CS230-HW6Sol

1. **[5 Pts] Haniyeh**
   Basis step: n=1
   Show $P(1) : 1 \cdot 1! = (1 + 1)! - 1.$
   Inductive step:
   Assume $P(k)$, $k \in \mathbb{Z}^+$
   $P(k) : 1 \cdot 1! + \cdots k \cdot k! = (k + 1)! - 1.$
   We prove that $P(k + 1)$ *is true, where*
   $P(k + 1) : 1 \cdot 1! + \cdots + k \cdot k! + (k + 1) \cdot (k + 1)! = (k + 2)! - 1.$

$$\begin{aligned}
&1 \cdot 1! + \cdots + k \cdot k! + (k + 1) \cdot (k + 1)! \\
&= (k + 1)! - 1 + (k + 1) \cdot (k + 1)!, \textit{ by inductive hypothesis} \\
&= (k + 1)! \cdot (k + 1 + 1) - 1 \\
&= (k + 1)! \cdot (k + 2) - 1 \\
&= (k + 2)! - 1
\end{aligned}$$

   Therefore, $P(k + 1)$ *is true.*

2. **[5 Pts] Haniyeh**
   Basis step: n=0
   Show $P(0) : 2(-7)^0 = (1 - (-7)^{0+1})/4$
   Inductive Step:
   Assume $P(k)$ *is true,* $k \in \mathbb{N}$
   $P(k) : 2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2(-7)^k = (1 - (-7)^{k+1})/4$
   We prove that $P(k + 1)$ *is also true, where*
   $P(k + 1) : 2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2(-7)^k + 2(-7)^{k+1} = (1 - (-7)^{k+2})/4.$

$$\begin{aligned}
&2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2(-7)^k + 2(-7)^{k+1} \\
&= \frac{1 - (-7)^{k+1}}{4} + 2(-7)^{k+1}, \textit{ by inductive hypothesis} \\
&= \frac{1 - (-7)^{k+1}}{4} + \frac{8(-7)^{k+1}}{4} \\
&= \frac{1 + 7(-7)^{k+1}}{4} \\
&= \frac{1 - (-7)(-7)^{k+1}}{4} \\
&= \frac{1 - (-7)^{k+2}}{4}
\end{aligned}$$

   Therefore, $P(k + 1)$ *is true.*

3. **[5 Pts] Haniyeh** *:*
   Basis step: n=1
   Show $\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1+1}$
   Inductive Step:
   Assume $P(k) : \sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1} \qquad k \geq 1$
   Prove $P(k + 1) : \sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{k+2}$

$$
\begin{aligned}
\sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\
&= \frac{1}{k+1}\left(\frac{k^2+2k+1}{k+2}\right) \\
&= \frac{(k+1)^2}{(k+1)(k+2)} \\
&= \frac{(k+1)}{(k+2)}
\end{aligned}
$$

Therefore $P(k+1)$ is true.

4. **[5 Pts] Nathan** *Let*

$$P(n) : n^3 + 3n^2 + 2n \textit{ is divisible by 3} \qquad n \in \mathbb{Z}^+$$

Basis: $n = 1$

$$1^3 + 3 \cdot 1^2 + 2 \cdot 1 = 6 = 3 \cdot 2$$

Therefore $P(1)$ *is true.*

Inductive Step: *Assume* $k^3 + 3k^2 + 2k$ *is divisible by 3*

Prove $P(k+1) : (k+1)^3 + 3(k+1)^2 + 2(k+1)$ *is divisible by 3*

$$
\begin{aligned}
(k+1)^3 + 3(k+1)^2 + 2(k+1) &= k^3 + 6k^2 + 11k + 6 \\
&= k^3 + (3k^2 + 3k^2) + (2k + 9k) + 6 \\
&= (k^3 + 3k^2 + 2k) + 3k^2 + 9k + 6 \\
&= 3l + 3k^2 + 9k + 6 \qquad [\textit{For some } l \in \mathbb{Z}^+ \textit{ by IH}] \\
&= 3(l + k^2 + 3k + 2)
\end{aligned}
$$

Since $(l + k^2 + 3k + 2) \in \mathbb{Z}^+$*, therefore* $(k+1)^3 + 3(k+1)^2 + 2(k+1)$ *is divisible by 3. Therefore* $P(k+1)$ *is true.*

5. **[5 Pts] Jacob** *We begin by identifying our base case:* $p(1)$ *is trivially true, and should be suspect as a base case, especially given the hint of Theorem 4.12 and the* $n = 2$ *case.*

Basis: $P(2)$

$A \cap (B_1 \cup B_2) = (A \cap B_1) \cup (A \cap B_2)$

Inductive step:

Assume $A \cap \bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} (A \cap B_i)$ *for* $n \geq 2$ ***(Inductive Hypothesis)***

Prove: $A \cap \bigcup_{i=1}^{n+1} B_i = \bigcup_{i=1}^{n+1} (A \cap B_i)$

$$
\begin{aligned}
A \cap \bigcup_{i=1}^{n+1} B_i &= A \cap (B_1 \cup B_2 \cup \ldots \cup B_{n+1}) && \textit{Def. of } \bigcup_{i=1}^{n} B_i \\
&= A \cap ((B_1 \cup B_2 \cup \ldots \cup B_n) \cup B_{n+1}) && \textit{Associative Law for Sets} \\
&= (A \cap (B_1 \cup B_2 \cup \ldots \cup B_n)) \cup (A \cap B_{n+1}) && \textit{Distributive Law for Sets} \\
&= ((A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_n)) \cup (A \cap B_{n+1}) && \textit{by IH} \\
&= (A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_n) \cup (A \cap B_{n+1}) \\
&= \bigcup_{i=1}^{n+1} (A \cap B_i) && \textit{Def. of } \bigcup_{i=1}^{n+1} (A \cap B_i)
\end{aligned}
$$

QED

6. **[9 Pts] Nathan**

   (a) *Basis step:* $P(14)$*: 14 cents can be made of two 3-cent stamps and an 8 cent stamp as* $2(3)+8 = 14$.
   Inductive step: Assume $P(k)$ *for* $k \geq 14$. *So,* k *cents can be made of only 3-cent and 8-cent stamps. To prove* $P(k) \to P(k+1)$, *we will break it into cases.*

   Case 1: k *contains an 8-cent stamp. Then we can remove the 8-cent stamp and replace it with three 3-cent stamps as* $k - 8 + 3(3) = k + 1$.

   Case 2: k *contains no 8-cent stamps, so* k *only contains 3-cent stamps, implying that* k *is a multiple of 3. Since* $k \geq 14$, *it follows that* $k \geq 15$ *because* $5(3) = 15 \geq 14$, *so at least five 3-cent stamps are used. Then five 3-cent stamps can be removed and replaced with two 8-cent stamps as* $k - 5(3) + 2(8) = k + 1$.

   Thus, $P(k) \to P(k+1)$.

   (b) *Basis steps:*
   $P(14)$*: 14 cents can be made of two 3-cent stamps and an 8-cent stamp as* $2(3) + 8 = 14$.
   $P(15)$*: 15 cents can be made from five 3-cent stamps as* $5(3) = 15$.
   $P(16)$*: 16 cents can be made from two 8-cent stamps as* $2(8) = 16$.

   Strong Inductive Step: For $k \geq 16$, *assume that* $P(j)$ *is true for all* j *where* $14 \leq j \leq k$. *We prove* $P(k+1)$. *Since* $k \geq 16$, *we have* $k - 2 \geq 14$, *so* $P(k-2)$ *is true. By adding another 3-cent stamp to* $k - 2$ *cent postage, we have postage for* $k - 2 + 3 = k + 1$. *So,* $P(k+1)$ *is true.*

7. **[6 Pts] Jacob**

   Suppose, for contradiction, there exists a positive integer ℓ *such that* $P(\ell)$ *is false. Since* $P(k+1) \to P(k)$ *for all positive integers* k, *by contrapositive, we have* $\neg P(k) \to \neg P(k+1)$ *for all positive integers* k.
   Since $P(\ell)$ *is false,* $\neg P(\ell)$ *is true, and by the principle of mathematical induction,* $\neg P(n)$ *is true for every integer* $n \geq \ell$. *This contradicts the fact that* $P(n)$ *is true for an infinite number of positive integers* n. *Therefore,* $P(n)$ *is true for all positive integers.*