Invitation to TDA – Theoretical Exercises

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Problem 1 Show that a unit square and unit circle are homeomorphic.

Problem 2 Show that an interval $[0, 1]$ is *not* homeomorphic to $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$.

Problem 3 For a given matrix, check if it represent a distance matrix of some discrete metric space. Search for algorithmic criteria that makes a given matrix a distance matrix of some metric space.

Problem 4 Show that a map between metric spaces $f\colon (X, d) \to (X', d')$ is continuous in the sense of epsilon-delta if and only if it is continuous in the sense that preimages of open sets are open.

Problem 5 Prove that a norm $\|\cdot\|$ on a real vector spaces induces a metric via $d(x, y) = \|x - y\|$.

Problem 6 Let X, Y be i.i.d. random variables sampled from the uniform distribution on $[0, 1]$. Show that $\mathbb{E}(|X - Y|) = 1/3$. (In the lecture, it was incorrectly stated that it would be $1/2$).

Problem 7 Search the literature for the proof that Peano curve indeed visits each point in a square.

Problem 8 Which of the axioms of metric are not satisfied by cosine similarity?

Problem 9 Show a deformation retraction from $[0, 1] \times [0, 1]$ to $\{0\} \times [0, 1]$.

Problem 10 Show that there is no deformation retraction from an interval $[0, 1]$ to the space $\{0, 1\}$.

Problem 11 Show that a convex set is contractible.

Problem 12 Show that star shape set is contractible.

Problem 13 (Wrapping effect) Consider a two dimensional cube (product of two intervals) $C = [-1, 1] \times [0, 1]$ and a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates

C by an angle α. The rotation is centered in $(0,0)$. Write down a formula for $f^n(C)$ using interval arithmetic.

Problem 14 (Dependency problem) Consider a function $f(x) = x^2 + x$ and give an exact bound on $f([-1,1])$. Compute $f([-1,1])$ using sequence of interval operations. Now, re-arrange the arithmetic representation of f so that x is present only once there and do the computations again. The phenomena you can observe is so called *dependency effect* that may give a considerable overestimation if the same variables are present in the formula multiple times (they are treated by the interval arithmetic as independent variables that may take all possible combinations of values from a given interval, while in reality they are the same variable).

Problem 15 Consider a map $f : [0,1]^2 \to [0,1]^2$. Which theorem guarantee existence of the fixed point of f?