Linear Algebra

Linear Algebra

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Notes

Linear Algebra Review

Notes

Basic

- Linear algebra provides a way of compactly representing and operating on sets of linear equations.
- For example, consider the following system of equations:

$$4x_1 - 5x_2 = -13$$
$$-2x_1 + 3x_2 = 9.$$

- In matrix notation, we can write the system more compactly as

$$Ax = b$$

with

$$A = \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix} \ and\ b = \begin{bmatrix} -13 \\ 9 \end{bmatrix}$$

Notes

Notations

- By $A \in \mathbb{R}_{m \times n}$ we denote a matrix with m rows and n columns, where the entries of A are real numbers.
- By $x \in \mathbb{R}_n$, we denote a vector with n entries. By convention, an n-dimensional vector is often thought of as a matrix with n rows and 1 column, known as a column vector.
- The ith element of a vector x is denoted x_i:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$

Notes

Notations

- We use the notation $a_{ij}(or A_{ij}, A_{i,j},\ etc)$ to denote the entry of A in the i^{th} row and jth column:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

- We denote the jth column of A by a_j or $A_{:,j}$ and the ith row of A by a_i^T or $A_{i,:}$:

$$A = \begin{bmatrix} | & | & & | \\ a_{11} & a_{12} & \dots & a_{1n} \\ | & | & & | \end{bmatrix} \qquad A = \begin{bmatrix} -- & a_1^T & -- \\ -- & a_1^T & -- \end{bmatrix}$$

Notes

Vector Space

- Let V be an arbitrary nonempty set of objects for which two operations are defined:
- addition and multiplication by numbers called scalars.
- addition implies that u and v in V an object u + v, called the sum of u and v.
- By scalar multiplication we mean a rule for associating with each scalar k and each object u in V an object ku, called the scalar multiple of u by k.
- If the following axioms are satisfied by all objects u, v, w in V and all scalars k and m, then we call V a vector space and we call the objects in V. vectors

Notes

Vector Space Axioms

- If u and v are objects in V, then $u + v \in V$.
- $u + v = v + u$
- $u + (v + w) = (u + v) + w$
- There exists zero vector, $0 + u = u + 0 = u \ \forall\ u \ \in \ V$.
- For each u in V, there is an object -u in V, called a negative of u, such that $u + (-u) = (-u) + u = 0$.
- If k is any scalar and u is any object in V, then ku is in V.
- $k(u + v) = ku + kv$
- $(k + m)u = ku + mu$
- $k(mu) = (km)(u)$
- $1u = u$

Notes

Example

- Given $u + v = (u_1 + v_1 + 1, u_2 + v_2 + 1), ku = (ku_1, ku_2)$
  a. Compute $u + v$ and ku for $u = (0, 4), v = (1, -3)$, and $k = 2$.
  b. Show that $(0, 0) \neq 0$.
  c. Show that $(-1, -1) = 0$.

Notes

SubSpaces

- If W is a nonempty set of vectors in a vector space V, then W is a subspace of V if and only if the following conditions are satisfied.
- If u and v are vectors in W, then u + v is in W.
- If k is a scalar and u is a vector in W, then ku is in W.

Notes

Example

- Check if following forms a subspace
- All matrices of the form

$$\begin{bmatrix} a & 1 \\ b & 1 \end{bmatrix}$$

- All matrices of the form

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

Notes

Row Echolen Form

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 3 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 3 & -4 \end{pmatrix} \times(-1)$$

$$\equiv$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & -5 \end{pmatrix} \times(1) \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -3 \end{pmatrix}$$

$$R_3 - 1 \cdot R_1 \rightarrow R_3 \qquad R_3 - (-1) \cdot R_2 \rightarrow R_3$$

Notes

Finding inverse of A (Gauss-Jordan)

- Find the inverse of the following matrix by Gauss-Jordan Algorithm

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 3 & -4 \end{bmatrix}$$

Notes

Finding inverse of A (Gauss-Jordan)

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 3 & -4 \end{pmatrix}^{(-1)} = ?$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 1 & 3 & -4 & 0 & 0 & 1 \end{array}\right) \times(-1) \underset{R_3 - 1 \cdot R_1 \to R_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 0 & 1 & -5 & -1 & 0 & 1 \end{array}\right) \times(-1) \underset{R_2/(-1) \to R_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & 1 & -5 & -1 & 0 & 1 \end{array}\right) \times(-1) \underset{R_3 - 1 \cdot R_2 \to R_3}{\sim}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & -3 & -1 & 1 & 1 \end{array}\right) \times\left(\frac{-1}{3}\right) \underset{R_3/(-3) \to R_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{-1}{3} & \frac{-1}{3} \end{array}\right) \times(2) \underset{R_2 - (-2) \cdot R_3 \to R_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{2}{3} & \frac{-5}{3} & \frac{-2}{3} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{-1}{3} & \frac{-1}{3} \end{array}\right) \times(-1) \underset{R_1 - 1 \cdot R_3 \to R_1}{\sim}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{2}{3} & \frac{-5}{3} & \frac{-2}{3} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{-1}{3} & \frac{-1}{3} \end{array}\right) \times(-2) \underset{R_1 - 2 \cdot R_2 \to R_1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-2}{3} & \frac{11}{3} & \frac{5}{3} \\ 0 & 1 & 0 & \frac{2}{3} & \frac{-5}{3} & \frac{-2}{3} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{-1}{3} & \frac{-1}{3} \end{array}\right)$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 3 & -4 \end{pmatrix}^{(-1)} = \begin{pmatrix} \frac{-2}{3} & \frac{11}{3} & \frac{5}{3} \\ \frac{2}{3} & \frac{-5}{3} & \frac{-2}{3} \\ \frac{1}{3} & \frac{-1}{3} & \frac{-1}{3} \end{pmatrix}$$

Notes

Row and Column Space of matrix A

- Let A be an m x n matrix
- The Column Space of matrix A is the vector space spanned by the column vectors of matrix A. i.e. all linear combinations of the column vectors.
- Since each column vector has m components, C(A) is a subspace of R^m
- The Row Space of matrix A is the vector space spanned by the row vectors of matrix A.
- Since each row vector has n components, R(A) is a subspace of R^n

Notes

Nullspace and Rank

- Let A be an m x n matrix
- The Nullspace of matrix A is the vector space spanned by all the vectors x which satisfy $Ax = 0$.
- Since the vector x has n components, N(A) is a subspace of R^n
- The Left Nullspace of matrix A, or the nullspace of A^T, is the vector space spanned by all the vectors y which satisfy $A^T y = 0$.
- Since the vector y has m components, $N(A^T)$ is a subspace of R^m.
- The common dimension of the row space and column space of a matrix A is called the rank of A and is denoted by rank(A)
- the dimension of the null space of A is called the nullity of A and is denoted by nullity(A).

Notes

- A is a 5×3 matrix.
- The rank of matrix A = 2.
- The dimension of C(A) =
- The dimension of R(A) =
- The C(A) is a subspace of R^{-}, so is __.
- The R(A) is a subspace of R^3, so is __.
- The dimension of N(A) is
- The dimension of $N(A^T)$=

Notes

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Example

- A is a 5×3 matrix.
- The rank of matrix A $= 2$.
- The dimension of C(A) $= 2$
- The dimension of R(A) $= 2$
- The C(A) is a subspace of R^5, so is$N(A^T)$.
- The R(A) is a subspace of R^3, so is N(A).
- The dimension of N(A) is 3-2=1
- The dimension of $N(A^T)$=5-2=3

Notes

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Calculating Nullspace

To find a basis for the null space, we form an augmented matrix by appending a column of zeros to the right, and then put this matrix in reduced row-echelon form.

We begin with the matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 4 & 6 & 0 \\ 3 & 6 & 9 & 0 \end{array}\right]$$

Add -2 times row 1 to row 2:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 6 & 9 & 0 \end{array}\right]$$

Add -3 times row 1 to row 3:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Convert the matrix equation back to an equivalent system:

$$x_1 + 2x_2 + 3x_3 = 0$$

Add an equation for each free variable:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ x_2 &= x_2 \\ x_3 &= x_3 \end{aligned}$$

Solve for each variable in terms of the free variables:

$$\begin{aligned} x_1 &= -2x_2 + -3x_3 \\ x_2 &= x_2 \\ x_3 &= x_3 \end{aligned}$$

Collect terms into vectors:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_3 \\ 0 \\ x_3 \end{bmatrix}$$

Factor out variables on the right side:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Thus, a basis for the null space is:

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Notes

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Calculating Row Space

We begin with the matrix:

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 5 \\ 0 & 4 & 5 & 6 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Add -2 times row 1 to row 2:

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -1 \\ 0 & 4 & 5 & 6 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Add -4 times row 1 to row 3:

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -3 & -6 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Swap rows 2 and 3:

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & -3 & -6 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Multiply row 2 by -1/3:

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Add -1 times row 2 to row 4:

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Multiply row 3 by -1:

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Add -3 times row 3 to row 4:

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Add -2 times row 3 to row 2:

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Add -3 times row 3 to row 1:

$$\begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Add -2 times row 2 to row 1:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The nonzero rows in the reduced row-echelon

$$\left\{ \begin{matrix} [\ 0, & 1, & 0, & 0\ ], \\ [\ 0, & 0, & 1, & 0\ ], \\ [\ 0, & 0, & 0, & 1\ ] \end{matrix} \right\}$$

Notes

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Calculating column Space

We begin with the matrix:

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 5 \\ 0 & 4 & 5 & 6 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Add -2 times row 1 to row 2:

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -1 \\ 0 & 4 & 5 & 6 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Add -4 times row 1 to row 3:

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -3 & -6 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Swap rows 2 and 3:

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & -3 & -6 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Multiply row 2 by -1/3:

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Add -1 times row 2 to row 4:

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Multiply row 3 by -1:

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Add -3 times row 3 to row 4:

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Add -2 times row 3 to row 2:

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Add -3 times row 3 to row 1:

$$\begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Add -2 times row 2 to row 1:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 6 \\ 5 \end{bmatrix} \right\}$$

The columns containing leading ones are the pivot columns. To obtain a basis for the column space, we just use the pivot columns from the original matrix:

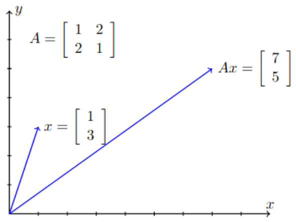
Notes

Eigen Values and Eigen Vectors, Diagonalization

Notes

Matrix Operations

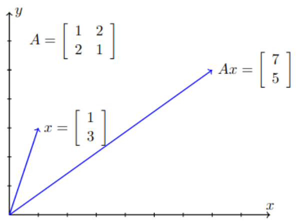
What happens when a matrix operates on a vector?



Notes

Matrix Operations

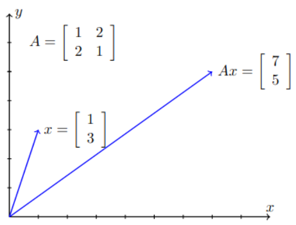
What happens when a matrix operates on a vector?
The vector gets transformed into a new vector (it strays from its path)



Notes

Matrix Operations

What happens when a matrix operates on a vector?
The vector gets transformed into a new vector (it strays from its path)
The vector may also get scaled (elongated or shortened) in the process.



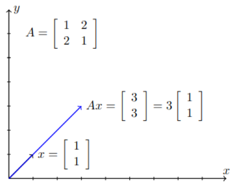
Notes

Eigen Values

- For a given square matrix A, there exist special vectors which refuse to stray from their path.
- These vectors are called eigenvectors.
- More formally,

$$Ax = \lambda x$$

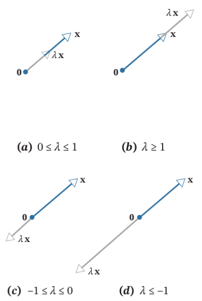
[direction remains the same]



Notes

Eigen Values

- In $\mathbb{R}^2$ or $\mathbb{R}^3$ multiplication by A maps each eigenvector x of A along the same line through the origin as x.
- Depending on the sign and magnitude of the eigenvalue λ corresponding to x,
- the operation $Ax = \lambda x$ compresses or stretches x by a factor of λ,
- Reversal of direction in the case where λ is negative



Notes

Eigen Values

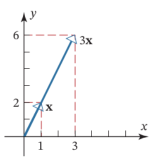
The vector $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

corresponding to the eigenvalue $\lambda = 3$, since

$$A\mathbf{x} = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\mathbf{x}$$

Geometrically, multiplication by A has stretched the vector $\mathbf{x}$ by a factor of 3



Notes

Characteristic Equation

- The basic equation for eigenvectors and eigenvalues is

$$Ax = \lambda$$

- then

$$(A - \lambda I)x = 0$$

- So, the matrix $(A - \lambda I)$ has a nontrivial nulispace, and therefore must be singular.
- So,

$$det(A - \lambda I) = 0.$$

- So, if A is an eigenvalue of A then $det(A - \lambda I) = 0$.

Notes

Example

Eigen values of the matrix

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

Notes

Example

Eigen values of the matrix

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

Solving $(\lambda I - A)x = 0$ we have $\lambda = 3, -1$

$$\begin{bmatrix} \lambda - 3 & 0 \\ -8 & \lambda + 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\lambda = 3 \qquad \begin{bmatrix} 0 & 0 \\ -8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} x_1 = \frac{1}{2}t \\ x_2 = t \end{matrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

$\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda = 3$

Notes

Example 2

Find bases for the eigenspaces of $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

Solution:

- The characteristic equation of matrix A is $\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$, or in factored form, $(\lambda - 1)(\lambda - 2)^2 = 0$; thus, the eigenvalues of A are $\lambda = 1$ and $\lambda = 2$, so there are two eigenspaces of A.

$$(\lambda I - A)\mathbf{x} = \mathbf{0} \Rightarrow \begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda - 2 & -1 \\ -1 & 0 & \lambda - 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If $\lambda = 2$, then (3) becomes $\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Notes

Example 2

Solving the system yield

$$x_1 = -s,\ x_2 = t,\ x_3 = s$$

Thus, the eigenvectors of A corresponding to $\lambda = 2$ are the nonzero vectors of the form

$$\mathbf{x} = \begin{bmatrix} -s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The vectors $[-1\ 0\ 1]^T$ and $[0\ 1\ 0]^T$ are linearly independent and form a basis for the eigenspace corresponding to $\lambda = 2$.

Similarly, the eigenvectors of A corresponding to $\lambda = 1$ are the nonzero vectors of the form $\mathbf{x} = s\ [-2\ 1\ 1]^T$

Thus, $[-2\ 1\ 1]^T$ is a basis for the eigenspace corresponding to $\lambda = 1$.

Notes

Diagonalization

From the definition of the eigenvector v corresponding to the eigenvalue λ we have

$Av = \lambda v$ Then: $Av - \lambda v = (A - \lambda I) \cdot v = 0$

Equation has a nonzero solution if and only if

$\det(A - \lambda I) = 0$

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -1 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & -1 & 3-\lambda \end{vmatrix}$$

$$= -\lambda^3 + 8\lambda^2 - 20\lambda + 16 = -(\lambda - 4)\cdot(\lambda^2 - 4\lambda + 4)$$

$$= -(\lambda - 4)\cdot(\lambda - 2)^2 = 0$$

$\lambda_1 = 4 \quad \lambda_2 = 2$

Notes

Diagonalization

For every λ we find its own vectors:

$\lambda_1 = 4$

$$A - \lambda_1 I = \begin{pmatrix} -1 & -1 & 1 \\ 0 & -2 & 0 \\ 1 & -1 & -1 \end{pmatrix}$$

$Av = \lambda v \quad (A - \lambda I)\cdot v = 0$

$\lambda_2 = 2$

$$A - \lambda_2 I = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

$Av = \lambda v \quad (A - \lambda I)\cdot v = 0$

The solution set: $\{x_3 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\}$: $\{x_2 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\}$

Notes

Diagonalization

- The diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \lambda_3$

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

- The matrix with the Eigenvectors v_1, v_2, v_3 as its columns.

$$P = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

- hence

$$A = PDP^{-1}$$
$$A^2 = PDP^{-1}PDP^{-1}$$
$$A^2 = PD.DP^{-1}$$

Notes

Diagonalization

- The diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \lambda_3$

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

- The matrix with the Eigenvectors v_1, v_2, v_3 as its columns.

$$P = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

- Verify $A = P^{-1}DP$
- Verify $D = P^{-1}AP$
- Find A^{100}

Notes

Inner Product Spaces

Notes

Definition of inner product

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- (Inner product) Let V be a vector space over $\mathbb{R}$. An inner product $<,>$ is a function $V \times V \to \mathbb{R}$ with the following properties
- $\forall\ u\ \in\ V,\ \ <u,u>\geq 0,\ and\ \ <u,u>=0\ iff\ u=0;$
- $\forall\ u,v\ \in\ V,\ holds\ <u,v>=<v,u>$
- $\forall\ u,v\ \in\ V,\ and\ \forall\ a,b\ \in\mathbb{R}$

$$<au+bv,w>=a<u,w>+b<v,w>$$

- Notation: V together with $<,>$ is called an inner product space.

Notes

Definition of inner product

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$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

If V is a real inner product space, then the ***norm*** (or ***length***) of a vector $\mathbf{v}$ in V is denoted by $\|\mathbf{v}\|$ and is defined by

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

and the ***distance*** between two vectors is denoted by $d(\mathbf{u}, \mathbf{v})$ and is defined by

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle}$$

A vector of norm 1 is called a ***unit vector***.

Notes

Weighted Euclidean Inner Product

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- Let $u=(u_1,u_2)$ and $v=(v_1,v_2)$ be vectors in $\mathbb{R}^2$. Verify that the weighted Euclidean inner product

$$<u,v>=3u_1v_1+2u_2v_2$$

satisfies the inner product axioms.

Notes

Example

Consider

$$\begin{aligned}\langle u-2v,3u+4v\rangle =&\langle u,3u+4v\rangle-\langle 2v,3u+4v\rangle\\ =&\langle u,3u\rangle+\langle u,4v\rangle-\langle 2v,3u\rangle-\langle 2v,4v\rangle\\ =&3\langle u,u\rangle+4\langle u,v\rangle-6\langle v,u\rangle-8\langle v,v\rangle\end{aligned}$$

Notes

Cauchy-Schwarz Inequality

Theorem Suppose $\langle \mathbf{x},\mathbf{y}\rangle$ is an inner product on a vector space V. Then

$$\langle \mathbf{x},\mathbf{y}\rangle^2 \le \langle \mathbf{x},\mathbf{x}\rangle\langle \mathbf{y},\mathbf{y}\rangle \quad \text{for all } \mathbf{x},\mathbf{y}\in V.$$

Proof: For any $t\in\mathbb{R}$ let $\mathbf{v}_t=\mathbf{x}+t\mathbf{y}$. Then

$$\langle \mathbf{v}_t,\mathbf{v}_t\rangle=\langle \mathbf{x},\mathbf{x}\rangle+2t\langle \mathbf{x},\mathbf{y}\rangle+t^2\langle \mathbf{y},\mathbf{y}\rangle.$$

The right-hand side is a quadratic polynomial in t (provided that $\mathbf{y}\neq\mathbf{0}$). Since $\langle \mathbf{v}_t,\mathbf{v}_t\rangle\ge 0$ for all t, the discriminant D is nonpositive. But $D=4\langle \mathbf{x},\mathbf{y}\rangle^2-4\langle \mathbf{x},\mathbf{x}\rangle\langle \mathbf{y},\mathbf{y}\rangle$.

Cauchy-Schwarz Inequality:

$$|\langle \mathbf{x},\mathbf{y}\rangle|\le\sqrt{\langle \mathbf{x},\mathbf{x}\rangle}\,\sqrt{\langle \mathbf{y},\mathbf{y}\rangle}.$$

Notes

Cauchy-Schwarz Inequality

Cauchy-Schwarz Inequality:

$$|\langle \mathbf{x},\mathbf{y}\rangle|\le\sqrt{\langle \mathbf{x},\mathbf{x}\rangle}\,\sqrt{\langle \mathbf{y},\mathbf{y}\rangle}.$$

Corollary 1 $|\mathbf{x}\cdot\mathbf{y}|\le|\mathbf{x}|\,|\mathbf{y}|$ for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$.
Equivalently, for all $x_i,y_i\in\mathbb{R}$,

$$(x_1y_1+\cdots+x_ny_n)^2\le(x_1^2+\cdots+x_n^2)(y_1^2+\cdots+y_n^2).$$

Corollary 2 For any $f,g\in C[a,b]$,

$$\left(\int_a^b f(x)g(x)\,dx\right)^2\le\int_a^b|f(x)|^2\,dx\cdot\int_a^b|g(x)|^2\,dx.$$

Notes

Norms induced by inner products

Examples. • The length of a vector in $\mathbb{R}^n$,

$$|\mathbf{x}|=\sqrt{x_1^2+x_2^2+\cdots+x_n^2},$$

is the norm induced by the dot product

$$\mathbf{x}\cdot\mathbf{y}=x_1y_1+x_2y_2+\cdots+x_ny_n.$$

• The norm $\|f\|_2=\left(\int_a^b|f(x)|^2\,dx\right)^{1/2}$ on the vector space $C[a,b]$ is induced by the inner product

$$\langle f,g\rangle=\int_a^b f(x)g(x)\,dx.$$

Notes

Angle

Since $|\langle \mathbf{x}, \mathbf{y}\rangle| \le \|\mathbf{x}\|\,\|\mathbf{y}\|$, we can define the *angle* between nonzero vectors in any vector space with an inner product (and induced norm):

$$\angle(\mathbf{x}, \mathbf{y}) = \arccos \frac{\langle \mathbf{x}, \mathbf{y}\rangle}{\|\mathbf{x}\|\,\|\mathbf{y}\|}.$$

Then $\langle \mathbf{x}, \mathbf{y}\rangle = \|\mathbf{x}\|\,\|\mathbf{y}\| \cos \angle(\mathbf{x}, \mathbf{y})$.

In particular, vectors $\mathbf{x}$ and $\mathbf{y}$ are **orthogonal** (denoted $\mathbf{x} \perp \mathbf{y}$) if $\langle \mathbf{x}, \mathbf{y}\rangle = 0$.

Notes

Angle

If $\mathbf{u} = U$ and $\mathbf{v} = V$ are matrices in the vector space M_{nn}, then the formula

$$\boxed{\langle \mathbf{u}, \mathbf{v}\rangle = \mathrm{tr}(U^T V)}$$

$$U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$$

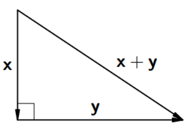
$$\langle \mathbf{u}, \mathbf{v}\rangle = \mathrm{tr}(U^T V) = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

Let M_{22} have the standard inner product. Find the cosine of the angle between the vectors

$$\mathbf{u} = U = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = V = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$$

Notes

Pythagorean Law



Pythagorean Law:

$$\mathbf{x} \perp \mathbf{y} \implies \|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$$

Proof: $\|\mathbf{x}+\mathbf{y}\|^2 = \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle$
$= \langle \mathbf{x}, \mathbf{x}\rangle + \langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{y}, \mathbf{x}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle$
$= \langle \mathbf{x}, \mathbf{x}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$.

Notes

Orthogonal sets

Let V be an inner product space with an inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$.

Definition. A nonempty set $S \subset V$ of nonzero vectors is called an **orthogonal set** if all vectors in S are mutually orthogonal. That is, $\mathbf{0} \notin S$ and $\langle \mathbf{x}, \mathbf{y}\rangle = 0$ for any $\mathbf{x}, \mathbf{y} \in S$, $\mathbf{x} \ne \mathbf{y}$.

An orthogonal set $S \subset V$ is called **orthonormal** if $\|\mathbf{x}\| = 1$ for any $\mathbf{x} \in S$.

Remark. Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$ form an orthonormal set if and only if

$$\langle \mathbf{v}_i, \mathbf{v}_j\rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}$$

Notes

Orthogonal sets

The vectors $\mathbf{u} = (1, 1)$ and $\mathbf{v} = (1, -1)$ are orthogonal

$$U = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

Notes

Let P_2 have the inner product $\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)\,dx$

and let $\mathbf{p} = x$ and $\mathbf{q} = x^2$. Then

$$\|\mathbf{p}\| = \langle \mathbf{p}, \mathbf{p} \rangle^{1/2} = \left[\int_{-1}^{1} xx\,dx\right]^{1/2} = \left[\int_{-1}^{1} x^2\,dx\right]^{1/2} = \sqrt{\frac{2}{3}}$$

$$\|\mathbf{q}\| = \langle \mathbf{q}, \mathbf{q} \rangle^{1/2} = \left[\int_{-1}^{1} x^2x^2\,dx\right]^{1/2} = \left[\int_{-1}^{1} x^4\,dx\right]^{1/2} = \sqrt{\frac{2}{5}}$$

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} xx^2\,dx = \int_{-1}^{1} x^3\,dx = 0$$

Because $\langle \mathbf{p}, \mathbf{q} \rangle = 0$, the vectors $\mathbf{p} = x$ and $\mathbf{q} = x^2$ are orthogonal

$$\|\mathbf{p} + \mathbf{q}\|^2 = \left(\sqrt{\frac{2}{3}}\right)^2 + \left(\sqrt{\frac{2}{5}}\right)^2 = \frac{2}{3} + \frac{2}{5} = \frac{16}{15}$$

$$\|\mathbf{p} + \mathbf{q}\|^2 = \langle \mathbf{p} + \mathbf{q}, \mathbf{p} + \mathbf{q} \rangle = \int_{-1}^{1} (x + x^2)(x + x^2)\,dx$$

$$= \int_{-1}^{1} x^2\,dx + 2\int_{-1}^{1} x^3\,dx + \int_{-1}^{1} x^4\,dx = \frac{2}{3} + 0 + \frac{2}{5} = \frac{16}{15}$$

Notes

Orthogonal set

Let $\mathbf{v}_1 = (0, 1, 0)$, $\mathbf{v}_2 = (1, 0, 1)$, $\mathbf{v}_3 = (1, 0, -1)$
and assume that R^3 has the Euclidean inner product.
$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set since $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \langle \mathbf{v}_1, \mathbf{v}_3 \rangle = \langle \mathbf{v}_2, \mathbf{v}_3 \rangle = 0$.

$$\|\mathbf{v}_1\| = 1, \quad \|\mathbf{v}_2\| = \sqrt{2}, \quad \|\mathbf{v}_3\| = \sqrt{2}$$

Consequently, normalizing $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ yields

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = (0, 1, 0), \quad \mathbf{u}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right),$$

$$\mathbf{u}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \langle \mathbf{u}_1, \mathbf{u}_3 \rangle = \langle \mathbf{u}_2, \mathbf{u}_3 \rangle = 0 \quad \text{and} \quad \|\mathbf{u}_1\| = \|\mathbf{u}_2\| = \|\mathbf{u}_3\| = 1$$

Notes

Orthogonal projection

Theorem Let V be an inner product space and V_0 be a finite-dimensional subspace of V. Then any vector $\mathbf{x} \in V$ is uniquely represented as $\mathbf{x} = \mathbf{p} + \mathbf{o}$, where $\mathbf{p} \in V_0$ and $\mathbf{o} \perp V_0$.

The component $\mathbf{p}$ is the **orthogonal projection** of the vector $\mathbf{x}$ onto the subspace V_0. We have

$$\|\mathbf{o}\| = \|\mathbf{x} - \mathbf{p}\| = \min_{\mathbf{v} \in V_0} \|\mathbf{x} - \mathbf{v}\|.$$

That is, the distance from $\mathbf{x}$ to the subspace V_0 is $\|\mathbf{o}\|$.

Notes

Orthogonal projection

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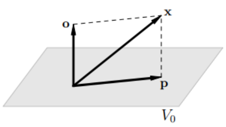

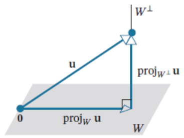

Notes

Orthogonal projection

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- Let $\mathbb{R}^3$ have the Euclidean inner product, and let W be the subspace spanned by the orthonormal vectors

$$v_1 = (0, 1, 0) and v_2 = (-45, 0, 35).$$

- Find the orthogonal projection of $u = (1, 1, 1)$ on W

$$\begin{aligned} proj_W u &=< u, v_1 > v_1 + < u, v_2 > v_2 \\ &= (1)(0, 1, 0) + (\frac{-1}{5})(\frac{-4}{5}, 0, \frac{3}{5}) \\ &= (\frac{4}{25}, 1, \frac{3}{25}) \end{aligned}$$

Notes

The Gram-Schmidt orthogonalization process

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Let V be a vector space with an inner product. Suppose $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ is a basis for V. Let

$$\mathbf{v}_1 = \mathbf{x}_1,$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1,$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2,$$

....

$$\mathbf{v}_n = \mathbf{x}_n - \frac{\langle \mathbf{x}_n, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 - \cdots - \frac{\langle \mathbf{x}_n, \mathbf{v}_{n-1} \rangle}{\langle \mathbf{v}_{n-1}, \mathbf{v}_{n-1} \rangle}\mathbf{v}_{n-1}.$$

Then $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is an orthogonal basis for V.

Notes

The Gram-Schmidt orthogonalization process

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The Gram–Schmidt Process

To convert a basis $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$, perform the following computations:

Step 1. $\mathbf{v}_1 = \mathbf{u}_1$

Step 2. $\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$

Step 3. $\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$

Step 4. $\mathbf{v}_4 = \mathbf{u}_4 - \frac{\langle \mathbf{u}_4, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_4, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 - \frac{\langle \mathbf{u}_4, \mathbf{v}_3 \rangle}{\|\mathbf{v}_3\|^2}\mathbf{v}_3$

$\vdots$

(continue for r steps)

Notes

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The Gram-Schmidt orthogonalization process

Assume that the vector space R^3 has the Euclidean inner product. Apply the Gram–Schmidt process to transform the basis vectors

$$\mathbf{u}_1 = (1,1,1), \quad \mathbf{u}_2 = (0,1,1), \quad \mathbf{u}_3 = (0,0,1)$$

into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, and then normalize the orthogonal basis vectors to obtain an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$.

Notes

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The Gram-Schmidt orthogonalization process

Step 1. $\mathbf{v}_1 = \mathbf{u}_1 = (1,1,1)$

Step 2. $\mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{W_1}\mathbf{u}_2 = \mathbf{u}_2 - \dfrac{\langle \mathbf{u}_2, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$

$$= (0,1,1) - \frac{2}{3}(1,1,1) = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

Step 3. $\mathbf{v}_3 = \mathbf{u}_3 - \text{proj}_{W_2}\mathbf{u}_3 = \mathbf{u}_3 - \dfrac{\langle \mathbf{u}_3, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \dfrac{\langle \mathbf{u}_3, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$

$$= (0,0,1) - \frac{1}{3}(1,1,1) - \frac{1/3}{2/3}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$= \left(0, -\frac{1}{2}, \frac{1}{2}\right)$$

$$\mathbf{v}_1 = (1,1,1), \quad \mathbf{v}_2 = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right), \quad \mathbf{v}_3 = \left(0, -\frac{1}{2}, \frac{1}{2}\right)$$

form an orthogonal basis for R^3.

The norms of these vectors are

$$\|\mathbf{v}_1\| = \sqrt{3}, \quad \|\mathbf{v}_2\| = \frac{\sqrt{6}}{3}, \quad \|\mathbf{v}_3\| = \frac{1}{\sqrt{2}}$$

so an orthonormal basis for R^3 is

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right),$$

$$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right),$$

$$\mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Notes

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QR-Decomposition

- If A is an m × n matrix with linearly independent column vectors, then A can be factored as

$$A = QR$$

where Q is an $m \times n$ matrix with orthonormal column vectors, and R is an $n \times n$ invertible upper triangular matrix.

Notes

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QR-Decomposition

Find a QR-decomposition of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

The column vectors of A are

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{q}_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \quad \mathbf{q}_2 = \begin{bmatrix} -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, \quad \mathbf{q}_3 = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$R = \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1\rangle & \langle \mathbf{u}_2, \mathbf{q}_1\rangle & \langle \mathbf{u}_3, \mathbf{q}_1\rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2\rangle & \langle \mathbf{u}_3, \mathbf{q}_2\rangle \\ 0 & 0 & \langle \mathbf{u}_3, \mathbf{q}_3\rangle \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Notes