National University of computer and emerging sciences

Foundation for Advancement of Science and Technology

# Lecture # 17 & 18

## Centered Differences (Stirling's Formula)

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James Stirling (1692–1770) published this and numerous other formulas in *Methodus Differentialis* in 1720. Techniques for accelerating the convergence of various series are included in this work.

For the centered-difference formulas, we choose $x_0$ near the point being approximated and label the nodes directly below $x_0$ as $x_1, x_2, \ldots$ and those directly above as $x_{-1}, x_{-2}, \ldots$. With this convention, **Stirling's formula** is given by

$$
\begin{aligned}
P_n(x) = P_{2m+1}(x) &= f[x_0] + \frac{sh}{2}(f[x_{-1}, x_0] + f[x_0, x_1]) + s^2h^2 f[x_{-1}, x_0, x_1] \qquad (3.14)\\
&+ \frac{s(s^2-1)h^3}{2} f[x_{-2}, x_{-1}, x_0, x_1] + f[x_{-1}, x_0, x_1, x_2])\\
&+ \cdots + s^2(s^2-1)(s^2-4)\cdots(s^2-(m-1)^2)h^{2m} f[x_{-m}, \ldots, x_m]\\
&+ \frac{s(s^2-1)\cdots(s^2-m^2)h^{2m+1}}{2}(f[x_{-m-1}, \ldots, x_m] + f[x_{-m}, \ldots, x_{m+1}]),
\end{aligned}
$$

if $n = 2m+1$ is odd. If $n = 2m$ is even, we use the same formula but delete the last line.

| $x$ | $f(x)$ | First divided differences | Second divided differences | Third divided differences | Fourth divided differences |
|---|---|---|---|---|---|
| $x_{-2}$ | $f[x_{-2}]$ | | | | |
| | | $f[x_{-2}, x_{-1}]$ | | | |
| $x_{-1}$ | $f[x_{-1}]$ | | $f[x_{-2}, x_{-1}, x_0]$ | | |
| | | $\underline{f[x_{-1}, x_0]}$ | | $\underline{f[x_{-2}, x_{-1}, x_0, x_1]}$ | |
| $x_0$ | $\underline{f[x_0]}$ | | $\underline{f[x_{-1}, x_0, x_1]}$ | | $\underline{f[x_{-2}, x_{-1}, x_0, x_1, x_2]}$ |
| | | $\underline{f[x_0, x_1]}$ | | $\underline{f[x_{-1}, x_0, x_1, x_2]}$ | |
| $x_1$ | $f[x_1]$ | | $f[x_0, x_1, x_2]$ | | |
| | | $f[x_1, x_2]$ | | | |
| $x_2$ | $f[x_2]$ | | | | |

$$P_n(x) = P_{2m+1}(x) = f[x_0] + \frac{sh}{2}(f[x_{-1}, x_0] + f[x_0, x_1]) + s^2h^2 f[x_{-1}, x_0, x_1] \qquad (3.14)$$

$$+ \frac{s(s^2-1)h^3}{2} f[x_{-2}, x_{-1}, x_0, x_1] + f[x_{-1}, x_0, x_1, x_2]) + \ldots$$

**Example 2** Consider the table of data given in the previous examples. Use Stirling's formula to approximate $f(1.5)$ with $x_0 = 1.6$.

| $x$ | $f(x)$ | First divided differences | Second divided differences | Third divided differences | Fourth divided differences |
|---|---|---|---|---|---|
| 1.0 | 0.7651977 | | | | |
| | | −0.4837057 | | | |
| 1.3 | 0.6200860 | | −0.1087339 | | |
| | | −0.5489460 | | 0.0658784 | |
| 1.6 | 0.4554022 | | −0.0494433 | | 0.0018251 |
| | | −0.5786120 | | 0.0680685 | |
| 1.9 | 0.2818186 | | 0.0118183 | | |
| | | −0.5715210 | | | |
| 2.2 | 0.1103623 | | | | |

The formula, with $h = 0.3$, $x_0 = 1.6$, and $s = -\frac{1}{3}$, becomes

$$f(1.5) \approx P_4\left(1.6 + \left(-\frac{1}{3}\right)(0.3)\right)$$

$$= 0.4554022 + \left(-\frac{1}{3}\right)\left(\frac{0.3}{2}\right)((-0.5489460) + (-0.5786120))$$

$$+ \left(-\frac{1}{3}\right)^2 (0.3)^2(-0.0494433)$$

$$+ \frac{1}{2}\left(-\frac{1}{3}\right)\left(\left(-\frac{1}{3}\right)^2 - 1\right)(0.3)^3(0.0658784 + 0.0680685)$$

$$+ \left(-\frac{1}{3}\right)^2\left(\left(-\frac{1}{3}\right)^2 - 1\right)(0.3)^4(0.0018251) = 0.5118200.$$



**Do Q 1,2,7,8 from Ex # 3.3**