neq y$.

Using the equation of $\text{prox}(x)$, we can calculate $\text{prox}(x_i) - \text{prox}(y_i)$. And there are 8 situations.
For $x, y \in \mathbb{R}$, $i \in \{1, 2, \ldots, n\}$

$$\psi(x_i, y_i) = \text{prox}(x_i) - \text{prox}(y_i) = \begin{cases} x_i - y_i, & x \geq 1, y \geq 1 \text{ } and \text{ } x \leq -1, y \leq -1 \\ x_i - 1, & x \geq 1, -1 \leq y \leq 1 \\ x_i - y_i - 2, & x \geq 1, y \leq -1 \\ 1 - y_i, & -1 \leq x \leq 1, y \geq 1 \\ 0, & -1 \leq x \leq 1, -1 \leq y \leq 1 \\ -y_i - 1, & -1 \leq x \leq 1, y \leq -1 \\ x_i - y_i + 2, & x \leq -1, y \geq 1 \\ x_i + 1, & x \leq -1, -1 \leq y \leq 1 \end{cases}$$

Let's look at every elements of $(\text{prox}(x_i) - \text{prox}(y_i))^2$ and $(\text{prox}(x_i) - \text{prox}(y_i))(x_i - y_i)$.

(a) $x \geq 1, y \geq 1$ *and* $x \leq -1, y \leq -1$
Easily find that $(\text{prox}(x_i) - \text{prox}(y_i))^2 = (x_i - y_i)^2 = (\text{prox}(x_i) - \text{prox}(y_i))(x_i - y_i)$.

(b) $x \geq 1, -1 \leq y \leq 1$
We get $x_i - 1 \geq 0$ and $x_i - y_i > 0$. Further we have $x_i - y_i > x_i - 1$. If we multiple $(x_i - 1)$ which is greater than 0, we get $(\text{prox}(x_i) - \text{prox}(y_i))^2 = (x_i - 1)^2 \leq (x_i - 1)(x_i - y_i) = (\text{prox}(x_i) - \text{prox}(y_i))(x_i - y_i)$.

(c) $x \geq 1, y \leq -1$
We get $x_i - y_i - 2 \geq 0$ and $x_i - y_i \geq 0$, so $x_i - y_i > x_i - y_i - 2$.
Then $(\text{prox}(x_i) - \text{prox}(y_i))^2 = (x_i - y_i - 2)^2 \leq (x_i - y_i - 2)(x_i - y_i) = (\text{prox}(x_i) - \text{prox}(y_i))(x_i - y_i)$.

(d) $-1 \leq x \leq 1, y \geq 1$
We get $1 - y_i \leq 0$ and $x_i - y_i < 0$, so $0 \geq 1 - y_i > x_i - y_i$.
Then $(\text{prox}(x_i) - \text{prox}(y_i))^2 = (1 - y_i)^2 \leq (1 - y_i)(x_i - y_i) = (\text{prox}(x_i) - \text{prox}(y_i))(x_i - y_i)$.

(e) $-1 \leq x \leq 1, -1 \leq y \leq 1$
If $x_i \neq y_i$, two side of it equals to 0. So $(\text{prox}(x_i) - \text{prox}(y_i))^2 = (\text{prox}(x_i) - \text{prox}(y_i))(x_i - y_i)$.

(f) $-1 \leq x \leq 1, y \leq -1$
We get $-y_i - 1 \geq 0$ and $x_i - y_i > 0$, so $x_i - y_i > -y_i - 1$.
Then $(\text{prox}(x_i) - \text{prox}(y_i))^2 = (-1 - y_i)^2 \leq (-1 - y_i)(x_i - y_i) = (\text{prox}(x_i) - \text{prox}(y_i))(x_i - y_i)$.

(g) $x \leq -1, y \geq 1$
We get $x_i - y_i + 2 \leq 0$ and $x_i - y_i \leq 0$, so $x_i - y_i + 2 > x_i - y_i$.
Then $(\text{prox}(x_i) - \text{prox}(y_i))^2 = (x_i - y_i + 2)^2 \leq (x_i - y_i + 2)(x_i - y_i) = (\text{prox}(x_i) - \text{prox}(y_i))(x_i - y_i)$.

(h) $x \leq -1, -1 \leq y \leq 1$
We get $x_i + 1 \leq 0$ and $x_i - y_i < 0$, so $x_i - y_i < x_i + 1$.
Then $(\text{prox}(x_i) - \text{prox}(y_i))^2 = (x_i + 1)^2 \leq (x_i + 1)(x_i - y_i) = (\text{prox}(x_i) - \text{prox}(y_i))(x_i - y_i)$.

So for $i \in \{1, 2, \ldots, n\}$ we have (2.4)

$$(\text{prox}(x_i) - \text{prox}(y_i))^2 = (\text{prox}(x_i) - \text{prox}(y_i))(x_i - y_i).$$

And if we add them together, also we have

$$\sum_{i=0}^{n} (\text{prox}(x_i) - \text{prox}(y_i))^2 \leq \sum_{i=0}^{n} (\text{prox}(x_i) - \text{prox}(y_i))(x_i - y_i)$$
$$\|\text{prox}_{\|\cdot\|_1}(\boldsymbol{x}) - \text{prox}_{\|\cdot\|_1}(\boldsymbol{y})\|^2 \leq \langle \text{prox}_{\|\cdot\|_1}(\boldsymbol{x}) - \text{prox}_{\|\cdot\|_1}(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y} \rangle.$$

That is the statement (2.1). So we can claim that $\text{prox}_{\|\cdot\|_1}$ is firmly nonexpansive. $\square$