CS230-HW8Sol

1. **[10 Pts] Nathan**

   (a) **[5 Pts]** $A \subseteq S$ by mathematical induction.
   We prove that for all $i \in \mathbf{Z}^+, 5^i \in S$.
   Base case: $i = 1$. $5^1 = 5$ and $5 \in S$ (Full answer expects more explanation/justification of this work. This is just the math that's necessary)

   Inductive step: Assume that $5^k \in S$ for some $k \in \mathbf{Z}^+$. We prove that $5^{k+1} \in S$.
   Proof: $5^{k+1} = (5^k)5$. $5^k \in S$ and $5 \in S$. Therefore $(5^k)5 \in S$.
   Therefore $5^{k+1} \in S$.
   Therefore, by the principle of mathematical induction, for all $i \in \mathbf{Z}^+$, $5^i \in S$.
   (Full answer expects more explanation/justification of this work. This is just the math that's necessary)

   (b) **[5 Pts]** $S \subseteq A$ by structural induction.
   We prove that for all $x \in S, x = 5^i$ for some $i \in \mathbf{Z}^+$.
   Base case: Since $5 \in S$ by the basis step of the inductive definition of S, we prove that $5 \in A$.
   Proof: This is true since $5 = 5^1$ and $1 \in \mathbf{Z}^+$. Therefore, $5 \in A$ and the base case holds.

   Inductive step: $s \in S$ and $t \in S$. We assume that $s \in A$ and $t \in A$. By the inductive step of the inductive definition of S, $st \in S$. We prove that $st \in A$.
   Proof: $s, t \in A$. $s = 5^j$ and $t = 5^k$ for some $j, k \in \mathbf{Z}^+$.
   (The final mathematical step is missing, as are justifying claims/phrases throughout the proofs)

2. **[5 Pts] Nathan** *We define the language consisting of Palindromes using our alphabet as the set $\boldsymbol{L}$.*
   *Base: We provide the inductive step, see if you can determine the base given that. (**Hint:** the base that matches the inductive step we've chosen claims that there are 4 starting elements in $\boldsymbol{L}$. Another option would be 8 starting elements, but that'd be more work than necessary if you can recall the special element that can be in any language regardless of its alphabet.)*

   Inductive Step: If $x \in \boldsymbol{L}$ then, $axa, bxb, cxc \in \boldsymbol{L}$

3. **[5 Pts] Haniyeh** *Base Case:*
   $6 \in S$
   Inductive Steps:
   if $a \in S$, then $2a \in S$ and $3a \in S$

4. **[8 Pts] Haniyeh**

   Defining $n(T)$: The vertices of T are the root of T and the vertices of the two subtrees T_1 and T_2. Formally, the number of the nodes in a tree T is $n(T)$, where

Basis step*: $n(T) = 1$ if T consists of only a root r,*
Inductive Step*: $n(T) = 1 + n(T_1) + n(T_2)$ if $T = T_1 \cdot T_2$ where T_1 and T_2 are subtrees of T.*

Defining $\ell(T)$: The leaves of T are the leaves of the subtrees T_1 and T_2. Formally, the number of the leaves in a tree T is $\ell(T)$, where
Basis step*: $\ell(T) = 1$ if T consists of only a leaf l,*
Inductive Step*: $\ell(T) = \ell(T_1) + \ell(T_2)$ if $T = T_1 \cdot T_2$ where T_1 and T_2 are subtrees of T.*

We prove that $n(T) = 2\ell(T) - 1$ by structural induction.

Base: in the basis step we have the tree consisting of just the root, so there is one node which is also a leaf. In other words, $\ell(T) = 1$ Substituting this value into the target equation is left as an exercise. Such substitution and evaluation of the equation would be expected in your submission.

Induction: For the inductive step, assume that this relationship holds for T_1 and T_2. Writing this relationship out in the context of T_1 and T_2 is left as an exercise (they should match the target equation of our proof, which we assume is true for T_1 and T_2).
Now, consider the tree T with a new root, whose children are the roots of T_1 and T_2. Thus we have $n(T) = n(T_1) + n(T_2) + 1$, by the inductive definition of $n(T)$ above.
Substitute $n(T_1)$ and $n(T_2)$ with their equations given by the inductive hypothesis (i.e. $n(T_1)$ and $n(T_2)$ in terms of $\ell(T_1)$ and $\ell(T_2)$), and show that this substituted form evaluates to $n(T) = 2\ell(T) - 1$.
You will need the inductive definition of $\ell(T)$ to do this. (i.e. the definition of $\ell(T)$ in terms of $\ell(T_1)$ and $\ell(T_2)$)

5. **[15 Pts]Jacob** *Let $L = \{(a, b) \mid a, b \in \mathcal{Z}, (a-b) \bmod 3 = 0\}$. We want to program a robot that can get to each point $(x, y) \in L$ starting at $(0, 0)$.*

   (a) **[5 Pts]**
   Base: $(0, 0) \in L'$.
   Inductive: *if $(a, b) \in L'$ then*
   $(a + 1, b + 1) \in L'$, $(a - 1, b - 1) \in L'$, $(a + 3, b) \in L'$, *and* $(a - 3, b) \in L'$.

   It is possible to have other inductive definitions, but notice the simplicity and elegance of this one. A single base element will make the base step of inductive proofs later on far easier.

   Additionally, we know we are meeting all of the needs of our robots movement. We exist in a 2D space, and can only visit coordinates whose x-y difference is divisible by 3. Since it's x-y, we know that for an unchanged y, our x can only change by increments of ± 3 if we want to keep x-y divisible by 3. Similarly, we know that if y changes, x has to change by the same *amount and sign to keep $(x-y)\%3 == 0$ true. These ideas will guide our proofs in parts b and c.*

$x\&y \pm 1$ or $x \pm 3$ are the smallest integer increments that meet our needs. Using constant integer increments is a good way to keep our definition simple. We don't want to introduce things that can vary in a non-inductive way, like "some integer n"

While it's not a proof, a good way to check yourself before trying to do the formal inductive proofs is to generate some values using the definition of L, and then see if you can generate them using L'.

For example: L contains pairs where x-y is divisible by 3, like (0,0), (0,3), (3,0), (1,1), (2,2), (5,2).

Check to see if you can generate these using our definition of L', or the definition you used in your submission.

(b) **[5 Pts]**

$L' \subseteq L$ means that every ordered pair (a,b) produced in definition (a) has the property that $a-b$ is divisible by 3, i.e., $(a,b) \in L$. We prove this by structural induction.

By the base case of the definition, $(0,0) \in L'$. Since $0-0 = 3 \times 0$, it follows that $(0,0) \in L$.

For the inductive step, assume that ordered pair $(a,b) \in L'$ is such that $(a,b) \in L$, i.e., $a-b$ is divisible by 3. The inductive step allows us to place $(a+1,b+1)$, $(a-1,b-1)$, $(a+3,b)$ and $(a-3,b)$ in L'. We prove that each of these are in L. There should be 4 cases that each substitute one of these inductive steps into the equation for L. For the sake of making the math more understandable, we re-write the equation of L as $a-b = 3k$ for $k \in \mathbf{Z}$. In every case, the inductive step should produce a new ordered pair based on the pair (a,b) that satisfies membership in L.

(c) **[5 Pts]**

For this one, we have chosen to give you the full solution format:

If $(m,n) \in L$ then $m-n = 3k$ for some integer k, so $m = n+3k$. So any element of L will have form $(n+3k,n) \in L$. We show that $(n+3k,n) \in L'$, i.e., $(n+3k,n)$ is reachable by the inductive definition in (a). There are four cases to consider.

- *if $n \geq 0$ and $k \geq 0$, we move from $(0,0)$ to (n,n) by using the rule $(a+1,b+1)$, n times, and then move from (n,n) to $(n+3k,n)$ by using the rule $(a+3,b)$, k times.*
- *if $n \geq 0$ and $k < 0$, we move from $(0,0)$ to (n,n) by using the rule $(a+1,b+1)$, n times, then move from (n,n) to $(n+3k,n)$ by using the rule $(a-3,b)$, $-k$ times.*
- *if $n < 0$ and $k \geq 0$, we move from $(0,0)$ to (n,n) by using the rule $(a-1,b-1)$, $-n$ times, and then move from (n,n) to $(n+3k,n)$ by using the rule $(a+3,b)$, k times.*
- *if $n < 0$ and $k < 0$, we move from $(0,0)$ to (n,n) by using the rule $(a-1,b-1)$, $-n$ times, then move from (n,n) to $(n+3k,n)$ by using the rule $(a-3,b)$, $-k$ times.*

As we can reach any element of L using the rules of L', $L \subseteq L'$.

Good luck on the Exam everyone!