CS230-HW9Sol

1. **[12 Pts] Jacob**

   (a) **[6 Pts]** *If a successful bijection is not readily apparent to you, it is recommended that you begin by enumerating some elements in your domain and co-domain. This can illustrate the goals for your function:*

   $$\mathbb{N} = 0, 1, 2, 3, 4, 5, ...$$

   $$f(x) = 0, -6, 6, -12, 12, ...$$

   A natural solution approach may try to create a function that maps the aligned elements in the enumerations above.
   I.e. $0 \to 0, 1 \to -6, 2 \to 6, 3 \to -12, 4 \to 12, ...$

   At this point, we know that our function needs to be able to produce the same value with opposite sign for two different inputs.
   To produce a function that alternates signs, there are two common math tricks:

   i. $f(n) = (-1)^n * (....)$

   ii. piecewise functions:

   $$f(n) = \left\{ \begin{array}{cc} -1 * (...) & \text{if ...} \\ 1 * (...) & \text{otherwise} \end{array} \right\} \tag{1}$$

   The remaining work to define this function with either method above is to account for the fact that $0 \to 0, 1 \to -6, 2 \to 6, ...$*, that is, that except for 0, for each odd n, the subsequent even value, n+1, needs to produce the same value.*

   In short, for some input n, how do we either force n and n+1 to the same value, or how do we selectively cause our function to calculate using only one of those options for both n and n+1 inputs? There are two math tricks available. One for each approach:

   i. ***Forcing n and n+1 to the same value:*** $\lceil \frac{n}{2} \rceil$ *These brackets are called a "ceiling" operation. If applied to a fraction or decimal number, it'll increase it to the next greatest whole number. It has no impact when used on a whole number.*

   ii. ***Selectively using one of the two values in calculation for both inputs:*** *A piecewise function that is multiplied by n for some condition and* $n + 1$ *otherwise can produce this behavior if the condition is chosen correctly (hint: condition should capture even vs. odd, the number-feature that changes between n and n+1)*

   (b) **[6 Pts]**
   Define bijection $f : \mathbb{Z}^+ \to \{0, 1, 2, 3\} \times \mathbb{Z}^+$

   Again, we start by enumerating.

   $$\{0, 1, 2, 3\} \times \mathbb{Z}^+ = \&\{(0, 1), (1, 1), (2, 1), (3, 1), (0, 2), (1, 2), (2, 2), (3, 2), (0, 3), (1, 3), (2, 3), (3, 3), ...\}$$

   $$\mathbb{Z}^+ = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ...$$

   Conceptually, this enumeration scheme (which can inform our function definition) goes "pair each positive integer with all four values from A, then increase to the next positive integer and repeat"

Again, writing out some of the mappings from $\mathbb{Z}^+$ to $\{0, 1, 2, 3\} \times \mathbb{Z}^+$ may help:

$$1 \to (0, 1), 2 \to (1, 1), 3 \to (2, 1), 4 \to (3, 1), 5 \to (0, 2), 6 \to (1, 2), 7 \to (2, 2), 8 \to (3, 2)$$

1st term: *We can see in our example mappings above that the 1st term in our output pairs starts at 0, increases by +1 for every new input, and resets at 0 after every 4th input. This* ***almost*** *matches the behavior of modulo 4, $n\%4$. However, since $n \in \mathbb{Z}^+$, and starts at 1, our first call to $n\%4$ will produce 1, not the 0 our mapping expects. So we do $(n-1)\%4$ to account for this.*

2nd term: *We can see in our example mappings that the 2nd term in our output pairs starts at 1, and only changes by +1 after every 4 inputs x.*
Since our inputs start at one, we can use the same trick in part a, $\lceil \frac{n}{4} \rceil$ will force all inputs 1 through 4 to the value 1, all inputs 5 through 8 to the value 2, etc.

The final solution should take the form

$$f(x) = (\textit{1st term equation}, \textit{2nd term equation})$$

2. **[7 Pts]Haniyeh** *Let $F_{\mathbb{N}\mapsto\mathbb{N}}$ be the set of functions from $\mathbb{N}$ to $\mathbb{N}$. Suppose, for the sake of contradiction, that $F_{\mathbb{N}\mapsto\mathbb{N}}$ is countable. Then we can enumerate it: $f_0, f_1, f_1, \ldots$.*
For our diagonalization argument, we need to construct a new function that disagrees with each function f_i on at least one input, such as $g(n) = f_n(n) + 1$.
You can also write an example to show this by construction and see that $f_0 \neq f_i$ for any f_i in our enumeration, since, in particular $f_0(i) \neq f_i(i)$, which is a contradiction. Thus, $F_{\mathbb{N}\mapsto\mathbb{N}}$ is uncountable.

3. **[12 Pts]**

   (a) **[6 Pts] Haniyeh** *We show that this set is countable by representing the numbers in a table where each row is labeled by the number of 9's before the decimal for any number in that row. Each column is labeled by the number of 9's after the decimal for any number in that column. (Note that we need a column for all the numbers with infinite recurring 9's after the decimal). We now have an infinite 2-dimensional table representing all the numbers. We cannot count all the numbers by counting off each row (or each column) since each row or column is infinite. Instead, we use dovetailing by counting off each diagonal, such as $9, .\overline{9}$ followed by $99, 9.\overline{9}, .9$, followed by $999, 99.\overline{9}, 9.9, .99$, and so on.*

   (b) **[6 Pts] Nathan** *We prove this set is uncountable. Let R be the set of real numbers with decimal representation consisting of 7's, 8's and 9's. We prove that R is uncountable by showing that it contains an uncountable subset. Let S be the set of real numbers in the interval $[0, 1]$ with infinite decimal representation consisting only of 7's, 8's and 9's, for example, $0.8888\overline{8}$, which has infinite number of 8's after the decimal, is included in S, but 0.8888, which has finite number of 8's after the decimal, is not included. Clearly, $S \subseteq R$. Assume, for contradiction, that S is countable. Then, S can be enumerated as the sequence $s_0, s_1, s_2, \ldots$. In other words, if $x \in S$, then $x = s_i$ for some i. We write $s_i[j]$ to mean "the jth digit of the number s_i."*
   We diagonalize against this countable sequence by defining a new number $t \in [0, 1]$ digit by digit, as follows, where $t[1]$ is the first digit of t after the decimal, $t[2]$ is the second

digit after the decimal, and so on.

$$t[i] = \begin{cases} 1 & \textit{if } s_i[i] = 7 \\ 2 & \textit{if } s_i[i] = 9 \\ 3 & \textit{if } s_i[i] = 8 \end{cases}$$

Note that $t \in S$ *because it is a number between 0 and 1 with decimal representation being an infinite sequence of 7's, 8's and 9's. Therefore,* $t = s_i$ *for some* i*. However,* $t \neq s_i$ *for any* i*, since* t *differs from every* s_i *in the* i*th digit (and perhaps elsewhere also). Therefore, we have a contradiction, implying that* S *is uncountable. Since a superset of an uncountable set is itself uncountable, we have proven that* R *is uncountable.*

4. **[9 Pts] Nathan**

   (a) finite
   Let A, B *be uncountable subsets of* $\mathbb{R}$ *such that* $A = (0, 1]$ *and* $B = [1, 2)$.
   $A \cap B = (0, 1] \cap [1, 2) = \{1\}$
   A, B *are uncountable subsets of* $\mathbb{R}$*. Since* A *includes one joint element than* B*,* $A \cap B$ *is finite.*

   (b) countably infinite
   Let $A = \mathbb{N} \cup (0, 1)$ *and let* $B = \mathbb{N} \cup (3, 4)$.
   $A \cap B = (\mathbb{N} \cup (0, 1)) \cap (\mathbb{N} \cup (3, 4)) = \mathbb{N}$
   A, B *both include uncountable subsets of* $\mathbb{R}$*. Since* A *is constructed from a union of a countably infinite set,* $\mathbb{N}$*, and the same uncountable set as* B*,* $A \cap B$ *is countably infinite.*

   (c) uncountably infinite
   Let $A = (0, 1)$ *and let* $B = (0, 1)$.
   $A \cap B = (0, 1) = A$
   A, B *are uncountable subsets of* $\mathbb{R}$*. Thus,* $A \cap B = A$ *which is uncountably infinite.*