

EXTENDS *Naturals*

VARIABLES

board, *board*[1 .. 3][1 .. 3] A 3x3 tic-tac-toe board

nextTurn who goes next

vars \triangleq $\langle \textit{board}, \textit{nextTurn} \rangle$

Pieces \triangleq { "X", "O", "-" } "-" represents a blank square

Init \triangleq

$\wedge \textit{nextTurn} = \text{"X"}$ X always goes first

Every space in the board states blank

$\wedge \textit{board} = [i \in 1 \dots 3 \mapsto [j \in 1 \dots 3 \mapsto "-"]]$

Move(*player*, *coordinate*) \triangleq

$\wedge \textit{board}[\textit{coordinate}[1]][\textit{coordinate}[2]] = \text{"-"}$

$\wedge \textit{board}' = [\textit{board} \text{ EXCEPT } \text{!}[\textit{coordinate}[1]][\textit{coordinate}[2]] = \textit{player}]$

MoveToEmpty(*player*) \triangleq

$\wedge \exists i \in 1 \dots 3 : \exists j \in 1 \dots 3 : \text{There exists a position on the board}$

$\wedge \textit{board}[i][j] = \text{"-"}$ Where the board is currently empty

$\wedge \textit{Move}(\textit{player}, \langle i, j \rangle)$

WinningPositions \triangleq {

Horizontal wins

$\langle \langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 1, 3 \rangle \rangle,$

$\langle \langle 2, 1 \rangle, \langle 2, 2 \rangle, \langle 2, 3 \rangle \rangle,$

$\langle \langle 3, 1 \rangle, \langle 3, 2 \rangle, \langle 3, 3 \rangle \rangle,$

Vertical wins

$\langle \langle 1, 1 \rangle, \langle 2, 1 \rangle, \langle 3, 1 \rangle \rangle,$

$\langle \langle 1, 2 \rangle, \langle 2, 2 \rangle, \langle 3, 2 \rangle \rangle,$

$\langle \langle 1, 3 \rangle, \langle 2, 3 \rangle, \langle 3, 3 \rangle \rangle,$

Diagonal wins

$\langle \langle 1, 1 \rangle, \langle 2, 2 \rangle, \langle 3, 3 \rangle \rangle,$

$\langle \langle 3, 1 \rangle, \langle 2, 2 \rangle, \langle 1, 3 \rangle \rangle$

}

Won(*player*) \triangleq

A player has won if there exists a winning position

$\exists \textit{winningPosition} \in \textit{WinningPositions} :$

Where all the needed spaces

$\forall i \in 1 \dots 3 :$

are occupied by one player
 $board[winningPosition[i][1]][winningPosition[i][2]] = player$

$MoveO \triangleq$
 $\wedge nextTurn = "O"$ Only enabled on O 's turn
 $\wedge \neg Won("X")$ And X has not won
 $\wedge MoveToEmpty("O")$ O still tries every empty space
 $\wedge nextTurn' = "X"$ The future state of next turn is X

$CenterEmpty \triangleq$
 $board[2][2] = "-"$

$Corners \triangleq \{$
 $\langle 1, 1 \rangle,$
 $\langle 3, 1 \rangle,$
 $\langle 1, 3 \rangle,$
 $\langle 3, 3 \rangle$
 $\}$

$PartialWins \triangleq \{$
 $\langle 1, 2, 3 \rangle,$
 $\langle 2, 3, 1 \rangle,$
 $\langle 3, 1, 2 \rangle$
 $\}$

$BoardEmpty \triangleq$
 There does not exist
 $\forall i \in 1 \dots 3, j \in 1 \dots 3 :$
 an empty space
 $LET\ space \triangleq board[i][j] IN$
 $space = "-"$

$StartInCorner \triangleq$
 $\exists corner \in Corners :$
 $Move("X", corner)$

$BoardIs(coordinate, player) \triangleq$
 $board[coordinate[1]][coordinate[2]] = player$

$CanWin \triangleq \exists winningPostion \in WinningPositions, partialWin \in PartialWins :$
 $\wedge BoardIs(winningPostion[partialWin[1]], "X")$
 $\wedge BoardIs(winningPostion[partialWin[2]], "X")$
 $\wedge BoardIs(winningPostion[partialWin[3]], "-")$

$Win \triangleq \exists winningPostion \in WinningPositions, partialWin \in PartialWins :$
 $\wedge BoardIs(winningPostion[partialWin[1]], "X")$

$$\begin{aligned}
& \wedge \text{BoardIs}(\text{winningPostion}[\text{partialWin}[2]], \text{"X"}) \\
& \wedge \text{BoardIs}(\text{winningPostion}[\text{partialWin}[3]], \text{"-"}) \\
& \wedge \text{Move}(\text{"X"}, \text{winningPostion}[\text{partialWin}[3]]) \text{ Move into the winning position} \\
\text{CanBlockWin} & \triangleq \exists \text{winningPostion} \in \text{WinningPositions}, \text{partialWin} \in \text{PartialWins} : \\
& \wedge \text{BoardIs}(\text{winningPostion}[\text{partialWin}[1]], \text{"O"}) \\
& \wedge \text{BoardIs}(\text{winningPostion}[\text{partialWin}[2]], \text{"O"}) \\
& \wedge \text{BoardIs}(\text{winningPostion}[\text{partialWin}[3]], \text{"-"}) \\
\text{BlockWin} & \triangleq \exists \text{winningPostion} \in \text{WinningPositions}, \text{partialWin} \in \text{PartialWins} : \\
& \wedge \text{BoardIs}(\text{winningPostion}[\text{partialWin}[1]], \text{"O"}) \\
& \wedge \text{BoardIs}(\text{winningPostion}[\text{partialWin}[2]], \text{"O"}) \\
& \wedge \text{BoardIs}(\text{winningPostion}[\text{partialWin}[3]], \text{"-"}) \\
& \wedge \text{Move}(\text{"X"}, \text{winningPostion}[\text{partialWin}[3]]) \text{ Move into the winning position} \\
\text{CanTakeCenter} & \triangleq \text{board}[2][2] = \text{"-"} \text{ precondition} \\
\text{TakeCenter} & \triangleq \\
& \wedge \text{Move}(\text{"X"}, \langle 2, 2 \rangle) \\
\text{CanSetupWin} & \triangleq \\
& \exists \text{winningPostion} \in \text{WinningPositions}, \text{partialWin} \in \text{PartialWins} : \\
& \wedge \text{BoardIs}(\text{winningPostion}[\text{partialWin}[1]], \text{"X"}) \\
& \wedge \text{BoardIs}(\text{winningPostion}[\text{partialWin}[2]], \text{"-"}) \\
& \wedge \text{BoardIs}(\text{winningPostion}[\text{partialWin}[3]], \text{"-"}) \\
\text{SetupWin} & \triangleq \\
& \exists \text{winningPostion} \in \text{WinningPositions}, \text{partialWin} \in \text{PartialWins} : \\
& \wedge \text{BoardIs}(\text{winningPostion}[\text{partialWin}[1]], \text{"X"}) \\
& \wedge \text{BoardIs}(\text{winningPostion}[\text{partialWin}[2]], \text{"-"}) \\
& \wedge \text{BoardIs}(\text{winningPostion}[\text{partialWin}[3]], \text{"-"}) \\
& \wedge \exists i \in 2 \dots 3 : \\
& \quad \text{Move}(\text{"X"}, \text{winningPostion}[\text{partialWin}[i]]) \text{ Move into one of the blank spots} \\
\text{MoveX} & \triangleq \\
& \wedge \text{nextTurn} = \text{"X"} \text{ Only enabled on X's turn} \\
& \wedge \neg \text{Won}(\text{"O"}) \text{ And X has not won} \\
& \text{This specifies the spots X will move on X's turn} \\
& \wedge \vee \wedge \text{BoardEmpty} \\
& \quad \wedge \text{StartInCorner} \\
& \vee \wedge \neg \text{BoardEmpty} \text{ If its not the start} \\
& \quad \wedge \vee \wedge \text{CanWin} \\
& \quad \quad \wedge \text{Win} \\
& \vee \wedge \neg \text{CanWin} \\
& \quad \wedge \vee \wedge \text{CanBlockWin} \\
& \quad \quad \wedge \text{BlockWin} \\
& \vee \wedge \neg \text{CanBlockWin}
\end{aligned}$$

$$\begin{aligned}
& \wedge \vee \wedge \text{CanTakeCenter} \\
& \quad \wedge \text{TakeCenter} \\
& \vee \wedge \neg \text{CanTakeCenter} \\
& \quad \wedge \vee \wedge \text{CanSetupWin} \\
& \quad \quad \wedge \text{SetupWin} \\
& \quad \vee \wedge \neg \text{CanSetupWin} \\
& \quad \quad \wedge \text{MoveToEmpty}(\text{"X"}) \quad \text{No more strategies. Pick spot} \\
\wedge \text{nextTurn}' = \text{"O"} \quad & \text{The future state of next turn is O}
\end{aligned}$$

Every state, X will move if X 's turn, O will move on O 's turn

$$\text{Next} \triangleq \text{MoveX} \vee \text{MoveO}$$

$$\text{Spec} \triangleq \text{Init} \wedge \Box[\text{Next}]_{\text{vars}}$$

Invariants: The things we are checking for.

$$\begin{aligned}
XHasNotWon & \triangleq \neg \text{Won}(\text{"X"}) \\
OHasNotWon & \triangleq \neg \text{Won}(\text{"O"})
\end{aligned}$$

$$\begin{aligned}
\text{BoardFilled} & \triangleq \\
& \text{There does not exist} \\
& \neg \exists i \in 1 \dots 3, j \in 1 \dots 3 : \\
& \quad \text{an empty space} \\
& \text{LET } \text{space} \triangleq \text{board}[i][j] \text{ IN} \\
& \text{space} = \text{"_"}
\end{aligned}$$

It's not a stalemate if one player has won or the board is not filled

$$\begin{aligned}
\text{NotStalemate} & \triangleq \\
& \vee \text{Won}(\text{"X"}) \\
& \vee \text{Won}(\text{"O"}) \\
& \vee \neg \text{BoardFilled}
\end{aligned}$$