CS230-HW7Sol

1. **[6 Pts] Nathan**
   Basis Step: *For* $n = 1$ *we see that* $P(1) : f_1 = 1 = f_2$.
   Inductive step: *Assume* $P(k) : f_1 + f_3 + \cdots + f_{2k-1} = f_{2k}$, *we prove* $P(k+1) : f_1 + f_3 + \cdots + f_{2k-1} + f_{2k+1} = f_{2k+2}$

$$\begin{aligned} P(k+1) &= f_1 + f_3 + \cdots + f_{2k-1} + f_{2k+1} \\ &= (f_1 + f_3 + \cdots + f_{2k-1}) + f_{2k+1} \\ &= f_{2k} + f_{2k+1} \textit{(by inductive hypothesis)} \\ &= f_{2k+2} \end{aligned}$$

2. **[8 Pts] Nathan**
   Using the above statement, Let $P(n)$ *be our induction hypothesis, where*
   $P(n)$*: if we take* n *steps in the state machine we will end up in state 0 iff* n *is divisible by 4.*
   There are 2 cases to consider:
   Case 1: $P(k)$ *is true and we are in state 0 after* k *steps, so by the induction hypothesis, 4 divides* k. *We can deduce that after* $k+1$ *steps we will be in state 1 or 2, and 4 does not divide* $k+1$, *which implies* $P(k+1)$ *is true.*
   Case 2: $P(k)$ *is true and we are not in state 0 after* k *steps, so by the induction hypothesis, 4 does not divide* k. *We get stuck here because we cannot deduce after* $k+1$ *steps which state we will be in.*

   We now strengthen the inductive hypothesis. Let Q(n) be the stronger inductive hypothesis (SIH) where
   Q(n) : if we take n *steps in the state machine we will end up in*
   (1) state 0 if $n \equiv 0 \mod 4$, *(2) state 1 or 2 if* $n \equiv 1 \mod 4$, *(3) state 3 or 4 if* $n \equiv 2 \mod 4$, *(4) state 5 if* $n \equiv 3 \mod 4$.
   Clearly, Q(n) implies P(n). Now, we prove the stronger inductive hypothesis:

   Base case: *After 0 steps, the machine is in state 0, and 0 is a multiple of 4.*
   Inductive step: *Assume that the SIH holds after* k *steps. There are 4 cases:*
   Case 1: After k *steps, the machine is in state 0, so* $k \equiv 0 \mod 4$ *by SIH. After one more step, the machine is in state 1 or 2, and* $(k+1) \equiv 1 \mod 4$.
   Case 2: After k *steps, the machine is in state 1 or 2, so* $k \equiv 1 \mod 4$ *by SIH. After one more step, the machine is in state 3 or 4, and* $(k+1) \equiv 2 \mod 4$.
   Case 3: After k *steps, the machine is in state 3 or 4, so* $k \equiv 2 \mod 4$ *by SIH. After one more step, the machine is in state 5, and* $(k+1) \equiv 3 \mod 4$.
   Case 4: After k *steps, the machine is in state 5, so* $k \equiv 3 \mod 4$ *by SIH. After one more step, the machine is in state 0, and* $(k+1) \equiv 0 \mod 4$.

3. **[8 Pts] Jacob**
   $P(n)$*: The total score obtained by dividing a stack of* n *bricks into* n *stacks of one brick each is* $\frac{n(n-1)}{2}$.
   Base case*:* $n(n-1)/2 = 0$, *so P(1) holds.*
   Inductive Step*: Assume P(1),* $\ldots$*,P(k). We need to prove P(k+1).*
   To prove P(k+1), we have to prove that with k+1 bricks, the total score will be $\frac{(k+1)((k+1)-1)}{2}$ *regardless of the order in which the bricks are split.*
   Suppose we have a stack with k+1 bricks. We split the stacks into two non-empty stacks of size a and b , where $1 \leq a, b \leq k$. *By Induction hypothesis , P(a) and P(b) hold. Since P(a) and P(b) holds,*

Total score =score for last split+score when we split a+ score when we split b

$$
\begin{aligned}
&= ab + \frac{a(a-1)}{2} + \frac{b(b-1)}{2} \\
&= \frac{(a+b)^2 - (a+b)}{2} \\
&= \frac{2ab + a^2 - a + b^2 - b}{2} \\
&= \frac{(a+b)^2 - (a+b)}{2} \\
&= \frac{(a+b)^2 - (a+b)}{2} \\
&= \frac{(a+b)((a+b)-1)}{2} \\
&= \frac{(k+1)((k+1)-1)}{2}
\end{aligned}
$$

therefore P(k+1) is true.

4. **[10 Pts] Haniyeh**
The state machine for the robot is an initial state of $(0, 0)$ *and if in state* (x, y)*, then there are transitions to states* $(x+1, y+3), (x+1, y-1), (x-4, y),$ *and* $(x, y+4)$.

The Preserved Invariant: when in state (x, y), $x + y$ *is divisible by 4.*
Base case: $(0, 0)$*, the initial state satisfies the invariant as* $0 + 0 = 0 = 4 * 0$.
Inductive step: *Assume when in state* (x, y) *the invariant holds:* $x+y = 4k, k \in \mathcal{Z}$*. Prove that states* $(x+1, y+3), (x+1, y-1), (x-4, y),$ *and* $(x, y+4)$ *satisfy the invariant. We will prove this by cases depending on which transition is taken.*

Case 1: Take transition $(x+1, y+3)$*. Then* $x+1+y+3 = x+y+4 \overset{IH}{=} 4k+4 = 4(k+1)$*, a multiple of 4. Other cases are proven similarly. Thus the preserved invariant holds for all states the robot can reach and since* $(1, 1)$ *yields* $1 + 1 = 2$*, which is not a multiple of 4,* $(1, 1)$ *cannot be reached.*

5. **[8 Pts] Haniyeh** *Let* $Q(n) : \wedge_{i=1}^{n} P(i) \qquad n \in \mathcal{Z}^+$
Basis: $n = 1$

$$Q(1) \equiv P(1)$$

$P(1)$ *is true, therefore* $Q(1)$ *is true.*
Inductive Step: *Assume* $Q(k) : \bigwedge_{i=1}^{k} P(k)$ *is true*
Prove $Q(k+1)$*:*

$$Q(k+1) \equiv \left(\bigwedge_{i=1}^{k} P(k) \right) \wedge P(k+1)$$

$$Q(k+1) \equiv Q(k) \wedge P(k+1)$$

Since we are assuming $Q(k)$ *is true, therefore,* $P(1) \wedge P(2) \wedge \cdots \wedge P(k)$ *is true. We only need to show* $P(k+1)$ *is true to prove* $Q(k+1)$*. Since we know that* $P(n)$ *can be proven by strong induction, in particular, the induction step can be proven. It follows that we can prove* $P(k+1)$ *given that* $P(1) \wedge P(2) \wedge \cdots \wedge P(k)$ *is true. So, we have* $Q(k) \wedge P(k+1)$*, which is* $Q(k+1)$.
Thus we have a strategy to reduce strong induction to weak induction by strengthening the induction hypothesis to the conjunction of all the previous terms.