National University
of computer and emerging sciences

DISCRETE STRUCTURES

COURSE INSTRUCTOR: MUHAMMAD SAIF UL ISLAM

Course Outline

- **Logic and Proofs** (Chapter 1)
- **Sets and Functions** (Chapter 2)
- **Relations** (Chapter 9)
- **Number Theory** (Chapter 4)
- **Combinatorics** (Chapter 6)
- **Graphs** (Chapter 10)
- **Trees** (Chapter 11)
- Discrete Probability

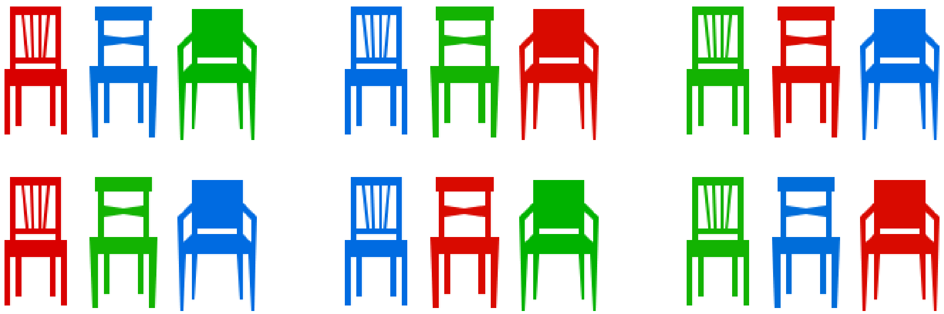
Lecture Outline

- The Sum Rule
- The Product Rule
- The Subtraction Rule
- The Division Rule
- Examples, Examples, and Examples
- Tree Diagrams
- The Pigeonhole Principle

COMBINATORICS

Combinatorics is the mathematics of counting and arranging objects. Counting of objects with certain properties (enumeration) is required to solve many different types of problem.

Applications, include topics as diverse as codes, circuit design and algorithm complexity [and gambling]



The Sum Rule

The Sum Rule:

If there are n(A) ways to do A and, **distinct** from them, n(B) ways to do B, then the number of ways to do A **or** B is n(A) + n(B).

This rule generalizes: there are n(A) + n(B) + n(C) ways to do A or B or C

Example: Sum Rule

- Suppose there are 7 different optional courses in Computer Science and 3 different optional courses in Mathematics. How many choices are there for a student who wants to take one optional course?
- The mathematics department must choose either a student or a faculty member as a representative for a university committee. How many choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student.
- A student can choose a computer project from one of the three lists. The three lists contain 23, 15 and 19 possible projects, respectively. How many possible projects are there to choose from?

The Product Rule

The Product Rule:

If there are n(A) ways to do A and n(B) ways to do B, then the number of ways to do A **and** B is $n(A) \times n(B)$. This is true if the number of ways of doing A and B are **independent**; the number of choices for doing B is the same regardless of which choice you made for A.

Again, this generalizes. There are $n(A) \times n(B) \times n(C)$ ways to do A and B and C

Example: Product Rule

- How many ways a student can choose one optional course each from computer science and mathematics courses if there are 7 different optional courses in Computer Science and 3 different optional courses in Mathematics.
- The chairs of an auditorium are to be labeled with two characters, a letter followed by a digit. What is the largest number of chairs that can be labeled differently?
- Find the number n of ways that an organization consisting of 15 members can elect a president, treasurer, and secretary. (assuming no person is elected to more than one position)
- A bit string is a sequence of 0's and 1's.How many bit string are there of length 4?

Example: Product Rule

- **There are four bus lines between A and B; and three bus lines between B and C. Find the number of ways a person can travel:**

a) By bus from A to C by way of B; $A \xrightarrow{4} B \xrightarrow{3} C$

b) Round trip by bus from A to C by way of B; $A \xrightarrow{4} B \xrightarrow{3} C \xrightarrow{3} B \xrightarrow{4} A$

c) Round trip by bus from A to C by way of B, if the person does not want to use a bus line more than once

$$A \xrightarrow{4} B \xrightarrow{3} C \xrightarrow{2} B \xrightarrow{3} A$$

- **How many bit strings of length 8:**

(i) begin with a 1?

(ii) begin and end with a 1?

Example: Product Rule

- How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?
- How many license plates could begin with A and end on 0?
- How many license plates begin with PQR?
- How many license plates are possible in which all the letters and digits are distinct?
- How many license plates could begin with AB and have all three letters and digits distinct.

Example: Product Rule

Example: The *North American numbering plan* (*NANP*) specifies that a telephone number consists of 10 digits, consisting of a three-digit area code, a three-digit office code, and a four-digit station code. There are some restrictions on the digits.

- Let *X* denote a digit from 0 through 9.
- Let *N* denote a digit from 2 through 9.
- Let *Y* denote a digit that is 0 or 1.
- In the old plan (in use in the 1960s) the format was *NYX-NNX-XXXX*.
- In the new plan, the format is *NXX-NXX-XXXX*.

How many different telephone numbers are possible under the old plan and the new plan?

Example: Product Rule

Example: Determine how many times the inner loop will be iterated when the following algorithm is implemented and run

For i: = 1 to 4

For j : = 1 to 3

[Statement in body of inner loop. None contain branching statements that lead out of the inner loop.]

next j

next i

Solution:

The outer loop is iterated four times, and during each iteration of the outer loop, there are three iterations of the inner loop. Hence, by product rules the total number of iterations of inner loop is $4 \cdot 3 = 12$

Combining the Sum and Product Rule

Calvin wants to go to Milwaukee. He can choose from 3 bus services or 2 train services to head from home to downtown Chicago.

From there, he can choose from 2 bus services or 3 train services to head to Milwaukee.

How many ways are there for Calvin to get to Milwaukee?

Solution:

He has **3+2=5** ways to get to downtown Chicago. (Rule of sum)

From there, he has **2+3=5** ways to get to Milwaukee. (Rule of sum)

Hence, he has **5×5=25** ways to get to Milwaukee in total. (Rule of product)

Combining the Sum and Product Rule

Example: Suppose statement labels in a programming language can be either a single letter or a letter followed by a digit. Find the number of possible labels.

Solution:

First consider variable names one character in length. Since such names consist of a single letter, there are 26 variable names of length 1.

Next, consider variable names two characters in length. Since the first character is a letter, there are 26 ways to choose it. The second character is a digit, there are 10 ways to choose it. Hence, to construct variable name of two characters in length, there are $26 \times 10 = 260$ ways.

Finally, by sum rule, there are $26 + 260 = 286$ possible variable names in the programming language.

Combining the Sum and Product Rule

Exercise: Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Solution: Let P be the total number of passwords, and let P_6, P_7, and P_8 be the passwords of length 6, 7, and 8.

- By the sum rule $P = P_6 + P_7 + P_8$.
- To find each of P_6, P_7, and P_8 , we find the number of passwords of the specified length composed of letters and digits and subtract the number composed only of letters. We find that:

$P_6 = 36^6 - 26^6 = 2{,}176{,}782{,}336 - 308{,}915{,}776 = 1{,}867{,}866{,}560.$

$P_7 = 36^7 - 26^7 = 78{,}364{,}164{,}096 - 8{,}031{,}810{,}176 = 70{,}332{,}353{,}920.$

$P_8 = 36^8 - 26^8 = 2{,}821{,}109{,}907{,}456 - 208{,}827{,}064{,}576 = 2{,}612{,}282{,}842{,}880.$

Consequently, $P = P_6 + P_7 + P_8 = 2{,}684{,}483{,}063{,}360.$

Combining the Sum and Product Rule

Example: Determine how many times the inner loop will be iterated when the following algorithm is implemented and run.

for i = 5 to 50

for j: = 10 to 20

[Statement in body of inner loop. None contain branching statements that lead out of the inner loop.]

next j

next i

Solution:

The outer loop is iterated 50 - 5 + 1 = 46 times and during each iteration of the outer loop there are 20 - 10 + 1 = 11 iterations of the inner loop. Hence by product rule, the total number of iterations of the inner loop is 46×11 = 506.

Subtraction Rule

Subtraction Rule: If a task can be done either in one of n_1 ways or in one of n_2 ways (but some *n_3 ways of the set n_1 ways are common with the n_2 ways*), then the total number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

Also known as, the *principle of inclusion-exclusion*:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Subtraction Rule: Example

Example: How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

Solution: Use the subtraction rule.

- Number of bit strings of length eight that start with a 1 bit: $2^7 = 128$
- Number of bit strings of length eight that start with bits 00: $2^6 = 64$
- Number of bit strings of length eight that start with a 1 bit and end with bits 00 : $2^5 = 32$

Hence, the number is $128 + 64 - 32 = 160$.

Basic Counting Principles: Division Rule

Division Rule: If a task has **n ways** to do it But for a specific **way w** has **d** ways for it to happen.

A rule to ignore "unimportant" differences when counting things.

Restated in terms of sets: If the finite set A is the union of n pairwise disjoint subsets each with d elements, then $n = |A|/d$.

In terms of functions: If f is a function from A to B, where both are finite sets, and for every value $y \in B$ there are exactly d values $x \in A$ such that $f(x) = y$, then $|B| = |A|/d$.

Basic Counting Principles: Division Rule

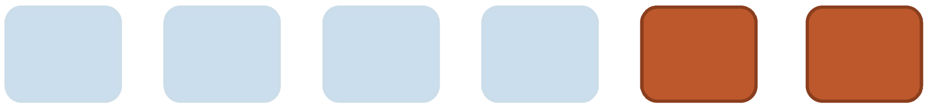
Example: How many ways are there to seat four people around a circular table, where two seating's are considered the same when each person has the same left and right neighbor?

Solution: Number the seats around the table from 1 to 4 proceeding clockwise. There are four ways to select the person for seat 1, 3 for seat 2, 2 for seat 3, and one way for seat 4. Thus there are $4! = 24$ ways to order the four people. But since two seating's are the same when each person has the same left and right neighbor, for every choice for seat 1, we get the same seating.

Therefore, by the division rule, there are $24/4 = 6$ different seating arrangements.

Basic Counting Principles: Division Rule

Example: Arrange 6 blocks in a row. There are **6!** ways form a line.



What if 4 are blue and 2 are red? **6!/ 4!x2!**

Exercise: Seating 5 people in a line. If the left-to-right order is important, then there are **5!** ways form a line.

However, if only the placements that result in **distinct neighborhood** of people are considered, then?

Basic Counting Principles: Division Rule

Solution:

P1 P2 P3 P4 P5 = P5 P4 P3 P2 P1

These 2 arrangements are the same as they have same neighborhood. **5!/2**

Exercise: There are 15 different sandwich and drink combinations for a lunch. If the choice of a drink does not matter, and there are three drink choices, what is the number of ways to pick a lunch?

Tree Diagrams

Exercise:

Calculate the number of different t-shirts possible if the t-shirts are available in five sizes -- S, M, L, XL and XXL -- and each size is available in four colors -- white, red, green and black.

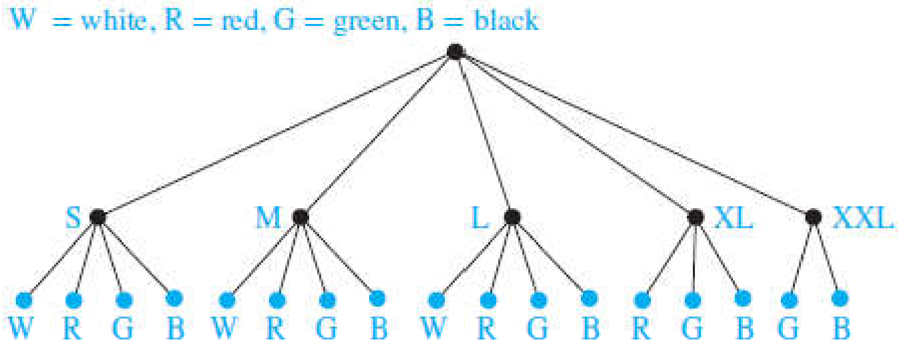
Simple application of **product rule**:

Number of different t-shirts = 5 * 4 = 20

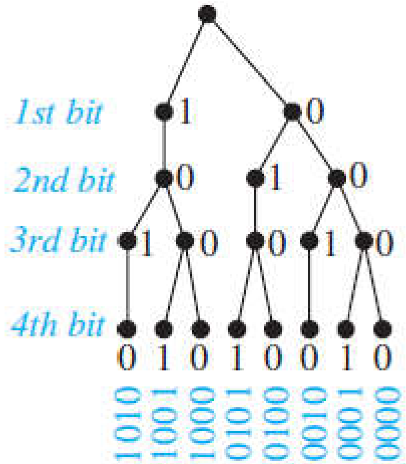
IF In the above problem, suppose XL comes in only red, green and black and XXL comes in only green and black. Recalculate the number of different t-shirts possible.

Tree Diagrams

Tree Diagram:



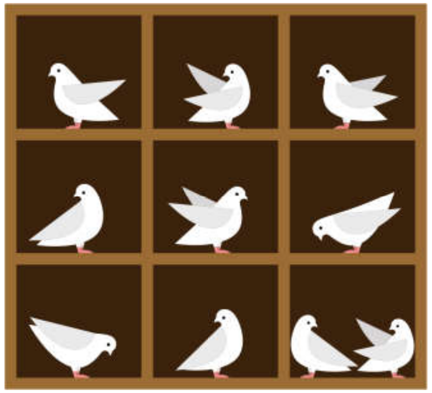
Example: How many bit strings of length four do not have two consecutive 1s?



The Pigeonhole Principle

Pigeonhole Principle: If k is a positive integer and $k + 1$ objects are placed into k boxes, then at least one box contains two or more objects.

Proof: We use a proof by contraposition. Suppose none of the k boxes has more than one object. Then the total number of objects would be at most k. This contradicts the statement that we have $k + 1$ objects.



The Pigeonhole Principle

Example: Among any group of 367 people, there must be at least two with the same birthday.

Example: Among 100 people there are at least 9 who were born in the same month.

Example: In any set of 27 English words, must be at least two that begin with the same letter, since there are 26 letters in the English.

Exercise: How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

Exercise: A drawer has a dozen brown socks and a dozen black socks, randomly placed. If you draw socks at random w/o looking at the drawer, how many socks do you need draw to have a pair of socks of the same color?

How many socks must be taken out to ensure at least two black socks?

Generalized Pigeonhole Principle

Generalized Pigeonhole Principle: If N, $N \geq 0$, objects are placed in k, $k \geq 1$, boxes, then at least one of the boxes has at least $[N/k]$ objects.

Proof: By contradiction. Assume that N objects are placed in k boxes, but that each and every box has fewer than $[N/k]$ objects.

$\therefore$ By assumption, each box has at most $[N/k]-1$ objects.

Total objects in k boxes $\leq k\,([N/k]-1)$.

Since $[N/k] < N/k + 1$ for all positive integer values of N and k, $k\,([N/k]-1) < k\,(N/k+1-1) = k\,(N/k) = N$

So, Total objects in k boxes $\leq k\,([N/k]-1) < N$ This is a contradiction since we started with the fact that N objects are placed in k boxes. Hence the theorem.

Generalized Pigeonhole Principle

Example: Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

Example: What is the minimum number of students required in a Discrete Mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F.

Solution:

The minimum number of students needed to guarantee that at least six students receive the same grade is the smallest integer N such that $\lceil N/K \rceil = \lceil N/5 \rceil = 6$.

Exercise: What is the minimum number of students that must take the test to ensure that at least three students receive the same grade? A grade can be A, B, C, D or F.

Generalized Pigeonhole Principle

Exercise: Consider a network of six computers. Each computer is directly connected to zero or more of other computers. Show that there are at least two computers that have the same number of direct connections.

Solution:

N = 6 (No of PCs)

K = 5 (No of possible connections)

6/5 = 2