─────────────────────── MODULE *tictactoexwin* ───────────────────────

EXTENDS *Naturals*

VARIABLES
    $board$, $board[1 .. 3][1 .. 3]$ A $3x3$ tic-tac-toe board
    $nextTurn$ who goes next

$vars \triangleq \langle board, nextTurn \rangle$

$Pieces \triangleq \{ \text{"X"}, \text{"O"}, \text{"\_"} \}$ "_" represents a blank square

$Init \triangleq$
      $\wedge nextTurn = \text{"X"}$ X always goes first
      Every space in the board states blank
      $\wedge board = [i \in 1 .. 3 \mapsto [j \in 1 .. 3 \mapsto \text{"\_"}]]$

$Move(player, coordinate) \triangleq$
      $\wedge board[coordinate[1]][coordinate[2]] = \text{"\_"}$
      $\wedge board' = [board \text{ EXCEPT}$
                   $![coordinate[1]][coordinate[2]] = player]$

$MoveToEmpty(player) \triangleq$
    $\wedge \exists i \in 1 .. 3 : \exists j \in 1 .. 3 :$ There exists a position on the board
      $\wedge board[i][j] = \text{"\_"}$ Where the board is currently empty
      $\wedge Move(player, \langle i, j \rangle)$

$WinningPositions \triangleq \{$
    Horizonal wins
    $\langle \langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 1, 3 \rangle \rangle,$
    $\langle \langle 2, 1 \rangle, \langle 2, 2 \rangle, \langle 2, 3 \rangle \rangle,$
    $\langle \langle 3, 1 \rangle, \langle 3, 2 \rangle, \langle 3, 3 \rangle \rangle,$
    Vertical wins
    $\langle \langle 1, 1 \rangle, \langle 2, 1 \rangle, \langle 3, 1 \rangle \rangle,$
    $\langle \langle 1, 2 \rangle, \langle 2, 2 \rangle, \langle 3, 2 \rangle \rangle,$
    $\langle \langle 1, 3 \rangle, \langle 2, 3 \rangle, \langle 3, 3 \rangle \rangle,$
    Diagonal wins
    $\langle \langle 1, 1 \rangle, \langle 2, 2 \rangle, \langle 3, 3 \rangle \rangle,$
    $\langle \langle 3, 1 \rangle, \langle 2, 2 \rangle, \langle 1, 3 \rangle \rangle$
$\}$

$Won(player) \triangleq$
      A player has won if there exists a winning position
      $\exists winningPosition \in WinningPositions :$
         Where all the needed spaces
         $\forall i \in 1 .. 3 :$

$$are\ occupied\ by\ one\ player$$
$$board[winningPosition[i][1]][winningPosition[i][2]] = player$$

$MoveO \triangleq$
    $\land\ nextTurn =$ "O" Only enabled on O's turn
    $\land\ \neg\, Won($ "X" $)$ And X has not won
    $\land\ MoveToEmpty($ "O" $)$ O still tries every empty space
    $\land\ nextTurn' =$ "X" The future state of next turn is X

$CenterEmpty \triangleq$
    $board[2][2] =$ "_"

$Corners \triangleq \{$
    $\langle 1, 1 \rangle,$
    $\langle 3, 1 \rangle,$
    $\langle 1, 3 \rangle,$
    $\langle 3, 3 \rangle$
$\}$

$PartialWins \triangleq \{$
    $\langle 1, 2, 3 \rangle,$
    $\langle 2, 3, 1 \rangle,$
    $\langle 3, 1, 2 \rangle$
$\}$

$BoardEmpty \triangleq$
    There does not exist
    $\forall\, i \in 1\,..\,3,\, j \in 1\,..\,3 :$
        an empty space
      LET $space \triangleq board[i][j]$ IN
      $space =$ "_"

$StartInCorner \triangleq$
    $\exists\, corner \in Corners :$
      $Move($ "X" $,\ corner)$

$BoardIs(coordinate,\ player) \triangleq$
    $board[coordinate[1]][coordinate[2]] = player$

$CanWin \triangleq \exists\, winningPostion \in WinningPositions,\ partialWin \in PartialWins :$
                  $\land\ BoardIs(winningPostion[partialWin[1]],$ "X" $)$
                  $\land\ BoardIs(winningPostion[partialWin[2]],$ "X" $)$
                  $\land\ BoardIs(winningPostion[partialWin[3]],$ "_" $)$

$Win \triangleq \exists\, winningPostion \in WinningPositions,\ partialWin \in PartialWins :$
                  $\land\ BoardIs(winningPostion[partialWin[1]],$ "X" $)$

$\qquad \wedge \; BoardIs(winningPostion[partialWin[2]], \; \text{“X”})$
$\qquad \wedge \; BoardIs(winningPostion[partialWin[3]], \; \text{“\_”})$
$\qquad \wedge \; Move(\text{“X”}, \; winningPostion[partialWin[3]])$ Move into the winning position

$CanBlockWin \; \triangleq \; \exists \, winningPostion \in WinningPositions, \; partialWin \in PartialWins :$
$\qquad\qquad\qquad \wedge \; BoardIs(winningPostion[partialWin[1]], \; \text{“O”})$
$\qquad\qquad\qquad \wedge \; BoardIs(winningPostion[partialWin[2]], \; \text{“O”})$
$\qquad\qquad\qquad \wedge \; BoardIs(winningPostion[partialWin[3]], \; \text{“\_”})$

$BlockWin \; \triangleq \; \exists \, winningPostion \in WinningPositions, \; partialWin \in PartialWins :$
$\qquad\qquad\qquad \wedge \; BoardIs(winningPostion[partialWin[1]], \; \text{“O”})$
$\qquad\qquad\qquad \wedge \; BoardIs(winningPostion[partialWin[2]], \; \text{“O”})$
$\qquad\qquad\qquad \wedge \; BoardIs(winningPostion[partialWin[3]], \; \text{“\_”})$
$\qquad\qquad\qquad \wedge \; Move(\text{“X”}, \; winningPostion[partialWin[3]])$ Move into the winning position

$CanTakeCenter \; \triangleq \; board[2][2] = \text{“\_”}$ precondition

$TakeCenter \; \triangleq$
$\quad \wedge \; Move(\text{“X”}, \; \langle 2, \, 2 \rangle)$

$CanSetupWin \; \triangleq$
$\quad \exists \, winningPostion \in WinningPositions, \; partialWin \in PartialWins :$
$\qquad \wedge \; BoardIs(winningPostion[partialWin[1]], \; \text{“X”})$
$\qquad \wedge \; BoardIs(winningPostion[partialWin[2]], \; \text{“\_”})$
$\qquad \wedge \; BoardIs(winningPostion[partialWin[3]], \; \text{“\_”})$

$SetupWin \; \triangleq$
$\quad \exists \, winningPostion \in WinningPositions, \; partialWin \in PartialWins :$
$\qquad \wedge \; BoardIs(winningPostion[partialWin[1]], \; \text{“X”})$
$\qquad \wedge \; BoardIs(winningPostion[partialWin[2]], \; \text{“\_”})$
$\qquad \wedge \; BoardIs(winningPostion[partialWin[3]], \; \text{“\_”})$
$\qquad \wedge \; \exists \, i \in 2 \, .. \, 3 :$
$\qquad\quad Move(\text{“X”}, \; winningPostion[partialWin[i]])$ Move into one of the blank spots

$MoveX \; \triangleq$
$\quad \wedge \; nextTurn = \text{“X”}$ Only enabled on X's turn
$\quad \wedge \; \neg \, Won(\text{“O”})$ And X has not won
$\quad$ This specifies the spots X will move on X's turn
$\quad \wedge \; \vee \; \wedge \; BoardEmpty$
$\qquad\qquad \wedge \; StartInCorner$
$\qquad \vee \; \wedge \; \neg BoardEmpty$ If its not the start
$\qquad\qquad \wedge \; \vee \; \wedge \; CanWin$
$\qquad\qquad\qquad\quad \wedge \; Win$
$\qquad\qquad\quad \vee \; \wedge \; \neg CanWin$
$\qquad\qquad\qquad\quad \wedge \; \vee \; \wedge \; CanBlockWin$
$\qquad\qquad\qquad\qquad\qquad \wedge \; BlockWin$
$\qquad\qquad\qquad\qquad \vee \; \wedge \; \neg CanBlockWin$

3

$$\wedge\ \vee\ \wedge\ CanTakeCenter$$
$$\wedge\ TakeCenter$$
$$\vee\ \wedge\ \neg\, CanTakeCenter$$
$$\wedge\ \vee\ \wedge\ CanSetupWin$$
$$\wedge\ SetupWin$$
$$\vee\ \wedge\ \neg\, CanSetupWin$$
$$\wedge\ MoveToEmpty(\text{``X''})\ \boxed{\text{No more strategies. Pick spot}}$$
$$\wedge\ nextTurn' = \text{``O''}\ \boxed{\text{The future state of next turn is } O}$$

Every state, X will move if X's turn, O will move on O's turn
$$Next\ \triangleq\ MoveX \vee MoveO$$

$$XMustEventuallyWin\ \triangleq\ \Diamond\, Won(\text{``X''})$$

$$Spec\ \triangleq\ Init \wedge \Box[Next]_{vars} \wedge \mathrm{WF}_{vars}(Next)$$

Invariants: The things we are checking for.

$$XHasNotWon\ \triangleq\ \neg\, Won(\text{``X''})$$
$$OHasNotWon\ \triangleq\ \neg\, Won(\text{``O''})$$

$$BoardFilled\ \triangleq$$
> There does not exist
> $\neg\exists\, i \in 1\,..\,3,\, j \in 1\,..\,3:$
>> an empty space
>> $\text{LET}\ space\ \triangleq\ board[i][j]\,\text{IN}$
>> $space = \text{``\_''}$

It's not a stalemate if one player has won or the board is not filled
$$NotStalemate\ \triangleq$$
$$\vee\ Won(\text{``X''})$$
$$\vee\ Won(\text{``O''})$$
$$\vee\ \neg\, BoardFilled$$