Interpolation

Topic 3.1 Practice Questions with Solution

1. For the given functions $f(x)$, let $x_0 = 0$, $x_1 = 0.6$, and $x_2 = 0.9$. Construct interpolation polynomials of degree at most one and at most two to approximate $f(0.45)$, and find the absolute error.

a. $f(x) = \cos x$ **b.** $f(x) = \sqrt{1+x}$

c. $f(x) = \ln(x+1)$ **d.** $f(x) = \tan x$

2. For the given functions $f(x)$, let $x_0 = 1$, $x_1 = 1.25$, and $x_2 = 1.6$. Construct interpolation polynomials of degree at most one and at most two to approximate $f(1.4)$, and find the absolute error.

a. $f(x) = \sin \pi x$ **b.** $f(x) = \sqrt[3]{x-1}$

c. $f(x) = \log_{10}(3x-1)$ **d.** $f(x) = e^{2x} - x$

Solution:

Question no 1

1. The interpolation polynomials are as follows.

(a) $P_1(x) = -0.148878x + 1$; $P_1(0.45) = 0.933005$;
$|f(0.45) - P_1(0.45)| = 0.032558$;
$P_2(x) = -0.452592x^2 - 0.0131009x + 1$; $P_2(0.45) = 0.902455$;
$|f(0.45) - P_2(0.45)| = 0.002008$

(b) $P_1(x) = 0.467251x + 1$; $P_1(0.45) = 1.210263$;
$|f(0.45) - P_1(0.45)| = 0.006104$;
$P_2(x) = -0.0780026x^2 + 0.490652x + 1$; $P_2(0.45) = 1.204998$;
$|f(0.45) - P_2(0.45)| = 0.000839$

(c) $P_1(x) = 0.874548x$; $P_1(0.45) = 0.393546$;
$|f(0.45) - P_1(0.45)| = 0.0212983$;
$P_2(x) = -0.268961x^2 + 0.955236x$; $P_2(0.45) = 0.375392$;
$|f(0.45) - P_2(0.45)| = 0.003828$

(d) $P_1(x) = 1.031121x$; $P_1(0.45) = 0.464004$;
$|f(0.45) - P_1(0.45)| = 0.019051$;
$P_2(x) = 0.615092x^2 + 0.846593x$; $P_2(0.45) = 0.505523$;
$|f(0.45) - P_2(0.45)| = 0.022468$

Question no 2:

The interpolation polynomials are as follows.

(a) $P_1(x) = -0.6969992408x + 0.1641422691$; $P_1(1.4) = -0.8116566680$;
$|f(1.4) - P_1(1.4)| = 0.1393998486$;
$P_2(x) = 3.552379809x^2 - 10.82128170x + 7.268901887$; $P_2(1.4) = -0.918228067$;
$|f(1.4) - P_2(1.4)| = 0.0328284496$

(b) $P_1(x) = 0.6099204008x - 0.1324399760$; $P_1(1.4) = 0.7214485851$;
$|f(1.4) - P_1(1.4)| = 0.0153577147$;
$P_2(x) = -3.183202832x^2 + 9.682048472x - 6.498845640$; $P_2(1.4) = 0.816944669$;
$|f(1.4) - P_2(1.4)| = 0.0801383692$

(c) $P_1(x) = 0.4012882937x - 0.0622776733$; $P_1(1.4) = 0.4995259379$;
$|f(1.4) - P_1(1.4)| = 0.0056240404$;
$P_2(x) = -0.2532041643x^2 + 1.122920162x - 0.5686860021$; $P_2(1.4) = 0.5071220629$;
$|f(1.4) - P_2(1.4)| = 0.0019720846$

(d) $P_1(x) = 34.28581783x - 31.92477833$; $P_1(1.4) = 16.07536663$;
$|f(1.4) - P_1(1.4)| = 1.03071986$;
$P_2(x) = 26.85344400x^2 - 42.24649756x + 21.78210966$; $P_2(1.4) = 15.26976332$;
$|f(1.4) - P_2(1.4)| = 0.22511655$

5. Use appropriate Lagrange interpolating polynomials of degrees one, two, and three to approximate each of the following:

a. $f(8.4)$ if $f(8.1) = 16.94410$, $f(8.3) = 17.56492$, $f(8.6) = 18.50515$, $f(8.7) = 18.82091$

b. $f\left(-\frac{1}{3}\right)$ if $f(-0.75) = -0.07181250$, $f(-0.5) = -0.02475000$, $f(-0.25) = 0.33493750$, $f(0) = 1.10100000$

c. $f(0.25)$ if $f(0.1) = 0.62049958$, $f(0.2) = -0.28398668$, $f(0.3) = 0.00660095$, $f(0.4) = 0.24842440$

d. $f(0.9)$ if $f(0.6) = -0.17694460$, $f(0.7) = 0.01375227$, $f(0.8) = 0.22363362$, $f(1.0) = 0.65809197$

6. Use appropriate Lagrange interpolating polynomials of degrees one, two, and three to approximate each of the following:

a. $f(0.43)$ if $f(0) = 1$, $f(0.25) = 1.64872$, $f(0.5) = 2.71828$, $f(0.75) = 4.48169$

b. $f(0)$ if $f(-0.5) = 1.93750$, $f(-0.25) = 1.33203$, $f(0.25) = 0.800781$, $f(0.5) = 0.687500$

c. $f(0.18)$ if $f(0.1) = -0.29004986$, $f(0.2) = -0.56079734$, $f(0.3) = -0.81401972$, $f(0.4) = -1.0526302$

d. $f(0.25)$ if $f(-1) = 0.86199480$, $f(-0.5) = 0.95802009$, $f(0) = 1.0986123$, $f(0.5) = 1.2943767$

Solution Q no 5

5. Interpolation polynomials give the following results.

(a)

n	$x_0, x_1, \ldots, x_n$	$P_n(8.4)$
1	8.3, 8.6	17.87833
2	8.3, 8.6, 8.7	17.87716
3	8.3, 8.6, 8.7, 8.1	17.87714

(b)

n	$x_0, x_1, \ldots, x_n$	$P_n(-1/3)$
1	−0.5, −0.25	0.21504167
2	−0.5, −0.25, 0.0	0.16988889
3	−0.5, −0.25, 0.0, −0.75	0.17451852

(c)

n	$x_0, x_1, \ldots, x_n$	$P_n(0.25)$
1	0.2, 0.3	−0.13869287
2	0.2, 0.3, 0.4	−0.13259734
3	0.2, 0.3, 0.4, 0.1	−0.13277477

(d)

n	$x_0, x_1, \ldots, x_n$	$P_n(0.9)$
1	0.8, 1.0	0.44086280
2	0.8, 1.0, 0.7	0.43841352
3	0.8, 1.0, 0.7, 0.6	0.44198500

6. Interpolation polynomials give the following results.

(a) $P_1(x) = 4.278240000x + 0.579160000; P_1(0.43) = 2.418803200$
$|f(0.43) - P_1(0.43)| = 0.055642506;$
$P_2(x) = 5.550800000x^2 + 0.115140000x + 1.273010000;\ P_2(0.43) = 2.348863120;$
$|f(0.43) - P_2(0.43)| = 0.014297574$
$P_3(x) = 2.912106668x^3 + 1.182639999x^2 + 2.117213334x + 1.0;\ P_3(0.43) = 2.360604734;$
$|f(0.43) - P_3(0.43)| = 0.002555960e$

(b) $P_1(x) = -1.062498000x + 1.066405500; P_1(0.0) = 1.066405500$
$|f(0.0) - P_1(0.0)| = 0.066405500;$
$P_2(x) = 1.812509334x^2 - 1.062497999x + 0.9531236670;\ P_2(0.0) = 0.9531236670;$
$|f(0.0) - P_2(0.0)| = 0.0468763330$
$P_3(x) = -1.000010667x^3 + 1.312504000x^2 - 0.9999973330x + 0.9843740000;\ P_3(0.0) = 0.9843740000;$
$|f(0.0) - P_3(0.0)| = 0.0156260000$

(c) $P_1(x) = -2.7074748x - 0.01930238; P_1(0.18) = -0.506647844$
$|f(0.18) - P_1(0.18)| = 0.0014756204;$
$P_2(x) = 0.8762550000x^2 - 2.970351300x - 0.0017772800;\ P_2(0.18) = -0.5080498520;$
$|f(0.18) - P_2(0.18)| = 0.0000736124$
$P_3(x) = -0.4855333334x^3 + 1.167575000x^2 - 3.023759967x + 0.0011359200;\ P_3(0.18) = -0.5081430745;$
$|f(0.18) - P_3(0.18)| = 0.0000196101$

(d) $P_1(x) = 0.3915288000x + 1.0986123; P_1(0.25) = 1.196494500$
$|f(0.25) - P_1(0.25)| = 0.007424569;$
$P_2(x) = 0.1103443800x^2 + 0.3363566100x + 1.098612300;\ P_2(0.25) = 1.189597976;$
$|f(0.25) - P_2(0.25)| = 0.000528045$
$P_3(x) = 0.01414036000x^3 + 0.1103443800x^2 + 0.3328215200x + 1.098612300;\ P_3(0.25) = 1.188935147;$
$|f(0.25) - P_3(0.25)| = 0.000134784$

13. Construct the Lagrange interpolating polynomials for the following functions, and find a bound for the absolute error on the interval $[x_0, x_n]$.

a. $f(x) = e^{2x}\cos 3x, \quad x_0 = 0, x_1 = 0.3, x_2 = 0.6, n = 2$

b. $f(x) = \sin(\ln x), \quad x_0 = 2.0, x_1 = 2.4, x_2 = 2.6, n = 2$

c. $f(x) = \ln x, \quad x_0 = 1, x_1 = 1.1, x_2 = 1.3, x_3 = 1.4, n = 3$

d. $f(x) = \cos x + \sin x, \quad x_0 = 0, x_1 = 0.25, x_2 = 0.5, x_3 = 1.0, n = 3$

14. Let $f(x) = e^x$, for $0 \le x \le 2$.

a. Approximate $f(0.25)$ using linear interpolation with $x_0 = 0$ and $x_1 = 0.5$.

b. Approximate $f(0.75)$ using linear interpolation with $x_0 = 0.5$ and $x_1 = 1$.

c. Approximate $f(0.25)$ and $f(0.75)$ by using the second interpolating polynomial with $x_0 = 0$, $x_1 = 1$, and $x_2 = 2$.

d. Which approximations are better and why?

Solution of Q no 13:

(a) $P_2(x) = -11.22388889x^2 + 3.810500000x + 1.$
An error bound is 0.11371294.

(b) $P_2(x) = -0.1306344167x^2 + 0.8969979335x - 0.63249693.$
An error bound is 9.45762×10^{-4}.

(c) $P_3(x) = 0.1970056667x^3 - 1.06259055x^2 + 2.532453189x - 1.666868305.$
An error bound is 10^{-4}.

(d) $P_3(x) = -0.07932x^3 - 0.545506x^2 + 1.0065992x + 1.$
An error bound is 1.591376×10^{-3}.

Calculation of Error bound is not inclusive

Solution Q no 14:

(a) 1.32436 (b) 2.18350 (c) 1.15277, 2.01191

(c) Parts (a) and (b) are better due to the spacing of the nodes.

19. It is suspected that the high amounts of tannin in mature oak leaves inhibit the growth of the winter moth (*Operophtera bromata L., Geometridae*) larvae that extensively damage these trees in certain years. The following table lists the average weight of two samples of larvae at times in the first 28 days after birth. The first sample was reared on young oak leaves, whereas the second sample was reared on mature leaves from the same tree.

a. Use Lagrange interpolation to approximate the average weight curve for each sample.

b. Find an approximate maximum average weight for each sample by determining the maximum of the interpolating polynomial.

Day	0	6	10	13	17	20	28
Sample 1 average weight (mg)	6.67	17.33	42.67	37.33	30.10	29.31	28.74
Sample 2 average weight (mg)	6.67	16.11	18.89	15.00	10.56	9.44	8.89

Solution Q no 19

29. (a) Sample 1: $P_6(x) = 6.67 - 42.6434x + 16.1427x^2 - 2.09464x^3 + 0.126902x^4 - 0.00367168x^5 + 0.0000409458x^6$;
Sample 2: $P_6(x) = 6.67 - 5.67821x + 2.91281x^2 - 0.413799x^3 + 0.0258413x^4 - 0.000752546x^5 + 0.00000836160x^6$

(b) Sample 1: 42.71 mg; Sample 2: 19.42 mg

Topic 3.3 Practice Questions with Solution

1. Use Eq. (3.10) or Algorithm 3.2 to construct interpolating polynomials of degree one, two, and three for the following data. Approximate the specified value using each of the polynomials.
   a. $f(8.4)$ if $f(8.1) = 16.94410$, $f(8.3) = 17.56492$, $f(8.6) = 18.50515$, $f(8.7) = 18.82091$
   b. $f(0.9)$ if $f(0.6) = -0.17694460$, $f(0.7) = 0.01375227$, $f(0.8) = 0.22363362$, $f(1.0) = 0.65809197$
2. Use Eq. (3.17) or Algorithm 3.2 to construct interpolating polynomials of degree one, two, and three for the following data. Approximate the specified value using each of the polynomials.
   a. $f(0.43)$ if $f(0) = 1$, $f(0.25) = 1.64872$, $f(0.5) = 2.71828$, $f(0.75) = 4.48169$
   b. $f(0)$ if $f(-0.5) = 1.93750$, $f(-0.25) = 1.33203$, $f(0.25) = 0.800781$, $f(0.5) = 0.687500$
3. Use the Newton forward-difference formula to construct interpolating polynomials of degree one, two, and three for the following data. Approximate the specified value using each of the polynomials.
   a. $f(-\frac{1}{3})$ if $f(-0.75) = -0.07181250$, $f(-0.5) = -0.0247500$, $f(-0.25) = 0.33493750$, $f(0) = 1.10100000$
   b. $f(0.25)$ if $f(0.1) = -0.62049958$, $f(0.2) = -0.28398668$, $f(0.3) = 0.00660095$, $f(0.4) = 0.24842440$
4. Use the Newton forward-difference formula to construct interpolating polynomials of degree one, two, and three for the following data. Approximate the specified value using each of the polynomials.
   a. $f(0.43)$ if $f(0) = 1$, $f(0.25) = 1.64872$, $f(0.5) = 2.71828$, $f(0.75) = 4.48169$
   b. $f(0.18)$ if $f(0.1) = -0.29004986$, $f(0.2) = -0.56079734$, $f(0.3) = -0.81401972$, $f(0.4) = -1.0526302$
5. Use the Newton backward-difference formula to construct interpolating polynomials of degree one, two, and three for the following data. Approximate the specified value using each of the polynomials.
   a. $f(-\frac{1}{3})$ if $f(-0.75) = -0.07181250$, $f(-0.5) = -0.02475000$, $f(-0.25) = 0.33493750$, $f(0) = 1.10100000$
   b. $f(0.25)$ if $f(0.1) = -0.62049958$, $f(0.2) = -0.28398668$, $f(0.3) = 0.00660095$, $f(0.4) = 0.24842440$
6. Use the Newton backward-difference formula to construct interpolating polynomials of degree one, two, and three for the following data. Approximate the specified value using each of the polynomials.
   a. $f(0.43)$ if $f(0) = 1$, $f(0.25) = 1.64872$, $f(0.5) = 2.71828$, $f(0.75) = 4.48169$
   b. $f(0.25)$ if $f(-1) = 0.86199480$, $f(-0.5) = 0.95802009$, $f(0) = 1.0986123$, $f(0.5) = 1.2943767$
7. a. Use Algorithm 3.2 to construct the interpolating polynomial of degree three for the unequally spaced points given in the following table:

x	$f(x)$
-0.1	5.30000
0.0	2.00000
0.2	3.19000
0.3	1.00000

   b. Add $f(0.35) = 0.97260$ to the table and construct the interpolating polynomial of degree four.
8. a. Use Algorithm 3.2 to construct the interpolating polynomial of degree four for the unequally spaced points given in the following table:

x	$f(x)$
0.0	−6.00000
0.1	−5.89483
0.3	−5.65014
0.6	−5.17788
1.0	−4.28172

b. Add $f(1.1) = -3.99583$ to the table, and construct the interpolating polynomial of degree five.

Solution

1. The interpolating polynomials are as follows.

   (a) $P_1(x) = 16.9441 + 3.1041(x-8.1)$; $P_1(8.4) = 17.87533$
   $P_2(x) = P_1(x) + 0.06(x-8.1)(x-8.3)$; $P_2(8.4) = 17.87713$
   $P_3(x) = P_2(x) + -0.00208333(x-8.1)(x-8.3)(x-8.6)$; $P_3(8.4) = 17.87714$

   (b) $P_1(x) = -0.1769446 + 1.9069687(x-0.6)$; $P_1(0.9) = 0.395146$
   $P_2(x) = P_1(x) + 0.959224(x-0.6)(x-0.7)$; $P_2(0.9) = 0.4526995$
   $P_3(x) = P_2(x) - 1.785741(x-0.6)(x-0.7)(x-0.8)$; $P_3(0.9) = 0.4419850$

2. The interpolating polynomials are as follows.

   (a) $P_1(x) = 1.0 + 2.594880000x$; $P_1(0.43) = 2.115798400$
   $P_2(x) = P_1(x) + 3.366720000x(x-0.25)$; $P_2(0.43) = 2.376382528$
   $P_3(x) = P_2(x) + 2.912106667x(x-0.25)(x-0.5)$; $P_3(0.43) = 2.360604734$

   (b) $P_1(x) = 0.726560000 - 2.421880000x$; $P_1(0) = 0.726560000$
   $P_2(x) = P_1(x) + 1.812509333(x+0.5)(x+0.25)$; $P_2(0) = 0.9531236666$
   $P_3(x) = P_2(x) - 1.000010666(x+0.5)(x+0.25)(x-0.25)$; $P_3(0) = 0.9843739999$

3. In the following equations, we have $s = (1/h)(x-x_0)$.

   (a) $P_1(s) = -0.718125 - 0.0470625s$; $P_1\left(-\frac{1}{3}\right) = -0.006625$
   $P_2(s) = P_1(s) + 0.312625s(s-1)/2$; $P_2\left(-\frac{1}{3}\right) = 0.1803056$
   $P_3(s) = P_2(s) + 0.09375s(s-1)(s-2)/6$; $P_3\left(-\frac{1}{3}\right) = 0.1745185$

   (b) $P_1(s) = -0.62049958 + 0.3365129s$; $P_1(0.25) = -0.1157302$
   $P_2(s) = P_1(s) - 0.04592527s(s-1)/2$; $P_2(0.25) = -0.1329522$
   $P_3(s) = P_2(s) - 0.00283891s(s-1)(s-2)/6$; $P_3(0.25) = -0.1327748$

4. In the following equations, we have $s = (1/h)(x-x_0)$.

   (a) $P_1(s) = 1.0 + 0.6487200000s$; $P_1(0.43) = 2.115798400$
   $P_2(s) = P_1(s) + 0.2104200000s(s-1)$; $P_2(0.43) = 2.376382528$
   $P_3(s) = P_2(s) + 0.04550166667s(s-1)(s-2)$; $P_3(0.43) = 2.360604734$

   (b) $P_1(s) = -0.29004986 - 0.2707474800s$; $P_1(0.18) = -0.5066478440$
   $P_2(s) = P_1(s) + 0.008762550000s(s-1)$; $P_2(0.18) = -0.5080498520$
   $P_3(s) = P_2(s) - 0.0004855333333s(s-1)(s-2)$; $P_3(0.18) = -0.5081430744$

5. In the following equations, we have $s = (1/h)(x-x_n)$.

   (a) $P_1(s) = 1.101 + 0.7660625s$; $f(-\frac{1}{3}) \approx P_1(-\frac{4}{3}) = 0.07958333$ $P_2(s) = P_1(s) + 0.406375s(s+1)/2$; $f(-\frac{1}{3}) \approx P_2(-\frac{4}{3}) = 0.1698889$ $P_3(s) = P_2(s) + 0.09375s(s+1)(s+2)/6$; $f(-\frac{1}{3}) \approx P_3(-\frac{4}{3}) = 0.1745185$

(b) $P_1(s) = 0.2484244 + 0.2418235s$; $f(0.25) \approx P_1(-1.5) = -0.1143108$ $P_2(s) = P_1(s) - 0.04876419s(s+1)/2$; $f(0.25) \approx P_2(-1.5) = -0.1325973$
$P_3(s) = P_2(s) - 0.00283891s(s+1)(s+2)/6$; $f(0.25) \approx P_3(-1.5) = -0.1327748$

6. In the following equations, we have $s = (1/h)(x - x_0)$.

(a) $P_1(s) = 4.48169 + 1.763410000s$; $P_1(0.43) = 2.224525200$
$P_2(s) = P_1(s) + 0.3469250000s(s+1)$; $P_2(0.43) = 2.348863120$
$P_3(s) = P_2(s) + 0.04550166667s(s+1)(s+2)$; $P_3(0.43) = 2.360604734$

(b) $P_1(s) = 1.2943767 + 0.1957644000s$; $P_1(0.25) = 1.196494500$
$P_2(s) = P_1(s) + 0.02758609500s(s+1)$; $P_2(0.25) = 1.189597976$
$P_3(s) = P_2(s) + 0.001767545000s(s+1)(s+2)$; $P_3(0.25) = 1.188935147$

7. (a) $P_3(x) = 5.3 - 33(x+0.1) + 129.8\overline{3}(x+0.1)x - 556.\overline{6}(x+0.1)x(x-0.2)$

(b) $P_4(x) = P_3(c) + 2730.243387(x+0.1)x(x-0.2)(x-0.3)$

8. (a) $P_4(x) = -6 + 1.05170x + 0.57250x(x-0.1) + 0.21500x(x-0.1)(x-0.3) + 0.063016x(x-0.1)(x-0.3)(x-0.6)$

(b) Add $0.014159x(x-0.1)(x-0.3)(x-0.6)(x-1)$ to the answer in part (a).