National University
of computer and emerging sciences

DISCRETE STRUCTURES

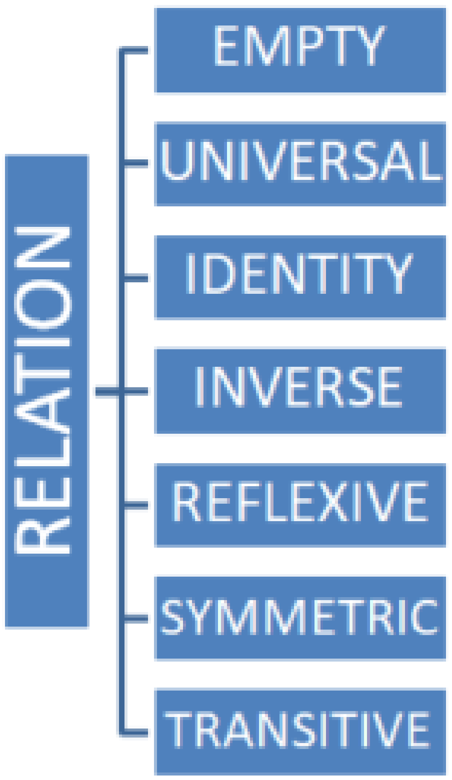
COURSE INSTRUCTOR: MUHAMMAD SAIF UL ISLAM

Course Outline

- **Logic and Proofs** (Chapter 1)
- **Sets and Functions** (Chapter 2)
- **Relations** (Chapter 9)
- Number Theory
- Combinatorics and Recurrence
- Graphs
- Trees
- Discrete Probability

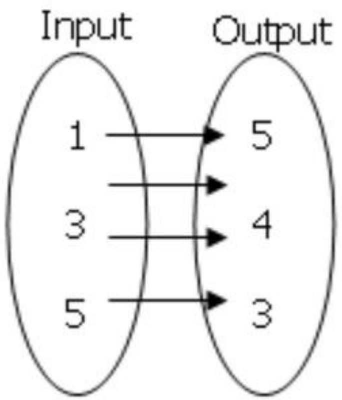
Lecture Outline

- Relations and Functions
- Properties of Relations
- Combining Relations
- Representing Relations using Matrices
- Representing Relations using Digraphs
- Equivalence Relations
- Equivalence Classes

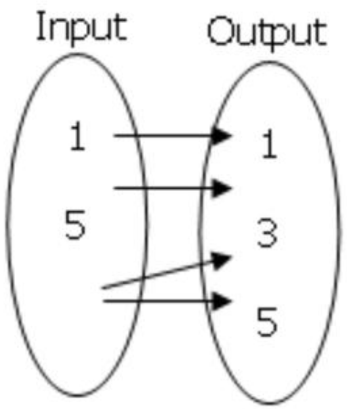


Function Vs Relation

Every function is a Relation BUT every Relation is not a Function



This ***is a function*** because there is exactly one output for every input.



This ***is not a function*** because there is more than one output for a given input. For the input number 7, there are two output values (5 and 7)

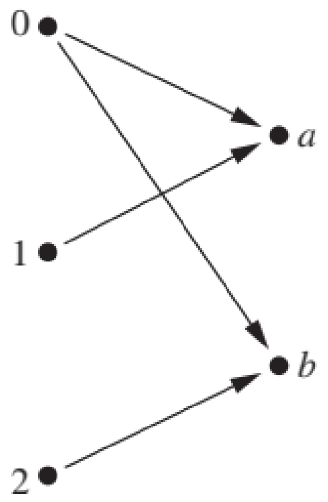
Relation is a subset of Cartesian product of two Non-Empty sets!

Binary Relations

Definition: A *binary relation R* from a set *A* to a set *B* is a subset $R \subseteq A \times B$.

Example:

- Let $A = \{0,1,2\}$ and $B = \{a,b\}$
- $\{(0, a), (0, b), (1,a), (2, b)\}$ is a relation from *A* to *B*.
- We can represent relations from a set *A* to a set *B* graphically or using a table:



R	a	b
0	×	×
1	×	
2		×

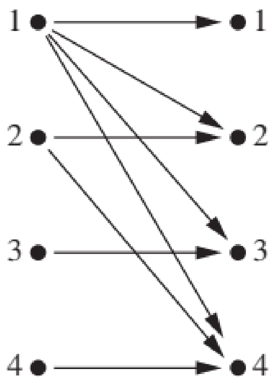
Relations are more general than functions. A function is a relation where exactly one element of *B* is related to each element of *A*.

Binary Relation on a Set

Definition: A binary relation R *on a set* A is a subset of $A \times A$ or a relation from A to A.

Example:

- Suppose that $A = \{a,b,c\}$. Then $R = \{(a,a),(a,b), (a,c)\}$ is a relation on A.
- Let $A = \{1, 2, 3, 4\}$. The ordered pairs in the relation R $= \{(a,b) \mid a \text{ divides } b\}$ are $(1,1), (1, 2), (1,3), (1, 4), (2, 2), (2, 4), (3, 3)$, and $(4, 4)$.



R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

Binary Relation on a Set (*cont.*)

Question: How many relations are there on a set A?

Solution: Because a relation on A is the same thing as a subset of $A \times A$, we count the subsets of $A \times A$. Since $A \times A$ has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are $2^{|A|^2}$ subsets of $A \times A$. Therefore, there are $2^{|A|^2}$ relations on a set A.

$A \times A = \boldsymbol{n^2}$

$R \subseteq A \times A = \boldsymbol{2^{n^2}}$

Binary Relations on a Set (*cont.*)

Example: Consider these relations on the set of integers:

$R_1 = \{(a,b) \mid a \leq b\}$,

$R_2 = \{(a,b) \mid a > b\}$,

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,

$R_4 = \{(a,b) \mid a = b\}$,

$R_5 = \{(a,b) \mid a = b + 1\}$,

$R_6 = \{(a,b) \mid a + b \leq 3\}$.

Which of these relations contain each of the pairs

(1,1), (1, 2), (2, 1), (1, −1), and (2, 2)?

Note that these relations are on an infinite set and each of these relations is an infinite set.

Solution: Checking the conditions that define each relation, we see that the pair (1,1) is in R_1, R_3, R_4 , and R_6: (1,2) is in R_1 and R_6: (2,1) is in R_2, R_5, and R_6: $(1, −1)$ is in R_2, R_3, and R_6 : (2,2) is in R_1, R_3, and R_4.

Reflexive Relations

Definition: R is ***reflexive*** iff $(a,a) \in R$ for every element $a \in A$.

Definition: R is ***irreflexive*** iff $(a,a) \notin R$ for every element $a \in A$.

Written symbolically, R is reflexive if and only if

$$\forall x[x \in U \longrightarrow (x,x) \in R]$$

Example: The following relations on the integers are reflexive:

$R_1 = \{(a,b) \mid a \leq b\}$,
$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,
$R_4 = \{(a,b) \mid a = b\}$.

The following relations are not reflexive:

$R_2 = \{(a,b) \mid a > b\}$ (note that $3 \ngtr 3$),
$R_5 = \{(a,b) \mid a = b + 1\}$ (note that $3 \neq 3 + 1$),
$R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that $4 + 4 \nleq 3$).

If $A = \emptyset$ then the empty relation is reflexive vacuously. That is the empty relation on an empty set is reflexive!

If every element of set A maps to itself, the relation is Reflexive Relation.

Symmetric Relations

Definition: R is *symmetric* iff $(b,a) \in R$ whenever $(a,b) \in R$ for all $a,b \in A$. Written symbolically, R is symmetric if and only if

$$\forall x \forall y\, [(x,y) \in R \longrightarrow (y,x) \in R]$$

Example: The following relations on the integers are symmetric:

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,

$R_4 = \{(a,b) \mid a = b\}$,

$R_6 = \{(a,b) \mid a + b \leq 3\}$.

The following are not symmetric:

$R_1 = \{(a,b) \mid a \leq b\}$ (note that $3 \leq 4$, but $4 \nleq 3$),

$R_2 = \{(a,b) \mid a > b\}$ (note that $4 > 3$, but $3 \ngtr 4$),

$R_5 = \{(a,b) \mid a = b + 1\}$ (note that $4 = 3 + 1$, but $3 \neq 4 + 1$).

Antisymmetric Relations

Definition: A relation R on a set A such that for all $a,b \in A$ if $(a,b) \in R$ and $(b,a) \in R$, then $a = b$ is called *antisymmetric*. Written symbolically, R is antisymmetric if and only if

$\forall x \forall y\, [(x,y) \in R \wedge (y,x) \in R \longrightarrow x = y]$

An anti-symmetric relation is one in which for any ordered pair (x,y) in R, the ordered pair (y,x) must NOT be in R, unless x = y.

Example: The following relations on the integers are antisymmetric:

$R_1 = \{(a,b) \mid a \leq b\}$,

$R_2 = \{(a,b) \mid a > b\}$,

$R_4 = \{(a,b) \mid a = b\}$,

$R_5 = \{(a,b) \mid a = b + 1\}$.

For any integer, if a $a \leq b$ and $b \leq a$, then $a = b$.

The following relations are not antisymmetric:

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$

(note that both $(1,-1)$ and $(-1,1)$ belong to R_3),

$R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that both $(1,2)$ and $(2,1)$ belong to R_6).

R1 = { (2,2), (2,4), (3,2) } is anti-symmetric. But if we add the ordered pair (4,2) to get R2 = { (2,2), (2,4), (3,2), (4,2) } then R2 is no longer anti-symmetric, because both (2,4), and (4,2) are in the relation.

Transitive Relations

Definition: A relation R on a set A is called transitive if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for all $a,b,c \in A$. Written symbolically, R is transitive if and only if

$$\forall x \forall y \forall z [(x,y) \in R \land (y,z) \in R \longrightarrow (x,z) \in R]$$

Example: The following relations on the integers are transitive:

$R_1 = \{(a,b) \mid a \leq b\}$,

$R_2 = \{(a,b) \mid a > b\}$,

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,

$R_4 = \{(a,b) \mid a = b\}$.

For every integer, $a \leq b$ and $b \leq c$, then $b \leq c$.

The following are not transitive:

$R_5 = \{(a,b) \mid a = b + 1\}$ (note that both (3,2) and (4,3) belong to R_5, but not (3,3)),

$R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that both (2,1) and (1,2) belong to R_6, but not (2,2)).

Example

Problem: Three friends A, B, and C live near each other at a distance of 5 km from one another. We define a relation R between the distances of their houses. What type of relation is R?

Solution:

- R is not reflexive as A cannot be 5 km away to itself.
- The relation, R is symmetric as the distance between A & B is 5 km which is the same as the distance between B & A.
- R is transitive as the distance between A & B is 5 km, the distance between B & C is 5 km and the distance between A & C is also 5 km.

Combining Relations

Given two relations R_1 and R_2, we can combine them using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, and $R_2 - R_1$.

Example: Let A = $\{1,2,3\}$ and B = $\{1,2,3,4\}$. The relations R_1 = $\{(1,1),(2,2),(3,3)\}$ and R_2 = $\{(1,1),(1,2),(1,3),(1,4)\}$ can be combined using basic set operations to form new relations:

$R_1 \cup R_2$ =$\{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}$

$R_1 \cap R_2$ =$\{(1,1)\}$

$R_2 - R_1$ =$\{(1,2),(1,3),(1,4)\}$ $R_1 - R_2$ =$\{(2,2),(3,3)\}$

Composition

Definition: Suppose

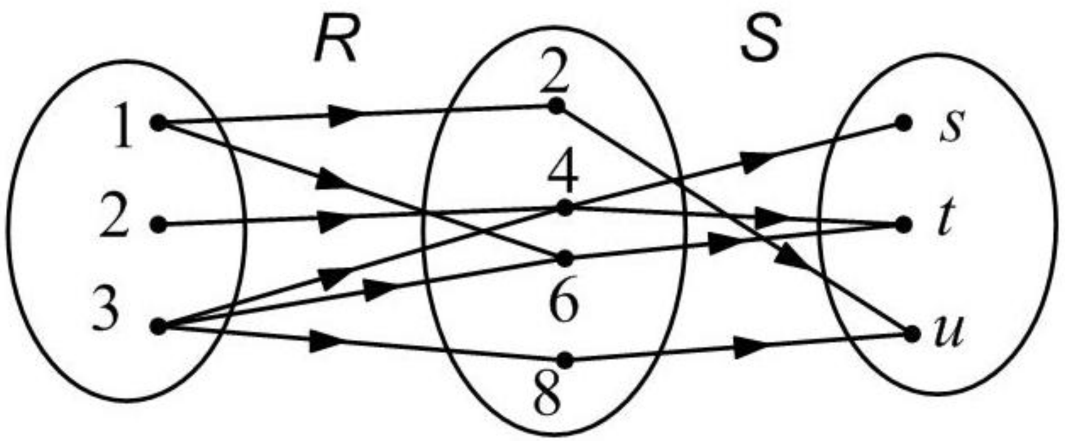
- R_1 is a relation from a set A to a set B.
- R_2 is a relation from B to a set C.

Then the *composition* (or *composite*) of R_2 with R_1, is a relation from A to C where

- if (x,y) is a member of R_1 and (y,z) is a member of R_2, then (x,z) is a member of $R_2 \circ R_1$.

Representing the Composition of a Relation

Composition of relations R and S can also be represented by using an arrow diagram:



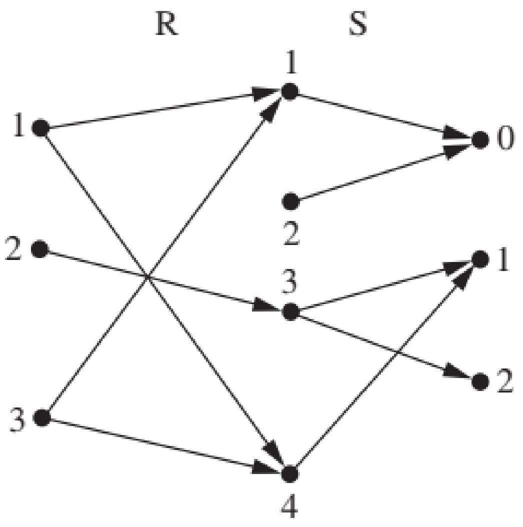
$R = \{(1,2),(1,6),(2,4),(3,4),(3,6),(3,8)\}$

$S = \{(2,u),(4,s),(4,t),(6,t),(8,u)\}$

$S \circ R = \{(1,u),(1,t),(2,s),(2,t),(3,s),(3,t),(3,u)\}$

Composition Example

What is the composite of the relations R and S, where R is the relation from {1, 2, 3} to {1, 2, 3, 4} with R = {(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)} and S is the relation from {1, 2, 3, 4} to {0, 1, 2} with S = {(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)}?



$1 \to 1 \to 0$	(1, 0)
$1 \to 4 \to 1$	(1, 1)
$2 \to 3 \to 1$	(2, 1)
$2 \to 3 \to 2$	(2, 2)
$3 \to 1 \to 0$	(3, 0)
$3 \to 4 \to 1$	(3, 1)

FIGURE 3 **Constructing $S \circ R$.**

Powers of a Relation

Definition: Let R be a binary relation on A. Then the powers R^n of the relation R can be defined inductively by:

- Basis Step: $R^1 = R$
- Inductive Step: $R^{n+1} = R^n \circ R$

(*see the slides for Section* 9.3 *for further insights*)

The powers of a transitive relation are subsets of the relation. This is established by the following theorem:

Theorem 1: The relation R on a set A is transitive iff $R^n \subseteq R$ for $n = 1,2,3 \ldots.$

(*see the text for a proof via mathematical induction*)

Representing Relations Using Matrices

A relation between finite sets can be represented using a zero-one matrix.

Suppose R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$.

- The elements of the two sets can be listed in any particular arbitrary order. When $A = B$, we use the same ordering.

The relation R is represented by the matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R, \\ 0 \text{ if } (a_i, b_j) \notin R. \end{cases}$$

The matrix representing R has a 1 as its (i,j) entry when a_i is related to b_j and a 0 if a_i is not related to b_j.

Examples of Representing Relations Using Matrices

Example 1: Suppose that $A = \{1,2,3\}$ and $B = \{1,2\}$. Let R be the relation from A to B containing (a,b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R (assuming the ordering of elements is the same as the increasing numerical order)?

Solution: Because $R = \{(2,1), (3,1),(3,2)\}$, the matrix is

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Examples of Representing Relations Using Matrices (*cont.*)

Example 2: Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

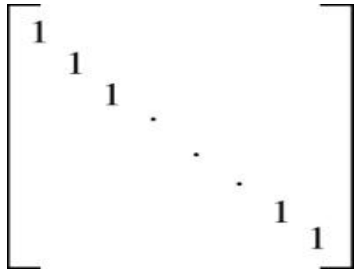
$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

Solution: Because R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$, it follows that:

$$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), \{(a_3, b_3), (a_3, b_5)\}.$$

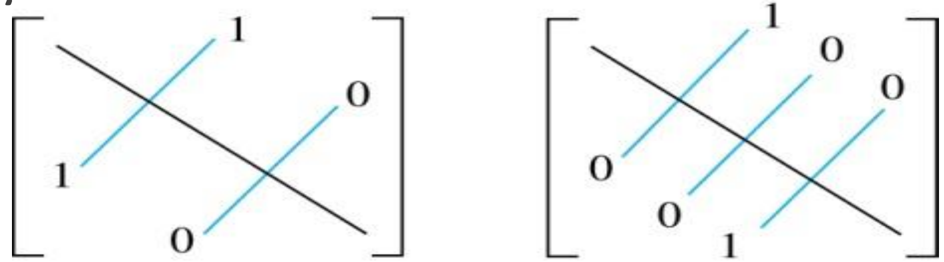
Matrices of Relations on Sets

If R is a **reflexive** relation, all the elements on the main diagonal of M_R are equal to 1.



R is a **symmetric** relation, if and only if $m_{ij} = 1$ whenever $m_{ji} = 1$.

R is an **antisymmetric** relation, if and only if $m_{ii} = 0$ or $m_{ii} = 0$ when $i \neq i$.



(a) Symmetric (b) Antisymmetric

Example of a Relation on a Set

Example 3: Suppose that the relation *R* on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Is *R* reflexive, symmetric, and/or antisymmetric?

Solution: Because all the diagonal elements are equal to 1, *R* is reflexive. Because M_R is symmetric, *R* is symmetric and not antisymmetric because both $m_{1,2}$ and $m_{2,1}$ are 1.

Exercise

Q. List the ordered pairs in the relations on {1, 2, 3} corresponding to these matrices

a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

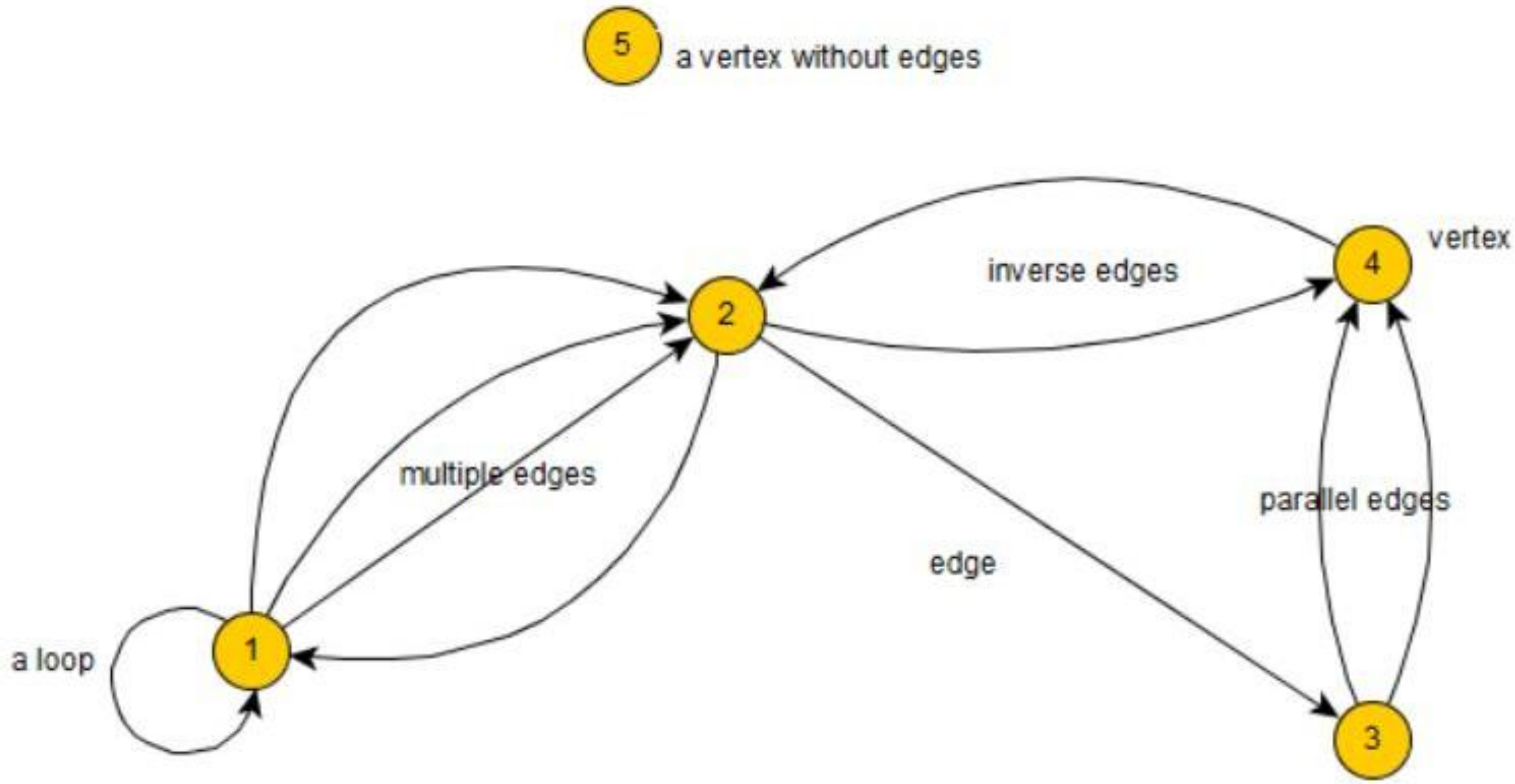
Application - Databases

- **Relational data model**, based on the concept of a relation.
- A database consists of **records**, which are n-tuples, made up of **fields**.
- Relations used to represent databases are also called **tables**
- A **domain** of an n-ary relation is called a **primary key** when the value of the n-tuple from this domain determines the n-tuple.

TABLE 1 **Students.**

Student_name	*ID_number*	*Major*	*GPA*
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

Graphs Representation

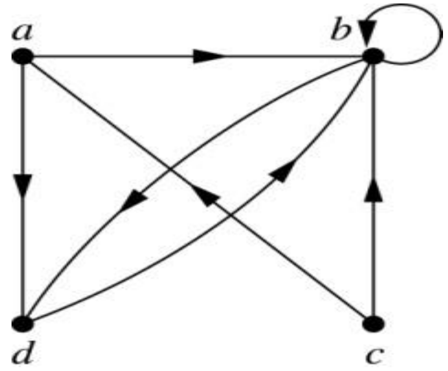


Representing Relations Using Digraphs

Definition: A *directed graph*, or *digraph*, consists of a set *V* of *vertices* (or *nodes*) together with a set *E* of ordered pairs of elements of *V* called *edges* (or *arcs*). The vertex *a* is called the *initial vertex* of the edge (*a*,*b*), and the vertex *b* is called the *terminal vertex* of this edge.

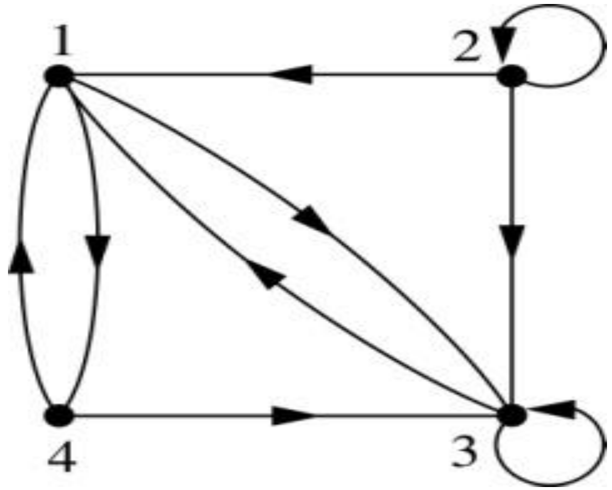
- An edge of the form (*a*,*a*) is called a *loop*.

Example 7: A drawing of the directed graph with vertices *a*, *b*, *c*, and *d*, and edges (*a*, *b*), (*a*, *d*), (*b*, *b*), (*b*, *d*), (*c*, a), (*c*, *b*), and (*d*, *b*) is shown here.



Examples of Digraphs Representing Relations

Example 8: What are the ordered pairs in the relation represented by this directed graph?



Solution: The ordered pairs in the relation are

$$(1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), \text{ and } (4, 3)$$

Determining which Properties a Relation has from its Digraph

Reflexivity: A loop must be present at all vertices in the graph.

Symmetry: If (x,y) is an edge, then so is (y,x).

Antisymmetry: If (x,y) with $x \neq y$ is an edge, then (y,x) is not an edge.

Transitivity: If (x,y) and (y,z) are edges, then so is (x,z).

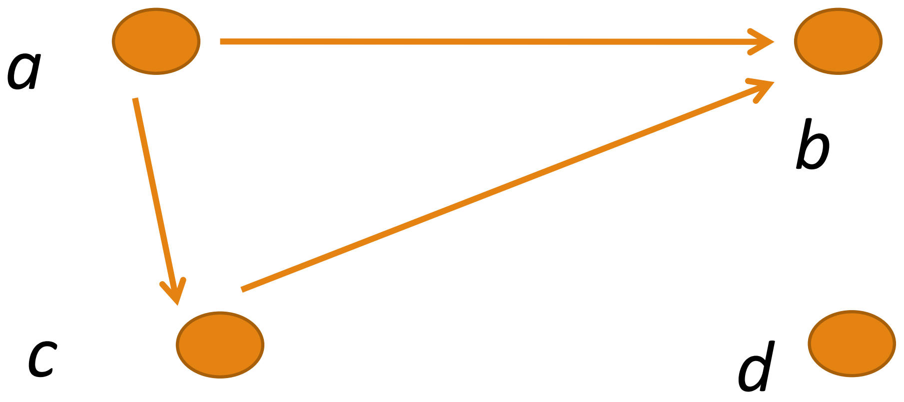
Determining which Properties a Relation has from its Digraph – Example 1

- *Reflexive?* No, not every vertex has a loop
- *Symmetric?* Yes (trivially), there is no edge from one vertex to another
- *Antisymmetric?* Yes (trivially), there is no edge from one vertex to another
- *Transitive?* Yes, (trivially) since there is no edge from one vertex to another

Determining which Properties a Relation has from its Digraph – Example 2

- *Reflexive?* No, there are no loops
- *Symmetric?* No, there is an edge from *a* to *b*, but not from *b* to *a*
- *Antisymmetric?* No, there is an edge from *d* to *b* and *b* to *d*
- *Transitive?* No, there are edges from *a* to *c* and from *c* to *b*, but there is no edge from *a* to *d*

Determining which Properties a Relation has from its Digraph – Example 3



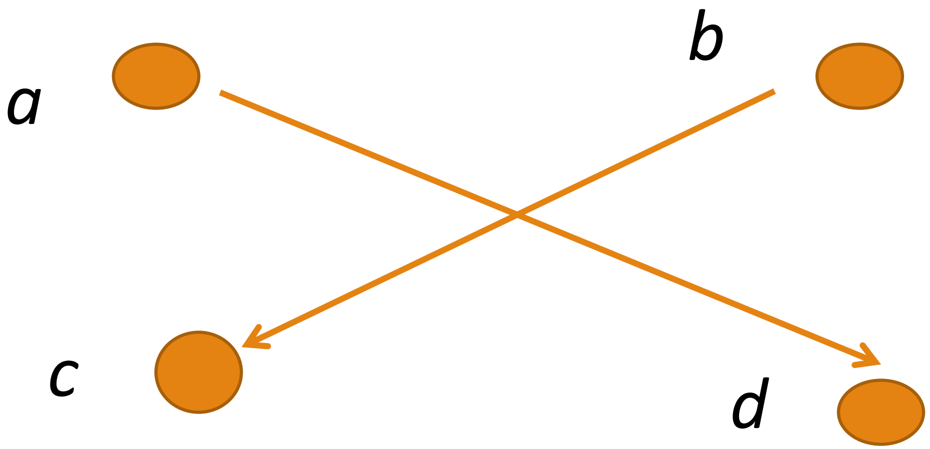
Reflexive? No, there are no loops

Symmetric? No, for example, there is no edge from *c* to *a*

Antisymmetric? Yes, whenever there is an edge from one vertex to another, there is not one going back

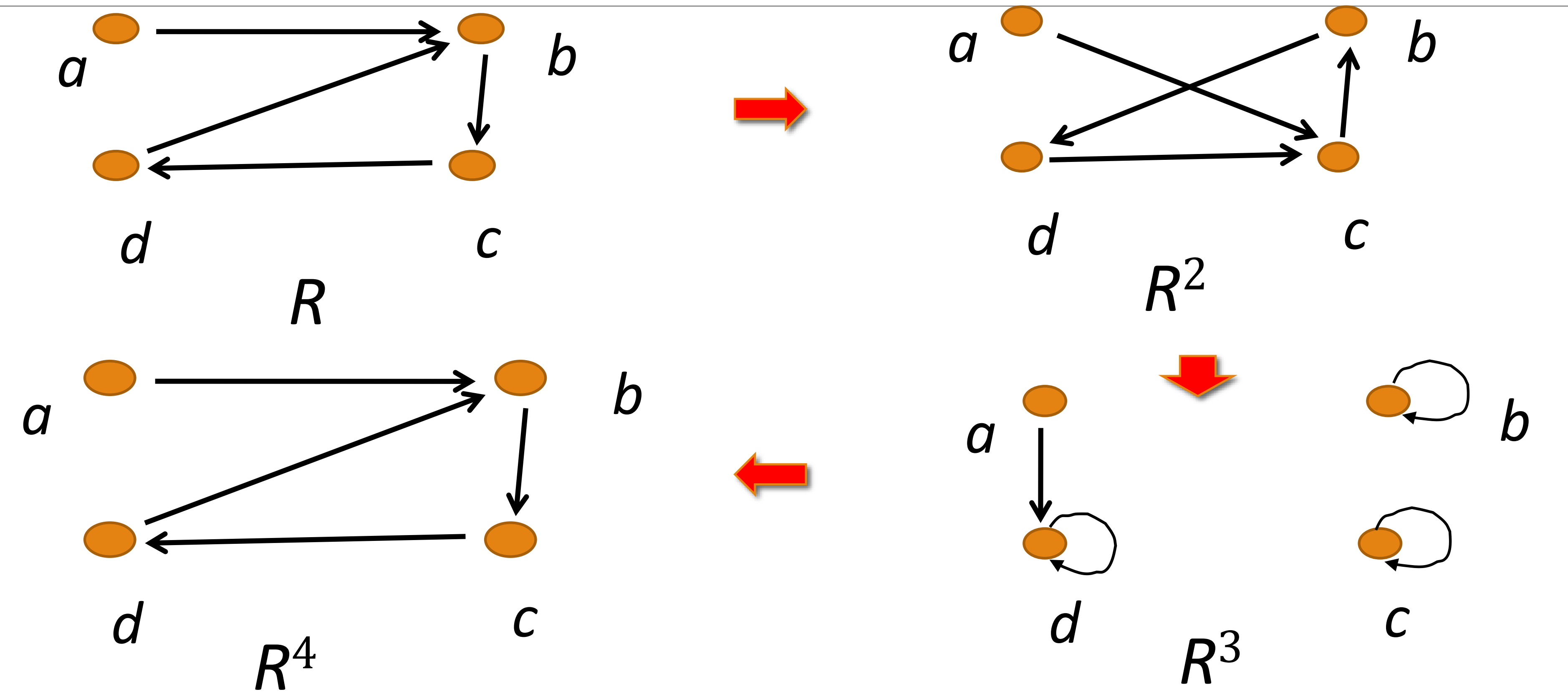
Transitive? Yes, there is edge from *a* to *c, c to b and a to b*

Determining which Properties a Relation has from its Digraph – Example 4



- *Reflexive?* No, there are no loops
- *Symmetric?* No, for example, there is no edge from *d* to *a*
- *Antisymmetric?* Yes, whenever there is an edge from one vertex to another, there is not one going back
- *Transitive?* Yes (trivially), there are no two edges where the first edge ends at the vertex where the second edge begins

Example of the Powers of a Relation



The pair (x,y) is in R^n if there is a path of length *n* from *x* to *y* in *R* (following the direction of the arrows).

Equivalence Relations

Definition 1: A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Definition 2: Two elements a, and b that are related by an equivalence relation are called *equivalent.* The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Equivalence - Example

Q. A relation R is defined on the set Z by "a R b if a – b is divisible by 5" for a, b ∈ Z. Examine if R is an equivalence relation on Z?

Solution:

(i) Let a ∈ Z. Then a – a is divisible by 5. Therefore aRa holds for all a in Z and R is **reflexive**.

(ii) Let a, b ∈ Z and aRb hold. Then a – b is divisible by 5 and therefore b – a is divisible by 5.

Thus, aRb ⇒ bRa and therefore R is **symmetric**.

(iii) Let a, b, c ∈ Z and aRb, bRc both hold. Then a – b and b – c are both divisible by 5.

Therefore a – c = (a – b) + (b – c) is divisible by 5.

Thus, aRb and bRc ⇒ aRc and therefore R is **transitive**.

Congruence Modulo *m*

If two numbers **a** and **b** have the property that their difference **a - b** is integrally divisible by a number **m** (i.e., **(a - b)/m** is an integer), then **a** and **b** are said to be "**congruent modulo**."

The number **m** is called the modulus, and the statement " **a** is congruent to **b** (modulo **m**)" is written mathematically as:

$$a \equiv b \pmod{m}$$

Congruence Modulo m

Example: Let m be an integer with $m > 1$. Show that the relation

$$R = \{(a,b) \mid a \equiv b \pmod{m}\}$$

is an equivalence relation on the set of integers.

Solution: Recall that $a \equiv b \pmod{m}$ if and only if m divides $a - b$.

- *Reflexivity*: $a \equiv a \pmod{m}$ since $a - a = 0$ is divisible by m since $0 = 0 \cdot m$.
- *Symmetry*: Suppose that $a \equiv b \pmod{m}$. Then $a - b$ is divisible by m, and so $a - b = km$, where k is an integer. It follows that $b - a = (-k)\, m$, so $b \equiv a \pmod{m}$.
- *Transitivity*: Suppose that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$. Then m divides both $a - b$ and $b - c$. Hence, there are integers k and l with $a - b = km$ and $b - c = lm$. We obtain by adding the equations:

$$a - c = (a - b) + (b - c) = km + lm = (k + l)\, m.$$

Therefore, $a \equiv c \pmod{m}$.

Divides ($a \mid b$ -> a divides b)

Example: Show that the "divides" relation on the set of positive integers is not an equivalence relation.

Solution: The properties of reflexivity, and transitivity do hold, but there relation is not transitive. Hence, "divides" is not an equivalence relation.

- *Reflexivity*: $a \mid a$ for all a.
- *Not Symmetric*: For example, $2 \mid 4$, but $4 \nmid 2$. Hence, the relation is not symmetric.
- *Transitivity*: Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c. Therefore, the relation is transitive.

Partial Orderings

Definition: A relation R on a set S is called a *partial ordering* or *partial order* if it is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R is called a *partially ordered set*, or *po-set*, and is denoted by (S, R). Members of S are called *elements* of the po-set.

Example: Show that the greater than or equal to relation **(≥)** is a partial ordering on the set of integers?

Because $a \geq a$ for every integer a, ≥ is **reflexive**.

If $a \geq b$ and $b \geq a$, then $a = b$. Hence, ≥ is **antisymmetric**.

Finally, ≥ is **transitive** because $a \geq b$ and $b \geq c$ imply that $a \geq c$. It follows that ≥ is a partial ordering on the set of integers and (**Z**, ≥) is a po-set

Thank you!!!

Understanding Math by reading slides is similar to Learning to swim by watching TV.

So, DO PRACTICE IT!