FAST- National University of Computer & Emerging Sciences, Karachi.
Department of Computer Science
Final Examinations, Fall 2018 -- Solution
December 19, 2018, 12:30 pm – 3:30 pm

Course Code: CS 211	**Course Name:** Discrete Structures
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Student's Roll No:	**Section:**

Instructions:

- Return the question paper.
- Read each question completely before answering it. There are **6 questions on 4 pages.**
- In case of any ambiguity, you may make assumption. But your assumption should not contradict any statement in the question paper.
- All the answers must be solved according to the sequence given in the question paper, otherwise marks will be deducted.

Time Allowed: 180 minutes. **Maximum Points**: 70 points

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Logical Proofs **[30 Minutes]** [4x2 = 8 points]

Q No. 1 (i) Express these system specifications using the propositions p "Swimming at the New Jersey shore is allowed" and q "Sharks have been spotted near the shore" together with logical connectives (including negation).

a) "If sharks have not been spotted near the shore, then swimming at the New Jersey shore is allowed."

Solution: $\neg q \rightarrow p$

b) "Swimming at the New Jersey shore is not allowed, and either swimming at the New Jersey shore is allowed or sharks have not been spotted near the shore."

Solution: $\neg p \wedge (p \vee \neg q)$

(ii) Prove the following logical equivalence using the laws of logic (Algebra of Proposition):

$\neg[r \vee (q \wedge (\neg r \rightarrow \neg p))] \equiv \neg r \wedge (p \vee \neg q)$

Solution: We will begin with $\neg [r \vee (q \wedge (\neg r \rightarrow \neg p))]$ and use rules of logic to show that this is equivalent to $\neg r \wedge (p \vee \neg q)$.

Here is one possible proof: $\neg[r \vee (q \wedge (\neg r \rightarrow \neg p))]$

$\equiv \neg r \wedge \neg (q \wedge (\neg r \rightarrow \neg p))$	**De Morgan's law**
$\equiv \neg r \wedge \neg (q \wedge (\neg\neg r \vee \neg p))$	**conditional rewritten as disjunction**
$\equiv \neg r \wedge \neg (q \wedge (r \vee \neg p))$	**double negation law**
$\equiv \neg r \wedge (\neg q \vee \neg(r \vee \neg p))$	**De Morgan's law**
$\equiv \neg r \wedge (\neg q \vee (\neg r \wedge p))$	**De Morgan's law and double negation**
$\equiv (\neg r \wedge \neg q) \vee (\neg r \wedge (\neg r \wedge p))$	**distributive law**
$\equiv (\neg r \wedge \neg q) \vee ((\neg r \wedge \neg r) \wedge p)$	**associative law**
$\equiv (\neg r \wedge \neg q) \vee (\neg r \wedge p)$	**idempotent law**
$\equiv \neg r \wedge (\neg q \vee p)$	**distributive law**
$\equiv \neg r \wedge (p \vee \neg q)$	**commutative law**

(iii) Suppose the variable x represents students and y represents courses, and:
M(y): y is a math course, F (x): x is a freshman, B(x): x is a full-time student and T (x, y): x is taking y.
Write the statement in good English without using variables in your answers.

a) $\forall x \exists y\, T(x, y)$.

Solution: Every student is taking a course.

b) $\forall x \exists y\, [(B(x) \wedge F(x)) \rightarrow (M(y) \wedge T(x, y))]$.

Solution: Every full-time freshman is taking a math course.

(iv) What relevant conclusion or conclusions can be drawn from the following premises? Also, explain the rules of inference used to obtain each conclusion from the premises.
“If you send me an email message, then I will finish writing the program,” “If you do not send me an email message, then I will go to sleep early,” and “If I go to sleep early, then I will wake up feeling refreshed”. Where:

p: “You send me an email message” q: “I will finish writing the program”
r: “I will go to sleep early” s: “I will wake up feeling refreshed.”

Solution:
Then the hypotheses are $p \rightarrow q$, $\neg p \rightarrow r$, $r \rightarrow s$ and the conclusion is $\neg q \rightarrow s$.

Step	Reason
1. $p \rightarrow q$	**Hypothesis**
2. $\neg q \rightarrow \neg p$	**Contrapositive of (1)**
3. $\neg p \rightarrow r$	**Hypothesis**
4. $\neg q \rightarrow r$	**Hypothetical Syllogism of (2) & (3)**
5. $r \rightarrow s$	**Hypothesis**
6. $\neg q \rightarrow s$	**Hypothetical Syllogism of (4) & (5)**

Sets, Functions, Relations and Mathematical Induction **[20 Minutes]** [3 x 2 + 4 = 10 points]

Q. No. 2 (i) Solve the following: Let A and B be sets. Using Set Builder form, show that:
$A \cap (B - A) = \emptyset$.

Solution:

L.H.S = $A \cap (B - A)$
= $\{x \mid (x \in A) \wedge (x \in B \wedge x \notin A)\}$
= $\{x \mid (x \in A \wedge x \notin A) \wedge (x \in B)\}$ **Associative Law**
= $\{x \mid (x \in \emptyset) \wedge (x \in B)\}$ **$A \cap A^c = \emptyset$**
= $\{x \mid x \in \emptyset\}$ **$\emptyset \cap A = \emptyset$**
Hence Proved

(ii) Define g: $R \rightarrow Z$ by the rule $g(x) = \lceil 2x - 1 \rceil$ for all integers x.

a) Is g one-to-one (injective)? Prove or give a counterexample.

Solution: Because g (0.3) = g(0.4)=0

b) Is g onto (surjective)? Prove or give a counterexample.

Solution: Because g (0.3) = g(0.4)=0 and Co-domain= range

(iii) Determine whether the relation R on the set of all Web pages is reflexive, symmetric, and/or transitive, where (a, b) ∈ R if and only if:

a) everyone who has visited Web page a has also visited Web page b.

Solution:

(I) **Reflexive:** **everyone who has visited web page *a* has also visited web page *a*.**

(II) **NOT Symmetric:** **if everyone who has visited web page *a* has visited web page *b*, this does not mean that everyone who has visited web page *b* has also visited web page *a*.**

(III) **Transitive:** **if everyone who has visited web page *a* also has visited web page *b* and everyone who has visited web page *b* also has visited web page *c*, then it follows that everyone who has visited web page *a* has also visited web page *c*.**

b) there is a Web page that includes links to both Web page a and Web page b.

(I) **NOT Reflexive:** **(a,b) ∈ R, if there exist a web-page (say web-page m) such that m has links for both web-page a and web-page b . Now,(a, a) need not belong to R, as there could be no web-page pointing to web-page a.**

(II) **Symmetric:** **if (a, b) ∈ R, i.e there is a web-page linking to web-page a and web-page b, then (b,a) ∈ R. Hence, Symmetric.**

(III) **NOT Transitive:** **If (a, b) ∈ R, and (b, c) ∈R then it is not necessary that (a,c) ∈ R. If web-page e points to web-page a and web-page b, web-page f points to web-page b and web-page c then it is not necessary that there exit a web-page which'll point to both web-page and web-page c.**

(iv) Consider $A_1, A_2, \ldots A_n$, and B are sets; and the following proposition:
$(A_1 - B) \cap (A_2 - B) \cap \ldots \cap (A_n - B) = (A_1 \cap A_2 \cap \ldots \cap A_n) - B$
Answer the following using the principles of Mathematical Induction.

a) Show that P (1) is true, completing the basis step of the proof.

b) What is the inductive hypothesis?

c) What do you need to prove in the inductive step? Complete the inductive step, identifying where you use the inductive hypothesis.

d) Explain why these steps show that this formula is true.

Solution:

If $n = 1$ there is nothing to prove, and the $n = 2$ case says that $(A_1 \cap \overline{B}) \cap (A_2 \cap \overline{B}) = (A_1 \cap A_2) \cap \overline{B}$, which is certainly true, since an element is in each side if and only if it is in all three of the sets A_1, A_2, and $\overline{B}$. Those take care of the basis step. For the inductive step, assume that

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) = (A_1 \cap A_2 \cap \cdots \cap A_n) - B;$$

we must show that

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) \cap (A_{n+1} - B) = (A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1}) - B.$$

We have

$$\begin{aligned}
&(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) \cap (A_{n+1} - B) \\
&\qquad = ((A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B)) \cap (A_{n+1} - B) \\
&\qquad = ((A_1 \cap A_2 \cap \cdots \cap A_n) - B) \cap (A_{n+1}) - B) \\
&\qquad = (A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1}) - B.
\end{aligned}$$

The third line follows from the inductive hypothesis, and the fourth line follows from the $n = 2$ case.

OR

Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) = (A_1 \cap A_2 \cap \cdots \cap A_n) - B$$

answer:

$P(n) : (A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) = (A_1 \cap A_2 \cap \cdots \cap A_n) - B$

Basis Step: $p(1)$ is trivial: $A_1 - B = A_1 - B$

Inductive Step: Assume $P(k)$ holds. We show $P(k+1)$ also holds.

$$\begin{aligned}
&(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_k - B) \cap (A_{k+1} - B) = \\
&\Big((A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_k - B)\Big) \cap (A_{k+1} - B) = \\
&\Big((A_1 \cap A_2 \cap \cdots \cap A_k) - B\Big) \cap (A_{k+1} - B) = \\
&\Big((A_1 \cap A_2 \cap \cdots \cap A_k) \cap \overline{B}\Big) \cap (A_{k+1} \cap \overline{B}) = \\
&(A_1 \cap A_2 \cap \cdots \cap A_k) \cap A_{k+1} \cap \overline{B} = \\
&(A_1 \cap A_2 \cap \cdots \cap A_k \cap A_{k+1}) - B
\end{aligned}$$

Number Theory **[25 Minutes]** [2 + 2 + 4 = 8 points]

Q. No. 3 (i) List all integers between −100 and 100 that are congruent to −1 modulo 25.

Solution:

SOLUTION

$$a \equiv -1 \bmod 25$$

We can find all values of a such that $-100 \leq a \leq 100$ by consecutively subtracting/adding 25 from -1 until we obtain all values between -100 and 100.

We first start by consecutively subtracting:

$$\begin{aligned}
a &\equiv -1 \bmod 25 \\
&\equiv -1 - 25 \bmod 25 \\
&\equiv -26 \bmod 25 \\
&\equiv -26 - 25 \bmod 25 \\
&\equiv -51 \bmod 25 \\
&\equiv -51 - 25 \bmod 25 \\
&\equiv -76 \bmod 25 \\
&\equiv -76 - 25 \bmod 25 \\
&\equiv -101 \bmod 25
\end{aligned}$$

Since -101 is the first value not in $-100 \leq a \leq 100$, a can then take on all previous values in the above derivation:

$$a = \{-76, -51, -26, -1\}$$

We next start by consecutively adding:

$$a \equiv -1 \bmod 25$$
$$\equiv -1 + 25 \bmod 25$$
$$\equiv 24 \bmod 25$$
$$\equiv 24 + 25 \bmod 25$$
$$\equiv 49 \bmod 25$$
$$\equiv 49 + 25 \bmod 25$$
$$\equiv 74 \bmod 25$$
$$\equiv 74 + 25 \bmod 25$$
$$\equiv 99 \bmod 25$$
$$\equiv 99 + 25 \bmod 25$$
$$\equiv 124 \bmod 25$$

Since 124 is the first value not in $-100 \leq a \leq 100$, a can then take on all previous values in the above derivation:

$$a = \{-1, 24, 49, 74, 99\}$$

Combining this set with the set obtained using consecutive subtractions:

$$a = \{-76, -51, -26, -1, 24, 49, 74, 99\}$$

(ii) Write C++ code for finding Multiplicate Inverse $\overline{a}$ of an integer a such that $a \,.\, \overline{a} = 1 \bmod m$.

Solution:

```
// A naive method to find modular multiplicative inverse of 'a' under modulo 'm'
int modInverse(int a, int m)
{
      a = a%m;
      for (int x=1; x<m; x++)
          if ((a*x) % m == 1)
              return x;
}
```

https://www.geeksforgeeks.org/multiplicative-inverse-under-modulo-m/

(iii) Chinese Reminder Theorem: An old woman goes to market and a horse steps on her basket and crushes the eggs. The rider offers to pay for the damages and asks her how many eggs she had brought. She does not remember the exact number, but when she had taken them out three at a time, there were two eggs left. When she took them four at a time, there was one egg left, and when she had taken them out five at a time, there were three eggs left. What is the smallest number of eggs she could have had?

We will follow the notation used in the proof of the Chinese remainder theorem.
We have $m=m_1 \times m_2 \times m_3 = 60$
Also, by simple inspection we see that:
$y_1 = 2$ is an inverse for $M_1 = 20$ modulo 3,
$y_2 = 3$ is an inverse for $M_2 = 15$ modulo 4, and
$y_3 = 3$ is an inverse for $M_3 = 12$ modulo 5.
The solutions to the system are then all numbers x such that
$x = a_1M_1y_1 + a_2M_2y_2 + a_3M_3y_3 = 2 * 20 * 2 + 1 * 15 * 3 + 3 * 12 * 3 = 233 \pmod{60} = 53$.
She could have 53 eggs.

Theorem Proofs: **[30 Minutes]** [5 x 2 = 10 points]

Q. No. 4 Prove the following theorems:

(i) If A = {x ∈ ℤ | x is a multiple of 5} and B = {x ∈ ℤ | x is a multiple of 7}. Then A ∩ B = {x ∈ ℤ | x is a multiple of 35}.

Solution: This question is more inclined towards evaluating whether the students understand set builder notation instead of evaluating their skills in direct proof. A full solution is given as follow:

Proof: Part 1. $A \cap B \subseteq \{x \in \mathbb{Z} \mid x \text{ is a multiple of } 35\}$

Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. This implies that x is a multiple of 5 and it is a multiple of 7. Therefore, $x = 5n$ and $x = 7m$ with n and m integer numbers.

If we combine these two equalities, we obtain $5n = 7m$. As 5 and 7 are prime numbers, $5n$ is divisible by 7 only if n is divisible by 7. Thus, $n = 7k$ for some integer number k. Therefore, $x = 5n = 5(7k) = 35k$ for some integer number k. This means that x is a multiple of 35.

Part 2. $\{x \in \mathbb{Z} \mid x \text{ is a multiple of } 35\} \subseteq A \cap B$

Let x be a multiple of 35. Therefore, $x = 35t$ for some integer number t. Thus, x is divisible by 5 (so $x \in A$) and it is divisible by 7 (so $x \in B$). This implies that $x \in A \cap B$. Therefore, the two sets are equal. ■

(ii) Let n be an integer and S is the following statement.

S: If $n^2 - (n-2)^2$ is not divisible by 8, then n is even.

a) Write down the contrapositive of S. Prove that the contrapositive of S is true.

b) Is the statement S true? Why?

(i) The contrapositive of a statement $p \implies q$ is $\neg q \implies \neg p$. So the contrapositive is:

If n is not even, then $n^2 - (n-2)^2$ is divisible by 8.

Equivalently, it is:

If n is odd, then $n^2 - (n-2)^2$ is divisible by 8.

This is true. For, suppose that n is odd, so that, for some integer k, $n = 2k+1$. We have

$$n^2 - (n-2)^2 = n^2 - (n^2 - 4n + 4) = 4n - 4 = 4(n-1) = 8k,$$

which is an integer multiple of 8. Therefore $n^2 - (n-2)^2$ is divisible by 8.

(ii) Statement S is true because its contrapositive is true and any statement is logically equivalent to its contrapositive. This is all we need to do here: there is no need to prove S is true, since we have already proved its contrapositive is true.

(iii) If n is an even integer, then n + 1 is odd. Give a proof by contraposition of this theorem.

Solution: Contraposition: If n + 1 is even, then n is an odd integer.

Let n + 1 = 2k be the odd number.
n + 1 = 2k
n = 2k -1 where n = (2k - 1) ∈ Z. This shows that n is odd.

(iv) Show that if a | b and b | a, where a and b are integers, then a = b or a = - b.

Solution: Suppose $a \mid b$ and $b \mid a$, where a, b are integers. By definition of divides, we have $b = ka, a = jb$ for some integers k, j. Combining these equations, we see that $a = j(ka)$. We go by cases on if a is zero or not.

Case 1: $a = 0$. Then, we have $b = k \cdot 0 = 0$. So, $a = b = 0$, and the theorem holds.

Case 2: $a \neq 0$. Then, dividing both sides by a, we get $1 = jk$. So, $\frac{1}{j} = k$. Note that j and k are integers, which is only possible if $j, k \in \{1, -1\}$. It follows that $b = -a$ or $b = a$, as required. Since the theorem is true in both cases, it is true.

OR

Answer:

1. $a|b \longrightarrow \exists c(b = ac)$
2. $b|a \longrightarrow \exists d(a = bd)$
3. From 1 and 2, $a = acd$
4. From 3 and the fact $a \neq 0$, $cd = 1$
5. From 4, either $c = d = 1$ or $c = d = -1$
6. From 5, 1 and 2, either $a = b$ or $a = -b$

(v) Let $n \in \mathbb{Z}^+$, with $n \geq 2$. If the sum of the divisors of n is equal to n+1, then n is prime.

PROPOSITION : Let $n \in \mathbb{Z}^+$, with $n \geq 2$.

If the sum of the divisors of n is equal to $n + 1$ then n is prime.

PROOF : We prove the contrapositive :

If n is *not* prime then the sum of the divisors can *not* equal $n + 1$.

So suppose that n is not prime.

Then n has divisors

$$1, \; n, \text{ and } m, \quad \text{for some } m \in \mathbb{Z}^+, \; m \neq 1, \; m \neq n \; ,$$

and possibly more.

Thus the sum of the divisors is greater than $n + 1$. QED !

Quod Erat Demonstrandum which **means** "that which was to be demonstrated", the proof is complete

Combinatorics and Discrete Probability **[35 Minutes]** [9 x 2 = 18 points]

Q. No. 5 Solve all the questions. Give an expression describing the number of different ways the following things can happen. No credit will be given for just the value, even if correct.

(i) Suppose that all license plates have three uppercase letters followed by three digits.

(a) How many license plates begin with A and end in 0?

Solution: Begins with A: 1*26*26*10*10*10=676000

End with 0: 26*26*26*10*10*1=1757600

Begins with A & end with 0: 1*26*26*10*10*1=67600

Hence,

676000+1757600-67600 =2366000

(b) How many license plates are possible in which all the letters and digits are distinct?

Solution:

26*25*24*10*9*8=1123200

(ii) The name of a variable in the JAVA programming language is a string of length between 1 and 65,535 characters, inclusive, where each character can be an uppercase or a lowercase letter, a dollar sign, an underscore, or a digit, except that the first character must not be a digit. Determine the number of different variable names in JAVA.

Solution

1. **Given that the first char must not be a digit, this first character can be chosen in 26(lowercase) + 26(uppercase) + 1(dollar sign) + 1(underscore) = 54 ways.**
2. **All subsequent chars can be chosen in 6465534 ways. As each subsequent digit cab be chosen in (26(uppercase) + 26(lowercase) + 1(dollar sign) + 1(underscore) + 10(digits)) = 64. As there are 65534 digits left, to get the number of possible combination 64 is multiplied 65534 times = 64^{65534}**
   Hence total number of possible names is: $54*64^{65534}$

(iii) Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have more women than men?

Explanation:

First we ask, how many ways are there to choose r objects out of n distinct objects?

The answer turns out to be

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

So for example, how many ways are there to choose 2 men out of 10 men?

The answer is

$$\binom{10}{2} = \frac{10!}{2!\cdot 8!} = 45$$

Now, with the condition that there must be more women than men, we are left with only 3 options: 4w+2m, 5w+1m and 6w+0m.

Number of ways with 4 women and 2 men

$$\binom{15}{4}\cdot\binom{10}{2} = \frac{15!}{4!\cdot 11!}\cdot\frac{10!}{2!\cdot 8!} = 1365\times 45$$

$$= 61425$$

Number of ways with 5 women and 1 man

$$\binom{15}{5}\cdot\binom{10}{1} = \frac{15!}{5!\cdot 10!}\cdot\frac{10!}{1!\cdot 9!} = 3003\times 10$$

$$= 30030$$

Number of ways with 6 women and 0 man

$$\binom{15}{6}\cdot\binom{10}{0} = \frac{15!}{6!\cdot 9!}\cdot\frac{10!}{0!\cdot 10!} = 5005\times 1$$

$$= 5005$$

Now all you have to do is to add the 3 cases up.

$$61425 + 30030 + 5005 = 96460$$

OR

Solution. Such a committee can be formed wither by 0 men and 6 women, or by 1 men and 5 women, or by 2 men and 4 women. There are

$$
\begin{aligned}
C(10,0)\cdot C(15,6)+C(10,1)\cdot C(15,5)+C(10,2)\cdot C(15,4) \\
&= \frac{10!}{0!10!}\cdot\frac{15!}{6!9!}+\frac{10!}{1!9!}\cdot\frac{15!}{5!10!}+\frac{10!}{2!8!}\cdot\frac{15!}{4!11!} \\
&= 1\cdot\frac{15!}{6!9!}+10\cdot\frac{15!}{5!10!}+\frac{10\cdot 9}{2}\frac{15!}{4!11!} \\
&= \frac{15!}{6!11!}(10\cdot 11+10\cdot 6\cdot 11+5\cdot 9\cdot 5\cdot 6) \\
&= \frac{15\cdot 14\cdot 13\cdot 12}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}(110+660+1350) \\
&= \frac{7\cdot 13}{2}\cdot 2120 = 96460.
\end{aligned}
$$

(iv) A professor teaching discrete structures is making up a final exam. He has a stash of 24 questions on probability, 16 questions on combinatorics, and 10 questions on logic. He wishes to put five questions on each topic on the exam. Give an expression describing the number of different ways the above things can happen.

Solution $\binom{24}{5}\binom{16}{5}\binom{10}{5}$ or equivalent

(v) For Q 5 (v and vi) use Pigeonhole Principle. A bowl contains 7 red balls and 7 blue balls. You select balls at random without looking at them.

a. How many balls must you select to be sure of having at least 3 balls of the same color?

Solution: 2 + 2 + 2 + 1 = 7
The colors are the holes and the balls are the pigeons. We need to force there to be 3 pigeons in at least one of the holes. If we select six or fewer balls, then there could be at most 2 pigeons in every hole. If we select 7 balls, then by the pigeonhole principle there must be a hole with at least 3 pigeons and so we have at least three balls of the same color.

b. How many balls must you select to be sure of having at least 2 blue balls?
Solution:
r + b + 3
Like part (a), the colors are the holes and the balls are the pigeons. This time, though, we need to force there to be 3 pigeons in the green hole. The way to do this is to fill up the red hole with r pigeons, the blue hole with b pigeons, and the have 3 more pigeons left over which have to go in the green hole. So we need to select r + b + 3 balls to be certain of having at least 3 green balls. **OR**

1. A bowl contains 7 red balls and 7 blue balls. You select balls at random without looking at them.

a. How many balls must you select to be sure of having at least 3 balls of the same color?

$\boxed{5}$ (pick 2 of one color, pick 2 of other color, the 5^{th} will get you 3 of the same color).

b. How many balls must you select to be sure of having at least 2 blue balls?

$\boxed{9}$ (pick all 7 red balls 1^{st}, then next two are blue).

(vi) How many people out of 100 people were born in the same month?

Solution:

$\lceil 100/12 \rceil = \lceil 8.333 \rceil = 9$

(vii) What is the probability that a positive integer less than 100 picked at random has all distinct digits?

Solution:

Eliminating 11, 22,33,44,55,66,77,88,99 then we have only 90 indistinct integer so 90/99 = 10/11.

(viii) A batch of 40 components contains 5 which are defective. If a component is drawn at random from the batch and tested and then a second component is drawn at random, calculate the probability of having one defective component, both with and without replacement.

Solution:

With replacement:

$$p = \frac{5}{40} = \frac{1}{8}$$

and $$q = \frac{35}{40} = \frac{7}{8}$$

Hence, probability of having one defective component is:

$$\frac{1}{8} \times \frac{7}{8} + \frac{7}{8} \times \frac{1}{8}$$

i.e.

$$\frac{7}{64} + \frac{7}{64} = \frac{7}{32} \text{ or } \mathbf{0.2188}$$

Without replacement:

$p_1 = \frac{1}{8}$ and $q_1 = \frac{7}{8}$ on the first of the two draws. The batch number is now 39 for the second draw, thus,

$$p_2 = \frac{5}{39} \text{ and } q_2 = \frac{35}{39}$$

$$p_1 q_2 + q_1 p_2 = \frac{1}{8} \times \frac{35}{39} + \frac{7}{8} \times \frac{5}{39}$$

$$= \frac{35 + 35}{312}$$

$$= \frac{70}{312} \text{ or } \mathbf{0.2244}$$

(ix) Write a C++ code which computes the combination $\binom{n}{k}$ by making use of the Pascal triangle. You should assume that in your program the Pascal triangle is not stored, and all the necessary values of the Pascal triangle needed to compute $\binom{n}{k}$ is calculated by using a recursive function..

Solution:

```
int pascal(int n, int k)
{
if ((k==0) || (n==k))
return 1;
return pascal(n − 1,k − 1) + pascal(n − 1,k);
}
```

Graph Theory **[35 Minutes]** [8 x 2 = 16 points]

Q. No. 6 Solve all question according to the instructions. No point will be given for just the value, even if correct.

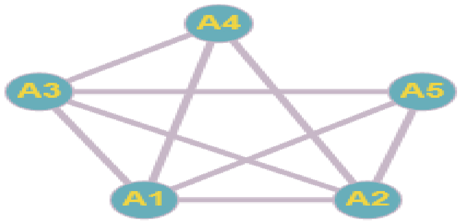
(i) The **intersection graph** of a collection of sets $A_1, A_2, \ldots, A_n$ is the graph that has a vertex for each of these sets and has an edge connecting the vertices representing two sets if these sets have a nonempty intersection. Construct the intersection graph of these collections of sets.

$A_1 = \{\ldots, -4, -3, -2, -1, 0\}$, $A_2 = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$,

$A_3 = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$, $A_4 = \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$,

$A_5 = \{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$

Solution:



(ii) Is there a graph with 102 vertices, such that exactly 49 vertices have degree 5, and the remaining 53 vertices have degree 6? How many vertices does a graph of degree four with 10 edges have?

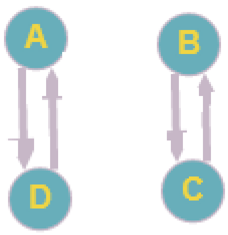
Solution: No, there is no graph with 102 vertices, such that exactly 49 vertices have degree 5, and the remaining 53 vertices have degree 6. Since

$$(49*5) + (53*6) \neq 2e$$

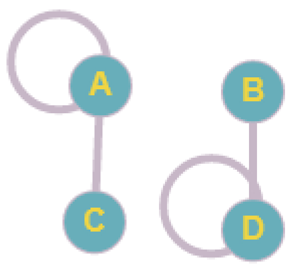
Let N be the total number of vertices. According to handshaking theorem $\sum_{u \in V} deg\ V = 2|E|$
Since degree of every vertices is 4, therefore sum of the degree of all vertices can be written as N *4.
Put the values in above equation,
N*4 = $2|E|$ **implies N= (2*10)/4 = 5. Hence total vertices are 5.**

(iii) Draw directed graphs of the below incidence matrixes.

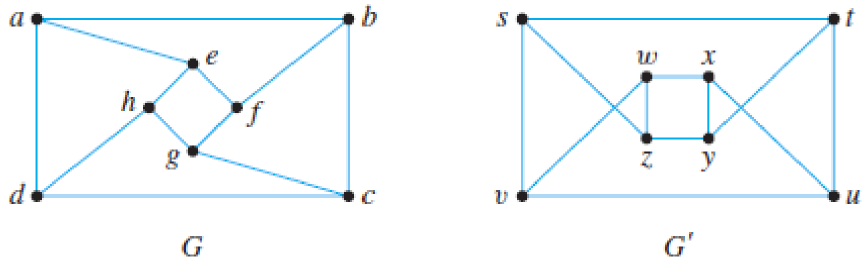
Solution:



$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

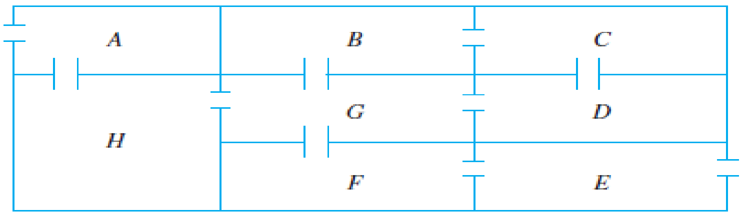


(iv) For given pair of graph G and G'. Determine whether G and G' are isomorphic. If they are, give functions (using vertices and edges) g: V(G) →V (G') and h: E(G) → E(G') that define the isomorphism. If they are not, give an invariant for graph isomorphism that they do not share.

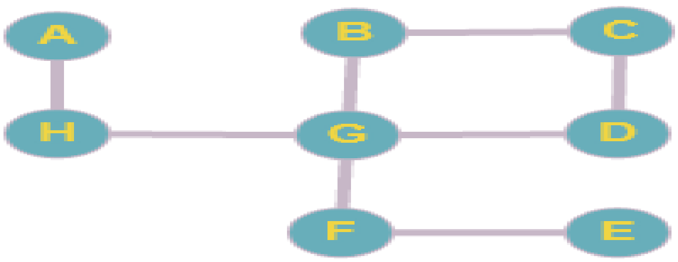


Solution:
Vertex function: F(a)=s, f(b)=t, F(c)=u, F(d)=v, F(e)=z, F(f)=y, F(g)=x, F(h)=w
Edge function: ab=st,bc=tu,cd=uv,ad=sv,ae=sz,ef=yz,fg=xy,eh=wz,gh=wx,bf=ty,cg=ux,dh=vw

(v) The following is a floor plan of a house. Is it possible to enter the house in room A, travel through every interior doorway of the house exactly once, and exit out of room E? If so, how can this be done?

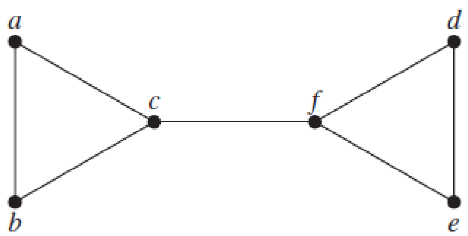


Solution:



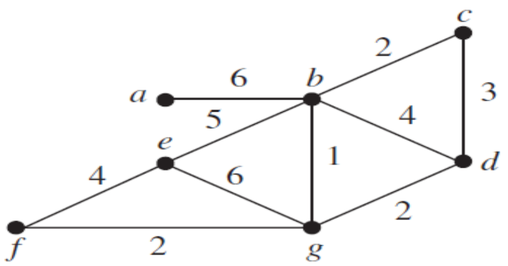
Solution: Not possible because from A,H,G,B,C,D,G is repeating.

(vi) Consider the following definition: “An Euler path in a multigraph G is a *simple path* that contains every edge of G. Simple path means every edge occurs exactly once in the path.
“An Euler circuit in an graph G is a simple circuit that contains every edge of G. So, every edge occurs exactly once in the circuit.”
Determine whether the graph has an (a) Euler path (b) Euler circuit. Construct such a path or circuit, if it exists.

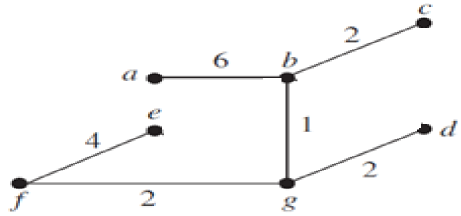


Solution: Euler circuit does not exists because deg(c) = deg (f)=3; which is odd hence Euler path exists.
EP1: c,b,a,c,f,d,e,f EP2:c,a,b,c,f,d,e,f EP3:c,a,b,c,f,e,d,f EP4:c,b,a,c,f,e,d,f

(vii) Find a minimum spanning tree for the following graph in Figure (iv), where the degree of each vertex in the spanning tree does not exceed 3.



Solution:



cost=17

Figure (iv)

(viii) Find the length (sum of weights) of a shortest path between a and z in the above figure (v) weighted graph. Use Dijkstra's Algorithm or otherwise.

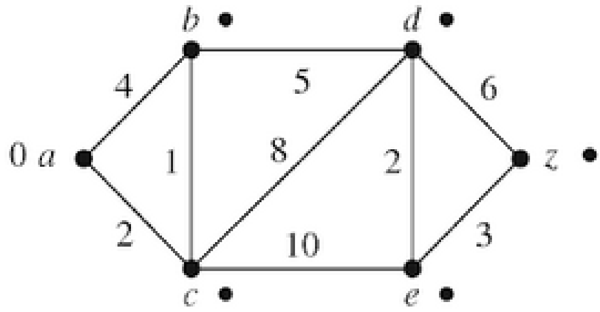


Figure (v)

Solution:
First iteration: distinguished vertices: a; labels: a : 0, b : 3, c : 7, d : 7, e : 7, z : 7.
Second iteration: distinguished vertices: a, b; labels: a : 0, b : 3, c : 5, d : 9, e : 7, z : 7.
Third iteration: distinguished vertices: a, b, c; labels: a : 0, b : 3, c : 5, d : 6, e : 11, z : 7.
Fourth iteration: distinguished vertices: a, b, c, d; labels: a : 0, b : 3, c : 5, d : 6, e : 8, z : 14.
Fifth iteration: distinguished vertices: a, b, c, d, e; labels: a : 0, b : 3, c : 5, d : 6, e : 8, z : 13.
Since at the next iteration z is a distinguished vertex, we conclude that the shortest path has length 13.