National University of computer and emerging sciences

DISCRETE STRUCTURES

COURSE INSTRUCTOR: MUHAMMAD SAIF UL ISLAM

Course Outline

- **Logic and Proofs** (Chapter 1)
- **Sets and Functions** (Chapter 2)
- **Relations** (Chapter 9)
- **Number Theory** (Chapter 4)
- **Combinatorics** (Chapter 6)
- **Graphs** (Chapter 10)
- **Trees** (Chapter 11)
- Discrete Probability

Lecture Outline

- Permutations
- Combinations
- Examples, Examples and Examples
- The Binomial Theorem
- Pascal's Identity and Triangle

$$C_{(n,r)} = \frac{n!}{r!\,(n-r)!}$$

$$P_{(n,r)} = \frac{n!}{(n-r)!}$$

n = set size:
the total number of
items in the sample

r = subset size:
the number of items to be
selected from the sample

Permutations vs. Combinations

Both are ways to count the possibilities

The difference between them is whether order matters or not

Consider a poker hand:

- A♦, 5♥, 7♣, 10♠, K♠

Is that the same hand as:

- K♠, 10♠, 7♣, 5♥, A♦

Does the order the cards are handed out matter?

- If yes, then we are dealing with permutations
- If no, then we are dealing with combinations

Permutations vs. Combinations

Consider the following two problems:

1) Consider the set { p , e , n} How many two-letter "words" (including nonsense words) can be formed from the members of this set?

   We will list all possibilities: pe, pn, en, ep, np, ne , a total of 6.

2) Now consider the set consisting of three males: {Paul, Ed, Nick} For simplicity, we will denote the set { p, e, n} How many two-man crews can be selected from this set?

   Answer: pe (Paul, Ed), pn (Paul, Nick) and en (Ed, Nick) and that is all!

Permutations

A permutation is an ordered arrangement of the elements of some set S

- Let S = {a, b, c}
- c, b, a is a permutation of S
- b, c, a is a *different* permutation of S

An r-permutation is an ordered arrangement of r elements of the set

- A♦, 5♥, 7♣, 10♠, K♠ is a 5-permutation of the set of cards

The notation for the number of r-permutations: $P(n,r)$

- The poker hand is one of P(52,5) permutations

Permutations: Sample question

How many permutations of {a, b, c, d, e, f, g} end with a?

- Note that the set has 7 elements

The last character must be a

- The rest can be in any order

Thus, we want a 6-permutation on the set {b, c, d, e, f, g}

$P(6,6) = 6! = 720$

Why is it not $P(7,6)$?

Permutations Examples

How many ways can 5 people sit on a park bench if the bench can only seat 3 people?

A bookshelf has space for exactly 5 books. How many different ways can 5 books be arranged on this bookshelf?

How many ways are there to select a first-prize winner, a second prize winner, and a third-prize winner from 100 different people who have entered a contest?

Suppose that there are eight runners in a race. The winner receives a gold medal, the second place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur and there are no ties?

Suppose that a salesperson has to visit eight different cities. He must begin the trip in a specified city, but he can visit the other seven cities in any order he wishes. How many possible orders can the salesperson use when visiting these cities?

Combination Examples

How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school?

A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission?

Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of **three** faculty members from the mathematics department and **four** from the computer science department?

Why is the answer 9!/(3!6!) * 11!/(4!7!) rather than 9!/(3!6!) + 11!/(4!7!)?

Combination Examples

Suppose we have an office of 5 women and 6 men and need to select a 4 person committee. How many ways can we select

a) 2 men and 2 women? b) 3 men and 1 woman? c) All women? d) All Men?

In how many ways can the season end with 8 wins, 4 losses, and 2 tie is a college football team plays 14 games?

The number of ways to tie 2 out of 14 games can be found by

C(14,2=91.

The number of ways to lose 4 out of 12 remaining games can be found by

C(12,4)=495.

The number of ways to win 8 out of 8 remaining games can be found by

C(8,8)=1

Hence, the total number of ways to have 8-4-2 record is

91·495·1=45045

Combination Examples

How many bit strings of length 10 contain:

a) exactly four 1's?

- Find the positions of the four 1's
- Does the order of these positions matter?
  - Nope!
  - Positions 2, 3, 5, 7 is the same as positions 7, 5, 3, 2
- Thus, the answer is $C(10,4) = 210$

b) at most four 1's?

- There can be 0, 1, 2, 3, **or** 4 occurrences of 1
- Thus, the answer is:
  - $C(10,0) + C(10,1) + C(10,2) + C(10,3) + C(10,4)$
  - $= 1+10+45+120+210$
  - $= 386$

Combination Examples

How many bit strings of length 10 contain:

c) at least four 1's?

- There can be 4, 5, 6, 7, 8, 9, **or** 10 occurrences of 1
- Thus, the answer is:
  - $C(10,4) + C(10,5) + C(10,6) + C(10,7) + C(10,8) + C(10,9) + C(10,10)$
  - $= 210+252+210+120+45+10+1$
  - $= 848$
- Alternative answer: subtract from 2^{10} the number of strings with 0, 1, 2, or 3 occurrences of 1

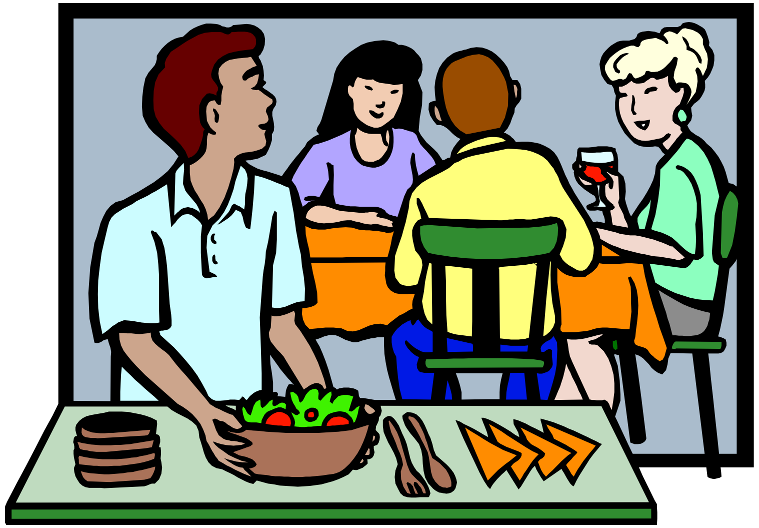
d) an equal number of 1's and 0's?

- Thus, there must be five 0's and five 1's
- Find the positions of the five 1's
- Thus, the answer is $C(10,5) = 252$

Combinations or Permutations?

1. In how many ways can you choose 5 out of 10 friends to invite to a dinner party?

Solution: Does the order of selection matter? If you choose friends in the order A,B,C,D,E or A,C,B,D,E the same set of 5 was chosen, so we conclude that the order of selection does not matter. We will use the formula for combinations since we are concerned with how many **subsets of size 5** we can select from a set of 10.



C(10,5) = $\frac{P(10,5)}{5!} = \frac{10(9)(8)(7)(6)}{5(4)(3)(2)(1)} = \frac{10(9)(8)(7)}{(5)(4)} = 2(9)(2)(7) = 252$

Combinatorial Proofs

Definition 1: A *combinatorial proof* of an identity is a proof that uses one of the following methods.

- A *double counting proof* uses counting arguments to prove that both sides of an identity count the same objects, but in different ways.
- *Count of* $C(n, n-r)$ is same as $C(n, r)$.

- A *bijective proof* shows that there is a bijection between the sets of objects counted by the two sides of the identity.

Binomial Theorem

Binomial Theorem: Let *x* and *y* be variables, and *n* a nonnegative integer. Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Proof: We use combinatorial reasoning . The terms in the expansion of $(x + y)^n$ are of the form $x^{n-j}y^j$ for $\binom{n}{n-j}$ $j = 0,1,2,\ldots,n$. To form the term $x^{n-j}y^j$, it is necessary to choose $n-j$ *x*s from the *n* sums. Therefore, the coefficient of $x^{n-j}y^j$ is $\binom{n}{j}$ which equals .

Powers of Binomial Expressions

Definition: A *binomial* expression is the sum of two terms, such as $x + y$. (More generally, these terms can be products of constants and variables.)

We can use counting principles to find the coefficients in the expansion of $(x + y)^n$ where n is a positive integer.

To illustrate this idea, we first look at the process of expanding $(x + y)^3$.

$(x + y)(x + y)(x + y)$ expands into a sum of terms that are the product of a term from each of the three sums.

Terms of the form x^3, x^2y, $x y^2$, y^3 arise. The question is what are the coefficients?

- To obtain x^3, an x must be chosen from each of the sums. There is only one way to do this. So, the coefficient of x^3 is 1.
- To obtain x^2y, an x must be chosen from two of the sums and a y from the other. There are $\binom{3}{2}$ ways to do this and so the coefficient of x^2y is 3.
- To obtain xy^2, an x must be chosen from of the sums and a y from the other two . There are $\binom{3}{1}$ ways to do this and so the coefficient of xy^2 is 3.
- To obtain y^3, a y must be chosen from each of the sums. There is only one way to do this. So, the coefficient of y^3 is 1.

We have used a counting argument to show that $(x + y)^3 = x^3 + 3x^2y + 3x y^2 + y^3$.

Next we present the binomial theorem gives the coefficients of the terms in the expansion of $(x + y)^n$.

Using the Binomial Theorem

Example:

What is the expansion of $(x + y)^4$?

Solution: From the binomial theorem it follows that

$$
\begin{aligned}
(x+y)^4 &= \sum_{j=0}^{4} \binom{4}{j} x^{4-j} y^j \\
&= \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4 \\
&= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.
\end{aligned}
$$

Using the Binomial Theorem: Example

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x + y)^{25}$?

Solution: From the binomial theorem it follows that this coefficient is

$$\binom{25}{13} = \frac{25!}{13!\,12!} = 5{,}200{,}300.$$

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Solution: We view the expression as $(2x + (-3y))^{25}$. By the binomial theorem

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j}(-3y)^j.$$

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$.

$$\binom{25}{13} 2^{12}(-3)^{13} = -\frac{25!}{13!12!}2^{12}3^{13}.$$

A Useful Identity

Corollary 1: With $n \geq 0$,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Proof (*using binomial theorem*): With $x = 1$ and $y = 1$, from the binomial theorem we see that:

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{(n-k)} = \sum_{k=0}^{n} \binom{n}{k}.$$

Proof (*combinatorial*): Consider the subsets of a set with n elements. There are $\binom{n}{0}$ subsets with zero elements, $\binom{n}{1}$ with one element, $\binom{n}{2}$ with two elements, …, and $\binom{n}{n}$ with n elements.

Therefore the total is

$$\sum_{k=0}^{n} \binom{n}{k}.$$

Since, we know that a set with n elements has 2^n subsets, we conclude:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Pascal's Identity

Blaise Pascal
(1623-1662)

Pascal's Identity: If n and k are integers with $n \geq k \geq 0$, then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Proof (*combinatorial*): Let T be a set where $|T| = n + 1$, $a \in T$, and $S = T - \{a\}$. There are $\binom{n+1}{k}$ subsets of T containing k elements. Each of these subsets either:

- contains a with $k - 1$ other elements, or
- contains k elements of S and not a.

There are

- $\binom{n}{k-1}$ subsets of k elements that contain a, since there are $\binom{n}{k-1}$ subsets of $k - 1$ elements of S,
- $\binom{n}{k}$ subsets of k elements of T that do not contain a, because there are $\binom{n}{k}$ subsets of k elements of S.

Hence,

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

See Exercise 19 for an algebraic proof.

Pascal’s Triangle

The nth row in the triangle consists of the binomial $\binom{n}{k}$ coefficients, $k = 0,1,\ldots,n$.

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$
$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$
$$\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$$
$$\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8}$$
$$\ldots$$

(a)

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$1$$
$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$
$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$
$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$
$$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1$$
$$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1$$
$$\ldots$$

(b)

By Pascal’s identity, adding two adjacent bionomial coefficients results is the binomial coefficient in the next row between these two coefficients.