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Lec # 30, 31 & 32

Gale-Shapley Algorithm

Input: Preference rankings of n women and n men.

Steps:

1. Each man proposes to the highest ranking woman on his list.
2. If every woman receives only one proposal, this matching is stable. Otherwise move to Step (3).
3. If a woman receives more than one proposal, she
   (a) accepts if it is from the man she prefers above all other currently available men and rejects the rest; or,
   (b) delays with a maybe to the highest ranked proposal and rejects the rest.
4. Each man now proposes to the highest ranking unmatched woman on his list who has not rejected him.
5. Repeat Steps (2)–(4) until all people have been paired.

Output: Stable Matching.

Example 5.17 Four women are to be paired as roommates. Each woman has ranked the other three as shown below. Find all possible pairings and determine if any are stable.

$$\begin{array}{rccccc} \text{Emma:} & l & > & m & > & z \\ \text{Leena:} & m & > & e & > & z \\ \text{Maggie:} & e & > & z & > & l \\ \text{Zara:} & e & > & l & > & m \end{array}$$

Solution: There are three possible pairings, only one of which is stable.

- Emma $\leftrightarrow$ Leena and Maggie $\leftrightarrow$ Zara
  This is stable since Emma is with her first choice and the only person Leena prefers over Emma is Maggie, but Maggie prefers Zara over Leena.

- Emma $\leftrightarrow$ Maggie and Leena $\leftrightarrow$ Zara
  This is not stable since Emma prefers Leena over Maggie and Leena prefers Emma over Zara.

- Emma $\leftrightarrow$ Zara and Leena $\leftrightarrow$ Maggie
  This is not stable since Emma prefers Maggie over Zara and Maggie prefers Emma over Leena.

Example 5.18 Before the four women from Example 5.17 are paired as roommates, Maggie and Zara get into an argument, causing them to adjust their preference lists. Determine if a stable matching exists.

$$\begin{array}{rccccc} \text{Emma:} & l & > & m & > & z \\ \text{Leena:} & m & > & e & > & z \\ \text{Maggie:} & e & > & l & > & z \\ \text{Zara:} & e & > & l & > & m \end{array}$$

Solution: There are three possible pairings, none of which are stable.

- Emma $\leftrightarrow$ Leena and Maggie $\leftrightarrow$ Zara
  This is not stable since Leena prefers Maggie over Emma and Maggie prefers Leena over Zara.
- Emma $\leftrightarrow$ Maggie and Leena $\leftrightarrow$ Zara
  This is not stable since Emma prefers Leena over Maggie and Leena prefers Emma over Zara.
- Emma $\leftrightarrow$ Zara and Leena $\leftrightarrow$ Maggie
  This is not stable since Emma prefers Maggie over Zara and Maggie prefers Emma over Leena.

Factors & Factorization of the graph

Definition 5.19 Let G be a graph with spanning subgraph H and let k be a positive integer. Then H is a ***k-factor*** of G if H is a k-regular.

Examples of Peterson & K_4

Factor

- A factor of a graph G is a spanning subgraph of G, not necessarily connected.
- G is the sum of factors G_i, if:
  - G is the edge-disjoint union of G_i's.

  Such a union is called factorization.
- n-factor: A regular factor of degree n.
- If G is the sum of n-factors:
  - The union of n-factors is called n-factorization.
  - G is n-factorable.

1-factor

- When G has a 1-factor, G_1,
  - |V| is even.
  - The edges of G_1 are edge disjoint.
- K_{2n+1} cannot have a 1-factor, but K_{2n} can.

Theorem: The complete graph K_{2n} is 1-factorable.

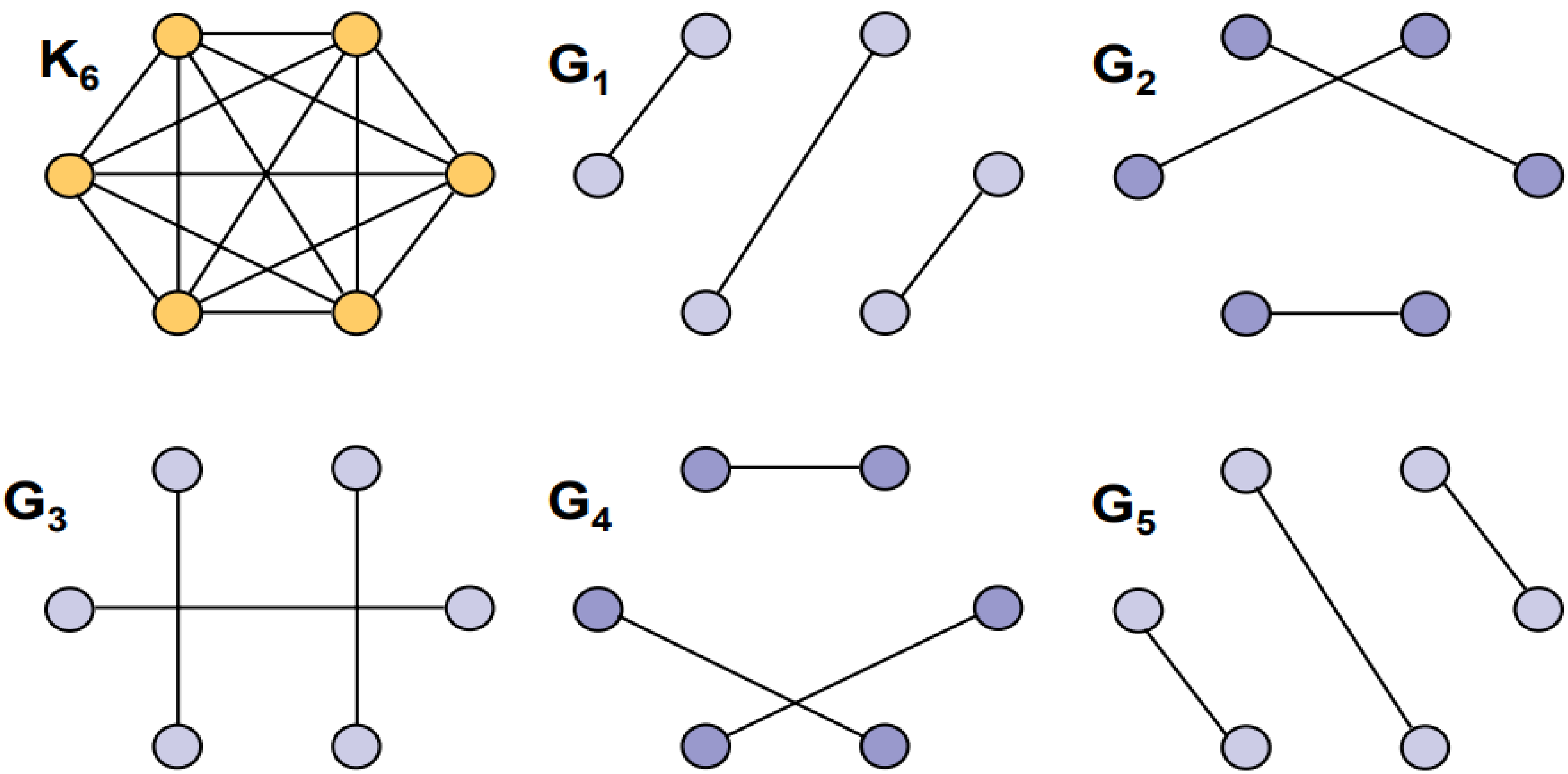
We need to display a partition of the set E of edges of K_{2n} into (2n – 1) 1-factors.

- Denote the vertices: $v_1, v_2, \ldots, v_{2n}$
- Define for $\mathrm{i} = 1, 2, \ldots, 2n - 1$

  The sets $\mathrm{E_i} = \{v_i v_{2n}\} \cup \{v_{i\text{-}j} v_{i+j} \mid j = 1, 2, n - 1\}$

  $i+1$ and $i - j$ are modulo$(2\mathrm{n} - 1)$ operations.

Example

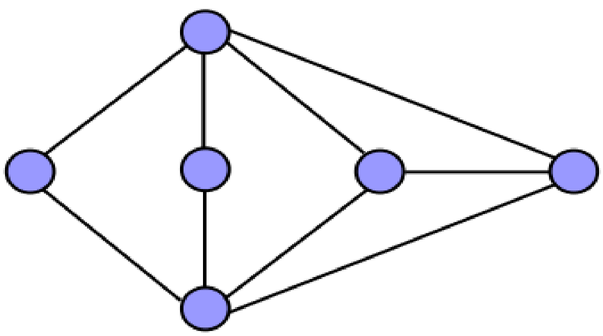


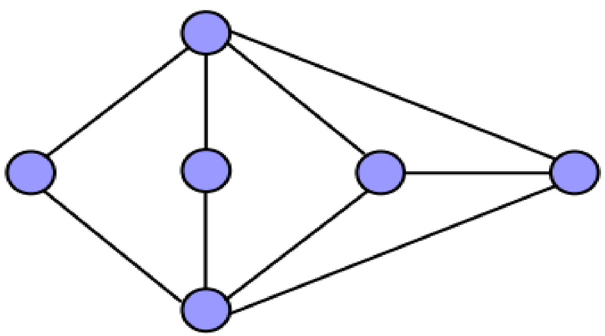
1-factors

- Complete bipartite graphs $K_{m,n}$ have no 1-factor if $n \neq m$.

Theorem: Every regular bipartite graph $K_{n,n}$ is 1-factorable.

Theorem: If a 2-connected graph has a 1-factor, then it has at least two different 1-factors.



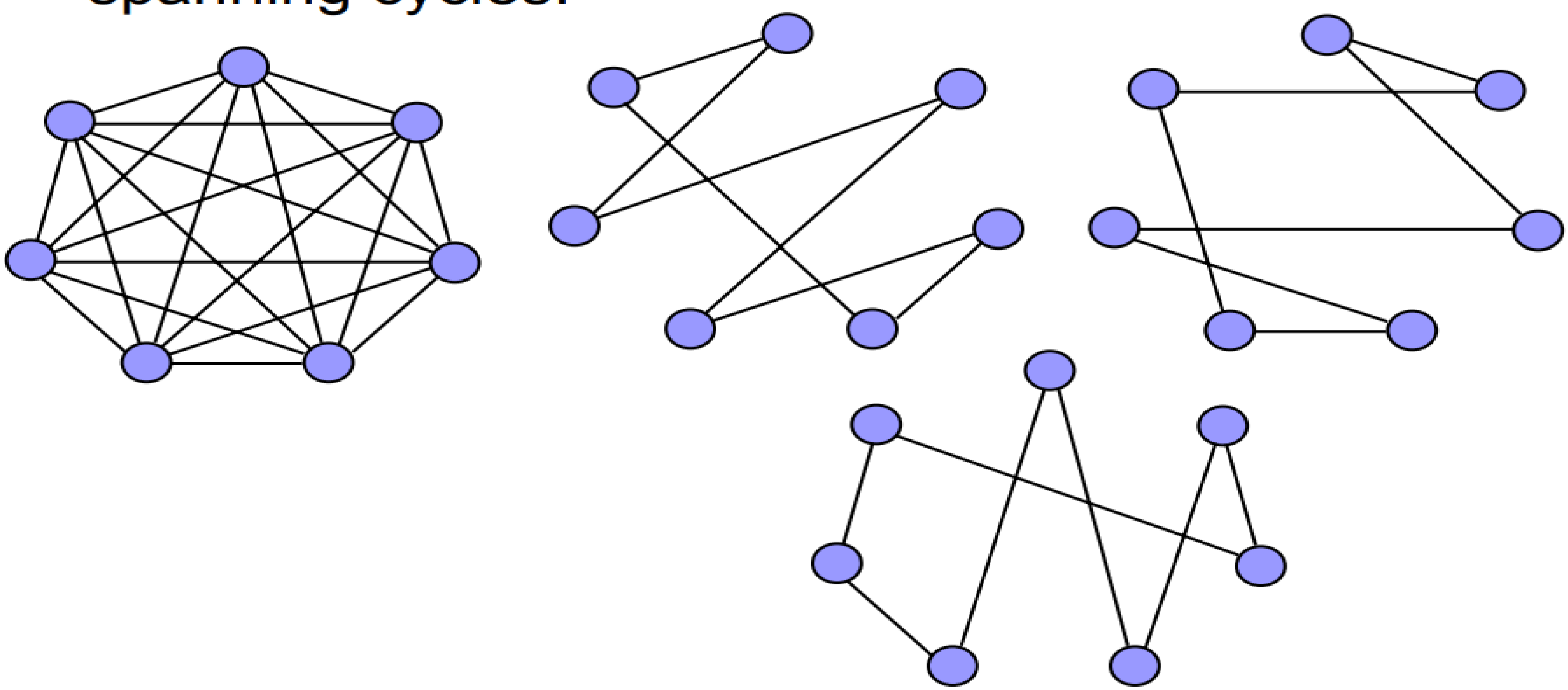


2-factorization

- If a graph is 2-factorable, then each factor is a union of disjoint cycles.
- If a 2-factor is connected, it is a spanning cycle.
- A 2-factorable graph must have all vertex degrees even.
- Complete graphs K_{2n} are not 2-factorable.
- K_{2n-1} complete graphs are 2-factorable.

2-factors

Theorem: The graph K_{2n+1} is the sum of n spanning cycles.



2-factors

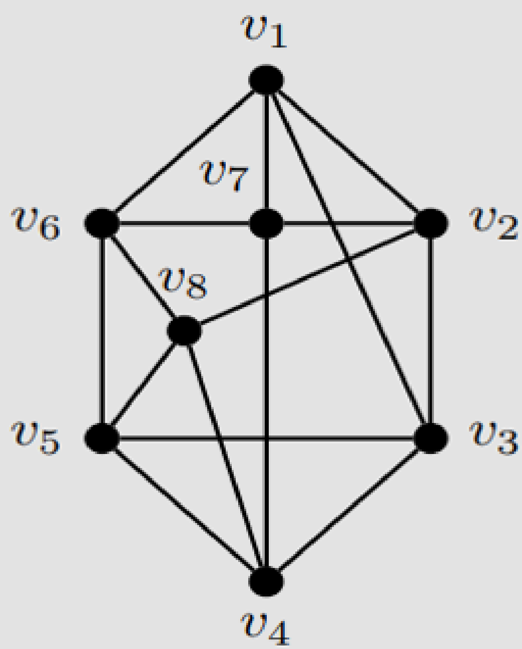
Theorem: The complete graph K_{2n} is the sum of a 1-factor and n – 1 spanning cycles.

- If every component of a regular graph G of degree 2 is an even-length cycle, then G is also 1-factorable.
  - It can be represented as the sum of two 1-factors.

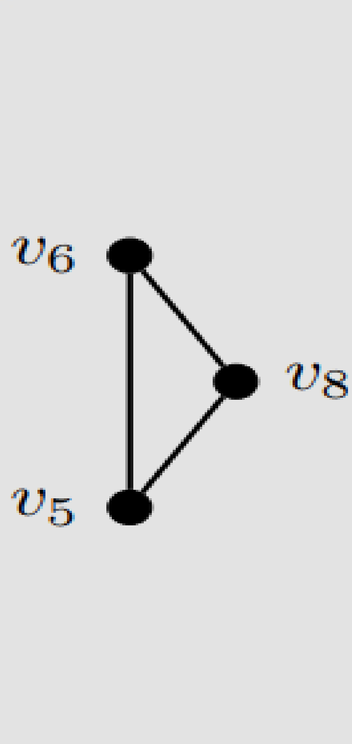
Theorem: Every bridgeless cubic graph is the sum of a 1-factor and a 2-factor.

- Example: Petersen graph.

Example 5.19 Find a 2-factor for the graph shown below.



Solution: More than one 2-factor exists for the graph above, as we simply need to find a spanning subgraph that is 2-regular. One such solution is given below. Note that this solution consists of two components, one of which is isomorphic to C_3 and the other to C_5.



Do it by your own

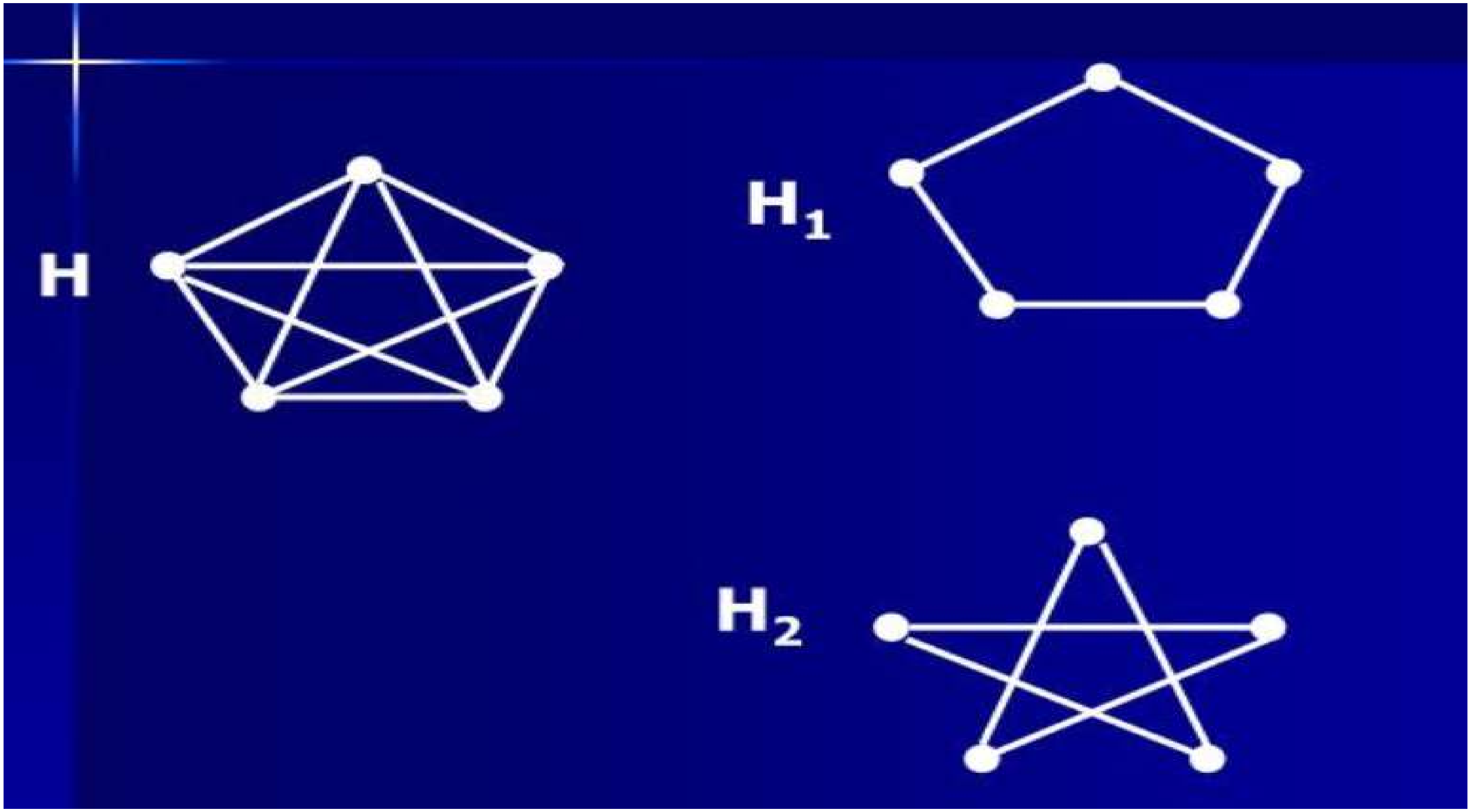
7.42 Show that an Eulerian graph cannot have a bridge.

Solution. If a graph is Eulerian, it is a connected graph in which the degree of each vertex is even. Suppose it has a bridge. If this bridge is deleted, there will be two components, and in each component will be exactly one vertex of odd degree. But in any graph, the number of odd vertices is always even.

7.43 If a cubic graph has a bridge, it is not 1-factorable.

Solution. Suppose a cubic graph with a bridge is 1-factorable. The bridge will be in one of the three 3-factors. The removal of the bridge gives two components. Each component will have an even number of odd vertices and a vertex of degree 2. So there cannot be one more 1-factor since the number of vertices in each component is odd.

1-factorization of $K_{3,3}$


H
H1
H2

Theorem 5.20 If G is a $2k$-regular graph, then G has a 2-factor.

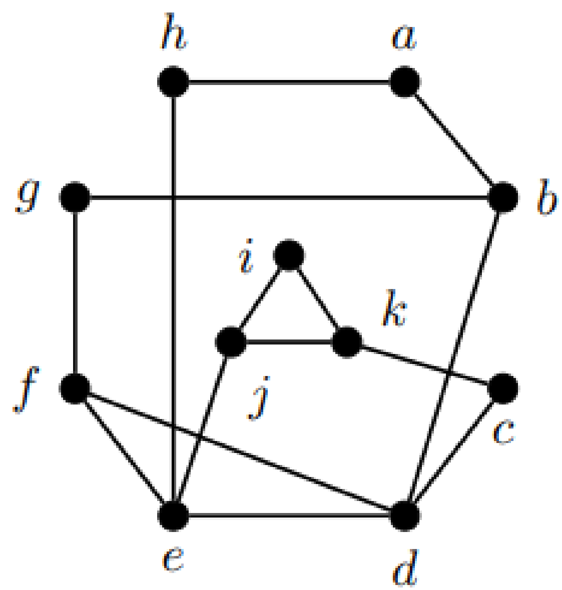
Definition 5.21 A ***k-factorization*** of G is a partition of the edges into disjoint k-factors.

Proposition 5.22 Every k-regular bipartite graph has a 1-factorization for all $k \geq 1$.

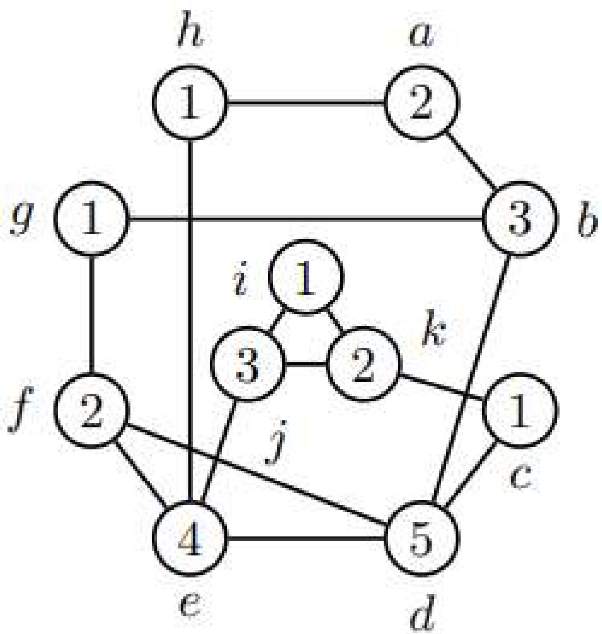
Theorem 5.23 A graph G has a k-factorization if and only if G is $2k$-regular.

Graph Coloring

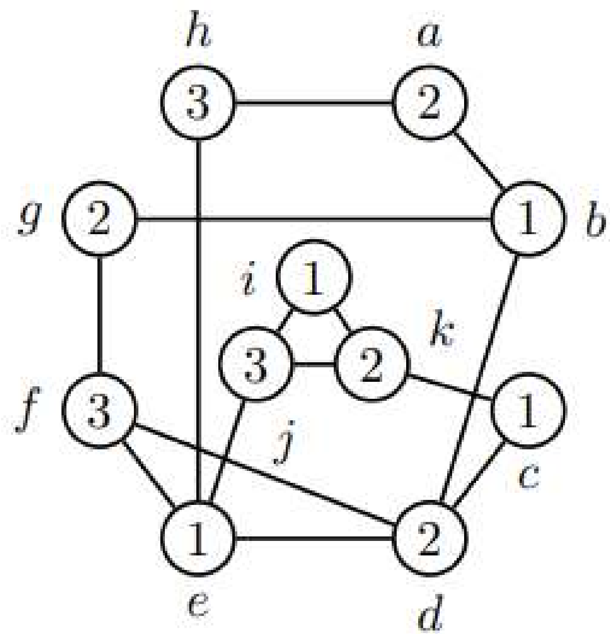
Definition 6.1 A proper ***k*-coloring** of a graph G is an assignment of colors to the vertices of G so that no two adjacent vertices are given the same color and exactly k colors are used.



G_1



proper 5-coloring



proper 3-coloring

Definition 6.2 Given a proper k-coloring of G, the ***color classes*** are sets $S_1, \ldots S_k$ where S_i consists of all vertices of color i.

Definition 6.3 The ***independence number*** of a graph G is $\alpha(G) = n$ if there exists a set of n vertices with no edges between them but every set of $n + 1$ vertices contains at least one edge.

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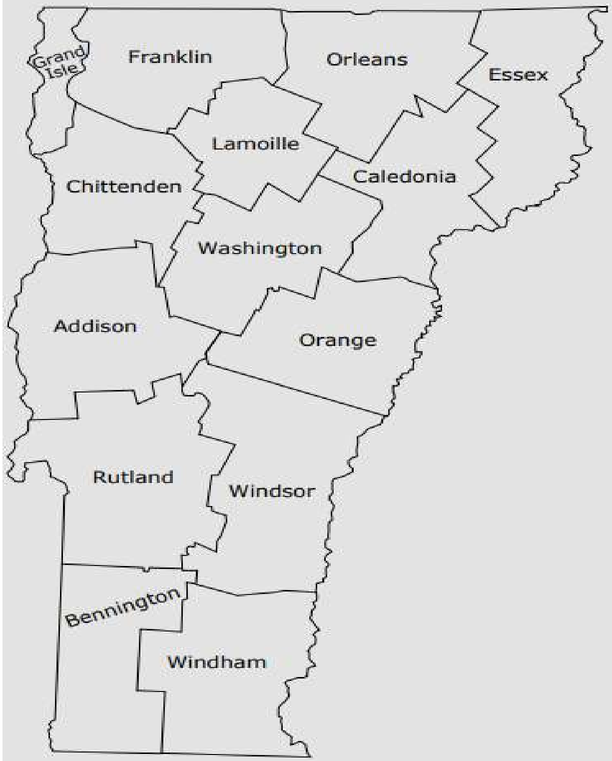
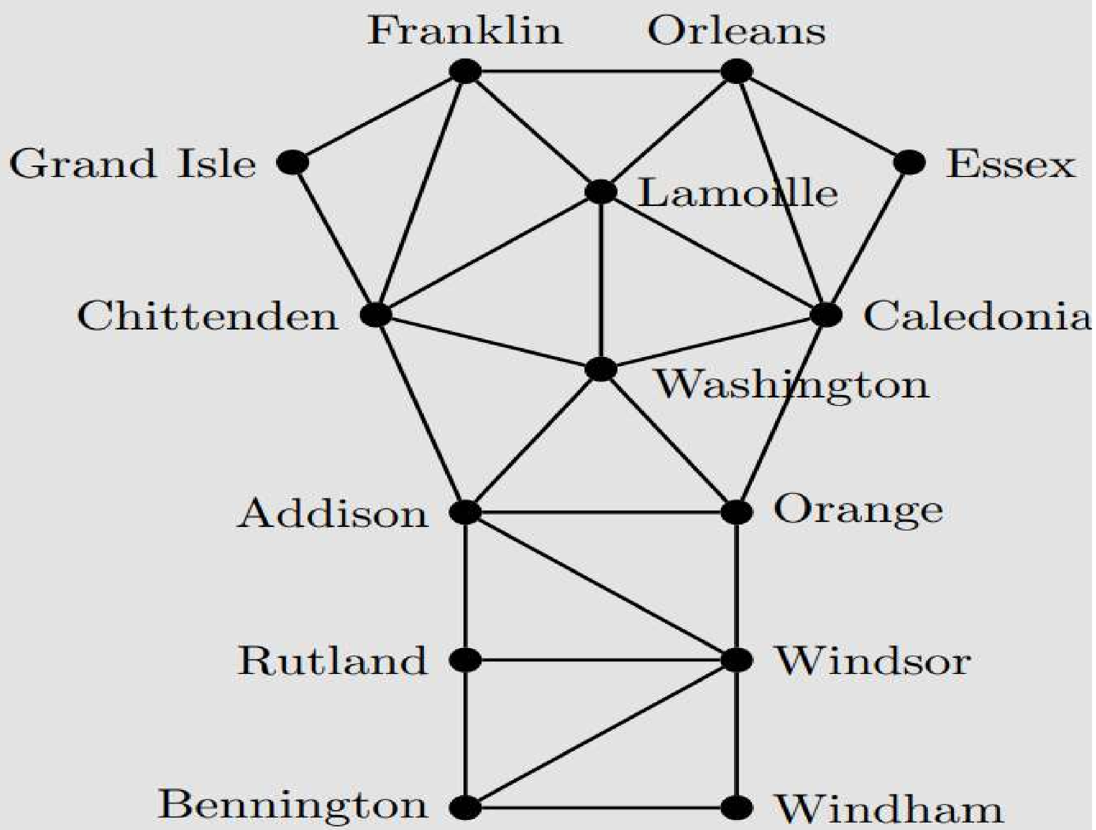
Example 6.1 Find a coloring of the map of the counties of Vermont and explain why three colors will not suffice.





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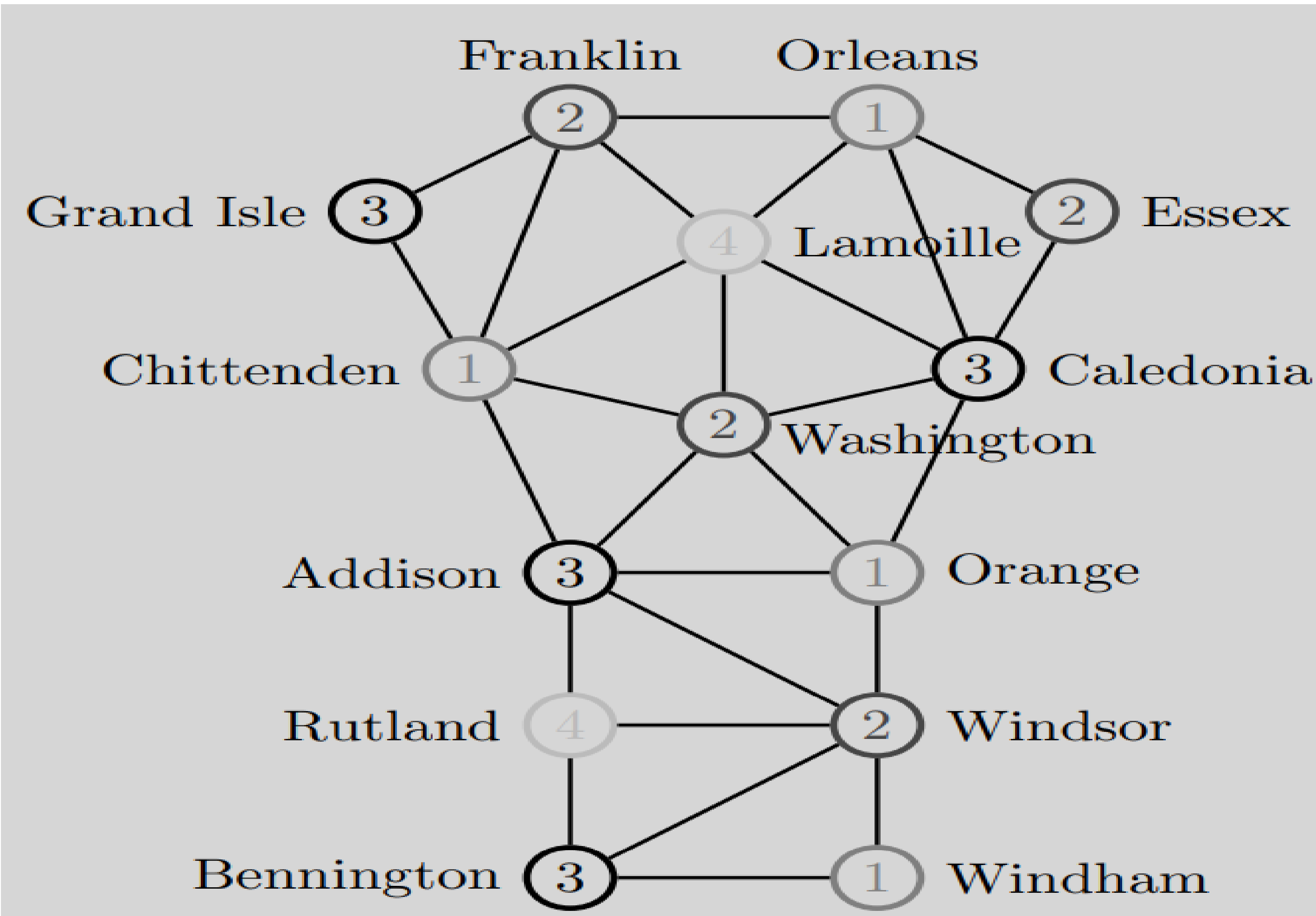

Note that Lamoille County is surrounded by five other counties. If we try to alternate colors amongst these five counties, for example Orleans – 1, Franklin – 2, Chittenden – 1, Washington – 2, we still need a third color for the fifth county (Caledonia – 3). Since Lamoille touches each of these counties, we know it needs a fourth color.

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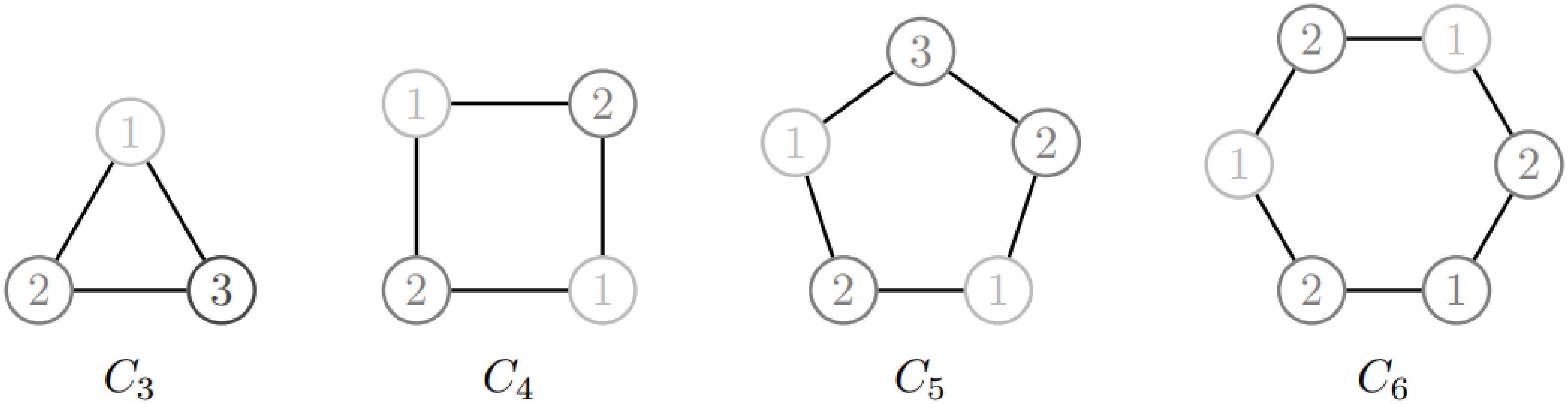
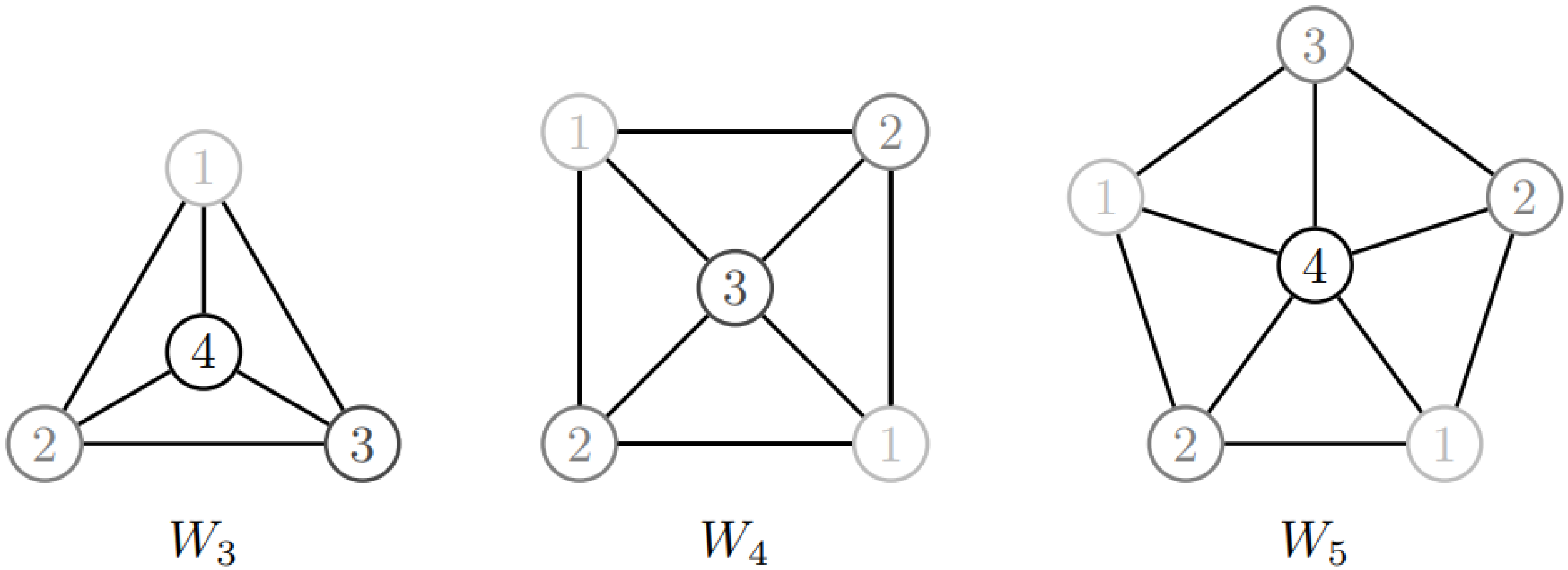
Definition 6.4 The ***chromatic number*** $\chi(G)$ of a graph is the smallest value k for which G has a proper k-coloring.



Definition 6.5 A ***wheel*** W_n is a graph in which n vertices form a cycle around a central vertex that is adjacent to each of the vertices in the cycle.



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Special Classes of Graphs with known $\chi(G)$

- $\chi(C_n) = 2$ if n is even $(n \geq 2)$
- $\chi(C_n) = 3$ if n is odd $(n \geq 3)$
- $\chi(K_n) = n$
- $\chi(W_n) = 4$ if n is odd $(n \geq 3)$