National University of computer and emerging sciences

DISCRETE STRUCTURES

COURSE INSTRUCTOR: MUHAMMAD SAIF UL ISLAM

Course Outline

- **Logic and Proofs** (Chapter 1)
- **Sets and Functions** (Chapter 2)
- Relations
- Number Theory
- Combinatorics and Recurrence
- Graphs
- Trees
- Discrete Probability

Lecture Outline

- Definition of a Function.
  - Domain, Cdomain
  - Image, Preimage
- Injection, Surjection, Bijection
- Inverse Function
- Function Composition
- Graphing Functions
- Floor, Ceiling, Factorial
- Partial Functions (optional)

Functions

Definition: Let A and B be nonempty sets. A *function* f from A to B, denoted $f: A \to B$ is an assignment of each element of A to exactly one element of B. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

Functions are sometimes called ***mappings*** or ***transformations***.

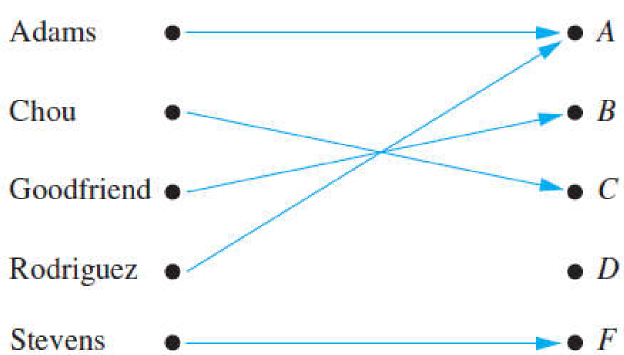
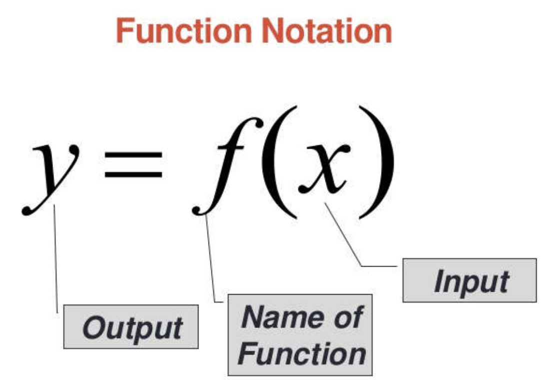


FIGURE 1 **Assignment of grades in a discrete mathematics class.**



Functions

Let **G** be the function that assigns a grade to a student in our discrete mathematics class.

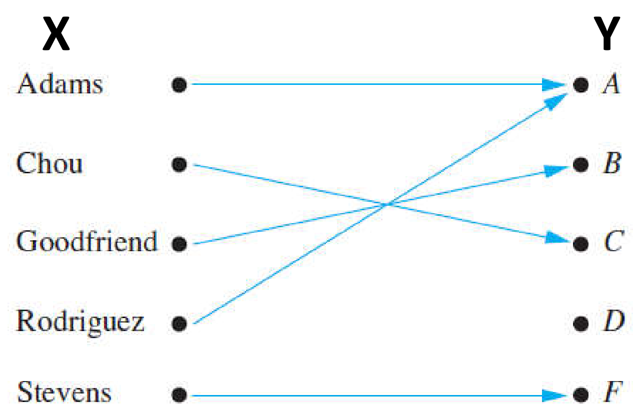
The domain of ***G*** is the set X= {Adams, Chou, Goodfriend, Rodriguez, Stevens}

The codomain of ***G*** is the set Y= {*A, B, C, D, F*}

The range of ***G*** is the set {*A, B,C, F*}

G(Adams) = *A*

G(Stevens) = *F*



Functions

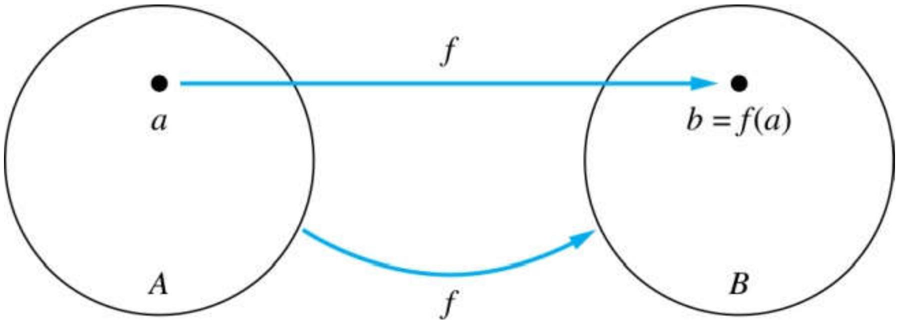
Given a function f: $A \rightarrow B$:

We say f *maps* A to B *or*

f is a *mapping* from A to B.

A is called the *domain* of f.

B is called the *co-domain* of f.



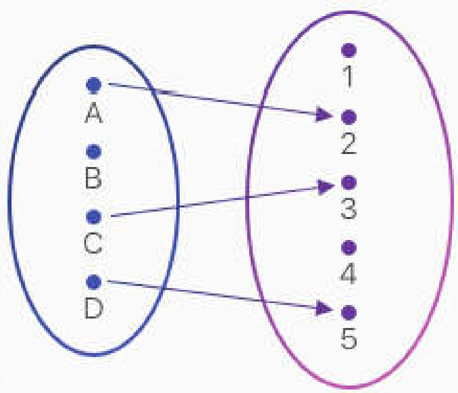
If $f(a) = b$,
- then b is called the *image* of a under f.
- a is called the *pre-image* of b.

The range of f is the set of all images of points in **A** under f. We denote it by $f(A)$.

Two functions are *equal* when they have the same domain, the same co-domain and map each element of the domain to the same element of the co-domain.

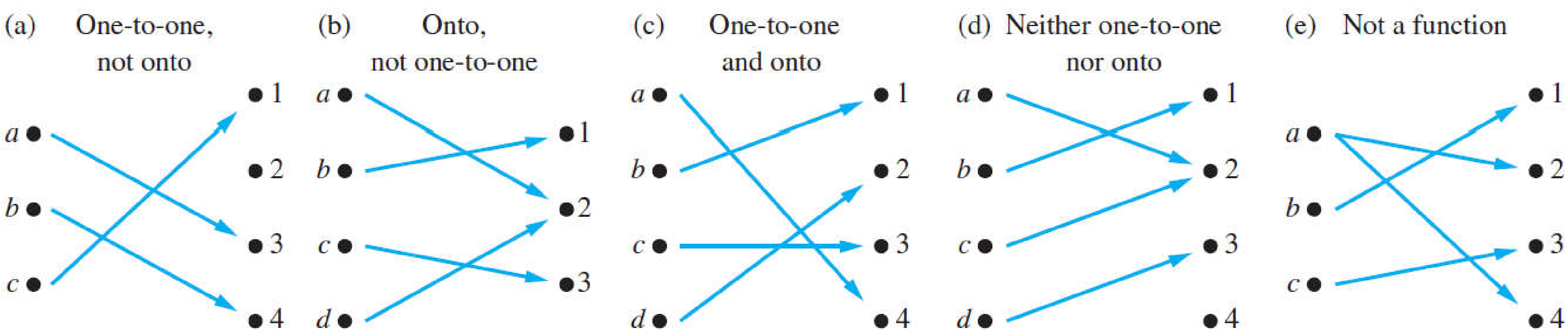
Functions

Set Function Definitions I



Domain	{ A,B,C,D }
Arguments	{ A,C,D }
Codomain	{ 1,2,3,4,5 }
Image	{ 2,3,5 }

Functions



(a) One-to-one, not onto
a
b
c
1
2
3
4
(b) Onto, not one-to-one
a
b
c
d
1
2
3
(c) One-to-one and onto
a
b
c
d
1
2
3
4
(d) Neither one-to-one nor onto
a
b
c
d
1
2
3
4
(e) Not a function
a
b
c
1
2
3
4

Representing Functions

Functions may be specified in different ways:

- An explicit statement of the assignment.
  - Students and grades example.
- A formula.
  - $f(x) = x + 1$
- A computer program.
  - A Java program that when given an integer *n*, produces the *n*th Fibonacci Number (covered in the next section and also in Chapter 5).

Questions

f(a) = ?   z

The image of d is ?   z

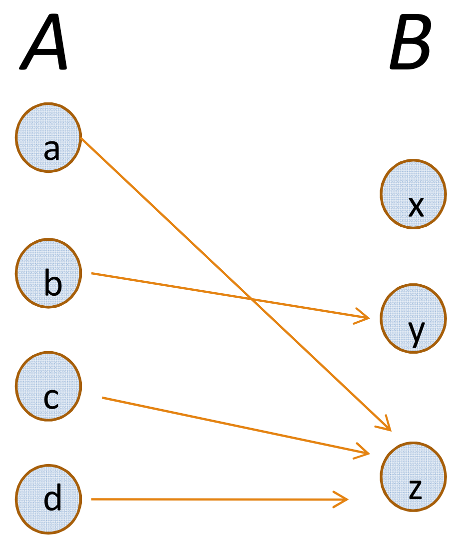
The domain of f is ?   *A*

The co-domain of f is ?   *B*

The pre-image of y is ?   b

$f(A)$ = ?

The pre-image(s) of z is (are) ?   {a,c,d}



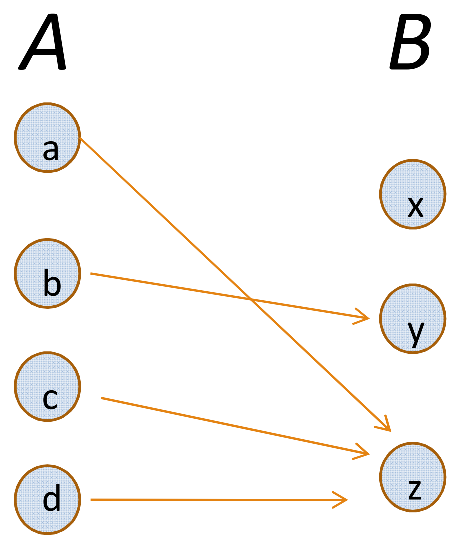
Question on Functions and Sets

If $f : A \to B$ and S is a subset of A, then

$$f(S) = \{f(s) | s \in S\}$$

f {a,b,c,} is ? {y,z}

f {c,d} is ? {z}

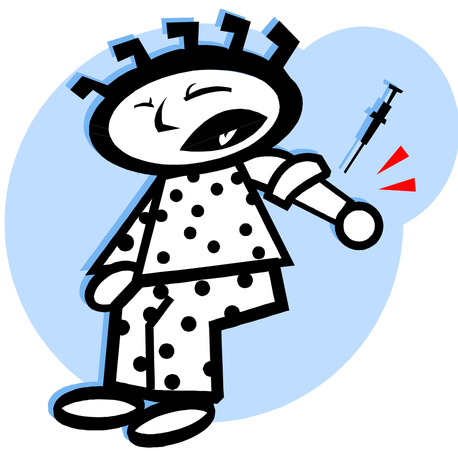


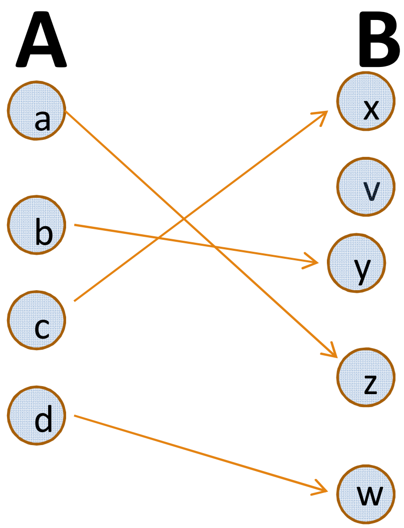
Injections

Definition: A function f is said to be ***one-to-one*** , or *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be an *injection* if it is one-to-one. $\forall a \forall b(a \neq b \rightarrow f(a) \neq f(b))$

Some functions never assign the same value to two different domain elements.

These functions are said to be **one-to-one**.



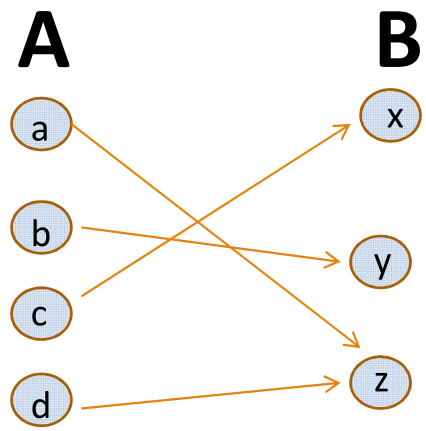


Surjections

Definition: A function *f* from *A* to *B* is called ***onto*** or *surjective*, if and only if for every element $a \in A$ there is an element $b \in B$ with $f(a) = b$. A function *f* is called a *surjection* if it is onto.

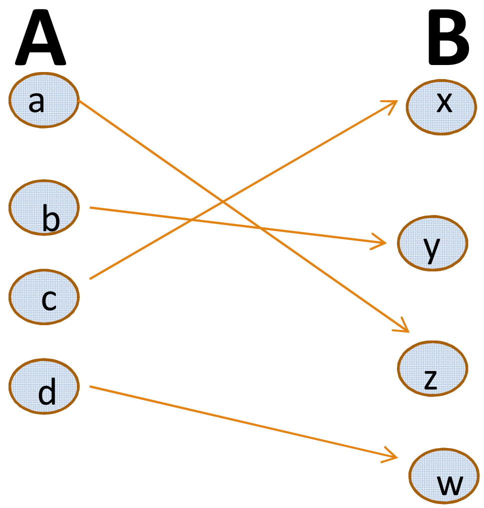
For some functions the range and the codomain are equal. That is, every member of the codomain is the image of some element of the domain.

Functions with this property are called **onto** functions.



Bijections

Definition: A function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto (surjective and injective).



Showing that f is one-to-one or onto

Suppose that $f : A \to B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

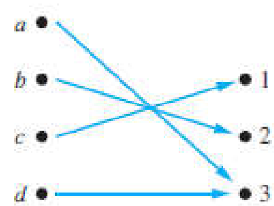
Showing that f is one-to-one or onto

Example 1: Let f be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

Solution: Yes, f is onto since all three elements of the codomain are images of elements in the domain. If the codomain were changed to $\{1,2,3,4\}$, f would not be onto.

Example 2: Is the function $f(x) = x^2$ from the set of integers onto?

Solution: No, f is not onto because there is no integer x with $x^2 = -1$, for example.



Showing that *f* is one-to-one or onto

Example 3: whether the function f from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, and $f(d) = 3$ is one-to-one.

Solution:

The function f is one-to-one because f takes on different values at the four elements of its domain. This is illustrated in Figure 3.

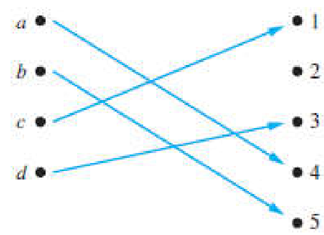


FIGURE 3 A one-to-one function.

Showing that f is bijection

Example 1: Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. **Is f a bijection?**

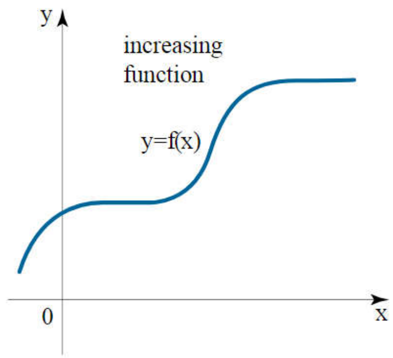
Solution:

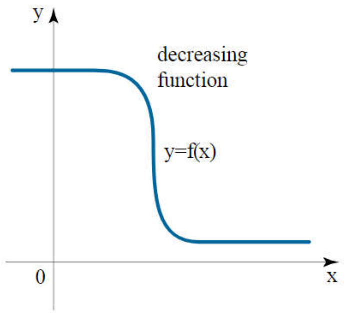
The function f is one-to-one and onto. It is one-to-one because no two values in the domain are assigned the same function value. It is onto because all four elements of the codomain are images of elements in the domain. Hence, f is a bijection.

Increasing/ decreasing functions

A function *f* whose domain and codomain are subsets of the set of real numbers is called *increasing* if $f(x) \leq f(y)$, and *strictly increasing* if $f(x) < f(y)$, whenever $x < y$ and x and y are in the domain of *f.*

Similarly, *f* is called *decreasing* if $f(x) \geq f(y)$, and *strictly decreasing* if $f(x) > f(y)$, whenever $x < y$ and x and y are in the domain of *f.*

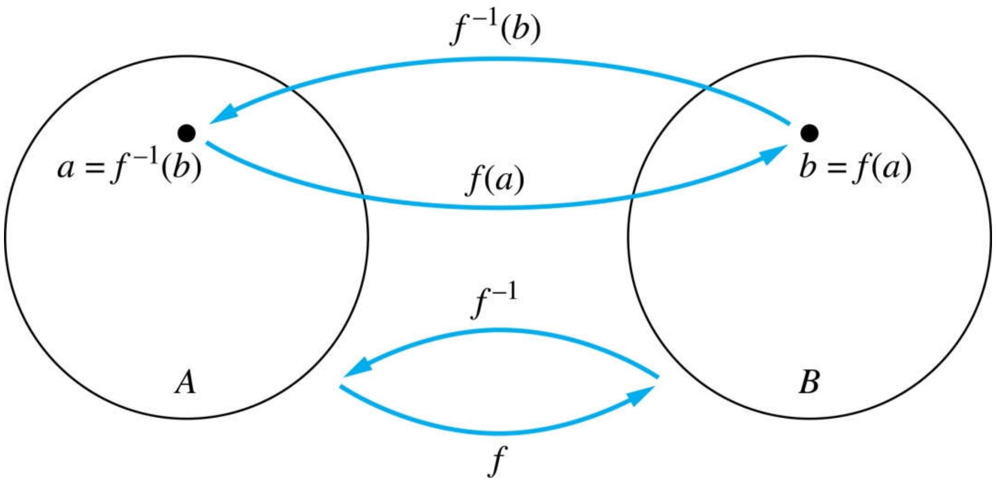




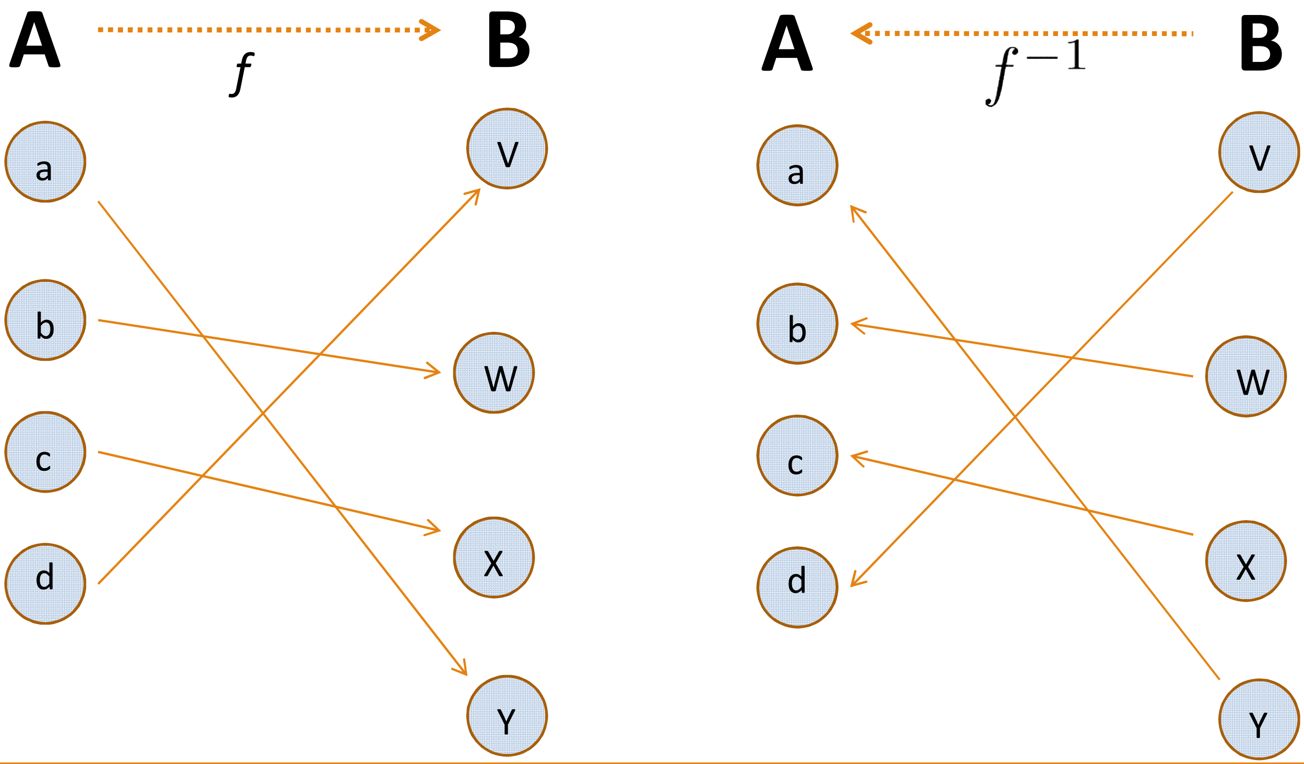
Inverse Functions

Definition: Let *f* be a bijection from *A* to *B*. Then the *inverse* of *f*, denoted f^{-1}, is the function from *B* to *A* defined as $f^{-1}(y) = x \text{ iff } f(x) = y$

No inverse exists unless *f* is a bijection. Why?



Inverse Functions



Questions

Example 1: Let f be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$.

Is f invertible and if so what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Questions

Example 2: Let f: **Z** $\rightarrow$ **Z** be such that $f(x) = x + 1$.

Is f invertible, and if so, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence so $f^{-1}(y) = y - 1$.

Questions

Example 3: Let *f*: **R** $\rightarrow$ **R** be such that $f(x) = x^2$

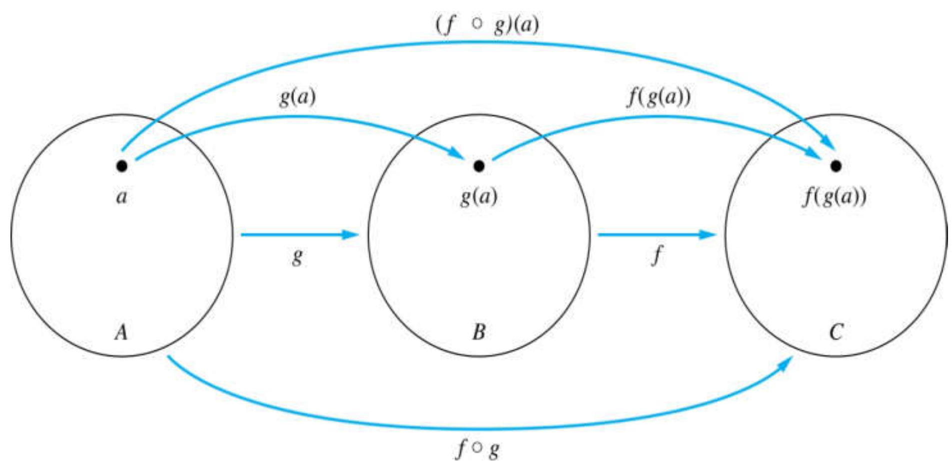
Is *f* invertible, and if so, what is its inverse?

Solution: The function *f* is not invertible because it is not one-to-one .

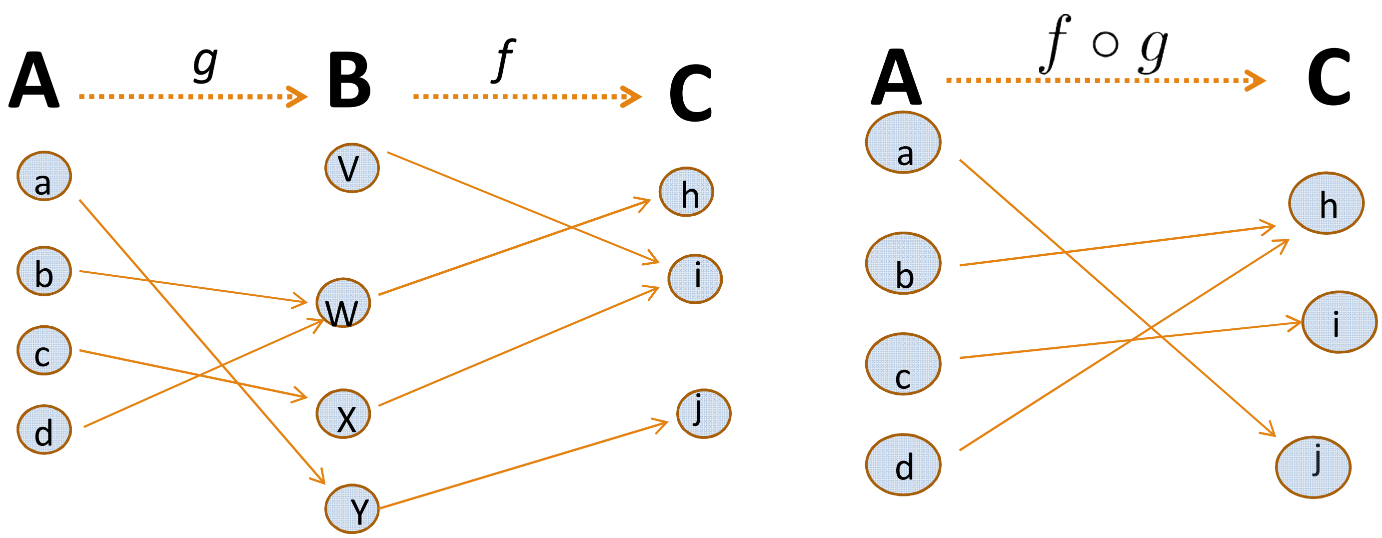
Composition

Definition: Let *f*: $B \rightarrow C$, *g*: $A \rightarrow B$. The *composition of f with g*, denoted $f \circ g$ is the function from *A* to *C* defined by

$$f \circ g(x) = f(g(x))$$



Composition



Composition

Example 1: If $f(x) = x^2$ and $g(x) = 2x + 1$, then

$$f(g(x)) = (2x + 1)^2$$

and

$$g(f(x)) = 2x^2 + 1$$

Composition Questions

Example 2: Let g be the function from the set $\{a,b,c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a,b,c\}$ to the set $\{1,2,3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.

What is the composition of f and g, and what is the composition of g and f.

Solution: The composition $f \circ g$ is defined by

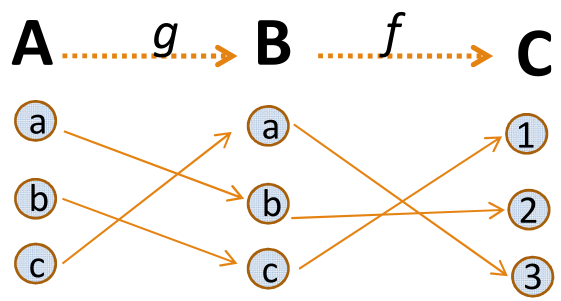
$f \circ g\ (a) = f(g(a)) = f(b) = 2.$

$f \circ g\ (b) = f(g(b)) = f(c) = 1.$

$f \circ g\ (c) = f(g(c)) = f(a) = 3.$

Note that $g \circ f$ is not defined,

because the range of f is not a subset of the domain of g.



Composition Questions

Example 2: Let f and g be functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$.

What is the composition of f and g, and also the composition of g and f ?

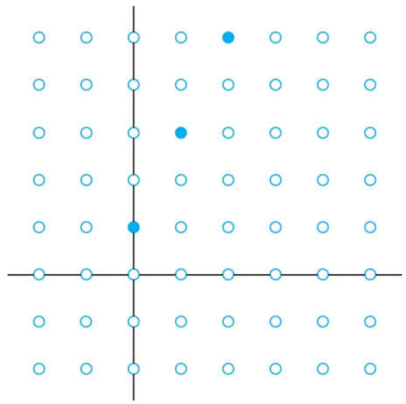
Solution:

$f \circ g\ (x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$

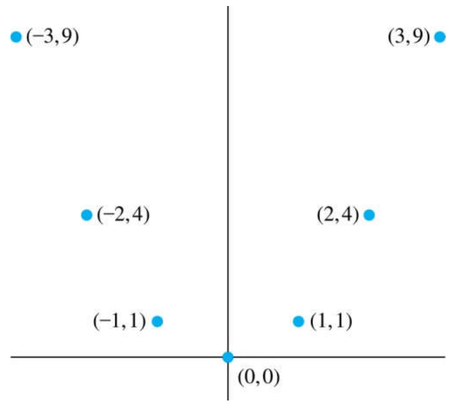
$g \circ f\ (x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$

Graphs of Functions

Let f be a function from the set A to the set B. The *graph* of the function f is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a) = b\}$.



Graph of $f(n) = 2n + 1$ from Z to Z



Graph of $f(x) = x^2$ from Z to Z

Some Important Functions

The *floor* function, denoted by

$$f(x) = \lfloor x \rfloor$$

is the largest integer less than or equal to *x*.

The *ceiling* function, denoted by

$$f(x) = \lceil x \rceil$$

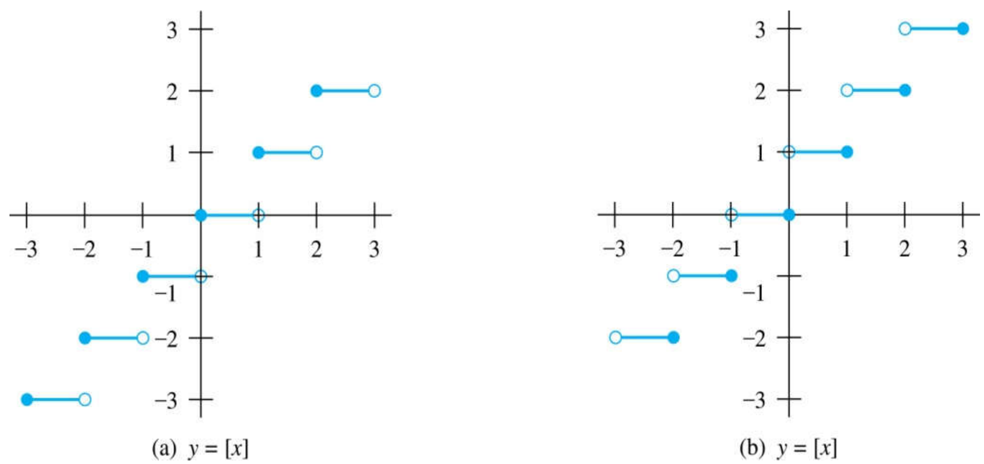
is the smallest integer greater than or equal to *x*

Example:

$$\lceil 3.5 \rceil = 4 \qquad \lfloor 3.5 \rfloor = 3$$

$$\lceil -1.5 \rceil = -1 \qquad \lfloor -1.5 \rfloor = -2$$

Floor and Ceiling Functions



Graph of (a) Floor and (b) Ceiling Functions

Floor and Ceiling Functions

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer, x is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \le x < n+1$
(1b)	$\lceil x \rceil = n$ if and only if $n-1 < x \le n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x-1 < n \le x$
(1d)	$\lceil x \rceil = n$ if and only if $x \le n < x+1$
(2)	$x-1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x+1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x+n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x+n \rceil = \lceil x \rceil + n$

Factorial Function

Definition: f: **N** → **Z**$^+$ **,** denoted by $f(n) = n!$ is the product of the first n positive integers when n is a nonnegative integer.

$f(n) = 1 \cdot 2 \cdots (n-1) \cdot n,\qquad f(0) = 0! = 1$

Examples:

$f(1) = 1! = 1$

$f(2) = 2! = 1 \cdot 2 = 2$

$f(6) = 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$

$f(20) = 2{,}432{,}902{,}008{,}176{,}640{,}000.$

Stirling's Formula:

$$n! \sim \sqrt{2\pi n}(n/e)^n$$

$$f(n) \sim g(n) \doteq lim_{n\to\infty} f(n)/g(n) = 1$$

Thank you!!!

Understanding Math by reading slides is similar to Learning to swim by watching TV.

So, DO PRACTICE IT!