CS230-HW11Sol

1. **[6 Pts] Jacob**

   (a) *Here, we see that we have to select a subset (12 cookies) from across multiple categories (5 types of cookies). Things in the same category are indistinguishable, and the order we choose in doesn't matter, so we can convert this to a stars and bars problem. A star is a cookie, and the bars indicate when we're moving to the next type of cookie.*

   Our 12 cookies are now 12 stars. We only need 4 bars to create 5 separations (before the 1st bar, all stars are type 1 cookies, after 1st bar they're type 2,..., after the 4th bar they're type 5.) We now create a 16 character string containing these 12 stars and 4 bars. We need to figure out all possible placements for either the 12 stars, or 4 bars.

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   This is a classic combinatoric problem. From our 16 blanks, how many ways can we choose 12 of them? (Here "choosing" a blank equates to making it a star, i.e. a cookie)
   Note: you could also ask how many ways to choose 4 of them (here, "choosing" a blank equates to making a it bar, i.e. a point in the string after which you're done buying the cookies you were buying before the bar.)

   Final answer should evaluate to 1820

   (b) *Must pick at least 2 of each:*
   *We account for the requirement: 2 * 5 = 10 cookies are chosen for us, and we don't worry about them. 12 - 10 = 2 cookies left to us to choose.*
   This is now stars and bars, like part a.

   5 categories: 5 - 1 = 4 bars
   2 cookies: 2 stars
   2 stars + 4 bars = string of length 6

   $$\textit{Choose spots for the stars:} \binom{6}{2} \textit{ or Choose spots for the bars:} \binom{6}{4}$$

   (c) *At least 4 oatmeal and at most 3 sugar:*
   Here, we borrow a concept from part b. Set aside 4 of our 12 cookies as already chosen (They're 4 oatmeal cookies).

   Let A = #ways to pick at least 4 oatmeal cookies.

   "At most" is a bit more tricky than "At least", so to account for the sugar cookies,

we are going to use subtraction rule to re-frame the problem:
Let $B = \#$ *ways to pick at least 4 oatmeal cookies and at least 3+1 sugar cookies.*

Now, A represents all possibilities that satisfy the oatmeal requirement. B represents all the possibilities that satisfy the oatmeal requirement ***and violate*** *the sugar requirement. Subtraction rule can be used with these values A and B to find out how many possibilities respect both rules.*

The mathematics are left as an exercise which should be very similar to parts a and b. If you'd like to check yourself, your final answer should evaluate to 425

Alternatively: *To handle the "At most" directly, you could set aside your 4 oatmeal cookies, and either 3, 2, 1, or 0 sugar cookies, then set up stars and bars to choose the remaining 5, 6, 7, or 8 cookies from 4 categories, not 5. (Sugar cookie is no longer a category option after you've set aside however many you're choosing for the given scenario, and you'll need to use addition rule to account for all 4 of these scenarios)*

2. **[4 Pts] Jacob** *Suppose the fruits were distinguishible, then the number of ways to distribute 4 distinguishable bananas, 3 distinguishable oranges, 5 distinguishable apples are* $(4+3+5)! = 12!$, *but fruits are identical so we divide by* $4! \cdot 3! \cdot 5!$. *Therefore,*

$$\frac{12!}{4! \cdot 3! \cdot 5!} = 27,720$$

It is a problem on permutation with indistinguishable objects.

3. **[6 Pts] Jacob**

   (a) *Boxes are distinguishable (labelled by address), therefore we keep picking sets of 3 till we have no books left*

$$\binom{24}{3}\binom{21}{3}\binom{18}{3}\binom{15}{3}\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3} = \frac{24!}{(3!)^8}$$

   (b) *In this case boxes are indistinguishable because their addresses are the same, divide by* $8!$ *because of double counting*

$$\frac{24!}{(3!)^8 \cdot 8!}$$

   (c) *In this case we only have four indistinguishable boxes, so we divide out those cases where the boxes are swapped. So the answer is:*

$$\frac{24!}{(3!)^8 \cdot 4!}$$

4. **[6 Pts] Haniyeh**

a *If the books and boxes are identical, then the problem is "how many ways can 5 be written as a sum of positive integers?" The ways are:*
5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1
for a total of 7 ways.

b *If the books are different, then the problem breaks into the following cases, where each number represents a box containing that many books (empty boxes are left out for convenience):*
1,1,1,1,1: There is only one way to put one book in each box.
2,1,1,1: There are $\binom{5}{2}$ *ways to pick a pair of books to share a box.*
Calculate all the other possibilities similarly (not too many, you can do this!) The sum gives us the number ways to put 5 different books into 5 identical boxes.

5. **[6 Pts] Haniyeh**

a *We use the "stars and bars" counting method, with 8 stars and 4 bars, because 4 bars are enough to divide the books into 5 separate areas.*

b *Order all the books first, then count the number of ways they can be put on the shelves in that order. The number of ways to order the books is* $8!$*, while the number of ways a single ordering can be put on the shelves is the answer to (a).*

6. **[4 Pts] Haniyeh** *No, it can't. By the Handshaking Theorem, the sum of the degrees of the vertices is twice the number of edges. But, by dividing number of edges in this potential graph, we get* 9.5*. So, such graph cannot exist!*

7. **[6 Pts] Nathan**

a *No. The graph with 4 vertices* $\{a, b, c, d\}$ *and 3 edges* $\{(a, b), (b, c), (a, c)\}$ *is not connected. Provide counterexample.*

b *No. The graph with 4 vertices* $\{a, b, c, d\}$ *and 3 edges* $\{(a, b), (b, c), (a, c)\}$ *is not acyclic. Provide counterexample*

8. **[12 Pts] Nathan** *First, suppose graph* G *is a tree. Then,* G *is connected and acyclic. We show that adding any edge* $(v, w) \notin E$ *will create a cycle in* G*. Since* G *is connected, there is a path* $v, u_1, u_2, \ldots, u_k, w$ *in* G*. So, adding the edge* (v, w) *will create the cycle* $v, u_1, u_2, \ldots, u_k, w, v$*. Therefore,* G *is acyclic but adding any edge will create a cycle.*

Now, suppose G *is acyclic, but adding any edge to* G *will create a cycle. We prove that* G *is connected. Choose any two vertices* x *and* y *in* G*. Either* $(x, y) \in E(G)$ *or* $(x, y) \notin E(G)$*. (i) If* $(x, y) \in E(G)$*, then clearly there is a path from* x *to* y *in* G*. (ii) If* $(x, y) \notin E(G)$*, then, by the assumption, adding the edge* (x, y) *to* G *creates a cycle* C *which contains* (x, y)*. Let* $C = \langle x, y, v_1, \ldots, v_k, x\rangle$*. The path* $\langle y, v_1, \ldots, v_k, x\rangle$ *is a path between* y *and* x *in* G*. So, in either case there is a path between* x *and* y *in* G*. Since this is true for any pair of vertices,* G *is connected. Since* G *is acyclic as well, therefore,* G *is a tree.*