

Principia Mathematica's Propositional Logic in *Coq*

Landon D. C. Elkind*

May 1, 2021

Abstract

This file contains the *Coq* code for the *Principia* Rewrite project's encoding of the propositional logic given in *1 – *5. The Github repository with this *Coq* file is here: <https://github.com/LogicalAtomist/principia>. To receive updates about the project, visit the *Principia Rewrite* project page: <https://www.principiarewrite.com/>. You can also follow the *Principia* Rewrite project on Twitter: <https://twitter.com/thePMrewrite>.

```
1  Require Import Unicode.Utf8.
2  Require Import Classical_Prop.
3  Require Import ClassicalFacts.
4  Require Import PropExtensionality.
5
6  Module No1.
7
8  Import Unicode.Utf8.
9  Import ClassicalFacts.
10 Import Classical_Prop.
11 Import PropExtensionality.
12
13   (*We first give the axioms of Principia in *1.*)
14
15 Theorem Impl1_01 :  $\forall$  P Q : Prop,
16   (P  $\rightarrow$  Q) = ( $\neg$ P  $\vee$  Q).
17 Proof. intros P Q.
18   apply propositional_extensionality.
19   split.
20   apply imply_to_or.
21   apply or_to_imply.
22 Qed.
```

*Department of Philosophy, University of Alberta, `elkind at ualberta dot ca`. This research was conducted as an Izaak Walton Killam Postdoctoral Fellow.

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23   (*This is a notational definition in Principia:
24     It is used to switch between " $\vee$ " and " $\rightarrow$ ".*)
25
26 Theorem MP1_1 :  $\forall$  P Q : Prop,
27   (P  $\rightarrow$  Q)  $\rightarrow$  P  $\rightarrow$  Q. (*Modus ponens*)
28 Proof. intros P Q.
29   intros iff_refl.
30   apply iff_refl.
31 Qed.
32   (*1.11 omitted: it is MP for propositions
33     containing variables. Likewise, omitted
34     the well-formedness rules 1.7, 1.71, 1.72*)
35
36 Theorem Taut1_2 :  $\forall$  P : Prop,
37   P  $\vee$  P  $\rightarrow$  P. (*Tautology*)
38 Proof. intros P.
39   apply imply_and_or.
40   apply iff_refl.
41 Qed.
42
43 Theorem Add1_3 :  $\forall$  P Q : Prop,
44   Q  $\rightarrow$  P  $\vee$  Q. (*Addition*)
45 Proof. intros P Q.
46   apply or_intror.
47 Qed.
48
49 Theorem Perm1_4 :  $\forall$  P Q : Prop,
50   P  $\vee$  Q  $\rightarrow$  Q  $\vee$  P. (*Permutation*)
51 Proof. intros P Q.
52   apply or_comm.
53 Qed.
54
55 Theorem Assoc1_5 :  $\forall$  P Q R : Prop,
56   P  $\vee$  (Q  $\vee$  R)  $\rightarrow$  Q  $\vee$  (P  $\vee$  R). (*Association*)
57 Proof. intros P Q R.
58   specialize or_assoc with P Q R.
59   intros or_assoc1.
60   replace (P $\vee$ Q $\vee$ R) with ((P $\vee$ Q) $\vee$ R).
61   specialize or_comm with P Q.
62   intros or_comm1.
63   replace (P $\vee$ Q) with (Q $\vee$ P).
64   specialize or_assoc with Q P R.

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65   intros or_assoc2.
66   replace ((Q∨P)∨R) with (Q∨P∨R).
67   apply iff_refl.
68   apply propositional_extensionality.
69   apply iff_sym.
70   apply or_assoc2.
71   apply propositional_extensionality.
72   apply or_comm.
73   apply propositional_extensionality.
74   apply or_assoc.
75   Qed.
76
77   Theorem Sum1_6 : ∀ P Q R : Prop,
78     (Q → R) → (P ∨ Q → P ∨ R). (*Summation*)
79   Proof. intros P Q R.
80     specialize imply_and_or2 with Q R P.
81     intros imply_and_or2a.
82     replace (P∨Q) with (Q∨P).
83     replace (P∨R) with (R∨P).
84     apply imply_and_or2a.
85     apply propositional_extensionality.
86     apply or_comm.
87     apply propositional_extensionality.
88     apply or_comm.
89     Qed.
90
91   Ltac MP H1 H2 :=
92     match goal with
93       | [ H1 : ?P → ?Q, H2 : ?P |- _ ] =>
94         specialize (H1 H2)
95     end.
96   (*We give this Ltac "MP" to make proofs
97     more human-readable and to more
98     closely mirror Principia's style.*)
99
100  End No1.
101
102  Module No2.
103
104  Import No1.
105
106  (*We proceed to the deductions of of Principia.*)

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107
108 Theorem Abs2_01 :  $\forall P : \text{Prop},$ 
109    $(P \rightarrow \neg P) \rightarrow \neg P.$ 
110 Proof. intros P.
111   specialize Taut1_2 with  $(\neg P).$ 
112   intros Taut1_2.
113   replace  $(\neg P \vee \neg P)$  with  $(P \rightarrow \neg P)$  in Taut1_2
114     by now rewrite Impl1_01.
115   exact Taut1_2.
116 Qed.
117
118 Theorem Simp2_02 :  $\forall P Q : \text{Prop},$ 
119    $Q \rightarrow (P \rightarrow Q).$ 
120 Proof. intros P Q.
121   specialize Add1_3 with  $(\neg P) Q.$ 
122   intros Add1_3.
123   replace  $(\neg P \vee Q)$  with  $(P \rightarrow Q)$  in Add1_3
124     by now rewrite Impl1_01.
125   exact Add1_3.
126 Qed.
127
128 Theorem Transp2_03 :  $\forall P Q : \text{Prop},$ 
129    $(P \rightarrow \neg Q) \rightarrow (Q \rightarrow \neg P).$ 
130 Proof. intros P Q.
131   specialize Perm1_4 with  $(\neg P) (\neg Q).$ 
132   intros Perm1_4.
133   replace  $(\neg P \vee \neg Q)$  with  $(P \rightarrow \neg Q)$  in Perm1_4
134     by now rewrite Impl1_01.
135   replace  $(\neg Q \vee \neg P)$  with  $(Q \rightarrow \neg P)$  in Perm1_4
136     by now rewrite Impl1_01.
137   exact Perm1_4.
138 Qed.
139
140 Theorem Comm2_04 :  $\forall P Q R : \text{Prop},$ 
141    $(P \rightarrow (Q \rightarrow R)) \rightarrow (Q \rightarrow (P \rightarrow R)).$ 
142 Proof. intros P Q R.
143   specialize Assoc1_5 with  $(\neg P) (\neg Q) R.$ 
144   intros Assoc1_5.
145   replace  $(\neg Q \vee R)$  with  $(Q \rightarrow R)$  in Assoc1_5
146     by now rewrite Impl1_01.
147   replace  $(\neg P \vee (Q \rightarrow R))$  with  $(P \rightarrow (Q \rightarrow R))$  in Assoc1_5
148     by now rewrite Impl1_01.

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149   replace ( $\neg P \vee R$ ) with ( $P \rightarrow R$ ) in Assoc1_5
150   by now rewrite Impl1_01.
151   replace ( $\neg Q \vee (P \rightarrow R)$ ) with ( $Q \rightarrow (P \rightarrow R)$ ) in Assoc1_5
152   by now rewrite Impl1_01.
153   exact Assoc1_5.
154 Qed.
155
156 Theorem Syll2_05 :  $\forall P Q R : \text{Prop}$ ,
157   ( $Q \rightarrow R$ )  $\rightarrow$  (( $P \rightarrow Q$ )  $\rightarrow$  ( $P \rightarrow R$ )).
158 Proof. intros P Q R.
159   specialize Sum1_6 with ( $\neg P$ ) Q R.
160   intros Sum1_6.
161   replace ( $\neg P \vee Q$ ) with ( $P \rightarrow Q$ ) in Sum1_6
162   by now rewrite Impl1_01.
163   replace ( $\neg P \vee R$ ) with ( $P \rightarrow R$ ) in Sum1_6
164   by now rewrite Impl1_01.
165   exact Sum1_6.
166 Qed.
167
168 Theorem Syll2_06 :  $\forall P Q R : \text{Prop}$ ,
169   ( $P \rightarrow Q$ )  $\rightarrow$  (( $Q \rightarrow R$ )  $\rightarrow$  ( $P \rightarrow R$ )).
170 Proof. intros P Q R.
171   specialize Comm2_04 with ( $Q \rightarrow R$ ) ( $P \rightarrow Q$ ) ( $P \rightarrow R$ ).
172   intros Comm2_04.
173   specialize Syll2_05 with P Q R.
174   intros Syll2_05.
175   MP Comm2_04 Syll2_05.
176   exact Comm2_04.
177 Qed.
178
179 Theorem n2_07 :  $\forall P : \text{Prop}$ ,
180    $P \rightarrow (P \vee P)$ .
181 Proof. intros P.
182   specialize Add1_3 with P P.
183   intros Add1_3.
184   exact Add1_3.
185 Qed.
186
187 Theorem Id2_08 :  $\forall P : \text{Prop}$ ,
188    $P \rightarrow P$ .
189 Proof. intros P.
190   specialize Syll2_05 with P ( $P \vee P$ ) P.

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191   intros Syll2_05.
192   specialize Taut1_2 with P.
193   intros Taut1_2.
194   MP Syll2_05 Taut1_2.
195   specialize n2_07 with P.
196   intros n2_07.
197   MP Syll2_05 n2_07.
198   exact Syll2_05.
199 Qed.
200
201 Theorem n2_1 :  $\forall P : \text{Prop},$ 
202    $(\neg P) \vee P.$ 
203 Proof. intros P.
204   specialize Id2_08 with P.
205   intros Id2_08.
206   replace  $(P \rightarrow P)$  with  $(\neg P \vee P)$  in Id2_08
207     by now rewrite Impl1_01.
208   exact Id2_08.
209 Qed.
210
211 Theorem n2_11 :  $\forall P : \text{Prop},$ 
212    $P \vee \neg P.$ 
213 Proof. intros P.
214   specialize Perm1_4 with  $(\neg P) P.$ 
215   intros Perm1_4.
216   specialize n2_1 with P.
217   intros n2_1.
218   MP Perm1_4 n2_1.
219   exact Perm1_4.
220 Qed.
221
222 Theorem n2_12 :  $\forall P : \text{Prop},$ 
223    $P \rightarrow \neg\neg P.$ 
224 Proof. intros P.
225   specialize n2_11 with  $(\neg P).$ 
226   intros n2_11.
227   replace  $(\neg P \vee \neg\neg P)$  with  $(P \rightarrow \neg\neg P)$  in n2_11
228     by now rewrite Impl1_01.
229   exact n2_11.
230 Qed.
231
232 Theorem n2_13 :  $\forall P : \text{Prop},$ 

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233   P  $\vee$   $\neg\neg\neg$ P.
234 Proof. intros P.
235   specialize Sum1_6 with P ( $\neg$ P) ( $\neg\neg\neg$ P).
236   intros Sum1_6.
237   specialize n2_12 with ( $\neg$ P).
238   intros n2_12.
239   MP Sum1_6 n2_12.
240   specialize n2_11 with P.
241   intros n2_11.
242   MP Sum1_6 n2_11.
243   exact Sum1_6.
244 Qed.
245
246 Theorem n2_14 :  $\forall$  P : Prop,
247    $\neg\neg$ P  $\rightarrow$  P.
248 Proof. intros P.
249   specialize Perm1_4 with P ( $\neg\neg\neg$ P).
250   intros Perm1_4.
251   specialize n2_13 with P.
252   intros n2_13.
253   MP Perm1_4 n2_13.
254   replace ( $\neg\neg\neg$ P  $\vee$  P) with ( $\neg\neg$ P  $\rightarrow$  P) in Perm1_4
255     by now rewrite Impl1_01.
256   exact Perm1_4.
257 Qed.
258
259 Theorem Transp2_15 :  $\forall$  P Q : Prop,
260   ( $\neg$ P  $\rightarrow$  Q)  $\rightarrow$  ( $\neg$ Q  $\rightarrow$  P).
261 Proof. intros P Q.
262   specialize Syll2_05 with ( $\neg$ P) Q ( $\neg\neg$ Q).
263   intros Syll2_05a.
264   specialize n2_12 with Q.
265   intros n2_12.
266   MP Syll2_05a n2_12.
267   specialize Transp2_03 with ( $\neg$ P) ( $\neg$ Q).
268   intros Transp2_03.
269   specialize Syll2_05 with ( $\neg$ Q) ( $\neg\neg$ P) P.
270   intros Syll2_05b.
271   specialize n2_14 with P.
272   intros n2_14.
273   MP Syll2_05b n2_14.
274   specialize Syll2_05 with ( $\neg$ P  $\rightarrow$  Q) ( $\neg$ P  $\rightarrow$   $\neg\neg$ Q) ( $\neg$ Q  $\rightarrow$   $\neg\neg$ P).

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275   intros Syll2_05c.
276   MP Syll2_05c Transp2_03.
277   MP Syll2_05c Syll2_05a.
278   specialize Syll2_05 with ( $\neg P \rightarrow Q$ ) ( $\neg Q \rightarrow \neg\neg P$ ) ( $\neg Q \rightarrow P$ ).
279   intros Syll2_05d.
280   MP Syll2_05d Syll2_05b.
281   MP Syll2_05d Syll2_05c.
282   exact Syll2_05d.
283   Qed.
284
285   Ltac Syll H1 H2 S :=
286     let S := fresh S in match goal with
287       | [ H1 : ?P  $\rightarrow$  ?Q, H2 : ?Q  $\rightarrow$  ?R |- _ ] =>
288         assert (S : P  $\rightarrow$  R) by (intros p; exact (H2 (H1 p)))
289     end.
290
291   Theorem Transp2_16 :  $\forall P Q : \text{Prop}$ ,
292     (P  $\rightarrow$  Q)  $\rightarrow$  ( $\neg Q \rightarrow \neg P$ ).
293   Proof. intros P Q.
294     specialize n2_12 with Q.
295     intros n2_12a.
296     specialize Syll2_05 with P Q ( $\neg\neg Q$ ).
297     intros Syll2_05a.
298     specialize Transp2_03 with P ( $\neg Q$ ).
299     intros Transp2_03a.
300     MP n2_12a Syll2_05a.
301     Syll Syll2_05a Transp2_03a S.
302     exact S.
303   Qed.
304
305   Theorem Transp2_17 :  $\forall P Q : \text{Prop}$ ,
306     ( $\neg Q \rightarrow \neg P$ )  $\rightarrow$  (P  $\rightarrow$  Q).
307   Proof. intros P Q.
308     specialize Transp2_03 with ( $\neg Q$ ) P.
309     intros Transp2_03a.
310     specialize n2_14 with Q.
311     intros n2_14a.
312     specialize Syll2_05 with P ( $\neg\neg Q$ ) Q.
313     intros Syll2_05a.
314     MP n2_14a Syll2_05a.
315     Syll Transp2_03a Syll2_05a S.
316     exact S.

```



```

317 Qed.
318
319 Theorem n2_18 :  $\forall$  P : Prop,
320   ( $\neg$ P  $\rightarrow$  P)  $\rightarrow$  P.
321 Proof. intros P.
322   specialize n2_12 with P.
323   intro n2_12a.
324   specialize Syll2_05 with ( $\neg$ P) P ( $\neg\neg$ P).
325   intro Syll2_05a.
326   MP Syll2_05a n2_12.
327   specialize Abs2_01 with ( $\neg$ P).
328   intros Abs2_01a.
329   Syll Syll2_05a Abs2_01a Sa.
330   specialize n2_14 with P.
331   intros n2_14a.
332   Syll H n2_14a Sb.
333   exact Sb.
334 Qed.
335
336 Theorem n2_2 :  $\forall$  P Q : Prop,
337   P  $\rightarrow$  (P  $\vee$  Q).
338 Proof. intros P Q.
339   specialize Add1_3 with Q P.
340   intros Add1_3a.
341   specialize Perm1_4 with Q P.
342   intros Perm1_4a.
343   Syll Add1_3a Perm1_4a S.
344   exact S.
345 Qed.
346
347 Theorem n2_21 :  $\forall$  P Q : Prop,
348    $\neg$ P  $\rightarrow$  (P  $\rightarrow$  Q).
349 Proof. intros P Q.
350   specialize n2_2 with ( $\neg$ P) Q.
351   intros n2_2a.
352   replace ( $\neg$ P $\vee$ Q) with (P $\rightarrow$ Q) in n2_2a
353     by now rewrite Impl1_01.
354   exact n2_2a.
355 Qed.
356
357 Theorem n2_24 :  $\forall$  P Q : Prop,
358   P  $\rightarrow$  ( $\neg$ P  $\rightarrow$  Q).

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359 Proof. intros P Q.
360   specialize n2_21 with P Q.
361   intros n2_21a.
362   specialize Comm2_04 with (¬P) P Q.
363   intros Comm2_04a.
364   MP Comm2_04a n2_21a.
365   exact Comm2_04a.
366 Qed.
367
368 Theorem n2_25 : ∀ P Q : Prop,
369   P ∨ ((P ∨ Q) → Q).
370 Proof. intros P Q.
371   specialize n2_1 with (P ∨ Q).
372   intros n2_1a.
373   specialize Assoc1_5 with (¬(P∨Q)) P Q.
374   intros Assoc1_5a.
375   MP Assoc1_5a n2_1a.
376   replace (¬(P∨Q)∨Q) with (P∨Q→Q) in Assoc1_5a
377     by now rewrite Impl1_01.
378   exact Assoc1_5a.
379 Qed.
380
381 Theorem n2_26 : ∀ P Q : Prop,
382   ¬P ∨ ((P → Q) → Q).
383 Proof. intros P Q.
384   specialize n2_25 with (¬P) Q.
385   intros n2_25a.
386   replace (¬P∨Q) with (P→Q) in n2_25a
387     by now rewrite Impl1_01.
388   exact n2_25a.
389 Qed.
390
391 Theorem n2_27 : ∀ P Q : Prop,
392   P → ((P → Q) → Q).
393 Proof. intros P Q.
394   specialize n2_26 with P Q.
395   intros n2_26a.
396   replace (¬P∨((P→Q)→Q)) with (P→(P→Q)→Q)
397     in n2_26a by now rewrite Impl1_01.
398   exact n2_26a.
399 Qed.
400

```

```

401 Theorem n2_3 :  $\forall$  P Q R : Prop,
402   (P  $\vee$  (Q  $\vee$  R))  $\rightarrow$  (P  $\vee$  (R  $\vee$  Q)).
403 Proof. intros P Q R.
404   specialize Perm1_4 with Q R.
405   intros Perm1_4a.
406   specialize Sum1_6 with P (Q $\vee$ R) (R $\vee$ Q).
407   intros Sum1_6a.
408   MP Sum1_6a Perm1_4a.
409   exact Sum1_6a.
410 Qed.
411
412 Theorem n2_31 :  $\forall$  P Q R : Prop,
413   (P  $\vee$  (Q  $\vee$  R))  $\rightarrow$  ((P  $\vee$  Q)  $\vee$  R).
414 Proof. intros P Q R.
415   specialize n2_3 with P Q R.
416   intros n2_3a.
417   specialize Assoc1_5 with P R Q.
418   intros Assoc1_5a.
419   specialize Perm1_4 with R (P $\vee$ Q).
420   intros Perm1_4a.
421   Syll Assoc1_5a Perm1_4a Sa.
422   Syll n2_3a Sa Sb.
423   exact Sb.
424 Qed.
425
426 Theorem n2_32 :  $\forall$  P Q R : Prop,
427   ((P  $\vee$  Q)  $\vee$  R)  $\rightarrow$  (P  $\vee$  (Q  $\vee$  R)).
428 Proof. intros P Q R.
429   specialize Perm1_4 with (P $\vee$ Q) R.
430   intros Perm1_4a.
431   specialize Assoc1_5 with R P Q.
432   intros Assoc1_5a.
433   specialize n2_3 with P R Q.
434   intros n2_3a.
435   specialize Syll2_06 with ((P $\vee$ Q) $\vee$ R) (R $\vee$ P $\vee$ Q) (P $\vee$ R $\vee$ Q).
436   intros Syll2_06a.
437   MP Syll2_06a Perm1_4a.
438   MP Syll2_06a Assoc1_5a.
439   specialize Syll2_06 with ((P $\vee$ Q) $\vee$ R) (P $\vee$ R $\vee$ Q) (P $\vee$ Q $\vee$ R).
440   intros Syll2_06b.
441   MP Syll2_06b Syll2_06a.
442   MP Syll2_06b n2_3a.

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443   exact Syll2_06b.
444 Qed.
445
446 Theorem Abb2_33 :  $\forall$  P Q R : Prop,
447   (P  $\vee$  Q  $\vee$  R) = ((P  $\vee$  Q)  $\vee$  R).
448 Proof. intros P Q R.
449   apply propositional_extensionality.
450   split.
451   specialize n2_31 with P Q R.
452   intros n2_31.
453   exact n2_31.
454   specialize n2_32 with P Q R.
455   intros n2_32.
456   exact n2_32.
457 Qed.
458   (*The default in Coq is right association.*)
459
460 Theorem n2_36 :  $\forall$  P Q R : Prop,
461   (Q  $\rightarrow$  R)  $\rightarrow$  ((P  $\vee$  Q)  $\rightarrow$  (R  $\vee$  P)).
462 Proof. intros P Q R.
463   specialize Perm1_4 with P R.
464   intros Perm1_4a.
465   specialize Syll2_05 with (P $\vee$ Q) (P $\vee$ R) (R $\vee$ P).
466   intros Syll2_05a.
467   MP Syll2_05a Perm1_4a.
468   specialize Sum1_6 with P Q R.
469   intros Sum1_6a.
470   Syll Sum1_6a Syll2_05a S.
471   exact S.
472 Qed.
473
474 Theorem n2_37 :  $\forall$  P Q R : Prop,
475   (Q  $\rightarrow$  R)  $\rightarrow$  ((Q  $\vee$  P)  $\rightarrow$  (P  $\vee$  R)).
476 Proof. intros P Q R.
477   specialize Perm1_4 with Q P.
478   intros Perm1_4a.
479   specialize Syll2_06 with (Q $\vee$ P) (P $\vee$ Q) (P $\vee$ R).
480   intros Syll2_06a.
481   MP Syll2_06a Perm1_4a.
482   specialize Sum1_6 with P Q R.
483   intros Sum1_6a.
484   Syll Sum1_6a Syll2_06a S.

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485     exact S.
486 Qed.
487
488 Theorem n2_38 :  $\forall P Q R : \text{Prop},$ 
489    $(Q \rightarrow R) \rightarrow ((Q \vee P) \rightarrow (R \vee P)).$ 
490 Proof. intros P Q R.
491   specialize Perm1_4 with P R.
492   intros Perm1_4a.
493   specialize Syll2_05 with  $(Q \vee P)$   $(P \vee R)$   $(R \vee P)$ .
494   intros Syll2_05a.
495   MP Syll2_05a Perm1_4a.
496   specialize Perm1_4 with Q P.
497   intros Perm1_4b.
498   specialize Syll2_06 with  $(Q \vee P)$   $(P \vee Q)$   $(P \vee R)$ .
499   intros Syll2_06a.
500   MP Syll2_06a Perm1_4b.
501   Syll Syll2_06a Syll2_05a H.
502   specialize Sum1_6 with P Q R.
503   intros Sum1_6a.
504   Syll Sum1_6a H S.
505   exact S.
506 Qed.
507
508 Theorem n2_4 :  $\forall P Q : \text{Prop},$ 
509    $(P \vee (P \vee Q)) \rightarrow (P \vee Q).$ 
510 Proof. intros P Q.
511   specialize n2_31 with P P Q.
512   intros n2_31a.
513   specialize Taut1_2 with P.
514   intros Taut1_2a.
515   specialize n2_38 with Q  $(P \vee P)$  P.
516   intros n2_38a.
517   MP n2_38a Taut1_2a.
518   Syll n2_31a n2_38a S.
519   exact S.
520 Qed.
521
522 Theorem n2_41 :  $\forall P Q : \text{Prop},$ 
523    $(Q \vee (P \vee Q)) \rightarrow (P \vee Q).$ 
524 Proof. intros P Q.
525   specialize Assoc1_5 with Q P Q.
526   intros Assoc1_5a.

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527   specialize Taut1_2 with Q.
528   intros Taut1_2a.
529   specialize Sum1_6 with P (Q $\vee$ Q) Q.
530   intros Sum1_6a.
531   MP Sum1_6a Taut1_2a.
532   Syll Assoc1_5a Sum1_6a S.
533   exact S.
534 Qed.
535
536 Theorem n2_42 :  $\forall$  P Q : Prop,
537   ( $\neg$ P  $\vee$  (P  $\rightarrow$  Q))  $\rightarrow$  (P  $\rightarrow$  Q).
538 Proof. intros P Q.
539   specialize n2_4 with ( $\neg$ P) Q.
540   intros n2_4a.
541   replace ( $\neg$ P $\vee$ Q) with (P $\rightarrow$ Q) in n2_4a
542     by now rewrite Impl1_01.
543   exact n2_4a.
544 Qed.
545
546 Theorem n2_43 :  $\forall$  P Q : Prop,
547   (P  $\rightarrow$  (P  $\rightarrow$  Q))  $\rightarrow$  (P  $\rightarrow$  Q).
548 Proof. intros P Q.
549   specialize n2_42 with P Q.
550   intros n2_42a.
551   replace ( $\neg$ P  $\vee$  (P $\rightarrow$ Q)) with (P $\rightarrow$ (P $\rightarrow$ Q))
552     in n2_42a by now rewrite Impl1_01.
553   exact n2_42a.
554 Qed.
555
556 Theorem n2_45 :  $\forall$  P Q : Prop,
557    $\neg$ (P  $\vee$  Q)  $\rightarrow$   $\neg$ P.
558 Proof. intros P Q.
559   specialize n2_2 with P Q.
560   intros n2_2a.
561   specialize Transp2_16 with P (P $\vee$ Q).
562   intros Transp2_16a.
563   MP n2_2 Transp2_16a.
564   exact Transp2_16a.
565 Qed.
566
567 Theorem n2_46 :  $\forall$  P Q : Prop,
568    $\neg$ (P  $\vee$  Q)  $\rightarrow$   $\neg$ Q.

```

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569 Proof. intros P Q.
570   specialize Add1_3 with P Q.
571   intros Add1_3a.
572   specialize Transp2_16 with Q (P ∨ Q).
573   intros Transp2_16a.
574   MP Add1_3a Transp2_16a.
575   exact Transp2_16a.
576 Qed.
577
578 Theorem n2_47 : ∀ P Q : Prop,
579   ¬(P ∨ Q) → (¬P ∨ Q).
580 Proof. intros P Q.
581   specialize n2_45 with P Q.
582   intros n2_45a.
583   specialize n2_2 with (¬P) Q.
584   intros n2_2a.
585   Syll n2_45a n2_2a S.
586   exact S.
587 Qed.
588
589 Theorem n2_48 : ∀ P Q : Prop,
590   ¬(P ∨ Q) → (P ∨ ¬Q).
591 Proof. intros P Q.
592   specialize n2_46 with P Q.
593   intros n2_46a.
594   specialize Add1_3 with P (¬Q).
595   intros Add1_3a.
596   Syll n2_46a Add1_3a S.
597   exact S.
598 Qed.
599
600 Theorem n2_49 : ∀ P Q : Prop,
601   ¬(P ∨ Q) → (¬P ∨ ¬Q).
602 Proof. intros P Q.
603   specialize n2_45 with P Q.
604   intros n2_45a.
605   specialize n2_2 with (¬P) (¬Q).
606   intros n2_2a.
607   Syll n2_45a n2_2a S.
608   exact S.
609 Qed.
610

```

```

611 Theorem n2_5 :  $\forall P Q : \text{Prop},$ 
612    $\neg(P \rightarrow Q) \rightarrow (\neg P \rightarrow Q).$ 
613 Proof. intros P Q.
614   specialize n2_47 with  $(\neg P) Q.$ 
615   intros n2_47a.
616   replace  $(\neg P \vee Q)$  with  $(P \rightarrow Q)$  in n2_47a
617     by now rewrite Impl1_01.
618   replace  $(\neg \neg P \vee Q)$  with  $(\neg P \rightarrow Q)$  in n2_47a
619     by now rewrite Impl1_01.
620   exact n2_47a.
621 Qed.
622
623 Theorem n2_51 :  $\forall P Q : \text{Prop},$ 
624    $\neg(P \rightarrow Q) \rightarrow (P \rightarrow \neg Q).$ 
625 Proof. intros P Q.
626   specialize n2_48 with  $(\neg P) Q.$ 
627   intros n2_48a.
628   replace  $(\neg P \vee Q)$  with  $(P \rightarrow Q)$  in n2_48a
629     by now rewrite Impl1_01.
630   replace  $(\neg P \vee \neg Q)$  with  $(P \rightarrow \neg Q)$  in n2_48a
631     by now rewrite Impl1_01.
632   exact n2_48a.
633 Qed.
634
635 Theorem n2_52 :  $\forall P Q : \text{Prop},$ 
636    $\neg(P \rightarrow Q) \rightarrow (\neg P \rightarrow \neg Q).$ 
637 Proof. intros P Q.
638   specialize n2_49 with  $(\neg P) Q.$ 
639   intros n2_49a.
640   replace  $(\neg P \vee Q)$  with  $(P \rightarrow Q)$  in n2_49a
641     by now rewrite Impl1_01.
642   replace  $(\neg \neg P \vee \neg Q)$  with  $(\neg P \rightarrow \neg Q)$  in n2_49a
643     by now rewrite Impl1_01.
644   exact n2_49a.
645 Qed.
646
647 Theorem n2_521 :  $\forall P Q : \text{Prop},$ 
648    $\neg(P \rightarrow Q) \rightarrow (Q \rightarrow P).$ 
649 Proof. intros P Q.
650   specialize n2_52 with P Q.
651   intros n2_52a.
652   specialize Transp2_17 with Q P.

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653   intros Transp2_17a.
654   Syll n2_52a Transp2_17a S.
655   exact S.
656 Qed.
657
658 Theorem n2_53 :  $\forall P Q : \text{Prop},$ 
659    $(P \vee Q) \rightarrow (\neg P \rightarrow Q).$ 
660 Proof. intros P Q.
661   specialize n2_12 with P.
662   intros n2_12a.
663   specialize n2_38 with Q P ( $\neg\neg P$ ).
664   intros n2_38a.
665   MP n2_38a n2_12a.
666   replace ( $\neg\neg P \vee Q$ ) with ( $\neg P \rightarrow Q$ ) in n2_38a
667     by now rewrite Impl1_01.
668   exact n2_38a.
669 Qed.
670
671 Theorem n2_54 :  $\forall P Q : \text{Prop},$ 
672    $(\neg P \rightarrow Q) \rightarrow (P \vee Q).$ 
673 Proof. intros P Q.
674   specialize n2_14 with P.
675   intros n2_14a.
676   specialize n2_38 with Q ( $\neg\neg P$ ) P.
677   intros n2_38a.
678   MP n2_38a n2_12a.
679   replace ( $\neg\neg P \vee Q$ ) with ( $\neg P \rightarrow Q$ ) in n2_38a
680     by now rewrite Impl1_01.
681   exact n2_38a.
682 Qed.
683
684 Theorem n2_55 :  $\forall P Q : \text{Prop},$ 
685    $\neg P \rightarrow ((P \vee Q) \rightarrow Q).$ 
686 Proof. intros P Q.
687   specialize n2_53 with P Q.
688   intros n2_53a.
689   specialize Comm2_04 with  $(P \vee Q) (\neg P) Q.$ 
690   intros Comm2_04a.
691   MP n2_53a Comm2_04a.
692   exact Comm2_04a.
693 Qed.
694

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```

695 Theorem n2_56 :  $\forall$  P Q : Prop,
696    $\neg Q \rightarrow ((P \vee Q) \rightarrow P)$ .
697 Proof. intros P Q.
698   specialize n2_55 with Q P.
699   intros n2_55a.
700   specialize Perm1_4 with P Q.
701   intros Perm1_4a.
702   specialize Syll2_06 with  $(P \vee Q) (Q \vee P) P$ .
703   intros Syll2_06a.
704   MP Syll2_06a Perm1_4a.
705   Syll n2_55a Syll2_06a Sa.
706   exact Sa.
707 Qed.
708
709 Theorem n2_6 :  $\forall$  P Q : Prop,
710    $(\neg P \rightarrow Q) \rightarrow ((P \rightarrow Q) \rightarrow Q)$ .
711 Proof. intros P Q.
712   specialize n2_38 with Q  $(\neg P) Q$ .
713   intros n2_38a.
714   specialize Taut1_2 with Q.
715   intros Taut1_2a.
716   specialize Syll2_05 with  $(\neg P \vee Q) (Q \vee Q) Q$ .
717   intros Syll2_05a.
718   MP Syll2_05a Taut1_2a.
719   Syll n2_38a Syll2_05a S.
720   replace  $(\neg P \vee Q)$  with  $(P \rightarrow Q)$  in S
721     by now rewrite Impl1_01.
722   exact S.
723 Qed.
724
725 Theorem n2_61 :  $\forall$  P Q : Prop,
726    $(P \rightarrow Q) \rightarrow ((\neg P \rightarrow Q) \rightarrow Q)$ .
727 Proof. intros P Q.
728   specialize n2_6 with P Q.
729   intros n2_6a.
730   specialize Comm2_04 with  $(\neg P \rightarrow Q) (P \rightarrow Q) Q$ .
731   intros Comm2_04a.
732   MP Comm2_04a n2_6a.
733   exact Comm2_04a.
734 Qed.
735
736 Theorem n2_62 :  $\forall$  P Q : Prop,

```

```

737   (P ∨ Q) → ((P → Q) → Q).
738 Proof. intros P Q.
739   specialize n2_53 with P Q.
740   intros n2_53a.
741   specialize n2_6 with P Q.
742   intros n2_6a.
743   Syll n2_53a n2_6a S.
744   exact S.
745 Qed.
746
747 Theorem n2_621 : ∀ P Q : Prop,
748   (P → Q) → ((P ∨ Q) → Q).
749 Proof. intros P Q.
750   specialize n2_62 with P Q.
751   intros n2_62a.
752   specialize Comm2_04 with (P ∨ Q) (P→Q) Q.
753   intros Comm2_04a.
754   MP Comm2_04a n2_62a.
755   exact Comm2_04a.
756 Qed.
757
758 Theorem n2_63 : ∀ P Q : Prop,
759   (P ∨ Q) → ((¬P ∨ Q) → Q).
760 Proof. intros P Q.
761   specialize n2_62 with P Q.
762   intros n2_62a.
763   replace (P→Q) with (¬P∨Q) in n2_62a
764     by now rewrite Impl1_01.
765   exact n2_62a.
766 Qed.
767
768 Theorem n2_64 : ∀ P Q : Prop,
769   (P ∨ Q) → ((P ∨ ¬Q) → P).
770 Proof. intros P Q.
771   specialize n2_63 with Q P.
772   intros n2_63a.
773   specialize Perm1_4 with P Q.
774   intros Perm1_4a.
775   Syll n2_63a Perm1_4a Ha.
776   specialize Syll2_06 with (P∨¬Q) (¬Q∨P) P.
777   intros Syll2_06a.
778   specialize Perm1_4 with P (¬Q).

```

```

779   intros Perm1_4b.
780   MP Syll2_06a Perm1_4b.
781   Syll Syll2_06a Ha S.
782   exact S.
783   Qed.
784
785   Theorem n2_65 :  $\forall P Q : \text{Prop},$ 
786      $(P \rightarrow Q) \rightarrow ((P \rightarrow \neg Q) \rightarrow \neg P).$ 
787   Proof. intros P Q.
788     specialize n2_64 with  $(\neg P) Q.$ 
789     intros n2_64a.
790     replace  $(\neg P \vee Q)$  with  $(P \rightarrow Q)$  in n2_64a
791       by now rewrite Impl1_01.
792     replace  $(\neg P \vee \neg Q)$  with  $(P \rightarrow \neg Q)$  in n2_64a
793       by now rewrite Impl1_01.
794     exact n2_64a.
795     Qed.
796
797   Theorem n2_67 :  $\forall P Q : \text{Prop},$ 
798      $((P \vee Q) \rightarrow Q) \rightarrow (P \rightarrow Q).$ 
799   Proof. intros P Q.
800     specialize n2_54 with P Q.
801     intros n2_54a.
802     specialize Syll2_06 with  $(\neg P \rightarrow Q) (P \vee Q) Q.$ 
803     intros Syll2_06a.
804     MP Syll2_06a n2_54a.
805     specialize n2_24 with P Q.
806     intros n2_24.
807     specialize Syll2_06 with P  $(\neg P \rightarrow Q) Q.$ 
808     intros Syll2_06b.
809     MP Syll2_06b n2_24a.
810     Syll Syll2_06b Syll2_06a S.
811     exact S.
812     Qed.
813
814   Theorem n2_68 :  $\forall P Q : \text{Prop},$ 
815      $((P \rightarrow Q) \rightarrow Q) \rightarrow (P \vee Q).$ 
816   Proof. intros P Q.
817     specialize n2_67 with  $(\neg P) Q.$ 
818     intros n2_67a.
819     replace  $(\neg P \vee Q)$  with  $(P \rightarrow Q)$  in n2_67a
820       by now rewrite Impl1_01.

```

```

821   specialize n2_54 with P Q.
822   intros n2_54a.
823   Syll n2_67a n2_54a S.
824   exact S.
825   Qed.
826
827   Theorem n2_69 :  $\forall P Q : \text{Prop},$ 
828      $((P \rightarrow Q) \rightarrow Q) \rightarrow ((Q \rightarrow P) \rightarrow P).$ 
829   Proof. intros P Q.
830     specialize n2_68 with P Q.
831     intros n2_68a.
832     specialize Perm1_4 with P Q.
833     intros Perm1_4a.
834     Syll n2_68a Perm1_4a Sa.
835     specialize n2_62 with Q P.
836     intros n2_62a.
837     Syll Sa n2_62a Sb.
838     exact Sb.
839   Qed.
840
841   Theorem n2_73 :  $\forall P Q R : \text{Prop},$ 
842      $(P \rightarrow Q) \rightarrow ((P \vee Q) \vee R) \rightarrow (Q \vee R).$ 
843   Proof. intros P Q R.
844     specialize n2_621 with P Q.
845     intros n2_621a.
846     specialize n2_38 with R (P $\vee$ Q) Q.
847     intros n2_38a.
848     Syll n2_621a n2_38a S.
849     exact S.
850   Qed.
851
852   Theorem n2_74 :  $\forall P Q R : \text{Prop},$ 
853      $(Q \rightarrow P) \rightarrow ((P \vee Q) \vee R) \rightarrow (P \vee R).$ 
854   Proof. intros P Q R.
855     specialize n2_73 with Q P R.
856     intros n2_73a.
857     specialize Assoc1_5 with P Q R.
858     intros Assoc1_5a.
859     specialize n2_31 with Q P R.
860     intros n2_31a. (*not cited*)
861     Syll Assoc1_5a n2_31a Sa.
862     specialize n2_32 with P Q R.

```

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863   intros n2_32a. (*not cited*)
864   Syll n2_32a Sa Sb.
865   specialize Syll2_06 with ((P∨Q)∨R) ((Q∨P)∨R) (P∨R).
866   intros Syll2_06a.
867   MP Syll2_06a Sb.
868   Syll n2_73a Syll2_05a H.
869   exact H.
870 Qed.
871
872 Theorem n2_75 : ∀ P Q R : Prop,
873   (P ∨ Q) → ((P ∨ (Q → R)) → (P ∨ R)).
874 Proof. intros P Q R.
875   specialize n2_74 with P (¬Q) R.
876   intros n2_74a.
877   specialize n2_53 with Q P.
878   intros n2_53a.
879   Syll n2_53a n2_74a Sa.
880   specialize n2_31 with P (¬Q) R.
881   intros n2_31a.
882   specialize Syll2_06 with (P∨(¬Q)∨R)((P∨(¬Q))∨R) (P∨R).
883   intros Syll2_06a.
884   MP Syll2_06a n2_31a.
885   Syll Sa Syll2_06a Sb.
886   specialize Perm1_4 with P Q.
887   intros Perm1_4a. (*not cited*)
888   Syll Perm1_4a Sb Sc.
889   replace (¬Q∨R) with (Q→R) in Sc
890     by now rewrite Impl1_01.
891   exact Sc.
892 Qed.
893
894 Theorem n2_76 : ∀ P Q R : Prop,
895   (P ∨ (Q → R)) → ((P ∨ Q) → (P ∨ R)).
896 Proof. intros P Q R.
897   specialize n2_75 with P Q R.
898   intros n2_75a.
899   specialize Comm2_04 with (P∨Q) (P∨(Q→R)) (P∨R).
900   intros Comm2_04a.
901   MP Comm2_04a n2_75a.
902   exact Comm2_04a.
903 Qed.
904

```

```

905 Theorem n2_77 :  $\forall$  P Q R : Prop,
906   (P  $\rightarrow$  (Q  $\rightarrow$  R))  $\rightarrow$  ((P  $\rightarrow$  Q)  $\rightarrow$  (P  $\rightarrow$  R)).
907 Proof. intros P Q R.
908   specialize n2_76 with ( $\neg$ P) Q R.
909   intros n2_76a.
910   replace ( $\neg$ P $\vee$ (Q $\rightarrow$ R)) with (P $\rightarrow$ Q $\rightarrow$ R) in n2_76a
911     by now rewrite Impl1_01.
912   replace ( $\neg$ P $\vee$ Q) with (P $\rightarrow$ Q) in n2_76a
913     by now rewrite Impl1_01.
914   replace ( $\neg$ P $\vee$ R) with (P $\rightarrow$ R) in n2_76a
915     by now rewrite Impl1_01.
916   exact n2_76a.
917 Qed.
918
919 Theorem n2_8 :  $\forall$  Q R S : Prop,
920   (Q  $\vee$  R)  $\rightarrow$  (( $\neg$ R  $\vee$  S)  $\rightarrow$  (Q  $\vee$  S)).
921 Proof. intros Q R S.
922   specialize n2_53 with R Q.
923   intros n2_53a.
924   specialize Perm1_4 with Q R.
925   intros Perm1_4a.
926   Syll Perm1_4a n2_53a Ha.
927   specialize n2_38 with S ( $\neg$ R) Q.
928   intros n2_38a.
929   Syll H n2_38a Hb.
930   exact Hb.
931 Qed.
932
933 Theorem n2_81 :  $\forall$  P Q R S : Prop,
934   (Q  $\rightarrow$  (R  $\rightarrow$  S))  $\rightarrow$  ((P  $\vee$  Q)  $\rightarrow$  ((P  $\vee$  R)  $\rightarrow$  (P  $\vee$  S))).
935 Proof. intros P Q R S.
936   specialize Sum1_6 with P Q (R $\rightarrow$ S).
937   intros Sum1_6a.
938   specialize n2_76 with P R S.
939   intros n2_76a.
940   specialize Syll2_05 with (P $\vee$ Q) (P $\vee$ (R $\rightarrow$ S)) ((P $\vee$ R) $\rightarrow$ (P $\vee$ S)).
941   intros Syll2_05a.
942   MP Syll2_05a n2_76a.
943   Syll Sum1_6a Syll2_05a H.
944   exact H.
945 Qed.
946

```

```

947 Theorem n2_82 :  $\forall P Q R S : \text{Prop},$ 
948    $(P \vee Q \vee R) \rightarrow ((P \vee \neg R \vee S) \rightarrow (P \vee Q \vee S)).$ 
949 Proof. intros P Q R S.
950   specialize n2_8 with Q R S.
951   intros n2_8a.
952   specialize n2_81 with P (Q $\vee$ R) ( $\neg$ R $\vee$ S) (Q $\vee$ S).
953   intros n2_81a.
954   MP n2_81a n2_8a.
955   exact n2_81a.
956 Qed.
957
958 Theorem n2_83 :  $\forall P Q R S : \text{Prop},$ 
959    $(P \rightarrow (Q \rightarrow R)) \rightarrow ((P \rightarrow (R \rightarrow S)) \rightarrow (P \rightarrow (Q \rightarrow S))).$ 
960 Proof. intros P Q R S.
961   specialize n2_82 with ( $\neg$ P) ( $\neg$ Q) R S.
962   intros n2_82a.
963   replace ( $\neg$ Q $\vee$ R) with (Q $\rightarrow$ R) in n2_82a
964     by now rewrite Impl1_01.
965   replace ( $\neg$ P $\vee$ (Q $\rightarrow$ R)) with (P $\rightarrow$ Q $\rightarrow$ R) in n2_82a
966     by now rewrite Impl1_01.
967   replace ( $\neg$ R $\vee$ S) with (R $\rightarrow$ S) in n2_82a
968     by now rewrite Impl1_01.
969   replace ( $\neg$ P $\vee$ (R $\rightarrow$ S)) with (P $\rightarrow$ R $\rightarrow$ S) in n2_82a
970     by now rewrite Impl1_01.
971   replace ( $\neg$ Q $\vee$ S) with (Q $\rightarrow$ S) in n2_82a
972     by now rewrite Impl1_01.
973   replace ( $\neg$ Q $\vee$ S) with (Q $\rightarrow$ S) in n2_82a
974     by now rewrite Impl1_01.
975   replace ( $\neg$ P $\vee$ (Q $\rightarrow$ S)) with (P $\rightarrow$ Q $\rightarrow$ S) in n2_82a
976     by now rewrite Impl1_01.
977   exact n2_82a.
978 Qed.
979
980 Theorem n2_85 :  $\forall P Q R : \text{Prop},$ 
981    $((P \vee Q) \rightarrow (P \vee R)) \rightarrow (P \vee (Q \rightarrow R)).$ 
982 Proof. intros P Q R.
983   specialize Add1_3 with P Q.
984   intros Add1_3a.
985   specialize Syll2_06 with Q (P $\vee$ Q) R.
986   intros Syll2_06a.
987   MP Syll2_06a Add1_3a.
988   specialize n2_55 with P R.

```



```

989   intros n2_55a.
990   specialize Syll2_05 with (P $\vee$ Q) (P $\vee$ R) R.
991   intros Syll2_05a.
992   Syll n2_55a Syll2_05a Ha.
993   specialize n2_83 with ( $\neg$ P) ((P $\vee$ Q) $\rightarrow$ (P $\vee$ R)) ((P $\vee$ Q) $\rightarrow$ R) (Q $\rightarrow$ R).
994   intros n2_83a.
995   MP n2_83a Ha.
996   specialize Comm2_04 with ( $\neg$ P) (P $\vee$ Q $\rightarrow$ P $\vee$ R) (Q $\rightarrow$ R).
997   intros Comm2_04a.
998   Syll Ha Comm2_04a Hb.
999   specialize n2_54 with P (Q $\rightarrow$ R).
1000  intros n2_54a.
1001  specialize Simp2_02 with ( $\neg$ P) ((P $\vee$ Q $\rightarrow$ R) $\rightarrow$ (Q $\rightarrow$ R)).
1002  intros Simp2_02a. (*Not cited*)
1003      (*Greg's suggestion per the BRS list on June 25, 2017.*)
1004  MP Syll2_06a Simp2_02a.
1005  MP Hb Simp2_02a.
1006  Syll Hb n2_54a Hc.
1007  exact Hc.
1008  Qed.
1009
1010  Theorem n2_86 :  $\forall$  P Q R : Prop,
1011    ((P  $\rightarrow$  Q)  $\rightarrow$  (P  $\rightarrow$  R))  $\rightarrow$  (P  $\rightarrow$  (Q  $\rightarrow$  R)).
1012  Proof. intros P Q R.
1013    specialize n2_85 with ( $\neg$ P) Q R.
1014    intros n2_85a.
1015    replace ( $\neg$ P $\vee$ Q) with (P $\rightarrow$ Q) in n2_85a
1016      by now rewrite Impl1_01.
1017    replace ( $\neg$ P $\vee$ R) with (P $\rightarrow$ R) in n2_85a
1018      by now rewrite Impl1_01.
1019    replace ( $\neg$ P $\vee$ (Q $\rightarrow$ R)) with (P $\rightarrow$ Q $\rightarrow$ R) in n2_85a
1020      by now rewrite Impl1_01.
1021    exact n2_85a.
1022  Qed.
1023
1024  End No2.
1025
1026  Module No3.
1027
1028  Import No1.
1029  Import No2.
1030

```

```

1031
1032 Theorem Prod3_01 :  $\forall P Q : \text{Prop},$ 
1033    $(P \wedge Q) = (\neg(\neg P \vee \neg Q)).$ 
1034 Proof. intros P Q.
1035   apply propositional_extensionality.
1036   split.
1037   specialize or_not_and with (P) (Q).
1038   intros or_not_and.
1039   specialize Transp2_03 with  $(\neg P \vee \neg Q) (P \wedge Q).$ 
1040   intros Transp2_03.
1041   MP Transp2_03 or_not_and.
1042   exact Transp2_03.
1043   specialize not_and_or with (P) (Q).
1044   intros not_and_or.
1045   specialize Transp2_15 with  $(P \wedge Q) (\neg P \vee \neg Q).$ 
1046   intros Transp2_15.
1047   MP Transp2_15 not_and_or.
1048   exact Transp2_15.
1049 Qed.
1050 (*This is a notational definition in Principia;
1051 it is used to switch between " $\wedge$ " and " $\neg\vee\neg$ ".*)
1052
1053 (*Axiom Abb3_02 :  $\forall P Q R : \text{Prop},$ 
1054  $(P \rightarrow Q \rightarrow R) = ((P \rightarrow Q) \wedge (Q \rightarrow R)).$ *)
1055 (*Since Coq forbids such strings as ill-formed, or
1056 else automatically associates to the right,
1057 we leave this notational axiom commented out.*)
1058
1059 Theorem Conj3_03 :  $\forall P Q : \text{Prop}, P \rightarrow Q \rightarrow (P \wedge Q).$ 
1060 Proof. intros P Q.
1061   specialize n2_11 with  $(\neg P \vee \neg Q).$ 
1062   intros n2_11a.
1063   specialize n2_32 with  $(\neg P) (\neg Q) (\neg(\neg P \vee \neg Q)).$ 
1064   intros n2_32a.
1065   MP n2_32a n2_11a.
1066   replace  $(\neg(\neg P \vee \neg Q))$  with  $(P \wedge Q)$  in n2_32a
1067     by now rewrite Prod3_01.
1068   replace  $(\neg Q \vee (P \wedge Q))$  with  $(Q \rightarrow (P \wedge Q))$  in n2_32a
1069     by now rewrite Impl1_01.
1070   replace  $(\neg P \vee (Q \rightarrow (P \wedge Q)))$  with  $(P \rightarrow Q \rightarrow (P \wedge Q))$  in n2_32a
1071     by now rewrite Impl1_01.
1072   exact n2_32a.

```

```

1073 Qed.
1074 (*3.03 is permits the inference from the theoremhood
1075    of P and that of Q to the theoremhood of P and Q. So:*)
1076
1077 Ltac Conj H1 H2 C :=
1078   let C := fresh C in match goal with
1079     | [ H1 : ?P, H2 : ?Q |- _ ] =>
1080       (specialize Conj3_03 with P Q;
1081        intros C;
1082        MP Conj3_03 P; MP Conj3_03 Q)
1083   end.
1084
1085 Theorem n3_1 :  $\forall P Q : \text{Prop},$ 
1086    $(P \wedge Q) \rightarrow \neg(\neg P \vee \neg Q).$ 
1087 Proof. intros P Q.
1088   specialize Id2_08 with (P $\wedge$ Q).
1089   intros Id2_08a.
1090   replace ((P $\wedge$ Q) $\rightarrow$ (P $\wedge$ Q)) with ((P $\wedge$ Q) $\rightarrow$  $\neg(\neg P \vee \neg Q)$ )
1091     in Id2_08a by now rewrite Prod3_01.
1092   exact Id2_08a.
1093 Qed.
1094
1095 Theorem n3_11 :  $\forall P Q : \text{Prop},$ 
1096    $\neg(\neg P \vee \neg Q) \rightarrow (P \wedge Q).$ 
1097 Proof. intros P Q.
1098   specialize Id2_08 with (P $\wedge$ Q).
1099   intros Id2_08a.
1100   replace ((P $\wedge$ Q) $\rightarrow$ (P $\wedge$ Q)) with ( $\neg(\neg P \vee \neg Q) \rightarrow$ (P $\wedge$ Q))
1101     in Id2_08a by now rewrite Prod3_01.
1102   exact Id2_08a.
1103 Qed.
1104
1105 Theorem n3_12 :  $\forall P Q : \text{Prop},$ 
1106    $(\neg P \vee \neg Q) \vee (P \wedge Q).$ 
1107 Proof. intros P Q.
1108   specialize n2_11 with ( $\neg P \vee \neg Q$ ).
1109   intros n2_11a.
1110   replace ( $\neg(\neg P \vee \neg Q)$ ) with (P $\wedge$ Q) in n2_11a
1111     by now rewrite Prod3_01.
1112   exact n2_11a.
1113 Qed.
1114

```

```

1115 Theorem n3_13 :  $\forall P Q : \text{Prop},$ 
1116    $\neg(P \wedge Q) \rightarrow (\neg P \vee \neg Q).$ 
1117 Proof. intros P Q.
1118   specialize n3_11 with P Q.
1119   intros n3_11a.
1120   specialize Transp2_15 with  $(\neg P \vee \neg Q) (P \wedge Q).$ 
1121   intros Transp2_15a.
1122   MP Transp2_15a n3_11a.
1123   exact Transp2_15a.
1124 Qed.
1125
1126 Theorem n3_14 :  $\forall P Q : \text{Prop},$ 
1127    $(\neg P \vee \neg Q) \rightarrow \neg(P \wedge Q).$ 
1128 Proof. intros P Q.
1129   specialize n3_1 with P Q.
1130   intros n3_1a.
1131   specialize Transp2_16 with  $(P \wedge Q) (\neg(\neg P \vee \neg Q)).$ 
1132   intros Transp2_16a.
1133   MP Transp2_16a n3_1a.
1134   specialize n2_12 with  $(\neg P \vee \neg Q).$ 
1135   intros n2_12a.
1136   Syll n2_12a Transp2_16a S.
1137   exact S.
1138 Qed.
1139
1140 Theorem n3_2 :  $\forall P Q : \text{Prop},$ 
1141    $P \rightarrow Q \rightarrow (P \wedge Q).$ 
1142 Proof. intros P Q.
1143   specialize n3_12 with P Q.
1144   intros n3_12a.
1145   specialize n2_32 with  $(\neg P) (\neg Q) (P \wedge Q).$ 
1146   intros n2_32a.
1147   MP n3_32a n3_12a.
1148   replace  $(\neg Q \vee P \wedge Q)$  with  $(Q \rightarrow P \wedge Q)$  in n2_32a
1149     by now rewrite Impl1_01.
1150   replace  $(\neg P \vee (Q \rightarrow P \wedge Q))$  with  $(P \rightarrow Q \rightarrow P \wedge Q)$ 
1151     in n2_32a by now rewrite Impl1_01.
1152   exact n2_32a.
1153 Qed.
1154
1155 Theorem n3_21 :  $\forall P Q : \text{Prop},$ 
1156    $Q \rightarrow P \rightarrow (P \wedge Q).$ 

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1157 Proof. intros P Q.
1158   specialize n3_2 with P Q.
1159   intros n3_2a.
1160   specialize Comm2_04 with P Q (P ∧ Q).
1161   intros Comm2_04a.
1162   MP Comm2_04a n3_2a.
1163   exact Comm2_04a.
1164 Qed.
1165
1166 Theorem n3_22 : ∀ P Q : Prop,
1167   (P ∧ Q) → (Q ∧ P).
1168 Proof. intros P Q.
1169   specialize n3_13 with Q P.
1170   intros n3_13a.
1171   specialize Perm1_4 with (¬Q) (¬P).
1172   intros Perm1_4a.
1173   Syll n3_13a Perm1_4a Ha.
1174   specialize n3_14 with P Q.
1175   intros n3_14a.
1176   Syll Ha n3_14a Hb.
1177   specialize Transp2_17 with (P ∧ Q) (Q ∧ P).
1178   intros Transp2_17a.
1179   MP Transp2_17a Hb.
1180   exact Transp2_17a.
1181 Qed.
1182
1183 Theorem n3_24 : ∀ P : Prop,
1184   ¬(P ∧ ¬P).
1185 Proof. intros P.
1186   specialize n2_11 with (¬P).
1187   intros n2_11a.
1188   specialize n3_14 with P (¬P).
1189   intros n3_14a.
1190   MP n3_14a n2_11a.
1191   exact n3_14a.
1192 Qed.
1193
1194 Theorem Simp3_26 : ∀ P Q : Prop,
1195   (P ∧ Q) → P.
1196 Proof. intros P Q.
1197   specialize Simp2_02 with Q P.
1198   intros Simp2_02a.

```

```

1199   replace (P→(Q→P)) with (¬P∨(Q→P)) in Simp2_02a
1200     by now rewrite <- Impl1_01.
1201   replace (Q→P) with (¬Q∨P) in Simp2_02a
1202     by now rewrite Impl1_01.
1203   specialize n2_31 with (¬P) (¬Q) P.
1204   intros n2_31a.
1205   MP n2_31a Simp2_02a.
1206   specialize n2_53 with (¬P∨¬Q) P.
1207   intros n2_53a.
1208   MP n2_53a Simp2_02a.
1209   replace (¬(¬P∨¬Q)) with (P∧Q) in n2_53a
1210     by now rewrite Prod3_01.
1211   exact n2_53a.
1212 Qed.
1213
1214 Theorem Simp3_27 : ∀ P Q : Prop,
1215   (P ∧ Q) → Q.
1216 Proof. intros P Q.
1217   specialize n3_22 with P Q.
1218   intros n3_22a.
1219   specialize Simp3_26 with Q P.
1220   intros Simp3_26a.
1221   Syll n3_22a Simp3_26a S.
1222   exact S.
1223 Qed.
1224
1225 Theorem Exp3_3 : ∀ P Q R : Prop,
1226   ((P ∧ Q) → R) → (P → (Q → R)).
1227 Proof. intros P Q R.
1228   specialize Id2_08 with ((P∧Q)→R).
1229   intros Id2_08a. (*This theorem isn't needed.*)
1230   replace (((P ∧ Q) → R) → ((P ∧ Q) → R)) with
1231     (((P ∧ Q) → R) → (¬(¬P ∨ ¬Q) → R)) in Id2_08a
1232     by now rewrite Prod3_01.
1233   specialize Transp2_15 with (¬P∨¬Q) R.
1234   intros Transp2_15a.
1235   Syll Id2_08a Transp2_15a Sa.
1236   specialize Id2_08 with (¬R → (¬P ∨ ¬Q)).
1237   intros Id2_08b. (*This theorem isn't needed.*)
1238   Syll Sa Id2_08b Sb.
1239   replace (¬P ∨ ¬Q) with (P → ¬Q) in Sb
1240     by now rewrite Impl1_01.

```

```

1241 specialize Comm2_04 with ( $\neg$ R) P ( $\neg$ Q).
1242 intros Comm2_04a.
1243 Syll Sb Comm2_04a Sc.
1244 specialize Transp2_17 with Q R.
1245 intros Transp2_17a.
1246 specialize Syll2_05 with P ( $\neg$ R  $\rightarrow$   $\neg$ Q) (Q  $\rightarrow$  R).
1247 intros Syll2_05a.
1248 MP Syll2_05a Transp2_17a.
1249 Syll Sa Syll2_05a Sd.
1250 exact Sd.
1251 Qed.
1252
1253 Theorem Imp3_31 :  $\forall$  P Q R : Prop,
1254   (P  $\rightarrow$  (Q  $\rightarrow$  R))  $\rightarrow$  (P  $\wedge$  Q)  $\rightarrow$  R.
1255 Proof. intros P Q R.
1256   specialize Id2_08 with (P  $\rightarrow$  (Q  $\rightarrow$  R)).
1257   intros Id2_08a.
1258   replace ((P  $\rightarrow$  (Q  $\rightarrow$  R)) $\rightarrow$ (P  $\rightarrow$  (Q  $\rightarrow$  R))) with
1259     ((P  $\rightarrow$  (Q  $\rightarrow$  R)) $\rightarrow$ ( $\neg$ P  $\vee$  (Q  $\rightarrow$  R))) in Id2_08a
1260     by now rewrite <- Impl1_01.
1261   replace ( $\neg$ P  $\vee$  (Q  $\rightarrow$  R)) with
1262     ( $\neg$ P  $\vee$  ( $\neg$ Q  $\vee$  R)) in Id2_08a
1263     by now rewrite Impl1_01.
1264   specialize n2_31 with ( $\neg$ P) ( $\neg$ Q) R.
1265   intros n2_31a.
1266   Syll Id2_08a n2_31a Sa.
1267   specialize n2_53 with ( $\neg$ P $\vee$  $\neg$ Q) R.
1268   intros n2_53a.
1269   replace ( $\neg$ ( $\neg$ P $\vee$  $\neg$ Q)) with (P $\wedge$ Q) in n2_53a
1270     by now rewrite Prod3_01.
1271   Syll Sa n2_53a Sb.
1272   exact Sb.
1273 Qed.
1274
1275 Theorem Syll3_33 :  $\forall$  P Q R : Prop,
1276   ((P  $\rightarrow$  Q)  $\wedge$  (Q  $\rightarrow$  R))  $\rightarrow$  (P  $\rightarrow$  R).
1277 Proof. intros P Q R.
1278   specialize Syll2_06 with P Q R.
1279   intros Syll2_06a.
1280   specialize Imp3_31 with (P $\rightarrow$ Q) (Q $\rightarrow$ R) (P $\rightarrow$ R).
1281   intros Imp3_31a.
1282   MP Imp3_31a Syll2_06a.

```

```

1283     exact Imp3_31a.
1284 Qed.
1285
1286 Theorem Syll3_34 :  $\forall P Q R : \text{Prop}$ ,
1287    $((Q \rightarrow R) \wedge (P \rightarrow Q)) \rightarrow (P \rightarrow R)$ .
1288 Proof. intros P Q R.
1289   specialize Syll2_05 with P Q R.
1290   intros Syll2_05a.
1291   specialize Imp3_31 with  $(Q \rightarrow R)$   $(P \rightarrow Q)$   $(P \rightarrow R)$ .
1292   intros Imp3_31a.
1293   MP Imp3_31a Syll2_05a.
1294   exact Imp3_31a.
1295 Qed.
1296
1297 Theorem Ass3_35 :  $\forall P Q : \text{Prop}$ ,
1298    $(P \wedge (P \rightarrow Q)) \rightarrow Q$ .
1299 Proof. intros P Q.
1300   specialize n2_27 with P Q.
1301   intros n2_27a.
1302   specialize Imp3_31 with P  $(P \rightarrow Q)$  Q.
1303   intros Imp3_31a.
1304   MP Imp3_31a n2_27a.
1305   exact Imp3_31a.
1306 Qed.
1307
1308 Theorem Transp3_37 :  $\forall P Q R : \text{Prop}$ ,
1309    $(P \wedge Q \rightarrow R) \rightarrow (P \wedge \neg R \rightarrow \neg Q)$ .
1310 Proof. intros P Q R.
1311   specialize Transp2_16 with Q R.
1312   intros Transp2_16a.
1313   specialize Syll2_05 with P  $(Q \rightarrow R)$   $(\neg R \rightarrow \neg Q)$ .
1314   intros Syll2_05a.
1315   MP Syll2_05a Transp2_16a.
1316   specialize Exp3_3 with P Q R.
1317   intros Exp3_3a.
1318   Syll Exp3_3a Syll2_05a Sa.
1319   specialize Imp3_31 with P  $(\neg R)$   $(\neg Q)$ .
1320   intros Imp3_31a.
1321   Syll Sa Imp3_31a Sb.
1322   exact Sb.
1323 Qed.
1324

```



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1325 Theorem n3_4 :  $\forall P Q : \text{Prop},$ 
1326    $(P \wedge Q) \rightarrow P \rightarrow Q.$ 
1327 Proof. intros P Q.
1328   specialize n2_51 with P Q.
1329   intros n2_51a.
1330   specialize Transp2_15 with  $(P \rightarrow Q) (P \rightarrow \neg Q).$ 
1331   intros Transp2_15a.
1332   MP Transp2_15a n2_51a.
1333   replace  $(P \rightarrow \neg Q)$  with  $(\neg P \vee \neg Q)$  in Transp2_15a
1334     by now rewrite Impl1_01.
1335   replace  $(\neg(\neg P \vee \neg Q))$  with  $(P \wedge Q)$  in Transp2_15a
1336     by now rewrite Prod3_01.
1337   exact Transp2_15a.
1338 Qed.
1339
1340 Theorem n3_41 :  $\forall P Q R : \text{Prop},$ 
1341    $(P \rightarrow R) \rightarrow (P \wedge Q \rightarrow R).$ 
1342 Proof. intros P Q R.
1343   specialize Simp3_26 with P Q.
1344   intros Simp3_26a.
1345   specialize Syll2_06 with  $(P \wedge Q) P R.$ 
1346   intros Syll2_06a.
1347   MP Simp3_26a Syll2_06a.
1348   exact Syll2_06a.
1349 Qed.
1350
1351 Theorem n3_42 :  $\forall P Q R : \text{Prop},$ 
1352    $(Q \rightarrow R) \rightarrow (P \wedge Q \rightarrow R).$ 
1353 Proof. intros P Q R.
1354   specialize Simp3_27 with P Q.
1355   intros Simp3_27a.
1356   specialize Syll2_06 with  $(P \wedge Q) Q R.$ 
1357   intros Syll2_06a.
1358   MP Syll2_06a Simp3_27a.
1359   exact Syll2_06a.
1360 Qed.
1361
1362 Theorem Comp3_43 :  $\forall P Q R : \text{Prop},$ 
1363    $(P \rightarrow Q) \wedge (P \rightarrow R) \rightarrow (P \rightarrow Q \wedge R).$ 
1364 Proof. intros P Q R.
1365   specialize n3_2 with Q R.
1366   intros n3_2a.

```

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1367 specialize Syll2_05 with P Q (R→Q∧R).
1368 intros Syll2_05a.
1369 MP Syll2_05a n2_2a.
1370 specialize n2_77 with P R (Q∧R).
1371 intros n2_77a.
1372 Syll Syll2_05a n2_77a Sa.
1373 specialize Imp3_31 with (P→Q) (P→R) (P→Q∧R).
1374 intros Imp3_31a.
1375 MP Sa Imp3_31a.
1376 exact Imp3_31a.
1377 Qed.
1378
1379 Theorem n3_44 : ∀ P Q R : Prop,
1380   (Q → P) ∧ (R → P) → (Q ∨ R → P).
1381 Proof. intros P Q R.
1382   specialize Syll3_33 with (¬Q) R P.
1383   intros Syll3_33a.
1384   specialize n2_6 with Q P.
1385   intros n2_6a.
1386   Syll Syll3_33a n2_6a Sa.
1387   specialize Exp3_3 with (¬Q→R) (R→P) ((Q→P)→P).
1388   intros Exp3_3a.
1389   MP Exp3_3a Sa.
1390   specialize Comm2_04 with (R→P) (Q→P) P.
1391   intros Comm2_04a.
1392   Syll Exp3_3a Comm2_04a Sb.
1393   specialize Imp3_31 with (Q→P) (R→P) P.
1394   intros Imp3_31a.
1395   Syll Sb Imp3_31a Sc.
1396   specialize Comm2_04 with (¬Q→R) ((Q→P)∧(R→P)) P.
1397   intros Comm2_04b.
1398   MP Comm2_04b Sc.
1399   specialize n2_53 with Q R.
1400   intros n2_53a.
1401   specialize Syll2_06 with (Q∨R) (¬Q→R) P.
1402   intros Syll2_06a.
1403   MP Syll2_06a n2_53a.
1404   Syll Comm2_04b Syll2_06a Sd.
1405   exact Sd.
1406 Qed.
1407
1408 Theorem Fact3_45 : ∀ P Q R : Prop,

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1409   (P → Q) → (P ∧ R) → (Q ∧ R).
1410 Proof. intros P Q R.
1411   specialize Syll2_06 with P Q (¬R).
1412   intros Syll2_06a.
1413   specialize Transp2_16 with (Q→¬R) (P→¬R).
1414   intros Transp2_16a.
1415   Syll Syll2_06a Transp2_16a Sa.
1416   specialize Id2_08 with (¬(P→R)→¬(Q→¬R)).
1417   intros Id2_08a.
1418   Syll Sa Id2_08a Sb.
1419   replace (P→¬R) with (¬P∨¬R) in Sb
1420     by now rewrite Impl1_01.
1421   replace (Q→¬R) with (¬Q∨¬R) in Sb
1422     by now rewrite Impl1_01.
1423   replace (¬(¬P∨¬R)) with (P∧R) in Sb
1424     by now rewrite Prod3_01.
1425   replace (¬(¬Q∨¬R)) with (Q∧R) in Sb
1426     by now rewrite Prod3_01.
1427   exact Sb.
1428 Qed.
1429
1430 Theorem n3_47 : ∀ P Q R S : Prop,
1431   ((P → R) ∧ (Q → S)) → (P ∧ Q) → R ∧ S.
1432 Proof. intros P Q R S.
1433   specialize Simp3_26 with (P→R) (Q→S).
1434   intros Simp3_26a.
1435   specialize Fact3_45 with P R Q.
1436   intros Fact3_45a.
1437   Syll Simp3_26a Fact3_45a Sa.
1438   specialize n3_22 with R Q.
1439   intros n3_22a.
1440   specialize Syll2_05 with (P∧Q) (R∧Q) (Q∧R).
1441   intros Syll2_05a.
1442   MP Syll2_05a n3_22a.
1443   Syll Sa Syll2_05a Sb.
1444   specialize Simp3_27 with (P→R) (Q→S).
1445   intros Simp3_27a.
1446   specialize Fact3_45 with Q S R.
1447   intros Fact3_45b.
1448   Syll Simp3_27a Fact3_45b Sc.
1449   specialize n3_22 with S R.
1450   intros n3_22b.

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1451 specialize Syll2_05 with (Q $\wedge$ R) (S $\wedge$ R) (R $\wedge$ S).
1452 intros Syll2_05b.
1453 MP Syll2_05b n3_22b.
1454 Syll Sc Syll2_05b Sd.
1455 clear Simp3_26a. clear Fact3_45a. clear Sa.
1456   clear n3_22a. clear Fact3_45b.
1457   clear Syll2_05a. clear Simp3_27a.
1458   clear Sc. clear n3_22b. clear Syll2_05b.
1459 Conj Sb Sd C.
1460 specialize n2_83 with ((P $\rightarrow$ R) $\wedge$ (Q $\rightarrow$ S)) (P $\wedge$ Q) (Q $\wedge$ R) (R $\wedge$ S).
1461 intros n2_83a. (*This with MP works, but it omits Conj3_03.*)
1462 specialize Imp3_31 with (((P $\rightarrow$ R) $\wedge$ (Q $\rightarrow$ S)) $\rightarrow$ ((P $\wedge$ Q) $\rightarrow$ (Q $\wedge$ R)))
1463   (((P $\rightarrow$ R) $\wedge$ (Q $\rightarrow$ S)) $\rightarrow$ ((Q $\wedge$ R) $\rightarrow$ (R $\wedge$ S)))
1464   (((P $\rightarrow$ R) $\wedge$ (Q $\rightarrow$ S)) $\rightarrow$ ((P $\wedge$ Q) $\rightarrow$ (R $\wedge$ S))).
1465 intros Imp3_31a.
1466 MP Imp3_31a n2_83a.
1467 MP Imp3_31a C.
1468 exact Imp3_31a.
1469 Qed.
1470
1471 Theorem n3_48 :  $\forall$  P Q R S : Prop,
1472   ((P  $\rightarrow$  R)  $\wedge$  (Q  $\rightarrow$  S))  $\rightarrow$  (P  $\vee$  Q)  $\rightarrow$  R  $\vee$  S.
1473 Proof. intros P Q R S.
1474   specialize Simp3_26 with (P $\rightarrow$ R) (Q $\rightarrow$ S).
1475   intros Simp3_26a.
1476   specialize Sum1_6 with Q P R.
1477   intros Sum1_6a.
1478   Syll Simp3_26a Sum1_6a Sa.
1479   specialize Perm1_4 with P Q.
1480   intros Perm1_4a.
1481   specialize Syll2_06 with (P $\vee$ Q) (Q $\vee$ P) (Q $\vee$ R).
1482   intros Syll2_06a.
1483   MP Syll2_06a Perm1_4a.
1484   Syll Sa Syll2_06a Sb.
1485   specialize Simp3_27 with (P $\rightarrow$ R) (Q $\rightarrow$ S).
1486   intros Simp3_27a.
1487   specialize Sum1_6 with R Q S.
1488   intros Sum1_6b.
1489   Syll Simp3_27a Sum1_6b Sc.
1490   specialize Perm1_4 with Q R.
1491   intros Perm1_4b.
1492   specialize Syll2_06 with (Q $\vee$ R) (R $\vee$ Q) (R $\vee$ S).

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1493   intros Syll2_06b.
1494   MP Syll2_06b Perm1_4b.
1495   Syll Sc Syll2_06a Sd.
1496   specialize n2_83 with ((P→R)∧(Q→S)) (P∨Q) (Q∨R) (R∨S).
1497   intros n2_83a.
1498   MP n2_83a Sb.
1499   MP n2_83a Sd.
1500   exact n2_83a.
1501   Qed.
1502
1503   End No3.
1504
1505   Module No4.
1506
1507   Import No1.
1508   Import No2.
1509   Import No3.
1510
1511   Theorem Equiv4_01 : ∀ P Q : Prop,
1512     (P ↔ Q) = ((P → Q) ∧ (Q → P)).
1513   Proof. intros P Q.
1514     apply propositional_extensionality.
1515     specialize iff_to_and with P Q.
1516     intros iff_to_and.
1517     exact iff_to_and.
1518     Qed.
1519     (*This is a notational definition in Principia;
1520     it is used to switch between "↔" and "→∧←".*)
1521
1522     (*Axiom Abb4_02 : ∀ P Q R : Prop,
1523     (P ↔ Q ↔ R) = ((P ↔ Q) ∧ (Q ↔ R)).*)
1524     (*Since Coq forbids ill-formed strings, or else
1525     automatically associates to the right, we leave
1526     this notational axiom commented out.*)"
1527
1528   Ltac Equiv H1 :=
1529     match goal with
1530     | [ H1 : (?P→?Q) ∧ (?Q→?P) |- _ ] =>
1531       replace ((P→Q) ∧ (Q→P)) with (P↔Q) in H1
1532       by now rewrite Equiv4_01
1533   end.
1534

```

```

1535 Theorem Transp4_1 :  $\forall$  P Q : Prop,
1536   (P  $\rightarrow$  Q)  $\leftrightarrow$  ( $\neg$ Q  $\rightarrow$   $\neg$ P).
1537 Proof. intros P Q.
1538   specialize Transp2_16 with P Q.
1539   intros Transp2_16a.
1540   specialize Transp2_17 with P Q.
1541   intros Transp2_17a.
1542   Conj Transp2_16a Transp2_17a C.
1543   Equiv C.
1544   exact C.
1545 Qed.
1546
1547 Theorem Transp4_11 :  $\forall$  P Q : Prop,
1548   (P  $\leftrightarrow$  Q)  $\leftrightarrow$  ( $\neg$ P  $\leftrightarrow$   $\neg$ Q).
1549 Proof. intros P Q.
1550   specialize Transp2_16 with P Q.
1551   intros Transp2_16a.
1552   specialize Transp2_16 with Q P.
1553   intros Transp2_16b.
1554   Conj Transp2_16a Transp2_16b Ca.
1555   specialize n3_47 with (P $\rightarrow$ Q) (Q $\rightarrow$ P) ( $\neg$ Q $\rightarrow$  $\neg$ P) ( $\neg$ P $\rightarrow$  $\neg$ Q).
1556   intros n3_47a.
1557   MP n3_47 Ca.
1558   specialize n3_22 with ( $\neg$ Q  $\rightarrow$   $\neg$ P) ( $\neg$ P  $\rightarrow$   $\neg$ Q).
1559   intros n3_22a.
1560   Syll n3_47a n3_22a Sa.
1561   replace ((P  $\rightarrow$  Q)  $\wedge$  (Q  $\rightarrow$  P)) with (P $\leftrightarrow$ Q) in Sa
1562   by now rewrite Equiv4_01.
1563   replace (( $\neg$ P  $\rightarrow$   $\neg$ Q)  $\wedge$  ( $\neg$ Q  $\rightarrow$   $\neg$ P)) with ( $\neg$ P $\leftrightarrow$  $\neg$ Q)
1564   in Sa by now rewrite Equiv4_01.
1565   clear Transp2_16a. clear Ca. clear Transp2_16b.
1566   clear n3_22a. clear n3_47a.
1567   specialize Transp2_17 with Q P.
1568   intros Transp2_17a.
1569   specialize Transp2_17 with P Q.
1570   intros Transp2_17b.
1571   Conj Transp2_17a Transp2_17b Cb.
1572   specialize n3_47 with ( $\neg$ P $\rightarrow$  $\neg$ Q) ( $\neg$ Q $\rightarrow$  $\neg$ P) (Q $\rightarrow$ P) (P $\rightarrow$ Q).
1573   intros n3_47a.
1574   MP n3_47a Cb.
1575   specialize n3_22 with (Q $\rightarrow$ P) (P $\rightarrow$ Q).
1576   intros n3_22a.

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1577 Syll n3_47a n3_22a Sb.
1578 clear Transp2_17a. clear Transp2_17b. clear Cb.
1579 clear n3_47a. clear n3_22a.
1580 replace ((P → Q) ∧ (Q → P)) with (P↔Q) in Sb
1581 by now rewrite Equiv4_01.
1582 replace ((¬P → ¬Q) ∧ (¬Q → ¬P)) with (¬P↔¬Q)
1583 in Sb by now rewrite Equiv4_01.
1584 Conj Sa Sb Cc.
1585 Equiv Cc.
1586 exact Cc.
1587 Qed.
1588
1589 Theorem n4_12 : ∀ P Q : Prop,
1590   (P ↔ ¬Q) ↔ (Q ↔ ¬P).
1591 Proof. intros P Q.
1592   specialize Transp2_03 with P Q.
1593   intros Transp2_03a.
1594   specialize Transp2_15 with Q P.
1595   intros Transp2_15a.
1596   Conj Transp2_03a Transp2_15a Ca.
1597   specialize n3_47 with (P→¬Q) (¬Q→P) (Q→¬P) (¬P→Q).
1598   intros n3_47a.
1599   MP n3_47a C.
1600   specialize Transp2_03 with Q P.
1601   intros Transp2_03b.
1602   specialize Transp2_15 with P Q.
1603   intros Transp2_15b.
1604   Conj Transp2_03b Transp2_15b Cb.
1605   specialize n3_47 with (Q→¬P) (¬P→Q) (P→¬Q) (¬Q→P).
1606   intros n3_47b.
1607   MP n3_47b H0.
1608   clear Transp2_03a. clear Transp2_15a. clear Ca.
1609   clear Transp2_03b. clear Transp2_15b. clear Cb.
1610   Conj n3_47a n3_47b Cc.
1611   rewrite <- Equiv4_01 in Cc.
1612   rewrite <- Equiv4_01 in Cc.
1613   rewrite <- Equiv4_01 in Cc.
1614   exact Cc.
1615 Qed.
1616
1617 Theorem n4_13 : ∀ P : Prop,
1618   P ↔ ¬¬P.

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1619   Proof. intros P.
1620   specialize n2_12 with P.
1621   intros n2_12a.
1622   specialize n2_14 with P.
1623   intros n2_14a.
1624   Conj n2_12a n2_14a C.
1625   Equiv C.
1626   exact C.
1627   Qed.
1628
1629   Theorem n4_14 :  $\forall P Q R : \text{Prop}$ ,
1630      $((P \wedge Q) \rightarrow R) \leftrightarrow ((P \wedge \neg R) \rightarrow \neg Q)$ .
1631   Proof. intros P Q R.
1632   specialize Transp3_37 with P Q R.
1633   intros Transp3_37a.
1634   specialize Transp3_37 with P ( $\neg R$ ) ( $\neg Q$ ).
1635   intros Transp3_37b.
1636   Conj Transp3_37a Transp3_37b C.
1637   specialize n4_13 with Q.
1638   intros n4_13a.
1639   apply propositional_extensionality in n4_13a.
1640   specialize n4_13 with R.
1641   intros n4_13b.
1642   apply propositional_extensionality in n4_13b.
1643   replace ( $\neg\neg Q$ ) with Q in C
1644     by now apply n4_13a.
1645   replace ( $\neg\neg R$ ) with R in C
1646     by now apply n4_13b.
1647   Equiv C.
1648   exact C.
1649   Qed.
1650
1651   Theorem n4_15 :  $\forall P Q R : \text{Prop}$ ,
1652      $((P \wedge Q) \rightarrow \neg R) \leftrightarrow ((Q \wedge R) \rightarrow \neg P)$ .
1653   Proof. intros P Q R.
1654   specialize n4_14 with Q P ( $\neg R$ ).
1655   intros n4_14a.
1656   specialize n3_22 with Q P.
1657   intros n3_22a.
1658   specialize Syll2_06 with  $(Q \wedge P)$   $(P \wedge Q)$  ( $\neg R$ ).
1659   intros Syll2_06a.
1660   MP Syll2_06a n3_22a.

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1661 specialize n4_13 with R.
1662 intros n4_13a.
1663 apply propositional_extensionality in n4_13a.
1664 replace ( $\neg\neg R$ ) with R in n4_14a
1665   by now apply n4_13a.
1666 rewrite Equiv4_01 in n4_14a.
1667 specialize Simp3_26 with  $((Q \wedge P \rightarrow \neg R) \rightarrow Q \wedge R \rightarrow \neg P)$ 
1668    $((Q \wedge R \rightarrow \neg P) \rightarrow Q \wedge P \rightarrow \neg R)$ .
1669 intros Simp3_26a.
1670 MP Simp3_26a n4_14a.
1671 Syll Syll2_06a Simp3_26a Sa.
1672 specialize Simp3_27 with  $((Q \wedge P \rightarrow \neg R) \rightarrow Q \wedge R \rightarrow \neg P)$ 
1673    $((Q \wedge R \rightarrow \neg P) \rightarrow Q \wedge P \rightarrow \neg R)$ .
1674 intros Simp3_27a.
1675 MP Simp3_27a n4_14a.
1676 specialize n3_22 with P Q.
1677 intros n3_22b.
1678 specialize Syll2_06 with  $(P \wedge Q) (Q \wedge P) (\neg R)$ .
1679 intros Syll2_06b.
1680 MP Syll2_06b n3_22b.
1681 Syll Syll2_06b Simp3_27a Sb.
1682 clear n4_14a. clear n3_22a. clear Syll2_06a.
1683   clear n4_13a. clear Simp3_26a. clear n3_22b.
1684   clear Simp3_27a. clear Syll2_06b.
1685 Conj Sa Sb C.
1686 Equiv C.
1687 exact C.
1688 Qed.
1689
1690 Theorem n4_2 :  $\forall P : \text{Prop},$ 
1691    $P \leftrightarrow P$ .
1692 Proof. intros P.
1693 specialize n3_2 with  $(P \rightarrow P) (P \rightarrow P)$ .
1694 intros n3_2a.
1695 specialize Id2_08 with P.
1696 intros Id2_08a.
1697 MP n3_2a Id2_08a.
1698 MP n3_2a Id2_08a.
1699 Equiv n3_2a.
1700 exact n3_2a.
1701 Qed.
1702

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1703 Theorem n4_21 :  $\forall$  P Q : Prop,
1704   (P  $\leftrightarrow$  Q)  $\leftrightarrow$  (Q  $\leftrightarrow$  P).
1705 Proof. intros P Q.
1706 specialize n3_22 with (P $\rightarrow$ Q) (Q $\rightarrow$ P).
1707 intros n3_22a.
1708 replace ((P  $\rightarrow$  Q)  $\wedge$  (Q  $\rightarrow$  P)) with (P $\leftrightarrow$ Q)
1709   in n3_22a by now rewrite Equiv4_01.
1710 replace ((Q  $\rightarrow$  P)  $\wedge$  (P  $\rightarrow$  Q)) with (Q $\leftrightarrow$ P)
1711   in n3_22a by now rewrite Equiv4_01.
1712 specialize n3_22 with (Q $\rightarrow$ P) (P $\rightarrow$ Q).
1713 intros n3_22b.
1714 replace ((P  $\rightarrow$  Q)  $\wedge$  (Q  $\rightarrow$  P)) with (P $\leftrightarrow$ Q)
1715   in n3_22b by now rewrite Equiv4_01.
1716 replace ((Q  $\rightarrow$  P)  $\wedge$  (P  $\rightarrow$  Q)) with (Q $\leftrightarrow$ P)
1717   in n3_22b by now rewrite Equiv4_01.
1718 Conj n3_22a n3_22b C.
1719 Equiv C.
1720 exact C.
1721 Qed.
1722
1723 Theorem n4_22 :  $\forall$  P Q R : Prop,
1724   ((P  $\leftrightarrow$  Q)  $\wedge$  (Q  $\leftrightarrow$  R))  $\rightarrow$  (P  $\leftrightarrow$  R).
1725 Proof. intros P Q R.
1726 specialize Simp3_26 with (P $\leftrightarrow$ Q) (Q $\leftrightarrow$ R).
1727 intros Simp3_26a.
1728 specialize Simp3_26 with (P $\rightarrow$ Q) (Q $\rightarrow$ P).
1729 intros Simp3_26b.
1730 replace ((P $\rightarrow$ Q)  $\wedge$  (Q $\rightarrow$ P)) with (P $\leftrightarrow$ Q)
1731   in Simp3_26b by now rewrite Equiv4_01.
1732 Syll Simp3_26a Simp3_26b Sa.
1733 specialize Simp3_27 with (P $\leftrightarrow$ Q) (Q $\leftrightarrow$ R).
1734 intros Simp3_27a.
1735 specialize Simp3_26 with (Q $\rightarrow$ R) (R $\rightarrow$ Q).
1736 intros Simp3_26c.
1737 replace ((Q $\rightarrow$ R)  $\wedge$  (R $\rightarrow$ Q)) with (Q $\leftrightarrow$ R)
1738   in Simp3_26c by now rewrite Equiv4_01.
1739 Syll Simp3_27a Simp3_26c Sb.
1740 specialize n2_83 with ((P $\leftrightarrow$ Q) $\wedge$ (Q $\leftrightarrow$ R)) P Q R.
1741 intros n2_83a.
1742 MP n2_83a Sa.
1743 MP n2_83a Sb.
1744 specialize Simp3_27 with (P $\leftrightarrow$ Q) (Q $\leftrightarrow$ R).

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1745   intros Simp3_27b.
1746   specialize Simp3_27 with (Q→R) (R→Q).
1747   intros Simp3_27c.
1748   replace ((Q→R) ∧ (R→Q)) with (Q↔R)
1749     in Simp3_27c by now rewrite Equiv4_01.
1750   Syll Simp3_27b Simp3_27c Sc.
1751   specialize Simp3_26 with (P↔Q) (Q↔R).
1752   intros Simp3_26d.
1753   specialize Simp3_27 with (P→Q) (Q→P).
1754   intros Simp3_27d.
1755   replace ((P→Q) ∧ (Q→P)) with (P↔Q)
1756     in Simp3_27d by now rewrite Equiv4_01.
1757   Syll Simp3_26d Simp3_27d Sd.
1758   specialize n2_83 with ((P↔Q) ∧ (Q↔R)) R Q P.
1759   intros n2_83b.
1760   MP n2_83b Sc. MP n2_83b Sd.
1761   clear Sd. clear Sb. clear Sc. clear Sa. clear Simp3_26a.
1762     clear Simp3_26b. clear Simp3_26c. clear Simp3_26d.
1763     clear Simp3_27a. clear Simp3_27b. clear Simp3_27c.
1764     clear Simp3_27d.
1765   Conj n2_83a n2_83b C.
1766   specialize Comp3_43 with ((P↔Q) ∧ (Q↔R)) (P→R) (R→P).
1767   intros Comp3_43a.
1768   MP Comp3_43a C.
1769   replace ((P→R) ∧ (R→P)) with (P↔R)
1770     in Comp3_43a by now rewrite Equiv4_01.
1771   exact Comp3_43a.
1772 Qed.
1773
1774 Theorem n4_24 : ∀ P : Prop,
1775   P ↔ (P ∧ P).
1776 Proof. intros P.
1777   specialize n3_2 with P P.
1778   intros n3_2a.
1779   specialize n2_43 with P (P ∧ P).
1780   intros n2_43a.
1781   MP n3_2a n2_43a.
1782   specialize Simp3_26 with P P.
1783   intros Simp3_26a.
1784   Conj n2_43a Simp3_26a C.
1785   Equiv C.
1786   exact C.

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1787 Qed.
1788
1789 Theorem n4_25 :  $\forall P : \text{Prop},$ 
1790    $P \leftrightarrow (P \vee P).$ 
1791 Proof. intros P.
1792   specialize Add1_3 with P P.
1793   intros Add1_3a.
1794   specialize Taut1_2 with P.
1795   intros Taut1_2a.
1796   Conj Add1_3a Taut1_2a C.
1797   Equiv C.
1798   exact C.
1799 Qed.
1800
1801 Theorem n4_3 :  $\forall P Q : \text{Prop},$ 
1802    $(P \wedge Q) \leftrightarrow (Q \wedge P).$ 
1803 Proof. intros P Q.
1804   specialize n3_22 with P Q.
1805   intros n3_22a.
1806   specialize n3_22 with Q P.
1807   intros n3_22b.
1808   Conj n3_22a n3_22b C.
1809   Equiv C.
1810   exact C.
1811 Qed.
1812
1813 Theorem n4_31 :  $\forall P Q : \text{Prop},$ 
1814    $(P \vee Q) \leftrightarrow (Q \vee P).$ 
1815 Proof. intros P Q.
1816   specialize Perm1_4 with P Q.
1817   intros Perm1_4a.
1818   specialize Perm1_4 with Q P.
1819   intros Perm1_4b.
1820   Conj Perm1_4a Perm1_4b C.
1821   Equiv C.
1822   exact C.
1823 Qed.
1824
1825 Theorem n4_32 :  $\forall P Q R : \text{Prop},$ 
1826    $((P \wedge Q) \wedge R) \leftrightarrow (P \wedge (Q \wedge R)).$ 
1827 Proof. intros P Q R.
1828   specialize n4_15 with P Q R.

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1829   intros n4_15a.
1830   specialize Transp4_1 with P ( $\neg(Q \wedge R)$ ).
1831   intros Transp4_1a.
1832   apply propositional_extensionality in Transp4_1a.
1833   specialize n4_13 with ( $Q \wedge R$ ).
1834   intros n4_13a.
1835   apply propositional_extensionality in n4_13a.
1836   specialize n4_21 with ( $\neg(P \wedge Q \rightarrow \neg R) \leftrightarrow \neg(P \rightarrow \neg(Q \wedge R))$ )
1837     ( $(P \wedge Q \rightarrow \neg R) \leftrightarrow (P \rightarrow \neg(Q \wedge R))$ ).
1838   intros n4_21a.
1839   apply propositional_extensionality in n4_21a.
1840   replace ( $\neg\neg(Q \wedge R)$ ) with ( $Q \wedge R$ ) in Transp4_1a
1841     by now apply n4_13a.
1842   replace ( $Q \wedge R \rightarrow \neg P$ ) with ( $P \rightarrow \neg(Q \wedge R)$ ) in n4_15a
1843     by now apply Transp4_1a.
1844   specialize Transp4_11 with ( $P \wedge Q \rightarrow \neg R$ ) ( $P \rightarrow \neg(Q \wedge R)$ ).
1845   intros Transp4_11a.
1846   apply propositional_extensionality in Transp4_11a.
1847   replace ( $(P \wedge Q \rightarrow \neg R) \leftrightarrow (P \rightarrow \neg(Q \wedge R))$ ) with
1848     ( $\neg(P \wedge Q \rightarrow \neg R) \leftrightarrow \neg(P \rightarrow \neg(Q \wedge R))$ ) in n4_15a
1849     by now apply Transp4_11a.
1850   replace ( $P \wedge Q \rightarrow \neg R$ ) with
1851     ( $\neg(P \wedge Q) \vee \neg R$ ) in n4_15a
1852     by now rewrite Impl1_01.
1853   replace ( $P \rightarrow \neg(Q \wedge R)$ ) with
1854     ( $\neg P \vee \neg(Q \wedge R)$ ) in n4_15a
1855     by now rewrite Impl1_01.
1856   replace ( $\neg(\neg(P \wedge Q) \vee \neg R)$ ) with
1857     ( $(P \wedge Q) \wedge R$ ) in n4_15a
1858     by now rewrite Prod3_01.
1859   replace ( $\neg(\neg P \vee \neg(Q \wedge R))$ ) with
1860     ( $P \wedge (Q \wedge R)$ ) in n4_15a
1861     by now rewrite Prod3_01.
1862   exact n4_15a.
1863   Qed.
1864   (*Note that the actual proof uses n4_12, but
1865     that transposition involves transforming a
1866     biconditional into a conditional. This citation
1867     of the lemma may be a misprint. Using
1868     Transp4_1 to transpose a conditional and
1869     then applying n4_13 to double negate does
1870     secure the desired formula.*)

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1871
1872 Theorem n4_33 :  $\forall$  P Q R : Prop,
1873   (P  $\vee$  (Q  $\vee$  R))  $\leftrightarrow$  ((P  $\vee$  Q)  $\vee$  R).
1874 Proof. intros P Q R.
1875   specialize n2_31 with P Q R.
1876   intros n2_31a.
1877   specialize n2_32 with P Q R.
1878   intros n2_32a.
1879   Conj n2_31a n2_32a C.
1880   Equiv C.
1881   exact C.
1882 Qed.
1883
1884 Theorem Abb4_34 :  $\forall$  P Q R : Prop,
1885   (P  $\wedge$  Q  $\wedge$  R) = ((P  $\wedge$  Q)  $\wedge$  R).
1886 Proof. intros P Q R.
1887   apply propositional_extensionality.
1888   specialize n4_21 with ((P  $\wedge$  Q)  $\wedge$  R) (P  $\wedge$  Q  $\wedge$  R).
1889   intros n4_21.
1890   replace (((P  $\wedge$  Q)  $\wedge$  R  $\leftrightarrow$  P  $\wedge$  Q  $\wedge$  R)  $\leftrightarrow$  (P  $\wedge$  Q  $\wedge$  R  $\leftrightarrow$  (P  $\wedge$  Q)  $\wedge$  R))
1891     with (((P  $\wedge$  Q)  $\wedge$  R  $\leftrightarrow$  P  $\wedge$  Q  $\wedge$  R)  $\rightarrow$  (P  $\wedge$  Q  $\wedge$  R  $\leftrightarrow$  (P  $\wedge$  Q)  $\wedge$  R))
1892      $\wedge$  ((P  $\wedge$  Q  $\wedge$  R  $\leftrightarrow$  (P  $\wedge$  Q)  $\wedge$  R)  $\rightarrow$  ((P  $\wedge$  Q)  $\wedge$  R  $\leftrightarrow$  P  $\wedge$  Q  $\wedge$  R)))
1893     in n4_21 by now rewrite Equiv4_01.
1894   specialize Simp3_26 with
1895     (((P  $\wedge$  Q)  $\wedge$  R  $\leftrightarrow$  P  $\wedge$  Q  $\wedge$  R)  $\rightarrow$  (P  $\wedge$  Q  $\wedge$  R  $\leftrightarrow$  (P  $\wedge$  Q)  $\wedge$  R))
1896     ((P  $\wedge$  Q  $\wedge$  R  $\leftrightarrow$  (P  $\wedge$  Q)  $\wedge$  R)  $\rightarrow$  ((P  $\wedge$  Q)  $\wedge$  R  $\leftrightarrow$  P  $\wedge$  Q  $\wedge$  R)).
1897   intros Simp3_26.
1898   MP Simp3_26 n4_21.
1899   specialize n4_32 with P Q R.
1900   intros n4_32.
1901   MP Simp3_26 n4_32.
1902   exact Simp3_26.
1903 Qed.
1904
1905 Theorem n4_36 :  $\forall$  P Q R : Prop,
1906   (P  $\leftrightarrow$  Q)  $\rightarrow$  ((P  $\wedge$  R)  $\leftrightarrow$  (Q  $\wedge$  R)).
1907 Proof. intros P Q R.
1908   specialize Fact3_45 with P Q R.
1909   intros Fact3_45a.
1910   specialize Fact3_45 with Q P R.
1911   intros Fact3_45b.
1912   Conj Fact3_45a Fact3_45b C.

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1913 specialize n3_47 with (P→Q) (Q→P)
1914       (P ∧ R → Q ∧ R) (Q ∧ R → P ∧ R).
1915 intros n3_47a.
1916 MP n3_47 C.
1917 replace ((P → Q) ∧ (Q → P)) with (P↔Q) in n3_47a
1918   by now rewrite Equiv4_01.
1919 replace ((P∧R→Q∧R)∧(Q∧R→P∧R)) with (P∧R↔Q∧R)
1920   in n3_47a by now rewrite Equiv4_01.
1921 exact n3_47a.
1922 Qed.
1923
1924 Theorem n4_37 : ∀ P Q R : Prop,
1925   (P ↔ Q) → ((P ∨ R) ↔ (Q ∨ R)).
1926 Proof. intros P Q R.
1927   specialize Sum1_6 with R P Q.
1928   intros Sum1_6a.
1929   specialize Sum1_6 with R Q P.
1930   intros Sum1_6b.
1931   Conj Sum1_6a Sum1_6b C.
1932   specialize n3_47 with (P → Q) (Q → P)
1933     (R ∨ P → R ∨ Q) (R ∨ Q → R ∨ P).
1934   intros n3_47a.
1935   MP n3_47 C.
1936   replace ((P → Q) ∧ (Q → P)) with (P↔Q) in n3_47a
1937     by now rewrite Equiv4_01.
1938   replace ((R∨P→R∨Q)∧(R∨Q→R∨P)) with (R∨P↔R∨Q)
1939     in n3_47a by now rewrite Equiv4_01.
1940   specialize n4_31 with Q R.
1941   intros n4_31a.
1942   apply propositional_extensionality in n4_31a.
1943   specialize n4_31 with P R.
1944   intros n4_31b.
1945   apply propositional_extensionality in n4_31b.
1946   replace (R ∨ P) with (P ∨ R) in n3_47a
1947     by now apply n4_31a.
1948   replace (R ∨ Q) with (Q ∨ R) in n3_47a
1949     by now apply n4_31b.
1950   exact n3_47a.
1951 Qed.
1952
1953 Theorem n4_38 : ∀ P Q R S : Prop,
1954   ((P ↔ R) ∧ (Q ↔ S)) → ((P ∧ Q) ↔ (R ∧ S)).

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1955 Proof. intros P Q R S.
1956   specialize n3_47 with P Q R S.
1957   intros n3_47a.
1958   specialize n3_47 with R S P Q.
1959   intros n3_47b.
1960   Conj n3_47a n3_47b Ca.
1961   specialize n3_47 with ((P→R) ∧ (Q→S))
1962     ((R→P) ∧ (S→Q)) (P ∧ Q → R ∧ S) (R ∧ S → P ∧ Q).
1963   intros n3_47c.
1964   MP n3_47c Ca.
1965   specialize n4_32 with (P→R) (Q→S) ((R→P) ∧ (S → Q)).
1966   intros n4_32a.
1967   apply propositional_extensionality in n4_32a.
1968   replace (((P → R) ∧ (Q → S)) ∧ (R → P) ∧ (S → Q)) with
1969     ((P → R) ∧ (Q → S) ∧ (R → P) ∧ (S → Q)) in n3_47c
1970     by now apply n4_32a.
1971   specialize n4_32 with (Q→S) (R→P) (S → Q).
1972   intros n4_32b.
1973   apply propositional_extensionality in n4_32b.
1974   replace ((Q → S) ∧ (R → P) ∧ (S → Q)) with
1975     (((Q → S) ∧ (R → P)) ∧ (S → Q)) in n3_47c
1976     by now apply n4_32b.
1977   specialize n3_22 with (Q→S) (R→P).
1978   intros n3_22a.
1979   specialize n3_22 with (R→P) (Q→S).
1980   intros n3_22b.
1981   Conj n3_22a n3_22b Cb.
1982   Equiv Cb.
1983   specialize n4_3 with (R→P) (Q→S).
1984   intros n4_3a.
1985   apply propositional_extensionality in n4_3a.
1986   replace ((Q → S) ∧ (R → P)) with
1987     ((R → P) ∧ (Q → S)) in n3_47c
1988     by now apply n4_3a.
1989   specialize n4_32 with (R → P) (Q → S) (S → Q).
1990   intros n4_32c.
1991   apply propositional_extensionality in n4_32c.
1992   replace (((R → P) ∧ (Q → S)) ∧ (S → Q)) with
1993     ((R → P) ∧ (Q → S) ∧ (S → Q)) in n3_47c
1994     by now apply n4_32c.
1995   specialize n4_32 with (P→R) (R → P) ((Q → S) ∧ (S → Q)).
1996   intros n4_32d.

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1997   apply propositional_extensionality in n4_32d.
1998   replace ((P → R) ∧ (R → P) ∧ (Q → S) ∧ (S → Q)) with
1999       ((P → R) ∧ (R → P)) ∧ (Q → S) ∧ (S → Q) in n3_47c
2000       by now apply n4_32d.
2001   replace ((P→R) ∧ (R → P)) with (P↔R) in n3_47c
2002       by now rewrite Equiv4_01.
2003   replace ((Q → S) ∧ (S → Q)) with (Q↔S) in n3_47c
2004       by now rewrite Equiv4_01.
2005   replace ((P↔R)↔(R↔S)) with ((P↔R)↔(R↔S))
2006       in n3_47c by now rewrite Equiv4_01.
2007   exact n3_47c.
2008 Qed.
2009
2010 Theorem n4_39 : ∀ P Q R S : Prop,
2011     ((P ↔ R) ∧ (Q ↔ S)) → ((P ∨ Q) ↔ (R ∨ S)).
2012 Proof. intros P Q R S.
2013     specialize n3_48 with P Q R S.
2014     intros n3_48a.
2015     specialize n3_48 with R S P Q.
2016     intros n3_48b.
2017     Conj n3_48a n3_48b Ca.
2018     specialize n3_47 with ((P → R) ∧ (Q → S))
2019         ((R → P) ∧ (S → Q)) (P ∨ Q → R ∨ S) (R ∨ S → P ∨ Q).
2020     intros n3_47a.
2021     MP n3_47a Ca.
2022     replace ((P↔R)↔(R↔S)) with ((P↔R)↔(R↔S))
2023         in n3_47a by now rewrite Equiv4_01.
2024     specialize n4_32 with ((P → R) ∧ (Q → S)) (R → P) (S → Q).
2025     intros n4_32a.
2026     apply propositional_extensionality in n4_32a.
2027     replace (((P → R) ∧ (Q → S)) ∧ (R → P) ∧ (S → Q)) with
2028         (((P → R) ∧ (Q → S)) ∧ (R → P)) ∧ (S → Q) in n3_47a
2029         by now apply n4_32a.
2030     specialize n4_32 with (P → R) (Q → S) (R → P).
2031     intros n4_32b.
2032     apply propositional_extensionality in n4_32b.
2033     replace (((P → R) ∧ (Q → S)) ∧ (R → P)) with
2034         ((P → R) ∧ (Q → S) ∧ (R → P)) in n3_47a
2035         by now apply n4_32b.
2036     specialize n3_22 with (Q → S) (R → P).
2037     intros n3_22a.
2038     specialize n3_22 with (R → P) (Q → S).

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2039   intros n3_22b.
2040   Conj n3_22a n3_22b Cb.
2041   Equiv Cb.
2042   apply propositional_extensionality in Cb.
2043   replace ((Q → S) ∧ (R → P)) with
2044     ((R → P) ∧ (Q → S)) in n3_47a
2045     by now apply Cb.
2046   specialize n4_32 with (P → R) (R → P) (Q → S).
2047   intros n4_32c.
2048   apply propositional_extensionality in n4_32c.
2049   replace ((P → R) ∧ (R → P) ∧ (Q → S)) with
2050     (((P → R) ∧ (R → P)) ∧ (Q → S)) in n3_47a
2051     by now apply n4_32c.
2052   replace ((P → R) ∧ (R → P)) with (P ↔ R) in n3_47a
2053     by now rewrite Equiv4_01.
2054   specialize n4_32 with (P ↔ R) (Q → S) (S → Q).
2055   intros n4_32d.
2056   apply propositional_extensionality in n4_32d.
2057   replace (((P ↔ R) ∧ (Q → S)) ∧ (S → Q)) with
2058     ((P ↔ R) ∧ (Q → S) ∧ (S → Q)) in n3_47a
2059     by now apply n4_32d.
2060   replace ((Q → S) ∧ (S → Q)) with (Q ↔ S) in n3_47a
2061     by now rewrite Equiv4_01.
2062   exact n3_47a.
2063   Qed.
2064
2065   Theorem n4_4 : ∀ P Q R : Prop,
2066     (P ∧ (Q ∨ R)) ↔ ((P ∧ Q) ∨ (P ∧ R)).
2067   Proof. intros P Q R.
2068     specialize n3_2 with P Q.
2069     intros n3_2a.
2070     specialize n3_2 with P R.
2071     intros n3_2b.
2072     Conj n3_2a n3_2b Ca.
2073     specialize Comp3_43 with P (Q → P ∧ Q) (R → P ∧ R).
2074     intros Comp3_43a.
2075     MP Comp3_43a Ca.
2076     specialize n3_48 with Q R (P ∧ Q) (P ∧ R).
2077     intros n3_48a.
2078     Syll Comp3_43a n3_48a Sa.
2079     specialize Imp3_31 with P (Q ∨ R) ((P ∧ Q) ∨ (P ∧ R)).
2080     intros Imp3_31a.

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2081 MP Imp3_31a Sa.
2082 specialize Simp3_26 with P Q.
2083 intros Simp3_26a.
2084 specialize Simp3_26 with P R.
2085 intros Simp3_26b.
2086 Conj Simp3_26a Simp3_26b Cb.
2087 specialize n3_44 with P (P $\wedge$ Q) (P $\wedge$ R).
2088 intros n3_44a.
2089 MP n3_44a Cb.
2090 specialize Simp3_27 with P Q.
2091 intros Simp3_27a.
2092 specialize Simp3_27 with P R.
2093 intros Simp3_27b.
2094 Conj Simp3_27a Simp3_27b Cc.
2095 specialize n3_48 with (P $\wedge$ Q) (P $\wedge$ R) Q R.
2096 intros n3_48b.
2097 MP n3_48b Cc.
2098 clear Cc. clear Simp3_27a. clear Simp3_27b.
2099 Conj n3_44a n3_48b Cdd. (*Cd is reserved*)
2100 specialize Comp3_43 with (P  $\wedge$  Q  $\vee$  P  $\wedge$  R) P (Q $\vee$ R).
2101 intros Comp3_43b.
2102 MP Comp3_43b Cdd.
2103 clear Cdd. clear Cb. clear n3_44a. clear n3_48b.
2104 clear Simp3_26a. clear Simp3_26b.
2105 Conj Imp3_31a Comp3_43b Ce.
2106 Equiv Ce.
2107 exact Ce.
2108 Qed.
2109
2110 Theorem n4_41 :  $\forall$  P Q R : Prop,
2111   (P  $\vee$  (Q  $\wedge$  R))  $\leftrightarrow$  ((P  $\vee$  Q)  $\wedge$  (P  $\vee$  R)).
2112 Proof. intros P Q R.
2113   specialize Simp3_26 with Q R.
2114   intros Simp3_26a.
2115   specialize Sum1_6 with P (Q  $\wedge$  R) Q.
2116   intros Sum1_6a.
2117   MP Simp3_26a Sum1_6a.
2118   specialize Simp3_27 with Q R.
2119   intros Simp3_27a.
2120   specialize Sum1_6 with P (Q  $\wedge$  R) R.
2121   intros Sum1_6b.
2122   MP Simp3_27a Sum1_6b.

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2123 clear Simp3_26a. clear Simp3_27a.
2124 Conj Sum1_6a Sum1_6a Ca.
2125 specialize Comp3_43 with (P ∨ Q ∧ R) (P ∨ Q) (P ∨ R).
2126 intros Comp3_43a.
2127 MP Comp3_43a Ca.
2128 specialize n2_53 with P Q.
2129 intros n2_53a.
2130 specialize n2_53 with P R.
2131 intros n2_53b.
2132 Conj n2_53a n2_53b Cb.
2133 specialize n3_47 with (P ∨ Q) (P ∨ R) (¬P → Q) (¬P → R).
2134 intros n3_47a.
2135 MP n3_47a Cb.
2136 specialize Comp3_43 with (¬P) Q R.
2137 intros Comp3_43b.
2138 Syll n3_47a Comp3_43b Sa.
2139 specialize n2_54 with P (Q ∧ R).
2140 intros n2_54a.
2141 Syll Sa n2_54a Sb.
2142 clear Sum1_6a. clear Sum1_6b. clear Ca. clear n2_53a.
2143 clear n2_53b. clear Cb. clear n3_47a. clear Sa.
2144 clear Comp3_43b. clear n2_54a.
2145 Conj Comp3_43a Sb Cc.
2146 Equiv Cc.
2147 exact Cc.
2148 Qed.
2149
2150 Theorem n4_42 : ∀ P Q : Prop,
2151   P ↔ ((P ∧ Q) ∨ (P ∧ ¬Q)).
2152 Proof. intros P Q.
2153   specialize n3_21 with P (Q ∨ ¬Q).
2154   intros n3_21a.
2155   specialize n2_11 with Q.
2156   intros n2_11a.
2157   MP n3_21a n2_11a.
2158   specialize Simp3_26 with P (Q ∨ ¬Q).
2159   intros Simp3_26a. clear n2_11a.
2160   Conj n3_21a Simp3_26a C.
2161   Equiv C.
2162   specialize n4_4 with P Q (¬Q).
2163   intros n4_4a.
2164   apply propositional_extensionality in C.

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2165   replace (P ∧ (Q ∨ ¬Q)) with P in n4_4a
2166   by now apply C.
2167   exact n4_4a.
2168 Qed.
2169
2170 Theorem n4_43 : ∀ P Q : Prop,
2171   P ↔ ((P ∨ Q) ∧ (P ∨ ¬Q)).
2172 Proof. intros P Q.
2173   specialize n2_2 with P Q.
2174   intros n2_2a.
2175   specialize n2_2 with P (¬Q).
2176   intros n2_2b.
2177   Conj n2_2a n2_2b Ca.
2178   specialize Comp3_43 with P (P ∨ Q) (P ∨ ¬Q).
2179   intros Comp3_43a.
2180   MP Comp3_43a Ca.
2181   specialize n2_53 with P Q.
2182   intros n2_53a.
2183   specialize n2_53 with P (¬Q).
2184   intros n2_53b.
2185   Conj n2_53a n2_53b Cb.
2186   specialize n3_47 with (P ∨ Q) (P ∨ ¬Q) (¬P → Q) (¬P → ¬Q).
2187   intros n3_47a.
2188   MP n3_47a Cb.
2189   specialize n2_65 with (¬P) Q.
2190   intros n2_65a.
2191   specialize n4_13 with P.
2192   intros n4_13a.
2193   apply propositional_extensionality in n4_13a.
2194   replace (¬¬P) with P in n2_65a by now apply n4_13a.
2195   specialize Imp3_31 with (¬P → Q) (¬P → ¬Q) (P).
2196   intros Imp3_31a.
2197   MP Imp3_31a n2_65a.
2198   Syll n3_47a Imp3_31a Sa.
2199   clear n2_2a. clear n2_2b. clear Ca. clear n2_53a.
2200   clear n2_53b. clear Cb. clear n2_65a.
2201   clear n3_47a. clear Imp3_31a. clear n4_13a.
2202   Conj Comp3_43a Sa Cc.
2203   Equiv Cc.
2204   exact Cc.
2205 Qed.
2206

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2207 Theorem n4_44 :  $\forall$  P Q : Prop,
2208   P  $\leftrightarrow$  (P  $\vee$  (P  $\wedge$  Q)).
2209 Proof. intros P Q.
2210   specialize n2_2 with P (P $\wedge$ Q).
2211   intros n2_2a.
2212   specialize Id2_08 with P.
2213   intros Id2_08a.
2214   specialize Simp3_26 with P Q.
2215   intros Simp3_26a.
2216   Conj Id2_08a Simp3_26a Ca.
2217   specialize n3_44 with P P (P  $\wedge$  Q).
2218   intros n3_44a.
2219   MP n3_44a Ca.
2220   clear Ca. clear Id2_08a. clear Simp3_26a.
2221   Conj n2_2a n3_44a Cb.
2222   Equiv Cb.
2223   exact Cb.
2224 Qed.
2225
2226 Theorem n4_45 :  $\forall$  P Q : Prop,
2227   P  $\leftrightarrow$  (P  $\wedge$  (P  $\vee$  Q)).
2228 Proof. intros P Q.
2229   specialize n2_2 with (P  $\wedge$  P) (P  $\wedge$  Q).
2230   intros n2_2a.
2231   specialize n4_4 with P P Q.
2232   intros n4_4a.
2233   apply propositional_extensionality in n4_4a.
2234   replace (P $\wedge$ P $\vee$ P $\wedge$ Q) with (P $\wedge$ (P $\vee$ Q)) in n2_2a
2235     by now apply n4_4a.
2236   specialize n4_24 with P.
2237   intros n4_24a.
2238   apply propositional_extensionality in n4_24a.
2239   replace (P  $\wedge$  P) with P in n2_2a
2240     by now apply n4_24a.
2241   specialize Simp3_26 with P (P  $\vee$  Q).
2242   intros Simp3_26a.
2243   clear n4_4a. clear n4_24a.
2244   Conj n2_2a Simp3_26a C.
2245   Equiv C.
2246   exact C.
2247 Qed.
2248

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2249 Theorem n4_5 :  $\forall P Q : \text{Prop},$ 
2250    $P \wedge Q \leftrightarrow \neg(\neg P \vee \neg Q).$ 
2251 Proof. intros P Q.
2252   specialize n4_2 with (P  $\wedge$  Q).
2253   intros n4_2a.
2254   replace ((P  $\wedge$  Q)  $\leftrightarrow$  (P  $\wedge$  Q)) with
2255     ((P  $\wedge$  Q)  $\leftrightarrow$   $\neg(\neg P \vee \neg Q)$ ) in n4_2a
2256     by now rewrite Prod3_01.
2257   exact n4_2a.
2258 Qed.
2259
2260 Theorem n4_51 :  $\forall P Q : \text{Prop},$ 
2261    $\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q).$ 
2262 Proof. intros P Q.
2263   specialize n4_5 with P Q.
2264   intros n4_5a.
2265   specialize n4_12 with (P  $\wedge$  Q) ( $\neg P \vee \neg Q$ ).
2266   intros n4_12a.
2267   specialize Simp3_26 with
2268     ((P  $\wedge$  Q  $\leftrightarrow$   $\neg(\neg P \vee \neg Q)$ )  $\rightarrow$  ( $\neg P \vee \neg Q \leftrightarrow \neg(P \wedge Q)$ ))
2269     (( $\neg P \vee \neg Q \leftrightarrow \neg(P \wedge Q)$ )  $\rightarrow$  ((P  $\wedge$  Q  $\leftrightarrow$   $\neg(\neg P \vee \neg Q)$ ))).
2270   intros Simp3_26a.
2271   replace ((P  $\wedge$  Q  $\leftrightarrow$   $\neg(\neg P \vee \neg Q)$ )  $\leftrightarrow$  ( $\neg P \vee \neg Q \leftrightarrow \neg(P \wedge Q)$ ))
2272     with (((P  $\wedge$  Q  $\leftrightarrow$   $\neg(\neg P \vee \neg Q)$ )  $\rightarrow$  ( $\neg P \vee \neg Q \leftrightarrow \neg(P \wedge Q)$ ))
2273        $\wedge$  (( $\neg P \vee \neg Q \leftrightarrow \neg(P \wedge Q)$ )  $\rightarrow$  ((P  $\wedge$  Q  $\leftrightarrow$   $\neg(\neg P \vee \neg Q)$ )))
2274     in n4_12a by now rewrite Equiv4_01.
2275   MP Simp3_26a n4_12a.
2276   MP Simp3_26a n4_5a.
2277   specialize n4_21 with ( $\neg(P \wedge Q)$ ) ( $\neg P \vee \neg Q$ ).
2278   intros n4_21a.
2279   specialize Simp3_27 with
2280     ((( $\neg(P \wedge Q) \leftrightarrow \neg P \vee \neg Q$ )  $\rightarrow$  (( $\neg P \vee \neg Q \leftrightarrow \neg(P \wedge Q)$ )))
2281     ((( $\neg P \vee \neg Q \leftrightarrow \neg(P \wedge Q)$ )  $\rightarrow$  (( $\neg(P \wedge Q) \leftrightarrow \neg P \vee \neg Q$ )))).
2282   intros Simp3_27a.
2283   MP Simp3_27a n4_21a.
2284   MP Simp3_27a Simp3_26a.
2285   exact Simp3_27a.
2286 Qed.
2287
2288 Theorem n4_52 :  $\forall P Q : \text{Prop},$ 
2289   (P  $\wedge$   $\neg Q$ )  $\leftrightarrow \neg(\neg P \vee Q).$ 
2290 Proof. intros P Q.

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2291 specialize n4_5 with P (¬Q).
2292 intros n4_5a.
2293 specialize n4_13 with Q.
2294 intros n4_13a.
2295 apply propositional_extensionality in n4_13a.
2296 replace (¬¬Q) with Q in n4_5a
2297   by now apply n4_13a.
2298 exact n4_5a.
2299 Qed.
2300
2301 Theorem n4_53 : ∀ P Q : Prop,
2302   ¬(P ∧ ¬Q) ↔ (¬P ∨ Q).
2303 Proof. intros P Q.
2304   specialize n4_52 with P Q.
2305   intros n4_52a.
2306   specialize n4_12 with (P ∧ ¬Q) ((¬P ∨ Q)).
2307   intros n4_12a.
2308   replace ((P ∧ ¬Q ↔ ¬(¬P ∨ Q)) ↔ (¬P ∨ Q ↔ ¬(P ∧ ¬Q)))
2309     with (((P ∧ ¬Q ↔ ¬(¬P ∨ Q)) → (¬P ∨ Q ↔ ¬(P ∧ ¬Q)))
2310       ∧ ((¬P ∨ Q ↔ ¬(P ∧ ¬Q)) → (P ∧ ¬Q ↔ ¬(¬P ∨ Q))))
2311     in n4_12a by now rewrite Equiv4_01.
2312   specialize Simp3_26 with
2313     ((P ∧ ¬Q ↔ ¬(¬P ∨ Q)) → (¬P ∨ Q ↔ ¬(P ∧ ¬Q)))
2314     ((¬P ∨ Q ↔ ¬(P ∧ ¬Q)) → (P ∧ ¬Q ↔ ¬(¬P ∨ Q))).
2315   intros Simp3_26a.
2316   MP Simp3_26a n4_12a.
2317   MP Simp3_26a n4_52a.
2318   specialize n4_21 with (¬(P ∧ ¬Q)) (¬P ∨ Q).
2319   intros n4_21a.
2320   replace ((¬(P ∧ ¬Q) ↔ ¬P ∨ Q) ↔ (¬P ∨ Q ↔ ¬(P ∧ ¬Q)))
2321     with (((¬(P ∧ ¬Q) ↔ ¬P ∨ Q) → (¬P ∨ Q ↔ ¬(P ∧ ¬Q)))
2322       ∧ ((¬P ∨ Q ↔ ¬(P ∧ ¬Q)) → (¬(P ∧ ¬Q) ↔ ¬P ∨ Q)))
2323     in n4_21a by now rewrite Equiv4_01.
2324   specialize Simp3_27 with
2325     ((¬(P ∧ ¬Q) ↔ ¬P ∨ Q) → (¬P ∨ Q ↔ ¬(P ∧ ¬Q)))
2326     ((¬P ∨ Q ↔ ¬(P ∧ ¬Q)) → (¬(P ∧ ¬Q) ↔ ¬P ∨ Q)).
2327   intros Simp3_27a.
2328   MP Simp3_27a n4_21a.
2329   MP Simp3_27a Simp3_26a.
2330   exact Simp3_27a.
2331 Qed.
2332

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2333 Theorem n4_54 :  $\forall$  P Q : Prop,
2334    $(\neg P \wedge Q) \leftrightarrow \neg(P \vee \neg Q)$ .
2335 Proof. intros P Q.
2336   specialize n4_5 with  $(\neg P)$  Q.
2337   intros n4_5a.
2338   specialize n4_13 with P.
2339   intros n4_13a.
2340   apply propositional_extensionality in n4_13a.
2341   replace  $(\neg\neg P)$  with P in n4_5a
2342     by now apply n4_13a.
2343   exact n4_5a.
2344 Qed.
2345
2346 Theorem n4_55 :  $\forall$  P Q : Prop,
2347    $\neg(\neg P \wedge Q) \leftrightarrow (P \vee \neg Q)$ .
2348 Proof. intros P Q.
2349   specialize n4_54 with P Q.
2350   intros n4_54a.
2351   specialize n4_12 with  $(\neg P \wedge Q)$   $(P \vee \neg Q)$ .
2352   intros n4_12a.
2353   apply propositional_extensionality in n4_12a.
2354   replace  $(\neg P \wedge Q \leftrightarrow \neg(P \vee \neg Q))$  with
2355      $(P \vee \neg Q \leftrightarrow \neg(\neg P \wedge Q))$  in n4_54a
2356     by now apply n4_12a.
2357   specialize n4_21 with  $(\neg(\neg P \wedge Q))$   $(P \vee \neg Q)$ .
2358   intros n4_21a. (*Not cited*)
2359   apply propositional_extensionality in n4_21a.
2360   replace  $(P \vee \neg Q \leftrightarrow \neg(\neg P \wedge Q))$  with
2361      $(\neg(\neg P \wedge Q) \leftrightarrow (P \vee \neg Q))$  in n4_54a
2362     by now apply n4_21a.
2363   exact n4_54a.
2364 Qed.
2365
2366 Theorem n4_56 :  $\forall$  P Q : Prop,
2367    $(\neg P \wedge \neg Q) \leftrightarrow \neg(P \vee Q)$ .
2368 Proof. intros P Q.
2369   specialize n4_54 with P  $(\neg Q)$ .
2370   intros n4_54a.
2371   specialize n4_13 with Q.
2372   intros n4_13a.
2373   apply propositional_extensionality in n4_13a.
2374   replace  $(\neg\neg Q)$  with Q in n4_54a

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2375         by now apply n4_13a.
2376     exact n4_54a.
2377 Qed.
2378
2379 Theorem n4_57 :  $\forall P Q : \text{Prop},$ 
2380    $\neg(\neg P \wedge \neg Q) \leftrightarrow (P \vee Q).$ 
2381 Proof. intros P Q.
2382     specialize n4_56 with P Q.
2383     intros n4_56a.
2384     specialize n4_12 with  $(\neg P \wedge \neg Q) (P \vee Q).$ 
2385     intros n4_12a.
2386     apply propositional_extensionality in n4_12a.
2387     replace  $(\neg P \wedge \neg Q \leftrightarrow \neg(P \vee Q))$  with
2388        $(P \vee Q \leftrightarrow \neg(\neg P \wedge \neg Q))$  in n4_56a
2389     by now apply n4_12a.
2390     specialize n4_21 with  $(\neg(\neg P \wedge \neg Q)) (P \vee Q).$ 
2391     intros n4_21a.
2392     apply propositional_extensionality in n4_21a.
2393     replace  $(P \vee Q \leftrightarrow \neg(\neg P \wedge \neg Q))$  with
2394        $(\neg(\neg P \wedge \neg Q) \leftrightarrow P \vee Q)$  in n4_56a
2395     by now apply n4_21a.
2396     exact n4_56a.
2397 Qed.
2398
2399 Theorem n4_6 :  $\forall P Q : \text{Prop},$ 
2400    $(P \rightarrow Q) \leftrightarrow (\neg P \vee Q).$ 
2401 Proof. intros P Q.
2402     specialize n4_2 with  $(\neg P \vee Q).$ 
2403     intros n4_2a.
2404     rewrite Impl1_01.
2405     exact n4_2a.
2406 Qed.
2407
2408 Theorem n4_61 :  $\forall P Q : \text{Prop},$ 
2409    $\neg(P \rightarrow Q) \leftrightarrow (P \wedge \neg Q).$ 
2410 Proof. intros P Q.
2411     specialize n4_6 with P Q.
2412     intros n4_6a.
2413     specialize Transp4_11 with  $(P \rightarrow Q) (\neg P \vee Q).$ 
2414     intros Transp4_11a.
2415     apply propositional_extensionality in Transp4_11a.
2416     replace  $((P \rightarrow Q) \leftrightarrow \neg P \vee Q)$  with

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2417      ( $\neg(P \rightarrow Q) \leftrightarrow \neg(\neg P \vee Q)$ ) in n4_6a
2418      by now apply Transp4_11a.
2419      specialize n4_52 with P Q.
2420      intros n4_52a.
2421      apply propositional_extensionality in n4_52a.
2422      replace ( $\neg(\neg P \vee Q)$ ) with ( $P \wedge \neg Q$ ) in n4_6a
2423      by now apply n4_52a.
2424      exact n4_6a.
2425      Qed.
2426
2427      Theorem n4_62 :  $\forall P Q : \text{Prop}$ ,
2428      ( $P \rightarrow \neg Q$ )  $\leftrightarrow$  ( $\neg P \vee \neg Q$ ).
2429      Proof. intros P Q.
2430      specialize n4_6 with P ( $\neg Q$ ).
2431      intros n4_6a.
2432      exact n4_6a.
2433      Qed.
2434
2435      Theorem n4_63 :  $\forall P Q : \text{Prop}$ ,
2436       $\neg(P \rightarrow \neg Q) \leftrightarrow (P \wedge Q)$ .
2437      Proof. intros P Q.
2438      specialize n4_62 with P Q.
2439      intros n4_62a.
2440      specialize Transp4_11 with ( $P \rightarrow \neg Q$ ) ( $\neg P \vee \neg Q$ ).
2441      intros Transp4_11a.
2442      specialize n4_5 with P Q.
2443      intros n4_5a.
2444      apply propositional_extensionality in n4_5a.
2445      replace ( $\neg(\neg P \vee \neg Q)$ ) with ( $P \wedge Q$ ) in Transp4_11a
2446      by now apply n4_5a.
2447      apply propositional_extensionality in Transp4_11a.
2448      replace (( $P \rightarrow \neg Q$ )  $\leftrightarrow$   $\neg P \vee \neg Q$ ) with
2449      (( $\neg(P \rightarrow \neg Q) \leftrightarrow P \wedge Q$ )) in n4_62a
2450      by now apply Transp4_11a.
2451      exact n4_62a.
2452      Qed.
2453      (*One could use Prod3_01 in lieu of n4_5.*)
2454
2455      Theorem n4_64 :  $\forall P Q : \text{Prop}$ ,
2456      ( $\neg P \rightarrow Q$ )  $\leftrightarrow$  ( $P \vee Q$ ).
2457      Proof. intros P Q.
2458      specialize n2_54 with P Q.

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2459     intros n2_54a.
2460     specialize n2_53 with P Q.
2461     intros n2_53a.
2462     Conj n2_54a n2_53a C.
2463     Equiv C.
2464     exact C.
2465 Qed.
2466
2467 Theorem n4_65 :  $\forall P Q : \text{Prop},$ 
2468    $\neg(\neg P \rightarrow Q) \leftrightarrow (\neg P \wedge \neg Q).$ 
2469 Proof. intros P Q.
2470 specialize n4_64 with P Q.
2471 intros n4_64a.
2472 specialize Transp4_11 with  $(\neg P \rightarrow Q) (P \vee Q).$ 
2473 intros Transp4_11a.
2474 specialize n4_21 with  $(\neg(\neg P \rightarrow Q) \leftrightarrow \neg(P \vee Q))$ 
2475    $((\neg P \rightarrow Q) \leftrightarrow (P \vee Q)).$ 
2476 intros n4_21a.
2477 apply propositional_extensionality in n4_21a.
2478 replace  $((\neg P \rightarrow Q) \leftrightarrow P \vee Q) \leftrightarrow (\neg(\neg P \rightarrow Q) \leftrightarrow \neg(P \vee Q))$  with
2479    $((\neg(\neg P \rightarrow Q) \leftrightarrow \neg(P \vee Q)) \leftrightarrow ((\neg P \rightarrow Q) \leftrightarrow P \vee Q))$  in Transp4_11a
2480   by now apply n4_21a.
2481 apply propositional_extensionality in Transp4_11a.
2482 replace  $((\neg P \rightarrow Q) \leftrightarrow P \vee Q)$  with
2483    $(\neg(\neg P \rightarrow Q) \leftrightarrow \neg(P \vee Q))$  in n4_64a
2484   by now apply Transp4_11a.
2485 specialize n4_56 with P Q.
2486 intros n4_56a.
2487 apply propositional_extensionality in n4_56a.
2488 replace  $(\neg(P \vee Q))$  with  $(\neg P \wedge \neg Q)$  in n4_64a
2489   by now apply n4_56a.
2490 exact n4_64a.
2491 Qed.
2492
2493 Theorem n4_66 :  $\forall P Q : \text{Prop},$ 
2494    $(\neg P \rightarrow \neg Q) \leftrightarrow (P \vee \neg Q).$ 
2495 Proof. intros P Q.
2496 specialize n4_64 with P  $(\neg Q).$ 
2497 intros n4_64a.
2498 exact n4_64a.
2499 Qed.
2500

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2501 Theorem n4_67 :  $\forall$  P Q : Prop,
2502    $\neg(\neg P \rightarrow \neg Q) \leftrightarrow (\neg P \wedge Q)$ .
2503 Proof. intros P Q.
2504 specialize n4_66 with P Q.
2505 intros n4_66a.
2506 specialize Transp4_11 with  $(\neg P \rightarrow \neg Q) (P \vee \neg Q)$ .
2507 intros Transp4_11a.
2508 apply propositional_extensionality in Transp4_11a.
2509 replace  $((\neg P \rightarrow \neg Q) \leftrightarrow P \vee \neg Q)$  with
2510    $(\neg(\neg P \rightarrow \neg Q) \leftrightarrow \neg(P \vee \neg Q))$  in n4_66a
2511   by now apply Transp4_11a.
2512 specialize n4_54 with P Q.
2513 intros n4_54a.
2514 apply propositional_extensionality in n4_54a.
2515 replace  $(\neg(P \vee \neg Q))$  with  $(\neg P \wedge Q)$  in n4_66a
2516   by now apply n4_54a.
2517 exact n4_66a.
2518 Qed.
2519
2520 Theorem n4_7 :  $\forall$  P Q : Prop,
2521    $(P \rightarrow Q) \leftrightarrow (P \rightarrow (P \wedge Q))$ .
2522 Proof. intros P Q.
2523 specialize Comp3_43 with P P Q.
2524 intros Comp3_43a.
2525 specialize Exp3_3 with
2526    $(P \rightarrow P) (P \rightarrow Q) (P \rightarrow P \wedge Q)$ .
2527 intros Exp3_3a.
2528 MP Exp3_3a Comp3_43a.
2529 specialize Id2_08 with P.
2530 intros Id2_08a.
2531 MP Exp3_3a Id2_08a.
2532 specialize Simp3_27 with P Q.
2533 intros Simp3_27a.
2534 specialize Syll2_05 with P  $(P \wedge Q)$  Q.
2535 intros Syll2_05a.
2536 MP Syll2_05a Simp3_27a.
2537 clear Id2_08a. clear Comp3_43a. clear Simp3_27a.
2538 Conj Syll2_05a Exp3_3a C.
2539 Equiv C.
2540 exact C.
2541 Qed.
2542

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2543 Theorem n4_71 :  $\forall$  P Q : Prop,
2544   (P  $\rightarrow$  Q)  $\leftrightarrow$  (P  $\leftrightarrow$  (P  $\wedge$  Q)).
2545 Proof. intros P Q.
2546 specialize n4_7 with P Q.
2547 intros n4_7a.
2548 specialize n3_21 with (P $\rightarrow$ (P $\wedge$ Q)) ((P $\wedge$ Q) $\rightarrow$ P).
2549 intros n3_21a.
2550 replace ((P $\rightarrow$ P $\wedge$ Q) $\wedge$ (P $\wedge$ Q $\rightarrow$ P)) with (P $\leftrightarrow$ (P $\wedge$ Q))
2551   in n3_21a by now rewrite Equiv4_01.
2552 specialize Simp3_26 with P Q.
2553 intros Simp3_26a.
2554 MP n3_21a Simp3_26a.
2555 specialize Simp3_26 with (P $\rightarrow$ (P $\wedge$ Q)) ((P $\wedge$ Q) $\rightarrow$ P).
2556 intros Simp3_26b.
2557 replace ((P $\rightarrow$ P $\wedge$ Q) $\wedge$ (P $\wedge$ Q $\rightarrow$ P)) with (P $\leftrightarrow$ (P $\wedge$ Q))
2558   in Simp3_26b by now rewrite Equiv4_01.
2559 clear Simp3_26a.
2560 Conj n3_21a Simp3_26b Ca.
2561 Equiv Ca.
2562 clear n3_21a. clear Simp3_26b.
2563 Conj n4_7a Ca Cb.
2564 specialize n4_22 with (P  $\rightarrow$  Q) (P  $\rightarrow$  P  $\wedge$  Q) (P  $\leftrightarrow$  P  $\wedge$  Q).
2565 intros n4_22a.
2566 MP n4_22a Cb.
2567 exact n4_22a.
2568 Qed.
2569
2570 Theorem n4_72 :  $\forall$  P Q : Prop,
2571   (P  $\rightarrow$  Q)  $\leftrightarrow$  (Q  $\leftrightarrow$  (P  $\vee$  Q)).
2572 Proof. intros P Q.
2573 specialize Transp4_1 with P Q.
2574 intros Transp4_1a.
2575 specialize n4_71 with ( $\neg$ Q) ( $\neg$ P).
2576 intros n4_71a.
2577 Conj Transp4_1a n4_71a Ca.
2578 specialize n4_22 with
2579   (P $\rightarrow$ Q) ( $\neg$ Q $\rightarrow$  $\neg$ P) ( $\neg$ Q $\leftrightarrow$  $\neg$ Q  $\wedge$   $\neg$ P).
2580 intros n4_22a.
2581 MP n4_22a Ca.
2582 specialize n4_21 with ( $\neg$ Q) ( $\neg$ Q  $\wedge$   $\neg$ P).
2583 intros n4_21a.
2584 Conj n4_22a n4_21a Cb.

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2585 specialize n4_22 with
2586   (P→Q) (¬Q ↔ ¬Q ∧ ¬P) (¬Q ∧ ¬P ↔ ¬Q).
2587 intros n4_22b.
2588 MP n4_22b Cb.
2589 specialize n4_12 with (¬Q ∧ ¬P) (Q).
2590 intros n4_12a.
2591 Conj n4_22b n4_12a Cc.
2592 specialize n4_22 with
2593   (P → Q) ((¬Q ∧ ¬P) ↔ ¬Q) (Q ↔ ¬(¬Q ∧ ¬P)).
2594 intros n4_22c.
2595 MP n4_22b Cc.
2596 specialize n4_57 with Q P.
2597 intros n4_57a.
2598 apply propositional_extensionality in n4_57a.
2599 replace (¬(¬Q ∧ ¬P)) with (Q ∨ P) in n4_22c
2600   by now apply n4_57a.
2601 specialize n4_31 with P Q.
2602 intros n4_31a.
2603 apply propositional_extensionality in n4_31a.
2604 replace (Q ∨ P) with (P ∨ Q) in n4_22c
2605   by now apply n4_22c.
2606 exact n4_22c.
2607 Qed.
2608 (*One could use Prod3_01 in lieu of n4_57.*)
2609
2610 Theorem n4_73 : ∀ P Q : Prop,
2611   Q → (P ↔ (P ∧ Q)).
2612 Proof. intros P Q.
2613 specialize Simp2_02 with P Q.
2614 intros Simp2_02a.
2615 specialize n4_71 with P Q.
2616 intros n4_71a.
2617 replace ((P → Q) ↔ (P ↔ P ∧ Q)) with
2618   (((P→Q)→(P↔P∧Q))∧((P↔P∧Q)→(P→Q)))
2619   in n4_71a by now rewrite Equiv4_01.
2620 specialize Simp3_26 with
2621   ((P → Q) → P ↔ P ∧ Q) (P ↔ P ∧ Q → P → Q).
2622 intros Simp3_26a.
2623 MP Simp3_26a n4_71a.
2624 Syll Simp2_02a Simp3_26a Sa.
2625 exact Sa.
2626 Qed.

```

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2627
2628 Theorem n4_74 :  $\forall P Q : \text{Prop},$ 
2629    $\neg P \rightarrow (Q \leftrightarrow (P \vee Q)).$ 
2630 Proof. intros P Q.
2631   specialize n2_21 with P Q.
2632   intros n2_21a.
2633   specialize n4_72 with P Q.
2634   intros n4_72a.
2635   apply propositional_extensionality in n4_72a.
2636   replace (P  $\rightarrow$  Q) with (Q  $\leftrightarrow$  P  $\vee$  Q) in n2_21a
2637     by now apply n4_72a.
2638   exact n2_21a.
2639 Qed.
2640
2641 Theorem n4_76 :  $\forall P Q R : \text{Prop},$ 
2642    $((P \rightarrow Q) \wedge (P \rightarrow R)) \leftrightarrow (P \rightarrow (Q \wedge R)).$ 
2643 Proof. intros P Q R.
2644   specialize n4_41 with ( $\neg P$ ) Q R.
2645   intros n4_41a.
2646   replace ( $\neg P \vee Q$ ) with (P $\rightarrow$ Q) in n4_41a
2647     by now rewrite Impl1_01.
2648   replace ( $\neg P \vee R$ ) with (P $\rightarrow$ R) in n4_41a
2649     by now rewrite Impl1_01.
2650   replace ( $\neg P \vee Q \wedge R$ ) with (P  $\rightarrow$  Q  $\wedge$  R) in n4_41a
2651     by now rewrite Impl1_01.
2652   specialize n4_21 with  $((P \rightarrow Q) \wedge (P \rightarrow R))$  (P  $\rightarrow$  Q  $\wedge$  R).
2653   intros n4_21a.
2654   apply propositional_extensionality in n4_21a.
2655   replace  $((P \rightarrow Q \wedge R) \leftrightarrow (P \rightarrow Q) \wedge (P \rightarrow R))$  with
2656      $((P \rightarrow Q) \wedge (P \rightarrow R) \leftrightarrow (P \rightarrow Q \wedge R))$  in n4_41a
2657     by now apply n4_41a.
2658   exact n4_41a.
2659 Qed.
2660
2661 Theorem n4_77 :  $\forall P Q R : \text{Prop},$ 
2662    $((Q \rightarrow P) \wedge (R \rightarrow P)) \leftrightarrow ((Q \vee R) \rightarrow P).$ 
2663 Proof. intros P Q R.
2664   specialize n3_44 with P Q R.
2665   intros n3_44a.
2666   specialize n2_2 with Q R.
2667   intros n2_2a.
2668   specialize Add1_3 with Q R.

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2669   intros Add1_3a.
2670   specialize Syll2_06 with Q (Q ∨ R) P.
2671   intros Syll2_06a.
2672   MP Syll2_06a n2_2a.
2673   specialize Syll2_06 with R (Q ∨ R) P.
2674   intros Syll2_06b.
2675   MP Syll2_06b Add1_3a.
2676   Conj Syll2_06a Syll2_06b Ca.
2677   specialize Comp3_43 with ((Q ∨ R) → P)
2678     (Q → P) (R → P).
2679   intros Comp3_43a.
2680   MP Comp3_43a Ca.
2681   clear n2_2a. clear Add1_3a. clear Ca.
2682   clear Syll2_06a. clear Syll2_06b.
2683   Conj n3_44a Comp3_43a Cb.
2684   Equiv Cb.
2685   exact Cb.
2686   Qed.
2687
2688   Theorem n4_78 : ∀ P Q R : Prop,
2689     ((P → Q) ∨ (P → R)) ↔ (P → (Q ∨ R)).
2690   Proof. intros P Q R.
2691   specialize n4_2 with ((P → Q) ∨ (P → R)).
2692   intros n4_2a.
2693   replace (((P → Q) ∨ (P → R)) ↔ ((P → Q) ∨ (P → R))) with
2694     (((P → Q) ∨ (P → R)) ↔ ((P → Q) ∨ ¬P ∨ R)) in n4_2a
2695     by now rewrite <- Impl1_01.
2696   replace (((P → Q) ∨ (P → R)) ↔ ((P → Q) ∨ ¬P ∨ R)) with
2697     (((P → Q) ∨ (P → R)) ↔ ((¬P ∨ Q) ∨ ¬P ∨ R)) in n4_2a
2698     by now rewrite <- Impl1_01.
2699   specialize n4_33 with (¬P) Q (¬P ∨ R).
2700   intros n4_33a.
2701   apply propositional_extensionality in n4_33a.
2702   replace ((¬P ∨ Q) ∨ ¬P ∨ R) with
2703     (¬P ∨ Q ∨ ¬P ∨ R) in n4_2a
2704     by now apply n4_33a.
2705   specialize n4_33 with Q (¬P) R.
2706   intros n4_33b.
2707   apply propositional_extensionality in n4_33b.
2708   replace (Q ∨ ¬P ∨ R) with
2709     ((Q ∨ ¬P) ∨ R) in n4_2a
2710     by now apply n4_33b.

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2711 specialize n4_31 with ( $\neg$ P) Q.
2712 intros n4_31a.
2713 specialize n4_37 with ( $\neg$ P $\vee$ Q) (Q  $\vee$   $\neg$ P) R.
2714 intros n4_37a.
2715 MP n4_37a n4_31a.
2716 apply propositional_extensionality in n4_37a.
2717 replace ((Q  $\vee$   $\neg$ P)  $\vee$  R) with
2718   (( $\neg$ P  $\vee$  Q)  $\vee$  R) in n4_2a
2719   by now apply n4_37a.
2720 specialize n4_33 with ( $\neg$ P) ( $\neg$ P $\vee$ Q) R.
2721 intros n4_33c.
2722 apply propositional_extensionality in n4_33c.
2723 replace ( $\neg$ P  $\vee$  ( $\neg$ P  $\vee$  Q)  $\vee$  R) with
2724   (( $\neg$ P  $\vee$  ( $\neg$ P  $\vee$  Q))  $\vee$  R) in n4_2a
2725   by now apply n4_33c.
2726 specialize n4_33 with ( $\neg$ P) ( $\neg$ P) Q.
2727 intros n4_33d.
2728 apply propositional_extensionality in n4_33d.
2729 replace ( $\neg$ P  $\vee$   $\neg$ P  $\vee$  Q) with
2730   (( $\neg$ P  $\vee$   $\neg$ P)  $\vee$  Q) in n4_2a
2731   by now apply n4_33d.
2732 specialize n4_33 with ( $\neg$ P  $\vee$   $\neg$ P) Q R.
2733 intros n4_33e.
2734 apply propositional_extensionality in n4_33e.
2735 replace ((( $\neg$ P  $\vee$   $\neg$ P)  $\vee$  Q)  $\vee$  R) with
2736   (( $\neg$ P  $\vee$   $\neg$ P)  $\vee$  Q  $\vee$  R) in n4_2a
2737   by now apply n4_33e.
2738 specialize n4_25 with ( $\neg$ P).
2739 intros n4_25a.
2740 specialize n4_37 with
2741   ( $\neg$ P) ( $\neg$ P  $\vee$   $\neg$ P) (Q  $\vee$  R).
2742 intros n4_37b.
2743 MP n4_37b n4_25a.
2744 apply propositional_extensionality in n4_25a.
2745 replace (( $\neg$ P  $\vee$   $\neg$ P)  $\vee$  Q  $\vee$  R) with
2746   (( $\neg$ P)  $\vee$  (Q  $\vee$  R)) in n4_2a
2747   by now rewrite <- n4_25a.
2748 replace ( $\neg$ P  $\vee$  Q  $\vee$  R) with
2749   (P  $\rightarrow$  (Q  $\vee$  R)) in n4_2a
2750   by now rewrite Impl1_01.
2751 exact n4_2a.
2752 Qed.

```

```

2753
2754 Theorem n4_79 :  $\forall P Q R : \text{Prop},$ 
2755    $((Q \rightarrow P) \vee (R \rightarrow P)) \leftrightarrow ((Q \wedge R) \rightarrow P).$ 
2756 Proof. intros P Q R.
2757   specialize Transp4_1 with Q P.
2758   intros Transp4_1a.
2759   specialize Transp4_1 with R P.
2760   intros Transp4_1b.
2761   Conj Transp4_1a Transp4_1b Ca.
2762   specialize n4_39 with
2763      $(Q \rightarrow P) (R \rightarrow P) (\neg P \rightarrow \neg Q) (\neg P \rightarrow \neg R).$ 
2764   intros n4_39a.
2765   MP n4_39a Ca.
2766   specialize n4_78 with  $(\neg P) (\neg Q) (\neg R).$ 
2767   intros n4_78a.
2768   rewrite Equiv4_01 in n4_78a.
2769   specialize Simp3_26 with
2770      $((\neg P \rightarrow \neg Q) \vee (\neg P \rightarrow \neg R)) \rightarrow (\neg P \rightarrow (\neg Q \vee \neg R))$ 
2771      $((\neg P \rightarrow (\neg Q \vee \neg R)) \rightarrow ((\neg P \rightarrow \neg Q) \vee (\neg P \rightarrow \neg R))).$ 
2772   intros Simp3_26a.
2773   MP Simp3_26a n4_78a.
2774   specialize Transp2_15 with P  $(\neg Q \vee \neg R).$ 
2775   intros Transp2_15a.
2776   specialize Simp3_27 with
2777      $((\neg P \rightarrow \neg Q) \vee (\neg P \rightarrow \neg R)) \rightarrow (\neg P \rightarrow (\neg Q \vee \neg R))$ 
2778      $((\neg P \rightarrow (\neg Q \vee \neg R)) \rightarrow ((\neg P \rightarrow \neg Q) \vee (\neg P \rightarrow \neg R))).$ 
2779   intros Simp3_27a.
2780   MP Simp3_27a n4_78a.
2781   specialize Transp2_15 with  $(\neg Q \vee \neg R)$  P.
2782   intros Transp2_15b.
2783   specialize Syll2_06 with  $((\neg P \rightarrow \neg Q) \vee (\neg P \rightarrow \neg R))$ 
2784      $(\neg P \rightarrow (\neg Q \vee \neg R)) (\neg (\neg Q \vee \neg R) \rightarrow P).$ 
2785   intros Syll2_06a.
2786   MP Syll2_06a Simp3_26a.
2787   MP Syll2_06a Transp2_15a.
2788   specialize Syll2_06 with  $(\neg (\neg Q \vee \neg R) \rightarrow P)$ 
2789      $(\neg P \rightarrow (\neg Q \vee \neg R)) ((\neg P \rightarrow \neg Q) \vee (\neg P \rightarrow \neg R)).$ 
2790   intros Syll2_06b.
2791   MP Syll2_06b Trans2_15b.
2792   MP Syll2_06b Simp3_27a.
2793   Conj Syll2_06a Syll2_06b Cb.
2794   Equiv Cb.

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2795   clear Transp4_1a. clear Transp4_1b. clear Ca.
2796   clear Simp3_26a. clear Syll2_06b. clear n4_78a.
2797   clear Transp2_15a. clear Simp3_27a.
2798   clear Transp2_15b. clear Syll2_06a.
2799   Conj n4_39a Cb Cc.
2800   specialize n4_22 with ((Q→P)∨(R→P))
2801   ((¬P→¬Q)∨(¬P→¬R)) (¬(¬Q∨¬R)→P).
2802   intros n4_22a.
2803   MP n4_22a Cc.
2804   specialize n4_2 with (¬(¬Q∨¬R)→P).
2805   intros n4_2a.
2806   Conj n4_22a n4_2a Cdd.
2807   specialize n4_22 with ((Q→P)∨(R→P))
2808   (¬(¬Q∨¬R)→P) (¬(¬Q∨¬R)→P).
2809   intros n4_22b.
2810   MP n4_22b Cdd.
2811   rewrite <- Prod3_01 in n4_22b.
2812   exact n4_22b.
2813   Qed.
2814
2815   Theorem n4_8 : ∀ P : Prop,
2816   (P → ¬P) ↔ ¬P.
2817   Proof. intros P.
2818     specialize Abs2_01 with P.
2819     intros Abs2_01a.
2820     specialize Simp2_02 with P (¬P).
2821     intros Simp2_02a.
2822     Conj Abs2_01a Simp2_02a C.
2823     Equiv C.
2824     exact C.
2825   Qed.
2826
2827   Theorem n4_81 : ∀ P : Prop,
2828   (¬P → P) ↔ P.
2829   Proof. intros P.
2830     specialize n2_18 with P.
2831     intros n2_18a.
2832     specialize Simp2_02 with (¬P) P.
2833     intros Simp2_02a.
2834     Conj n2_18a Simp2_02a C.
2835     Equiv C.
2836     exact C.

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2837 Qed.
2838
2839 Theorem n4_82 :  $\forall P Q : \text{Prop},$ 
2840    $((P \rightarrow Q) \wedge (P \rightarrow \neg Q)) \leftrightarrow \neg P.$ 
2841 Proof. intros P Q.
2842   specialize n2_65 with P Q.
2843   intros n2_65a.
2844   specialize Imp3_31 with  $(P \rightarrow Q) (P \rightarrow \neg Q) (\neg P).$ 
2845   intros Imp3_31a.
2846   MP Imp3_31a n2_65a.
2847   specialize n2_21 with P Q.
2848   intros n2_21a.
2849   specialize n2_21 with P  $(\neg Q).$ 
2850   intros n2_21b.
2851   Conj n2_21a n2_21b Ca.
2852   specialize Comp3_43 with  $(\neg P) (P \rightarrow Q) (P \rightarrow \neg Q).$ 
2853   intros Comp3_43a.
2854   MP Comp3_43a Ca.
2855   clear n2_65a. clear n2_21a.
2856   clear n2_21b. clear Ca.
2857   Conj Imp3_31a Comp3_43a Cb.
2858   Equiv Cb.
2859   exact Cb.
2860 Qed.
2861
2862 Theorem n4_83 :  $\forall P Q : \text{Prop},$ 
2863    $((P \rightarrow Q) \wedge (\neg P \rightarrow Q)) \leftrightarrow Q.$ 
2864 Proof. intros P Q.
2865   specialize n2_61 with P Q.
2866   intros n2_61a.
2867   specialize Imp3_31 with  $(P \rightarrow Q) (\neg P \rightarrow Q) (Q).$ 
2868   intros Imp3_31a.
2869   MP Imp3_31a n2_61a.
2870   specialize Simp2_02 with P Q.
2871   intros Simp2_02a.
2872   specialize Simp2_02 with  $(\neg P) Q.$ 
2873   intros Simp2_02b.
2874   Conj Simp2_02a Simp2_02b Ca.
2875   specialize Comp3_43 with  $Q (P \rightarrow Q) (\neg P \rightarrow Q).$ 
2876   intros Comp3_43a.
2877   MP Comp3_43a H.
2878   clear n2_61a. clear Simp2_02a.

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2879     clear Simp2_02b. clear Ca.
2880     Conj Imp3_31a Comp3_43a Cb.
2881     Equiv Cb.
2882     exact Cb.
2883 Qed.
2884
2885 Theorem n4_84 :  $\forall P Q R : \text{Prop},$ 
2886    $(P \leftrightarrow Q) \rightarrow ((P \rightarrow R) \leftrightarrow (Q \rightarrow R)).$ 
2887 Proof. intros P Q R.
2888   specialize Syll2_06 with P Q R.
2889   intros Syll2_06a.
2890   specialize Syll2_06 with Q P R.
2891   intros Syll2_06b.
2892   Conj Syll2_06a Syll2_06b Ca.
2893   specialize n3_47 with
2894      $(P \rightarrow Q) (Q \rightarrow P) ((Q \rightarrow R) \rightarrow P \rightarrow R) ((P \rightarrow R) \rightarrow Q \rightarrow R).$ 
2895   intros n3_47a.
2896   MP n3_47a Ca.
2897   replace  $((P \rightarrow Q) \wedge (Q \rightarrow P))$  with  $(P \leftrightarrow Q)$ 
2898     in n3_47a by now rewrite Equiv4_01.
2899   replace  $((Q \rightarrow R) \rightarrow P \rightarrow R) \wedge ((P \rightarrow R) \rightarrow Q \rightarrow R)$  with
2900      $((Q \rightarrow R) \leftrightarrow (P \rightarrow R))$  in n3_47a by
2901     now rewrite Equiv4_01.
2902   specialize n4_21 with  $(P \rightarrow R) (Q \rightarrow R).$ 
2903   intros n4_21a.
2904   apply propositional_extensionality in n4_21a.
2905   replace  $((Q \rightarrow R) \leftrightarrow (P \rightarrow R))$  with
2906      $((P \rightarrow R) \leftrightarrow (Q \rightarrow R))$  in n3_47a
2907     by now apply n4_21a.
2908   exact n3_47a.
2909 Qed.
2910
2911 Theorem n4_85 :  $\forall P Q R : \text{Prop},$ 
2912    $(P \leftrightarrow Q) \rightarrow ((R \rightarrow P) \leftrightarrow (R \rightarrow Q)).$ 
2913 Proof. intros P Q R.
2914   specialize Syll2_05 with R P Q.
2915   intros Syll2_05a.
2916   specialize Syll2_05 with R Q P.
2917   intros Syll2_05b.
2918   Conj Syll2_05a Syll2_05b Ca.
2919   specialize n3_47 with
2920      $(P \rightarrow Q) (Q \rightarrow P) ((R \rightarrow P) \rightarrow R \rightarrow Q) ((R \rightarrow Q) \rightarrow R \rightarrow P).$ 

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2921   intros n3_47a.
2922   MP n3_47a Ca.
2923   replace ((P→Q) ∧ (Q → P)) with (P↔Q) in n3_47a
2924   by now rewrite Equiv4_01.
2925   replace (((R→P)→R→Q)∧((R→Q)→R→P)) with
2926     ((R→P)↔(R→Q)) in n3_47a
2927     by now rewrite Equiv4_01.
2928   exact n3_47a.
2929 Qed.
2930
2931 Theorem n4_86 : ∀ P Q R : Prop,
2932   (P ↔ Q) → ((P ↔ R) ↔ (Q ↔ R)).
2933 Proof. intros P Q R.
2934   specialize n4_22 with Q P R.
2935   intros n4_22a.
2936   specialize Exp3_3 with (Q↔P) (P↔R) (Q↔R).
2937   intros Exp3_3a. (*Not cited*)
2938   MP Exp3_3a n4_22a.
2939   specialize n4_22 with P Q R.
2940   intros n4_22b.
2941   specialize Exp3_3 with (P↔Q) (Q↔R) (P↔R).
2942   intros Exp3_3b.
2943   MP Exp3_3b n4_22b.
2944   specialize n4_21 with P Q.
2945   intros n4_21a.
2946   apply propositional_extensionality in n4_21a.
2947   replace (Q↔P) with (P↔Q) in Exp3_3a
2948     by now apply n4_21a.
2949   clear n4_22a. clear n4_22b. clear n4_21a.
2950   Conj Exp3_3a Exp3_3b Ca.
2951   specialize Comp3_43 with (P↔Q)
2952     ((P↔R)→(Q↔R)) ((Q↔R)→(P↔R)).
2953   intros Comp3_43a. (*Not cited*)
2954   MP Comp3_43a Ca.
2955   replace (((P↔R)→(Q↔R))∧((Q↔R)→(P↔R)))
2956     with ((P↔R)↔(Q↔R)) in Comp3_43a
2957     by now rewrite Equiv4_01.
2958   exact Comp3_43a.
2959 Qed.
2960
2961 Theorem n4_87 : ∀ P Q R : Prop,
2962   (((P ∧ Q) → R) ↔ (P → Q → R)) ↔

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2963      ((Q → (P → R)) ↔ (Q ∧ P → R)).
2964 Proof. intros P Q R.
2965 specialize Exp3_3 with P Q R.
2966 intros Exp3_3a.
2967 specialize Imp3_31 with P Q R.
2968 intros Imp3_31a.
2969 Conj Exp3_3a Imp3_31a Ca.
2970 Equiv Ca.
2971 specialize Exp3_3 with Q P R.
2972 intros Exp3_3b.
2973 specialize Imp3_31 with Q P R.
2974 intros Imp3_31b.
2975 Conj Exp3_3b Imp3_31b Cb.
2976 Equiv Cb.
2977 specialize n4_21 with (Q→P→R) (Q∧P→R).
2978 intros n4_21a.
2979 apply propositional_extensionality in n4_21a.
2980 replace ((Q∧P→R)↔(Q→P→R)) with
2981   ((Q→P→R)↔(Q∧P→R)) in Cb
2982   by now apply n4_21a.
2983 specialize Simp2_02 with ((P∧Q→R)↔(P→Q→R))
2984   ((Q→P→R)↔(Q∧P→R)).
2985 intros Simp2_02a.
2986 MP Simp2_02a Cb.
2987 specialize Simp2_02 with ((Q→P→R)↔(Q∧P→R))
2988   ((P∧Q→R)↔(P→Q→R)).
2989 intros Simp2_02b.
2990 MP Simp2_02b Ca.
2991 Conj Simp2_02a Simp2_02b Cc.
2992 Equiv Cc.
2993 exact Cc.
2994 Qed.
2995   (*The proof sketch cites Comm2_04. This
2996     bit of the sketch was indecipherable.*)
2997
2998 End No4.
2999
3000 Module No5.
3001
3002 Import No1.
3003 Import No2.
3004 Import No3.

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3005 Import No4.
3006
3007 Theorem n5_1 :  $\forall P Q : \text{Prop},$ 
3008    $(P \wedge Q) \rightarrow (P \leftrightarrow Q).$ 
3009 Proof. intros P Q.
3010   specialize n3_4 with P Q.
3011   intros n3_4a.
3012   specialize n3_4 with Q P.
3013   intros n3_4b.
3014   specialize n3_22 with P Q.
3015   intros n3_22a.
3016   Syll n3_22a n3_4b Sa.
3017   clear n3_22a. clear n3_4b.
3018   Conj n3_4a Sa C.
3019   specialize n4_76 with  $(P \wedge Q) (P \rightarrow Q) (Q \rightarrow P).$ 
3020   intros n4_76a. (*Not cited*)
3021   apply propositional_extensionality in n4_76a.
3022   replace  $((P \wedge Q \rightarrow P \rightarrow Q) \wedge (P \wedge Q \rightarrow Q \rightarrow P))$  with
3023      $(P \wedge Q \rightarrow (P \rightarrow Q) \wedge (Q \rightarrow P))$  in C
3024     by now apply n4_76a.
3025   replace  $((P \rightarrow Q) \wedge (Q \rightarrow P))$  with  $(P \leftrightarrow Q)$  in C
3026     by now rewrite Equiv4_01.
3027   exact C.
3028 Qed.
3029
3030 Theorem n5_11 :  $\forall P Q : \text{Prop},$ 
3031    $(P \rightarrow Q) \vee (\neg P \rightarrow Q).$ 
3032 Proof. intros P Q.
3033   specialize n2_5 with P Q.
3034   intros n2_5a.
3035   specialize n2_54 with  $(P \rightarrow Q) (\neg P \rightarrow Q).$ 
3036   intros n2_54a.
3037   MP n2_54a n2_5a.
3038   exact n2_54a.
3039 Qed.
3040   (*The proof sketch cites n2_51,
3041     but this may be a misprint.*)
3042
3043 Theorem n5_12 :  $\forall P Q : \text{Prop},$ 
3044    $(P \rightarrow Q) \vee (P \rightarrow \neg Q).$ 
3045 Proof. intros P Q.
3046   specialize n2_51 with P Q.

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3047   intros n2_51a.
3048   specialize n2_54 with ((P → Q)) (P → ¬Q).
3049   intros n2_54a.
3050   MP n2_54a n2_5a.
3051   exact n2_54a.
3052 Qed.
3053   (*The proof sketch cites n2_52,
3054     but this may be a misprint.*)
3055
3056 Theorem n5_13 : ∀ P Q : Prop,
3057   (P → Q) ∨ (Q → P).
3058 Proof. intros P Q.
3059   specialize n2_521 with P Q.
3060   intros n2_521a.
3061   replace (¬(P → Q) → Q → P) with
3062     (¬¬(P → Q) ∨ (Q → P)) in n2_521a
3063     by now rewrite <- Impl1_01.
3064   specialize n4_13 with (P→Q).
3065   intros n4_13a. (*Not cited*)
3066   apply propositional_extensionality in n4_13a.
3067   replace (¬¬(P→Q)) with (P→Q)
3068     in n2_521a by now apply n4_13a.
3069   exact n2_521a.
3070 Qed.
3071
3072 Theorem n5_14 : ∀ P Q R : Prop,
3073   (P → Q) ∨ (Q → R).
3074 Proof. intros P Q R.
3075   specialize Simp2_02 with P Q.
3076   intros Simp2_02a.
3077   specialize Transp2_16 with Q (P→Q).
3078   intros Transp2_16a.
3079   MP Transp2_16a Simp2_02a.
3080   specialize n2_21 with Q R.
3081   intros n2_21a.
3082   Syll Transp2_16a n2_21a Sa.
3083   replace (¬(P→Q)→(Q→R)) with
3084     (¬¬(P→Q)∨(Q→R)) in Sa
3085     by now rewrite <- Impl1_01.
3086   specialize n4_13 with (P→Q).
3087   intros n4_13a.
3088   apply propositional_extensionality in n4_13a.

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3089   replace ( $\neg\neg(P \rightarrow Q)$ ) with ( $P \rightarrow Q$ ) in Sa
3090   by now apply n4_13a.
3091   exact Sa.
3092 Qed.
3093
3094 Theorem n5_15 :  $\forall P Q : \text{Prop},$ 
3095   ( $P \leftrightarrow Q$ )  $\vee$  ( $P \leftrightarrow \neg Q$ ).
3096 Proof. intros P Q.
3097 specialize n4_61 with P Q.
3098 intros n4_61a.
3099 replace ( $\neg(P \rightarrow Q) \leftrightarrow P \wedge \neg Q$ ) with
3100   ( $(\neg(P \rightarrow Q) \rightarrow P \wedge \neg Q) \wedge ((P \wedge \neg Q) \rightarrow \neg(P \rightarrow Q))$ ) in n4_61a
3101   by now rewrite Equiv4_01.
3102 specialize Simp3_26 with
3103   ( $\neg(P \rightarrow Q) \rightarrow P \wedge \neg Q$ ) ( $(P \wedge \neg Q) \rightarrow \neg(P \rightarrow Q)$ ).
3104 intros Simp3_26a.
3105 MP Simp3_26a n4_61a.
3106 specialize n5_1 with P ( $\neg Q$ ).
3107 intros n5_1a.
3108 Syll Simp3_26a n5_1a Sa.
3109 specialize n2_54 with ( $P \rightarrow Q$ ) ( $P \leftrightarrow \neg Q$ ).
3110 intros n2_54a.
3111 MP n2_54a Sa.
3112 specialize n4_61 with Q P.
3113 intros n4_61b.
3114 replace ( $(\neg(Q \rightarrow P)) \leftrightarrow (Q \wedge \neg P)$ ) with
3115   ( $((\neg(Q \rightarrow P)) \rightarrow (Q \wedge \neg P)) \wedge ((Q \wedge \neg P) \rightarrow (\neg(Q \rightarrow P)))$ )
3116   in n4_61b by now rewrite Equiv4_01.
3117 specialize Simp3_26 with
3118   ( $\neg(Q \rightarrow P) \rightarrow (Q \wedge \neg P)$ ) ( $(Q \wedge \neg P) \rightarrow (\neg(Q \rightarrow P))$ ).
3119 intros Simp3_26b.
3120 MP Simp3_26b n4_61b.
3121 specialize n5_1 with Q ( $\neg P$ ).
3122 intros n5_1b.
3123 Syll Simp3_26b n5_1b Sb.
3124 specialize n4_12 with P Q.
3125 intros n4_12a.
3126 apply propositional_extensionality in n4_12a.
3127 replace ( $Q \leftrightarrow \neg P$ ) with ( $P \leftrightarrow \neg Q$ ) in Sb
3128   by now apply n4_12a.
3129 specialize n2_54 with ( $Q \rightarrow P$ ) ( $P \leftrightarrow \neg Q$ ).
3130 intros n2_54b.

```

```

3131 MP n2_54b Sb.
3132 replace ( $\neg(P \rightarrow Q) \rightarrow P \leftrightarrow \neg Q$ ) with
3133   ( $\neg\neg(P \rightarrow Q) \vee (P \leftrightarrow \neg Q)$ ) in Sa
3134   by now rewrite <- Impl1_01.
3135 specialize n4_13 with  $(P \rightarrow Q)$ .
3136 intros n4_13a.
3137 apply propositional_extensionality in n4_13a.
3138 replace ( $\neg\neg(P \rightarrow Q)$ ) with  $(P \rightarrow Q)$  in Sa
3139   by now apply n4_13a.
3140 replace ( $\neg(Q \rightarrow P) \rightarrow (P \leftrightarrow \neg Q)$ ) with
3141   ( $\neg\neg(Q \rightarrow P) \vee (P \leftrightarrow \neg Q)$ ) in Sb
3142   by now rewrite <- Impl1_01.
3143 specialize n4_13 with  $(Q \rightarrow P)$ .
3144 intros n4_13b.
3145 apply propositional_extensionality in n4_13b.
3146 replace ( $\neg\neg(Q \rightarrow P)$ ) with  $(Q \rightarrow P)$  in Sb
3147   by now apply n4_13b.
3148 clear n4_61a. clear Simp3_26a. clear n5_1a.
3149   clear n2_54a. clear n4_61b. clear Simp3_26b.
3150   clear n5_1b. clear n4_12a. clear n2_54b.
3151   clear n4_13a. clear n4_13b.
3152 Conj Sa Sb C.
3153 specialize n4_31 with  $(P \rightarrow Q) (P \leftrightarrow \neg Q)$ .
3154 intros n4_31a.
3155 apply propositional_extensionality in n4_31a.
3156 replace  $((P \rightarrow Q) \vee (P \leftrightarrow \neg Q))$  with
3157    $((P \leftrightarrow \neg Q) \vee (P \rightarrow Q))$  in C
3158   by now apply n4_31a.
3159 specialize n4_31 with  $(Q \rightarrow P) (P \leftrightarrow \neg Q)$ .
3160 intros n4_31b.
3161 apply propositional_extensionality in n4_31b.
3162 replace  $((Q \rightarrow P) \vee (P \leftrightarrow \neg Q))$  with
3163    $((P \leftrightarrow \neg Q) \vee (Q \rightarrow P))$  in C
3164   by now apply n4_31b.
3165 specialize n4_41 with  $(P \leftrightarrow \neg Q) (P \rightarrow Q) (Q \rightarrow P)$ .
3166 intros n4_41a.
3167 apply propositional_extensionality in n4_41a.
3168 replace  $((P \leftrightarrow \neg Q) \vee (P \rightarrow Q)) \wedge ((P \leftrightarrow \neg Q) \vee (Q \rightarrow P))$ 
3169   with  $((P \leftrightarrow \neg Q) \vee (P \rightarrow Q) \wedge (Q \rightarrow P))$  in C
3170   by now apply n4_41a.
3171 replace  $((P \rightarrow Q) \wedge (Q \rightarrow P))$  with  $(P \leftrightarrow Q)$  in C
3172   by now rewrite Equiv4_01.

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```

3173     specialize n4_31 with (P ↔ ¬Q) (P ↔ Q).
3174     intros n4_31c.
3175     apply propositional_extensionality in n4_31c.
3176     replace ((P ↔ ¬Q) ∨ (P ↔ Q)) with
3177         ((P ↔ Q) ∨ (P ↔ ¬Q)) in C
3178     by now apply n4_31c.
3179     exact C.
3180 Qed.
3181
3182 Theorem n5_16 : ∀ P Q : Prop,
3183     ¬((P ↔ Q) ∧ (P ↔ ¬Q)).
3184 Proof. intros P Q.
3185     specialize Simp3_26 with ((P→Q) ∧ (P → ¬Q)) (Q→P).
3186     intros Simp3_26a.
3187     specialize Id2_08 with ((P ↔ Q) ∧ (P → ¬Q)).
3188     intros Id2_08a.
3189     specialize n4_32 with (P→Q) (P→¬Q) (Q→P).
3190     intros n4_32a.
3191     apply propositional_extensionality in n4_32a.
3192     replace (((P → Q) ∧ (P → ¬Q)) ∧ (Q → P)) with
3193         ((P→Q) ∧ ((P→¬Q) ∧ (Q→P))) in Simp3_26a
3194     by now apply n4_32a.
3195     specialize n4_3 with (Q→P) (P→¬Q).
3196     intros n4_3a.
3197     apply propositional_extensionality in n4_3a.
3198     replace ((P → ¬Q) ∧ (Q → P)) with
3199         ((Q → P) ∧ (P → ¬Q)) in Simp3_26a
3200     by now apply n4_3a.
3201     specialize n4_32 with (P→Q) (Q→P) (P→¬Q).
3202     intros n4_32b.
3203     apply propositional_extensionality in n4_32b.
3204     replace ((P→Q) ∧ (Q → P) ∧ (P → ¬Q)) with
3205         (((P→Q) ∧ (Q → P)) ∧ (P → ¬Q)) in Simp3_26a
3206     by now apply n4_32b.
3207     replace ((P → Q) ∧ (Q → P)) with (P↔Q)
3208     in Simp3_26a by now rewrite Equiv4_01.
3209     Syll Id2_08a Simp3_26a Sa.
3210     specialize n4_82 with P Q.
3211     intros n4_82a.
3212     apply propositional_extensionality in n4_82a.
3213     replace ((P → Q) ∧ (P → ¬Q)) with (¬P) in Sa
3214     by now apply n4_82a.

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3215 specialize Simp3_27 with
3216   (P→Q) ((Q→P) ∧ (P → ¬Q)).
3217 intros Simp3_27a.
3218 replace ((P→Q) ∧ (Q → P) ∧ (P → ¬Q)) with
3219   ((P→Q) ∧ (Q → P)) ∧ (P → ¬Q) in Simp3_27a
3220   by now apply n4_32b.
3221 replace ((P → Q) ∧ (Q → P)) with (P↔Q)
3222   in Simp3_27a by now rewrite Equiv4_01.
3223 specialize Syll3_33 with Q P (¬Q).
3224 intros Syll3_33a.
3225 Syll Simp3_27a Syll2_06a Sb.
3226 specialize Abs2_01 with Q.
3227 intros Abs2_01a.
3228 Syll Sb Abs2_01a Sc.
3229 clear Sb. clear Simp3_26a. clear Id2_08a.
3230   clear n4_82a. clear Simp3_27a. clear Syll3_33a.
3231   clear Abs2_01a. clear n4_32a. clear n4_32b. clear n4_3a.
3232 Conj Sa Sc C.
3233 specialize Comp3_43 with
3234   ((P ↔ Q) ∧ (P → ¬Q)) (¬P) (¬Q).
3235 intros Comp3_43a.
3236 MP Comp3_43a C.
3237 specialize n4_65 with Q P.
3238 intros n4_65a.
3239 specialize n4_3 with (¬P) (¬Q).
3240 intros n4_3a.
3241 apply propositional_extensionality in n4_3a.
3242 replace (¬Q ∧ ¬P) with (¬P ∧ ¬Q) in n4_65a
3243   by now apply n4_3a.
3244 apply propositional_extensionality in n4_65a.
3245 replace (¬P ∧ ¬Q) with (¬(¬Q→P)) in Comp3_43a
3246   by now apply n4_65a.
3247 specialize Exp3_3 with
3248   (P↔Q) (P→¬Q) (¬(¬Q→P)).
3249 intros Exp3_3a.
3250 MP Exp3_3a Comp3_43a.
3251 replace ((P→¬Q)→¬(¬Q→P)) with
3252   (¬(P→¬Q) ∨ ¬(¬Q→P)) in Exp3_3a
3253   by now rewrite <- Impl1_01.
3254 specialize n4_51 with (P→¬Q) (¬Q→P).
3255 intros n4_51a.
3256 apply propositional_extensionality in n4_51a.

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3257   replace ( $\neg(P \rightarrow \neg Q) \vee \neg(\neg Q \rightarrow P)$ ) with
3258     ( $\neg((P \rightarrow \neg Q) \wedge (\neg Q \rightarrow P))$ ) in Exp3_3a
3259     by now apply n4_51a.
3260   replace ( $(P \rightarrow \neg Q) \wedge (\neg Q \rightarrow P)$ ) with ( $P \leftrightarrow \neg Q$ )
3261     in Exp3_3a by now rewrite Equiv4_01.
3262   replace ( $(P \leftrightarrow Q) \rightarrow \neg(P \leftrightarrow \neg Q)$ ) with
3263     ( $\neg(P \leftrightarrow Q) \vee \neg(P \leftrightarrow \neg Q)$ ) in Exp3_3a
3264     by now rewrite Impl1_01.
3265   specialize n4_51 with ( $P \leftrightarrow Q$ ) ( $P \leftrightarrow \neg Q$ ).
3266   intros n4_51b.
3267   apply propositional_extensionality in n4_51b.
3268   replace ( $\neg(P \leftrightarrow Q) \vee \neg(P \leftrightarrow \neg Q)$ ) with
3269     ( $\neg((P \leftrightarrow Q) \wedge (P \leftrightarrow \neg Q))$ ) in Exp3_3a
3270     by now apply n4_51b.
3271   exact Exp3_3a.
3272 Qed.
3273
3274 Theorem n5_17 :  $\forall P Q : \text{Prop}$ ,
3275   ( $(P \vee Q) \wedge \neg(P \wedge Q) \leftrightarrow (P \leftrightarrow \neg Q)$ ).
3276 Proof. intros P Q.
3277   specialize n4_64 with Q P.
3278   intros n4_64a.
3279   specialize n4_21 with ( $Q \vee P$ ) ( $\neg Q \rightarrow P$ ).
3280   intros n4_21a.
3281   apply propositional_extensionality in n4_21a.
3282   replace ( $(\neg Q \rightarrow P) \leftrightarrow (Q \vee P)$ ) with
3283     ( $(Q \vee P) \leftrightarrow (\neg Q \rightarrow P)$ ) in n4_64a
3284     by now apply n4_21a.
3285   specialize n4_31 with P Q.
3286   intros n4_31a.
3287   apply propositional_extensionality in n4_31a.
3288   replace ( $Q \vee P$ ) with ( $P \vee Q$ ) in n4_64a
3289     by now apply n4_31a.
3290   specialize n4_63 with P Q.
3291   intros n4_63a.
3292   specialize n4_21 with ( $P \wedge Q$ ) ( $\neg(P \rightarrow \neg Q)$ ).
3293   intros n4_21b.
3294   apply propositional_extensionality in n4_21b.
3295   replace ( $\neg(P \rightarrow \neg Q) \leftrightarrow P \wedge Q$ ) with
3296     ( $P \wedge Q \leftrightarrow \neg(P \rightarrow \neg Q)$ ) in n4_63a
3297     by now apply n4_21b.
3298   specialize Transp4_11 with ( $P \wedge Q$ ) ( $\neg(P \rightarrow \neg Q)$ ).

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3299   intros Transp4_11a.
3300   specialize n4_13 with (P → ¬Q).
3301   intros n4_13a.
3302   apply propositional_extensionality in n4_13a.
3303   replace (¬¬(P → ¬Q)) with (P → ¬Q)
3304     in Transp4_11a by now apply n4_13a.
3305   apply propositional_extensionality in Transp4_11a.
3306   replace (P ∧ Q ↔ ¬(P → ¬Q)) with
3307     (¬(P ∧ Q) ↔ (P → ¬Q)) in n4_63a
3308     by now apply Transp4_11a.
3309   clear Transp4_11a. clear n4_21a.
3310   clear n4_31a. clear n4_21b. clear n4_13a.
3311   Conj n4_64a n4_63a C.
3312   specialize n4_38 with
3313     (P ∨ Q) (¬(P ∧ Q)) (¬Q → P) (P → ¬Q).
3314   intros n4_38a.
3315   MP n4_38a C.
3316   replace ((¬Q → P) ∧ (P → ¬Q)) with (¬Q ↔ P)
3317     in n4_38a by now rewrite Equiv4_01.
3318   specialize n4_21 with P (¬Q).
3319   intros n4_21c.
3320   apply propositional_extensionality in n4_21c.
3321   replace (¬Q ↔ P) with (P ↔ ¬Q) in n4_38a
3322     by now apply n4_21c.
3323   exact n4_38a.
3324   Qed.
3325
3326   Theorem n5_18 : ∀ P Q : Prop,
3327     (P ↔ Q) ↔ ¬(P ↔ ¬Q).
3328   Proof. intros P Q.
3329   specialize n5_15 with P Q.
3330   intros n5_15a.
3331   specialize n5_16 with P Q.
3332   intros n5_16a.
3333   Conj n5_15a n5_16a C.
3334   specialize n5_17 with (P ↔ Q) (P ↔ ¬Q).
3335   intros n5_17a.
3336   rewrite Equiv4_01 in n5_17a.
3337   specialize Simp3_26 with
3338     (((P ↔ Q) ∨ (P ↔ ¬Q)) ∧ ¬((P ↔ Q) ∧ (P ↔ ¬Q)))
3339     →((P ↔ Q) ↔ ¬(P ↔ ¬Q)) ((P ↔ Q) ↔ ¬(P ↔ ¬Q)) →
3340     (((P ↔ Q) ∨ (P ↔ ¬Q)) ∧ ¬((P ↔ Q) ∧ (P ↔ ¬Q))).

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3341   intros Simp3_26a. (*not cited*)
3342   MP Simp3_26a n5_17a.
3343   MP Simp3_26a C.
3344   exact Simp3_26a.
3345 Qed.
3346
3347 Theorem n5_19 :  $\forall P : \text{Prop},$ 
3348    $\neg(P \leftrightarrow \neg P).$ 
3349 Proof. intros P.
3350 specialize n5_18 with P P.
3351 intros n5_18a.
3352 specialize n4_2 with P.
3353 intros n4_2a.
3354 rewrite Equiv4_01 in n5_18a.
3355 specialize Simp3_26 with  $(P \leftrightarrow P \rightarrow \neg(P \leftrightarrow \neg P))$ 
3356    $(\neg(P \leftrightarrow \neg P) \rightarrow P \leftrightarrow P).$ 
3357 intros Simp3_26a. (*not cited*)
3358 MP Simp3_26a n5_18a.
3359 MP Simp3_26a n4_2a.
3360 exact Simp3_26a.
3361 Qed.
3362
3363 Theorem n5_21 :  $\forall P Q : \text{Prop},$ 
3364    $(\neg P \wedge \neg Q) \rightarrow (P \leftrightarrow Q).$ 
3365 Proof. intros P Q.
3366 specialize n5_1 with  $(\neg P) (\neg Q).$ 
3367 intros n5_1a.
3368 specialize Transp4_11 with P Q.
3369 intros Transp4_11a.
3370 apply propositional_extensionality in Transp4_11a.
3371 replace  $(\neg P \leftrightarrow \neg Q)$  with  $(P \leftrightarrow Q)$  in n5_1a
3372   by now apply Transp4_11a.
3373 exact n5_1a.
3374 Qed.
3375
3376 Theorem n5_22 :  $\forall P Q : \text{Prop},$ 
3377    $\neg(P \leftrightarrow Q) \leftrightarrow ((P \wedge \neg Q) \vee (Q \wedge \neg P)).$ 
3378 Proof. intros P Q.
3379 specialize n4_61 with P Q.
3380 intros n4_61a.
3381 specialize n4_61 with Q P.
3382 intros n4_61b.

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3383 Conj n4_61a n4_61b C.
3384 specialize n4_39 with
3385   ( $\neg(P \rightarrow Q)$ ) ( $\neg(Q \rightarrow P)$ ) ( $P \wedge \neg Q$ ) ( $Q \wedge \neg P$ ).
3386 intros n4_39a.
3387 MP n4_39a C.
3388 specialize n4_51 with ( $P \rightarrow Q$ ) ( $Q \rightarrow P$ ).
3389 intros n4_51a.
3390 apply propositional_extensionality in n4_51a.
3391 replace ( $\neg(P \rightarrow Q) \vee \neg(Q \rightarrow P)$ ) with
3392   ( $\neg((P \rightarrow Q) \wedge (Q \rightarrow P))$ ) in n4_39a
3393   by now apply n4_51a.
3394 replace (( $P \rightarrow Q$ )  $\wedge$  ( $Q \rightarrow P$ )) with ( $P \leftrightarrow Q$ )
3395   in n4_39a by now rewrite Equiv4_01.
3396 exact n4_39a.
3397 Qed.
3398
3399 Theorem n5_23 :  $\forall P Q : \text{Prop}$ ,
3400   ( $P \leftrightarrow Q$ )  $\leftrightarrow$  (( $P \wedge Q$ )  $\vee$  ( $\neg P \wedge \neg Q$ )).
3401 Proof. intros P Q.
3402 specialize n5_18 with P Q.
3403 intros n5_18a.
3404 specialize n5_22 with P ( $\neg Q$ ).
3405 intros n5_22a.
3406 Conj n5_18a n5_22a C.
3407 specialize n4_22 with ( $P \leftrightarrow Q$ ) ( $\neg(P \leftrightarrow \neg Q)$ )
3408   ( $P \wedge \neg \neg Q \vee \neg Q \wedge \neg P$ ).
3409 intros n4_22a.
3410 MP n4_22a C.
3411 specialize n4_13 with Q.
3412 intros n4_13a.
3413 apply propositional_extensionality in n4_13a.
3414 replace ( $\neg \neg Q$ ) with Q in n4_22a by now apply n4_13a.
3415 specialize n4_3 with ( $\neg P$ ) ( $\neg Q$ ).
3416 intros n4_3a.
3417 apply propositional_extensionality in n4_3a.
3418 replace ( $\neg Q \wedge \neg P$ ) with ( $\neg P \wedge \neg Q$ ) in n4_22a
3419   by now apply n4_3a.
3420 exact n4_22a.
3421 Qed.
3422 (The proof sketch in Principia offers n4_36.
3423   This seems to be a misprint. We used n4_3.*)
3424

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3425 Theorem n5_24 :  $\forall P Q : \text{Prop},$ 
3426    $\neg((P \wedge Q) \vee (\neg P \wedge \neg Q)) \leftrightarrow ((P \wedge \neg Q) \vee (Q \wedge \neg P)).$ 
3427 Proof. intros P Q.
3428 specialize n5_23 with P Q.
3429 intros n5_23a.
3430 specialize Transp4_11 with
3431    $(P \leftrightarrow Q) (P \wedge Q \vee \neg P \wedge \neg Q).$ 
3432 intros Transp4_11a. (*Not cited*)
3433 rewrite Equiv4_01 in Transp4_11a.
3434 specialize Simp3_26 with  $((P \leftrightarrow Q) \leftrightarrow P \wedge Q \vee \neg P \wedge \neg Q)$ 
3435    $\rightarrow (\neg(P \leftrightarrow Q) \leftrightarrow \neg(P \wedge Q \vee \neg P \wedge \neg Q))$ 
3436    $((\neg(P \leftrightarrow Q) \leftrightarrow \neg(P \wedge Q \vee \neg P \wedge \neg Q))$ 
3437    $\rightarrow ((P \leftrightarrow Q) \leftrightarrow P \wedge Q \vee \neg P \wedge \neg Q)).$ 
3438 intros Simp3_26a.
3439 MP Simp3_26a Transp4_11a.
3440 MP Simp3_26a n5_23a.
3441 specialize n5_22 with P Q.
3442 intros n5_22a.
3443 clear n5_23a. clear Transp4_11a.
3444 Conj Simp3_26a n5_22a C.
3445 specialize n4_22 with  $(\neg(P \wedge Q \vee \neg P \wedge \neg Q))$ 
3446    $(\neg(P \leftrightarrow Q)) (P \wedge \neg Q \vee Q \wedge \neg P).$ 
3447 intros n4_22a.
3448 specialize n4_21 with  $(\neg(P \wedge Q \vee \neg P \wedge \neg Q)) (\neg(P \leftrightarrow Q)).$ 
3449 intros n4_21a.
3450 apply propositional_extensionality in n4_21a.
3451 replace  $((\neg(P \leftrightarrow Q)) \leftrightarrow (\neg((P \wedge Q) \vee (\neg P \wedge \neg Q))))$ 
3452   with  $((\neg((P \wedge Q) \vee (\neg P \wedge \neg Q))) \leftrightarrow (\neg(P \leftrightarrow Q)))$  in C
3453   by now apply n4_21a.
3454 MP n4_22a C.
3455 exact n4_22a.
3456 Qed.
3457
3458 Theorem n5_25 :  $\forall P Q : \text{Prop},$ 
3459    $(P \vee Q) \leftrightarrow ((P \rightarrow Q) \rightarrow Q).$ 
3460 Proof. intros P Q.
3461 specialize n2_62 with P Q.
3462 intros n2_62a.
3463 specialize n2_68 with P Q.
3464 intros n2_68a.
3465 Conj n2_62a n2_68a C.
3466 Equiv C.

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3467     exact C.
3468 Qed.
3469
3470 Theorem n5_3 :  $\forall P Q R : \text{Prop}$ ,
3471    $((P \wedge Q) \rightarrow R) \leftrightarrow ((P \wedge Q) \rightarrow (P \wedge R))$ .
3472 Proof. intros P Q R.
3473   specialize Comp3_43 with (P  $\wedge$  Q) P R.
3474   intros Comp3_43a.
3475   specialize Exp3_3 with
3476     (P  $\wedge$  Q  $\rightarrow$  P) (P  $\wedge$  Q  $\rightarrow$  R) (P  $\wedge$  Q  $\rightarrow$  P  $\wedge$  R).
3477   intros Exp3_3a. (*Not cited*)
3478   MP Exp3_3a Comp3_43a.
3479   specialize Simp3_26 with P Q.
3480   intros Simp3_26a.
3481   MP Exp3_3a Simp3_26a.
3482   specialize Syll2_05 with (P  $\wedge$  Q) (P  $\wedge$  R) R.
3483   intros Syll2_05a.
3484   specialize Simp3_27 with P R.
3485   intros Simp3_27a.
3486   MP Syll2_05a Simp3_27a.
3487   clear Comp3_43a. clear Simp3_27a.
3488     clear Simp3_26a.
3489   Conj Exp3_3a Syll2_05a C.
3490   Equiv C.
3491   exact C.
3492 Qed.
3493
3494 Theorem n5_31 :  $\forall P Q R : \text{Prop}$ ,
3495    $(R \wedge (P \rightarrow Q)) \rightarrow (P \rightarrow (Q \wedge R))$ .
3496 Proof. intros P Q R.
3497   specialize Comp3_43 with P Q R.
3498   intros Comp3_43a.
3499   specialize Simp2_02 with P R.
3500   intros Simp2_02a.
3501   specialize Exp3_3 with
3502     (P  $\rightarrow$  R) (P  $\rightarrow$  Q) (P  $\rightarrow$  (Q  $\wedge$  R)).
3503   intros Exp3_3a. (*Not cited*)
3504   specialize n3_22 with (P  $\rightarrow$  R) (P  $\rightarrow$  Q). (*Not cited*)
3505   intros n3_22a.
3506   Syll n3_22a Comp3_43a Sa.
3507   MP Exp3_3a Sa.
3508   Syll Simp2_02a Exp3_3a Sb.

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3509   specialize Imp3_31 with R (P→Q) (P→(Q ∧ R)).
3510   intros Imp3_31a. (*Not cited*)
3511   MP Imp3_31a Sb.
3512   exact Imp3_31a.
3513 Qed.
3514
3515 Theorem n5_32 : ∀ P Q R : Prop,
3516   (P → (Q ↔ R)) ↔ ((P ∧ Q) ↔ (P ∧ R)).
3517 Proof. intros P Q R.
3518   specialize n4_76 with P (Q→R) (R→Q).
3519   intros n4_76a.
3520   specialize Exp3_3 with P Q R.
3521   intros Exp3_3a.
3522   specialize Imp3_31 with P Q R.
3523   intros Imp3_31a.
3524   Conj Exp3_3a Imp3_31a Ca.
3525   Equiv Ca.
3526   specialize Exp3_3 with P R Q.
3527   intros Exp3_3b.
3528   specialize Imp3_31 with P R Q.
3529   intros Imp3_31b.
3530   Conj Exp3_3b Imp3_31b Cb.
3531   Equiv Cb.
3532   specialize n5_3 with P Q R.
3533   intros n5_3a.
3534   specialize n5_3 with P R Q.
3535   intros n5_3b.
3536   apply propositional_extensionality in Ca.
3537   replace (P→Q→R) with (P∧Q→R) in n4_76a
3538     by now apply Ca.
3539   apply propositional_extensionality in Cb.
3540   replace (P→R→Q) with (P∧R→Q) in n4_76a
3541     by now apply Cb.
3542   apply propositional_extensionality in n5_3a.
3543   replace (P∧Q→R) with (P∧Q→P∧R) in n4_76a
3544     by now apply n5_3a.
3545   apply propositional_extensionality in n5_3b.
3546   replace (P∧R→Q) with (P∧R→P∧Q) in n4_76a
3547     by now apply n5_3b.
3548   replace ((P∧Q→P∧R)∧(P∧R→P∧Q)) with
3549     ((P∧Q)↔(P∧R)) in n4_76a
3550     by now rewrite Equiv4_01.

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3551 specialize n4_21 with
3552   (P → ((Q → R) ∧ (R → Q))) ((P ∧ Q) ↔ (P ∧ R)).
3553 intros n4_21a.
3554 apply propositional_extensionality in n4_21a.
3555 replace ((P ∧ Q ↔ P ∧ R) ↔ (P → (Q → R) ∧ (R → Q))) with
3556   ((P → (Q → R) ∧ (R → Q)) ↔ (P ∧ Q ↔ P ∧ R)) in n4_76a
3557   by now apply n4_21a.
3558 replace ((Q → R) ∧ (R → Q)) with (Q ↔ R) in n4_76a
3559   by now rewrite Equiv4_01.
3560 exact n4_76a.
3561 Qed.
3562
3563 Theorem n5_33 : ∀ P Q R : Prop,
3564   (P ∧ (Q → R)) ↔ (P ∧ ((P ∧ Q) → R)).
3565 Proof. intros P Q R.
3566   specialize n5_32 with P (Q → R) ((P ∧ Q) → R).
3567   intros n5_32a.
3568   replace
3569     ((P → (Q → R) ↔ (P ∧ Q → R)) ↔ (P ∧ (Q → R) ↔ P ∧ (P ∧ Q → R)))
3570     with
3571     (((P → (Q → R) ↔ (P ∧ Q → R)) → (P ∧ (Q → R) ↔ P ∧ (P ∧ Q → R)))
3572      ∧
3573      ((P ∧ (Q → R) ↔ P ∧ (P ∧ Q → R) → (P → (Q → R) ↔ (P ∧ Q → R))))))
3574     in n5_32a by now rewrite Equiv4_01.
3575   specialize Simp3_26 with
3576     ((P → (Q → R) ↔ (P ∧ Q → R)) → (P ∧ (Q → R) ↔ P ∧ (P ∧ Q → R)))
3577     ((P ∧ (Q → R) ↔ P ∧ (P ∧ Q → R) → (P → (Q → R) ↔ (P ∧ Q → R))).
3578   intros Simp3_26a. (*Not cited*)
3579   MP Simp3_26a n5_32a.
3580   specialize n4_73 with Q P.
3581   intros n4_73a.
3582   specialize n4_84 with Q (Q ∧ P) R.
3583   intros n4_84a.
3584   Syll n4_73a n4_84a Sa.
3585   specialize n4_3 with P Q.
3586   intros n4_3a.
3587   apply propositional_extensionality in n4_3a.
3588   replace (Q ∧ P) with (P ∧ Q) in Sa
3589     by now apply n4_3a. (*Not cited*)
3590   MP Simp3_26a Sa.
3591   exact Simp3_26a.
3592   Qed.

```

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3593
3594 Theorem n5_35 :  $\forall$  P Q R : Prop,
3595   ((P  $\rightarrow$  Q)  $\wedge$  (P  $\rightarrow$  R))  $\rightarrow$  (P  $\rightarrow$  (Q  $\leftrightarrow$  R)).
3596 Proof. intros P Q R.
3597   specialize Comp3_43 with P Q R.
3598   intros Comp3_43a.
3599   specialize n5_1 with Q R.
3600   intros n5_1a.
3601   specialize Syll2_05 with P (Q $\wedge$ R) (Q $\leftrightarrow$ R).
3602   intros Syll2_05a.
3603   MP Syll2_05a n5_1a.
3604   Syll Comp3_43a Syll2_05a Sa.
3605   exact Sa.
3606 Qed.
3607
3608 Theorem n5_36 :  $\forall$  P Q : Prop,
3609   (P  $\wedge$  (P  $\leftrightarrow$  Q))  $\leftrightarrow$  (Q  $\wedge$  (P  $\leftrightarrow$  Q)).
3610 Proof. intros P Q.
3611   specialize Id2_08 with (P $\leftrightarrow$ Q).
3612   intros Id2_08a.
3613   specialize n5_32 with (P $\leftrightarrow$ Q) P Q.
3614   intros n5_32a.
3615   apply propositional_extensionality in n5_32a.
3616   replace (P $\leftrightarrow$ Q $\rightarrow$ P $\leftrightarrow$ Q) with
3617     ((P $\leftrightarrow$ Q) $\wedge$ P $\leftrightarrow$ (P $\leftrightarrow$ Q) $\wedge$ Q) in Id2_08a
3618     by now apply n5_32a.
3619   specialize n4_3 with P (P $\leftrightarrow$ Q).
3620   intros n4_3a.
3621   apply propositional_extensionality in n4_3a.
3622   replace ((P $\leftrightarrow$ Q) $\wedge$ P) with (P $\wedge$ (P $\leftrightarrow$ Q)) in Id2_08a
3623     by now apply n4_3a.
3624   specialize n4_3 with Q (P $\leftrightarrow$ Q).
3625   intros n4_3b.
3626   apply propositional_extensionality in n4_3b.
3627   replace ((P $\leftrightarrow$ Q) $\wedge$ Q) with (Q $\wedge$ (P $\leftrightarrow$ Q)) in Id2_08a
3628     by now apply n4_3b.
3629   exact Id2_08a.
3630 Qed.
3631   (*The proof sketch cites Ass3_35 and n4_38,
3632     but the sketch was indecipherable.*)
3633
3634 Theorem n5_4 :  $\forall$  P Q : Prop,

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3635   (P → (P → Q)) ↔ (P → Q).
3636 Proof. intros P Q.
3637 specialize n2_43 with P Q.
3638 intros n2_43a.
3639 specialize Simp2_02 with (P) (P→Q).
3640 intros Simp2_02a.
3641 Conj n2_43a Simp2_02a C.
3642 Equiv C.
3643 exact C.
3644 Qed.
3645
3646 Theorem n5_41 : ∀ P Q R : Prop,
3647   ((P → Q) → (P → R)) ↔ (P → Q → R).
3648 Proof. intros P Q R.
3649 specialize n2_86 with P Q R.
3650 intros n2_86a.
3651 specialize n2_77 with P Q R.
3652 intros n2_77a.
3653 Conj n2_86a n2_77a C.
3654 Equiv C.
3655 exact C.
3656 Qed.
3657
3658 Theorem n5_42 : ∀ P Q R : Prop,
3659   (P → Q → R) ↔ (P → Q → P ∧ R).
3660 Proof. intros P Q R.
3661 specialize n5_3 with P Q R.
3662 intros n5_3a.
3663 specialize n4_87 with P Q R.
3664 intros n4_87a.
3665 specialize Imp3_31 with P Q R.
3666 intros Imp3_31a.
3667 specialize Exp3_3 with P Q R.
3668 intros Exp3_3a.
3669 Conj Imp3_31a Exp3_3 Ca.
3670 Equiv Ca.
3671 apply propositional_extensionality in Ca.
3672 replace ((P∧Q)→R) with (P→Q→R) in n5_3a
3673   by now apply Ca.
3674 specialize n4_87 with P Q (P∧R).
3675 intros n4_87b.
3676 specialize Imp3_31 with P Q (P∧R).

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3677   intros Imp3_31b.
3678   specialize Exp3_3 with P Q (P $\wedge$ R).
3679   intros Exp3_3b.
3680   Conj Imp3_31b Exp3_3b Cb.
3681   Equiv Cb.
3682   apply propositional_extensionality in Cb.
3683   replace ((P $\wedge$ Q) $\rightarrow$ (P $\wedge$ R)) with
3684     (P $\rightarrow$ Q $\rightarrow$ (P $\wedge$ R)) in n5_3a by now apply Cb.
3685   exact n5_3a.
3686 Qed.
3687
3688 Theorem n5_44 :  $\forall$  P Q R : Prop,
3689   (P $\rightarrow$ Q)  $\rightarrow$  ((P  $\rightarrow$  R)  $\leftrightarrow$  (P  $\rightarrow$  (Q  $\wedge$  R))).
3690 Proof. intros P Q R.
3691   specialize n4_76 with P Q R.
3692   intros n4_76a.
3693   rewrite Equiv4_01 in n4_76a.
3694   specialize Simp3_26 with
3695     (((P $\rightarrow$ Q) $\wedge$ (P $\rightarrow$ R)) $\rightarrow$ (P $\rightarrow$ (Q $\wedge$ R)))
3696     ((P $\rightarrow$ (Q $\wedge$ R)) $\rightarrow$ ((P $\rightarrow$ Q) $\wedge$ (P $\rightarrow$ R))).
3697   intros Simp3_26a.
3698   MP Simp3_26a n4_76a.
3699   specialize Simp3_27 with
3700     (((P $\rightarrow$ Q) $\wedge$ (P $\rightarrow$ R)) $\rightarrow$ (P $\rightarrow$ (Q $\wedge$ R)))
3701     ((P $\rightarrow$ (Q $\wedge$ R)) $\rightarrow$ ((P $\rightarrow$ Q) $\wedge$ (P $\rightarrow$ R))).
3702   intros Simp3_27a.
3703   MP Simp3_27a n4_76a.
3704   specialize Simp3_27 with (P $\rightarrow$ Q) (P $\rightarrow$ Q $\wedge$ R).
3705   intros Simp3_27d.
3706   Syll Simp3_27d Simp3_27a Sa.
3707   specialize n5_3 with (P $\rightarrow$ Q) (P $\rightarrow$ R) (P $\rightarrow$ (Q $\wedge$ R)).
3708   intros n5_3a.
3709   rewrite Equiv4_01 in n5_3a.
3710   specialize Simp3_26 with
3711     (((P $\rightarrow$ Q) $\wedge$ (P $\rightarrow$ R)) $\rightarrow$ (P $\rightarrow$ (Q $\wedge$ R))) $\rightarrow$ 
3712     (((P $\rightarrow$ Q) $\wedge$ (P $\rightarrow$ R)) $\rightarrow$ ((P $\rightarrow$ Q) $\wedge$ (P $\rightarrow$ (Q $\wedge$ R))))
3713     (((P $\rightarrow$ Q) $\wedge$ (P $\rightarrow$ R)) $\rightarrow$ ((P $\rightarrow$ Q) $\wedge$ (P $\rightarrow$ (Q $\wedge$ R))))
3714      $\rightarrow$ ((P $\rightarrow$ Q) $\wedge$ (P $\rightarrow$ R)) $\rightarrow$ (P $\rightarrow$ (Q $\wedge$ R))).
3715   intros Simp3_26b.
3716   MP Simp3_26b n5_3a.
3717   specialize Simp3_27 with
3718     (((P $\rightarrow$ Q) $\wedge$ (P $\rightarrow$ R)) $\rightarrow$ (P $\rightarrow$ (Q $\wedge$ R))) $\rightarrow$ 

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3719  (((P→Q) ∧ (P→R)) → ((P→Q) ∧ (P→(Q ∧ R))))
3720  (((((P→Q) ∧ (P→R)) → ((P→Q) ∧ (P→(Q ∧ R))))
3721  → (((P→Q) ∧ (P→R)) → (P→(Q ∧ R))))).
3722  intros Simp3_27b.
3723  MP Simp3_27b n5_3a.
3724  MP Simp3_26a Simp3_26b.
3725  MP Simp3_27a Simp3_27b.
3726  clear n4_76a. clear Simp3_26a. clear Simp3_27a.
3727  clear Simp3_27b. clear Simp3_27d. clear n5_3a.
3728  Conj Simp3_26b Sa C.
3729  Equiv C.
3730  specialize n5_32 with (P→Q) (P→R) (P→(Q ∧ R)).
3731  intros n5_32a.
3732  rewrite Equiv4_01 in n5_32a.
3733  specialize Simp3_27 with
3734    (((P → Q) → (P → R) ↔ (P → Q ∧ R))
3735     → (P → Q) ∧ (P → R) ↔ (P → Q) ∧ (P → Q ∧ R))
3736    (((P → Q) ∧ (P → R) ↔ (P → Q) ∧ (P → Q ∧ R)
3737     → (P → Q) → (P → R) ↔ (P → Q ∧ R))).
3738  intros Simp3_27c.
3739  MP Simp3_27c n5_32a.
3740  specialize n4_21 with
3741    ((P→Q) ∧ (P→R)) ((P→Q) ∧ (P→(Q ∧ R))).
3742  intros n4_21a.
3743  apply propositional_extensionality in n4_21a.
3744  replace (((P→Q) ∧ (P→(Q ∧ R))) ↔ ((P→Q) ∧ (P→R)))
3745    with (((P→Q) ∧ (P→R)) ↔ ((P→Q) ∧ (P→(Q ∧ R))))
3746    in C by now apply n4_21a.
3747  MP Simp3_27c C.
3748  exact Simp3_27c.
3749  Qed.
3750
3751  Theorem n5_5 : ∀ P Q : Prop,
3752    P → ((P → Q) ↔ Q).
3753  Proof. intros P Q.
3754  specialize Ass3_35 with P Q.
3755  intros Ass3_35a.
3756  specialize Exp3_3 with P (P→Q) Q.
3757  intros Exp3_3a.
3758  MP Exp3_3a Ass3_35a.
3759  specialize Simp2_02 with P Q.
3760  intros Simp2_02a.

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3761 specialize Exp3_3 with P Q (P→Q).
3762 intros Exp3_3b.
3763 specialize n3_42 with P Q (P→Q). (*Not cited*)
3764 intros n3_42a.
3765 MP n3_42a Simp2_02a.
3766 MP Exp3_3b n3_42a.
3767 clear n3_42a. clear Simp2_02a. clear Ass3_35a.
3768 Conj Exp3_3a Exp3_3b C.
3769 specialize n3_47 with P P ((P→Q)→Q) (Q→(P→Q)).
3770 intros n3_47a.
3771 MP n3_47a C.
3772 specialize n4_24 with P.
3773 intros n4_24a. (*Not cited*)
3774 apply propositional_extensionality in n4_24a.
3775 replace (P∧P) with P in n3_47a by now apply n4_24a.
3776 replace (((P→Q)→Q)∧(Q→(P→Q))) with ((P→Q)↔Q)
3777   in n3_47a by now rewrite Equiv4_01.
3778 exact n3_47a.
3779 Qed.
3780
3781 Theorem n5_501 : ∀ P Q : Prop,
3782   P → (Q ↔ (P ↔ Q)).
3783 Proof. intros P Q.
3784 specialize n5_1 with P Q.
3785 intros n5_1a.
3786 specialize Exp3_3 with P Q (P↔Q).
3787 intros Exp3_3a.
3788 MP Exp3_3a n5_1a.
3789 specialize Ass3_35 with P Q.
3790 intros Ass3_35a.
3791 specialize Simp3_26 with (P∧(P→Q)) (Q→P).
3792 intros Simp3_26a. (*Not cited*)
3793 Syll Simp3_26a Ass3_35a Sa.
3794 specialize n4_32 with P (P→Q) (Q→P).
3795 intros n4_32a. (*Not cited*)
3796 apply propositional_extensionality in n4_32a.
3797 replace ((P∧(P→Q))∧(Q→P)) with
3798   (P∧((P→Q)∧(Q→P))) in Sa by now apply n4_32a.
3799 replace ((P→Q)∧(Q→P)) with (P↔Q) in Sa
3800   by now rewrite Equiv4_01.
3801 specialize Exp3_3 with P (P↔Q) Q.
3802 intros Exp3_3b.

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3803 MP Exp3_3b Sa.
3804 clear n5_1a. clear Ass3_35a. clear n4_32a.
3805 clear Simp3_26a. clear Sa.
3806 Conj Exp3_3a Exp3_3b C.
3807 specialize n4_76 with P (Q → (P ↔ Q)) ((P ↔ Q) → Q).
3808 intros n4_76a. (*Not cited*)
3809 apply propositional_extensionality in n4_76a.
3810 replace ((P → Q → P ↔ Q) ∧ (P → P ↔ Q → Q)) with
3811 ((P → (Q → P ↔ Q) ∧ (P ↔ Q → Q))) in C
3812 by now apply n4_76a.
3813 replace ((Q → (P ↔ Q)) ∧ ((P ↔ Q) → Q)) with
3814 (Q ↔ (P ↔ Q)) in C by now rewrite Equiv4_01.
3815 exact C.
3816 Qed.
3817
3818 Theorem n5_53 : ∀ P Q R S : Prop,
3819 ((P ∨ Q) ∨ R) → S ↔ ((P → S) ∧ (Q → S)) ∧ (R → S)).
3820 Proof. intros P Q R S.
3821 specialize n4_77 with S (P ∨ Q) R.
3822 intros n4_77a.
3823 specialize n4_77 with S P Q.
3824 intros n4_77b.
3825 apply propositional_extensionality in n4_77b.
3826 replace (P ∨ Q → S) with
3827 ((P → S) ∧ (Q → S)) in n4_77a
3828 by now apply n4_77b. (*Not cited*)
3829 specialize n4_21 with ((P ∨ Q) ∨ R → S)
3830 ((P → S) ∧ (Q → S)) ∧ (R → S)).
3831 intros n4_21a. (*Not cited*)
3832 apply propositional_extensionality in n4_21a.
3833 replace (((P → S) ∧ (Q → S)) ∧ (R → S)) ↔ (((P ∨ Q) ∨ R) → S))
3834 with
3835 (((P ∨ Q) ∨ R) → S) ↔ ((P → S) ∧ (Q → S)) ∧ (R → S))
3836 in n4_77a by now apply n4_21.
3837 exact n4_77a.
3838 Qed.
3839
3840 Theorem n5_54 : ∀ P Q : Prop,
3841 ((P ∧ Q) ↔ P) ∨ ((P ∧ Q) ↔ Q).
3842 Proof. intros P Q.
3843 specialize n4_73 with P Q.
3844 intros n4_73a.

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3845 specialize n4_44 with Q P.
3846 intros n4_44a.
3847 specialize Transp2_16 with Q (P ↔ (P ∧ Q)).
3848 intros Transp2_16a.
3849 MP n4_73a Transp2_16a.
3850 specialize n4_3 with P Q.
3851 intros n4_3a. (*Not cited*)
3852 apply propositional_extensionality in n4_3a.
3853 replace (Q ∧ P) with (P ∧ Q) in n4_44a
3854   by now apply n4_3a.
3855 specialize Transp4_11 with Q (Q ∨ (P ∧ Q)).
3856 intros Transp4_11a.
3857 apply propositional_extensionality in Transp4_11a.
3858 replace (Q ↔ Q ∨ P ∧ Q) with
3859   (¬Q ↔ ¬(Q ∨ P ∧ Q)) in n4_44a by now apply Transp4_11a.
3860 apply propositional_extensionality in n4_44a.
3861 replace (¬Q) with (¬(Q ∨ P ∧ Q)) in Transp2_16a
3862   by now apply n4_44a.
3863 specialize n4_56 with Q (P ∧ Q).
3864 intros n4_56a. (*Not cited*)
3865 apply propositional_extensionality in n4_56a.
3866 replace (¬(Q ∨ P ∧ Q)) with
3867   (¬Q ∧ ¬(P ∧ Q)) in Transp2_16a
3868   by now apply n4_56a.
3869 specialize n5_1 with (¬Q) (¬(P ∧ Q)).
3870 intros n5_1a.
3871 Syll Transp2_16a n5_1a Sa.
3872 replace (¬(P ↔ P ∧ Q) → (¬Q ↔ ¬(P ∧ Q))) with
3873   (¬¬(P ↔ P ∧ Q) ∨ (¬Q ↔ ¬(P ∧ Q))) in Sa
3874   by now rewrite Impl1_01. (*Not cited*)
3875 specialize n4_13 with (P ↔ (P ∧ Q)).
3876 intros n4_13a. (*Not cited*)
3877 apply propositional_extensionality in n4_13a.
3878 replace (¬¬(P ↔ P ∧ Q)) with (P ↔ P ∧ Q) in Sa
3879   by now apply n4_13a.
3880 specialize Transp4_11 with Q (P ∧ Q).
3881 intros Transp4_11b.
3882 apply propositional_extensionality in Transp4_11b.
3883 replace (¬Q ↔ ¬(P ∧ Q)) with (Q ↔ (P ∧ Q)) in Sa
3884   by now apply Transp4_11b.
3885 specialize n4_21 with (P ∧ Q) Q.
3886 intros n4_21a. (*Not cited*)

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3887   apply propositional_extensionality in n4_21a.
3888   replace (Q ↔ (P ∧ Q)) with ((P ∧ Q) ↔ Q) in Sa
3889   by now apply n4_21a.
3890   specialize n4_21 with (P ∧ Q) P.
3891   intros n4_21b. (*Not cited*)
3892   apply propositional_extensionality in n4_21b.
3893   replace (P ↔ (P ∧ Q)) with ((P ∧ Q) ↔ P) in Sa
3894   by now apply n4_21b.
3895   exact Sa.
3896 Qed.
3897
3898 Theorem n5_55 : ∀ P Q : Prop,
3899   ((P ∨ Q) ↔ P) ∨ ((P ∨ Q) ↔ Q).
3900 Proof. intros P Q.
3901   specialize Add1_3 with (P ∧ Q) (P).
3902   intros Add1_3a.
3903   specialize n4_3 with P Q.
3904   intros n4_3a. (*Not cited*)
3905   apply propositional_extensionality in n4_3a.
3906   specialize n4_41 with P Q P.
3907   intros n4_41a. (*Not cited*)
3908   replace (Q ∧ P) with (P ∧ Q) in n4_41a
3909   by now apply n4_3a.
3910   specialize n4_31 with (P ∧ Q) P.
3911   intros n4_31a.
3912   apply propositional_extensionality in n4_31a.
3913   replace (P ∨ P ∧ Q) with (P ∧ Q ∨ P) in n4_41a
3914   by now apply n4_31a.
3915   apply propositional_extensionality in n4_41a.
3916   replace ((P ∧ Q) ∨ P) with ((P ∨ Q) ∧ (P ∨ P)) in Add1_3a
3917   by now apply n4_4a.
3918   specialize n4_25 with P.
3919   intros n4_25a. (*Not cited*)
3920   apply propositional_extensionality in n4_25a.
3921   replace (P ∨ P) with P in Add1_3a
3922   by now apply n4_25a.
3923   specialize n4_31 with P Q.
3924   intros n4_31b.
3925   apply propositional_extensionality in n4_31b.
3926   replace (Q ∨ P) with (P ∨ Q) in Add1_3a
3927   by now apply n4_31b.
3928   specialize n5_1 with P (P ∨ Q).

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3929   intros n5_1a.
3930   specialize n4_3 with (P ∨ Q) P.
3931   intros n4_3b.
3932   apply propositional_extensionality in n4_3b.
3933   replace ((P ∨ Q) ∧ P) with (P ∧ (P ∨ Q)) in Add1_3a
3934     by now apply n4_3b.
3935   Syll Add1_3a n5_1a Sa.
3936   specialize n4_74 with P Q.
3937   intros n4_74a.
3938   specialize Transp2_15 with P (Q ↔ P ∨ Q).
3939   intros Transp2_15a. (*Not cited*)
3940   MP Transp2_15a n4_74a.
3941   Syll Transp2_15a Sa Sb.
3942   replace (¬ (Q ↔ P ∨ Q) → P ↔ P ∨ Q) with
3943     (¬¬(Q ↔ P ∨ Q) ∨ (P ↔ P ∨ Q)) in Sb
3944     by now rewrite Impl1_01.
3945   specialize n4_13 with (Q ↔ P ∨ Q).
3946   intros n4_13a. (*Not cited*)
3947   apply propositional_extensionality in n4_13a.
3948   replace (¬¬(Q ↔ (P ∨ Q))) with (Q ↔ (P ∨ Q)) in Sb
3949     by now apply n4_13a.
3950   specialize n4_21 with (P ∨ Q) Q.
3951   intros n4_21a. (*Not cited*)
3952   apply propositional_extensionality in n4_21a.
3953   replace (Q ↔ (P ∨ Q)) with ((P ∨ Q) ↔ Q) in Sb
3954     by now apply n4_21a.
3955   specialize n4_21 with (P ∨ Q) P.
3956   intros n4_21b. (*Not cited*)
3957   apply propositional_extensionality in n4_21b.
3958   replace (P ↔ (P ∨ Q)) with ((P ∨ Q) ↔ P) in Sb
3959     by now apply n4_21b.
3960   apply n4_31 in Sb.
3961   exact Sb.
3962   Qed.
3963
3964   Theorem n5_6 : ∀ P Q R : Prop,
3965     ((P ∧ ¬Q) → R) ↔ (P → (Q ∨ R)).
3966   Proof. intros P Q R.
3967   specialize n4_87 with P (¬Q) R.
3968   intros n4_87a.
3969   specialize n4_64 with Q R.
3970   intros n4_64a.

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3971 specialize n4_85 with P Q R.
3972 intros n4_85a.
3973 replace (((P ∧ ¬Q → R) ↔ (P → ¬Q → R)) ↔ ((¬Q → P → R) ↔ (¬Q ∧ P → R)))
3974 with
3975 (((P ∧ ¬Q → R) ↔ (P → ¬Q → R)) → ((¬Q → P → R) ↔ (¬Q ∧ P → R)))
3976 ∧
3977 (((¬Q → P → R) ↔ (¬Q ∧ P → R)) → (((P ∧ ¬Q → R) ↔ (P → ¬Q → R))))
3978 in n4_87a by now rewrite Equiv4_01.
3979 specialize Simp3_27 with
3980 (((P ∧ ¬Q → R) ↔ (P → ¬Q → R) → (¬Q → P → R) ↔ (¬Q ∧ P → R)))
3981 (((¬Q → P → R) ↔ (¬Q ∧ P → R) → (P ∧ ¬Q → R) ↔ (P → ¬Q → R))).
3982 intros Simp3_27a.
3983 MP Simp3_27a n4_87a.
3984 specialize Imp3_31 with (¬Q) P R.
3985 intros Imp3_31a.
3986 specialize Exp3_3 with (¬Q) P R.
3987 intros Exp3_3a.
3988 Conj Imp3_31a Exp3_3a C.
3989 Equiv C.
3990 MP Simp3_27a C.
3991 apply propositional_extensionality in n4_64a.
3992 replace (¬Q → R) with (Q ∨ R) in Simp3_27a
3993 by now apply n4_64a.
3994 exact Simp3_27a.
3995 Qed.
3996
3997 Theorem n5_61 : ∀ P Q : Prop,
3998 ((P ∨ Q) ∧ ¬Q) ↔ (P ∧ ¬Q).
3999 Proof. intros P Q.
4000 specialize n4_74 with Q P.
4001 intros n4_74a.
4002 specialize n5_32 with (¬Q) P (Q ∨ P).
4003 intros n5_32a.
4004 apply propositional_extensionality in n5_32a.
4005 replace (¬Q → P ↔ Q ∨ P) with
4006 (¬Q ∧ P ↔ ¬Q ∧ (Q ∨ P)) in n4_74a
4007 by now apply n5_32a.
4008 specialize n4_3 with P (¬Q).
4009 intros n4_3a. (*Not cited*)
4010 apply propositional_extensionality in n4_3a.
4011 replace (¬Q ∧ P) with (P ∧ ¬Q) in n4_74a
4012 by now apply n4_3a.

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4013 specialize n4_3 with (Q ∨ P) (¬Q).
4014 intros n4_3b. (*Not cited*)
4015 apply propositional_extensionality in n4_3b.
4016 replace (¬Q ∧ (Q ∨ P)) with ((Q ∨ P) ∧ ¬Q) in n4_74a
4017   by now apply n4_3b.
4018 specialize n4_31 with P Q.
4019 intros n4_31a. (*Not cited*)
4020 apply propositional_extensionality in n4_31a.
4021 replace (Q ∨ P) with (P ∨ Q) in n4_74a
4022   by now apply n4_31a.
4023 specialize n4_21 with ((P ∨ Q) ∧ ¬Q) (P ∧ ¬Q).
4024 intros n4_21a. (*Not cited*)
4025 apply propositional_extensionality in n4_21a.
4026 replace (P ∧ ¬Q ↔ (P ∨ Q) ∧ ¬Q) with
4027   ((P ∨ Q) ∧ ¬Q ↔ P ∧ ¬Q) in n4_74a
4028   by now apply n4_21a.
4029 exact n4_74a.
4030 Qed.
4031
4032 Theorem n5_62 : ∀ P Q : Prop,
4033   ((P ∧ Q) ∨ ¬Q) ↔ (P ∨ ¬Q).
4034 Proof. intros P Q.
4035 specialize n4_7 with Q P.
4036 intros n4_7a.
4037 replace (Q → P) with (¬Q ∨ P) in n4_7a
4038   by now rewrite Impl1_01.
4039 replace (Q → (Q ∧ P)) with (¬Q ∨ (Q ∧ P)) in n4_7a
4040   by now rewrite Impl1_01.
4041 specialize n4_31 with (Q ∧ P) (¬Q).
4042 intros n4_31a. (*Not cited*)
4043 apply propositional_extensionality in n4_31a.
4044 replace (¬Q ∨ (Q ∧ P)) with ((Q ∧ P) ∨ ¬Q) in n4_7a
4045   by now apply n4_31a.
4046 specialize n4_31 with P (¬Q).
4047 intros n4_31b. (*Not cited*)
4048 apply propositional_extensionality in n4_31b.
4049 replace (¬Q ∨ P) with (P ∨ ¬Q) in n4_7a
4050   by now apply n4_31b.
4051 specialize n4_3 with P Q.
4052 intros n4_3a. (*Not cited*)
4053 apply propositional_extensionality in n4_3a.
4054 replace (Q ∧ P) with (P ∧ Q) in n4_7a

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4055     by now apply n4_3a.
4056 specialize n4_21 with (P ∧ Q ∨ ¬Q) (P ∨ ¬Q).
4057 intros n4_21a. (*Not cited*)
4058 apply propositional_extensionality in n4_21a.
4059 replace (P ∨ ¬Q ↔ P ∧ Q ∨ ¬Q) with
4060     (P ∧ Q ∨ ¬Q ↔ P ∨ ¬Q) in n4_7a
4061     by now apply n4_21a.
4062 exact n4_7a.
4063 Qed.
4064
4065 Theorem n5_63 : ∀ P Q : Prop,
4066     (P ∨ Q) ↔ (P ∨ (¬P ∧ Q)).
4067 Proof. intros P Q.
4068 specialize n5_62 with Q (¬P).
4069 intros n5_62a.
4070 specialize n4_13 with P.
4071 intros n4_13a. (*Not cited*)
4072 apply propositional_extensionality in n4_13a.
4073 replace (¬¬P) with P in n5_62a
4074     by now apply n4_13a.
4075 specialize n4_31 with P Q.
4076 intros n4_31a. (*Not cited*)
4077 apply propositional_extensionality in n4_31a.
4078 replace (Q ∨ P) with (P ∨ Q) in n5_62a
4079     by now apply n4_31a.
4080 specialize n4_31 with P (Q ∧ ¬P).
4081 intros n4_31b. (*Not cited*)
4082 apply propositional_extensionality in n4_31b.
4083 replace ((Q ∧ ¬P) ∨ P) with (P ∨ (Q ∧ ¬P)) in n5_62a
4084     by now apply n4_31b.
4085 specialize n4_21 with (P ∨ Q) (P ∨ (Q ∧ ¬P)).
4086 intros n4_21a. (*Not cited*)
4087 apply propositional_extensionality in n4_21a.
4088 replace (P ∨ Q ∧ ¬P ↔ P ∨ Q) with
4089     (P ∨ Q ↔ P ∨ Q ∧ ¬P) in n5_62a
4090     by now apply n4_21a.
4091 specialize n4_3 with (¬P) Q.
4092 intros n4_3a. (*Not cited*)
4093 apply propositional_extensionality in n4_3a.
4094 replace (Q ∧ ¬P) with (¬P ∧ Q) in n5_62a
4095     by now apply n4_3a.
4096 exact n5_62a.

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4097 Qed.
4098
4099 Theorem n5_7 :  $\forall P Q R : \text{Prop}$ ,
4100    $((P \vee R) \leftrightarrow (Q \vee R)) \leftrightarrow (R \vee (P \leftrightarrow Q))$ .
4101 Proof. intros P Q R.
4102   specialize n4_74 with R P.
4103   intros n4_74a.
4104   specialize n4_74 with R Q.
4105   intros n4_74b. (*Greg's suggestion*)
4106   Conj n4_74a n4_74b Ca.
4107   specialize Comp3_43 with
4108      $(\neg R) (P \leftrightarrow R \vee P) (Q \leftrightarrow R \vee Q)$ .
4109   intros Comp3_43a.
4110   MP Comp3_43a Ca.
4111   specialize n4_22 with P  $(R \vee P) (R \vee Q)$ .
4112   intros n4_22a.
4113   specialize n4_22 with P  $(R \vee Q) Q$ .
4114   intros n4_22b.
4115   specialize Exp3_3 with  $(P \leftrightarrow (R \vee Q))$ 
4116      $((R \vee Q) \leftrightarrow Q) (P \leftrightarrow Q)$ .
4117   intros Exp3_3a.
4118   MP Exp3_3a n4_22b.
4119   Syll n4_22a Exp3_3a Sa.
4120   specialize Imp3_31 with  $((P \leftrightarrow (R \vee P)) \wedge$ 
4121      $((R \vee P) \leftrightarrow (R \vee Q))) ((R \vee Q) \leftrightarrow Q) (P \leftrightarrow Q)$ .
4122   intros Imp3_31a.
4123   MP Imp3_31a Sa.
4124   specialize n4_32 with  $(P \leftrightarrow R \vee P) (R \vee P \leftrightarrow R \vee Q) (R \vee Q \leftrightarrow Q)$ .
4125   intros n4_32a.
4126   apply propositional_extensionality in n4_32a.
4127   replace  $((P \leftrightarrow (R \vee P)) \wedge ((R \vee P) \leftrightarrow$ 
4128      $(R \vee Q))) \wedge ((R \vee Q) \leftrightarrow Q)$  with
4129      $((P \leftrightarrow (R \vee P)) \wedge ((R \vee P) \leftrightarrow$ 
4130      $(R \vee Q))) \wedge ((R \vee Q) \leftrightarrow Q))$  in Imp3_31a
4131   by now apply n4_32a.
4132   specialize n4_3 with  $(R \vee Q \leftrightarrow Q) (R \vee P \leftrightarrow R \vee Q)$ .
4133   intros n4_3a.
4134   apply propositional_extensionality in n4_3a.
4135   replace  $((R \vee P \leftrightarrow R \vee Q) \wedge (R \vee Q \leftrightarrow Q))$  with
4136      $((R \vee Q \leftrightarrow Q) \wedge (R \vee P \leftrightarrow R \vee Q))$  in Imp3_31a
4137   by now apply n4_3a.
4138   specialize n4_32 with  $(P \leftrightarrow R \vee P) (R \vee Q \leftrightarrow Q) (R \vee P \leftrightarrow R \vee Q)$ .

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4139 intros n4_32b.
4140 apply propositional_extensionality in n4_32b.
4141 replace ((P↔(R∨P)) ∧
4142         ((R ∨ Q ↔ Q) ∧ (R ∨ P ↔ R ∨ Q))) with
4143         (((P↔(R∨P)) ∧ (R ∨ Q ↔ Q)) ∧
4144          (R ∨ P ↔ R ∨ Q)) in Imp3_31a
4145 by now apply n4_32b.
4146 specialize Exp3_3 with
4147         ((P↔(R∨P)) ∧ (R∨Q↔Q))
4148         (R ∨ P ↔ R ∨ Q) (P ↔ Q).
4149 intros Exp3_3b.
4150 MP Exp3_3b Imp3_31a.
4151 specialize n4_21 with Q (R∨Q).
4152 intros n4_21a.
4153 apply propositional_extensionality in n4_21a.
4154 replace (Q↔R∨Q) with (R∨Q↔Q) in Comp3_43a
4155 by now apply n4_21a.
4156 Syll Comp3_43a Exp3_3b Sb.
4157 specialize n4_31 with P R.
4158 intros n4_31a.
4159 apply propositional_extensionality in n4_31a.
4160 replace (R∨P) with (P∨R) in Sb by now apply n4_31a.
4161 specialize n4_31 with Q R.
4162 intros n4_31b.
4163 apply propositional_extensionality in n4_31b.
4164 replace (R∨Q) with (Q∨R) in Sb by now apply n4_31b.
4165 specialize Imp3_31 with (¬R) (P∨R↔Q∨R) (P↔Q).
4166 intros Imp3_31b.
4167 MP Imp3_31b Sb.
4168 specialize n4_3 with (P ∨ R ↔ Q ∨ R) (¬R).
4169 intros n4_3b.
4170 apply propositional_extensionality in n4_3b.
4171 replace (¬R ∧ (P ∨ R ↔ Q ∨ R)) with
4172         ((P ∨ R ↔ Q ∨ R) ∧ ¬R) in Imp3_31b
4173 by now apply n4_3b.
4174 specialize Exp3_3 with
4175         (P ∨ R ↔ Q ∨ R) (¬R) (P ↔ Q).
4176 intros Exp3_3c.
4177 MP Exp3_3c Imp3_31b.
4178 replace (¬R→(P↔Q)) with (¬¬R∨(P↔Q))
4179 in Exp3_3c by now rewrite Impl1_01.
4180 specialize n4_13 with R.

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4181  intros n4_13a.
4182  apply propositional_extensionality in n4_13a.
4183  replace ( $\neg\neg R$ ) with R in Exp3_3c
4184    by now apply n4_13a.
4185  specialize Add1_3 with P R.
4186  intros Add1_3a.
4187  specialize Add1_3 with Q R.
4188  intros Add1_3b.
4189  Conj Add1_3a Add1_3b Cb.
4190  specialize Comp3_43 with (R) (P $\vee$ R) (Q $\vee$ R).
4191  intros Comp3_43b.
4192  MP Comp3_43b Cb.
4193  specialize n5_1 with (P  $\vee$  R) (Q  $\vee$  R).
4194  intros n5_1a.
4195  Syll Comp3_43b n5_1a Sc.
4196  specialize n4_37 with P Q R.
4197  intros n4_37a.
4198  Conj Sc n4_37a Cc.
4199  specialize n4_77 with (P  $\vee$  R  $\leftrightarrow$  Q  $\vee$  R)
4200    R (P $\leftrightarrow$ Q).
4201  intros n4_77a.
4202  rewrite Equiv4_01 in n4_77a.
4203  specialize Simp3_26 with
4204    ((R  $\rightarrow$  P  $\vee$  R  $\leftrightarrow$  Q  $\vee$  R)  $\wedge$ 
4205     (P  $\leftrightarrow$  Q  $\rightarrow$  P  $\vee$  R  $\leftrightarrow$  Q  $\vee$  R)
4206      $\rightarrow$  R  $\vee$  (P  $\leftrightarrow$  Q)  $\rightarrow$  P  $\vee$  R  $\leftrightarrow$  Q  $\vee$  R)
4207    ((R  $\vee$  (P  $\leftrightarrow$  Q)  $\rightarrow$  P  $\vee$  R  $\leftrightarrow$  Q  $\vee$  R)
4208      $\rightarrow$  (R  $\rightarrow$  P  $\vee$  R  $\leftrightarrow$  Q  $\vee$  R)  $\wedge$ 
4209     (P  $\leftrightarrow$  Q  $\rightarrow$  P  $\vee$  R  $\leftrightarrow$  Q  $\vee$  R)).
4210  intros Simp3_26a.
4211  MP Simp3_26 n4_77a.
4212  MP Simp3_26a Cc.
4213  clear n4_77a. clear Cc. clear n4_37a. clear Sa.
4214    clear n5_1a. clear Comp3_43b. clear Cb.
4215    clear Add1_3a. clear Add1_3b. clear Ca. clear Imp3_31b.
4216    clear n4_74a. clear n4_74b. clear Comp3_43a.
4217    clear Imp3_31a. clear n4_22a. clear n4_22b.
4218    clear Exp3_3a. clear Exp3_3b. clear Sb. clear Sc.
4219    clear n4_13a. clear n4_3a. clear n4_3b. clear n4_21a.
4220    clear n4_31a. clear n4_31b. clear n4_32a. clear n4_32b.
4221  Conj Exp3_3c Simp3_26a Cdd.
4222  Equiv Cdd.

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4223     exact Cdd.
4224 Qed.
4225
4226 Theorem n5_71 :  $\forall P Q R : \text{Prop}$ ,
4227    $(Q \rightarrow \neg R) \rightarrow ((P \vee Q) \wedge R) \leftrightarrow (P \wedge R)$ .
4228 Proof. intros P Q R.
4229   specialize n4_62 with Q R.
4230   intros n4_62a.
4231   specialize n4_51 with Q R.
4232   intros n4_51a.
4233   specialize n4_21 with  $(\neg(Q \wedge R)) (\neg(Q \vee \neg R))$ .
4234   intros n4_21a.
4235   rewrite Equiv4_01 in n4_21a.
4236   specialize Simp3_26 with
4237      $((\neg(Q \wedge R) \leftrightarrow (\neg(Q \vee \neg R))) \rightarrow ((\neg(Q \vee \neg R) \leftrightarrow \neg(Q \wedge R))))$ 
4238      $((\neg(Q \vee \neg R) \leftrightarrow \neg(Q \wedge R)) \rightarrow (\neg(Q \wedge R) \leftrightarrow (\neg(Q \vee \neg R))))$ .
4239   intros Simp3_26a.
4240   MP Simp3_26a n4_21a.
4241   MP Simp3_26a n4_51a.
4242   clear n4_21a. clear n4_51a.
4243   Conj n4_62a Simp3_26a C.
4244   specialize n4_22 with
4245      $(Q \rightarrow \neg R) (\neg(Q \vee \neg R) (\neg(Q \wedge R)))$ .
4246   intros n4_22a.
4247   MP n4_22a C.
4248   replace  $((Q \rightarrow \neg R) \leftrightarrow \neg(Q \wedge R))$  with
4249      $((Q \rightarrow \neg R) \rightarrow \neg(Q \wedge R))$ 
4250      $\wedge$ 
4251      $(\neg(Q \wedge R) \rightarrow (Q \rightarrow \neg R))$  in n4_22a
4252     by now rewrite Equiv4_01.
4253   specialize Simp3_26 with
4254      $((Q \rightarrow \neg R) \rightarrow \neg(Q \wedge R)) (\neg(Q \wedge R) \rightarrow (Q \rightarrow \neg R))$ .
4255   intros Simp3_26b.
4256   MP Simp3_26b n4_22a.
4257   specialize n4_74 with  $(Q \wedge R) (P \wedge R)$ .
4258   intros n4_74a.
4259   Syll Simp3_26a n4_74a Sa.
4260   specialize n4_31 with  $(Q \wedge R) (P \wedge R)$ .
4261   intros n4_31a. (*Not cited*)
4262   apply propositional_extensionality in n4_31a.
4263   replace  $((P \wedge R) \vee (Q \wedge R))$  with  $((Q \wedge R) \vee (P \wedge R))$ 
4264     in Sa by now rewrite n4_31a.

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4265 specialize n4_31 with (R $\wedge$ Q) (R $\wedge$ P).
4266 intros n4_31b. (*Not cited*)
4267 apply propositional_extensionality in n4_31b.
4268 specialize n4_21 with ((P $\vee$ Q) $\wedge$ R) (P $\wedge$ R).
4269 intros n4_21a. (*Not cited*)
4270 apply propositional_extensionality in n4_21a.
4271 specialize n4_4 with R P Q.
4272 intros n4_4a.
4273 replace (R  $\wedge$  P  $\vee$  R  $\wedge$  Q) with (R  $\wedge$  Q  $\vee$  R  $\wedge$  P)
4274   in n4_4a by now apply n4_31b.
4275 specialize n4_3 with P R.
4276 intros n4_3a.
4277 apply propositional_extensionality in n4_3a.
4278 replace (R  $\wedge$  P) with (P  $\wedge$  R) in n4_4a
4279   by now apply n4_3a.
4280 specialize n4_3 with Q R.
4281 intros n4_3b.
4282 apply propositional_extensionality in n4_3b.
4283 replace (R  $\wedge$  Q) with (Q  $\wedge$  R) in n4_4a
4284   by now apply n4_3b.
4285 apply propositional_extensionality in n4_4a.
4286 replace ((Q $\wedge$ R) $\vee$ (P $\wedge$ R)) with (R $\wedge$ (P $\vee$ Q)) in Sa
4287   by now apply n4_4a.
4288 specialize n4_3 with (P $\vee$ Q) R.
4289 intros n4_3c. (*Not cited*)
4290 apply propositional_extensionality in n4_3c.
4291 replace (R $\wedge$ (P $\vee$ Q)) with ((P $\vee$ Q) $\wedge$ R) in Sa
4292   by now apply n4_3c.
4293 replace ((P $\wedge$ R) $\leftrightarrow$ ((P $\vee$ Q) $\wedge$ R)) with
4294   (((P $\vee$ Q) $\wedge$ R) $\leftrightarrow$ (P $\wedge$ R)) in Sa
4295   by now apply n4_21a.
4296 exact Sa.
4297 Qed.
4298
4299 Theorem n5_74 :  $\forall$  P Q R : Prop,
4300   (P  $\rightarrow$  (Q  $\leftrightarrow$  R))  $\leftrightarrow$  ((P  $\rightarrow$  Q)  $\leftrightarrow$  (P  $\rightarrow$  R)).
4301 Proof. intros P Q R.
4302 specialize n5_41 with P Q R.
4303 intros n5_41a.
4304 specialize n5_41 with P R Q.
4305 intros n5_41b.
4306 Conj n5_41a n5_41b C.

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4307 specialize n4_38 with
4308   ((P→Q)→(P→R)) ((P→R)→(P→Q))
4309   (P→Q→R) (P→R→Q).
4310 intros n4_38a.
4311 MP n4_38a C.
4312 replace (((P→Q)→(P→R))∧((P→R)→(P→Q)))
4313   with ((P→Q)↔(P→R)) in n4_38a
4314   by now rewrite Equiv4_01.
4315 specialize n4_76 with P (Q→R) (R→Q).
4316 intros n4_76a.
4317 replace ((Q→R)∧(R→Q)) with (Q↔R) in n4_76a
4318   by now rewrite Equiv4_01.
4319 apply propositional_extensionality in n4_76a.
4320 replace ((P→Q→R)∧(P→R→Q)) with
4321   (P→(Q↔R)) in n4_38a by now apply n4_76a.
4322 specialize n4_21 with (P→Q↔R)
4323   ((P→Q)↔(P→R)).
4324 intros n4_21a. (*Not cited*)
4325 apply propositional_extensionality in n4_21a.
4326 replace (((P→Q)↔(P→R))↔(P→Q↔R)) with
4327   ((P→(Q↔R))↔((P→Q)↔(P→R))) in n4_38a
4328   by now apply n4_21a.
4329 exact n4_38a.
4330 Qed.
4331
4332 Theorem n5_75 : ∀ P Q R : Prop,
4333   ((R → ¬Q) ∧ (P ↔ Q ∨ R)) → ((P ∧ ¬Q) ↔ R).
4334 Proof. intros P Q R.
4335 specialize n5_6 with P Q R.
4336 intros n5_6a.
4337 replace ((P∧¬Q→R)↔(P→Q∨R)) with
4338   (((P∧¬Q→R)→(P→Q∨R)) ∧
4339    ((P→Q∨R)→(P∧¬Q→R))) in n5_6a
4340   by now rewrite Equiv4_01.
4341 specialize Simp3_27 with
4342   ((P∧¬Q→R)→(P→Q∨R))
4343   ((P→Q∨R)→(P∧¬Q→R)).
4344 intros Simp3_27a.
4345 MP Simp3_27a n5_6a.
4346 specialize Simp3_26 with
4347   (P→(Q∨R)) ((Q∨R)→P).
4348 intros Simp3_26a.

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4349  replace ((P→(Q∨R))∧((Q∨R)→P)) with
4350      (P↔(Q∨R)) in Simp3_26a
4351      by now rewrite Equiv4_01.
4352  Syll Simp3_26a Simp3_27a Sa.
4353  specialize Simp3_27 with
4354      (R→¬Q) (P↔(Q∨R)).
4355  intros Simp3_27b.
4356  Syll Simp3_27b Sa Sb.
4357  specialize Simp3_27 with
4358      (P→(Q∨R)) ((Q∨R)→P).
4359  intros Simp3_27c.
4360  replace ((P→(Q∨R))∧((Q∨R)→P)) with
4361      (P↔(Q∨R)) in Simp3_27c
4362      by now rewrite Equiv4_01.
4363  Syll Simp3_27b Simp3_27c Sc.
4364  specialize n4_77 with P Q R.
4365  intros n4_77a.
4366  apply propositional_extensionality in n4_77a.
4367  replace (Q∨R→P) with ((Q→P)∧(R→P)) in Sc
4368      by now apply n4_77a.
4369  specialize Simp3_27 with (Q→P) (R→P).
4370  intros Simp3_27d.
4371  Syll Sa Simp3_27d Sd.
4372  specialize Simp3_26 with (R→¬Q) (P↔(Q∨R)).
4373  intros Simp3_26b.
4374  Conj Sd Simp3_26b Ca.
4375  specialize Comp3_43 with
4376      ((R→¬Q)∧(P↔(Q∨R))) (R→P) (R→¬Q).
4377  intros Comp3_43a.
4378  MP Comp3_43a Ca.
4379  specialize Comp3_43 with R P (¬Q).
4380  intros Comp3_43b.
4381  Syll Comp3_43a Comp3_43b Se.
4382  clear n5_6a. clear Simp3_27a.
4383      clear Simp3_27c. clear Simp3_27d.
4384      clear Simp3_26a. clear Comp3_43b.
4385      clear Simp3_26b. clear Comp3_43a.
4386      clear Sa. clear Sc. clear Sd. clear Ca.
4387      clear n4_77a. clear Simp3_27b.
4388  Conj Sb Se Cb.
4389  specialize Comp3_43 with
4390      ((R→¬Q)∧(P↔Q∨R))

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4391      (P $\wedge$  $\neg$ Q $\rightarrow$ R) (R $\rightarrow$ P $\wedge$  $\neg$ Q) .
4392  intros Comp3_43c.
4393  MP Comp3_43c Cb.
4394  replace ((P $\wedge$  $\neg$ Q $\rightarrow$ R) $\wedge$ (R $\rightarrow$ P $\wedge$  $\neg$ Q)) with
4395      (P $\wedge$  $\neg$ Q $\leftrightarrow$ R) in Comp3_43c
4396      by now rewrite Equiv4_01.
4397  exact Comp3_43c.
4398  Qed.
4399
4400  End No5.

```