National University
of computer and emerging sciences

DISCRETE STRUCTURES

COURSE INSTRUCTOR: MUHAMMAD SAIF UL ISLAM

Course Outline

- **Logic and Proofs** (Chapter 1)
- **Sets and Functions** (Chapter 2)
- Relations
- Number Theory
- Combinatorics and Recurrence
- Graphs
- Trees
- Discrete Probability

Lecture Outline

- Definition of sets
- Describing Sets
  - Roster Method
  - Set-Builder Notation
- Some Important Sets in Mathematics
- Empty Set and Universal Set
- Subsets and Set Equality
- Cardinality of Sets
- Tuples
- Cartesian Product

Continued…

Lecture Outline

- Set Operations
  - Union
  - Intersection
  - Complementation
  - Difference
- More on Set Cardinality
- Set Identities
- Proving Identities
- Membership Tables

Introduction

Sets are one of the basic building blocks for the types of objects considered in discrete mathematics.

- Important for counting.
- Programming languages have set operations.

Set theory is an important branch of mathematics.

- Many different systems of axioms have been used to develop set theory.
- Here we are not concerned with a formal set of axioms for set theory. Instead, we will use what is called naïve set theory.

Sets

A *set* is an unordered, well defined and distinct collection of objects.

- the students in this class
- the chairs in this room

The objects in a set are called the ***elements***, or ***members*** of the set. A set is said to *contain* its elements.

The notation $a \in A$ denotes that a is an element of the set A.

If a is not a member of A, write $a \notin A$

Describing a Set: Roster Method

$S = \{a,b,c,d\}$

Order not important

$$S = \{a,b,c,d\} = \{b,c,a,d\}$$

Each distinct object is either a member or not; listing more than once does not change the set.

$$S = \{a,b,c,d\} = \{a,b,c,b,c,d\}$$

Elipses (…) may be used to describe a set without listing all of the members when the pattern is clear.

$$S = \{a,b,c,d, \ldots\ldots,z\}$$

Roster Method

Set of all vowels in the English alphabet:

$$V = \{a,e,i,o,u\}$$

Set of all odd positive integers less than 10:

$$O = \{1,3,5,7,9\}$$

Set of all positive integers less than 100:

$$S = \{1,2,3,\ldots\ldots,99\}$$

Set of all integers less than 0:

$$S = \{\ldots., -3,-2,-1\}$$

Some Important Sets

N = *natural numbers* = {0,1,2,3….}

Z = *integers* = {…,-3,-2,-1,0,1,2,3,…}

$\mathbf{Z}^+$ = *positive integers* = {1,2,3,…..}

R = set of *real numbers*

$\mathbf{R}^+$ = set of *positive real numbers*

C = set of *complex numbers*.

Q = set of rational numbers

Set-Builder Notation

Specify the property or properties that all members must satisfy:

$S = \{x \mid x \text{ is a positive integer less than } 100\}$

$O = \{x \mid x \text{ is an odd positive integer less than } 10\}$

$O = \{x \in \mathbf{Z}^{+} \mid x \text{ is odd and } x < 10\}$

A predicate may be used:

$$S = \{x \mid \mathrm{P}(x)\}$$

Example: $S = \{x \mid \mathrm{Prime}(x)\}$

Positive rational numbers:

$$\mathbf{Q}^{+} = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p,q\}$$

Interval Notation

$$[a,b] = \{x \mid a \leq x \leq b\}$$

$$[a,b) = \{x \mid a \leq x < b\}$$

$$(a,b] = \{x \mid a < x \leq b\}$$

$$(a,b) = \{x \mid a < x < b\}$$

closed interval [a,b]

open interval (a,b)

Universal Set and Empty Set

The ***universal set U*** is the set containing everything currently under consideration.

- Sometimes implicit
- Sometimes explicitly stated.
- Contents depend on the context.

The **empty set** is the set with no elements.

Symbolized ∅, but

{} also used.

A set with one element is called a **singleton set**.

Some things to remember

Sets can be elements of sets.

$X = \{\{1,2,3\}, a, \{b,c\}\}$

$Y = \{\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}\}$

The **empty set** is different from a set containing the empty set.

$\emptyset \neq \{ \emptyset \}$

Venn Diagram

- ➢Sets can be represented **graphically** using Venn diagrams, named after the English mathematician **John Venn**
- ➢In Venn diagrams the **universal set** *U,* which contains all the objects under consideration, is represented by a rectangle.
- ➢Draw a Venn diagram that represents *V,* the set of vowels in the English alphabet.

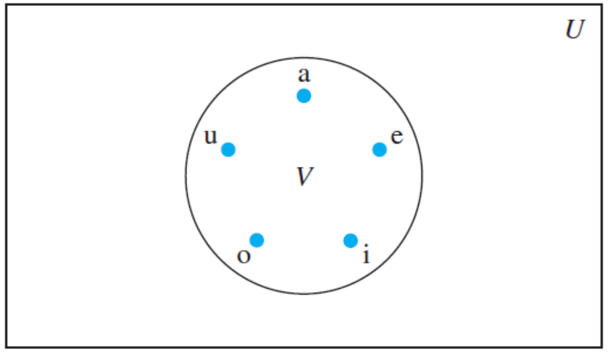


FIGURE 1 **Venn diagram for the set of vowels.**

John Venn (1834-1923)
Cambridge, UK

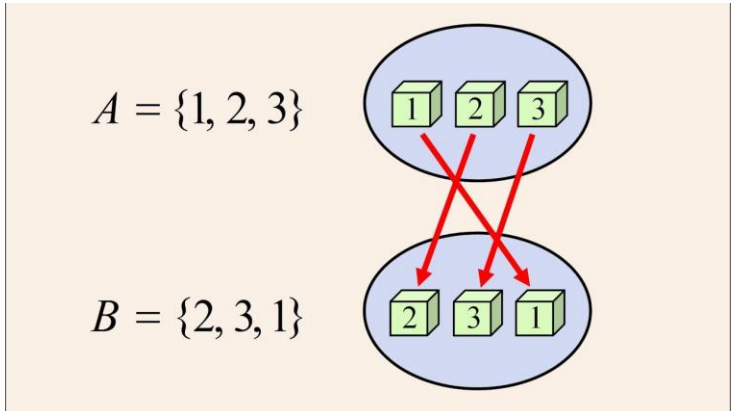
Set Equality

Definition: Two sets are *equal* if and only if they have the same elements.

- Therefore if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$
- We write *A* = *B* if *A* and *B* are equal sets.

$$\{1,2,3\} = \{2, 3, 1\}$$

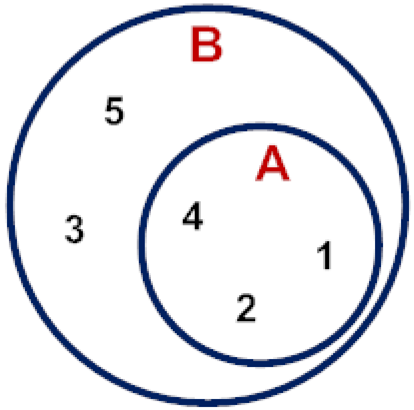
$$\{1,5,5,5,3,3,1\} = \{1,3,5\}$$



Subsets

Definition: The set A is a *subset* of B, if and only if every element of A is also an element of B.

- The notation $A \subseteq B$ is used to indicate that A is a subset of the set B.
- $A \subseteq B$ holds if and only if $\forall x(x \in A \to x \in B)$ is true.
  1. Because $a \in \emptyset$ is always false, $\emptyset \subseteq S$,for every set S.
  2. Because $a \in S \to a \in S$, $S \subseteq S$, for every set S.



Showing a Set is or is not a Subset of Another Set

Showing that A is a Subset of B: To show that $A \subseteq B$, show that if x belongs to A, then x also belongs to B.

Showing that A is not a Subset of B: To show that A is not a subset of B, $A \nsubseteq B$, find an element $x \in A$ with $x \notin B$. (Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.)

Examples:

1. The set of all computer science majors at your school is a subset of all students at your school.
2. The set of integers with squares less than 100 is not a subset of the set of nonnegative integers.

Another look at Equality of Sets

Recall that two sets *A* and *B* are *equal*, denoted by *A* = *B*, iff

$$\forall x(x \in A \leftrightarrow x \in B)$$

Using logical equivalences we have that *A* = *B* iff

$$\forall x[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$$

This is equivalent to

$A \subseteq B$ **and** $B \subseteq A$

Proper Subsets

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset* of B, denoted by $A \subset B$. If $A \subset B$, then

$$\forall x(x \in A \rightarrow x \in B) \land \exists x(x \in B \land x \notin A)$$

is true.

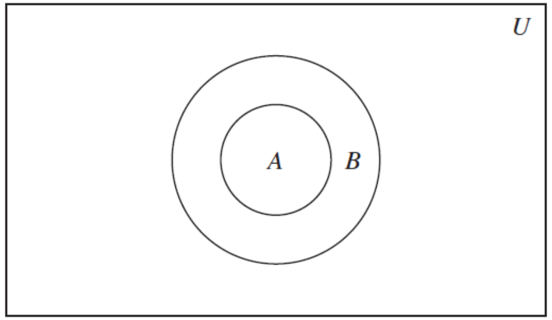


FIGURE 2 Venn diagram showing that A is a subset of B.

Set Cardinality

Definition: If there are exactly n distinct elements in *S* where *n* is a nonnegative integer, we say that *S* is *finite*. Otherwise it is *infinite*.

Definition: The *cardinality* of a finite set *A,* denoted by $|A|$, is the number of (distinct) elements of *A*.

Examples:

1. $|ø| = 0$
2. Let S be the letters of the English alphabet. Then $|S| = 26$
3. $|\{1,2,3\}| = 3$
4. $|\{ø\}| = 1$
5. The set of integers is infinite.

Set Cardinality - Exercise

Q: Compute each cardinality

1. |{1, -13, 4, -13, 1}|
2. |{3, {1,2,3,4}, ∅}|
3. |{}|
4. |{ {}, {{}}, {{{}}} }|

$A = \{2, 3, 5, 7, 11, 13, 17, 19\}; \ |A$

$A = N$ (natural numbers); $|N| =$

$A = Q$ (rational numbers); $|Q| =$

$A = \{2n \mid n$ is an integer$\}; \ |A| =$
(the set of even integers)

Power Sets

Definition: The set of all subsets of a set *A*, denoted P(*A*), is called the *power set* of *A*.

Example: If $A = \{a,b\}$ then

$$\mathcal{P}(A) = \{\emptyset, \{a\},\{b\},\{a,b\}\}$$

$$\mathcal{S} = \{A, B, C\}$$

$$P(\mathcal{S}) = \Big\{\{\}, \{A\}, \{B\}, \{C\}, \{A, B\}, \{A, C\}, \{B, C\}, \{A, B, C\}\Big\}$$

If a set has *n* elements, then the cardinality of the power set is 2^n. (In Chapters 5 and 6, we will discuss different ways to show this.)

Tuples

- The *ordered n-tuple* $(a_1, a_2, \ldots, a_n)$ is the **ordered collection** that has a_1 as its first element and a_2 as its second element and so on until a_n as its last element.
- Two n-tuples are equal if and only if their corresponding elements are equal.
- 2-tuples are called *ordered pairs*.
- The ordered pairs (a,b) and (c,d) are equal if and only if $a = c$ and $b = d$.

Cartesian Product

René Descartes
(1596-1650)

Definition: The *Cartesian Product* of two sets *A* and *B*, denoted by $A \times B$ is the set of ordered pairs (a,b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a, b)|a \in A \wedge b \in B\}$$

Example:

$A = \{a,b\}$ $B = \{1,2,3\}$

$A \times B = \{(a,1),(a,2),(a,3), (b,1),(b,2),(b,3)\}$

Definition: A subset *R* of the Cartesian product $A \times B$ is called a *relation* from the set A to the set B. (Relations will be covered in depth in Chapter 9.)

Cartesian Product

Definition: The cartesian products of the sets $A_1, A_2, \ldots\ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots\ldots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots\ldots, a_n)$ where a_i belongs to A_i for $i = 1, \ldots\ n$.

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Example: What is $A \times B \times C$ where $A = \{0,1\}$, $B = \{1,2\}$ and $C = \{0,1,2\}$

Solution: $A \times B \times C$ = {(0,1,0), (0,1,1), (0,1,2),(0,2,0), (0,2,1), (0,2,2),(1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,1,2)}

Truth Sets of Quantifiers

Given a predicate *P* and a domain *D*, we define the *truth set* of *P* to be the set of elements in *D* for which *P*(*x*) is true. The truth set of *P*(x) is denoted by

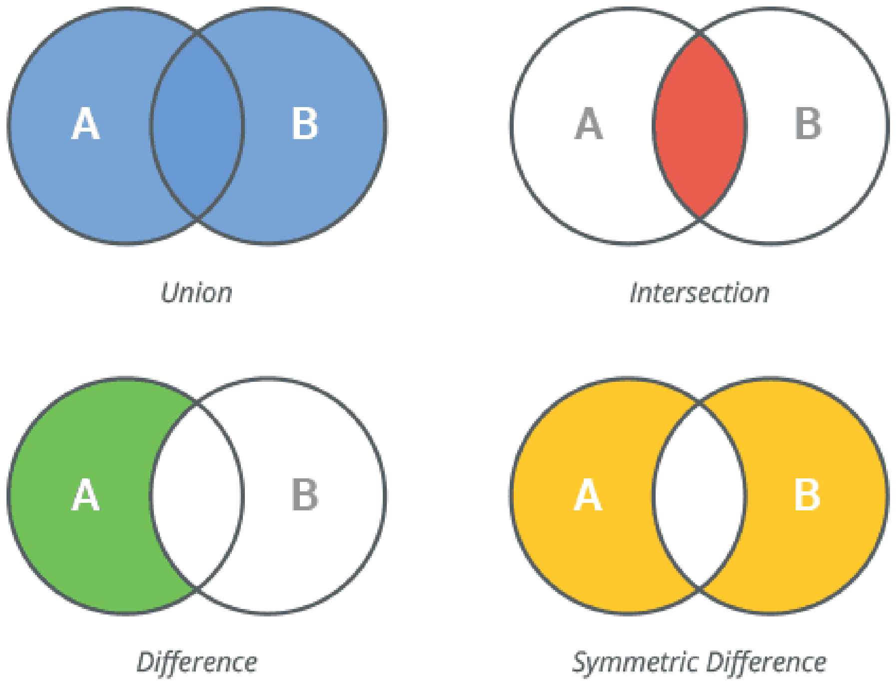
$$\{x \in D | P(x)\}$$

Example: The truth set of *P*(*x*) where the domain is the integers and

P(*x*) is "|*x*| = 1" is the set {-1,1}

Set Operations

- Set Operations
  - Union
  - Intersection
  - Complementation
  - Difference
- More on Set Cardinality
- Set Identities
- Proving Identities
- Membership Tables



Union

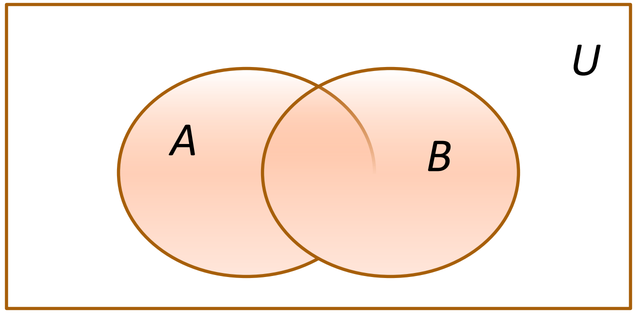
Definition: Let *A* and *B* be sets. The *union* of the sets *A* and *B*, denoted by $A \cup B$, is the set:

$$\{x | x \in A \lor x \in B\}$$

Example: What is $\{1,2,3\} \cup \{3, 4, 5\}$?

Solution: $\{1,2,3,4,5\}$

Venn Diagram for $A \cup B$



Intersection

Definition: The *intersection* of sets *A* and *B*, denoted by $A \cap B$, is

$$\{x | x \in A \land x \in B\}$$

Note if the intersection is empty, then *A* and *B* are said to be *disjoint*.

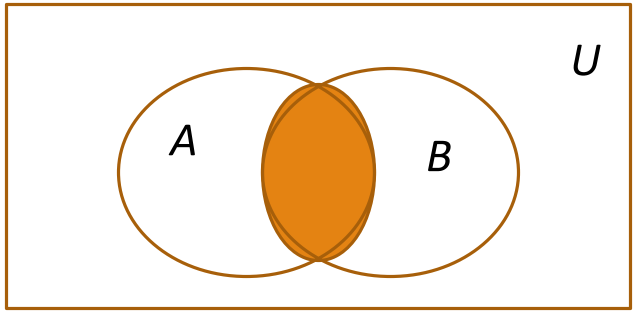
Example: What is? $\{1,2,3\} \cap \{3,4,5\}$?

Solution: $\{3\}$

Example: What is? $\{1,2,3\} \cap \{4,5,6\}$?

Solution: $\emptyset$

Venn Diagram for $A \cap B$



Complement

Definition: If A is a set, then the complement of the A (with respect to U), denoted by $\bar{A}$ is the set $U - A$

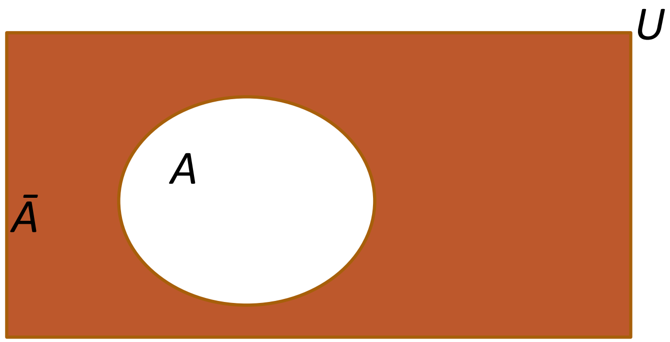
$$\bar{A} = \{x \in U \mid x \notin A\}$$

(The complement of A is sometimes denoted by A^c .)

Example: If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$

Solution: $\{x \mid x \leq 70\}$

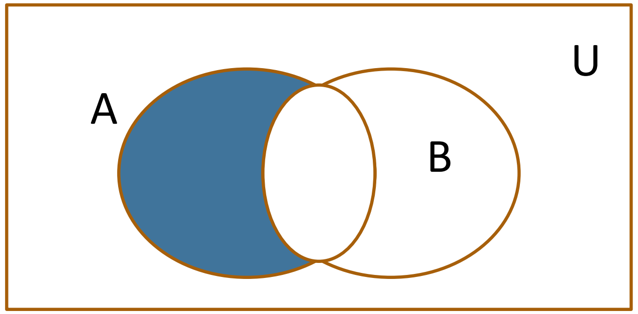
Venn Diagram for Complement



Difference

Definition: Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing the elements of A that are not in B. The difference of A and B is also called the complement of B with respect to A.

$$A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \overline{B}$$

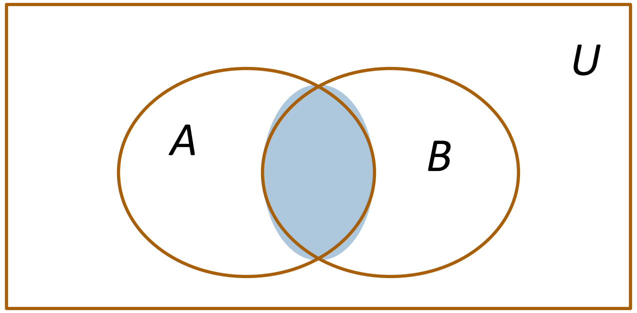


Venn Diagram for $A - B$

The Cardinality of the Union of Two Sets

- Inclusion-Exclusion

$|A \cup B| = |A| + |B| - |A \cap B|$



Venn Diagram for A, B, $A \cap B$, $A \cup B$

- **Example**: Let A be the math majors in your class and B be the CS majors. To count the number of students who are either math majors or CS majors, add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.
- We will return to this principle in Chapter 6 and Chapter 8 where we will derive a formula for the cardinality of the union of n sets, where n is a positive integer.

Review Questions

Example: $U = \{0,1,2,3,4,5,6,7,8,9,10\}$ $A = \{1,2,3,4,5\}$, $B = \{4,5,6,7,8\}$

1. $A \cup B$

   Solution: $\{1,2,3,4,5,6,7,8\}$

2. $A \cap B$

   Solution: $\{4,5\}$

3. $\bar{A}$

   Solution: $\{0,6,7,8,9,10\}$

4. $\bar{B}$

   Solution: $\{0,1,2,3,9,10\}$

5. $A - B$

   Solution: $\{1,2,3\}$

6. $B - A$

   Solution: $\{6,7,8\}$

Symmetric Difference (*optional*)

Definition: The *symmetric difference* of **A** and **B**, denoted by $A \oplus B$ is the set

$$(A - B) \cup (B - A)$$

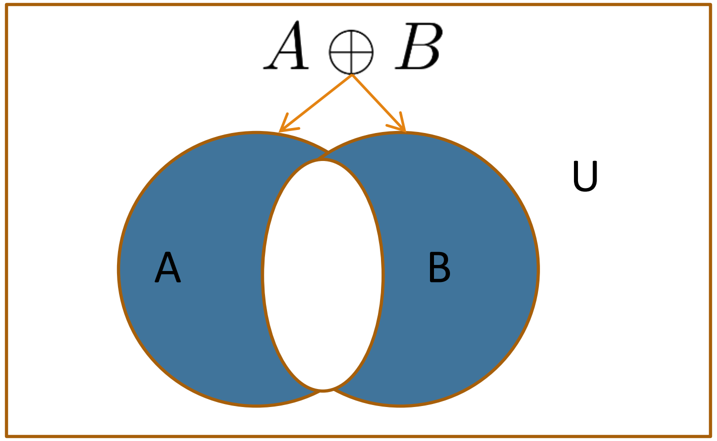
Example:

$U = \{0,1,2,3,4,5,6,7,8,9,10\}$

$A = \{1,2,3,4,5\}$ $B = \{4,5,6,7,8\}$

What is:

- **Solution**: $\{1,2,3,6,7,8\}$



Venn Diagram

Set Identities

TABLE 1 Set Identities.

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law

$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Proving Set Identities

Different ways to prove set identities:

1. Prove that each set (side of the identity) is a subset of the other.
2. Use set builder notation and propositional logic.
3. Membership Tables: Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity. Use 1 to indicate it is in the set and a 0 to indicate that it is not.

Proof of Second De Morgan Law

Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution: We prove this identity by showing that:

1) $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

and

2) $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

Continued on next slide →

Proof of Second De Morgan Law

These steps show that:

$$\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$$

$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	defn. of complement $\bar{A} = \{x \in U \mid x \notin A\}$
$\neg((x \in A) \wedge (x \in B))$	defn. of intersection $\{x \mid x \in A \wedge x \in B\}$
$\neg(x \in A) \vee \neg(x \in B)$	1st De Morgan Law for Prop Logic
$x \notin A \vee x \notin B$	defn. of negation
$x \in \overline{A} \vee x \in \overline{B}$	defn. of complement $\bar{A} = \{x \in U \mid x \notin A\}$
$x \in \overline{A} \cup \overline{B}$	defn. of union $\{x \mid x \in A \vee x \in B\}$

Continued on next slide →

Proof of Second De Morgan Law

These steps show that:

$$\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$$

$x \in \overline{A} \cup \overline{B}$	by assumption
$(x \in \overline{A}) \vee (x \in \overline{B})$	defn. of union
$(x \notin A) \vee (x \notin B)$	defn. of complement
$\neg(x \in A) \vee \neg(x \in B)$	defn. of negation
$\neg((x \in A) \wedge (x \in B))$	by 1st De Morgan Law for Prop Logic
$\neg(x \in A \cap B)$	defn. of intersection
$x \in \overline{A \cap B}$	defn. of complement

Set-Builder Notation: Second De Morgan Law

$$
\begin{array}{rcll}
\overline{A \cap B} & = & \{x | x \notin A \cap B\} & \text{by defn. of complement} \\
 & = & \{x | \neg(x \in (A \cap B))\} & \text{by defn. of does not belong symbol} \\
 & = & \{x | \neg(x \in A \wedge x \in B\} & \text{by defn. of intersection} \\
 & = & \{x | \neg(x \in A) \vee \neg(x \in B)\} & \text{by 1st De Morgan law} \\
 & & & \text{for Prop Logic} \\
 & = & \{x | x \notin A \vee x \notin B\} & \text{by defn. of not belong symbol} \\
 & = & \{x | x \in \overline{A} \vee x \in \overline{B}\} & \text{by defn. of complement} \\
 & = & \{x | x \in \overline{A} \cup \overline{B}\} & \text{by defn. of union} \\
 & = & \overline{A} \cup \overline{B} & \text{by meaning of notation}
\end{array}
$$

Membership Table

Example: Construct a membership table to show that the distributive law holds.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Solution:

A	**B**	**C**	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

Thank you!!!

Understanding Math by reading slides is similar to Learning to swim by watching TV.

So, DO PRACTICE IT!