Numerical Differentiation

1. Forward Difference Formula:

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} - \boldsymbol{\frac{h}{2} f''(\zeta)}$$

$$\zeta \text{ lies between } x_0 \text{ and } x_0 + h$$

2. Backward Difference Formula:

$$f'(x_0) = \frac{f(x_0) - f(x_0 - h)}{h} + \boldsymbol{\frac{h}{2} f''(\zeta)}$$

$$\zeta \text{ lies between } x_0 - h \text{ and } x_0$$

3. Three Point Endpoint Formula:

$$f'(x_0) = \frac{1}{2h}\left(-3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h)\right) + \boldsymbol{\frac{h^2}{3} f'''(\zeta_0)}$$

$$\text{for right end point approximation replace } h \text{ by } -h$$

$$\zeta \text{ lies between } x_0 \text{ and } x_0 + 2h$$

4. Three point Midpoint Formula:

$$f'(x_0) = \frac{1}{2h}\left(f(x_0 + h) - f(x_0 - h)\right) + \boldsymbol{\frac{h^2}{6} f'''(\zeta_1)}$$

$$\zeta \text{ lies between } x_0 + h \text{ and } x_0 - h$$

5. Five point Endpoint Formula:

$$f'(x_0) = \frac{1}{12h}\left(-25f(x_0) + 48f(x_0 + h) - 36f(x_0 + 2h) + 16f(x_0 + 3h) - 3f(x_0 + 4h)\right) + \boldsymbol{\frac{h^4}{5} f^{(5)}(\zeta)}$$

$$\text{for right end point approximation replace } h \text{ by } -h$$

$$\zeta \text{ lies between } x_0 \text{ and } x_0 + 4h$$

6. Five point Midpoint Formula:

$$f'(x_0) = \frac{1}{12h}\left(f(x_0 - 2h) - 8f(x_0 - h) + 8f(x_0 + h) - f(x_0 + 2h)\right) + \boldsymbol{\frac{h^4}{30} f^{(5)}(\zeta)}$$

$$\zeta \text{ lies between } x_0 - 2h \text{ and } x_0 + 2h$$

7. Second Derivative Midpoint Formula:

$$f''(x_0) = \frac{1}{h^2}\left(f(x_0 - h) - 2f(x_0) + f(x_0 + h)\right) - \boldsymbol{\frac{h^2}{12} f^{(4)}(\zeta)}$$

$$\zeta \text{ lies between } x_0 - h \text{ and } x_0 + h$$

Numerical Integration

1. Newton Cotes Closed Integration Formulas:

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}[f(x_0) + f(x_1)] - \frac{h^3}{12} f''(\xi), \quad \text{where} \quad x_0 < \xi < x_1. \tag{4.25}$$

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{90} f^{(4)}(\xi), \quad \text{where} \quad x_0 < \xi < x_2. \tag{4.26}$$

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)] - \frac{3h^5}{80} f^{(4)}(\xi), \quad \text{where} \quad x_0 < \xi < x_3. \tag{4.27}$$

2. Newton Cotes Open Integration Formulas:

$$\int_{x_{-1}}^{x_1} f(x)\,dx = 2hf(x_0) + \frac{h^3}{3} f''(\xi), \quad \text{where} \quad x_{-1} < \xi < x_1. \tag{4.29}$$

$$\int_{x_{-1}}^{x_2} f(x)\,dx = \frac{3h}{2}[f(x_0) + f(x_1)] + \frac{3h^3}{4} f''(\xi), \quad \text{where} \quad x_{-1} < \xi < x_2. \tag{4.30}$$

3. Composite Integration Formulas:

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(x_0) + 2\sum_{j=1}^{(n/2)-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(x_n)\right] - \frac{h^5}{90}\sum_{j=1}^{n/2} f^{(4)}(\xi_j).$$

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \frac{b-a}{12}h^2 f''(\mu).$$

$h = (b-a)/n$, and $x_j = a + jh$, for each $j = 0, 1, \ldots, n$.

Stirling Formula:

$$\begin{aligned} P_n(x) = P_{2m+1}(x) = {} & f[x_0] + \frac{sh}{2}(f[x_{-1}, x_0] + f[x_0, x_1]) + s^2h^2 f[x_{-1}, x_0, x_1] \qquad (3.14) \\ & + \frac{s(s^2-1)h^3}{2} f[x_{-2}, x_{-1}, x_0, x_1] + f[x_{-1}, x_0, x_1, x_2]) \\ & + \cdots + s^2(s^2-1)(s^2-4)\cdots(s^2-(m-1)^2)h^{2m} f[x_{-m}, \ldots, x_m] \\ & + \frac{s(s^2-1)\cdots(s^2-m^2)h^{2m+1}}{2}(f[x_{-m-1}, \ldots, x_m] + f[x_{-m}, \ldots, x_{m+1}]), \end{aligned}$$

Euler's Formula:

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(\xi_i),$$

Where h= step size