National University
of computer and emerging sciences

DISCRETE STRUCTUERS

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Course Outline

- **Logic and Proofs** (Chapter 1)
- Sets and Functions
- Relations
- Number Theory
- Combinatorics and Recurrence
- Graphs
- Trees
- Discrete Probability

Lecture Outline

- Valid Arguments
- Inference Rules for Propositional Logic
- Using Rules of Inference to Build Arguments
- Fallacies
- Rules of Inference for Quantified Statements
- Building Arguments for Quantified Statements

The Argument

We have the two premises:

- “All students are present.”
- “Ali is a student.”

And the conclusion:

- “Ali is present”

How do we get the conclusion from the premises?

The Argument

"If you have a current password, then you can log onto the network."

premises

"You have a current password."

Therefore,

"You can log onto the network."

conclusion

- We would like to determine whether this is a **valid argument**.
- That is, we would like to determine whether the **conclusion** must be **true** when the **premises** are both **true**.

p = "You have a current password" and ***q*** ="You can log onto the network." Then, the argument has the form

$$\begin{array}{l} p \to q \\ \underline{p \quad\quad} \\ \therefore q \end{array}$$

where $\therefore$ is the symbol that denotes "therefore."

Valid Arguments

We will show how to construct valid arguments in two stages; first for propositional logic and then for predicate logic. The rules of inference are the essential building block in the construction of valid arguments.

1. Propositional Logic

   Inference Rules

2. Predicate Logic

   Inference rules for propositional logic plus additional inference rules to handle variables and quantifiers.

Arguments in Propositional Logic

- A *argument* in propositional logic is a sequence of propositions. All but the final proposition are called ***premises***. The last statement is the ***conclusion***.
- The argument is valid if the premises imply the conclusion. An *argument form* is an argument that is valid no matter what propositions are substituted into its propositional variables.
- If the premises are $p_1, p_2, \ldots, p_n$ and the conclusion is q then
- $(p_1 \land p_2 \land \ldots \land p_n) \rightarrow q$ is a tautology.
- Inference rules are all argument simple argument forms that will be used to construct more complex argument forms.

Rules of Inference for Propositional Logic

Rule of Inference	Tautology	Name
$\begin{array}{l} p \\ \underline{p \to q} \\ q \end{array}$	$(p \land (p \to q)) \to q$	**Modus ponens (MP)**
$\begin{array}{l} \neg q \\ \underline{p \to q} \\ \neg p \end{array}$	$(\neg q \land (p \to q)) \to \neg p$	**Modus tonens (MT)**
$\begin{array}{l} p \to q \\ \underline{q \to r} \\ p \to r \end{array}$	$((p \to q) \land (q \to r)) \to (p \to r)$	**Hypothetical syllogism (HS)**
$\begin{array}{l} p \lor q \\ \underline{\neg p} \\ q \end{array}$	$((p \lor q) \land \neg p) \to q$	**Disjunctive syllogism (DS)**
$\begin{array}{l} \underline{p} \\ p \lor q \end{array}$	$p \to (p \lor q)$	**Addition**
$\begin{array}{l} \underline{p \land q} \\ p \end{array}$	$(p \land q) \to p$	**Simplification**
$\begin{array}{l} p \\ \underline{q} \\ p \land q \end{array}$	$((p) \land (q)) \to (p \land q)$	**Conjunction**
$\begin{array}{l} p \lor q \\ \underline{\neg p \lor r} \\ q \lor r \end{array}$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	**Resolution**

Rules of Inference for Propositional Logic

Example 1:

Suppose that the conditional statement "If it snows today, then we will go skiing"

And its hypothesis, "It is snowing today".

Therefore, "We will go skiing"

Rule of inference: modus ponens

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

Corresponding Tautology:

$(p \wedge (p \rightarrow q)) \rightarrow q$

Rules of Inference for Propositional Logic

Example 2:

Let *p* be “it is snowing.” Let *q* be “I will study discrete math.”

“If it is snowing, then I will study discrete math.”

“I will not study discrete math.”

“*Therefore* , it is not snowing.”

Rule of inference: Modus Tollens

$$\begin{array}{l} p \to q \\ \underline{\neg q \qquad} \\ \therefore \neg p \end{array}$$

Corresponding Tautology:

$(\neg q \wedge (p \to q)) \to \neg p$

Rules of Inference for Propositional Logic

Example 3:

“If it rains today, then we will not have a barbecue today”. “If we do not have a barbecue today, then we will have a barbecue tomorrow”.

Therefore, “if it rains today, then we will have a barbecue tomorrow.”

Let ***p*** be the proposition “It is raining today,” let ***q*** be the proposition “We will not have a barbecue today,” and let ***r*** be the proposition “We will have a barbecue tomorrow.”

Rule of inference: hypothetical syllogism.

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

Corresponding Tautology:

$((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)$

Rules of Inference for Propositional Logic

Example 4:

"I will study discrete math or I will study English literature."

"I will not study discrete math."

"*Therefore* , I will study English literature."

Let ***p*** be "I will study discrete math." And ***q*** be "I will study English literature."

Rule of inference: Disjunctive syllogism.

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

Corresponding Tautology:
$(\neg p \wedge (p \vee q)) \rightarrow q$

Rules of Inference for Propositional Logic

Example 5:

"It is below freezing and raining now. Therefore, it is below freezing now."

Let *p* be "It is below freezing now."

Let *q* be "It is raining now."

Rule of inference: Simplification rule

$$\frac{p \wedge q}{\therefore q}$$

Corresponding Tautology:

$(p \wedge q) \rightarrow p$

Rules of Inference for Propositional Logic

Example 6:

"It is below freezing now."

Therefore, it is below freezing or raining now."

Let ***p*** be the proposition "It is below freezing now,"

Let ***q*** be the proposition "It is raining now."

Rule of inference: Addition rule

$$\frac{p}{\therefore p \vee q}$$

Corresponding Tautology:

$p \rightarrow (p \vee q)$

Rules of Inference for Propositional Logic

Example 7:

"I will study discrete math."

"I will study English literature."

"*Therefore*, I will study discrete math and I will study English literature."

Let ***p*** be "I will study discrete math."

Let ***q*** be "I will study English literature."

Rule of inference: Conjunction

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

Corresponding Tautology:

$((p) \wedge (q)) \rightarrow (p \wedge q)$

Rules of Inference for Propositional Logic

Example 8:

"I will not study discrete math or I will study English literature."

"I will study discrete math or I will study databases."

"Therefore, I will study databases or I will English literature."

Let ***p*** be "I will study discrete math."

Let ***r*** be "I will study English literature."

Let **q** be "I will study databases."

Rule of inference: Resolution

$$\begin{array}{l} \neg p \vee r \\ p \vee q \\ \hline \therefore q \vee r \end{array}$$

Corresponding Tautology:

$((\neg p \vee r) \wedge (p \vee q)) \rightarrow (q \vee r)$

Using the Rules of Inference to Build Valid Arguments

A *valid argument* is a sequence of statements. Each statement is either a premise or follows from previous statements by rules of inference. The last statement is called conclusion.

A valid argument takes the following form:

$$
\begin{array}{l}
S_1 \\
S_2 \\
\;. \\
\;. \\
\;. \\
S_n \\
\\
\therefore C
\end{array}
$$

Valid Arguments

Example 1:

With these hypotheses:

"It is not sunny this afternoon and it is colder than yesterday."

"We will go swimming only if it is sunny."

"If we do not go swimming, then we will take a canoe trip."

"If we take a canoe trip, then we will be home by sunset."

Using the inference rules, construct a valid argument for the conclusion:

"We will be home by sunset."

Solution:

1. Choose propositional variables:

   p : "It is sunny this afternoon." r : "We will go swimming." t : "We will be home by sunset."

   q : "It is colder than yesterday." s : "We will take a canoe trip."

2. Translation into propositional logic:

$$\text{Hypotheses: } \neg p \wedge q,\ r \rightarrow p,\ \neg r \rightarrow s,\ s \rightarrow t$$
$$\text{Conclusion: } t$$

Continued on next slide →

Valid Arguments: Proof

3. Construct the Valid Argument

Step	Reason
1. $\neg p \land q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \to p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \to s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \to t$	Premise
8. t	Modus ponens using (6) and (7)

Rule of Inference
$\begin{array}{l} p \\ p \to q \\ \hline q \end{array}$
$\begin{array}{l} \neg q \\ p \to q \\ \hline \neg p \end{array}$
$\begin{array}{l} p \to q \\ q \to r \\ \hline p \to r \end{array}$
$\begin{array}{l} p \lor q \\ \neg p \\ \hline q \end{array}$
$\begin{array}{l} p \\ \hline p \lor q \end{array}$
$\begin{array}{l} p \land q \\ \hline p \end{array}$
$\begin{array}{l} p \\ q \\ \hline p \land q \end{array}$
$\begin{array}{l} p \lor q \\ \neg p \lor r \\ \hline q \lor r \end{array}$

Valid Arguments: Proof

Hypotheses: $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, and $s \rightarrow t$; *Conclusion:* t

	$(\neg p \wedge q) \wedge (r \rightarrow p) \wedge (\neg r \rightarrow s) \wedge (s \rightarrow t)$
$\dfrac{p \wedge q}{\therefore q}$	$(\neg p \wedge q) \wedge (r \rightarrow p) \wedge (\neg r \rightarrow s) \wedge (s \rightarrow t)$ $\neg p \wedge (r \rightarrow p) \wedge (\neg r \rightarrow s) \wedge (s \rightarrow t)$
$\dfrac{p \rightarrow q \quad \neg q}{\therefore \neg p}$	$\neg p \wedge (r \rightarrow p) \wedge (\neg r \rightarrow s) \wedge (s \rightarrow t)$ $\neg r \wedge (\neg r \rightarrow s) \wedge (s \rightarrow t)$
$\dfrac{p \rightarrow q \quad p}{\therefore q}$	$\neg r \wedge (\neg r \rightarrow s) \wedge (s \rightarrow t)$ $s \wedge (s \rightarrow t)$
$\dfrac{p \rightarrow q \quad p}{\therefore q}$	$s \wedge (s \rightarrow t)$ t

Conclusion

Rule of Inference
$\dfrac{p \quad p \rightarrow q}{q}$
$\dfrac{\neg q \quad p \rightarrow q}{\neg p}$
$\dfrac{p \rightarrow q \quad q \rightarrow r}{p \rightarrow r}$
$\dfrac{p \vee q \quad \neg p}{q}$
$\dfrac{p}{p \vee q}$
$\dfrac{p \wedge q}{p}$
$\dfrac{p \quad q}{p \wedge q}$
$\dfrac{p \vee q \quad \neg p \vee r}{q \vee r}$

Valid Arguments: Proof

Example 2:

With these **hypotheses**:

"If it is Saturday today, then we play soccer or basketball."

"If the soccer field is occupied, we don't play soccer. "

"It is Saturday today, and the soccer field is occupied. "

Prove: "we play basketball or volleyball".

Solution: First we formalize the problem:

P: It is Saturday today. **Q**: We play soccer. **R**: We play basketball.

S: The soccer field is occupied. **T**: We play volleyball.

Premise: **P → (Q V R), S → ¬ Q, P, S** Need to prove: **R V T**.

Rule of Inference
$\begin{array}{l} p \\ p \to q \\ \hline q \end{array}$
$\begin{array}{l} \neg q \\ p \to q \\ \hline \neg p \end{array}$
$\begin{array}{l} p \to q \\ q \to r \\ \hline p \to r \end{array}$
$\begin{array}{l} p \lor q \\ \neg p \\ \hline q \end{array}$
$\begin{array}{l} p \\ \hline p \lor q \end{array}$
$\begin{array}{l} p \land q \\ \hline p \end{array}$
$\begin{array}{l} p \\ q \\ \hline p \land q \end{array}$
$\begin{array}{l} p \lor q \\ \neg p \lor r \\ \hline q \lor r \end{array}$

Valid Arguments: Proof

Premise: **$P \rightarrow (Q \lor R)$, $S \rightarrow \neg Q$, P, S** Need to prove: **$R \lor T$**.

(1) $P \rightarrow (Q \lor R)$ Premise

(2) P Premise

(3) $Q \lor R$ Apply MP rule to (1)(2)

(4) $S \rightarrow \neg Q$ Premise

(5) S Premise

(6) $\neg Q$ Apply MP rule to (4)(5)

(7) R Apply DS rule to (3)(6)

(8) $R \lor T$ Apply Addition rule to (7)

Rule of Inference
$\dfrac{p \quad p \rightarrow q}{q}$
$\dfrac{\neg q \quad p \rightarrow q}{\neg p}$
$\dfrac{p \rightarrow q \quad q \rightarrow r}{p \rightarrow r}$
$\dfrac{p \lor q \quad \neg p}{q}$
$\dfrac{p}{p \lor q}$
$\dfrac{p \land q}{p}$
$\dfrac{p \quad q}{p \land q}$
$\dfrac{p \lor q \quad \neg p \lor r}{q \lor r}$

Fallacies

- Arguments are based on tautologies.
- Fallacies are based on contingencies.
- The proposition $((p \rightarrow q) \wedge q) \rightarrow p$ is not a tautology, because it is false when p is false and q is true.
- This type of incorrect reasoning is called the **fallacy of affirming the conclusion**.

Example:

If you do every problem in this book, then you will learn discrete mathematics. You learned discrete mathematics. Therefore, you did every problem in this book.

Fallacy of affirming the conclusion

- **A fallacy of affirming the conclusion** is an incorrect reasoning in proving **p → q** by starting with assuming q and proving p.

- **For example**: Show that if x+y is odd, then either x or y is odd, but not both.

 A fallacy of arming the conclusion argument would start with:

“Assume that either x or y is odd, but not both. Then..”

Fallacy of denying the hypothesis.

- The proposition $((p \to q) \land \neg p) \to \neg q$ is not a tautology, because it is false when p is false and q is true.
- Is it correct to assume that you did not learn discrete mathematics if you did not do every problem in the book, assuming that if you do every problem in this book, then you will learn discrete mathematics?
- It is possible that you learned discrete mathematics even if you did not do every problem in this book. This incorrect argument is of the form $p \to q$ and $\neg p$ imply $\neg q$, which is an example of the fallacy of denying the hypothesis.

Fallacy of denying the hypothesis.

- **A fallacy of denying the hypothesis** is an incorrect reasoning in proving $p \rightarrow q$ by starting with assuming $\neg p$ and proving $\neg q$.
- **For example:** Show that if x is irrational, then x=2 is irrational.

A fallacy of denying the hypothesis argument would start with:

"Assume that x is rational. Then..."

Handling Quantified Statements

Valid arguments for quantified statements are a sequence of statements. Each statement is either a premise or follows from previous statements by rules of inference which include:

- Rules of Inference for Propositional Logic
- Rules of Inference for Quantified Statements

The rules of inference for quantified statements are introduced in the next several slides.

Valid Arguments (Predicate Logic)

TABLE 2 Rules of Inference for Quantified Statements.

Rule of Inference	*Name*
$\dfrac{\forall x P(x)}{\therefore\ P(c)}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore\ \forall x P(x)}$	Universal generalization
$\dfrac{\exists x P(x)}{\therefore\ P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore\ \exists x P(x)}$	Existential generalization

Universal Instantiation (UI)

$$\frac{\forall x P(x)}{\therefore P(c)}$$

Example:

Our domain consists of all students and Ali is a student.

“All students are present.”

“Therefore, Ali is present.”

Universal Generalization (UG)

$$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$$

Used often implicitly in Mathematical Proofs.

Explanation:
What this rule says is that if P(c) holds for any arbitrary element c of the universe, then we can conclude that $\forall$x P(x). If, however, c is supposed to represent some specific element of the universe that has the property P, then one can not generalize it to all the elements. For example, if P(x) means "x is fast", then all it means is that an unspecified element represented by x is fast. It does not necessarily mean that everything in the universe is fast.

This rule is something we can use when we want to prove that a **certain property holds for every element of the universe**. That is when we want to prove $\forall$x P(x), we take an abrbitrary element x in the universe and prove P(x). Then by this Universal Generalization we can conclude $\forall$x P(x).

Existential Instantiation (EI)

$$\frac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$$

Example:

“There is someone who got an A in the course.”
“Let’s call her *X* and say that *X* got an A”

Existential Generalization (EG)

$$\frac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$$

Example:

“Michelle got an A in the class.”
“Therefore, someone got an A in the class.”

Using Rules of Inference

Example 1: Show that the premises

"Everyone in this discrete mathematics class has taken a course in computer science" and

"Maria is a student in this class"

imply the conclusion **"Maria has taken a course in computer science."**

Solution: Let ***D*(*x*)** denote "*x* is in this discrete mathematics class," and let ***C*(*x*)** denote "*x* has taken a course in computer science."

Valid Argument:

Step	**Reason**
1. $\forall x(D(x) \rightarrow C(x))$	Premise
2. $D(\text{Marla}) \rightarrow C(\text{Marla})$	Universal instantiation from (1)
3. $D(\text{Marla})$	Premise
4. $C(\text{Marla})$	Modus ponens from (2) and (3)

Using Rules of Inference

Example 2: Use the rules of inference to construct a valid argument showing that the conclusion
"Someone who passed the first exam has not read the book."

follows from the premises
"A student in this class has not read the book."
"Everyone in this class passed the first exam."

Solution: Let *C*(*x*) denote "*x* is in this class," *B*(*x*) denote " *x* has read the book," and *P*(*x*) denote "*x* passed the first exam."

First we translate the premises and conclusion into symbolic form.

$$\begin{array}{l} \exists x(C(x) \land \neg B(x)) \\ \forall x(C(x) \to P(x)) \\ \hline \therefore\ \exists x(P(x) \land \neg B(x)) \end{array}$$

Continued on next slide →

Using Rules of Inference

Valid Argument:

Step	**Reason**
1. $\exists x(C(x) \wedge \neg B(x))$	Premise
2. $C(a) \wedge \neg B(a)$	EI from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \rightarrow P(x))$	Premise
5. $C(a) \rightarrow P(a)$	UI from (4)
6. $P(a)$	MP from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \wedge \neg B(a)$	Conj from (6) and (7)
9. $\exists x(P(x) \wedge \neg B(x))$	EG from (8)

Rule of Inference
$\begin{array}{l} p \\ p \rightarrow q \\ \hline q \end{array}$
$\begin{array}{l} \neg q \\ p \rightarrow q \\ \hline \neg p \end{array}$
$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline p \rightarrow r \end{array}$
$\begin{array}{l} p \vee q \\ \neg p \\ \hline q \end{array}$
$\begin{array}{l} p \\ \hline p \vee q \end{array}$
$\begin{array}{l} p \wedge q \\ \hline p \end{array}$
$\begin{array}{l} p \\ q \\ \hline p \wedge q \end{array}$
$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline q \vee r \end{array}$

Thank you!!!

Understanding Math by reading slides is similar to Learning to swim by watching TV.

So, DO PRACTICE IT!