Relations and their Properties

1. List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$, where $(a, b) \in R$ if and only if **a)** $a = b$.
b) $a + b = 4$.

c) $a > b$. **d)** $a \mid b$.

e) $\gcd(a, b) = 1$. **f)** $\mathrm{lcm}(a, b) = 2$.

2. a) List all the ordered pairs in the relation $R = \{(a, b) \mid a \text{ divides } b\}$ on the set $\{1, 2, 3, 4, 5, 6\}$.

b) Display this relation graphically, as was done in Example 4.

c) Display this relation in tabular form, as was done in Example 4.

3. For each of these relations on the set $\{1, 2, 3, 4\}$, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive. **a)** $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$

b) $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$

c) $\{(2, 4), (4, 2)\}$

d) $\{(1, 2), (2, 3), (3, 4)\}$

e) $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$ **f)** $\{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)\}$

4. Determine whether the relation R on the set of all people is reflexive, symmetric, antisymmetric, and/or transitive, where $(a, b) \in R$ if and only if **a)** a is taller than b.

b) a and b were born on the same day.

c) a has the same first name as b.

d) a and b have a common grandparent.

5. Determine whether the relation R on the set of all Web pages is reflexive, symmetric, antisymmetric, and/or transitive, where $(a, b) \in R$ if and only if **a)** everyone who has visited Web page a has also visited

Web page b.

b) there are no common links found on both Web page a and Web page b.

c) there is at least one common link on Web page a and Web page b.

d) there is a Web page that includes links to both Web page a and Web page b.

6. Determine whether the relation R on the set of all real numbers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if **a)** $x + y = 0$.

b) $x = \pm y$.

c) $x - y$ is a rational number.

d) $x = 2y$.

e) $xy \geq 0$. **f)** $xy = 0$. **g)** $x = 1$.

h) $x = 1$ or $y = 1$.

7. Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if **a)** $x \neq y$.

b) $xy \geq 1$.

c) $x = y + 1$ or $x = y - 1$.

d) $x \equiv y \pmod 7$.

e) x is a multiple of y.

f) x and y are both negative or both nonnegative. **g)** $x = y2$.

h) $x \geq y2$

10. Give an example of a relation on a set that is

a) both symmetric and antisymmetric.

b) neither symmetric nor antisymmetric.

A relation R on the set A is **irreflexive** if for every\ $a \in A$, $(a, a) \notin R$. That is, R is irreflexive if no element in A is related to itself.

11. Which relations in Exercise 3 are irreflexive?

12. Which relations in Exercise 4 are irreflexive?

13. Which relations in Exercise 5 are irreflexive?

14. Which relations in Exercise 6 are irreflexive?

15. Can a relation on a set be neither reflexive nor irreflexive?

16. Use quantifiers to express what it means for a relation to be irreflexive

31. Let A be the set of students at your school and B the set of books in the school library. Let R_1 and R_2 be the relations consisting of all ordered pairs (a, b), where student a is required to read book b in a course, and where student a has read book b, respectively. Describe the ordered pairs in each of these relations. **a)** $R_1 \cup R_2$ **b)** $R_1 \cap R_2$

c) $R_1 \oplus R_2$ **d)** $R_1 - R_2$

e) $R_2 - R_1$

3 2. Let R be the relation $\{(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)\}$, and let S be the relation $\{(2, 1), (3, 1), (3, 2), (4, 2)\}$. Find $S \circ R$

44. List the 16 different relations on the set $\{0, 1\}$.

45. How many of the 16 different relations on $\{0, 1\}$ contain the pair $(0, 1)$?

58. Let R be the relation on the set $\{1, 2, 3, 4, 5\}$ containing the ordered pairs (1, 1), (1, 2), (1, 3), (2, 3), (2, 4), (3, 1), (3, 4), (3, 5), (4, 2), (4, 5), (5, 1), (5, 2), and (5, 4). Find **a)** R^2.

b) R^3.

c) R^4.

d) R^5.

Answers to Odd No Exercise

1. a) $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$ b) $\{(1, 3), (2, 2), (3, 1), (4, 0)\}$ c) $\{(1, 0), (2, 0), (2, 1), (3, 0), (3, 1), (3, 2), (4, 0), (4, 1), (4, 2), (4, 3)\}$ d) $\{(1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (3, 0), (3, 3), (4, 0)\}$ e) $\{(0, 1), (1, 0), (1, 1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2), (4, 1), (4, 3)\}$ f) $\{(1, 2), (2, 1), (2, 2)\}$ **3.** a) Transitive b) Reflexive, symmetric, transitive c) Symmetric d) Antisymmetric e) Reflexive, symmetric, antisymmetric, transitive f) None of these properties **5.** a) Reflexive, transitive b) Symmetric c) Symmetric d) Symmetric **7.** a) Symmetric b) Symmetric, transitive c) Symmetric d) Reflexive, symmetric, transitive e) Reflexive, transitive f) Reflexive, symmetric, transitive g) Antisymmetric h) Antisymmetric, transitive **9.** Each of the three properties is vacuously satisfied. **11.** (c), (d), (f) **13.** a) Not irreflexive b) Not irreflexive c) Not irreflexive d) Not irreflexive **15.** Yes, for instance $\{(1, 1)\}$ on $\{1, 2\}$ **17.** $(a, b) \in R$ if and only if a is taller than b **19.** (a) **21.** None **23.** $\forall a \forall b\,[(a, b) \in R \to (b, a) \notin R]$ **25.** 2^{mn} **27.** a) $\{(a, b) \mid b$ divides $a\}$ b) $\{(a, b) \mid a$ does not divide $b\}$ **29.** The graph of f^{-1} **31.** a) $\{(a, b) \mid a$ is required to read or has read $b\}$ b) $\{(a, b) \mid a$ is required to read and has read $b\}$ c) $\{(a, b) \mid$ either a is required to read b but has not read it or a has read b but is not required to$\}$ d) $\{(a, b) \mid a$ is required to read b but has not read it$\}$ e) $\{(a, b) \mid a$ has read b but is not required to$\}$ **33.** $S \circ R = \{(a, b) \mid a$ is a parent of b and b has a sibling$\}$, $R \circ S = \{(a, b) \mid a$ is an aunt or uncle of $b\}$ **35.** a) R_2 b) R_6 c) R_3 d) R_3 e) $\varnothing$ f) R_1 g) R_4 h) R_4 **37.** a) R_1 b) R_2 c) R_3 d) R_2 e) R_3 f) R_2 g) R_2 h) R_2 **39.** $S_1^2 = \{(a, b) \in \mathbb{Z}^2 \mid a > b + 1\}$, $S_2^2 = S_2$, $S_3^2 = \{(a, b) \in \mathbb{Z}^2 \mid a < b - 1\}$, $S_4^2 = S_4$, $S_5^2 = S_5$, $S_6^2 = \mathbb{Z}^2$ **41.** b got his or her doctorate under someone who got his or her doctorate under a; there is a sequence of $n + 1$ people, starting with a and ending with b, such that each is the advisor of the next person in the sequence **43.** a) $\{(a, b) \mid a - b \equiv 0, 3, 4, 6, 8, \text{ or } 9 \pmod{12}\}$

b) $\{(a, b) \mid a \equiv b \pmod{12}\}$ c) $\{(a, b) \mid a - b \equiv 3, 6,$ or $9 \pmod{12}\}$ d) $\{(a, b) \mid a - b \equiv 4$ or $8 \pmod{12}\}$ e) $\{(a, b) \mid a - b \equiv 3, 4, 6, 8,$ or $9 \pmod{12}\}$ **45.** 8 **47.** a) 65,536 b) 32,768 **49.** a) $2^{n(n+1)/2}$ b) $2^n 3^{n(n-1)/2}$ c) $3^{n(n-1)/2}$ d) $2^{n(n-1)}$ e) $2^{n(n-1)/2}$ f) $2^{n^2} - 2 \cdot 2^{n(n-1)}$ **51.** There may be no such b. **53.** If R is symmetric and $(a, b) \in R$, then $(b, a) \in R$, so $(a, b) \in R^{-1}$. Hence, $R \subseteq R^{-1}$. Similarly, $R^{-1} \subseteq R$. So $R = R^{-1}$. Conversely, if $R = R^{-1}$ and $(a, b) \in R$, then $(a, b) \in R^{-1}$, so $(b, a) \in R$. Thus, R is symmetric. **55.** R is reflexive if and only if $(a, a) \in R$ for all $a \in A$ if and only if $(a, a) \in R^{-1}$ [because $(a, a) \in R$ if and only if $(a, a) \in R^{-1}$] if and only if R^{-1} is reflexive. **57.** Use mathematical induction. The result is trivial for $n = 1$. Assume R^n is reflexive and transitive. By Theorem 1, $R^{n+1} \subseteq R$. To see that $R \subseteq R^{n+1} = R^n \circ R$, let $(a, b) \in R$. By the inductive hypothesis, $R^n = R$ and hence, is reflexive. Thus, $(b, b) \in R^n$. Therefore, $(a, b) \in R^{n+1}$. **59.** Use mathematical induction. The result is trivial for $n = 1$. Assume R^n is reflexive. Then $(a, a) \in R^n$ for all $a \in A$ and $(a, a) \in R$. Thus, $(a, a) \in R^n \circ R = R^{n+1}$ for all $a \in A$. **61.** No, for instance, take $R = \{(1, 2), (2, 1)\}$.

Representation of relations

1. Represent each of these relations on {1, 2, 3} with a matrix (with the elements of this set listed in increasing order). **a)** {(1, 1), (1, 2), (1, 3)}

b) {(1, 2), (2, 1), (2, 2), (3, 3)}

c) {(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)}

d) {(1, 3), (3, 1)}

2. Represent each of these relations on {1, 2, 3, 4} with a matrix (with the elements of this set listed in increasing order).

a) {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}

b) {(1, 1), (1, 4), (2, 2), (3, 3), (4, 1)}

c) {(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3)} **d)** {(2, 4), (3, 1), (3, 2), (3, 4)}

3. List the ordered pairs in the relations on {1, 2, 3} corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order). **a)** 1 0 1

0 1 0

1 0 1

b) 0 1 0

0 1 0

0 1 0

c) 1 1 1

1 0 1

1 1 1

4. List the ordered pairs in the relations on {1, 2, 3, 4} corresponding to these matrices

(where the rows and columns
correspond to the integers listed in increasing order). **a)** 1 1
0 1

1 0 1 0

0 1 1 1

1 0 1 1

b) 1 1 1 0

0 1 0 0

0 0 1 1

1 0 0 1

c) 0 1 0 1

1 0 1 0

0 1 0 1

1 0 1 0

5. How can the matrix representing a relation *R* on a set *A* be used to determine whether the relation is irreflexive?

6. How can the matrix representing a relation *R* on a set *A* be used to determine whether the relation is asymmetric?

7. Determine whether the relations represented by the matrices in Exercise 3 are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.

8. Determine whether the relations represented by the matrices in Exercise 4 are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive

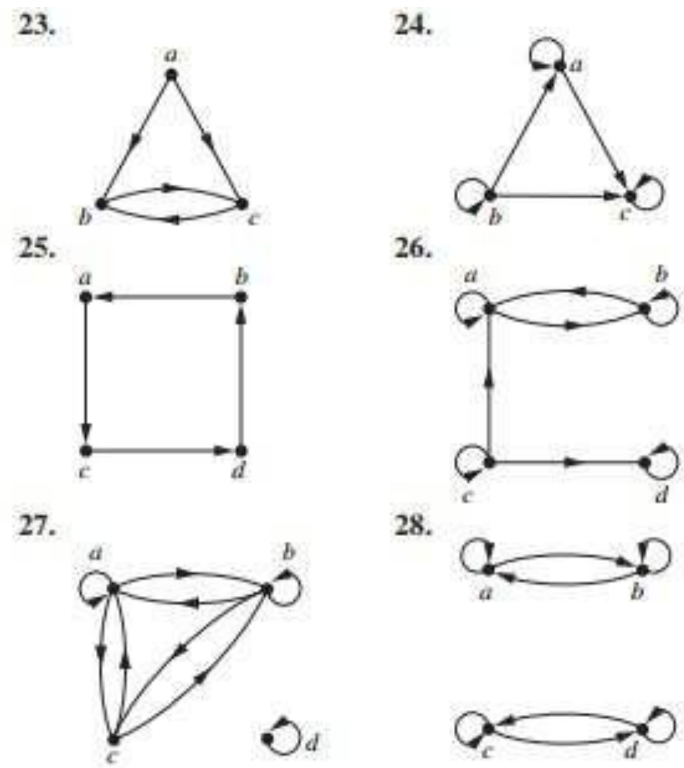
19. Draw the directed graphs representing each of the relations from Exercise 2.

20. Draw the directed graph representing each of the relations from Exercise 3.

21. Draw the directed graph representing each of the relations from Exercise 4.

22. Draw the directed graph that represents the relation $\{(a, a), (a, b), (b, c), (c, b), (c, d), (d, a), (d, b)\}$.

In Exercises 23–28 list the ordered pairs in the relations represented by the directed graphs



31. Determine whether the relations represented by the directed graphs shown in Exercises 23– 25 are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.

32. Determine whether the relations represented by the directed graphs shown in Exercises 26– 28 are reflexive, irreflexive, symmetric, antisymmetric, asymmetric, and/or transitive.

Answers to Odd No Exercise

1. a)
1 1 1
0 0 0
0 0 0
b
0 1 0
1 1 0
0 0 1
c)
1 1 1
0 1 1
0 0 1
d
0 0 1
0 0 0
1 0 0
3. a) (1, 1), (1, 3), (2, 2), (3, 1), (3, 3) b) (1, 2), (2, 2), (3, 2) c) (1, 1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2), (3, 3)
5. The relation is irreflexive if and only if the main diagonal of the matrix contains only 0s. 7. a) Reflexive, symmetric, transitive b) Antisymmetric, transitive c) Symmetric
9. a) 4950 b) 9900 c) 99 d) 100 e) 1 11. Change each 0 to a 1 and each 1 to a 0.
13. a)
0 1 1
1 1 0
1 0 1
b)
1 0 0
0 0 1
0 1 0
c)
1 1 1
1 1 1
1 1 1
15. a)

0 0 1
1 1 0
0 1 1
b)
1 1 0
0 1 1
1 1 1

c)

0 1 1
1 1 1
1 1 1

17. $n^2 - k$

19. a)

3

1 4

2

b)

2 3

1 4

c) 1 4

2 3

d) 1 4

2 3

21. For simplicity we have indicated pairs of edges between the same two vertices in opposite directions by using a double arrowhead, rather than drawing two separate lines.

a)

1 2

3 4

b)

1 2

3 4

c) 1 2

3 4

23. $\{(a, b), (a, c), (b, c), (c, b)\}$ 25. (a, c), (b, a), (c, d), (d, b) 27. $\{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (d, d)\}$ 29. The relation is asymmetric if and only if the directed graph has no loops and no closed paths of length 2. 31. Exercise 23: irreflexive. Exercise 24: reflexive, antisymmetric, transitive. Exercise 25: irreflexive, antisymmetric. 33. Reverse the direction on every edge in the digraph for R. 35. Proof by mathematical induction. *Basis step:* Trivial for $n = 1$. *Inductive step:* Assume true for k. Because $R^{k+1} = R^k \circ R$, its matrix is $\mathbf{M}_R \odot$