National University
of computer and emerging sciences

Foundation for Advancement of Science and Technology

LECTURE # 04, 05 & 06

Theorem 1.33 Let $G = (V, A)$ be a digraph and $|A|$ denote the number of arcs in G. Then both the sum of the in-degrees of the vertices and the sum of the out-degrees equals the number of arcs; that is, if $V = \{v_1, v_2, \ldots, v_n\}$, then

$$\begin{aligned} \deg^-(v_1) + \cdots + \deg^-(v_n) &= |A| \\ &= \deg^+(v_1) + \cdots + \deg^+(v_n) \end{aligned}$$

Isomorphisms

Two graphs G and H are said to be **isomorphic** if there exists a one-one and onto mapping $f : V(G) \to V(H)$ such that two vertices u, v are adjacent in G when and only when their images $f(u)$ and $f(v)$ under f are adjacent in H (i.e., **the adjaceny is preserved under** f).
In this case, we shall write $G \cong H$ and call the mapping f an **isomorphism** from G to H.

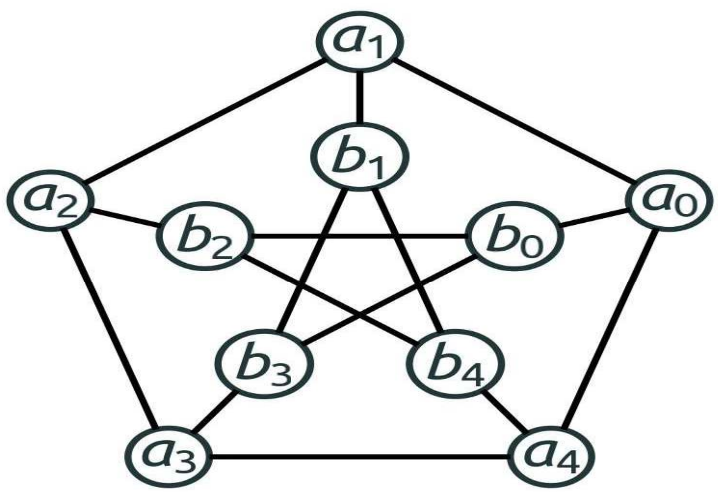
(1) The phrase 'when and only when' used above means that if u and v are adjacent in G, then $f(u)$ and $f(v)$ **must be** adjacent in H; and if u and v are **not** adjacent in G, then $f(u)$ and $f(v)$ **must not be** adjacent in H.

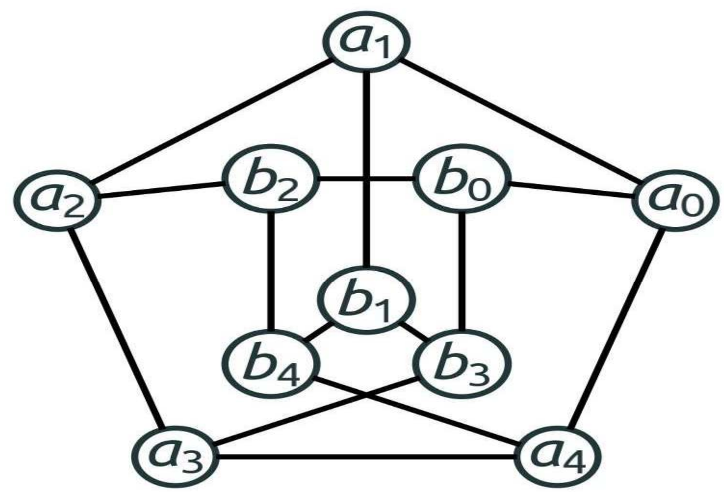
(2) The word **isomorphism** is derived from the Greek words *isos* (meaning 'equal') and *morphe* (meaning 'form').

Recall Lecture # 01

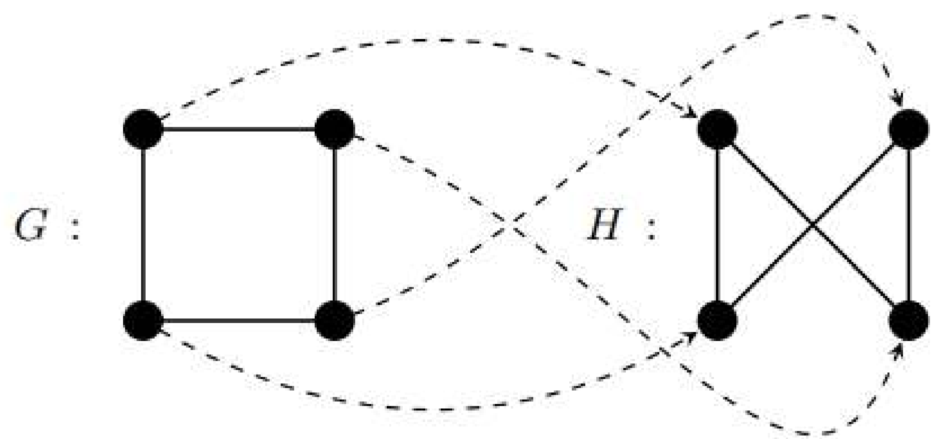
Many Ways to Draw a Graph

Are these graphs the same?

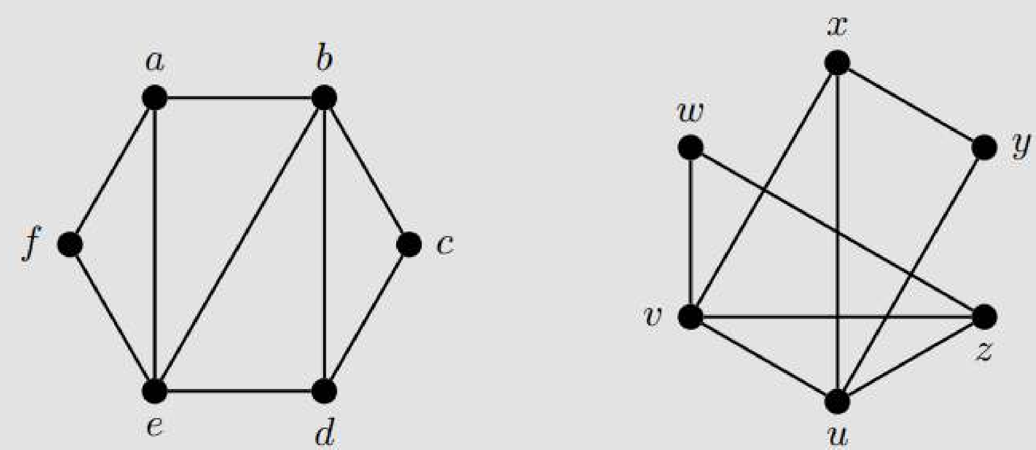




- number of vertices
- number of edges
- vertex degrees

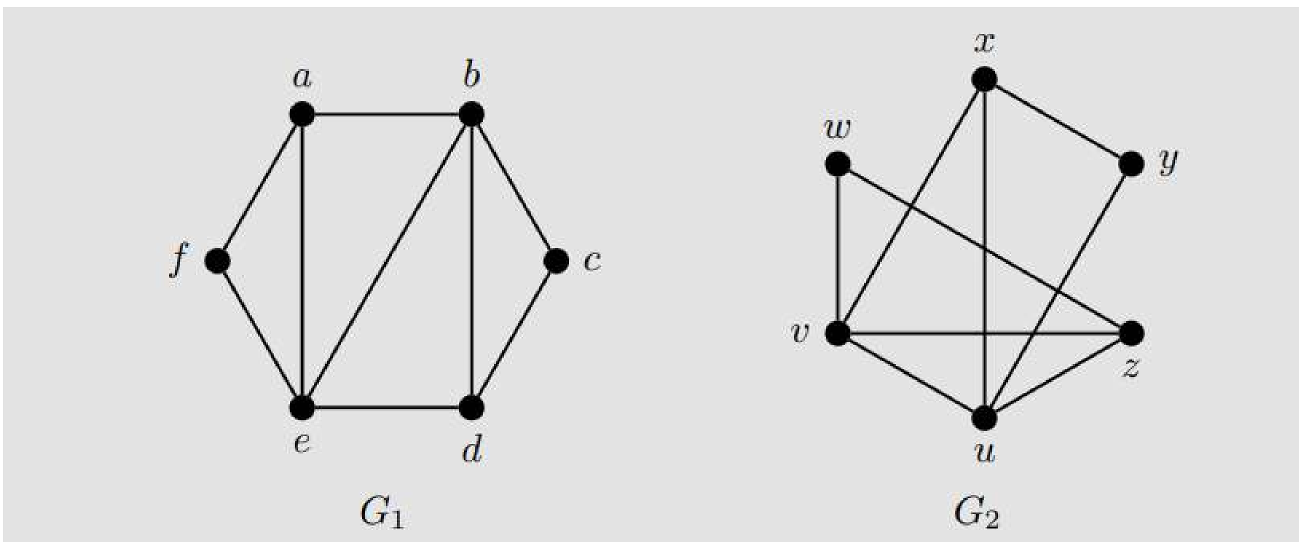


Example 1.14 Determine if the following pair of graphs are isomorphic. If so, give the vertex pairings; if not, explain what property is different among the graphs.

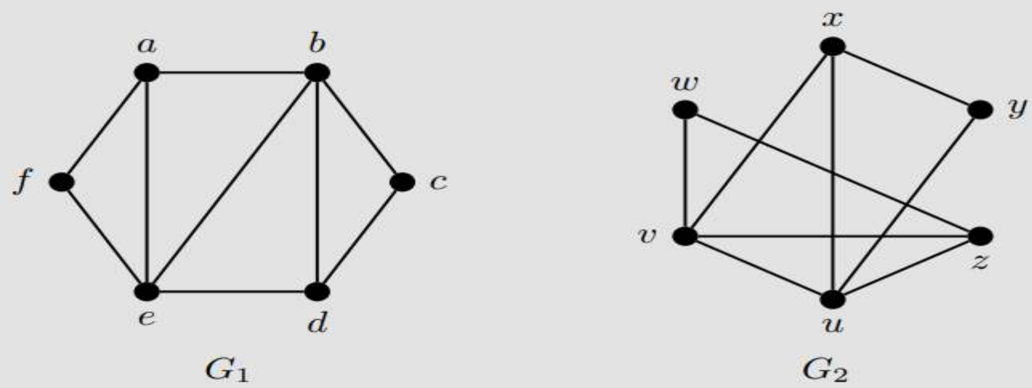


G_1 G_2

- number of vertices
- number of edges
- vertex degrees



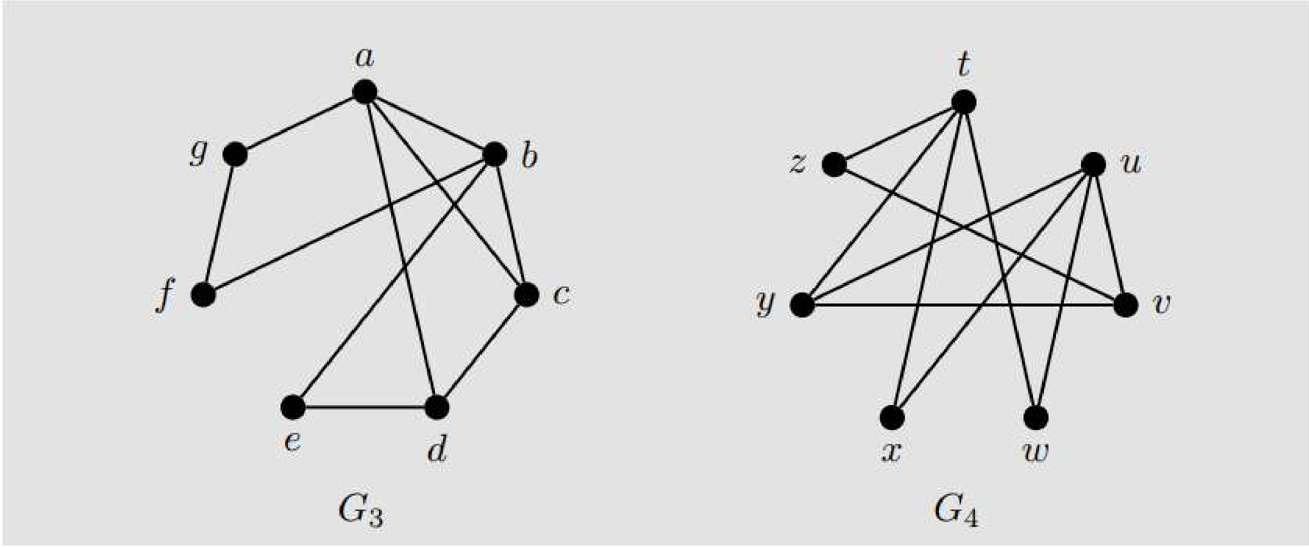
Solution: First note that both graphs have six vertices and nine edges, with two vertices each of degrees 4, 3, and 2. Since corresponding vertices must have the same degree, we know b must map to either u or v. We start by trying to map b to v. By looking at vertex adjacencies and degree, we must have e map to u, c map to w, and a map to x. This leaves f and d, which must be mapped to y and z, respectively. The chart below show the vertex pairings and checks for corresponding edges.



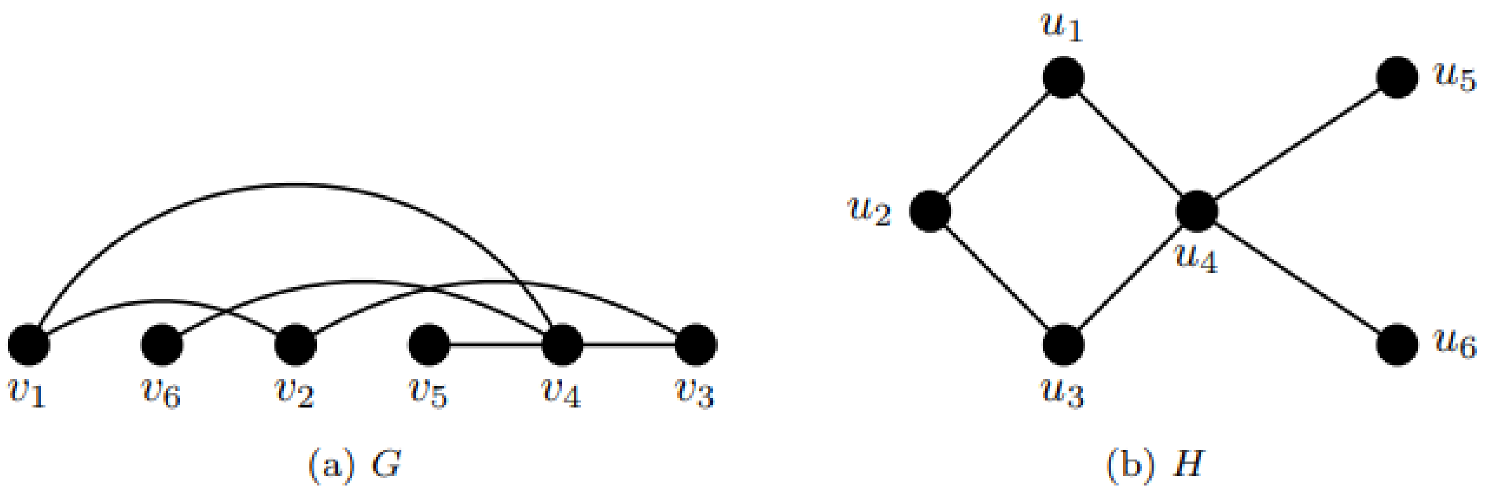
$V(G_1) \longleftrightarrow V(G_2)$	Edges	
$a \longleftrightarrow x$	$ab \longleftrightarrow xv$	✓
$b \longleftrightarrow v$	$ae \longleftrightarrow xu$	✓
$c \longleftrightarrow w$	$af \longleftrightarrow xy$	✓
$d \longleftrightarrow z$	$bc \longleftrightarrow vw$	✓
$e \longleftrightarrow u$	$bd \longleftrightarrow vz$	✓
$f \longleftrightarrow y$	$be \longleftrightarrow vu$	✓
	$cd \longleftrightarrow wz$	✓
	$de \longleftrightarrow zu$	✓
	$ef \longleftrightarrow uy$	✓

Since all edge relationships are maintained, we know G_1 and G_2 are isomorphic.

Example 1.15 Determine if the following pair of graphs are isomorphic. If so, give the vertex pairings; if not, explain what property is different among the graphs.



~~**Example 1.15**~~ Determine if the following pair of graphs are isomorphic. If so, give the vertex pairings; if not, explain what property is different among the graphs.



Theorem 1.18 Assume G_1 and G_2 are isomorphic graphs. Then G_1 and G_2 must satisfy any of the properties listed below; that is, if G_1

- is connected
- has n vertices
- has m edges
- has m vertices of degree k
- has a cycle of length k (see Section 2.1.2)
- has an eulerian circuit (see Section 2.1.3)
- has a hamiltonian cycle (see Section 2.2)

then so too must G_2 (where n, m, and k are non-negative integers).

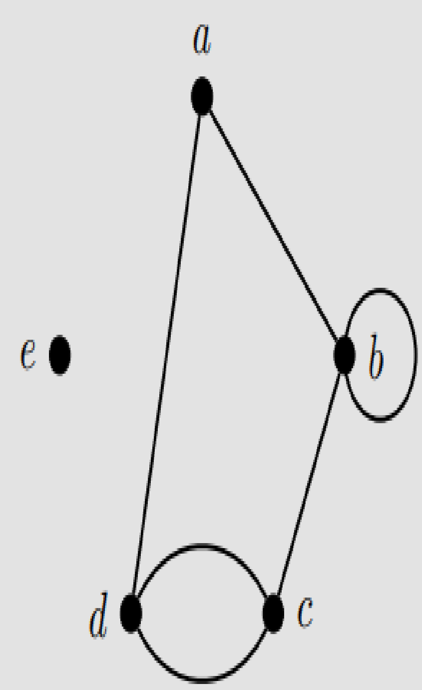
Matrix Representation

WHY Matrix Representation??

Matrix Representation

Definition 1.19 The ***adjacency matrix*** $A(G)$ of the graph G is the $n \times n$ matrix where vertex v_i is represented by row i and column i and the entry a_{ij} denotes the number of edges between v_i and v_j.

Example 1.16 Find the adjacency matrix for the graph G_4 from Example 1.1.

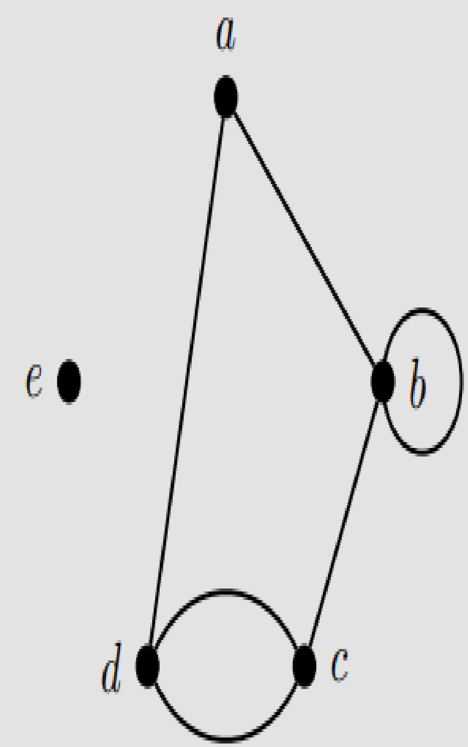


Example 1.16 Find the adjacency matrix for the graph G_4 from Example 1.1.

Solution:

$$\begin{array}{c c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Note that the entry $(2,2)$ represents the loop at b and the entries $(3,4)$ and $(4,3)$ show that there are two edges between c and d. The column for e has all 0's since e is an isolated vertex.



NOTE:

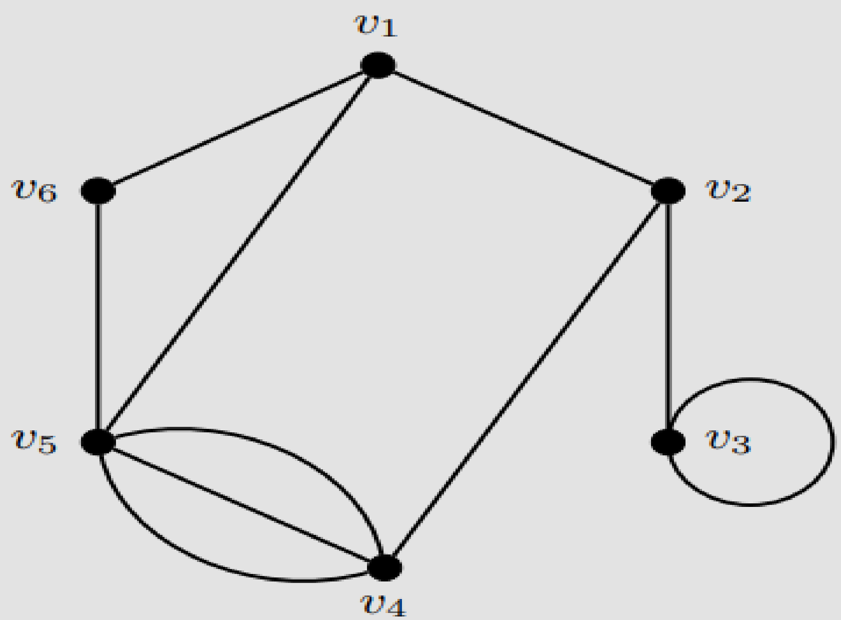
A few interesting properties of the adjacency matrix can be seen. First, the matrix is symmetric along the main diagonal since if there is an edge $v_i v_j$ then it will be accounted for in both the entry (i, j) and (j, i) in the matrix. Second, the main diagonal represents all loops in the graph. Finally, the degree of a vertex can be easily calculated from the adjacency matrix by adding the entries along the row (or column) representing the vertex but double any item along the diagonal. In the matrix above, we would get $\deg(a) = 2$ and $\deg(b) = 4$, which matches the graph representation from Example 1.1.

Example 1.17 Draw the graph whose adjacency matrix is shown below.

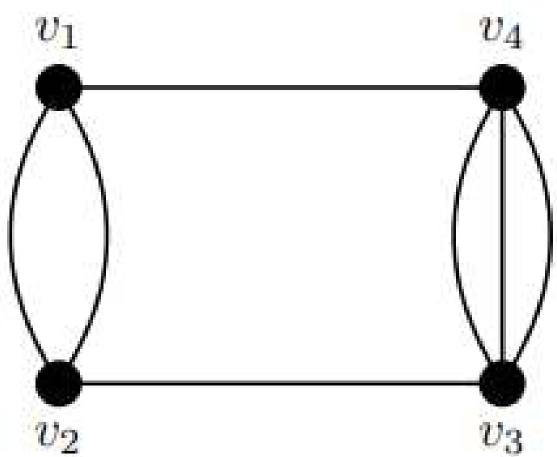
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 0 \\ 1 & 0 & 0 & 3 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Example 1.17 Draw the graph whose adjacency matrix is shown below.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 0 \\ 1 & 0 & 0 & 3 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$



Example 1.2.11. Let G be the multigraph shown in Fig. 1.2.9, where its four vertices are named v_1, v_2, v_3 and v_4.



$$\boldsymbol{A} = \begin{pmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & 3 & 0 \end{pmatrix}.$$

Let G be an (n, m)-multigraph with

$$V(G) = \{v_1, v_2, ..., v_n\} \text{ and } E(G) = \{e_1, e_2, ..., e_m\}.$$

The **incidence matrix** of G is the $n \times m$ matrix

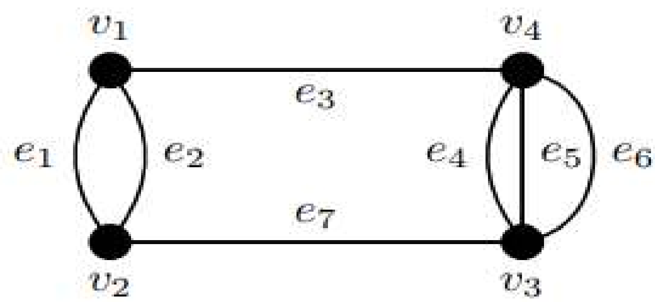
$$\boldsymbol{B}(G) = (b_{ij})_{n \times m},$$

where

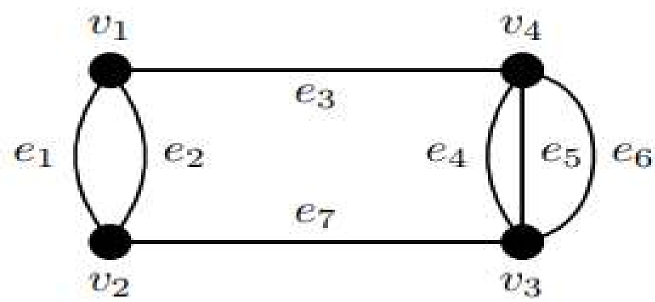
$$b_{ij} = \begin{cases} 1 \text{ if } v_i \text{ is incident with } e_j, \\ 0 \text{ otherwise.} \end{cases}$$

2 For Loop

Example 1.2.14. Let G be the multigraph discussed in Example 1.2.11, where its seven edges are named $e_1, e_2, ..., e_7$ as shown in Fig. 1.2.11:



Example 1.2.14. Let G be the multigraph discussed in Example 1.2.11, where its seven edges are named $e_1, e_2, ..., e_7$ as shown in Fig. 1.2.11:



$$\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

PROOF TECHNIQUES

Handshaking

- Before a business meeting, some people shook hands. Then the number of people who made an odd number of handshakes is even
- In graph terms: A graph has an even number of odd nodes

PROOF TECHNIQUES

Proposition 1.20 The sum of two odd integers is even.

Proof: Assume x and y are odd integers. Then there exist integers n and m such that $x = 2n+1$ and $y = 2m+1$. Thus $x+y = (2n+1)+(2m+1) = 2(n+m+1) = 2k$, where k is the integer given by $n+m+1$. Therefore $x+y$ is even.

Degree Sum Formula

Lemma

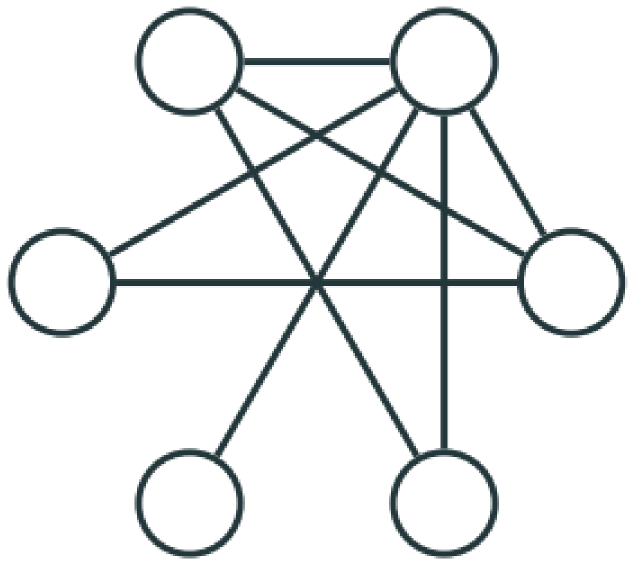
For any graph $G(V, E)$, the sum of degrees of all its nodes is twice the number of edges:

$$\sum_{v \in V} \text{degree}(v) = 2 \cdot |E| \, .$$

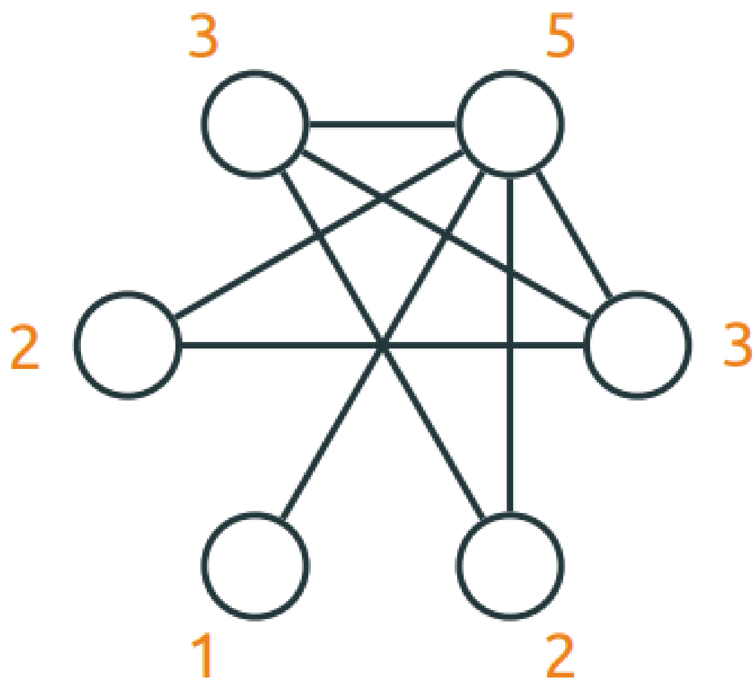
Theorem 1.21 (Handshaking Lemma) Let $G = (V, E)$ be a graph and $|E|$ denote the number of edges in G. Then the sum of the degrees of the vertices equals twice the number of edges; that is if $V = \{v_1, v_2, \ldots, v_n\}$, then

$$\sum_{i=1}^{n} \deg(v_i) = \deg(v_1) + \deg(v_2) + \cdots + \deg(v_n) = 2|E|.$$

Example

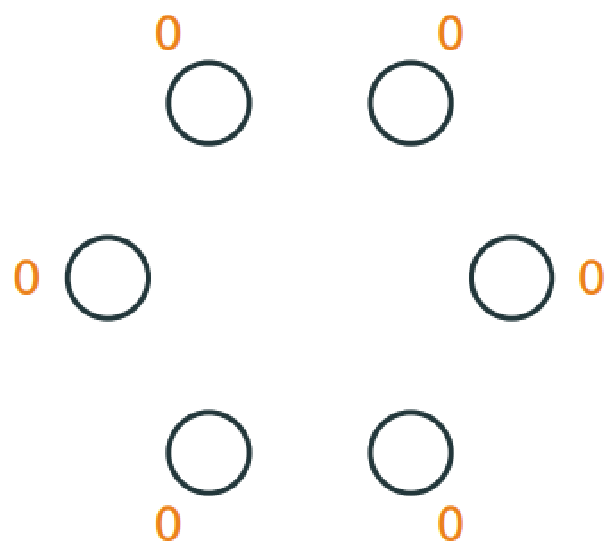


Example



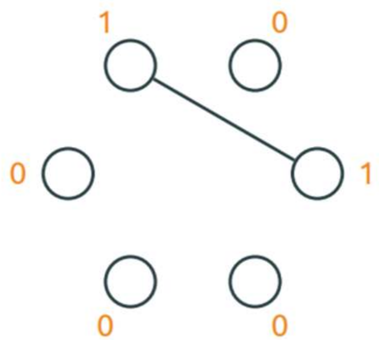
$$3 + 5 + 3 + 2 + 1 + 2 = 2 \cdot 8$$

Proof



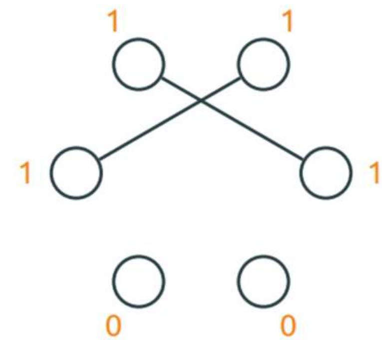
now, start adding them back, one by one

Proof



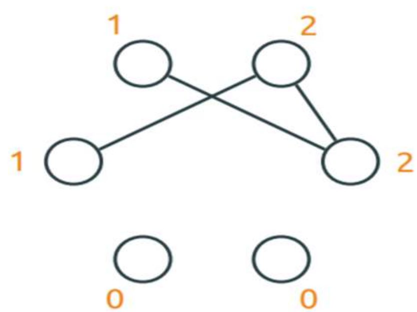
now, start adding them back, one by one

Proof



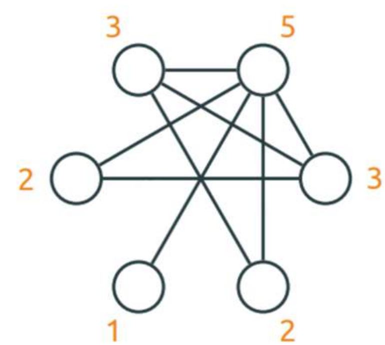
now, start adding them back, one by one

Proof



now, start adding them back, one by one

Proof



each edge contributes 2 to the sum of degrees

National University of computer and emerging sciences

Summary

- Essentially, we proved the formula by induction on the number of edges
- Base case: the formula holds if there are no edges (all degrees are equal to 0)
- Induction step: when we add an edge, the sum of degrees increases by 2

Theorem 1.4 **(The First Theorem of Graph Theory)** *If G is a graph of size m, then*

$$\sum_{v \in V(G)} \deg v = 2m.$$

Proof. When summing the degrees of the vertices of G, each edge of G is counted twice, once for each of its two incident vertices. ■

The sum of the degrees of the vertices of the graph G of Figure 1.4 is 12, which is twice the size 6 of G, as expected from Theorem 1.4. The **average degree** of a graph G of order n and size m is

$$\frac{\sum_{v \in V(G)} \deg v}{n} = \frac{2m}{n}.$$

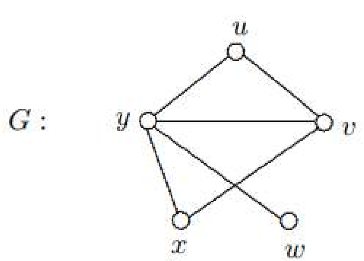


Figure 1.4: A graph

Even and Odd Vertices

A vertex in a graph G is **even** or **odd**, according to whether its degree in G is even or odd. Thus, the graph G of Figure 1.4 has three even vertices and two odd vertices. While a graph can have either an even or an odd number of even vertices, this is not the case for odd vertices.

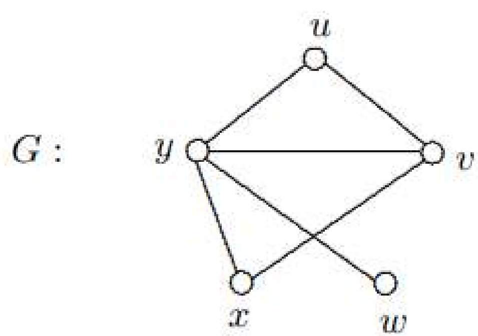


Figure 1.4: A graph

Try to understand it!

Corollary 1.22 Every graph has an even number of vertices of odd degree.

Prove Theorem 1.2: Every graph has an even number of odd vertices.

Solution. Suppose the sum of the degrees of the odd vertices is x and the sum of the degrees of the even vertices is y. The number y is even, and the number $x + y$, being twice the number of edges, is also even. So x is necessarily even. If there are p odd vertices, the even number x is the sum of p odd numbers. So p is even.

Proposition 1.23 For any integer n, if n^2 is odd then n is odd.

Proof: Suppose for a contradiction that n^2 is odd but n is even. Then $n = 2k$ for some integer k and $n^2 = (2k)^2 = 4k^2 = 2j$ where j is the integer $2k^2$. Thus n^2 is both even and odd, a contradiction. Therefore if n^2 is odd then n is also odd.

Proof: Suppose n is not odd. Then n is even and $n = 2k$ for some integer k. Then $n^2 = (2k)^2 = 4k^2 = 2j$ where j is the integer $2k^2$, and so n^2 is even. Thus if n^2 is odd, it must be that n is also odd.

Proposition 1.24 For every simple graph G on at least 2 vertices, there exist two vertices of the same degree.

Proof: Suppose for a contradiction that G is a simple graph on n vertices, with $n \geq 2$, in which no two vertices have the same degree. Since there are no loops and each vertex can have at most one edge to any other vertex, we know the maximum degree for any vertex is $n - 1$ and the minimum degree is 0. Since there are exactly n integers from 0 to $n - 1$, we know there must be exactly one vertex for each degree between 0 and $n-1$. But the vertex of degree $n - 1$ must then be adjacent to every other vertex of G, which contradicts the fact that a vertex has degree 0. Thus G must have at least two vertices of the same degree.

National University of computer and emerging sciences

Foundation for Advancement of Science and Technology

Mathematical Induction

Proposition 1.25 The complete graph K_n has $\dfrac{n(n-1)}{2}$ edges.

Proof: Argue by induction on n. If $n = 1$ then K_1 is just a single vertex and has $0 = \frac{1(0)}{2}$ edges.

Suppose for some $n \geq 1$ that K_n has $\dfrac{n(n-1)}{2}$ edges. We can form K_{n+1} by adding a new vertex v to K_n and adjoining v to all the vertices from K_n. Thus K_{n+1} has n more edges than K_n and so by the induction hypothesis has

$$n + \frac{n(n-1)}{2} = \frac{2n + n(n-1)}{2} = \frac{n(2+n-1)}{2} = \frac{n(n+1)}{2} = \frac{n(n-1)+2n}{2}$$

edges.

Thus by induction we know K_n has $\dfrac{n(n-1)}{2}$ edges for all $n \geq 1$.

The power of induction is that we are proving a statement that holds for an infinite number of objects but only need to prove two very specific items.

Mathematical Induction is useful when studying discrete objects, especially objects that can easily be transformed into ones of smaller size.

Definition 1.26 The ***degree sequence*** of a graph is a listing of the degrees of the vertices. It is customary to write these in decreasing order. If a sequence is a degree sequence of a simple graph then we call it ***graphical***.

Remark 2.5.1. All graphs considered for graphic sequences need not be connected, but are always assumed to be simple.

Any graph G of order n with $V(G) = \{v_1, v_2, ..., v_n\}$ has its **degree sequence** defined to be $(d(v_1), d(v_2), ..., d(v_n))$. We usually assume that the sequence is non-increasing, that is, $d(v_1) \geq d(v_2) \geq ... \geq d(v_n)$.

A non-increasing sequence $\boldsymbol{d} = (d_1, d_2, ..., d_n)$ of non-negative integers is called a **graphic sequence** if $\boldsymbol{d}$ is the degree sequence of some graph. If $\boldsymbol{d}$ is the degree sequence of a graph G, we say that $\boldsymbol{d}$ is **representable** by G or G **represents** $\boldsymbol{d}$.

Remark 2.5.1. All graphs considered for graphic sequences need not be connected, but are always assumed to be simple.

Example 1.18 Explain why neither $4, 4, 2, 1, 0$ nor $4, 4, 3, 1, 0$ can be graphical.

Example 1.18 Explain why neither $4, 4, 2, 1, 0$ nor $4, 4, 3, 1, 0$ can be graphical.

Solution: The first sequence sums to 11, but we know the sum of the degrees of a graph must be even by the Handshaking Lemma. Thus it cannot be a degree sequence.

The second sequence sums to 12, so it is at least even. However, in a simple graph with 5 vertices, a vertex with degree 4 must be adjacent to all the other vertices, which would mean no vertex could have degree 0. Thus the second sequence cannot be a degree sequence.

Which of the following sequences are graphic?

(i) $(0, 0, 0, 0)$
(ii) $(1, 0, 0, 0)$
(iii) $(1, 1, 0, 0)$
(iv) $(1, 1, 1, 0)$
(v) $(1, 1, 1, 1)$
(vi) $(2, 1, 1, 0)$
(vii) $(2, 2, 0, 0)$
(viii) $(2, 1, 1, 1, 0)$
(ix) $(2, 1, 1, 1, 1)$
(x) $(3, 2, 2, 1, 1)$
(xi) $(4, 3, 2, 1, 0)$
(xii) $(4, 3, 2, 2, 1)$

Which of the following sequences are graphic?

(i) $(0, 0, 0, 0)$
(ii) $(1, 0, 0, 0)$
(iii) $(1, 1, 0, 0)$
(iv) $(1, 1, 1, 0)$
(v) $(1, 1, 1, 1)$
(vi) $(2, 1, 1, 0)$
(vii) $(2, 2, 0, 0)$
(viii) $(2, 1, 1, 1, 0)$
(ix) $(2, 1, 1, 1, 1)$
(x) $(3, 2, 2, 1, 1)$
(xi) $(4, 3, 2, 1, 0)$
(xii) $(4, 3, 2, 2, 1)$

It is clear that the sequences (ii), (iv), (viii) and (x) are non-graphic. The reason is that the number of odd vertices in any graph must be even.

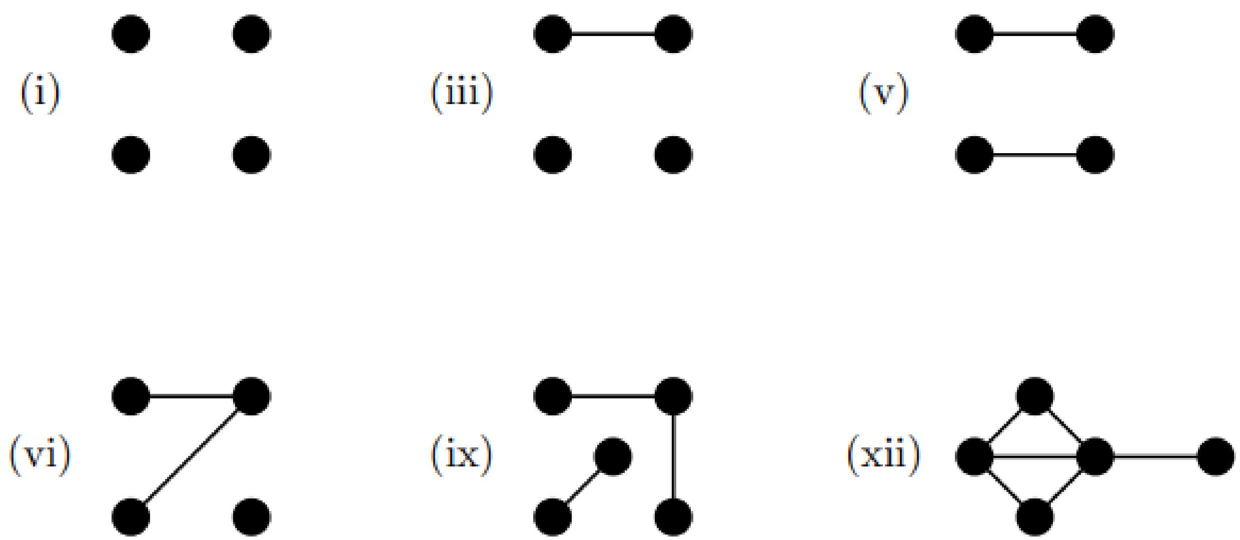
Which of the following sequences are graphic?

(i) $(0, 0, 0, 0)$
(ii) $(1, 0, 0, 0)$
(iii) $(1, 1, 0, 0)$
(iv) $(1, 1, 1, 0)$
(v) $(1, 1, 1, 1)$
(vi) $(2, 1, 1, 0)$
(vii) $(2, 2, 0, 0)$
(viii) $(2, 1, 1, 1, 0)$
(ix) $(2, 1, 1, 1, 1)$
(x) $(3, 2, 2, 1, 1)$
(xi) $(4, 3, 2, 1, 0)$
(xii) $(4, 3, 2, 2, 1)$

The sequence (vii) is not graphic. For if it were representable by, say G, then G would have two isolated vertices, and thus have its remaining two vertices joined by '2' parallel edges, which is not allowed for 'simple' graphs.

(i) $(0,0,0,0)$ (ii) $(1,0,0,0)$
(iii) $(1,1,0,0)$ (iv) $(1,1,1,0)$
(v) $(1,1,1,1)$ (vi) $(2,1,1,0)$
(vii) $(2,2,0,0)$ (viii) $(2,1,1,1,0)$
(ix) $(2,1,1,1,1)$ (x) $(3,2,2,1,1)$
(xi) $(4,3,2,1,0)$ (xii) $(4,3,2,2,1)$

The remaining sequences are graphic, and the graphs representing them



Theorem 1.27 (Havel-Hakimi Theorem) An increasing sequence S : $s_1, s_2, \ldots, s_n$ (for $n \geq 2$) of nonnegative integers is graphical if and only if the sequence

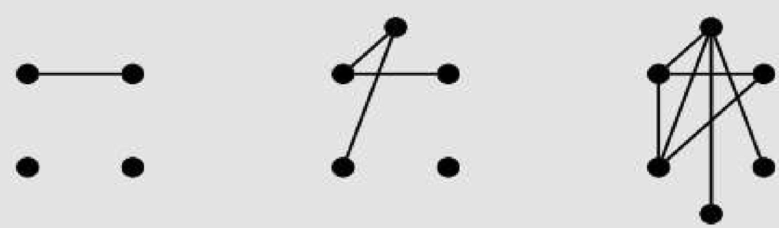
$$S' : s_2 - 1, s_3 - 1, \ldots, s_{s_1} - 1, s_{s_{n+1}}, \ldots, s_n$$

is graphical.

Example 1.19 Determine if either of $S: 4, 4, 2, 1, 1, 0$ or $T: 4, 3, 3, 2, 1, 1$ is graphical.

Solution: Applying the Havel-Hakimi Theorem to S, we note the first term of the sequence is 4, and so we eliminate the first term and subtract 1 from the next 4 terms and leaving the last one alone. This gives $S_1: 3, 1, 0, 0, 0$. In this new sequence the first term is 3, so we eliminate it and subtract 1 from the next 3 terms of S_1, producing $S_2: 0, -1, -1, 0$. This last sequence cannot be graphical since degrees cannot be negative. Thus S_1 and S cannot be graphical either.

Using the same procedure on T, after the first iteration we get $2, 2, 1, 0, 1$. We reorder this to make it decreasing as $T_1: 2, 2, 1, 1, 0$. After the second iteration we have $1, 0, 1, 0$, which is again reordered to $T_2: 1, 1, 0, 0$. At this point we can stop since it is not too difficult to see this sequence is graphical: the two vertices of degree 1 are adjacent and the other two vertices are isolated (see below). This means that T_2, T_1, and the original sequence T are all graphical.



T_2 T_1 T

Complete other examples covered in class

National University
of computer and emerging sciences

Foundation for Advancement of Science and Technology

EX #: 1.8
Problems:
1.1-1.7, 1.12,1.14, 1.15, 1.16, 1.17, 1.20, 1.22