CS 211: Discrete structures
Fall 2018
Solutions final exam

Question 1 **(Marks: 10)**

Prove using mathematical induction: $\sum_{k=0}^{n}\binom{n}{k} = 2^n$. Also, write down all identities you use.

Solution

Let the predicate P(t): $\sum_{k=0}^{t}\binom{t}{k} = 2^t$

Base case

for t=0

LHS: $\sum_{k=0}^{0}\binom{0}{k} = 1$

RHS: $2^0 = 1$

Hence P(0) is true

Inductive hypothesis: Let P(t) be true for an arbitrary integer t.

Inductive step

We have to prove that P(t)→P(t+1)

That is we have to prove $\sum_{k=0}^{t+1}\binom{t+1}{k} = 2^{t+1}$ is true if P(t) is true

Taking the LHS

$\sum_{k=0}^{t+1}\binom{t+1}{k}$

$= \binom{t+1}{0} + \sum_{k=1}^{t}\binom{t+1}{k} + \binom{t+1}{t+1}$

$= \binom{t+1}{0} + \sum_{k=1}^{t}\left(\binom{t}{k} + \binom{t}{k-1}\right) + \binom{t+1}{t+1}$ (using Pascal's identity)

$= \binom{t}{0} + \sum_{k=1}^{t}\binom{t}{k} + \sum_{k=1}^{t}\binom{t}{k-1} + \binom{t+1}{t+1}$

$= \sum_{k=0}^{t}\binom{t}{k} + \sum_{k=1}^{t}\binom{t}{k-1} + \binom{t+1}{t+1}$ (merge first two terms in the sum)

$= \sum_{k=0}^{t}\binom{t}{k} + \sum_{r=0}^{t-1}\binom{t}{r} + \binom{t}{t}$ (substitute r=k-1 and use $\binom{t+1}{t+1} = \binom{t}{t}$)

$= \sum_{k=0}^{t}\binom{t}{k} + \sum_{r=0}^{t}\binom{t}{r}$ (merge the last two terms)

$= 2^t + 2^t$ (from the inductive hypothesis)

$= 2 * 2^t$

$= 2^{t+1}$

(which is the RHS)

If the inductive hypothesis is true then P(t+1) is also true. Hence proved that the given identity is true for all n≥0

Question 2 **(Marks: 5+5)**

i. Assume that there are 51 different types of animals having integral weights less than 100 kg (starting from one kg) and all weights are unique. Using the concept of Pigeon hole principle, prove that there is a pair whose sum of weights is 100 kg.

Solution

Let $w_1, w_2, w_3, \dots, w_{51}$ be the weights of 51 animals. These can be considered as pigeons. The Pigeon holes can be defined as $B_r = \{r, 100 - r\}, r = 1,2, \dots, 50$. ■

OR

create 50 bins. Bin 1 can hold number 1 or 99, Bin 2 can hold numbers 2 or 98, …, Bin 49 can hold numbers 49 or 51, and Bin 50 can hold numbers 50 or 100.

Atleast one bin will have 2 weights.

Q2, ii. For section A,C,D,E,F,G

Roll number: ________________________ Section: _______

A bag has 200 socks. There are 60 red, 60 blue, 60 orange, and 20 green ones. If socks are taken out one at a time, what is the minimum number of socks one must draw from the bag to ensure that at least 20 of them are of the same color. Show working/formula.

Solution

ceil (x/4) = 20

Here x can be 77, 78, 79 and 80. The smallest number is 77.

OR

We have 19 socks of each color = 19*4 = 76. Hence 77th will ensure we have 20 of the same color.

Question 3 **(Marks: 4+4)**

i. It is required that a number plate has three English capital letters, followed by four digits. In how many ways can a policeman trace a car whose **number starts from L and ends with digit 5 if:** **(for each part give formula/reasoning)**

(a) a Letters and digits both can be repeated. (a) 1*26*26*10*10*10*1
(b) Letters and digits both are distinct. (b) 1*25*24*9*8*7*1
(c) Letters can be repeated but digits have to be unique. (c) 1*26*26*9*8*7*1
(d) Letters are distinct but digits are repeated. (d) 1*25*24*10*10*10*1

ii. How many arrangements of the letters A,S,T,I,O,N,M can be made with no repetition if: **(for each part give formula/reasoning)**

a. A is to be first letter in each arrangement

6!

b. A and T is fixed at first and last place respectively

5!

c. MAS appears as a string at any place

5!

d. A is fixed at second place and MT, NS appear as strings

2*3*2*1 = 12

Question 4 **(Marks: 10)**

Prove that $\sqrt{7}$ is an irrational number by giving a proof by contradiction.

Solution

If one does not agree that sqrt(7) is not rational, then let sqrt(7) = p/q, p and q relatively coprime integers and q not 0.

Then

$7 = p^2/q^2$ ie. $p^2 = 7q$

so 7 divides p^2 so 7 divides p. thus p = 7k. this yields:

$(7k)^2 = 7.q^2$

$7k^2 = q^2$

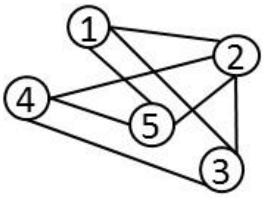
thus 7 divides q^2 so 7 divides q. Thus q =7r.

This implies 7 divides p as well as q, which contradicts the choice of p and q (relatively coprime0.

Hence sqrt 7 can not be rational.Proved.

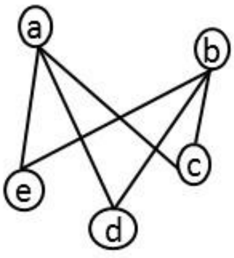
Question 5 **(Marks: 3+3+3+3+3)**

i. Is the following graph isomorphic to W_4 (wheel of order 4). If yes, transform the given graph to W_4 or show mapping. If not, then explain why?

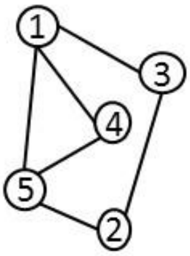


Yes. Place node 2 in the centre of the wheel.

ii. Are the following graphs G_1 and G_2 isomorphic? If yes, show their mapping. If not, then explain why?



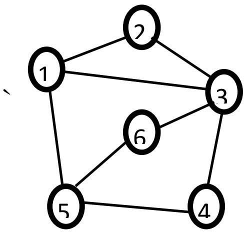
G_1



G_2

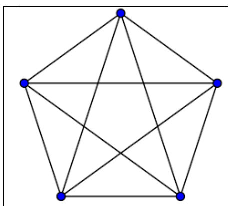
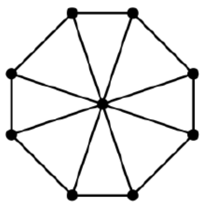
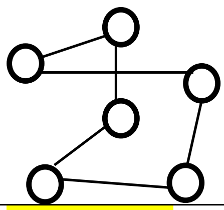
No they are not. Graph 2 has a 3-node cycle.

iii. Is the following graph bi-partite? If yes prove it by redrawing the graph, otherwise explain why it isn't?

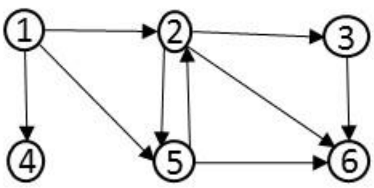


No they are not. If node 1 is placed in set 1, then nodes 2 and 3 must be in set 2. But they both can't be in the same set either.

iv. Name the following simple graphs:

		
a. K5 complete	**b. W8, wheel**	**c. C6, cylce**

v. Write down all the simple paths from **1** to **6.**



Solution: 1 2 6, 1 2 3 6, 1 2 5 6, 1 5 6, 1 5 2 6, 1 5 2 3 6

Roll number: ________________________ Section: _______

Question 6 FOR SECTIONS B, D, E, F & G (Marks: 2+8)

i. Suppose the roots of the characteristic equation of the associated linear homogeneous recurrence relation are {1,2,2,2} for some constants c_1, c_2, c_3, c_4. The recurrence relation is given by:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + c_4 a_{n-4} + n2^n$$

What is the form of the particular solution for the above recurrence? You are not required to solve the recurrence?

Solution: $a_n^p = n^3(p_1 n + p_0)2^n$

ii. Solve the following recurrence relation and give the final solution for initial conditions: $a_0=3, a_1=6$

$a_n = 6a_{n-1} - 9a_{n-2}$ NOTE: Clearly write down the final answer also.

Solution

roots of the characteristic polynomial are {3,3}

The solution is given by: $a_n = (3-n)3^n$

Question 7 (Marks: 4+2+2+4+3+2+5+5)

For all questions (where applicable) you can give the formula and do not have to compute the exact number.

i. Tick the correct options. C is the choose function:

a. C(5000,100) = C(4999,99)+C(4999,100) X true ☐ false
b. C(5000,100) = C(5000,4900) x true ☐ false
c. P(5000,100) = P(5000,4900) ☐ true x false
d. C(5000,100) = C(100,90)*C(4900,10) ☐ true x false

ii. GCD(100,190) = 10

iii. Give the smallest positive integer x that satisfies the following congruence:

$3x \equiv 2 \pmod 8$

x = 6

iv. Tick the correct option?

a. {apples, oranges, bananas} x countable ☐ uncountable
b. {x | 0 ≤ x ≤ 1 and x is a real number with 100 digits after the decimal} x countable ☐ uncountable
c. 2.2222 <= x <= 0.2223 and x is a real number ☐ countable x uncountable
d. $\{2^x \mid x \in Z\}$ x countable ☐ uncountable

v. Find the transitive closure of the following relation R defined on {a,b,c,d}:

R = {(a,b),(a,d),(c,b),(d,b),(d,c),(c,a)}

Solution: transitive closure = {(a,a),(a,b),(a,c),(a,d),(c,a),(c,b),(c,c),(c,d),(d,a),(d,b),(d,c),(d,d)}

vi. Find the inverse of the following function $f : Z \to Z$

$f(x) = x + 5$

$f^{-1}(x) = x-5$

vii. Given the following knowledge base

cat(mano); cat(chotto); puppy(kalu); puppy(ragy); puppy(goldy);
color(mano,black); color(chotto,black);
color(kalu,black); color(ragy,brown); color(goldy,black)

Tick the correct option given the above facts.

a. ∀x (¬color(x,black) → ¬cat(x)) x true ☐ false
b. ∀x (puppy(x) ∧ (color(x,brown) ∨ color(x,black))) ☐ true x false
c. ∀x (color(x,black) → cat(x)) ☐ true x false
d. ∃x (puppy(x) ∧ color(x,brown)) x true ☐ false
e. ∀y ∃x (puppy(y) → color (y,x)) x true ☐ false

viii. Use modular exponentiation algorithm to calculate the value of 4^{281} mod 11. No marks without proper working.

(281)10 = (100011001)2

4 mod 11 is 4,	**1**	**running_product = 4**
16 mod 11 is 5	**0**	**running_product = 4**
25 mod 11 is 3,	**0**	**running_product = 4**
9 mod 11 is 9,	**1**	**running_product = 4*9**
81 mod 11 is 4,	**1**	**running_product = 4*9*4**
16 mod 11 is 5,	**0**	**running_product = 4*9*4**
25 mod 11 is 3,	**0**	**running_product = 4*9*4**
9 mod 11 is 9,	**0**	**running_product = 4*9*4**
81 mod 11 is 4,	**1**	**running_product = 4*9*4*4 mod 11 =4**

Answer = 4.

SHORTER ANSWER (using Fermat's little theorem):
281 = 28*10+1
4^{281} mod 11 = $(4^{10})^{28}$ * 4 mod 11 = 1^{28} * 4 mod 11 = 4