CS230-HW3Sol

1. **Haniyeh 5 pts**

   Since the statement is a biconditional, we have to prove two parts: if p is odd, then p^3 is odd, and if p^3 is odd, then p is odd.

   i Prove if p is odd, then p^3 is odd. We can use direct proof. Let p be an odd number. Then $p = 2k + 1$, for some integer k. Now we form p^3 and see if we can apply algebraic manipulation to find a form of an odd number.

   ii Prove if p^3 is odd, then p is odd. We can prove this using contrapositive. The statement we will prove is if p is even, then p^3 is even.

   Since both implications have been proven true, then p is odd if and only if p^3 is odd.

2. **Haniyeh 6 pts**

   Let x and y be non-zero rational numbers and z be irrational. We need to prove that $x + yz$ is irrational. It is not possible to use a direct proof because there is no way to represent an irrational number. We will prove this using a proof by contradiction.

   Let x and y be non-zero rational numbers and z be irrational. Assume for contradiction that $x + yz$ is a rational number.

   We can write rational numbers as a fraction of integers where the denominator is non-negative. So, let $x = \frac{a}{b}$ and $y = \frac{c}{d}$, where a, b, c and d are all some integer but not zero. Similarly, $x + yz = \frac{p}{q}$, where p and q are integers and $q \neq 0$. Now form $x + yz$ by substituting x and y, and set it equal to $\frac{p}{q}$. If we solve for z, we will see it as a form of a rational number and that is a contradiction. So, our assumption is incorrect and $x + yz$ is not a rational number, thus, it is an irrational number.

3. **Nathan 6 pts**

   We can prove this using the contrapositive of the statement. The contrapositive is: if "$m < 6 \wedge n < 8 \rightarrow mn \leq 35$".

   Let $m < 6$ and $n < 8$. We need to show that $mn \leq 35$. Since m and n are both positive integers, then $m \leq 5$ and $n \leq 7$. This means the largest value m can be is 5 and the largest value n can be is 7. Then the largest value mn can be is 35, so $mn \leq 35$. By showing the contapositive of the statement being true, we proved the original statement is true as well.

4. **Nathan 6 pts**

   We prove this using the contrapositive. Negate two sides of the statement, we will have: the maximum number of freshmen there can be is 4, the maximum number of sophomores is 7, the maximum number of juniors is 9, and the maximum number of seniors is 6. Then the maximum number of total students in the organization is $4+7+9+6 = 26$. So it is not possible for the total number of students in the organization to be 32, which would cause the contrapositive to be false. Thus if there are less than 5 freshmen, less than 8 sophomores, less than 10 juniors, and less than 7 seniors, then the total number of students in the organization is not equal to 32 is true.

5. **Jacob**

   Let $p \geq 3$ or $p \leq -7$. We need to prove that $(p+2)^2 \geq 25$. Since it isn't known which of the two possible conditions for p is true, both need to be examined individually. This gives two cases.

   A common mistake is that students will try and make cases for $p = 3$, $p = 3.1$ (or some other number > 3), $p = -7$, and $p = -7.1$ (or some other number < -7). Proving like this only covers those 4 cases of equality. The solution needs to cover the 2 larger cases, $p \geq 3$ and $p \leq -7$. The proof employed needs to maintain the broadness of an inequality (it's similar to the idea of proving a universal or existential claim). The attempt to do the four cases shown hints at an idea called inductive reasoning, but is not enough by itself to be an inductive proof. We will cover this later in the course.

   One easy way to build out these cases that maintains the use of the inequalities is to just work with the root of both sides of the problem statement. Example cases below.

   Case 1: Have $p \geq 3$. Then $p+2 \geq 5 \Rightarrow (p+2)^2 \geq 25$.

   Case 2: Have $p \leq -7$. Then $p+2 \leq -5 \Rightarrow |p+2| \geq 5 \Rightarrow (p+2)^2 \geq 25$. Note! Doing this case requires a solid understanding of how signs/arithmetic work when dealing with inequalities.

   Both cases reach the conclusion that $(p+2)^2 \geq 25$. Therefore if $p \geq 3$ or $p \leq -7$, then $(p+2)^2 \geq 25$.

6. **Jacob**

   Prove that $\sqrt{5}$ is irrational.

   The only good way to formally describe irrational numbers that is covered in this class is "non-rational real number". This makes it necessary to use contradiction, which allows us to work from the negation of what we are trying to prove. I.e. we can claim "$\sqrt{5}$ is rational" and use deductive reasoning to come to a contradiction based on the implications of that claim.

   By claiming $\sqrt{5}$ is rational, we can then say $\sqrt{5} = \frac{p}{q}$ for some integers p and q such that $q \neq 0$ and q and p share no common factors (i.e. $\frac{p}{q}$ is in simplest form).

   From this point, algebra can be used to illustrate some of the contradictions the claim poses. The first step for almost any proof from here is squaring both sides of the above equation.

   At this stage, any people will try to show that the above equation implies that p and q have a common factor of 5, which contradicts them being in. This method has some pitfalls as far as our class is concerned though. Namely, people make unsubstantiated claims like "5 is prime" or since p^2 is divisible by 5, p is divisible by 5". Others make take these claims at face value. We do not. Making such claims puts you on the hook for proving them!

Another common method that technically works but requires information that we consider dubious in the context of this course deals with making assertions about the number of prime factors the two sides of the equation have. This similarly requires the claim "5 is prime" to be taken at face value.

One method that relies only on topics covered in class is evaluating the even/oddness of p and q, and coming to the conclusion they can't be both. Since a number can be even or odd, but not both, there are 4 broad cases:

P & Q both even: trivial contradiction of base claim, as even numbers share a common factor of 2 based on our definition of even numbers (that they equal $2k$ for some integer k), and p and q are defined to not have common factors.

P even Q odd or Q even P odd: these are technically two cases, but very similar in nature. Without giving specifics, they can be evaluated to show that if one value is even, the other must also be, meaning these cases contradict.

P & Q odd: This one is a bit tricky, and will require actually re-writing p^2 as $(2k+1)^2$ and q^2 as $(2j+1)^2$ in the earlier equation. Using algebra/arithmetic to do some further expansion will eventually reach a contradiction in which an expression that fits the definition of even is equal to an expression that fits the definition of odd. Note! This will also rely on the ability to identify and explain a rule of parity (that is, $j(j+1)$ will be even for any integer j).

Since all four possibilities for how to describe p and q cause our rational $\sqrt{5}$ equation to result in a contradiction, the rational $\sqrt{5}$ claim must be false.

7. **Jacob**

   This is an open ended question with many answers. The easiest one is to pick values for x and y such that x^y is irrational, thus performing a constructive proof, since it involves selecting specific values that satisfy the claim. Some people confuse a constructive proof with proving the method of construction. Construction is the method by which satisfactory values are generated. A constructive proof is far simpler, and just requires that you accurately identify satisfactory inputs.

   $x = 5; y = \frac{1}{2}$ Results in $x^y = \sqrt{5}$. Conveniently, $\sqrt{5}$ was proven irrational in an earlier question.