Invitation to TDA – Theoretical Exercises

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Problem 1 Show that a unit square and unit circle are homeomorphic.

Problem 2 Show that an interval $[0, 1]$ is *not* homeomorphic to $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$.

Problem 3 For a given matrix, check if it represent a distance matrix of some discrete metric space. Search for algorithmic criteria that makes a given matrix a distance matrix of some metric space.

Problem 4 Show that a map between metric spaces $f\colon (X, d) \to (X', d')$ is continuous in the sense of epsilon-delta if and only if it is continuous in the sense that preimages of open sets are open.

Problem 5 Prove that a norm $\|\cdot\|$ on a real vector spaces induces a metric via $d(x, y) = \|x - y\|$.

Problem 6 Let X, Y be i.i.d. random variables sampled from the uniform distribution on $[0, 1]$. Show that $\mathbb{E}(|X - Y|) = 1/3$. (In the lecture, it was incorrectly stated that it would be $1/2$).

Problem 7 Search the literature for the proof that Peano curve indeed visits each point in a square.

Problem 8 Which of the axioms of metric are not satisfied by cosine similarity?

Problem 9 Show a deformation retraction from $[0, 1] \times [0, 1]$ to $\{0\} \times [0, 1]$.

Problem 10 Show that there is no deformation retraction from an interval $[0, 1]$ to the space $\{0, 1\}$.

Problem 11 Show that a convex set is contractible.

Problem 12 Show that star shape set is contractible.

Problem 13 (Wrapping effect) Consider a two dimensional cube (product of two intervals) $C = [-1, 1] \times [0, 1]$ and a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates

C by an angle α. The rotation is centered in $(0,0)$. Write down a formula for $f^n(C)$ using interval arithmetic.

Problem 14 (Dependency problem) Consider a function $f(x) = x^2 + x$ and give an exact bound on $f([-1,1])$. Compute $f([-1,1])$ using sequence of interval operations. Now, re-arrange the arithmetic representation of f so that x is present only once there and do the computations again. The phenomena you can observe is so called *dependency effect* that may give a considerable overestimation if the same variables are present in the formula multiple times (they are treated by the interval arithmetic as independent variables that may take all possible combinations of values from a given interval, while in reality they are the same variable).

Problem 15 Consider a map $f : [0,1]^2 \to [0,1]^2$. Which theorem guarantee existence of the fixed point of f?

Problem 16 Show that simplices are convex.

Problem 17 Find triangulations of the following spaces:

1. 2-sphere (at least three different triangulations),
2. Möbius band,
3. real projective plane,
4. torus,
5. Klein bottle.
6. * n-fold torus.

Problem 18 For each pair $i \leq k$ of non-negative integers, how many faces of codimension i does the solid k-simplex $\Delta(k)$ have?

Problem 19 Let $P \subset \mathbb{R}^d$ be a finite point cloud. Show that for each $\varepsilon > 0$, we have inclusions of subcomplexes

$$\mathcal{C}_\varepsilon(P) \subseteq \mathcal{R}_{2\varepsilon}(P) \subseteq \mathcal{C}_{2\varepsilon}(P),$$

where $\mathcal{C}$ and $\mathcal{R}$ denote the Čech and Vietoris-Rips complexes, respectively.

Problem 20

a) Show that the intersection of two subcomplexes is again a subcomplex.

b) Show that the complement of a subcomplex need not be a subcomplex.

Problem 21 Consider an abstract simplicial complex K with n vertices. Realize it as a geometric simplicial complex in $\mathbb{R}^n$.

Problem 22 * Look up a proof of the nerve lemma, e.g. in Hatcher 4.G.

Problem 23 Show that every Voronoi cell in a Voronoi diagram is convex.

Problem 24 Show formally that the Alpha complex computed for point cloud P and radius parameter ϵ is homotopically equivalent to to a Čech complex of P with the radius ϵ.

Problem 25 Show an example of a point cloud in $\mathbb{R}^2$ whose Vietoris-Rips and Čech complex has dimension greater than 2. What is the maximal dimension of a complex that can be generated given n points in $\mathbb{R}^2$?

Problem 26 Show a case of non-generic point clouds whose Delaunay triangulation / alpha complex has dimension higher than the dimension of the ambient space.

Problem 27

a) Find the smallest open cover of the circle with contractible intersections. What is the nerve of this good cover?

b) Find a cover of the circle containing at least two open sets which violates the hypotheses of the nerve lemma. What is the nerve of this bad cover?

Problem 28 For a finite abstract simplicial complex K, let K_i denote its faces of dimension i (for $i \geq 0$). The *Euler characteristic* of K is

$$\chi(K) = \sum_{i=0}^{\dim K} (-1)^i |K_i|,$$

where $|.|$ denotes the cardinality of a set. Show that

$$\chi(K) = \sum_{i=0}^{\dim K} (-1)^i \beta_i,$$

where $\beta_i = \dim(H_i(K))$ is the ith Betti number.

Problem 29 Compute the $\mathbb{Z}/2$ homology and Euler characteristic of the hollow and the solid 2-simplex.

Problem 30 Perform the matrix reduction algorithm to compute the $\mathbb{Z}/2$ persistent homology of the filtered simplicial complex displayed in Figure 1. Draw the persistence diagram.

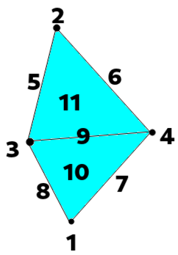

Figure 1: Filtered Complex for Problem 30