Program of Spectral volume method for 1D hyperbolic conservation law

1 One-dimensional formulations

Consider the following one-dimensional hyperbolic conservation law:

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \tag{1.1}$$

where u is the state variable and f if the flux.

Given a partition of the domain $[a,b]$, $\{x_{i+1/2}\}_{i=0}^{N}$, the domain is then divided into N nonoverlapping spectral volumes(SVs)

$$[a,b] = \bigcup_{i=1}^{N} S_i, \quad S_i = [x_{i-1/2}, x_{i+1/2}].$$

with $\Delta x_i = x_{i+1/2} - x_{i-1/2}$ and $x_{1/2} = a, x_{N+1/2} = b$.

Given a desired numerical order of accuracy $k+1$ (degree k polynomail) for (1.1), each spectral volume S_i is subdivided into $k+1$ control volumes using the following parting $\{x_{i,j+1/2}\}_{j=0}^{k+1}$ with $x_{i,1/2} = x_{i-1/2}$ and $x_{i,k+1+1/2} = x_{i+1/2}$, the j-th control volumes(CVs) $C_{i,j}$ is then

$$C_{i,j} = (x_{i,j-1/2}, x_{i,j+1/2}), \quad j = 1, \ldots, k+1.$$

Denote $\Delta x_{i,j} = x_{i,j+1/2} - x_{i,j-1/2}$, the cell-average at time t for control volume $C_{i,j}$ is defined as

$$\bar{u}_{i,j} \triangleq \frac{\int_{C_{i,j}} u(x,t)dx}{\Delta x_{i,j}}, \quad j = 1, \cdots, k+1, \quad i = 1, \cdots, N.$$

Integrating (1.1) over each CV and applying the Gauss theorem then yields:

$$\frac{d\bar{u}_{i,j}}{dt} = -\frac{1}{\Delta x_{i,j}} \left(f(u_{i,j+1/2}) - f(u_{i,j-1/2}) \right), \tag{1.2}$$

where $\bar{u}_{i,j}$ are the CV-averaged solutions, and they are the DOFs. The DOFs are then used to reconstruct a degree k polynomail in each SV_i:

$$u(x,t) \approx U_i(x,t) = \sum_{j=1}^{k+1} \bar{u}_{i,j}(t) \bar{L}_{i,j}(x), \tag{1.3}$$

where $\bar{L}_{i,j}$ are the degree k polynomials called shape functions satisfying the following equations:

$$\frac{1}{\Delta x_{i,j}} \int_{C_{i,j}} \bar{L}_{i,p}(x)dx = \delta_{jp}$$

with δ_{jp} the Kronecker delta function. Using this solution polynomail, the fluxs at the CV-boundaries can be approximated and the right-hand-side of (1.2) can be evaluated.

It is obvious that the solution is continuous within each SV, but discontinuous across SV boundaries. Therefor, at internal CV interfaces, the analytical flux is computed as follows:

$$f(u_{i,j+1/2}) \approx F_{i,j+1/2} = f(U_i(x_{i,j+1/2})).$$

At the element interface, a Riemann flux is computed since the numerical solution is discontinuous

$$F_{i,1/2} = f_{Riem}\left(U_{i-1}(x_{i,k+1+1/2}), U_i(x_{i,1/2})\right), \quad F_{i,k+1+1/2} = f_{Riem}\left(U_i(x_{i,k+1+1/2}), U_{i+1}(x_{i,1/2})\right).$$

Finally the semidiscrete SV scheme becomes

$$\frac{d\bar{u}_{i,j}}{dt} = -\frac{1}{\Delta x_{i,j}}\left(F_{i,j+1/2} - F_{i,j-1/2}\right), \tag{1.4}$$

Furthermore, uses SSP Runge-Kutta time discretization to obtain the full discrete schemes. Let

$$\mathcal{L}_{i,j}(U) = -\frac{1}{\Delta x_{i,j}}\left(F_{i,j+1/2} - F_{i,j-1/2}\right)$$

and

$$U = \begin{bmatrix} \bar{u}_{1,1} \\ \cdots \\ \bar{u}_{i,j} \\ \cdots \\ \bar{u}_{N,k+1} \end{bmatrix}, \qquad \mathcal{L}(U) = \begin{bmatrix} \mathcal{L}_{1,1}(U) \\ \cdots \\ \mathcal{L}_{i,j}(U) \\ \cdots \\ \mathcal{L}_{N,k+1}(U) \end{bmatrix}$$

Then the semi-discrete scheme will be written as

$$\frac{dU}{dt} = \mathcal{L}(U)$$

From the time level n to n+1, the SSP 3rd order Runge-Kutta scheme:

$$\begin{aligned} U^{(1)} &= U^n + \Delta t \mathcal{L}(U^n), \\ U^{(2)} &= \frac{3}{4}U^n + \frac{1}{4}U^{(1)} + \frac{1}{4}\Delta t \mathcal{L}(U^{(1)}), \\ U^{n+1} &= \frac{1}{3}U^n + \frac{2}{3}U^{(2)} + \frac{2}{3}\Delta t \mathcal{L}(U^{(2)}). \end{aligned}$$

Remark

★ One of choice for the flux is used upwind Riemann flux, that is

$$F_{i,1/2} = f(U^-(x_{i,1/2})) = f(U_{i-1}(x_{i-1,k+1+1/2})).$$

★ Now, we give one of the construction of the SV basis function $\bar{L}_{i,j}$. On each S_i, let $p_i(x)$ is the degree k polynomail function approximation to the function $u(x)$:

$$p_i(x) = u(x) + O(h^{k+1}), \quad x \in S_i, \quad i = 1, \cdots, N,$$

where $h = \min \Delta x_{i,j}$. And we are looking for the function p_i whose cell average in each of the CVs in S_i agrees with that of $u(x)$

$$\frac{\int_{C_{i,j}} p_i(x)dx}{\Delta x_{i,j}} = \bar{u}_{i,j}, \quad j = 1, \cdots, k+1.$$

Denote the primitive function of $u(x)$,

$$U(x) \equiv \int_{x_{i,1/2}}^{x} u(\xi)d\xi, \quad x \in S_i.$$

It is obvious that $U(x_{i,j+1/2})$ can be expressed exaclty by the averages of $u(x)$ at the control volume boundaries of S_i,

$$U(x_{i,j+1/2}) = \sum_{l=1}^{j} \int_{x_{i,l-1/2}}^{x_{i,l+1/2}} u(\xi)d\xi = \sum_{l=1}^{j} \bar{u_{i,l}} \Delta x_{i,l}, \quad j = 1, \cdots, k+1. \tag{1.5}$$

Note that by definition, we have $U(x_{i,1/2}) = 0$. Thus with the cell-averaged variables for the CVs, we know the primitive function $U(x)$ at CV boundaries exactly. If we denote the polynomial of degree at most $k+1$, which interpolates $U(x_{i,j+1/2})$ at the following $k+2$ points

$$x_{i,1/2}, \cdots, x_{i,k+1+1/2}$$

by $P_i(x)$ and denotes its derivative by $p_i(x)$, i.e. $p_i(x) \equiv P_i'(x)$. Then it is easy to verify that

$$\begin{aligned}
\frac{1}{\Delta x_{i,j}} \int_{x_{i,j-1/2}}^{x_{i,j+1/2}} p_i(x)dx = \quad & \frac{1}{\Delta x_{i,j}} \int_{x_{i,j-1/2}}^{x_{i,j+1/2}} P_i'(x)dx = \frac{1}{\Delta x_{i,j}} [P(x_{i,j+1/2}) - P(x_{i,j-1/2})] \\
& = \frac{1}{\Delta x_{i,j}} [U(x_{i,j+1/2}) - U(x_{i,j-1/2})] \\
& = \frac{1}{\Delta x_{i,j}} \left(\int_{x_{i,1/2}}^{x_{i,j+1/2}} u(x)dx - \int_{x_{i,1/2}}^{x_{i,j-1/2}} u(x)dx \right) \\
& = \frac{1}{\Delta x_{i,j}} \int_{x_{i,j-1/2}}^{x_{i,j+1/2}} u(x)dx = \bar{u}_{i,j}, \quad j = 1, \cdots, k+1.
\end{aligned}$$

This implies that $p_i(x)$ is what we are looking for.Standard approximation theory tell us that

$$P_i'(x) = U'(x) + O(h^{k+1}), \quad x \in S_i,$$

i.e.

$$p_i(x) = u(x) + O(h^{k+1}), \quad x \in S_i,$$

which is actually the accuracy requirement. For this we use the Lagrange interpolation polynomial

$$P_i(x) = \sum_{r=0}^{k+1} U_{i,r+1/2} \varphi_{i,r+1/2}(x), \quad x \in S_i,$$

where $\varphi_{i,r+1/2}(x)$ are the Lagrange interpolation coefficients, $U_{i,r+1/2} = U(x_{i,r+1/2})$, and

$$\varphi_{i,r+1/2}(x) = \prod_{l=0, l\neq r}^{k+1} \frac{x - x_{i,l+1/2}}{x_{i,r+1/2} - x_{i,l+1/2}}.$$

Note that $U_{i,1/2} = 0$, then

$$p_i(x) = P_i'(x) = \sum_{r=0}^{k+1} U_{i,r+1/2} \varphi'_{i,r+1/2}(x) = \sum_{r=1}^{k+1} U_{i,r+1/2} \varphi'_{i,r+1/2}(x),$$

Substitute (1.5) into the above equation, we obtain the polynomial based on the cell-averaged variables

$$p_i(x) = \sum_{r=1}^{k+1}\sum_{l=1}^{r} \bar{u}_{i,l}\Delta x_{i,l}\varphi'_{i,r+1/2}(x) = \sum_{l=1}^{k+1}\left(\sum_{r=l}^{k+1}\varphi'_{i,r+1/2}(x)\Delta x_{i,l}\right)\bar{u}_{i,l} = \sum_{j=1}^{k+1}\bar{u}_{i,j}\bar{L}_{i,j}(x),$$

where

$$\bar{L}_{i,j}(x) = \Delta x_{i,j}\sum_{r=j}^{k+1}\varphi'_{i,r+1/2}(x)$$

are the SV basis function. Further calculation shows that

$$\bar{L}_{i,j}(x) = \Delta x_{i,j}\sum_{r=j}^{k+1}\sum_{s=0,s\neq r}^{k+1}\frac{1}{x_{i,r+1/2}-x_{i,s+1/2}}\prod_{l=0,l\neq r,s}^{k+1}\frac{x-x_{i,l+1/2}}{x_{i,r+1/2}-x_{i,l+1/2}}, \quad j=1,\cdots,k+1.$$

and

$$\bar{L}'_{i,j}(x) = \Delta x_{i,j}\sum_{r=j}^{k+1}\sum_{s=0,s\neq r}^{k+1}\frac{1}{x_{i,r+1/2}-x_{i,s+1/2}}\sum_{t=0,t\neq r,s}^{k+1}\frac{1}{x_{i,r+1/2}-x_{i,t+1/2}}\prod_{l=0,l\neq r,s,t}^{k+1}\frac{x-x_{i,l+1/2}}{x_{i,r+1/2}-x_{i,l+1/2}}.$$

Note: In matlab code, the subscript r is from $j+1$ to $k+2$, and s,t,l are from 1 to $k+2$ when compute the $\bar{L}_{i,j}$ and $\bar{L}'_{i,j}$.

2 Error formulas

Let u and u_h be the exact and approximate solution, relatively. we will test the convergence in the whole domain. We compute Max error, L_1 error and L_2 error, denote by

$$e_{max} := \max(|u-u_h|), \quad e_{l1} := \int_a^b |u-u_h|dx, \quad e_{l2} := \left(\int_a^b (u-u_h)^2 dx\right)^{1/2}.$$

We also will test the superconvergence at some specific points. For this purpose, we denote the error at downwind point by

$$e_n := \left(\frac{1}{N}\sum_{i=1}^{N}(u-u_h)^2(x^-_{i+1/2},t)\right)^{1/2},$$

the error of the cell average by

$$e_c := \left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{h_i}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}(u-u_h)(x,t)dx\right)^2\right)^{1/2},$$

the error at the interior flux points by

$$e_i := \left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{k}(u-u_h)^2(x_{i,j+1/2},t)\right)^{1/2},$$

and the derivative error at the interior flux points by

$$e_{di} := \left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{k}\partial_x^2(u-u_h)(x_{i,j+1/2},t)\right)^{1/2}.$$