National University of computer and emerging sciences

DISCRETE STRUCTURES

COURSE INSTRUCTOR: MUHAMMAD SAIF UL ISLAM

Course Outline

- **Logic and Proofs** (Chapter 1)
- **Sets and Functions** (Chapter 2)
- **Relations** (Chapter 9)
- **Number Theory** (Chapter 4)
- **Combinatorics** (Chapter 6)
- **Graphs** (Chapter 10)
- **Trees** (Chapter 11)
- **Discrete Probability** (Chapter 7)

Lecture Outline

- Probability Rules
- Complementary Events
- Mutually Exclusive Vs Non-M.E
- Dependent Vs Independent
- Conditional Probability
- Random Variables and Probability Distributions

Probability

Experiment: It is a situation involving chance or probability that leads to results called outcomes.

Outcome: An **outcome** is the result of a single trial of an experiment.

Event: An **event** is one or more outcomes of an experiment.

Probability: It is the measure of how likely an event is.

The probability of an event occurring is a number between 0 and 1, and represents essentially how often that event occurs.

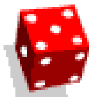
Sample Space: A sample space is the set of all possible outcomes of a random process.

The probability of an event E is

$$P(E) = \frac{\text{Number of outcomes in E}}{\text{Number of outcomes in the sample space}} = \frac{n(E)}{n(S)}$$

In “Discrete Probability”, we focus on finite and countable sample spaces.

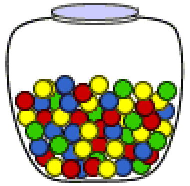
Probability

Experiment 1: What is the probability of each outcome when a single 6-sided die is rolled? 

Sample Space: {1, 2, 3, 4, 5, 6}

Probabilities: P(1)= 1/6, P(2)= 1/6, P(3)= 1/6, P(4)= 1/6, P(5)= 1/6, P(6)= 1/6

Experiment 2: A glass jar contains 1 red, 3 green, 2 blue and 4 yellow marbles. If a single marble is chosen at random from the jar, what is the probability of each outcome?



Sample Space: {red, green, blue, yellow}

Probabilities: P(red)= 1/10, P(green)= 3/10, P(blue)= 2/10, P(yellow)= 4/10

What is the sample space for choosing 1 jelly bean at random from a jar containing 5 red, 7 blue and 2 green jelly beans?

What is the sample space for choosing a prime number less than 15 at random?

Probability Rules

Probability Rules

1. The Probability of an event E must be a number between 0 and 1. i.e., $0 \leq P(E) \leq 1$.

2. If an event E **cannot** occur, then its probability is **0**.

3. If an event E **must** occur, then its probability is **1**.

4. The sum of all probabilities of all the outcomes in the sample space is 1.

Probability

Example: Find the sample space for drawing one card from an ordinary deck of cards.

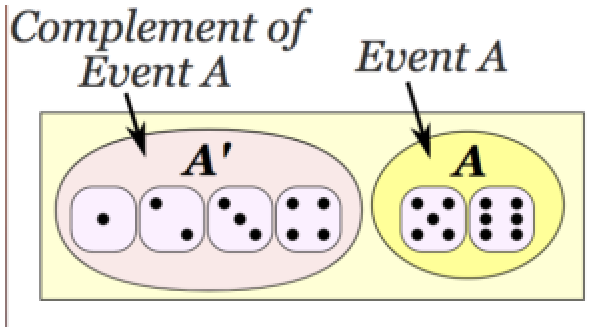
A	2	3	4	5	6	7	8	9	10	J	Q	K
♥	♥	♥	♥	♥	♥	♥	♥	♥	♥	♥	♥	♥
A	2	3	4	5	6	7	8	9	10	J	Q	K
♦	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦
A	2	3	4	5	6	7	8	9	10	J	Q	K
♠	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠
A	2	3	4	5	6	7	8	9	10	J	Q	K
♣	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣

Sample space consists of all possible 13x4=52 outcomes:
A♥, 2♥,...,K♥,..., A♣, 2♣,...,K♣

Example: Two dice are tossed.

		1st Die					
		1	**2**	**3**	**4**	**5**	**6**
2nd Die	**1**	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

Complementary Events

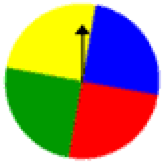


Definition: The **complement** of an event A is the set of all outcomes in the sample space that are not included in the outcomes of event A. The complement of event A is represented by $\bar{A}$ *(read as A bar)*.

Rule for Complementary Events:

$$P(\bar{E}) = 1 - P(E) \text{ or } P(E) = 1 - P(\bar{E}) \text{ or } P(E) + P(\bar{E}) = 1.$$

Experiment: A spinner has 4 equal sectors colored yellow, blue, green and red. What is the probability of landing on a sector that is not red after spinning this spinner?



Sample Space: {yellow, blue, green, red}

Probability: The probability of each outcome in this experiment is one fourth. The probability of landing on a sector that is not red is the same as the probability of landing on all the other colors except red.

$$P(\text{not red}) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$$

Complementary Events

The collection of all events is non-empty and satisfies the following:

1. If A is an event, so is A^c, the event that A doesn't happen.
2. If A and B are events, so is $A \cup B$, the event that (either or both) A or B happens.
3. If A and B are events, so is $A \cap B$, the event that A and B happen.

Example, let U be the sample space of all sequences of three coin tosses described above, and consider the following events:

$A = \{\text{HTT, HTH, HHT, HHH}\}$	The first flip was heads.
$A^c = \{\text{TTT, TTH, THT, TTT}\}$	The first flip was not a head.
$B = \{\text{TTH, THH, HTH, HHH}\}$	The third coin flip is heads.
$A \cup B = \{\text{TTH, THH, HTT, HTH, HHT, HHH}\}$	The first or third flip was heads.
$A \cap B = \{\text{HTH, HHH}\}$	The first and third flip were heads.
$C = \{\text{THH, HTH, HHT, HHH}\}$	Their were more heads than tails.

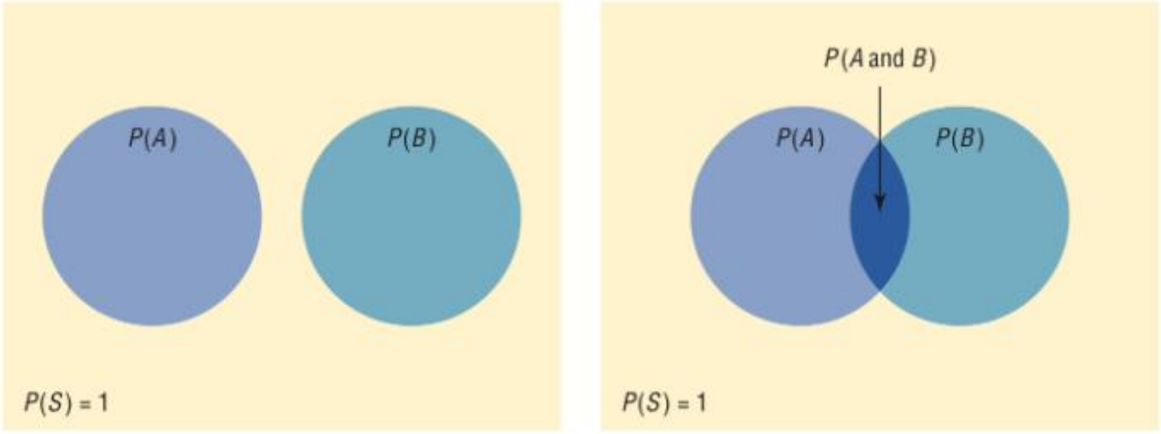
Mutually Vs Non-Mutually Exclusive Events

Mutually Exclusive Events

Two events are **mutually exclusive** if they cannot occur at the same time (i.e., they have no outcomes in common).

Non-Mutually Exclusive Events

Two events are non-mutually exclusive if they have one or more outcomes in common.

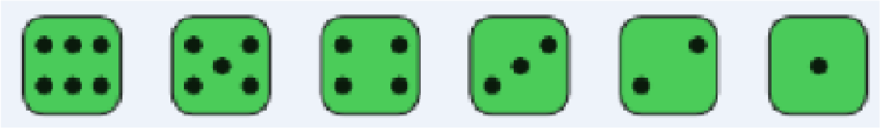


(a) Mutually exclusive events
$P(A \text{ or } B) = P(A) + P(B)$

(b) Nonmutually exclusive events
$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

Non-Mutually Exclusive Events

<u>Example</u>: Suppose we roll a six-sided die. Let A be that we roll an even number. Let B be that we roll a number greater than 3.



A A A

B B B

What is the intersection between A and B?

Rolling a 6 or 4

What is the union of A and B?

Rolling a 6, 5, 4, or 2

Mutually Exclusive Events

Example: At a political rally, there are 20 Republicans, 13 Democrats, and 6 Independents. If a person is selected at random, find the probability that he or she is either a Democrat or an Independent.

Event A = a person is a democrat
Event B = a person is an independent
These are mutually exclusive since you can NOT be both.

$$
\begin{aligned}
P(\text{a person is a Democrat or an Independent}) &= P(\text{A or B}) \\
&= P(A) + P(B) \\
&= \frac{13}{20+13+6} + \frac{6}{20+13+6} \\
&= \frac{13}{39} + \frac{6}{39} \\
&= \frac{19}{39} \approx 0.487
\end{aligned}
$$

Mutually Exclusive Events

The probabilities of three teams A, B and C winning a badminton competition are $\frac{1}{3}, \frac{1}{5}$ and $\frac{1}{9}$ respectively.

Calculate the probability that
a) either A or B will win
b) either A or B or C will win
c) none of these teams will win
d) neither A nor B will win

Solution:

a) $P(\text{A or B will win}) = \frac{1}{3} + \frac{1}{5} = \frac{8}{15}$

b) $P(\text{A or B or C will win}) = \frac{1}{3} + \frac{1}{5} + \frac{1}{9} = \frac{29}{45}$

c) P(none will win) = 1 – P(A or B or C will win) $= 1 - \frac{29}{45} = \frac{16}{45}$

d) P(neither A nor B will win) = 1 – P(either A or B will win) $= 1 - \frac{8}{15} = \frac{7}{15}$

Dependent Vs Independent Events

Independent - two events *A* and *B* are independent events if the fact that *A* occurs does not affect the probability of *B* occurring.

Example: Rolling one die and getting a six, rolling a second die and getting a three.

Example: Draw a card from a deck and replacing it, drawing a second card from the deck and getting a queen.

In each example, the first event has no effect on the probability of the second event.

Probability of Independent Events

When two events, A and B, are independent the probability of both occurring is:

P(A and B) = P(A) x P(B)

Example:

If a dice is thrown twice, find the probability of getting two 5's.

$$P(\text{getting a 5 on the first throw}) = \frac{1}{6}$$

$$P(\text{getting a 5 on the second throw}) = \frac{1}{6}$$

$$P(\text{two 5's}) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

Dependent Events

Dependent - Two outcomes are said to be *dependent* if knowing that one of the outcomes has occurred affects the probability that the other occurs.

Examples:

- Drawing a card from a deck, not replacing it, and then drawing a second card.
- Being a lifeguard and getting a suntan
- Having high grades and getting a scholarship
- Parking in a no-parking zone and getting a ticket

Dependent Events

The outcome of one event affects the outcome of the other.

If A and B are dependent events then the probability of both occurring is

$$P(A \text{ and } B) = P(A) \times \underbrace{P(B|A)}_{\text{Probability of B given A}}$$

Dependent Events

Example:

A purse contains four \$5 bills, five \$10 bills and three \$20 bills. Two bills are selected without the first selection being replaced. Find P(\$5, then \$5)

Solution:

There are four \$5 bills.

There are a total of twelve bills.

P(\$5) = $\frac{4}{12}$

The result of the first draw affected the probability of the second draw.

There are three \$5 bills left.

There are a total of eleven bills left.

P(\$5 after \$5) = $\frac{3}{11}$

P(\$5, then \$5) = P(\$5) · P(\$5 after \$5) = $\frac{4}{12} \times \frac{3}{11} = \frac{1}{11}$

The probability of drawing a \$5 bill and then a \$5 bill is $\frac{1}{11}$

A bag contains 6 red, 5 blue and 4 yellow marbles. Two marbles are drawn, but the first marble drawn is not replaced.
a) Find P(red, then blue)
b) Find P(blue, then blue)

We have a box with 10 red marbles and 10 blue marbles. Find P(drawing two blue marbles).

Conditional Probability

The probability of an event occurring given that another event has already occurred is called a **conditional probability**.

Recall that when two events, A and B, are dependent, the probability of both occurring is

P(A and B) = P(A) × P(B given A)
or P(A and B) = P(A) × P(B | A)

If we divide both sides of the equation by P(*A*) we get the **Formula for Conditional Probability**

$$\frac{P(A \text{ and } B)}{P(A)} = \frac{P(A) \times P(B|A)}{P(A)}$$

$$P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)}$$

How to find the Conditional Probability from a word problem?
Step 1: Write out the Conditional Probability Formula in terms of the problem
Step 2: Substitute in the values and solve

Conditional Probability

Example:

Susan took two tests. The probability of her passing both tests is 0.6. The probability of her passing the first test is 0.8. What is the probability of her passing the second test given that she has passed the first test?

$$P(\text{second} \mid \text{first}) = \frac{P(\text{first and second})}{P(\text{first})} = \frac{0.6}{0.8} = 0.75$$

A bag contains red and blue marbles. Two marbles are drawn without replacement. The probability of selecting a red marble and then a blue marble is 0.28. The probability of selecting a red marble on the first draw is 0.5. What is the probability of selecting a blue marble on the second draw, given that the first marble drawn was red

$$P(Blue \mid \text{Red}) = \frac{P(\text{Blue and Red})}{P(\text{Red})} = \frac{0.28}{0.5} = 0.56$$

Conditional Probability

Example:

What is the probability that the total of two dice will be greater than 9, given that the first die is a 5?

Let A = first die is 5
Let B = total of two dice is greater than 9

P(A) = $\frac{1}{6}$

Possible outcomes for A and B: (5, 5), (5, 6)
P(A and B) = $\frac{2}{36} = \frac{1}{18}$

$$P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)} = \frac{1}{18} \div \frac{1}{6} = \frac{1}{3}$$

Random Variables

A variable is a characteristic or attribute that can assume different values.

A random variable is a variable whose values are determined by chance.

Discrete variables are countable.

Example: Roll a die and let X represent the outcome

so X = {1,2,3,4,5,6}

Discrete probability distribution - the values a random variable can assume and the corresponding probabilities of the values.

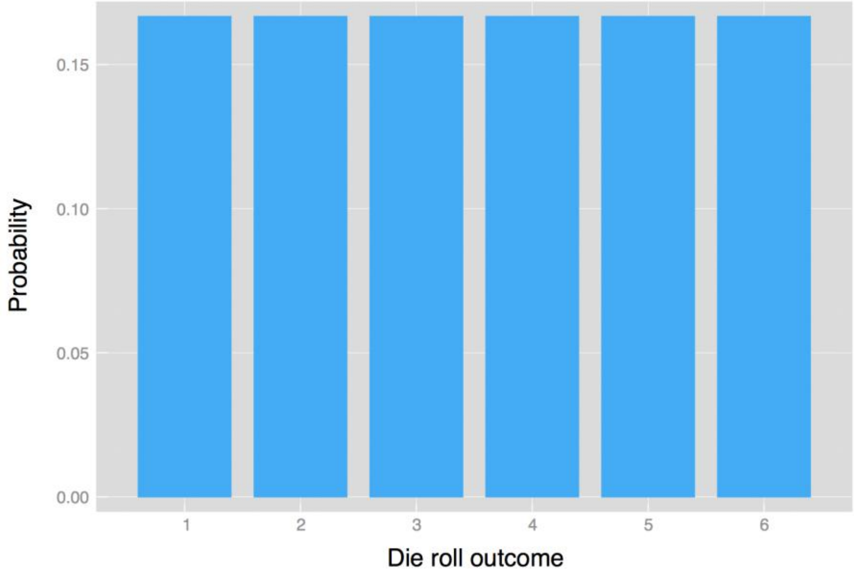
- They **can be displayed by a graph or a table.**

Probability Distribution

*A **probability distribution** is a list of all of the possible outcomes of a random variable along with their corresponding probability values.*

To give a concrete example, here is the probability distribution of a fair 6-sided die.

Outcome of die roll	1	2	3	4	5	6
Probability	1/6	1/6	1/6	1/6	1/6	1/6

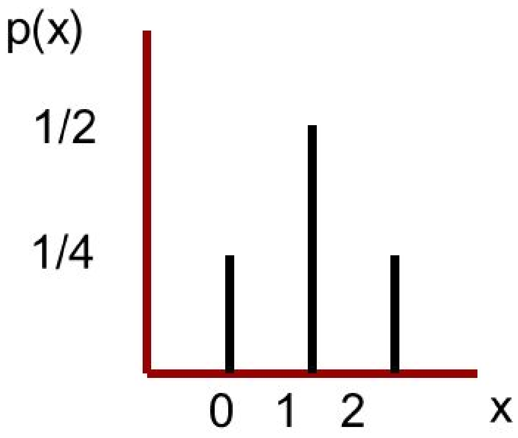


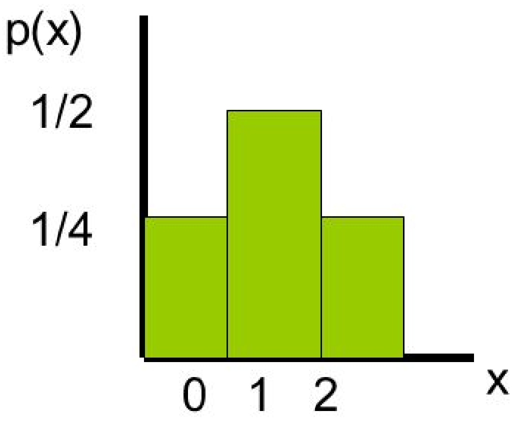
Probability Distribution

Example : Tossing two coins and let x be the number of heads observed.

Solution:

X	0	1	2
p(x)	1/4	1/2	1/4





Cumulative Probability

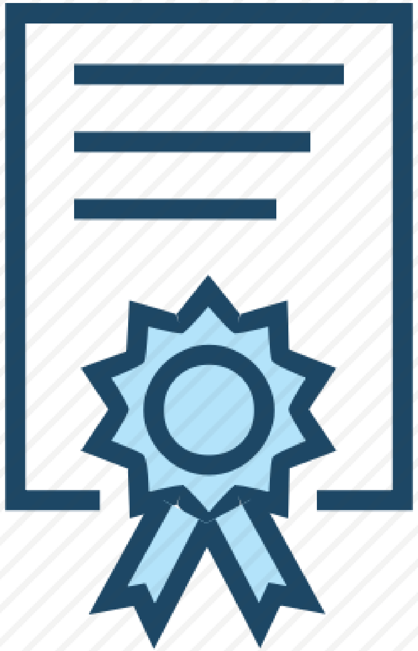
A **cumulative probability** refers to the probability that the value of a random variable falls within a specified range. Frequently, cumulative probabilities refer to the probability that a random variable is less than or equal to a specified value.

Consider a coin flip experiment. If we flip a coin two times, we might ask: What is the probability that the coin flips would result in one or fewer heads? The answer would be a cumulative probability. It would be the probability that the coin flip results in zero heads plus the probability that the coin flip results in one head. Thus, the cumulative probability would equal:

$$P(X \leq 1) = P(X = 0) + P(X = 1) = 0.25 + 0.50 = 0.75$$

Number of heads	Probability	Cumulative Probability
0	0.25	0.25
1	0.50	0.75
2	0.25	1.00

COURSE COMPLETED



A Teacher
Doesn't Teach
You
To Think Like Him
But
To Think without
Him