DISCRETE STRUCTUERS

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Course Outline

- **Logic and Proofs** (Chapter 1)
- Sets and Functions
- Relations
- Number Theory
- Combinatorics and Recurrence
- Graphs
- Trees
- Discrete Probability

Lecture Outline

- Limitations of Propositional Logic
- Predicates
- Quantifiers
- Logical Equivalences Involving Quantifiers
- Translating from English to Logic
- Examples
- Nested Quantifiers
- Order of Quantifiers
- Translating Nested Quantifiers

Limitations of Propositional Logic

- Propositional logic cannot adequately express the meaning of all statements in mathematics and in natural language

Example:

If we want to say "*All students are present.*" OR "*Some students are present*", We need **N** number of statement to represent it. Where **N** refers to the *number of students*.

- Can't be represented in propositional logic. Need a language that talks about objects, their properties, and their relations.

Predicates

- Predicate logic can be used to express the meaning of a wide range of statements in mathematics and computer science in ways that permit us to reason and explore relationships between objects
- Predicate logic uses the following new features:
  - Variables: x, y, z
  - Predicates: $P(x)$, $M(x)$
  - Quantifiers (*to be covered in a few slides*):
- *Propositional functions* are a generalization of propositions.
  - They contain variables and a predicate, e.g., $P(x)$
  - Variables can be replaced by elements from their *domain*.

Predicates

Example:

The statement **"*x* is greater than 3"** has two parts.

1- The variable ***x***, is the **subject** of the statement.

2- The **predicate**, **"is greater than 3"** refers to a property that the subject of the statement can have.

We can denote the statement "*x* is greater than 3" by ***P*(*x*),**

where ***P*** denotes the predicate "is greater than 3" and ***x*** is the variable.

Propositional Functions

- Once a value has been assigned to the variable x, the statement $P(x)$ becomes a proposition and has a truth value

Example:

Let $P(x)$ denote the statement "$\boldsymbol{x > 3}$." What are the truth values of $P(4)$ and $P(2)$?

We obtain the statement $P(4)$ by setting $x = 4$ in the statement "$x > 3$."

$P(4)$, which is the statement "$4 > 3$," is **true**.

$P(2)$, which is the statement "$2 > 3$," is **false**.

Propositional Functions

Example:

Let $Q(x, y)$ denote the statement "$\boldsymbol{x = y + 3}$." What are the truth values of the propositions

$Q(1, 2)$ and $Q(3, 0)$?

To obtain $Q(1, 2)$, set $x = 1$ and $y = 2$ in the statement $Q(x, y)$.

$Q(1, 2)$ is the proposition "$1 = 2 + 3$," which is **false**.

$Q(3, 0)$ is the proposition "$3 = 0 + 3$," which is **true**.

Compound Expressions

- Connectives from propositional logic carry over to predicate logic.

If $P(x)$ denotes "$x > 0$," find these truth values:

$P(3) \vee P(-1)$ **Solution**: T

$P(3) \wedge P(-1)$ **Solution**: F

$P(3) \rightarrow P(-1)$ **Solution**: F

$P(3) \rightarrow P(-1)$ **Solution**: T

- Expressions with variables are not propositions and therefore do not have truth values. For example,

$P(3) \wedge P(y)$

$P(x) \rightarrow P(y)$

- When used with quantifiers (to be introduced next), these expressions (propositional functions) become propositions.

Quantifiers

- The area of logic that deals with predicates and quantifiers is called the **predicate calculus**.
- We need *quantifiers* to express the meaning of English words including ***all*** and ***some***:
  - "All Students have identity cards."
  - "Some students are absent."
- The two most important quantifiers are:
  - ***Universal** Quantifier,* **"For all,"** symbol: $\forall$
  - ***Existential** Quantifier*, **"There exists,"** symbol: $\exists$

We write as in $\forall x\, P(x)$ and $\exists x\, P(x)$.

$\forall x\, P(x)$ asserts $P(x)$ is true for every x in the *domain*.

$\exists x\, P(x)$ asserts $P(x)$ is true for some x in the *domain*.

The quantifiers are said to bind the variable x in these expressions.

Quantifiers

Example: Universal Quantifier

Let $P(x)$ be the statement "$x + 1 > x$." What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real numbers?

***Solution*:** Because $P(x)$ is true for all real numbers x, the quantification $\forall x P(x)$ is **true**.

* Besides "for all" and "for every," universal quantification can be expressed in many other ways, including "all of," "for each," "given any," "for arbitrary," and "for any."

TABLE 1 Quantifiers.

Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Quantifiers

Example: Universal Quantifier

Let $Q(x)$ be the statement "$x < 2$." What is the truth value of the quantification $\forall x Q(x)$, where the domain consists of all real numbers?

Solution: $Q(x)$ is not true for every real number x, because, for instance, $Q(3)$ is false.

That is, $x = 3$ is a **counterexample** for the statement $\forall x Q(x)$. Thus, $\forall x Q(x)$ is false.

* $\exists x\, P(x)$ is read as "For some x, $P(x)$", or as "There is an x such that $P(x)$," or "For at least one x, $P(x)$."

Exercise:

What does the statement $\forall x N(x)$ mean if $N(x)$ is "Computer x is connected to the network" and the domain consists of all computers on campus?

Quantifiers

Example: Existential Quantifier

Let $P(x)$ denote the statement "$x > 3$." What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all real numbers?

Solution: Because "$x > 3$" is sometimes true—for instance,

When $x = 4$—the existential quantification of $P(x)$, which is $\exists x P(x)$, is true.

Exercise:

Let $Q(x)$ denote the statement "$x = x + 1$." What is the truth value of the quantification $\exists x Q(x)$,

where the domain consists of all real numbers?

Uniqueness Quantifier

- $\exists!x\ P(x)$ means that $P(x)$ is true for one and only one x in the universe of discourse.
- This is commonly expressed in English in the following equivalent ways:
  - "There is a **unique** x such that $P(x)$."
  - "There is **one and only one** x such that $P(x)$"

Examples:

1. If *P(x)* denotes "$x + 1 = 0$" and U is the integers, then $\exists!x\ P(x)$ is **true**.
2. But if *P(x)* denotes "$x > 0$," then $\exists!x\ P(x)$ is **false**.

Thinking about Quantifiers

- When the domain of discourse is finite, we can think of quantification as looping through the elements of the domain.
- To evaluate $\forall x\, P(x)$ loop through all x in the domain.
  - If at every step P(x) is true, then $\forall x\, P(x)$ is true.
  - If at a step P(x) is false, then $\forall x\, P(x)$ is false and the loop terminates.
- To evaluate $\exists x\, P(x)$ loop through all x in the domain.
  - If at some step, P(x) is true, then $\exists x\, P(x)$ is true and the loop terminates.
  - If the loop ends without finding an x for which P(x) is true, then $\exists x\, P(x)$ is false.
- Even if the domains are infinite, we can still think of the quantifiers this fashion, but the loops will not terminate in some cases.

Properties of Quantifiers

The truth value of $\exists x\, P(x)$ and $\forall x\, P(x)$ depend on both the propositional function $P(x)$ and on the domain U.

Examples:

1. If U is the positive integers and $P(x)$ is the statement "$x < 2$", then $\exists x\, P(x)$ is true, but $\forall x\, P(x)$ is false.
2. If U is the negative integers and $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true.
3. If U consists of 3, 4, and 5, and $P(x)$ is the statement "$x > 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true.
4. But if $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are false.

Precedence of Quantifiers

- The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.
- For example, $\forall x\, P(x) \vee Q(x)$ means $(\forall x\, P(x)) \vee Q(x)$
- $\forall x\, (P(x) \vee Q(x))$ means something different.
- Unfortunately, often people write $\forall x\, P(x) \vee Q(x)$ when they mean $\forall x\, (P(x) \vee Q(x))$.

Logical Equivalences Involving Quantifiers

- Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value
  - for every predicate substituted into these statements and
  - for every domain of discourse used for the variables in the expressions.
- The notation $S \equiv T$ indicates that S and T are logically equivalent.
- **Example:** $\forall x \neg\neg S(x) \equiv \forall x\, S(x)$

Logical Equivalences Involving Quantifiers

Quantifiers as Conjunctions and Disjunctions

- If the domain is finite, a universally quantified proposition is equivalent to a conjunction of propositions without quantifiers and an existentially quantified proposition is equivalent to a disjunction of propositions without quantifiers.

- If *U* consists of the integers 1,2, and 3:

$$\forall x P(x) \equiv P(1) \land P(2) \land P(3)$$

$$\exists x P(x) \equiv P(1) \lor P(2) \lor P(3)$$

- Even if the domains are infinite, you can still think of the quantifiers in this fashion, but the equivalent expressions without quantifiers will be infinitely long.

Negating Quantified Expressions

- Consider $\forall x\, J(x)$
  - "Every student in your class has taken a course in Java."
  - Here $J(x)$ is "x has taken a course in Java" and
  - the domain is students in your class.
- Negating the original statement gives "It is not the case that every student in your class has taken Java." This implies that "There is a student in your class who has not taken java."
- Symbolically $\neg\forall x\, J(x)$ and $\exists x\, \neg J(x)$ are equivalent
- Now Consider $\exists\, x\, J(x)$
  - "There is a student in this class who has taken a course in Java."
  - Where $J(x)$ is "x has taken a course in Java."
- Negating the original statement gives "It is not the case that there is a student in this class who has taken Java." This implies that "Every student in this class has not taken Java"
- Symbolically $\neg\exists\, x\, J(x)$ and $\forall\, x\, \neg J(x)$ are equivalent

De Morgan's Laws for Quantifiers

The reasoning in the table shows that:

$$\neg\forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg\exists x P(x) \equiv \forall x \neg P(x)$$

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Translating from English to Logic

Example 1: Translate the following sentence into predicate logic: "Every student in this class has taken a course in Java."

Solution:

First decide on the domain *U*.

Solution 1: If ***U*** is all students in this class, define a propositional function J(x) denoting "**x has taken a course in Java**" and translate as $\forall x\, J(x)$.

Solution 2: If we change the domain to consist of all people, we will need to express our statement as:

"For every person ***x***, if person *x* is a student in this class, then *x* has studied java."

statement can be expressed as $\forall x\, (S(x) \rightarrow J(x))$.

$\forall x\, (S(x) \wedge J(x))$ is not correct. What does it mean?

Because this statement says that all people are students in this class and have studied java

Translating from English to Logic

Example 2: Translate the following sentence into predicate logic: "**Some student in this class has taken a course in Java**."

Solution:

First decide on the domain *U*.

Solution 1: If ***U*** is all students in this class, translate as

$$\exists x\, J(x)$$

Solution 2: But if *U* is all people, then translate as

$$\exists x\, (S(x) \land J(x))$$

$\exists x\, (S(x) \rightarrow J(x))$ is not correct. What does it mean?

Because it is true when there is someone not in the class because, in that case, for such a person *x*, $S(x) \rightarrow J(x)$ becomes either **F** → **T** or **F** →**F**, both of which are true.

Translating from English to Logic

Example 3: "Some student in this class has visited Mexico."

Solution: Let $M(x)$ denote "x has visited Mexico" and $S(x)$ denote "x is a student in this class," and U be all people.

$$\exists x \ (S(x) \wedge M(x))$$

Example 4: "Every student in this class has visited Canada or Mexico."

Solution: Add $C(x)$ denoting "x has visited Canada."

$$\forall x \ (S(x) \rightarrow (M(x) \vee C(x)))$$

System Specification Example

Predicate logic is used for specifying properties that systems must satisfy.

For example, translate into predicate logic:

- “Every mail message larger than one megabyte will be compressed.”
- “If a user is active, at least one network link will be available.”

Decide on predicates and domains (left implicit here) for the variables:

- Let $L(m, y)$ be “Mail message m is larger than y megabytes.”
- Let $C(m)$ denote “Mail message m will be compressed.”

$$\forall m(L(m,1) \rightarrow C(m))$$

Lewis Carroll Example

The first two are called *premises* and the third is called the *conclusion*.

1. "All lions are fierce."
2. "Some lions do not drink coffee."
3. "Some fierce creatures do not drink coffee."

Here is one way to translate these statements to predicate logic. Let P(x), Q(x), and R(x) be the propositional functions "x is a lion," "x is fierce," and "x drinks coffee," respectively.

1. $\forall x\, (P(x) \rightarrow Q(x))$
2. $\exists x\, (P(x) \wedge \neg R(x))$
3. $\exists x\, (Q(x) \wedge \neg R(x))$

Later we will see how to prove that the conclusion follows from the premises.

Nested Quantifiers

Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.

Example: "for every real number *x* there is a real number *y* such that *x* + *y* = 0." This states that every real number has an additive inverse." is

$$\forall x \exists y(x + y = 0)$$

where the domains of x and y are the real numbers.

We can also think of nested propositional functions:

$\forall x \exists y(x + y = 0)$ can be viewed as $\forall x\, Q(x)$

where $Q(x)$ is $\exists y\, P(x, y)$

where $P(x, y)$ is $(x + y = 0)$

Thinking of Nested Quantification

Nested Loops

- To see if $\forall x \forall y P(x,y)$ is true, loop through the values of x :
  - At each step, loop through the values for y.
  - If for some pair of x and y, $P(x,y)$ is false, then $\forall x \forall y P(x,y)$ is false and both the outer and inner loop terminate.

  $\forall x \forall y P(x,y)$ is true if the outer loop ends after stepping through each x.
- To see if $\forall x \exists y P(x,y)$ is true, loop through the values of x:
  - At each step, loop through the values for y.
  - The inner loop ends when a pair x and y is found such that $P(x, y)$ is true.
  - If no y is found such that $P(x, y)$ is true the outer loop terminates as $\forall x \exists y P(x,y)$ has been shown to be false.

  $\forall x \exists y P(x,y)$ is true if the outer loop ends after stepping through each x.

If the domains of the variables are infinite, then this process can not actually be carried out.

Order of Quantifiers

Examples:

1. Let $P(x,y)$ be the statement "$x + y = y + x$." Assume that U is the real numbers. Then $\forall x \forall y P(x,y)$ and $\forall y \forall x P(x,y)$ have the same truth value.

2. Let $Q(x,y)$ be the statement "$x + y = 0$." Assume that U is the real numbers. Then $\forall x \exists y P(x,y)$ is true, but $\exists y \forall x P(x,y)$ is false.

Questions on Order of Quantifiers

Example 1: Let *U* be the real numbers,

Define $P(x,y) : x \cdot y = 0$

What is the truth value of the following:

1. $\forall x \forall y P(x,y)$

   Answer: False

2. $\forall x \exists y P(x,y)$

   Answer: True

3. $\exists x \forall y\, P(x,y)$

   Answer: True

4. $\exists x \exists\, y\, P(x,y)$

   Answer: True

Questions on Order of Quantifiers

Example 2: Let *U* be the real numbers,

Define $P(x,y) : x / y = 1$

What is the truth value of the following:

1. $\forall x \forall y P(x,y)$

   Answer: False

2. $\forall x \exists y P(x,y)$

   Answer: False

3. $\exists x \forall y \, P(x,y)$

   Answer: False

4. $\exists x \, \exists \, y \, P(x,y)$

   Answer: True

Quantifications of Two Variables

Statement	When True?	When False
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	$P(x,y)$ is true for every pair x,y.	There is a pair x, y for which $P(x,y)$ is false.
$\forall x \exists y P(x,y)$	For every x there is a y for which $P(x,y)$ is true.	There is an x such that $P(x,y)$ is false for every y.
$\exists x \forall y P(x,y)$	There is an x for which $P(x,y)$ is true for every y.	For every x there is a y for which P(x,y) is false.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair x, y for which $P(x,y)$ is true.	P(x,y) is false for every pair x,y

TRANSLATION EXAMPLES

Translating Nested Quantifiers into English

Example 1: Translate the statement

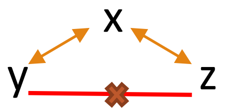
$$\forall x \ (C(x) \lor \exists y \ (C(y) \land F(x, y)))$$

where C(x) is "*x* has a computer," and *F*(*x*,*y*) is "*x* and *y* are friends," and the domain for both *x* and *y* consists of all students in your school.

Solution: Every student in your school has a computer or has a friend who has a computer.

Example 2: Translate the statement

$$\exists x \ \forall y \ \forall z \ ((F(x, y) \land F(x,z) \land (y \neq z)) \rightarrow \neg F(y,z))$$



Solution: There is a student none of whose friends are also friends with each other

Translating Mathematical Statements into Predicate Logic

Example : Translate "The sum of two positive integers is always positive" into a logical expression.

Solution:

1. Rewrite the statement to make the implied quantifiers and domains explicit:

   "For every two integers, if these integers are both positive, then the sum of these integers is positive."

2. Introduce the variables *x* and *y*, and specify the domain, to obtain:

   "For all positive integers *x* and *y*, *x* + *y* is positive."

3. The result is:

   $$\forall x \forall y((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$$

   where the domain of both variables consists of all integers

Translating English into Logical Expressions Example

Example: Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."

Solution:

1. Let $P(w,f)$ be "w has taken f" and $Q(f,a)$ be "f is a flight on a."
2. The domain of w is all women, the domain of f is all flights, and the domain of a is all airlines.
3. Then the statement can be expressed as:

   $$\exists w\, \forall a\, \exists f\, (P(w,f) \land Q(f,a))$$

Questions on Translation from English

Choose the obvious predicates and express in predicate logic.

Example 1: "Brothers are siblings."

Solution: $\forall x\ \forall y\ (B(x,y) \rightarrow S(x,y))$

Example 2: "Siblinghood is symmetric."

Solution: $\forall x\ \forall y\ (S(x,y) \rightarrow S(y,x))$

Example 3: "Everybody loves somebody."

Solution: $\forall x\ \exists y\ L(x,y)$

Example 4: "There is someone who is loved by everyone."

Solution: $\exists y\ \forall x\ L(x,y)$

Example 5: "There is someone who loves someone."

Solution: $\exists x\ \exists y\ L(x,y)$

Example 6: "Everyone loves himself"

Solution: $\forall x\ L(x,x)$

Negating Nested Quantifiers

Example 1: Recall the logical expression developed three slides back:

$$\exists w \forall a \exists f\,(P(w,f) \wedge Q(f,a))$$

Part 1: Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."

Solution: $\neg\exists w \forall a \exists f\,(P(w,f) \wedge Q(f,a))$

Part 2: Now use De Morgan's Laws to move the negation as far inwards as possible.

Solution:

1. $\neg\exists w \forall a \exists f\,(P(w,f) \wedge Q(f,a))$
2. $\forall w \neg \forall a \exists f\,(P(w,f) \wedge Q(f,a))$ by De Morgan's for $\exists$
3. $\forall w \exists a \neg \exists f\,(P(w,f) \wedge Q(f,a))$ by De Morgan's for $\forall$
4. $\forall w \exists a \forall f \neg (P(w,f) \wedge Q(f,a))$ by De Morgan's for $\exists$
5. $\forall w \exists a \forall f (\neg P(w,f) \vee \neg Q(f,a))$ by De Morgan's for $\wedge$.

Part 3: Can you translate the result back into English?

Solution:
"For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline"

Thank you!!!

Understanding Math by reading slides is similar to Learning to swim by watching TV.

So, DO PRACTICE IT!