计算几何

Computational Geometry

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一：基本公式

1.1 三角形

```
半周长 P = ( a + b + c ) / 2
面积 S = absin(C) / 2 = sqrt( P (P – a)(P - b)(P - c) )
余弦定理 2bc*cos(A) = b^2 + c^2 – a^2
中线 Ma = sqrt( 2 ( b^2 + c^2 ) – a^2 ) / 2
       = sqrt( b^2 + c^2 + 2bc*cos(A) ) / 2
角平分线 Ta = sqrt( bc ( (b + c)^2 - a^2) ) / (b + c)
           = 2bc*cos(A/2) / (b + c)
高线 Ha = bsin(C) = csin(B)
       = sqrt( b^2 - (( a^2 + b^2 - c^2 ) / (2a)) ^ 2 )
内切圆半径 r = S/P = a*sin(B/2) * sin(C/2) / sin((B+C)/2)
         = 4R*sin(A/2) * sin(B/2) * sin(C/2)
         = sqrt( (P – a) * (P – b) * (P – c) / P )
         = P * tan(A/2) * tan(B/2) * tan(C/2)
外接圆半径 R = abc / (4S) = a / (2sin(A))
          = b / (2sin(B)) = c / (2sin(C))
```

1.2 四边形

```
D1,D2 为对角线,M 对角线中点连线,A 为对角线夹角
1. a^2 + b^2 + c^2 + d^2 = D1^2 + D2^2 + 4M^2
2. S = D1D2sin(A)/2
(以下对圆的内接四边形)
3. ac + bd = D1 * D2
4. S = sqrt((P - a)(P - b)(P - c)(P - d)), P 为半周长
```

1.3 正 n 边形

```
R 为外接圆半径,r 为内切圆半径
1. 中心角 A = 2 * PI / n
2. 内角 C = (n - 2) * PI / n
3. 边长 a = 2sqrt(R^2 - r^2) = 2R * sin(A/2) = 2r * tan(A/2)
4. 面积 S = nar / 2 = nr^2 * tan(A/2)
     = nR^2 * sin(A) / 2 = na^2 / (4tan(A/2))
```

1.4 圆

```
r 为半径, A 为角度
弧长 l = rA
弦长 a = 2 * sqrt(2hr - h^2) = 2rsin(A/2)
弓形高 h = r - sqrt(r^2 - a^2/4) = r(1 - cos(A/2)) = atan(A/4) / 2
扇形面积 S1 = r * l / 2 = r^2 * A / 2
弓形面积 S2 = (rl - a(r - h)) / 2 = r^2 * (A - sin(A)) / 2
```

1.5 棱柱

1. 体积 V = Ah, A 为底面积,h 为高
2. 侧面积 S = lp, l 为棱长, p 为直截面周长
3. 全面积 T=S+2A

1.6 棱锥

1. 体积 V = Ah/3, A 为底面积,h 为高

(以下对正棱锥)

2. 侧面积 S = lp/2, l 为斜高,p 为底面周长
3. 全面积 T = S + A

1.7 棱台

1. 体积 V = (A1 + A2 + sqrt(A1A2)) * h / 3, A1.A2 为上下底面积,h 为高

(以下为正棱台)

2. 侧面积 S = (p1 + p2) l / 2, p1.p2 为上下底面周长,l 为斜高
3. 全面积 T = S + A1 + A2

1.8 圆柱

1. 侧面积 S = 2 * PI * r * h
2. 全面积 T = 2 * PI * r * (h + r)
3. 体积 V = PI * r^2 * h

1.9 圆锥

母线 l = sqrt(h^2 + r^2)

侧面积 S = PI * r * l

全面积 T = PI * r (l + r)

体积 V = PI * r^2 * h / 3

1.10 圆台

母线 l = sqrt(h^2 + (r1 - r2)^2)

侧面积 S = PI * (r1 + r2) * l

全面积 T = PI * r1 * (l + r1) + PI * r2 * (l + r2)

体积 V = PI * (r1^2 + r2^2 + r1r2) * h / 3

1.11 球

1. 全面积 T = 4 * PI * r^2
2. 体积 V = 4 * PI * r^3 / 3

1.12 球台

1. 侧面积 S = 2 * PI * r * h
2. 全面积 T = PI * (2rh + r1^2 + r2^2)
3. 体积 V = PI * h(3 * (r1^2 + r2^2) + h^2) / 6

1.13 球扇形

1. 全面积 T = PI * r(2h + r0), h 为球冠高,r0 为球冠底面半径
2. 体积 V = 2 * PI * r^2 * h / 3
3. 球缺 V = V 球扇 - V 圆锥 = PI * h * h * (3r - h) / 3

二: 点、线

2.1 结构定义

```
#define MP make_pair
#define LL long long
#define uLL unsigned long long
const double PI = acos(-1.0);
const double eps = 1e-8;
struct point {
    double x, y, z;
    point(){}
    point( double x, double y, double z ) : x(x), y(y), z(z) {}
    point operator - ( const point b ) const {
        return point( x - b.x, y - b.y, z - b.z );
    }
    point operator + ( const point b ) const {
        return point( x + b.x, y + b.y, z + b.z );
    }
    point operator * ( double d ) const {
        return point( x * d, y * d, z * d );
    }
    point operator / ( double d ) const {
        return point( x / d, y / d, z / d );
    }
    double len() {
        return sqrt( x * x + y * y + z * z );
    }
    void input() {
        scanf( "%lf%lf%lf", &x, &y, &z );
    }
};
struct Line { point a, b; };
struct Nline { int a, b, c; }; // ax + by + c = 0 一般方程
struct Circle { double r; point c; };
struct Sphere { double r; point3 c; }; //球体
int dcmp( double x ){
    return (x > eps) - (x < -eps) ;
}
```

2.2 向量 p 绕着圆点转动 radian（弧度） 返回得到的点

```
point rotate(point p, double radian) {
    double c = cos(radian), s = sin(radian);
    point res;
    res.x = p.x * c - p.y * s;
    res.y = p.y * c + p.x * s;
    return res;
}
```

2.3 二维叉乘 返回 a × b

```
double cross( point a, point b ) {
    return a.x * b.y - a.y * b.x;
}
```

2.4 二维点乘 返回 a · b

```
double dot( point a, point b ) {
    return a.x * b.x + a.y * b.y;
}
```

2.5 二维两点距离

```
double dis(point a, point b) {
    return sqrt(dot(a – b, a - b));
}
```

2.6 向量 a, b 夹角的余弦值(弧度制)

```
double cos(point a, point b) {
    return dot(a, b) / a.len() / b.len();
}
```

2.7 向量 a, b 夹角的正弦值(弧度制)

```
double sin(point a, point b) {
    return fabs( cross(a, b) / a.len() / b.len() );
}
```

2.8 判断 a,b,c 三点共线

```
bool in_line(point a, point b, point b) {
    return dcmp( cross(b - a, c - a) ) == 0 ;
}
```

2.9 判断点在线段的位置

前提假设 a、b、x 共线

    返回:
        x 在 seg(a,b)内: -1
        x 在 seg(a,b)上: 0

```
        x 在 seg(a,b)外: 1
int btw(point x, point a, point b) {
    return dcmp( dot(a - x, b - x) );
}
```

2.10 判断线段 ab 和 cd 是否相交

```
    类型          返回          res
----------------------------------------
1.  不相交            0        不变
2.  规范相交          1        交点  （交叉）
3.  非规范相交        2        不变  （端点在另一线段，有重叠段）
int segCross(point a, point b, point c, point d, point &res) {
    double s1, s2;
    int d1, d2, d3, d4;
    d1 = dcmp( s1 = cross(b - a, c - a) );
    d2 = dcmp( s2 = cross(b - a, d - a) );
    d3 = dcmp( cross(d - c, a - c) );
    d4 = dcmp( cross(d - c, b - c) );
    if( (d1^d2) == -2 && (d3^d4) == -2 ){
        res.x = (c.x * s2 - d.x * s1) / (s2 - s1);
        res.y = (c.y * s2 - d.y * s1) / (s2 - s1);
        return 1;
    }
    if( d1 == 0 && btw(c, a, b) <= 0 ||
        d2 == 0 && btw(d, a, b) <= 0 ||
        d3 == 0 && btw(a, c, d) <= 0 ||
        d4 == 0 && btw(b, c, d) <= 0)
        return 2;
    return 0;
}
```

2.11 判断直线 ab 和线段 cd 是否相交

```
    类型          返回          res
----------------------------------------
1.  不相交            0        不变
2.  规范相交          1        交点  （交叉）
3.  非规范相交        2        不变  （线段端点在直线，有重叠段）
int segLineCross(point a, point b, point c, point d, point &res) {
    double s1, s2;
    int d1, d2;
    d1 = dcmp( s1 = cross(b - a, c - a) );
    d2 = dcmp( s2 = cross(b - a, d - a) );
    if( (d1^d2) == -2 ) {
        res.x = (c.x * s2 - d.x * s1) / (s2 - s1);
```

```
        res.y = (c.y * s2 - d.y * s1) / (s2 - s1);
        return 1;
    }
    if( d1 == 0 || d2 == 0 ) return 2;
    return 0;
}
```

2.12 判断直线 ab 和直线 cd 是否相交

```
    类型              返回        res
--------------------------------------
1.  不相交（平行）       0        不变
2.  规范相交             1        交点
3.  非规范相交（重合）    2        不变
int lineCross(point a, point b, point c, point d, point &res) {
    double s1, s2;
    s1 = cross(b - a, c - a);
    s2 = cross(b - a, d - a);
    if( dcmp(s1) == 0 && dcmp(s2) == 0 ) return 2;
    if( dcmp(s2 - s1) == 0 ) return 0;
    res.x = (c.x * s2 - d.x * s1) / (s2 - s1);
    res.y = (c.y * s2 - d.y * s1) / (s2 - s1);
    return 1;
}
```

2.13 判断两点在线段的同侧或异侧..............

```
    类型              返回
----------------------------
1.  某点在线段上       0
2.  同侧              1
3.  异侧              -1
int pointside( point a, point b, Line l ) {
    return dcmp( cross(a – l.a, l.b – l.a) * cross(b – l.a, l.b – l.a) );
}
```

2.14 求线段所在直线一般方程

```
Nline lfs(point p1, point p2) //line from segment
{
    Nline tmp;
    tmp.a = p2.y - p1.y;
    tmp.b = p1.x - p2.x;
    tmp.c = p2.x * p1.y - p1.x * p2.y;
    return tmp;
}
```

2.15 求点关于直线的对称点

```
point spl(point p, Nline L) { // symmetrical point of Line
    point p2;
    double d;
    d = L.a * L.a + L.b * L.b;
    p2.x = (L.b * L.b * p.x - L.a * L.a * p.x -
           2 * L.a * L.b * p.y - 2 * L.a * L.c) / d;
    p2.y = (L.a * L.a * p.y - L.b * L.b * p.y -
           2 * L.a * L.b * p.x - 2 * L.b * L.c) / d;
    return p2;
}
```

2.16 点到直线的最近距离

```
double ptoline( point p, point a, point b ){
    return fabs(cross(p - a, b - a)) / dis(a, b);
}
double ptoline( point p, Nline l ){
    return ( p.x * l.a + p.y * l.b + l.c ) / sqrt( l.a * l.a + l.b * l.b );
}
```

2.17 点到线段的最近距离

```
double ptoseg( point p, point a, point b ) {
    if( dcmp( dot(p - a, b - a) ) <= 0 ) return dis(p, a);
    if( dcmp( dot(p - b, a - b) ) <= 0 ) return dis(p, b);
    return fabs(cross(p - a, b - a)) / dis(a, b);
}
```

2.18 两线段最近距离

相交距离为 0，否则枚举两条线段的端点到另一线段的距离

2.19 向量夹角

```
double angle( point a, point b ) {
    double k = dot(a, b) / a.len() / b.len();
    k = max(k, -1.0); k = min(k, 1.0);
    return acos( k );
}
```

2.20 动点共线方程

```
//动点 point( pnt[i].x + dx[i] * t, pnt[i].y + dy[i] * t ) 求大于 0 的解
void cal( int &i, int &j, int &k, double &t1, double &t2 ) {
    double a1 = pnt[i].x - pnt[j].x, b1 = dx[i] - dx[j];
    double a2 = pnt[k].y - pnt[j].y, b2 = dy[k] - dy[j];
    double a3 = pnt[i].y - pnt[j].y, b3 = dy[i] - dy[j];
    double a4 = pnt[k].x - pnt[j].x, b4 = dx[k] - dx[j];
```

```
    double a = b1 * b2 - b3 * b4;
    double b = a1 * b2 + a2 * b1 - a3 * b4 - a4 * b3;
    double c = a1 * a2 - a3 * a4;
    double dlt = b * b - 4.0 * a * c;
    if( dcmp(a) == 0 ) {
        if( dcmp(b) == 0 ) { //c == 0 无穷解  c != 0 无解
            t1 = t2 = -1.0;
        }
        else t1 = -c / b, t2 = -1.0;
    }
    else if( dcmp(dlt) == 0 ) t1 = -b / (2 * a), t2 = -1.0;
    else if( dlt > 0 ) {
        t1 = (-b - sqrt(dlt)) / (2 * a);
        t2 = (-b + sqrt(dlt)) / (2 * a);
    }
    else t1 = t2 = -1.0;
}
```

2.21 最近点对

```
//调用 closep(p, 0, n)
point p[mxn], py[mxn];

bool cmpy( point a, point b ) {
    return a.y < b.y;
}

double closep( point *p, int ll, int rr ) {
    if( rr - ll <= 1 ) return inf; //周长最小三角形返回 MP(inf,-1)
    int m = (ll + rr) >> 1;
    double midx = p[m].x;
    double res = min( closep(p, ll, m), closep(p, m, rr) );
    inplace_merge(p + ll, p + m, p + rr, cmpy);
    double x1 = midx - res, x2 = midx + res;
    int len = 0;
    for( int i = ll; i < rr; ++i )
        if( p[i].x > x1 && p[i].x < x2 )
            py[len++] = p[i];
    for( int i = 0; i < len; ++i )
        for( int j = i + 1; j < len && py[j].y < py[i].y + res; ++j )
            res = min(res, dis(py[i], py[j]));
    //可以再加一层 for(k = j + 1 -> len && py[k].y < py[i].y + res) 求周长最小的三角形  res 初始化为当前最优三角形周长一半  函数返回 pair<周长, id>
    return res;
}
```

三：三角形

3.1. 结构定义

```
struct triangle {
    point a, b, c;
    void input() {
        a.input(); b.input(); c.input();
        if( dcmp(cross(b - a, c - a)) < 0 ) //保障逆时针序
            swap(b, c);
    }
};
```

3.2 点在三角形内判定

```
//判断点 o 是否在△abc 内
bool intrian( point o, point a, point b, point c ) {
    if( dcmp(cross(b - a, o - a)) < 0 ) return false;
    if( dcmp(cross(c - b, o - b)) < 0 ) return false;
    if( dcmp(cross(a - c, o - c)) < 0 ) return false;
    return true;
}
```

3.3 三角形覆盖 k 次面积并（可扩展为任意多边形，二叉空间划分）

```
//初始无穷大平面  递归切割
struct polygon {
    int n;
    vector<point> p;
    double area() {
        double s = 0;
        for( int i = 2; i < n; ++i )
            s += cross(p[i-1] - p[0], p[i] - p[0]);
        return fabs(s) / 2;
    }
    point center() { //多边形重心
        double s = 0, sx = 0, sy = 0;
        for( int i = 2; i < n; ++i ) {
            double x = p[0].x + p[i-1].x + p[i].x;
            double y = p[0].y + p[i-1].y + p[i].y;
            double tmps = cross(p[i-1] - p[0], p[i] - p[0]) / 2;
            s += tmps;
            sx += x * tmps;
            sy += y * tmps;
        }
        return point( sx / s / 3, sy / s / 3 );
    }
```

```
}g[mxn]; Line L[mxn]; triangle sjx[mxn]; int cnt;

void add( point a, point b, int id, int cur ) {
    g[cur].p.clear();
    for( int i = 0; i < g[id].n; ++i ) {
        int d1 = dcmp( cross(b - a, g[id].p[i] - a) );
        int d2 = dcmp( cross(b - a, g[id].p[i+1] - a) );
        if( d1 >= 0 ) g[cur].p.push_back( g[id].p[i] );
        if( (d1 ^ d2) == -2 ) {
            point x = linecross( a, b, g[id].p[i], g[id].p[i+1] );
            g[cur].p.push_back(x);
        }
    }
    g[cur].n = g[cur].p.size();
}

void dfs( int dep, int id ) {
    g[id].p.push_back(g[id].p[0]);
    if( dep == m ) return ;
    point a = L[dep].s, b = L[dep].t;
    bool nolft = true, norht = true;
    for( int i = 0; i < g[id].n; ++i ) {
        int d = dcmp( cross(b - a, g[id].p[i] - a) );
        if( d > 0 ) nolft = false;
        if( d < 0 ) norht = false;
    }
    if( nolft || norht ) {
        dfs( dep + 1, id );
        return ;
    }
    bad[id] = true;
    ++cnt;
    add( a, b, id, cnt );
    dfs( dep + 1, cnt );
    ++cnt;
    add( b, a, id, cnt );
    dfs( dep + 1, cnt );
}

int t, n;
scanf( "%d", &t );
while( t-- ) {
    scanf( "%d", &n );
    m = cnt = 0;
```

```
    memset( ans, 0, sizeof(ans) );
    memset( bad, 0, sizeof(bad) );
    for( int i = 0; i < n; ++i ) {
        sjx[i].input();
        L[m++] = Line( sjx[i].a, sjx[i].b );
        L[m++] = Line( sjx[i].b, sjx[i].c );
        L[m++] = Line( sjx[i].c, sjx[i].a );
    }
    g[0].p.clear();
    g[0].p.push_back( point(-200, -200) ); //确保足够大
    g[0].p.push_back( point(200, -200) );
    g[0].p.push_back( point(200, 200) );
    g[0].p.push_back( point(-200, 200) );
    g[0].n = g[0].p.size();
    dfs(0, 0);
    for( int i = 0; i <= cnt; ++i ) {
        if( bad[i] ) continue;
        point x = g[i].center();
        int num = 0;
        for( int j = 0; j < n; ++j )
            if( intrian(x, sjx[j].a, sjx[j].b, sjx[j].c) )
                ++num;
        ans[num] += g[i].area();
    }
    for( int i = 1; i <= n; ++i )
        printf( "%.10lf\n", ans[i] );
}
```

3.4 三角形四心

重心：中线交点，近边三等分点，三角形内到三边距离之积最大，到三顶点距离的平方和最小

```
point zhong(point a, point b, point c) {
    return (a + b + c) / 3;
}
```

外心：三条垂直平分线交点，外接圆圆心

```
point wai( point& a, point& b, point& c ) {
    point res;
    double a1 = b.x - a.x, b1 = b.y - a.y, c1 = (a1 * a1 + b1 * b1)/2;
    double a2 = c.x - a.x, b2 = c.y - a.y, c2 = (a2 * a2 + b2 * b2)/2;
    double d = a1 * b2 - a2 * b1;
    res.x = a.x + (c1 * b2 - c2 * b1) / d;
    res.y = a.y + (a1 * c2 - a2 * c1) / d;
    return res;
}
```

内心：三内角平分线交点，内接圆圆心

```
point nei(point a, point b, point c) {
    double A = dis(b, c), B = dis(a, c), C = dis(a, b);
    double P = A + B + C;
    return a * (A/P) + b * (B/P) + c * (C/P);
}
```

旁心：一内角平分线和另外两角的外角平分线（好像木有啥用）

3.5 三角形费马点

费马点：到所有点距离之和最小的点

有△ABC，设∠A 大于 120 度，则点 A 为费马点

否则，费马点到三点的连线等分费马点周角，故此类三角形费马点也是三角形等角中心

从三角形三边向外做等边三角形 A’BC，AB’C，ABC’，则 AA’,BB’,CC’三线共点于费马点

//凸四边形费马点：对角线交点

//凹四边形费马点：凹点

```
//Find three numbers r + s + t = 1, which make p = r * a + s * b + t * c
void parametric(point p, point a, point b, point c) {
    double d = cross(b – a, c - a);
    r = cross(b – p, c - p) / d;
    s = cross(p – a, c - a) / d;
    t = cross(b – a, p - a) / d;
}
```

四：圆

4.1 结构定义

```
struct circle {
    point c;
    double r;
    int id;
    circle(){}
    circle(point c, double r) : c(c), r(r) {}
    point getp(double ang) { //圆上相对圆心以 ang 为极角的点
        return point(c.x + r * cos(ang), c.y + r * sin(ang), id);
    }
    void input(int k) {
        id = k;
        c.input(); scanf( "%lf", &r );
    }
}
```

4.2 点与圆的切点

```
//前提点在圆外
int ptancircle( point k, circle a ) {
    point u = k - a.c;
    double len = u.len();
    double ang = acos( a.r / len );
    double bas = atan2( u.y, u.x );
    pnt[num++] = a.getp( bas + ang );
    pnt[num++] = a.getp( bas - ang );
    return 2;
}
```

4.3 圆的公切线

```
//精度曾卡 1e-15, pnt 保存所有切点, 可分别保存在另外两个数组
int getTangents( circle a, circle b ) {
    int cnt = 0;
    if( a.r < b.r ) swap(a, b);
    double d2 = dis2(a.c, b.c);
    double rcha = a.r - b.r;
    double rsum = a.r + b.r;
    if( dcmp(d2 - rcha * rcha) < 0 ) return 0;
    double bas = atan2(b.c.y - a.c.y, b.c.x - a.c.x);
    if( dcmp(d2) == 0 && dcmp(a.r - b.r) == 0 ) return -1;
    if( dcmp(d2 - rcha * rcha) == 0 ) {
        pnt[num++] = a.getp(bas);
        pnt[num++] = b.getp(bas);
        cnt++;
        return 1;
    }
    double ang = acos( (a.r - b.r) / sqrt(d2) );
    pnt[num++] = a.getp(bas + ang); pnt[num++] = a.getp(bas - ang); cnt++;
    pnt[num++] = b.getp(bas + ang); pnt[num++] = b.getp(bas - ang); cnt++;
    if( dcmp(d2 - rsum * rsum) == 0 ) {
        pnt[num++] = a.getp(bas); pnt[num++] = b.getp(bas + pi);
        cnt++;
    }
    else if( dcmp(d2 - rsum * rsum) > 0 ) {
        double ang = acos( (a.r + b.r) / sqrt(d2) );
        pnt[num++] = a.getp(bas + ang); pnt[num++] = a.getp(bas - ang); cnt++;
        pnt[num++] = b.getp(pi + bas + ang); pnt[num++] = b.getp(pi + bas - ang);
        cnt++;
    }
    return cnt;
}
```

4.4 线段与圆交点

```
//圆心 c，半径 r，线段 ab，交点为 res1，res2，返回 k 是交点个数
//#define sqr(x) ((x)*(x))
int seg_cir(point c, double r, point a, point b, point &res1, point &res2) {
    int k = 0;
    double aa = sqr(a.x - b.x) + sqr(a.y - b.y);
    double bb = 2 * ((b.x - a.x)*(a.x - c.x) + (b.y - a.y)*(a.y - c.y));
    double cc = sqr(c.x) + sqr(c.y) + sqr(a.x) + sqr(a.y) - r * r - 2 * (c.x
* a.x + c.y * a.y);
    if( dcmp( bb * bb - 4 * aa * cc ) >= 0 ) {
        double u1 = (-bb + sqrt(bb * bb - 4 * aa * cc)) / 2.0 / aa;
        double u2 = (-bb - sqrt(bb * bb - 4 * aa * cc)) / 2.0 / aa;
        if( u1 > u2 && dcmp(u2) >= 0 ) swap(u1, u2);
        if( dcmp(u1) >= 0 && dcmp(u1-1) <= 0 ) {
            res1.x = a.x + u1 * (b.x - a.x);
            res1.y = a.y + u1 * (b.y - a.y);
            //if( dcmp(res1.y - c.y) <= 0 ) res1.ok = true; 下半圆判定
            ++k;
        }
        if( dcmp(u1-u2) && dcmp(u2) >= 0 && dcmp(u2-1) <= 0 ) {
            res2.x = a.x + u2 * (b.x - a.x);
            res2.y = a.y + u2 * (b.y - a.y);
            //if( dcmp(res2.y - c.y) <= 0 ) res2.ok = true; 下半圆判定
            ++k;
        }
    }
    return k;
}
```

4.5 圆与圆交点

```
//两圆圆心为 c1,c2，半径为 r1,r2，交点保存在 k1,k2
//两圆重合需要自行特判,返回交点个数
int CirCrossCir(point c1, double r1, point c2, double r2, point &k1, point &k2)
{
    double mx = c2.x - c1.x, sx = c2.x + c1.x, mx2 = mx * mx;
    double my = c2.y - c1.y, sy = c2.y + c1.y, my2 = my * my;
    double sq = mx2 + my2, d = -(sq - sqr(r1 - r2)) * (sq - sqr(r1 + r2));
    if (dcmp(d) < 0) return 0;
    if (dcmp(d) == 0) d = 0; else d = sqrt(d);
    double x = mx * ((r1 + r2) * (r1 - r2) + mx * sx) + sx * my2;
    double y = my * ((r1 + r2) * (r1 - r2) + my * sy) + sy * mx2;
    double dx = mx * d, dy = my * d; sq *= 2;
    k1.x = (x - dy) / sq; k1.y = (y + dx) / sq;
    k2.x = (x + dy) / sq; k2.y = (y - dx) / sq;
```

```
    if (d > eps) return 2;
    else return 1;
}
```

4.6 圆的面积并

把下面那个覆盖k次加起来，都是O(n^2)的复杂度

4.7 圆覆盖k次面积并

```
//area[i]保存覆盖i次的面积
#define sqr(x) ((x)*(x))
struct Circle {
    point c;
    double r, ang;
    int d;
    Circle(){}
    Circle(point c, double ang = 0, int d = 0):c(c), ang(ang), d(d) {}
    void input() {
        c.input(); d = 1;
        scanf( "%lf", &r );
    }
}cir[mxn], tp[mxn * 2];

bool circmp(const Circle& a, const Circle& b) {
    return dcmp(a.r - b.r) < 0;
}
bool cmp(const Circle& a, const Circle& b) {
    if( dcmp(a.ang - b.ang) )
        return a.ang < b.ang;
    return a.d > b.d;
}

double calc(Circle o, Circle a, Circle b) {
    double ans = (b.ang - a.ang) * sqr(o.r)
        - cross(a.c - o.c, b.c - o.c) + cross(a.c - point(0,0), b.c - point(0,0));
    return ans / 2;
}

void CirUnion(Circle cir[], int n) {
    Circle res1, res2;
    sort( cir, cir + n, circmp );
    for( int i = 0; i < n; ++i )
        for( int j = i + 1; j < n; ++j )
            if( dcmp(dis(cir[i].c, cir[j].c) + cir[i].r - cir[j].r) <= 0 )
                cir[i].d++;
```

```
    for( int i = 0; i < n; ++i ) {
        int tn = 0, cnt = 0;
        for( int j = 0; j < n; ++j ) {
            if( i == j ) continue;
            if( CirCrossCir(cir[i].c, cir[i].r, cir[j].c, cir[j].r,
            res2.c, res1.c) < 2) continue; //res2和res1不能交换
            res1.ang = atan2(res1.c.y - cir[i].c.y, res1.c.x - cir[i].c.x);
            res2.ang = atan2(res2.c.y - cir[i].c.y, res2.c.x - cir[i].c.x);
            res1.d = 1;    tp[tn++] = res1;
            res2.d = -1;   tp[tn++] = res2;
            if( dcmp(res1.ang - res2.ang) > 0 ) cnt++;
        }
        tp[tn++] = Circle(point(cir[i].c.x - cir[i].r, cir[i].c.y), pi, -cnt);
        tp[tn++] = Circle(point(cir[i].c.x - cir[i].r, cir[i].c.y), -pi, cnt);
        sort( tp, tp + tn, cmp );
        int p, s = cir[i].d + tp[0].d;
        for( int j = 1; j < tn; ++j ) {
            p = s;  s += tp[j].d;
            area[p] += calc( cir[i], tp[j - 1], tp[j] );
        }
    }
}

void solve() {
    for (int i = 0; i < n; ++i)
        cir[i].input();
    memset(area, 0, sizeof(area));
    CirUnion(cir, n);
    for (int i = 1; i <= n; ++i)
        area[i] -= area[i + 1];
    for (int i = 1; i <= n; ++i)
        printf("[%d] = %.3lf\n", i, area[i]);
}
```

4.8 圆与多边形面积交

```
//圆的圆心固定为(0,0)，如果不是进行坐标变换，半径是R
double R;

point point::change() { //加到点结构体的函数
    return point( R * x / d, R * y / d );
}

double calang(point a, point b) { //有方向的极角差，不同于向量夹角
    double t = atan2(a.y, a.x) - atan2(b.y, b.x);
```

```
    while( dcmp(t - pi) > 0 ) t -= pi*2;
    while( dcmp(t + pi) < 0 ) t += pi*2;
    return t;
}

double solve( int n, point *p ) {
    double ans = 0, ang = 0;
    point res1, res2, o(0, 0);

    p[n] = p[0]; //点加一个变量 d 保存点到原点的距离
    for( int i = 0; i <= n; ++i )
        p[i].d = p[i].len();

    for( int i = 1; i <= n; ++i ) {
        if( dcmp(p[i-1].d - R) < 0 ) {
            if( dcmp(p[i].d - R) < 0 )
                ans += cross(p[i-1], p[i]);
            else {
                seg_cir(o, R, p[i-1], p[i], res1, res2); //线段与圆交
                ans += cross(p[i-1], res1);
                ang += calang(p[i].change(), res1);
            }
        }
        else {
            if( dcmp(p[i].d - R) < 0 ) {
                seg_cir(o, R, p[i-1], p[i], res1, res2);
                ans += cross(res1, p[i]);
                ang += calang(res1, p[i-1].change());
            }
            else {
                if( seg_cir(o, R, p[i-1], p[i], res1, res2) == 2 ) {
                    ang += calang(res1, p[i-1].change());
                    ans += cross(res1, res2);
                    ang += calang(p[i].change(), res2);
                }
                else
                    ang += calang(p[i].change(), p[i-1].change());
            }
        }
    }
    ans = ans / 2 + ang * R * R / 2;
    return fabs(ans);
}
```

4.9 点集最小圆覆盖

```
//期望复杂度是线性的
void minCirle( int n, point *p, point &c, double &r ) {
    random_shuffle( p, p + n );
    c = p[0]; r = 0;
    for( int i = 1; i < n; ++i ) if( dcmp(dis(p[i], c) - r) > 0 ) {
        c = p[i]; r = 0;
        for( int j = 0; j < i; ++j ) if( dcmp(dis(p[j], c) - r) > 0 ) {
            c.x = 0.5 * (p[i].x + p[j].x);
            c.y = 0.5 * (p[i].y + p[j].y);
            r = dis(p[j], c);
            for( int k = 0; k < j; ++k ) if( dcmp(dis(p[k], c) - r) > 0 ) {
                c = wai(p[i], p[j], p[k]); //三角形外心
                r = dis(p[i], c);
            }
        }
    }
}
```

五：凸包多边形

5.1 andrew 求凸包

```
//点按 x 坐标从小到大排序，相同按 y 排序，double 要加 dcmp,PS 先按 y 排序也可以
//凸包边上无共线点，如果要保留共线点，去掉 cross 后面的等号
int andrew( int n ) {
    sort( pnt, pnt + n );
    int m = 0;
    for( int i = 0; i < n; ++i ) {
        while( m > 1 && cross( res[m-1] - res[m-2], pnt[i] - res[m-1] ) <= 0 )
            --m;
        res[m++] = pnt[i];
    }
    int k = m;
    for( int i = n - 2; i >= 0; --i ) {
        while( m > k && cross( res[m-1] - res[m-2], pnt[i] - res[m-1] ) <= 0 )
            --m;
        res[m++] = pnt[i];
    }
    if( n > 1 ) --m;
    return m;
}
```

5.2 点在多边形内判定

```
//double 要 dcmp
bool ponseg( point p, point a, point b ) {
    return cross( a - p, b - p ) == 0 && dot( a - p, b - p ) <= 0;
}
// 0:外, 1:内, 2:边
int pointInPolygon( point cp, point* p, int n ) {
    int w = 0;
    p[n] = p[0];
    for( int i = 0; i < n; ++i ) {
        if( ponseg(cp, p[i], p[i+1]) )
            return 2;
        int k = dcmp(cross(p[i+1] - p[i], cp - p[i]));
        int d1 = dcmp(p[i].y - cp.y);
        int d2 = dcmp(p[i+1].y - cp.y);
        if( k > 0 && d1 <= 0 && d2 > 0 )    w++;
        if( k < 0 && d2 <= 0 && d1 > 0 )    w--;
    }
    return w != 0;
}
```

5.3 旋转卡壳求凸包直径

```
double maxd( point* p, int n ) {
    double ret = 0;
    int j = 0;
    for( int i = 0; i < n; ++i ) {
        while( (j + 1) % n != i && cross(p[(i+1)%n] - p[i], p[(j+1)%n] - p[i])
            >= cross(p[(i+1)%n] - p[i], p[j] - p[i]) )
            j = (j + 1) % n;
        ret = max(ret, dis(p[i], p[j]));
        ret = max(ret, dis(p[i+1], p[j]));
    }
    return ret;
}
```

5.4 旋转卡壳求凸包上最大三角形面积

```
double maxarea( point* p, int n ) {
    int j = 1, k = 2;
    LL ans = 0;
    p[n] = p[0];
    p[n+1] = p[1];
    p[n+2] = p[2];
    for( int i = 0; i < n; ++i ) {
        if( j == i ) j = (j+1)%n;
```

```
        if( k == j ) k = (k+1)%n;
        while( cross(p[j] - p[i], p[k] - p[i]) <
        cross(p[j] - p[i], p[(k+1)%n] - p[i]) )
            k = (k + 1) % n;
        ans = max(ans, cross(p[j] - p[i], p[k] - p[i]));
        while( cross(p[j] - p[i], p[k] - p[i]) <
        cross(p[(j+1)%n] - p[i], p[k] - p[i]) )
            j = (j + 1) % n;
        ans = max(ans, cross(p[j] - p[i], p[k] - p[i]));
    }
    return ans / 2;
}
```

5.5 旋转卡壳求凸包最近距离

```
double mind( point *p, int np, point *q, int nq ) {
    int sp = 0, sq = 0;
    for( int i = 1; i < np; ++i )
        if( dcmp(p[i].y - p[sp].y) < 0 || dcmp(p[i].y - p[sp].y) == 0 &&
dcmp(p[i].x - p[sp].x) < 0 )
            sp = i;
    for( int i = 1; i < nq; ++i )
        if( dcmp(q[i].y - q[sq].y) > 0 || dcmp(q[i].y - q[sq].y) == 0 &&
dcmp(q[i].x - q[sq].x) > 0 )
            sq = i;
    int tp = sp, tq = sq;
    double ans = dis(p[sp], q[sq]);
    do {
        double len = cross(p[(sp+1)%np] - p[sp], q[sq] - q[(sq+1)%nq]);
        if( dcmp(len) == 0 ) {
            ans = min(ans, ptoseg(p[sp], q[sq], q[(sq+1)%nq]));
            ans = min(ans, ptoseg(p[(sp+1)%np], q[sq], q[(sq+1)%nq]));
            ans = min(ans, ptoseg(q[sq], p[sp], p[(sp+1)%np]));
            ans = min(ans, ptoseg(q[(sq+1)%nq], p[sp], p[(sp+1)%np]));
            sp = (sp + 1) % np; sq = (sq + 1) % nq;
        }
        else if( dcmp(len) > 0 ) {
            ans = min(ans, ptoseg(q[sq], p[sp], p[(sp+1)%np]));
            sp = (sp + 1) % np;
        }
        else {
            ans = min(ans, ptoseg(p[sp], q[sq], q[(sq+1)%nq]));
            sq = (sq + 1) % nq;
        }
    } while( tp != sp || tq != sq );
```

```
    return ans;
}
```

5.6 logn 直线切割凸包

```
//点结构重载小于号运算符 return ang < b.ang, res 是凸包点集
//andrew 排序务必先按 y 轴，保障凸包点集第一个点是 y 坐标最小，逆时针序
double cal_ang( point& a, point& b ) {
    double ang = atan2(b.y - a.y, b.x - a.x);
    if( ang < 0 ) ang += 2 * pi;
    return ang;
}

double sum[mxn];
void init( point *p, int n ) {
    p[n] = p[0];
    for( int i = 0; i < n; ++i )
        p[i].ang = cal_ang(p[i], p[i+1]);
    sum[0] = cross(p[0], p[1]);
    for( int i = 1; i < n; ++i )
        sum[i] = sum[i-1] + cross(p[i], p[i+1]);
}

double get( int a, int b ) {
    if( (--b) < 0 ) return 0;
    if( (--a) < 0 ) return sum[b];
    return sum[b] - sum[a];
}

int find( int beg, int maxlen, point s, point t, point *p, int n ) {
    int sign = dcmp(cross(t - s, p[beg] - s));
    int l = 0, r = maxlen + 1;
    while( r - l > 1 ) {
        int m = (l + r) / 2;
        if( dcmp(cross(t - s, p[(beg+m)%n] - s)) * sign >= 0 )
            l = m;
        else
            r = m;
    }
    return (beg + l) % n;
}

double line_cut_con( point s, point t, point *p, int n ) {
    double ang = cal_ang(s, t), res;
    point tmp, res1, res2;
```

```
    p[n] = p[0];
    tmp.ang = ang;
    int a = upper_bound(p, p + n, tmp) - p;   a %= n;
    tmp.ang = (ang + pi > 2 * pi) ? ang - pi : ang + pi;
    int b = upper_bound(p, p + n, tmp) - p; b %= n;

    int d1 = dcmp(cross(t - s, p[a] - s));
    int d2 = dcmp(cross(t - s, p[b] - s));
    if( d1 * d2 != -1 )
        return 0;

    d1 = find(a, (b-a+n)%n, s, t, p, n);
    d2 = find(b, (a-b+n)%n, s, t, p, n);
    if( d1 > d2 )  swap(d1, d2);

    lineCross(s, t, p[d1], p[d1+1], res1);
    lineCross(s, t, p[d2], p[d2+1], res2);
    res = cross(p[d2], res2) + cross(res2, res1) + cross(res1, p[d1]);
    res += get(d1, d2);

    return fabs(res);
}

//SGU 345
int main()
{
    int n, m;
    point s, t;
    while( scanf( "%d", &n ) != EOF ) {
        for( int i = 0; i < n; ++i )
            pnt[i].input();
        n = andrew(n);
        init(res, n);
        double area = fabs(get(0, n)), tmp;
        scanf("%d", &m);
        while( m-- ) {
            s.input(); t.input();
            tmp = line_cut_con(s, t, res, n);
            tmp = min(tmp, area - tmp);
            printf("%.10lf\n", tmp * 0.5);
        }
    }
    return 0;
}
```

5.7 动态凸包

```
#define spit set<point>::iterator
//部分函数省略
struct point {
    LL x, y;
    double ang;
    bool operator < (const point &b) const {
        return ang < b.ang;
    }
    double angle(double X, double Y) {
        return atan2(y - Y, x - X);
    }
};

bool cmp( point a, point b ) {
    return a.x < b.x || a.x == b.x && a.y < b.y;
}

set<point> st;
vector<point> vec;
LL area;
double X, Y;

void init(point a, point b, point c) { //abc 不共线
    st.clear();
    X = (a.x + b.x + c.x + 0.0) / 3;
    Y = (a.y + b.y + c.y + 0.0) / 3;
    a.ang = a.angle(X, Y); st.insert(a);
    b.ang = b.angle(X, Y); st.insert(b);
    c.ang = c.angle(X, Y); st.insert(c);
    area = cross(a, b) + cross(b, c) + cross(c, a);
    if( area < 0 ) area = -area;
}

spit pre( spit it ) {
    if( it == st.begin() ) it = st.end();
    return --it;
}

spit nxt( spit it ) {
    if( ++it == st.end() ) it = st.begin();
    return it;
}
```

```
void update( point p ) {
    p.ang = p.angle(X, Y);
    spit it = pre(st.lower_bound(p));
    if( cross(*nxt(it) - *it, p - *it) > 0 ) return ;
    spit lft = it, rht = it;
    while( cross(*nxt(rht) - p, *rht - p) >= 0 ) rht = nxt(rht);
    while( cross(*lft - p, *pre(lft) - p) >= 0 ) lft = pre(lft);
    it = lft;
    while( it != rht ) area -= cross(*it, *(nxt(it))), it = nxt(it);
    it = nxt(lft);
    while( it != rht ) it = nxt(it), st.erase(pre(it));
    area += cross(*lft, p) + cross(p, *rht);
    st.insert(p);
}

//SGU 277
int main()
{
    int n;
    point a, b, c, p;
    while( scanf( "%I64d%I64d", &a.x, &a.y ) == 2 ) {
        area = 0; b.input(); c.input();
        bool hav = false;
        if( cross(b - a, c - a) != 0 ) init(a, b, c), hav = true;
        else {
            point t[3] = {a, b, c};
            sort(t, t + 3, cmp);
            a = t[0], b = t[2];
        }
        scanf( "%d", &n );
        while( n-- ) {
            p.input();
            if( hav ) update(p);
            else if( cross(b - a, p - a) == 0 ) {
                point t[3] = {a, b, p};
                sort(t, t + 3, cmp);
                a = t[0], b = t[2];
            }
            else init(a, b, p), hav = true;
            printf( "%I64d\n", area );
        }
    }
    return 0;
}
```

5.8 任意多边形最大内切圆(点+线=3 限制内切圆)

```
//省略部分函数
struct point {
    double x, y;
    point perp() {
        return point(-y, x);
    }
};

struct Line {
    point s, t; bool seg;
    Line(){}
    Line(point s, point t, bool f = true):s(s), t(t), seg(f){}
    bool intersectLine(const Line &L, point* r = NULL) const {
        point v1 = t - s, v2 = L.t - L.s;
        point dS = L.s - s;
        double D = v2.x * v1.y - v1.x * v2.y;
        if (D == 0) return false;
        double u1 = (dS.y * v2.x - dS.x * v2.y) / D;
        double u2 = (dS.y * v1.x - dS.x * v1.y) / D;
        if (r != NULL) *r = s + v1 * u1;
        return ((!seg || (0 <= u1 && u1 <= 1))
            && (!L.seg || (0 <= u2 && u2 <= 1)));
    }
};

double pointToLineDist(const point &p, const Line &L) {
    point v = L.t - L.s;
    double u = ((p.x - L.s.x) * v.x + (p.y - L.s.y) * v.y)
        / (v.x * v.x + v.y * v.y);
    if (L.seg) u = max(min(u, 1.0), 0.0);
    return (L.s + v * u - p).len();
}

struct Quadr { // Ax^2 + By^2 + Cxy + Dx + Ey + F = 0
    double A, B, C, D, E, F;
    Quadr(){}
    Quadr(double a, double b, double c, double d, double e, double f) {
        A = a; B = b; C = c; D = d; E = e; F = f;
    }
};

Line getBisector(const point &p1, const point &p2) {
    point mid = (p1 + p2) / 2;
```

```
    return Line(mid, mid + (p2 - p1).perp(), false);
}

Line getBisector(const Line &L1, const Line &L2) {
    point v1 = L1.s - L1.t, v2 = L2.t - L2.s;
    v1 = v1 / v1.len(); v2 = v2 / v2.len();
    point v = (v1 + v2) / 2;
    point p;
    if (L1.intersectLine(L2, &p)) return Line(p, p + v, false);
    else {
        double u = ((L1.s.x - L2.s.x) * v2.x + (L1.s.y - L2.s.y) * v2.y)
            / (v2.x * v2.x + v2.y * v2.y);
        p = L2.s + v2 * u; v1 = (p - L1.s) / 2;
        return Line(L1.s + v1, L1.t + v1);
    }
}

Quadr getBisector(const point &p, const Line &L) {
    point v = L.t - L.s; v = v / v.len(); v = v.perp();
    double C = -v.x * L.s.x - v.y * L.s.y; // v.x * x + v.y * y + C = 0
    return Quadr(1.0 - v.x * v.x, 1.0 - v.y * v.y, -2.0 * v.x * v.y, -2.0 *
    (p.x + v.x * C), -2.0 * (p.y + v.y * C), p.x * p.x + p.y * p.y - C * C);
}

vector<point> intersect(const Line &L, const Quadr &Q) {
    vector<point> V;
    point v = L.t - L.s; v = v / v.len();
    double A = Q.A * v.x * v.x + Q.B * v.y * v.y + Q.C * v.x * v.y;
    double B = 2.0 * (Q.A * L.s.x * v.x + Q.B * L.s.y * v.y)
        + Q.C * (L.s.x * v.y + L.s.y * v.x) + Q.D * v.x + Q.E * v.y;
    double C = Q.A * L.s.x * L.s.x + Q.B * L.s.y * L.s.y
        + Q.C * L.s.x * L.s.y + Q.D * L.s.x + Q.E * L.s.y + Q.F;
    if (A == 0) {
        if (B != 0.0) {
            double u = -C/B;
            V.push_back(L.s + v * u);
        }
        return V;
    }
    double D = B * B - 4.0 * A * C;
    if (D < 0.0) return V;
    D = sqrt(D);
    double u1 = (-B + D)/(2.0 * A);
    double u2 = (-B - D)/(2.0 * A);
```

```
    V.push_back(L.s + v * u1); V.push_back(L.s + v * u2);
    return V;
}

int N;
point P[25];
double maxR;

double fitCircle(const point &p) {
    if (!pointInPoly(p)) return 0.0;
    double R = 1000000;
    for (int i = 0; i < N; i++) {
        int j = (i+1)%N;
        R = min(R, pointToLineDist(p, Line(P[i], P[j])));
    }
    return R;
}

void check(const point &p1, const point &p2, const point &p3) {
    point r;
    if(getBisector(p1, p2).intersectLine(getBisector(p2, p3), &r))
        maxR = max(maxR, fitCircle(r));
}

void check(const point &p1, const point &p2, const Line &L) {
    vector<point> V = intersect(getBisector(p1, p2), getBisector(p1, L));
    for(int i = 0; i < V.size(); i++) maxR = max(maxR, fitCircle(V[i]));
}

void check(const point &p, const Line &L1, const Line &L2) {
    vector<point> V = intersect(getBisector(L1, L2), getBisector(p, L1));
    for (int i = 0; i < V.size(); i++) maxR = max(maxR, fitCircle(V[i]));
}

void check(const Line &L1, const Line &L2, const Line &L3) {
    point r;
    if(getBisector(L1, L2).intersectLine(getBisector(L2, L3), &r))
        maxR = max(maxR, fitCircle(r));
}

void solve() {
    cin >> N;
    for (int i = 0; i < N; i++) cin >> P[i].x >> P[i].y;
```

```
    maxR = 0.0;
    for (int i = 0; i < N; i++) for (int j = 0; j < N; j++) {
        if (i == j) continue;
        for (int k = 0; k < N; k++) {
            if (k == i || k == j) continue;
            int i2 = (i+1)%N, j2 = (j+1)%N, k2 = (k+1)%N;
            check(P[i], P[j], P[k]);
            if (k2 != i && k2 != j)
                check(P[i], P[j], Line(P[k], P[k2], 0));
            if (k2 != i && j2 != i)
                check(P[i], Line(P[j], P[j2], 0), Line(P[k], P[k2], 0));
            check(Line(P[i], P[i2], 0), Line(P[j], P[j2], 0), Line(P[k], P[k2],
0));
        }
    }
    printf( "%.2lf\n", maxR );
}
```

5.9 多边形面积并

```
//输入点集为逆时针，输入后调用 init()
struct polygon {
    point p[500];
    int sz;
    void init() {
        p[sz] = p[0];
    }
}g[505];
pair<double, int> c[100000];

double segP( point a, point b, point c ) {
    if( dcmp(b.x - c.x) )
        return (a.x - b.x) / (c.x - b.x);
    return (a.y - b.y) / (c.y - b.y);
}

double polyUnion( int n )
{
    double sum = 0;
    for( int i = 0; i < n; ++i )
    for( int ii = 0; ii < g[i].sz; ++ii ) {
        int tot = 0;
        c[tot++] = MP(0, 0);
        c[tot++] = MP(1, 0);
        for( int j = 0; j < n; ++j ) if( i != j )
```

```
        for( int jj = 0; jj < g[j].sz; ++jj ) {
            int d1 = dcmp(cross(g[i].p[ii+1] - g[i].p[ii],
                            g[j].p[jj] - g[i].p[ii]));
            int d2 = dcmp(cross(g[i].p[ii+1] - g[i].p[ii],
                            g[j].p[jj+1] - g[i].p[ii]));
            if( !d1 && !d2 ) {
                point t1 = g[j].p[jj+1] - g[j].p[jj];
                point t2 = g[i].p[ii+1] - g[i].p[ii];
                if( dcmp( dot(t1, t2) ) > 0 && j < i ) {
                c[tot++]=MP(segP(g[j].p[jj], g[i].p[ii], g[i].p[ii+1]), 1);
                c[tot++]=MP(segP(g[j].p[jj+1],g[i].p[ii],g[i].p[ii+1]), -1);
                }
            }
            else if( d1 >= 0 && d2 < 0 || d1 < 0 && d2 >= 0 ) {
                double tc = cross(g[j].p[jj+1] - g[j].p[jj],
                                g[i].p[ii] - g[j].p[jj]);
                double td = cross(g[j].p[jj+1] - g[j].p[jj],
                                g[i].p[ii+1] - g[j].p[jj]);
                if( d2 < 0 )
                    c[tot++] = MP(tc / (tc - td), 1);
                else c[tot++] = MP(tc / (tc - td), -1);
            }
        }
        sort(c, c + tot);
        double cur = min(max(c[0].first, 0.0), 1.0);
        int sgn = c[0].second;
        double s = 0;
        for( int j = 1; j < tot; ++j ) {
            double nxt = min(max(c[j].first, 0.0), 1.0);
            if( !sgn ) s += nxt - cur;
            sgn += c[j].second;
            cur = nxt;
        }
        sum += cross(g[i].p[ii], g[i].p[ii+1]) * s;
    }
    return sum / 2;
}
```

六：概率算法

6.1 最小球覆盖（模拟退火）

```
int cal( point t, int n ) {
    int id = -1;
    double r = 0;
    for( int i = 0; i < n; ++i ) {
        double d = (p[i] - t).len();
        if( d > r )
            r = d, id = i;
    }
    return id;
}

double solve( int n ) {
    double r = 0.0;
    point t = point(0, 0, 0);
    for( int i = 0; i < n; ++i )
        r = max(r, p[i].len());
    double dlt = r;
    while( dlt > eps ) {
        int id = cal(t, n);
        double d = (p[id] - t).len();
        r = min(r, d);
        t.x += (p[id].x - t.x) / d * dlt;
        t.y += (p[id].y - t.y) / d * dlt;
        t.z += (p[id].z - t.z) / d * dlt;
        dlt *= 0.98;
    }
    return r;
}
```

6.2 平面费马点

```
double solve( int n ) {
    point t = point(0, 0);
    double r = 0;
    for( int i = 0; i < n; ++i )
        r += p[i].len();
    double dlt = 10000;
    while( dlt > eps ) {
        for( int i = 0; i < 30; ++i ) {
            double ang = (rand() % 20000 * pi) / 10000 - pi;
            point k = point(t.x + dlt * cos(ang), t.y + dlt * sin(ang));
            double rk = 0;
```

```
            for( int i = 0; i < n; ++i )
                rk += (p[i] - k).len();
            if( rk < r )
                r = rk, t = k;
        }
        dlt *= 0.98;
    }
    return r;
}
```

七：平面问题

7.1 半平面交

```
//直线用向量法 p-v 表示，半平面为直线左侧平面
struct line {
    point p, v;
    double ang;
    line(){}
    line( point p, point v ) :p(p), v(v) { ang = atan2(v.y, v.x); }
    bool operator < ( const line &b ) const {
        return ang < b.ang;
    }
}L[mxn], q[mxn];
point p[mxn], poly[mxn];

point normal( point a ) {
    double l = a.len();
    return point( -a.y / l, a.x / l );
}

bool onleft( line l, point p ) {
    return dcmp( cross( l.v, p - l.p ) ) > 0;
}

point lineinter( line a, line b ) {
    point u = a.p - b.p;
    double t = cross( b.v, u ) / cross( a.v, b.v );
    return a.p + a.v * t;
}

int halfplane( int n ) {
    sort( L, L + n);
    int head = 0, tail = 0;
```

```
    q[tail] = L[0];
    for( int i = 0; i < n; ++i ) {
        while( head < tail && !onleft( L[i], p[tail-1] ) ) --tail;
        while( head < tail && !onleft( L[i], p[head] ) ) ++head;
        q[++tail] = L[i];
        if( dcmp( cross( q[tail].v, q[tail-1].v ) ) == 0 ) {
            --tail;
            if( onleft( q[tail], L[i].p ) ) q[tail] = L[i];
        }
        if( head < tail )
            p[tail-1] = lineinter( q[tail-1], q[tail] );
    }
    while( head < tail && !onleft( q[head], p[tail-1] ) ) --tail;
    if( tail - head <= 1 ) return 0;
    p[tail] = lineinter( q[tail], q[head] );
    int m = 0;
    for( int i = head; i <= tail; ++i ) poly[m++] = p[i];
    return m;
}
```

7.2 PSLG 平面直线图

```
#define Polygon vector<point>
//省略部分函数
struct point {
    double x, y;
    bool operator < (const point &b) const {
        return dcmp(x - b.x) < 0 || dcmp(x - b.x) == 0 && dcmp(y - b.y) < 0;
    }
    bool operator == (const point &b) const {
        return dcmp(x - b.x) == 0 && dcmp(y - b.y) == 0;
    }
};

point LineCross(point &P, point &Pv, point &Q, point &Qw) {
    point u = P - Q;
    point v = Pv - P, w = Qw - Q;
    double t = cross(w, u) / cross(v, w);
    return P + v * t;
}

bool SegInter(point &a1, point &a2, point &b1, point &b2) {
    double c1 = dcmp(cross(a2 - a1,b1 - a1));
    double c2 = dcmp(cross(a2 - a1,b2 - a1));
    double c3 = dcmp(cross(b2 - b1,a1 - b1));
```

```
    double c4 = dcmp(cross(b2 - b1,a2 - b1));
    return c1 * c2 < 0 && c3 * c4 < 0;
}

bool ponseg(point p, point a, point b) {
  return dcmp(cross(a - p, b - p)) == 0
      && dcmp(dot(a - p, b - p)) < 0;
}

double PolygonArea(Polygon poly) {
    double area = 0;
    int n = poly.size();
    for( int i = 1; i < n - 1; ++i )
        area += cross(poly[i] - poly[0], poly[(i+1)%n] - poly[0]);
    return area / 2;
}

struct Edge {
    int from, to;
    double ang;
    Edge(){}
    Edge(int f, int t, double a):from(f), to(t), ang(a){}
};

const int mxn = 10000 + 10; // 最大边数

struct PSLG {
    int n, m, face_cnt;
    double x[mxn], y[mxn];
    vector<Edge> edges;
    vector<int> G[mxn];
    int vis[mxn*2];  // 每条边是否已经访问过
    int left[mxn*2]; // 左面的编号
    int prev[mxn*2];
    // prev 相同起点的上一条边（即顺时针旋转碰到的下一条边）d 编号

    vector<Polygon> faces;
    double area[mxn]; // 每个 polygon 的面积

    void init(int n) {
        this->n = n;
        for( int i = 0; i < n; ++i ) G[i].clear();
        edges.clear();
        faces.clear();
```

```
}

// 有向线段 from->to 的极角
double getAngle(int from, int to) {
    return atan2(y[to] - y[from], x[to] - x[from]);
}

void AddEdge(int from, int to) {
    edges.push_back(Edge(from, to, getAngle(from, to)));
    edges.push_back(Edge(to, from, getAngle(to, from)));
    m = edges.size();
    G[from].push_back(m - 2);
    G[to].push_back(m - 1);
}

// 找出 faces 并计算面积
void Build() {
    for( int u = 0; u < n; ++u ) {
        // 给从 u 出发的各条边按极角排序
        int d = G[u].size();
        for( int i = 0; i < d; ++i )
            for( int j = i+1; j < d; ++j ) //假设从每点出发的线段不多
                if(edges[G[u][i]].ang > edges[G[u][j]].ang)
                    swap(G[u][i], G[u][j]);
        //必要时把 edges 拿出去，写索引 sort
        for(int i = 0; i < d; i++)
            prev[G[u][(i+1)%d]] = G[u][i];
    }

    memset(vis, 0, sizeof(vis));
    face_cnt = 0;
    for( int u = 0; u < n; ++u )
    for( int i = 0; i < G[u].size(); ++i ) {
        int e = G[u][i];
        if( !vis[e] ) { // 逆时针找圈
            face_cnt++;
            Polygon poly;
            for(;;) {
                vis[e] = 1; left[e] = face_cnt;
                int from = edges[e].from;
                poly.push_back(point(x[from], y[from]));
                e = prev[e^1];
                if(e == G[u][i]) break;
                //assert(vis[e] == 0);
```

```
                }
                faces.push_back(poly);
            }
        }

        for(int i = 0; i < faces.size(); i++)
            area[i] = PolygonArea(faces[i]);
    }
};

PSLG g;

const int mxp = 100 + 5;
int n, c;
point P[mxp];

point V[mxp*(mxp-1)/2+mxp];

// 在 V 数组里找到点 p
int ID(point p) {
  return lower_bound(V, V + c, p) - V;
}

// 假定 poly 没有相邻点重合，只需删除三点共线
Polygon simplify(const Polygon &poly) {
    Polygon ans;
    int n = poly.size();
    for( int i = 0; i < n; ++i ) {
        point a = poly[i];
        point b = poly[(i+1)%n];
        point c = poly[(i+2)%n];
        if(dcmp(cross(a-b, c-b)) != 0)
            ans.push_back(b);
    }
    return ans;
}

void build_graph() {
    c = n;
    for( int i = 0; i < n; ++i )
        V[i] = P[i];

    vector<double> dist[mxp]; // dist[i][j]是第 i 条线段上的第 j 个点离起点(P[i])
的距离
```

```
for( int i = 0; i < n; ++i )
for( int j = i+1; j < n; ++j )
if(SegInter(P[i], P[(i+1)%n], P[j], P[(j+1)%n])) {
    point p = LineCross(P[i], P[(i+1)%n], P[j], P[(j+1)%n]);
    V[c++] = p;
    dist[i].push_back((p - P[i]).len());
    dist[j].push_back((p - P[j]).len());
}

sort(V, V + c);
c = unique(V, V + c) - V;

g.init(c); // c 是平面图的点数
for( int i = 0; i < c; ++i ) {
    g.x[i] = V[i].x;
    g.y[i] = V[i].y;
}
for( int i = 0; i < n; ++i ) {
    point v = P[(i+1)%n] - P[i];
    double len = v.len();
    dist[i].push_back(0);
    dist[i].push_back(len);
    sort(dist[i].begin(), dist[i].end());
    int sz = dist[i].size();
    for( int j = 1; j < sz; ++j ) {
        point a = P[i] + v * (dist[i][j-1] / len);
        point b = P[i] + v * (dist[i][j] / len);
        if(a == b) continue;
        g.AddEdge(ID(a), ID(b));
    }
}

g.Build();

Polygon poly;
for( int i = 0; i < g.faces.size(); ++i ) if(g.area[i] < 0) {
    // 对于连通图，惟一一个面积小于零的面是无限面
    poly = g.faces[i];
    reverse(poly.begin(), poly.end());
    // 对于内部区域来说，无限面多边形的各个顶点是顺时针的
    poly = simplify(poly); // 无限面多边形上可能会有相邻共线点
    break;
}
```

```
    int m = poly.size();
    printf("%d\n", m);

    // 挑选坐标最小的点作为输出的起点
    int start = 0;
    for( int i = 0; i < m; ++i )
        if(poly[i] < poly[start])
            start = i;
    for( int i = start; i < m; ++i )
        printf("%.4lf %.4lf\n", poly[i].x, poly[i].y);
    for( int i = 0; i < start; ++i )
        printf("%.4lf %.4lf\n", poly[i].x, poly[i].y);
}

//LA3218 自交多边形找不自交边界
int main()
{
    while(scanf("%d", &n) == 1 && n) {
        for(int i = 0; i < n; i++) {
            int x, y;
            scanf("%d%d", &x, &y);
            P[i] = point(x, y);
        }
        build_graph();
    }
    return 0;
}
```

八：三维几何

8.1 三维叉乘

```
point cross( point a, point b ) {
    point res;
    res.x = a.y * b.z - b.y * a.z;
    res.y = a.z * b.x - b.z * a.x;
    res.z = a.x * b.y - b.x * a.y;
    return res;
}
```

8.2 三维旋转矩阵

```
//向量 p 绕向量 v 逆时针旋转 af 弧度，多重多种旋转可以用矩阵快速幂加速
point rot( point p, point v, double af ) {
    af = af * pi / 180;
    double c = cos( af ), s = sin( af );
    double l = v.len();
    double x = v.x / l, y = v.y / l, z = v.z / l;
    double a[3][3] = {
        { x * x + ( 1 - x * x ) * c, x * y * ( 1 - c ) - z * s, x * z * ( 1 - c ) + y * s },
        { y * x * ( 1 - c ) + z * s, y * y + ( 1 - y * y ) * c, y * z * ( 1 - c ) - x * s },
        { z * x * ( 1 - c ) - y * s, z * y * ( 1 - c ) + x * s, z * z + ( 1 - z * z ) * c }
    };
    point res = point (
        p.x * a[0][0] + p.y * a[0][1] + p.z * a[0][2],
        p.x * a[1][0] + p.y * a[1][1] + p.z * a[1][2],
        p.x * a[2][0] + p.y * a[2][1] + p.z * a[2][2]
    );
    return res;
}
```

8.3 三维旋转模型

1.将三维凸包一个面贴在地上

```
plane pl = vp[i]; //凸包的一个面
point z = point( 0, 0, 1 ); //z 轴
point x = cross( pl.f, z ); //旋转轴
double af = angle( z, pl.f ); //旋转角度
```

把每个点都 rot(p[], x, af)

2.多重旋转矩阵加速

旋转方式

translate tx ty tz

Everything in (x, y, z) must be moved to (x+tx, y+ty, z+tz)

scale a b c

Everything in (x, y, z) will be moved to (ax, by, cz)

rotate a b c d

Everything 绕向量 v(a, b, c) 逆时针旋转 d 弧度

定义一系列上述旋转序列，循环 k 次，求点 p 旋转后的位置

每种操作可以得到转化为一个矩阵，把旋转序列的矩阵都乘起来，再求 k 次幂得到矩阵 tmp

设点 p 最终的位置是点 o

则有[p.x, p.y, p.z, 1] * tmp[4][4] = [o.x, o.y, o.z, not_use]

下述矩阵未赋值的位置均为 0

```
void trans( point p ) { // p = point(tx, ty, tz)
    m[0][0] = m[1][1] = m[2][2] = m[3][3] = 1;
    m[3][0] = p.x, m[3][1] = p.y, m[3][2] = p.z;
```

```
}
void scal( point p ) { // p = point(a, b, c)
    m[0][0] = p.x, m[1][1] = p.y, m[2][2] = p.z, m[3][3] = 1;
}
void rot( point v, double af ) {
    m[][]左上角 3*3 赋值为 8.2 中 rot(v, af)的矩阵 a
    m[3][3] = 1;
}
```

8.4 三维凸包相关

```
const int mxn = 550;
const double eps = 1e-8;
int n;

struct face {
    int a, b, c;
    point v;
    //表示该面是否属于最终凸包上的面
    bool ok;
    void init() {
        v = cross( p[b] - p[a], p[c] - p[a] );
    }
};

struct CH3D {
    int num; //凸包表面的三角形数
    face F[8*mxn]; //凸包表面的三角形
    int g[mxn][mxn]; //凸包表面的边
    double area( point a, point b, point c ) {
        return cross( b - a, c - a ).len() / 2;
    }
    //四面体有向体积
    double volume( point a, point b, point c, point d ) {
        return dot( cross( b - a, c - a ), d - a ) / 6;
    }
    //点与面法向量方向关系 1 同向 0 点在面上 -1 反向
    int pside( point pt, face f ) {
        f.init();
        return dcmp( dot( f.v, pt - p[f.a] ) );
    }
    void deal( int pt, int a, int b ) {
        int f = g[a][b]; //搜索与该边相邻的另一个平面
        face add;
        if( F[f].ok ) {
```

```
            if( pside( p[pt], F[f] ) == 1 )
                dfs( pt, f );
            else {
                add.a = b;
                add.b = a;
                add.c = pt; //顺序要成右手系
                add.ok = true;
                g[pt][b] = g[a][pt] = g[b][a]  =num;
                F[num++] = add;
            }
        }
    }
    void dfs( int pt, int now ) { //递归搜索所有应该从凸包内删除的面
         F[now].ok = 0;
         deal( pt, F[now].b, F[now].a );
         deal( pt, F[now].c, F[now].b );
         deal( pt, F[now].a, F[now].c );
    }
    bool same( int s, int t ) {
        point &a = p[F[s].a];
        point &b = p[F[s].b];
        point &c = p[F[s].c];
        return dcmp( volume( a, b, c, p[F[t].a] ) ) == 0 &&
              dcmp( volume( a, b, c, p[F[t].b] ) ) == 0 &&
              dcmp( volume( a, b, c, p[F[t].c] ) ) == 0;
    }
    void create() { //构建三维凸包
        face add;
        num = 0;
        if( n < 4 ) return; //hehe
        bool flag = true;
        for( int i = 1; i < n; ++i ) { //前两点不共点
            if( dcmp( (p[0] - p[i]).len() ) > 0 ) {
                swap( p[1], p[i] );
                flag = false;
                break;
            }
        }
        if( flag ) return;
        flag = true;
        for( int i = 2; i < n; ++i ) { //前三点不共线
            if( dcmp( cross( p[0] - p[1], p[i] - p[1] ).len() ) != 0 ) {
                swap( p[2] ,p[i] );
                flag  =false;
```

```
                break;
            }
        }
        if( flag )return;
        flag = true;
        for( int i = 3; i < n; ++i ) { //前四点不共面
            if( dcmp( volume( p[0], p[1], p[2], p[i] ) ) != 0 ) {
                swap( p[3], p[i] );
                flag = false;
                break;
            }
        }
        if( flag ) return;
        for( int i = 0; i < 4; ++i ) {
            add.a = (i + 1) % 4;
            add.b = (i + 2) % 4;
            add.c = (i + 3) % 4;
            add.ok = true;
            if( pside( p[i], add ) == 1 ) swap( add.b, add.c );
            g[add.a][add.b] = g[add.b][add.c] = g[add.c][add.a] = num;
            F[num++] = add;
        }
        for( int i = 4; i < n; ++i ) {
            for( int j = 0; j < num; ++j) {
                if( F[j].ok && pside( p[i], F[j] ) == 1 ) {
                    dfs( i, j );
                    break;
                }
            }
        }
        int tmp = num; num = 0;
        for( int i = 0; i < tmp; ++i )
            if( F[i].ok )
                F[num++] = F[i];
    }
    double calarea() { //表面积
        double res=0;
        if( n == 3 )
            return area( p[0], p[1], p[2] );
        for( int i = 0; i < num; ++i )
            res += area( p[F[i].a], p[F[i].b], p[F[i].c] );
        return res;
    }
    double calvol() { //体积
```

```
        double res = 0;
        point o( 0, 0, 0 );
        for( int i = 0; i < num; ++i )
            res += volume( p[F[i].a], p[F[i].b], p[F[i].c], o );
        return fabs( res );
    }
    //表面多边形个数
    int polygon() {
        int res = 0;
        for( int i = 0; i < num; ++i ) {
            int flag = 1;
            for( int j = 0; j < i; ++j ) {
                if( same( i, j ) ) {
                    flag = 0;
                    break;
                }
            }
            res += flag;
        }
        return res;
    }
    //三维凸包重心
    point barycenter()
    {
        point ans( 0, 0, 0 ), o( 0, 0, 0 );
        double all = 0;
        for( int i = 0; i < num; ++i ) {
            double vol = volume( p[F[i].a], p[F[i].b], p[F[i].c], o) * 6;
            ans = ans + ( o + p[F[i].a] + p[F[i].b] + p[F[i].c]) * vol / 4;
            all += vol;
        }
        ans = ans / all;
        return ans;
    }
    double ptoface( point pt, int i ) {
        face tmp;
        tmp.a = F[i].a; tmp.b = F[i].b; tmp.c = F[i].c;
        tmp.init();
        return fabs( dot( pt - p[tmp.a], tmp.v ) ) / tmp.v.len();
    }
};

CH3D hull; //内有大数组，不宜定义在函数内
```

```
int main()
{
    while( scanf( "%d", &n ) == 1 ) {
        for( int i = 0; i < n; ++i )
            p[i].input();
        hull.create();
        point pt = hull.barycenter();
        double opt = 1e20;
        for( int i = 0; i < hull.num; ++i )
            opt = min( opt, hull.ptoface( pt, i ) );
        printf( "%.3lf\n", opt );
    }
    return 0;
}
```

8.5 三维光线反射

1.平面反射

射线起点 s，方向 v，平面 p0-n

```
void reflect(point s, point v, point p0, point n, point &rs, point &rv) {
    rs = LinePlaneInter(s, s + v, p0, n);
    point tmp = p_plane_q(s, p0, n);
    rv = rs - tmp;
}
```

2.球面反射

射线起点 s，方向 v，球心 p，半径 r

```
bool reflect(point s, point v, point p, double r, point &rs, point &rv) {
    double a = dot(v, v);
    double b = dot(s - p, v) * 2;
    double c = dot(s - p, s - p) - r * r;
    double dlt = b * b - 4 * a * c;
    if( dlt < 0 ) return false;
    double t = (-b - sqrt(dlt)) / a / 2;
    rs = s + v * t;
    point tn = p - rs;
    rv = v - tn * (dot(v, tn) * 2 / dot(tn, tn));
    return true;
}
```

8.6 点到直线距离

```
double ptoline( point p, point a, point b ) {
    return (cross( p - a, b - a )).len() / dis( a, b );
}
```

8.7 点到线段距离

```
double ptoseg( point p, point a, point b ) {
    if( dcmp(dot( p - a, b - a )) < 0 ) return dis( p, a );
    if( dcmp(dot( p - b, a - b )) < 0 ) return dis( p, b );
    return (cross( p - a, b - a )).len() / dis( a, b );
}
```

8.8 两直线距离

```
//n.len()为0说明直线平行
double LineDis( point a, point b, point c, point d ) {
    point n = cross(a - b, c - d);
    if( dcmp(n.len()) == 0 ) return ptoline(a, c, d);
    return fabs(dot(a - c, n)) / n.len();
}
```

8.9 两线段距离

```
double SegDis( point a, point b, point c, point d ) {
    point n = cross(a - b, c - d);
    if( dcmp(n.len()) != 0 ) {
        point cc = ptoplane(c, a, n);
        point dd = ptoplane(d, a, n);
        point res;
        if( SegCross(a, b, cc, dd, res) == 1 )
            return LineDis(a, b, c, d);
    }
    double ret = ptoseg(a, c, d);
    ret = min(ret, ptoseg(b, c, d));
    ret = min(ret, ptoseg(c, a, b));
    ret = min(ret, ptoseg(d, a, b));
    return ret;
}
```

8.10 直线相交判定

```
    类型              返回        res
--------------------------------------
1.  不相交（平行）       0         不变
2.  规范相交            1         交点
3.  非规范相交（重合）    2         不变
4.  异面不相交          3         不变
int LineCross( point a, point b, point c, point d, point &res ) {
    point n = cross(a - b, c - d);
    if( dcmp(n.len()) == 0 ) {
        if( dcmp(cross(a - b, c - b).len()) == 0 ) return 2;
        return 0;
```

```
    }
    if( dcmp(ptoline(a, c, d)) == 0 ) {res = a; return 1;}
    if( dcmp(ptoline(b, c, d)) == 0 ) {res = b; return 1;}
    if( dcmp(ptoline(c, a, b)) == 0 ) {res = c; return 1;}
    if( dcmp(ptoline(d, a, b)) == 0 ) {res = d; return 1;}
    if( dcmp(dot( cross( b - a, c - a ), d - a )) != 0 ) return 3;
    n = d + n;
    point f = cross(d - c, n - c);
    double t = dot(f, c - a) / dot(f, b - a);
    res = a + (b - a) * t;
    return 1;
}
```

8.11 线段相交判定

```
    类型            返回        res
--------------------------------------
1.  不相交              0       不变
2.  规范相交            1       交点
3.  非规范相交          2       不变
int SegCross(point a, point b, point c, point d, point &res) {
    int k = LineCross(a, b, c, d, res);
    if( k == 0 || k == 3 ) return 0;
    if( k == 1 ) {
        double d1 = dot(a - res, b - res);
        double d2 = dot(c - res, d - res);
        if( d1 < 0 && d2 < 0 ) return 1;
        if( d1 == 0 && d2 <= 0 || d2 == 0 && d1 <= 0 ) return 2;
        return 0;
    }
    if( dot(a - c, b - c) <= 0 || dot(a - d, b - d) <= 0
     || dot(c - a, d - a) <= 0 || dot(c - b, d - b) <= 0)
        return 2;
    return 0;
}
```

8.12 点关于直线的对称点

```
point p_line_q(point p, point a, point b) {
    point k = cross(b - a, p - a);
    if( dcmp(k.len()) == 0 ) return p;
    k = cross(k, b - a);
    return p_plane_q(p, a, k);
}
```

8.13 点到平面距离

```
//点 p 到平面 p0-n 的距离，不加 fabs 是有向距离
double distoplane(point p, point p0, point n) {
    return fabs(dot(p - p0, n)) / n.len();
}
```

8.14 点在平面投影

```
//点 p 在平面 p0-n 上的投影
point ptoplane(point p, point p0, point n) {
    double d = dot(p - p0, n) / n.len();
    return p - n * d;
}
```

8.15 点关于平面的对称点

```
//点 p 关于平面 p0-n 的对称点
point p_plane_q(point p, point p0, point n) {
    double d = 2 * dot(p - p0, n) / n.len();
    return p - n * d;
}
```

8.16 直线与平面交点

```
//直线 p1-p2 到平面 p0-n 的交点
//分母(dot(n, p2 - p1))为 0 说明直线与平面平行或直线在平面上
point LinePlaneInter(point p1, point p2, point p0, point n) {
    point v = p2 - p1;
    double t = dot(n, p0 - p1) / dot(n, p2 - p1);
    return p1 + v * t;
}
```

8.17 线段与平面交点

```
//线段 p1-p2 到平面 p0-n 的交点，返回 0 说明无交点
//分母(dot(n, p2 - p1))为 0 说明线段与平面平行或直线在平面上
int SegPlaneInter(point p1, point p2, point p0, point n, point &res) {
    point v = p2 - p1;
    double t = dot(n, p0 - p1) / dot(n, p2 - p1);
    if( dcmp(t) < 0 || dcmp(t - 1) > 0 ) return 0;
    res = p1 + v * t;
    return 1;
}
```

8.18 直线与平面位置关系判定

```
//直线 p1-p2 与平面 p0-n 的位置关系
//0:相交  1:平行  2:垂直
int LineAndPlane(point p1, point p2, point p0, point n) {
```

```
    point v = p2 - p1;
    if( dcmp(dot(v, n)) == 0 ) return 1;
    if( dcmp(cross(v, n).len()) == 0 ) return 2;
    return 0;
}
```

8.19 两平面位置关系判定

```
//平面 p0-n0 和 p1-n1 的位置关系
//0:有唯一交线   1:两平面垂直   2:两平面重合    3:两平面平行不重合
int PlaneAndPlane(point p0, point n0, point p1, point n1) {
    if( dcmp(dot(n0, n1)) == 0 ) return 1;
    if( dcmp(cross(n0, n1).len()) == 0 ) {
        if( dcmp(dot(n0, p1 - p0)) == 0 ) return 2;
        return 3;
    }
    return 0;
}
```

8.20 平面交线

```
//平面 p0-n0 和 p1-n1 的交线，仅知道这 4 个量的时候，返回直线是向量式
bool PlaneCross(point p0, point n0, point p1, point n1, point &s, point &v) {
    v = cross(n0, n1);
    if( dcmp(v.len()) == 0 ) return false;
    point tmp = p0 + rot(n0, v, 90);
    s = LinePlaneInter(p0, tmp, p1, n1);
    return true;
}
```

8.21 平面距离

```
//平面 p0-n0 和 p1-n1 的距离
double PlaneDis(point p0, point n0, point p1, point n1) {
    if( PlaneAndPlane(p0, n0, p1, n1) != 3 ) return 0;
    return fabs(dot(p1 - p0, n0)) / n0.len();
}
```

8.22 点在空间三角形内判定

```
//判断点 p 是否在△abc 中，包括边界
bool PointInTri(point p, point a, point b, point c) {
    double area0 = cross(b - a, c - a).len();
    double area1 = cross(a - p, b - p).len();
    double area2 = cross(b - p, c - p).len();
    double area3 = cross(c - p, a - p).len();
    return dcmp(area1 + area2 + area3 - area0) == 0;
}
```

8.23 线段和空间三角形的位置关系

```
//线段 p1-p2 是否与三角形 abc 相交
bool SegTriInter(point p1, point p2, point a, point b, point c, point &res) {
    point n = cross(b - a, c - a);
    if( dcmp(dot(n, p2 - p1)) == 0 ) return false;
    //线段与三角形所在平面平行或重合，如果这种情况也算相交再求线段交点即可
    double t = dot(n, a - p1) / dot(n, p2 - p1);
    if( dcmp(t) < 0 || dcmp(t - 1) > 0 ) return false;
    res = p1 + (p2 - p1) * t;
    return PointInTri(res, a, b, c);
}
```

8.24 经纬度坐标转笛卡尔坐标

```
//lat 纬度 -90 ~ 90   lng 经度 -180 ~ 180   R 球体半径
void get(double lat, double lng, double &x, double &y, double &z) {
    lat = lat * pi / 180;
    lng = lng * pi / 180;
    x = R * cos(lat) * cos(lng);
    y = R * cos(lat) * sin(lng);
    z = R * sin(lat);
}
```

8.25 球面距离

```
//ab 是笛卡尔坐标
double cal(point a, point b, double R) {
    double d = (a - b).len();
    return 2 * R * asin(d/(2*R));
}
```

九：数据结构优化算法

9.1 K-D 树

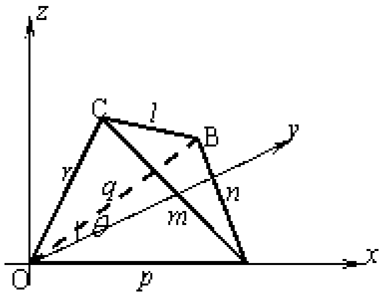
```
int K = 2; //维数
struct kdNode {
    LL x[5];
    int div, id;
};//优先队列里保存的 pair 带有点 id，有了 id 干什么都随便了

int cmpNo;
int cmp( kdNode a, kdNode b ) {
    return a.x[cmpNo] < b.x[cmpNo];
}
```

```
LL dis2( kdNode& a, kdNode& b ) {
    LL res = 0;
    for( int i = 0; i < K; ++i )
        res += (a.x[i] - b.x[i]) * (a.x[i] - b.x[i]);
    return res;
}

void buildKD( int l, int r, kdNode* p, int d ) {
    if( l > r ) return;
    int m = (l + r) >> 1;
    cmpNo = d;
    nth_element( p + l, p + m, p + r + 1, cmp);
    p[m].div = d;
    buildKD( l, m - 1, p, (d + 1) % K );
    buildKD( m + 1, r, p, (d + 1) % K );
}
//n 个点 编号 0 ~ n-1，建树调用 buildKD(0,n-1,kp,0); kp 是 kdNode 点集

priority_queue<pair<LL,int> > Q; //(距离平方，点的 id)
void KNN( int l, int r, kdNode tar, kdNode* p, int k ) {
    if( l > r ) return;
    int m = (l + r) >> 1;
    pair<LL,int> v = MP(dis2(p[m], tar), p[m].id);
    if( Q.size() == k && v < Q.top() ) Q.pop();
    if( Q.size() < k ) Q.push(v);

    LL t = tar.x[ p[m].div ] - p[m].x[ p[m].div ];
    if( t <= 0 ) {
        KNN(l, m - 1, tar, p, k);
        if( Q.top().first > t * t )
            KNN( m + 1, r, tar, p, k);
    }
    else if( t > 0 ) {
        KNN( m + 1, r, tar, p, k);
        if( Q.top().first > t * t )
            KNN( l, m - 1, tar, p, k);
    }
}
//调用 KNN(0,n-1,tar,kp,k) 查找 tar 点的 k 个临近点，存在 Q 中
```

十：其他

10.1 欧拉四面体公式

建立 x，y，z 直角坐标系。设 A、B、C 坐标分别为(a1,b,1,c1),(a2,b2,c2),(a3,b3,c3), 四面体 O-ABC 的六条棱长分别为 l,m,n,p,q,r （有向体积，注意取 fabs）



$$V^2 = \frac{1}{36}\begin{vmatrix} p^2 & \frac{p^2+q^2-n^2}{2} & \frac{p^2+r^2-m^2}{2} \\ \frac{p^2+q^2-n^2}{2} & q^2 & \frac{q^2+r^2-l^2}{2} \\ \frac{p^2+r^2-m^2}{2} & \frac{q^2+r^2-l^2}{2} & r^2 \end{vmatrix}$$

10.2 simpson 数值积分

```
double f(double x) {
    return y = //函数值
}
double simpson(double x, double y) {
    return (f(x) + f(y) + f((x+y) / 2) * 4.0) / 6.0 * (y - x);
}
double rsimpson(double x, double y) { //积分区域是[x,y]
    double m = (x + y) / 2;
    double res = simpson(x, y), l = simpson(x, m), r = simpson(m, y);
    if( fabs(res-l-r) < eps*15 )
        return res;
    return rsimpson(x, m) + rsimpson(m, y);
}
```

10.3 常用积分公式

1.弧长公式

有弧 y = f(x)，则弧长 L = ∫sqrt(1 + y’^2)

2.第一类曲线积分

设函数 f(x,y)在曲线 L 上有定义，L 的参数方程为

x = x(t), y = y(t)

则∫f(x,y)ds = ∫f[x(t),y(t)] * sqrt(x’(t)^2 + y’(t)^2) dt

3.不定积分

1) ∫kdx = kx+c

2) ∫x^udx = (x^(u+1))/(u+1)+c

3) ∫1/xdx = ln|x|+c

4) ∫a^xdx = (a^x)/lna+c

5) ∫e^xdx = e^x+c

6) ∫sinxdx = -cosx+c

7) ∫cosxdx = sinx+c

8) ∫1/(cosx)^2dx = tanx+c

9) ∫1/(sinx)^2dx = -cotx+c

10) ∫1/√(a^2-x^2)dx = arcsin(x/a)+c

11) ∫1/(a^2+x^2)dx = 1/a*arctan(x/a)+c

12) ∫1/(a^2-x^2)dx = (1/(2a))ln|(a+x)/(a-x)|+c

13) ∫secxdx = ln|secx+tanx|+c

14) ∫sec^2 x dx = tanx+c;

15) ∫shx dx = chx+c;

16) ∫chx dx = shx+c;

17) ∫thx dx = ln(chx)+c;

18) ∫1/(1+x^2) dx = arctanx+c

19) ∫1/√(1-x^2) dx = arcsinx+c

20) ∫tanx dx = -In|cosx|+c

21) ∫cotx dx = In|sinx|+c

22) ∫secx dx = In|secx+tanx|+c

23) ∫cscx dx = In|cscx-cotx|+c

24) ∫1/√(x^2+a^2) dx = In(x+√(x^2+a^2))+c

25) ∫1/√(x^2-a^2) dx = |In(x+√(x^2-a^2))|+c

10.4 三角函数

稀有函数

正割(sec)等于斜边比邻边; secA = c/b = 1 / cosA

余割(csc)等于斜边比对边。cscA = c/a = 1 / sinA

sinh / 双曲正弦: sh(x) = [e^x - e^(-x)] / 2

cosh / 双曲余弦: ch(x) = [e^x + e^(-x)] / 2

tanh / 双曲正切: th(x) = sh(x) / ch(x)=[e^x - e^(-x)] / [e^x + e^(-x)]

coth / 双曲余切: coth(x) = ch(x) / sh(x) = [e^x + e^(-x)] / [e^(x) - e^(-x)]

sech / 双曲正割: sech(x) = 1 / ch(x) = 2 / [e^x + e^(-x)]

csch / 双曲余割: csch(x) = 1 / sh(x) = 2 / [e^x - e^(-x)]

两角和与差的三角函数:

$\cos(\alpha + \beta) = \cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta$

$\cos(\alpha - \beta) = \cos\alpha\cdot\cos\beta + \sin\alpha\cdot\sin\beta$

$\sin(\alpha + \beta) = \sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta$

$\sin(\alpha - \beta) = \sin\alpha\cdot\cos\beta - \cos\alpha\cdot\sin\beta$

$\tan(\alpha + \beta) = (\tan\alpha + \tan\beta) / (1 - \tan\alpha\cdot\tan\beta)$

$\tan(\alpha - \beta) = (\tan\alpha - \tan\beta) / (1 + \tan\alpha\cdot\tan\beta)$

二倍角公式:

$\sin(2\alpha) = 2\sin\alpha\cdot\cos\alpha$

$\cos(2\alpha) = \cos^2(\alpha)-\sin^2(\alpha) = 2\cos^2(\alpha)-1 = 1-2\sin^2(\alpha)$

$\tan(2\alpha) = 2\tan\alpha/[1-\tan^2(\alpha)]$

倍角公式:

$\sin3\alpha = 3\sin\alpha - 4\sin^3(\alpha)$

$\cos3\alpha = 4\cos^3(\alpha) - 3\cos\alpha$

半角公式:

$\sin^2(\alpha/2) = (1-\cos\alpha)/2$

$\cos^2(\alpha/2) = (1+\cos\alpha)/2$

$\tan^2(\alpha/2) = (1-\cos\alpha)/(1+\cos\alpha)$

$\tan(\alpha/2) = \sin\alpha/(1+\cos\alpha) = (1-\cos\alpha)/\sin\alpha$

万能公式:

半角的正弦、余弦和正切公式（降幂扩角公式）

$\sin\alpha = 2\tan(\alpha/2) / [1+\tan^2(\alpha/2)]$

$\cos\alpha = [1-\tan^2(\alpha/2)] / [1+\tan^2(\alpha/2)]$

$\tan\alpha = 2\tan(\alpha/2) / [1-\tan^2(\alpha/2)]$

积化和差公式:

$\sin\alpha\cdot\cos\beta = (1/2)[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

$\cos\alpha\cdot\sin\beta = (1/2)[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$

$\cos\alpha\cdot\cos\beta = (1/2)[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$

$\sin\alpha\cdot\sin\beta = -(1/2)[\cos(\alpha+\beta) - \cos(\alpha-\beta)]$

和差化积公式:

$\sin\alpha + \sin\beta = 2\sin[(\alpha+\beta)/2]\cos[(\alpha-\beta)/2]$

$\sin\alpha - \sin\beta = 2\cos[(\alpha+\beta)/2]\sin[(\alpha-\beta)/2]$

$\cos\alpha + \cos\beta = 2\cos[(\alpha+\beta)/2]\cos[(\alpha-\beta)/2]$

$\cos\alpha - \cos\beta = -2\sin[(\alpha+\beta)/2]\sin[(\alpha-\beta)/2]$