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Proofs

A proof is a **valid argument** that establishes the truth of a mathematical statement

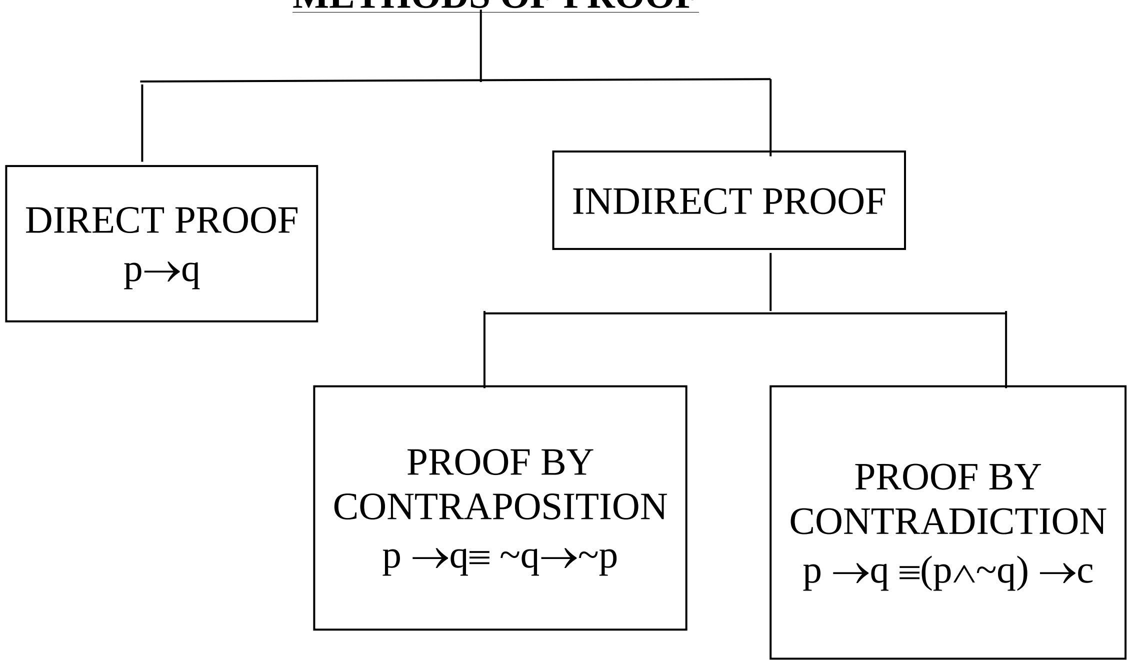
A proof can use the hypotheses of the theorem

theorem is a statement that can be shown to be true.

To understand written mathematics, one must understand what makes up a correct mathematical argument, that is, a **proof**

Many theorems in mathematics are implications, $p \rightarrow q$. The techniques of proving implications give rise to different methods of proofs.

METHODS OF PROOF



DIRECT PROOF:

The implication $p \rightarrow q$ can be proved by showing that if p is true, the q must also be true. This shows that the combination p true and q false never occurs. A proof of this kind is called a direct proof.

p	q	$p\rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

1. An integer n is even if, and only if, $n = 2k$ for some integer k.
2. An integer n is odd if, and only if, $n = 2k + 1$ for some integer k.
3. An integer n is prime if, and only if, $n > 1$ and for all positive integers r and s, if $n = r\cdot s$, then $r = 1$ or $s = 1$.

4. An integer n > 1 is composite if, and only if, $n = r \cdot s$ for some positive integers r and s with $r \neq 1$ and $s \neq 1$.

5. A real number r is rational if, and only if, $r = \frac{a}{b}$ for some integers a and b with $b \neq 0$.

6. If n and d are integers and $d \neq 0$, then d divides n, written d/n if, and only if, $n = d.k$ for some integers k.

7. An integer n is called a perfect square if, and only if, $n = k^2$ for some integer k.

EXERCISE:

Prove that the sum of two odd integers is even.

SOLUTION:

Let **m** and **n** be two odd integers. Then by definition of odd numbers

$m = 2k + 1$ for some $k \in Z$

$n = 2l + 1$ for some $l \in Z$

Now $m + n = (2k + 1) + (2l + 1)$

$= 2k + 2l + 2$

$= 2(k + l + 1)$

$= 2r$ where $r = (k + l + 1) \in Z$

Hence m + n is even.

EXERCISE:

Prove that if n is any even integer, then $(-1)^n = 1$

SOLUTION:

Suppose n is an even integer. Then n = 2k for some integer k.

Now

$$\begin{aligned}(-1)^n &= (-1)^{2k} \\ &= [(-1)^2]^k \\ &= (1)^k \\ &= 1\end{aligned}$$ (proved)

EXERCISE:

Prove that the product of an even integer and an odd integer is even.

SOLUTION:

Suppose m is an even integer and n is an odd integer. Then

$m = 2k$ for some integer k

and $n = 2l + 1$ for some integer l

Now

$$m \cdot n = 2k \cdot (2l + 1)$$
$$= 2 \cdot k\,(2l + 1)$$
$$= 2 \cdot r \quad \text{where } r = k(2l + 1) \text{ is an integer}$$

Hence m·n is even. (Proved)

EXERCISE:

Prove that the square of an even integer is even.

SOLUTION:

Suppose n is an even integer. Then $n = 2k$

Now

$$\text{square of } n = n^2 = (2 \cdot k)^2$$
$$= 4k^2$$
$$= 2 \cdot (2k^2)$$
$$= 2 \cdot p \quad \text{where } p = 2k^2 \in Z$$

Hence, n^2 is even. (proved)

EXERCISE:

Prove that if n is an odd integer, then $n^3 + n$ is even.

SOLUTION:

Let n be an odd integer, then $n = 2k + 1$for some $k \in Z$

Now $n^3 + n = n\,(n^2 + 1)$

$$= (2k + 1)\,((2k+1)^2 + 1)$$
$$= (2k + 1)\,(4k^2 + 4k + 1 + 1)$$
$$= (2k + 1)\,(4k^2 + 4k + 2)$$
$$= (2k + 1)\,2.\,(2k^2 + 2k + 1)$$
$$= 2 \cdot (2k + 1)\,(2k^2 + 2k + 1) \qquad k \in Z$$
$$= \text{an even integer}$$

EXERCISE:

Prove that, if the sum of any two integers is even, then so is their difference.

SOLUTION:

Suppose m and n are integers so that m + n is even. Then by definition of even numbers

$m + n = 2k$ for some integer k

$\Rightarrow \quad m = 2k - n$ ……………….(1)

Now $m - n = (2k - n) - n$ using (1)

$= 2k - 2n$

$= 2(k - n) = 2r$ where $r = k - n$ is an integer

Hence m - n is even.

EXERCISE:

Prove that the sum of any two rational numbers is rational.

SOLUTION:

Suppose r and s are rational numbers.

Then by definition of rational

$$r = \frac{a}{b} \quad \text{and} \quad s = \frac{c}{d}$$

for some integers a, b, c, d with $b \neq 0$ and $d \neq 0$

Now
$$\begin{aligned} r + s &= \frac{a}{b} + \frac{c}{d} \\ &= \frac{ad + bc}{bd} \\ &= \frac{p}{q} \end{aligned}$$

where $p = ad + bc \in Z$ and $q = bd \in Z$ and $q \neq 0$

Hence r + s is rational.

EXERCISE:

Given any two distinct rational numbers r and s with r < s. Prove that there is a rational number x such that r < x < s.

SOLUTION:

Given two distinct rational numbers r and s such that

$r < s$ ………………….(1)

Adding r to both sides of (1), we get

$$r + r < r + s$$
$$2r < r + s$$
$$\Rightarrow \quad r < \frac{r+s}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Next adding s to both sides of (1), we get

$$r + s < s + s$$
$$\Rightarrow \quad r + s < 2s$$
$$\Rightarrow \quad \frac{r+s}{2} < s \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Combining (2) and (3), we may write

$$r < \frac{r+s}{2} < s \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(4)$$

Since the sum of two rationals is rational, therefore r + s is rational. Also the quotient of a rational by a non-zero rational, is rational, therefore $\frac{r+s}{2}$ is rational and by (4) it lies between r & s. Hence, we have found a rational number such that r < x < s. (proved)

EXERCISE:

Prove that for all integers a, b and c, if a|b and b|c then a|c.

PROOF:

Suppose a|b and b|c where a, b, c $\in$Z. Then by definition of divisibility b=a·r and c=b·s for some integers r and s.

Now c = b·s
= (a·r)·s (substituting value of b)
= a·(r·s) (associative law)
= a·k where k = r·s $\in$ Z
$\Rightarrow$ a|c by definition of divisibility

EXERCISE:

Prove that for all integers a, b and c if a|b and a|c then a|(b+c)

PROOF:

Suppose a|b and a|c where a, b, c $\in$Z

By definition of divides

b = a·r and c = a·s for some r, s $\in$Z

Now

$$\begin{aligned} b + c &= a\cdot r + a\cdot s && \text{(substituting values)} \\ &= a\cdot(r+s) && \text{(by distributive law)} \\ &= a\cdot k && \text{where } k = (r + s) \in Z \end{aligned}$$

Hence $a|(b + c)$ by definition of divides.

EXERCISE:

Prove that the sum of any three consecutive integers is divisible by 3.

PROOF:

Let n, n + 1 and n + 2 be three consecutive integers.

Now

$$\begin{aligned} n + (n + 1) + (n + 2) &= 3n + 3 \\ &= 3(n + 1) \\ &= 3\cdot k \quad \text{where } k=(n+1)\in Z \end{aligned}$$

Hence, the sum of three consecutive integers is divisible by 3.

EXERCISE:

Prove the statement:

There is an integer $n > 5$ such that $2^n - 1$ is prime

PROOF:

Here we are asked to show a single integer for which $2^n -1$is prime. First of all we will check the integers from 1 and check whether the answer is prime or not by putting these values in 2^n-1.when we got the answer is prime then we will stop our process of checking the integers and we note that,

Let n = 7, then

$$2^n - 1 = 2^7 - 1 = 128 - 1 = 127$$

and we know that 127 is prime.

EXERCISE:

Prove the statement: There are real numbers a and b such that

$$\sqrt{a+b} = \sqrt{a} + \sqrt{b}$$

PROOF:

Let $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$

Squaring,we get $a + b = a + b + 2\sqrt{a}\sqrt{b}$

$\Rightarrow \quad 0 = 2\sqrt{a}\sqrt{b}$ canceling a+b

$\Rightarrow \quad 0 = \sqrt{ab}$

$\Rightarrow \quad 0 = ab$ squaring

$\Rightarrow$ either a = 0 or b = 0

It means that if we want to find out the integers which satisfy the given condition then one of them must be zero.

Hence if we let a = 0 and b = 3 then

$$R.H.S = \sqrt{a+b} = \sqrt{0+3}$$

$$R.H.S = \sqrt{3}$$

$$Now\ L.H.S = \sqrt{a} + \sqrt{b} \quad by\ putting\ the\ values\ of\ a\ and\ b\ we\ get$$

$$= \sqrt{0} + \sqrt{3}$$

$$L.H.S = \sqrt{3}$$

From above it quite clear that the given condition is satisfied if we take a=0 and b=3.

PROOF BY COUNTER EXAMPLE:

Disprove the statement by giving a counter example.

For all real numbers a and b, if a < b then $a^2 < b^2$.

SOLUTION:

Suppose a = -5 and b = -2

then clearly - 5 < - 2

But $a^2 = (-5)^2 = 25$ and $b^2 = (-2)^2 = 4$

But 25 > 4

This disproves the given statement.

EXERCISE:

Prove or give counter example to disprove the statement.

For all integers n, $n^2 - n + 11$ is a prime number.

SOLUTION:

The statement is not true

For n = 11

we have , $n^2 - n + 11 = (11)^2 - 11 + 11$

$= (11)^2$

$= (11)(11)$

$= 121$

which is obviously not a prime number.

EXERCISE:

Prove or disprove that the product of any two irrational numbers is an irrational number.

SOLUTION:

We know that $\sqrt{2}$ is an irrational number. Now

$$(\sqrt{2})(\sqrt{2}) = (\sqrt{2})^2 = 2 = \frac{2}{1}$$

which is a rational number. Hence the statement is disproved.

EXERCISE:

Find a counter example to the proposition:
For every prime number n, n + 2 is prime.

SOLUTION:

Let the prime number n be 7 then
n + 2 = 7 + 2 = 9
which is not prime.