CS230-HW10Sol

1. **[2 Pts] Haniyeh** *Let D_k be the set of integers between 1 and 60000, inclusive, that are divisible by k. We need to compute the size of $D_3 \cup D_5 \cup D_7$. By the Inclusion-Exclusion Principle, $|D_3 \cup D_5 \cup D_7| = |D_3| + |D_5| + |D_7| - |D_3 \cap D_5| - |D_3 \cap D_7| - |D_5 \cap D_7| + |D_3 \cap D_5 \cap D_7|$. As an instance we can have see there are* $60000/3 = 20000$ *numbers between* 1 *and* 60000 *that are divisible by* 3*, so* $|D_3| = 20000$.
   You should do all the calculations and derive the final result which is 32571.

2. **[2 Pts] Haniyeh**
   (a) By product rule, we have $7 \times 5 \times 2 \times 2$ different types of shirts
   (b) For red and gold shirts we have $2 \times 5 \times 1 \times 2$ different shirts. For shirts that are neither red nor gold we have $(7-2) \times 5 \times 2 \times 2$ different shirts. Sum of these two values gives us the result we are looking for.

3. **[8 Pts] Haniyeh** Let A and B be sets of 7 elements and 9 elements, respectively. (a) How many different functions possible from A to B? from B to A? (b) How many different relations possible from A to B? (c) How many of the functions from A to B are one-to-one? (d) How many of the functions from B to A are onto?
   a Each function from A to B maps each of the 7 elements of A to one of the 9 elements of B. There are 9 choices for the first element in A, times 9 choices for the second, times 9 choices for the third, and so on, for a total of 9^7 functions. We have a similar calculation from B to A.
   b A relation from A to B is a subset of $A \times B$. The set of all subsets of a set S is the powerset of S. The size of the powerset of S is $2^{|S|}$. The size of $A \times B$ is 63. Using this information we can calculate the number of relations.
   c If a function from A to B is one-to-one, each element of A is mapped to a different element of B. So, we have 9 choices for mapping the first element of A, we have 8 choices for the second element of A, then only 7 choices for the third, and so on down to 3 choices for the seventh, so $9!/2$
   d Since each element of A must be mapped to, and there are two more elements in B than in A, there are two possibilities: (i)three elements in B mapped to the same element in A, (ii)two elements in B mapped to one element in A, and two mapped to a different element in A.
   (i) We group 3 elements in B together, while the other 5 are single, creating 7 groups. We then map the 7 elements in A to the 7 groups in B. How many ways are there to select the group of 3? And and how many ways to assign the 7 elements of A to the 7 groups of B?
   (ii) We group 2 elements in B together, another 2 elements of B together, and the remaining 5 are single, creating 7 groups. We then map the 7 elements in A to the 7 groups in B. In how many ways can we select the groups of 2?(don't forget to divide by 2) And in how many ways can we assign the 7 groups of B to the 7 elements of A?

   Now add the values you get in parts (i) and (ii) to get what you are looking for.

4. **[4 Pts] Jacob** The department is ordering new office chairs that will be placed in six labs. How many chairs should we buy to make sure that some lab gets at least five chairs? Use the Pigeonhole Principle to justify your answer.

   I think of this problem from the perspective of an acquisitions administrator (the guy in charge of buying stuff).

   I want to save money. My best case scenario for meeting the ***some lab*** *(aka 1 lab) requirement is to just buy 5 chairs and put them in one lab.*

   The equity officer (in charge of making sure things are fair) doesn't like this idea. If I buy 1 chair, he lets me put it wherever I want. But if I buy 2 or more, he makes me disperse them evenly across the labs. (From a mathematical perspective, the equity officer is why I have to buy enough to ***guarantee*** *one of the labs will get 5, rather than enough for it to be a* ***possibility****)*

   So to ***guarantee*** *one of the labs will 5 chairs while keeping the equity officer happy, I have to buy enough chairs for all 6 labs to get 4 chairs. That's 6x4 = 24 chairs. Then, since I want a lab with 5 chairs, I'll buy 1 more, and put it in whatever lab I want. This makes 25 total chairs purchased.*

   In pigeon hole terms, I have $n = 6$ holes (labs), and will need 25 pigeons (chairs) to ***guarantee*** *1 of the holes contains $r = 5$ pigeons. $n = 6$; $r = 5$;* $25 = 6(5 - 1) + 1$

5. **[4 Pts] Jacob** In how many ways can a photographer arrange 8 people in a row from a family of 12 people, if (a) Mom and Dad are in the photo, (b) Mom and Dad are next to each other in the center of the photo (three people on each side of them).

   (a) *Since Mom and Dad are in the photo, we choose* 6 *more people from the* 8 *remaining people. There are* $\binom{8}{6} = 28$ *different ways to do this. Then, we arrange the* 8 *people (including Mom and Dad) in a row in* $8!$ *ways, giving a total of*

   $$\binom{8}{6} \times 8! = 28 \times 8! = 1128960$$

   (b) *Mom and Dad are next to each other but Mom could be to the left or to the right of Dad, so there are* $2! = 2$ *ways to arrange them. There are* $P(8, 6) = \frac{8!}{2!}$ *ways to pick the 6 people for the remaining slots from the remaining 8 people. This gives a total of*

   $$2 \times P(8, 6) = 2 \times \frac{8!}{2!} = 40320$$

   Note: This is the same as choosing and ordering the 3 family-members left of the Mom and Dad, and then choosing and ordering the 3 family-members right of the Mom and Dad from those who remain after the left 3 are selected.

   $$P(8, 3) * 2! * P(8 - 3, 3) = 40320$$

6. **[4 Pts] Jacob** A coin is flipped nine times where each flip comes up either head or tails. How many possible outcomes contain at least five heads?

   Solution:
   Here, use the addition rule between each combinatoric that represents an outcome of 5 or more heads.

Hint: there are $C\binom{9}{5}$ *sequences of 9 coin flips that contain exactly 5 heads. Use the addition rule to collect the other possibilities, since we are interested in options with* ***at least****, not* ***exactly*** *5 heads.*

7. **[6 pts] Nathan** *(a) There are 22 people total, so a total of* $\binom{22}{6}$ *possible committees that can be formed if we don't restrict the committees in any way. If Ann and Beth are both on the committee, we choose the 4 remaining committee members from 20 people in* $\binom{20}{4}$ *ways. So, by Subtraction Rule, the number of committees where Ann and Beth do not serve together is* $\binom{22}{6} - \binom{20}{4}$.
(b) There are $\binom{10}{6}$ *ways to pick an all-male committee. Therefore, the answer is* $\binom{22}{6} - \binom{10}{6}$.
(c) As we did with (a) and (b), we subtract the "problem" cases from the total number of possibilities. There are $\binom{12}{6}$ *ways of creating an all-women committee, and* $\binom{10}{6}$ *ways of creating an all-male committee. Therefore, the answer is* $\binom{22}{6} - \binom{12}{6} - \binom{10}{6}$.

8. **[4 Pts] Nathan** *Let us consider having two groups of, say m men and n women, and we need to pick a committee of size 2. If we choose 2 people from a group of m men and n women, that is, $m+n$ total people, we can count that as* $\binom{m+n}{2}$*. This is the lhs.*
If we choose 2 people from a group of m men and n women, either we pick 2 women, or 2 men, or 1 man and 1 woman. There are $\binom{n}{2}$ *ways to pick 2 women,* $\binom{m}{2}$ *ways to pick 2 men, and mn ways to choose 1 of each, for* $\binom{m}{2} + \binom{n}{2} + mn$ *ways to pick 2 people. This is the rhs.*
Therefore, $\binom{m+n}{2} = \binom{m}{2} + \binom{n}{2} + mn$.

9. **[6 Pts] Nathan** Prove that $C(n, 2k)C(2k, k) = C(n, k)C(n-k, k)$, where $n \geq 2k > 0$, by using (a)an algebraic proof, (b) a combinatorial proof.

   Solution:
   (a) $C(n, 2k)C(2k, k) = \frac{n!}{2k!(n-2k)!}\frac{2k!}{k!(2k-k)!} = \frac{n!}{(n-2k)!}\frac{1}{k!k!}\frac{(n-k)!}{(n-k)!} = \frac{n!}{k!(n-k)!}\frac{(n-k)!}{k!(n-k-k)!} = C(n,k)C(n-k,k)$
   (b) Suppose we have a group of n people and we want to choose two groups of k people from that group. Then we could either choose $2k$ and then split the group in half, which is the left side of the equation. Or we could choose k from the size n group, then choose another k, which would be the right side.