question 1.1

(a)

Step 1. Ask the question

There three stages in step 1.

Variables:	x = population of blue whales y = population of fin whales $r_1 r_2$ = the intrinsic growth rates of each species $K_1 K_2$ = the the maximum sustainable population in the absence of competition α = the effects of competition
Assumptions:	$r_1 = 0.05$ $r_2 = 0.08$ $K_1 = 150000$ $K_2 = 400000$ $\alpha = 10^{-8}$ $f = \frac{dx}{dt} + \frac{dy}{dt}$ $x, y \geq 0$
Objective:	Maximize f

Step 2. Select the modeling approach

We need to maximize the number of new whales each years, that is to maximize the growth rate of the whales. Then is to maximize the function f in the set of $S = \{x, y \geq 0\}$. We need to different the function f and let the gradient equals to 0.

Step 3. Formulate the model

Let

$$\begin{aligned} f &= \frac{dx}{dt} + \frac{dy}{dt} \\ &= r_1 \cdot x(1 - \frac{x}{K_1}) - \alpha xy + r_2 \cdot x(1 - \frac{y}{K_2}) - \alpha xy \\ &= 0.05 \cdot x(1 - \frac{x}{150000}) - 10^{-8}xy + 0.08 \cdot y(1 - \frac{y}{400000}) - 10^{-8}xy \end{aligned}$$

And then we find the gradient of it we get

$$\begin{aligned} \frac{\partial f}{\partial x} &= 0.05 - \frac{x}{1500000} - 2 \cdot 10^{-8}y = 0, \\ \frac{\partial f}{\partial y} &= 0.08 - \frac{y}{2500000} - 2 \cdot 10^{-8}x = 0. \end{aligned}$$

Step 4. Solve the model

By solving the equations in Step 3, we can easily get
$x = \frac{138000000}{1997} \approx 69103.66$ and $y = \frac{392500000}{1997} \approx 196544.82$.

Step 5. Answer the question

By solving this model we can find the max new-whales-born number where the number of blue whales is approaching to 69103, and the number of fin whales is approaching to 196544.

(b)

Sensitivity analysis

Treat r_1 as an unknown number we get

$$\begin{aligned} f &= r_1 \cdot x(1-\frac{x}{150000}) - 10^{-8}xy + 0.08 \cdot y(1-\frac{y}{400000}) - 10^{-8}xy, \\ \frac{\partial f}{\partial x} &= r_1 - \frac{2r_1 \cdot x}{150000} - 2 \cdot 10^{-8} y = 0, \\ \frac{\partial f}{\partial y} &= 0.08 - \frac{y}{2500000} - 2 \cdot 10^{-8} x = 0. \end{aligned}$$

Solving for x and y as before yields

$$\begin{aligned} x &= \frac{12000000(250r_1 - 1)}{40000r_1 - 3}, \\ y &= \frac{7850000000r_1}{40000r_1 - 3}. \end{aligned}$$

We can also get

$$\begin{aligned} \frac{dx}{dr_1} &= \frac{471000000000}{(40000r_1 - 3)^2}, \\ \frac{dy}{dr_1} &= -\frac{23550000000}{(40000r_1 - 3)^2}. \end{aligned}$$

At $r_1 = 0.05$, so that

$$\begin{aligned} S(x, r_1) &= \frac{dx}{dr_1} \cdot \frac{r_1}{x} \approx 0.0855, \\ S(y, r_1) &= \frac{dy}{dr_1} \cdot \frac{r_1}{y} \approx -0.0015. \end{aligned}$$

Now let's treat r_2 as an unknown number, then we get

$$\begin{aligned} f &= 0.05 \cdot x(1-\frac{x}{150000}) - 10^{-8}xy + r_2 \cdot y(1-\frac{y}{400000}) - 10^{-8}xy, \\ \frac{\partial f}{\partial x} &= 0.05 - \frac{x}{1500000} - 2 \cdot 10^{-8} y = 0, \\ \frac{\partial f}{\partial y} &= r_2 - \frac{2r_2 \cdot y}{400000} - 2 \cdot 10^{-8} x = 0. \end{aligned}$$

Solving for x and y as before yields

$$\begin{aligned} x &= \frac{1725000000r_2}{25000r_2 - 3}, \\ y &= \frac{5000000000r_2 - 7500000}{25000r_2 - 3}. \end{aligned}$$

We can also get

$$\begin{aligned} \frac{dx}{dr_2} &= -\frac{5175000000}{(25000r_2 - 3)^2}, \\ \frac{dy}{dr_2} &= \frac{172500000000}{(25000r_2 - 3)^2}. \end{aligned}$$

At $r_2 = 0.08$, so that

$$\begin{aligned} S(x, r_2) &= \frac{dx}{dr_2} \cdot \frac{r_1}{x} \approx -0.0150, \\ S(y, r_2) &= \frac{dy}{dr_2} \cdot \frac{r_1}{y} \approx 0.1761. \end{aligned}$$

So we can get $S(x, r_1) = 0.0855, S(y, r_1) = -0.0015, S(x, r_2) = -0.0150 \; and \; S(y, r_2) = 0.1761.$

question 1.6

(a)
Step 1. Ask the question
There three stages in step 1.

Variables:	p = the sale price of each unit (\$)
	q = the cost of advertising (\$)
	n = the number of selling units
	R = the revenue per month (\$/month)
	C = the cost per month (\$/month)
	P = the profit per month (\$/month)
Assumptions:	$n = 10000 + 50(950 - p) + 0.02(q - 50000)$
	$C = 700 \cdot n + q$
	$R = p \cdot n$
	$F = R - C$
	$700 \leq p \leq 950$
	$50000 \leq q \leq 100000$
Objective:	Maximize F

Step 2. Select the modeling approach
This problem will be modeled as a multivariable constrained optimization problem and solved using the method of Lagrange multipliers.

Step 3. Formulate the model
We can get

$$\begin{aligned} F &= R - C = (p \cdot n) - (700n + q) \\ &= (p(10000 + 50(950 - p) + 0.02(q - 50000))) - (700(10000 + 50(950 - p) + 0.02(q - 50000)) + q) \\ &= -50p^2 + 91500p + 0.02p \cdot q - 15q - 39550000. \end{aligned}$$

We wish to maximize F satisfying the constraints $700 \leq p \leq 950$ and $50000 \leq q \leq 100000$.

Step 4. Solve the model
Then we will apply Lagrange multiplier methods to find the maximum of $y = F(p, q)$ over the set previously showing. Compute

$$\nabla F = (-100p + 91500 + 0.02q, 0.02p - 15).$$

First, let $\nabla F = 0$, we can find that the solution of q is not in the feasible region. So the maximum F is on the boundary of the feasible region. We can find that if p is a fixed parameter, then $F(p, q)$ increases, as $q \longrightarrow 100000$. Since the profit F and the cost of advertising q shows a positive correlation. So we can claim that F is optimal when $q = 100000$. And we set $g(p, q) = q = 100000$. So we get $\nabla g = (0, 1)$.

From $\nabla F = \lambda \cdot \nabla g$, so the Lagrange multiplier equations are

$$\begin{aligned} -100p + 91500 + 0.02q &= 0 \\ 0.02p - 15 &= \lambda. \end{aligned}$$

We get

$$\begin{aligned} &p = 935, \\ &q = 100000, \\ &\lambda = 3.7, \\ &F = 2661250. \end{aligned}$$

Step 5. Answer the question
So we need to set the price to 935\$, and pay 100000\$ into the advertising. Then we can get the maximum of the final profit which is 2661250\$.

(b)
Let's treat the price elasticity as an unknown variable denoted as e_1. Then we can get

$$\begin{aligned} n &= 10000 + e_1(950 - p) + 0.02(q - 50000), \\ F &= R - C \\ &= -e_1p^2 + 0.02pq + (9000 + 1650e_1)p - 15q - (6300000 + 665000e_1) \end{aligned}$$

And $\nabla F = (-2e_1p + 0.02q + 9000 + 1650e_1, 0.02p - 15)$.
Using the Lagrange multiplier method we can get

$$\begin{aligned} p &= \frac{825e_1 + 5500}{e_1}, \\ q &= 100000, \\ \lambda &= \frac{3e_1 + 220}{2e_1}. \end{aligned}$$

so at the point $p = 935$, $q = 100000$, $e_1 = 50$, we can get

$$\begin{aligned} S(p, e_1) &= \frac{dp}{de_1} \cdot \frac{e_1}{p} \approx -0.1176, \\ S(q, e_1) &= 0. \end{aligned}$$

Then let's treat the advertising elasticity as an unknown variable denoted as e_2. Then we can get

$$\begin{aligned} n &= 10000 + 50(950 - p) + e_2(q - 50000), \\ F &= R - C \\ &= -50p^2 + e_2pq + (92500 - 50000e_2)p - (700e_2 + 1)q + (35000000e_2 - 40250000) \end{aligned}$$

And $\nabla F = (-100p + e_2q + 92500 - 50000e_2, e_2p - 700e_2 - 1)$.
Using the Lagrange multiplier method we can get

$$\begin{aligned} p &= 500e_2 + 925, \\ q &= 100000, \\ \lambda &= 500{e_2}^2 + 225e_2 - 1. \end{aligned}$$

so at the point $p = 935$, $q = 100000$, $e_2 = 0.02$, we can get

$$\begin{aligned} S(p, e_2) &= \frac{dp}{de_2} \cdot \frac{e_2}{p} \approx 0.0107, \\ S(q, e_2) &= 0. \end{aligned}$$

(d)
The value of the multiplier found in part (a) is $\lambda = 3.7$. That means we can get

$$\frac{\partial F}{\partial c} = \lambda = 3.7,$$

where c is equals to q since $g = q = 100000$. That means that the addition of 1 \$ of the cost in advertising $\Delta q = 1$, the profit $\Delta F = 3.7$ \$. So if the manufacturer wants to get more profit, they should increase the cost of advertising.

Question 2.

We can write the objective function as Maximize $l = -2sx_1^2 - 2x_2^2 + 2x_1x_2 + 6x_1$. So we can get that

$$\nabla l = (-4x_1 + 2x_2 + 6, 2x_1 - 4x_2).$$

By solving $\nabla l = 0$, we can find that the solution is not in feasible region. So the maximum must occur on the boundary. And let's add relaxation parameter x_3 and x_4. Then we can write the question as

$$\begin{array}{ll} Maximize & l(x_1, x_2, x_3, x_4) = -2x_1^2 - 2x_2^2 + 2x_1x_2 + 6x_1 + 0x_3 + 0x_4 \\ Subject\ to & \begin{cases} g_1(x_1, x_2, x_3, x_4) = 3x_1 + 4x_2 + x_3^2 - 6 = 0 \\ g_2(x_1, x_2, x_3, x_4) = -x_1 + 4x_2 + x_4^2 - 2 = 0 \end{cases} \end{array}$$

Compute

$$\begin{aligned} &\nabla l = (-4x_1 + 2x_2 + 6, -4x_2 + 2x_1, 0, 0), \\ &\nabla g_1 = (3, 4, 2x_3, 0), \\ &\nabla g_2 = (-1, 4, 0, 2x_4). \end{aligned}$$

Then the Lagrange multiplier formula $\nabla l = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$ gives

$$\begin{cases} -4x_1 + 2x_2 + 6 = 3\lambda_1 - \lambda_2 \\ -4x_2 + 2x_1 = 4\lambda_1 + 4\lambda_2 \\ 0 = 2x_3\lambda_1 + 0 \\ 0 = 0 + 2x_4\lambda_2 \end{cases}$$

Observing the equations we can find that λ_1, λ_2 cannot both being 0s.
i. If $x_3 = x_4 = 0$.
It becomes to

$$\begin{cases} 3x_1 + 4x_2 - 6 = 0 \\ -x_1 + 4x_2 - 2 = 0 \end{cases}$$

We get $x_1 = 1, x_2 = \frac{3}{4}$, and $f = -\frac{35}{8}$.
ii. If $x_3 = 0, x_4 \neq 0$. This can imply $\lambda_1 \neq 0, \lambda_2 = 0$.
It becomes to

$$\begin{cases} 3x_1 + 4x_2 - 6 = 0 \\ -4x_1 + 2x_2 + 6 = 3\lambda_1 \\ -4x_2 + 2x_1 = 4\lambda_1 \end{cases}$$

We get $x_1 = \frac{54}{37}, x_2 = \frac{15}{37}, \lambda_1 = \frac{12}{37}$, and $f = -\frac{198}{37}$.
iii. If $x_3 \neq 0, x_4 = 0$. This can imply $\lambda_2 \neq 0, \lambda_1 = 0$.
It becomes to

$$\begin{cases} -x_1 + 4x_2 - 2 = 0 \\ -4x_1 + 2x_2 + 6 = -\lambda_2 \\ -4x_2 + 2x_1 = 4\lambda_2 \end{cases}$$

We get $\lambda_2 = 0$, that is a contradiction.

So we can get that $-\frac{198}{37} \leq -\frac{35}{8}$, so the minimum of f is $-\frac{198}{37}$.