

Principia Mathematica's Propositional Logic in *Coq*

Landon D. C. Elkind*

February 12, 2021

Abstract

This file contains the *Coq* code for the *Principia* Rewrite project's encoding of the propositional logic given in *1 – *5. The Github repository with this *Coq* file is here: <https://github.com/LogicalAtomist/principia>. To receive updates about the project, visit the *Principia Rewrite* project page: <https://www.principiarewrite.com/>. You can also follow the *Principia* Rewrite project on Twitter: <https://twitter.com/thePMrewrite>.

```
1  Require Import Unicode.Utf8.
2
3  Module No1.
4  Import Unicode.Utf8.
5      (*We first give the axioms of Principia
6      for the propositional calculus in *1.*)
7
8  Axiom Impl1_01 :  $\forall$  P Q : Prop,
9      (P  $\rightarrow$  Q) = ( $\neg$ P  $\vee$  Q).
10      (*This is a definition in Principia: there  $\rightarrow$  is a
11      defined sign and  $\vee$ ,  $\neg$  are primitive ones. So
12      we will use this axiom to switch between
13      disjunction and implication.*)
14
15  Axiom MP1_1 :  $\forall$  P Q : Prop,
16      (P  $\rightarrow$  Q)  $\rightarrow$  P  $\rightarrow$  Q. (*Modus ponens*)
17
18      (*1.11 omitted: it is MP for propositions
19      containing variables. Likewise, omitted
20      the well-formedness rules 1.7, 1.71, 1.72*)
21
22  Axiom Taut1_2 :  $\forall$  P : Prop,
```

*Department of Philosophy, University of Alberta, `elkind at ualberta dot ca`. This research was conducted as an Izaak Walton Killam Postdoctoral Fellow.

```

23   P  $\vee$  P  $\rightarrow$  P. (*Tautology*)
24
25   Axiom Add1_3 :  $\forall$  P Q : Prop,
26     Q  $\rightarrow$  P  $\vee$  Q. (*Addition*)
27
28   Axiom Perm1_4 :  $\forall$  P Q : Prop,
29     P  $\vee$  Q  $\rightarrow$  Q  $\vee$  P. (*Permutation*)
30
31   Axiom Assoc1_5 :  $\forall$  P Q R : Prop,
32     P  $\vee$  (Q  $\vee$  R)  $\rightarrow$  Q  $\vee$  (P  $\vee$  R). (*Association*)
33
34   Axiom Sum1_6 :  $\forall$  P Q R : Prop,
35     (Q  $\rightarrow$  R)  $\rightarrow$  (P  $\vee$  Q  $\rightarrow$  P  $\vee$  R). (*Summation*)
36
37   (*These are all the propositional axioms of Principia.*)
38
39   End No1.
40
41   Module No2.
42   Import No1.
43
44   (*We proceed to the deductions of of Principia.*)
45
46   Theorem Abs2_01 :  $\forall$  P : Prop,
47     (P  $\rightarrow$   $\neg$ P)  $\rightarrow$   $\neg$ P.
48   Proof. intros P.
49     specialize Taut1_2 with ( $\neg$ P).
50     replace ( $\neg$ P  $\vee$   $\neg$ P) with (P  $\rightarrow$   $\neg$ P).
51     apply MP1_1.
52     apply Impl1_01.
53   Qed.
54
55   Theorem Simp2_02 :  $\forall$  P Q : Prop,
56     Q  $\rightarrow$  (P  $\rightarrow$  Q).
57   Proof. intros P Q.
58     specialize Add1_3 with ( $\neg$ P) Q.
59     replace ( $\neg$ P  $\vee$  Q) with (P  $\rightarrow$  Q).
60     apply (MP1_1 Q (P  $\rightarrow$  Q)).
61     apply Impl1_01.
62   Qed.
63
64   Theorem Transp2_03 :  $\forall$  P Q : Prop,

```

```

65   (P → ¬Q) → (Q → ¬P).
66 Proof. intros P Q.
67   specialize Perm1_4 with (¬P) (¬Q).
68   replace (¬P ∨ ¬Q) with (P → ¬Q).
69   replace (¬Q ∨ ¬P) with (Q → ¬P).
70   apply (MP1_1 (P → ¬Q) (Q → ¬P)).
71   apply Impl1_01.
72   apply Impl1_01.
73 Qed.
74
75 Theorem Comm2_04 : ∀ P Q R : Prop,
76   (P → (Q → R)) → (Q → (P → R)).
77 Proof. intros P Q R.
78   specialize Assoc1_5 with (¬P) (¬Q) R.
79   replace (¬Q ∨ R) with (Q → R).
80   replace (¬P ∨ (Q → R)) with (P → (Q → R)).
81   replace (¬P ∨ R) with (P → R).
82   replace (¬Q ∨ (P → R)) with (Q → (P → R)).
83   apply (MP1_1 (P → Q → R) (Q → P → R)).
84   apply Impl1_01.
85   apply Impl1_01.
86   apply Impl1_01.
87   apply Impl1_01.
88 Qed.
89
90 Theorem Syll2_05 : ∀ P Q R : Prop,
91   (Q → R) → ((P → Q) → (P → R)).
92 Proof. intros P Q R.
93   specialize Sum1_6 with (¬P) Q R.
94   replace (¬P ∨ Q) with (P → Q).
95   replace (¬P ∨ R) with (P → R).
96   apply (MP1_1 (Q → R) ((P → Q) → (P → R))).
97   apply Impl1_01.
98   apply Impl1_01.
99 Qed.
100
101 Theorem Syll2_06 : ∀ P Q R : Prop,
102   (P → Q) → ((Q → R) → (P → R)).
103 Proof. intros P Q R.
104   specialize Comm2_04 with (Q → R) (P → Q) (P → R).
105   intros Comm2_04.
106   specialize Syll2_05 with P Q R.

```

```

107   intros Syll2_05.
108   specialize MP1_1 with ((Q → R) → (P → Q) → P → R)
109       ((P → Q) → ((Q → R) → (P → R))).
110   intros MP1_1.
111   apply MP1_1.
112   apply Comm2_04.
113   apply Syll2_05.
114   Qed.
115
116   Theorem n2_07 : ∀ P : Prop,
117       P → (P ∨ P).
118   Proof. intros P.
119       specialize Add1_3 with P P.
120       apply MP1_1.
121   Qed.
122
123   Theorem Id2_08 : ∀ P : Prop,
124       P → P.
125   Proof. intros P.
126       specialize Syll2_05 with P (P ∨ P) P.
127       intros Syll2_05.
128       specialize Taut1_2 with P.
129       intros Taut1_2.
130       specialize MP1_1 with ((P ∨ P) → P) (P → P).
131       intros MP1_1.
132       apply Syll2_05.
133       apply Taut1_2.
134       apply n2_07.
135   Qed.
136
137   Theorem n2_1 : ∀ P : Prop,
138       (¬P) ∨ P.
139   Proof. intros P.
140       specialize Id2_08 with P.
141       replace (¬P ∨ P) with (P → P).
142       apply MP1_1.
143       apply Impl1_01.
144   Qed.
145
146   Theorem n2_11 : ∀ P : Prop,
147       P ∨ ¬P.
148   Proof. intros P.

```

```

149   specialize Perm1_4 with (¬P) P.
150   intros Perm1_4.
151   specialize n2_1 with P.
152   intros n2_1.
153   apply (MP1_1 (¬P∨P) (P∨¬P)).
154   apply Perm1_4.
155   apply n2_1.
156   Qed.
157
158   Theorem n2_12 : ∀ P : Prop,
159     P → ¬¬P.
160   Proof. intros P.
161     specialize n2_11 with (¬P).
162     intros n2_11.
163     rewrite Impl1_01.
164     apply n2_11.
165     Qed.
166
167   Theorem n2_13 : ∀ P : Prop,
168     P ∨ ¬¬¬P.
169   Proof. intros P.
170     specialize Sum1_6 with P (¬P) (¬¬¬P).
171     intros Sum1_6.
172     specialize n2_12 with (¬P).
173     intros n2_12.
174     apply (MP1_1 (¬P→¬¬¬P) ((P∨¬P)→(P∨¬¬¬P))).
175     apply Sum1_6.
176     apply n2_12.
177     specialize n2_11 with P.
178     intros n2_11.
179     apply n2_11.
180     Qed.
181
182   Theorem n2_14 : ∀ P : Prop,
183     ¬¬P → P.
184   Proof. intros P.
185     specialize Perm1_4 with P (¬¬P).
186     intros Perm1_4.
187     specialize n2_13 with P.
188     intros n2_13.
189     rewrite Impl1_01.
190     apply (MP1_1 (P∨¬¬P) (¬¬P∨P)).

```

```

191   apply Perm1_4.
192   apply n2_13.
193 Qed.
194
195 Theorem Transp2_15 :  $\forall P Q : \text{Prop},$ 
196    $(\neg P \rightarrow Q) \rightarrow (\neg Q \rightarrow P).$ 
197 Proof. intros P Q.
198   specialize Syll2_05 with  $(\neg P) Q (\neg\neg Q).$ 
199   intros Syll2_05a.
200   specialize n2_12 with Q.
201   intros n2_12.
202   specialize Transp2_03 with  $(\neg P) (\neg Q).$ 
203   intros Transp2_03.
204   specialize Syll2_05 with  $(\neg Q) (\neg\neg P) P.$ 
205   intros Syll2_05b.
206   specialize Syll2_05 with  $(\neg P \rightarrow Q) (\neg P \rightarrow \neg\neg Q) (\neg Q \rightarrow \neg\neg P).$ 
207   intros Syll2_05c.
208   specialize Syll2_05 with  $(\neg P \rightarrow Q) (\neg Q \rightarrow \neg\neg P) (\neg Q \rightarrow P).$ 
209   intros Syll2_05d.
210   apply Syll2_05d.
211   apply Syll2_05b.
212   specialize n2_14 with P.
213   intros n2_14.
214   apply n2_14.
215   apply Syll2_05c.
216   apply Transp2_03.
217   apply (MP1_1  $(Q \rightarrow \neg\neg Q) ((\neg P \rightarrow Q) \rightarrow (\neg P \rightarrow \neg\neg Q))$ ).
218   apply Syll2_05a.
219   apply n2_12.
220 Qed.
221
222 Ltac Syll H1 H2 S :=
223   let S := fresh S in match goal with
224     | [ H1 : ?P  $\rightarrow$  ?Q, H2 : ?Q  $\rightarrow$  ?R |- _ ] =>
225       assert (S : P  $\rightarrow$  R) by (intros p; apply (H2 (H1 p)))
226   end.
227
228 Ltac MP H1 H2 :=
229   match goal with
230     | [ H1 : ?P  $\rightarrow$  ?Q, H2 : ?P |- _ ] => specialize (H1 H2)
231   end.
232

```

```

233 Theorem Transp2_16 :  $\forall P Q : \text{Prop},$ 
234    $(P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P).$ 
235 Proof. intros P Q.
236   specialize n2_12 with Q.
237   intros n2_12a.
238   specialize Syll2_05 with P Q ( $\neg\neg Q$ ).
239   intros Syll2_05a.
240   specialize Transp2_03 with P ( $\neg Q$ ).
241   intros Transp2_03a.
242   MP n2_12a Syll2_05a.
243   Syll Syll2_05a Transp2_03a S.
244   apply S.
245 Qed.
246
247 Theorem Transp2_17 :  $\forall P Q : \text{Prop},$ 
248    $(\neg Q \rightarrow \neg P) \rightarrow (P \rightarrow Q).$ 
249 Proof. intros P Q.
250   specialize Transp2_03 with ( $\neg Q$ ) P.
251   intros Transp2_03a.
252   specialize n2_14 with Q.
253   intros n2_14a.
254   specialize Syll2_05 with P ( $\neg\neg Q$ ) Q.
255   intros Syll2_05a.
256   MP n2_14a Syll2_05a.
257   Syll Transp2_03a Syll2_05a S.
258   apply S.
259 Qed.
260
261 Theorem n2_18 :  $\forall P : \text{Prop},$ 
262    $(\neg P \rightarrow P) \rightarrow P.$ 
263 Proof. intros P.
264   specialize n2_12 with P.
265   intro n2_12a.
266   specialize Syll2_05 with ( $\neg P$ ) P ( $\neg\neg P$ ).
267   intro Syll2_05a.
268   MP Syll2_05a n2_12.
269   specialize Abs2_01 with ( $\neg P$ ).
270   intros Abs2_01a.
271   Syll Syll2_05a Abs2_01a Sa.
272   specialize n2_14 with P.
273   intros n2_14a.
274   Syll H n2_14a Sb.

```

```

275     apply Sb.
276 Qed.
277
278 Theorem n2_2 :  $\forall P Q : \text{Prop},$ 
279    $P \rightarrow (P \vee Q).$ 
280 Proof. intros P Q.
281   specialize Add1_3 with Q P.
282   intros Add1_3a.
283   specialize Perm1_4 with Q P.
284   intros Perm1_4a.
285   Syll Add1_3a Perm1_4a S.
286   apply S.
287 Qed.
288
289 Theorem n2_21 :  $\forall P Q : \text{Prop},$ 
290    $\neg P \rightarrow (P \rightarrow Q).$ 
291 Proof. intros P Q.
292   specialize n2_2 with ( $\neg P$ ) Q.
293   intros n2_2a.
294   specialize Impl1_01 with P Q.
295   intros Impl1_01a.
296   replace ( $\neg P \vee Q$ ) with ( $P \rightarrow Q$ ) in n2_2a.
297   apply n2_2a.
298 Qed.
299
300 Theorem n2_24 :  $\forall P Q : \text{Prop},$ 
301    $P \rightarrow (\neg P \rightarrow Q).$ 
302 Proof. intros P Q.
303   specialize n2_21 with P Q.
304   intros n2_21a.
305   specialize Comm2_04 with ( $\neg P$ ) P Q.
306   intros Comm2_04a.
307   apply Comm2_04a.
308   apply n2_21a.
309 Qed.
310
311 Theorem n2_25 :  $\forall P Q : \text{Prop},$ 
312    $P \vee ((P \vee Q) \rightarrow Q).$ 
313 Proof. intros P Q.
314   specialize n2_1 with (P  $\vee$  Q).
315   intros n2_1a.
316   specialize Assoc1_5 with ( $\neg(P \vee Q)$ ) P Q.

```



```

317   intros Assoc1_5a.
318   MP Assoc1_5a n2_1a.
319   replace ( $\neg(P \vee Q) \vee Q$ ) with ( $P \vee Q \rightarrow Q$ ) in Assoc1_5a.
320   apply Assoc1_5a.
321   apply Impl1_01.
322   Qed.
323
324   Theorem n2_26 :  $\forall P Q : \text{Prop}$ ,
325      $\neg P \vee ((P \rightarrow Q) \rightarrow Q)$ .
326   Proof. intros P Q.
327     specialize n2_25 with ( $\neg P$ ) Q.
328     intros n2_25a.
329     replace ( $\neg P \vee Q$ ) with ( $P \rightarrow Q$ ) in n2_25a.
330     apply n2_25a.
331     apply Impl1_01.
332     Qed.
333
334   Theorem n2_27 :  $\forall P Q : \text{Prop}$ ,
335      $P \rightarrow ((P \rightarrow Q) \rightarrow Q)$ .
336   Proof. intros P Q.
337     specialize n2_26 with P Q.
338     intros n2_26a.
339     replace ( $\neg P \vee ((P \rightarrow Q) \rightarrow Q)$ ) with ( $P \rightarrow (P \rightarrow Q) \rightarrow Q$ ) in n2_26a.
340     apply n2_26a.
341     apply Impl1_01.
342     Qed.
343
344   Theorem n2_3 :  $\forall P Q R : \text{Prop}$ ,
345      $(P \vee (Q \vee R)) \rightarrow (P \vee (R \vee Q))$ .
346   Proof. intros P Q R.
347     specialize Perm1_4 with Q R.
348     intros Perm1_4a.
349     specialize Sum1_6 with P (Q  $\vee$  R) (R  $\vee$  Q).
350     intros Sum1_6a.
351     MP Sum1_6a Perm1_4a.
352     apply Sum1_6a.
353     Qed.
354
355   Theorem n2_31 :  $\forall P Q R : \text{Prop}$ ,
356      $(P \vee (Q \vee R)) \rightarrow ((P \vee Q) \vee R)$ .
357   Proof. intros P Q R.
358     specialize n2_3 with P Q R.

```

```

359   intros n2_3a.
360   specialize Assoc1_5 with P R Q.
361   intros Assoc1_5a.
362   specialize Perm1_4 with R (P∨Q).
363   intros Perm1_4a.
364   Syll Assoc1_5a Perm1_4a Sa.
365   Syll n2_3a Sa Sb.
366   apply Sb.
367   Qed.
368
369   Theorem n2_32 : ∀ P Q R : Prop,
370     ((P ∨ Q) ∨ R) → (P ∨ (Q ∨ R)).
371   Proof. intros P Q R.
372     specialize Perm1_4 with (P∨Q) R.
373     intros Perm1_4a.
374     specialize Assoc1_5 with R P Q.
375     intros Assoc1_5a.
376     specialize n2_3 with P R Q.
377     intros n2_3a.
378     specialize Syll2_06 with ((P∨Q)∨R) (R∨P∨Q) (P∨R∨Q).
379     intros Syll2_06a.
380     MP Syll2_06a Perm1_4a.
381     MP Syll2_06a Assoc1_5a.
382     specialize Syll2_06 with ((P∨Q)∨R) (P∨R∨Q) (P∨Q∨R).
383     intros Syll2_06b.
384     MP Syll2_06b Syll2_06a.
385     MP Syll2_06b n2_3a.
386     apply Syll2_06b.
387     Qed.
388
389   Axiom Abb2_33 : ∀ P Q R : Prop,
390     (P ∨ Q ∨ R) = ((P ∨ Q) ∨ R).
391     (*This definition makes the default left association.
392       The default in Coq is right association.*)
393
394   Theorem n2_36 : ∀ P Q R : Prop,
395     (Q → R) → ((P ∨ Q) → (R ∨ P)).
396   Proof. intros P Q R.
397     specialize Perm1_4 with P R.
398     intros Perm1_4a.
399     specialize Syll2_05 with (P∨Q) (P∨R) (R∨P).
400     intros Syll2_05a.

```

```

401   MP Syll2_05a Perm1_4a.
402   specialize Sum1_6 with P Q R.
403   intros Sum1_6a.
404   Syll Sum1_6a Syll2_05a S.
405   apply S.
406   Qed.
407
408   Theorem n2_37 :  $\forall P Q R : \text{Prop}$ ,
409      $(Q \rightarrow R) \rightarrow ((Q \vee P) \rightarrow (P \vee R))$ .
410   Proof. intros P Q R.
411     specialize Perm1_4 with Q P.
412     intros Perm1_4a.
413     specialize Syll2_06 with  $(Q \vee P)$   $(P \vee Q)$   $(P \vee R)$ .
414     intros Syll2_06a.
415     MP Syll2_06a Perm1_4a.
416     specialize Sum1_6 with P Q R.
417     intros Sum1_6a.
418     Syll Sum1_6a Syll2_06a S.
419     apply S.
420     Qed.
421
422   Theorem n2_38 :  $\forall P Q R : \text{Prop}$ ,
423      $(Q \rightarrow R) \rightarrow ((Q \vee P) \rightarrow (R \vee P))$ .
424   Proof. intros P Q R.
425     specialize Perm1_4 with P R.
426     intros Perm1_4a.
427     specialize Syll2_05 with  $(Q \vee P)$   $(P \vee R)$   $(R \vee P)$ .
428     intros Syll2_05a.
429     MP Syll2_05a Perm1_4a.
430     specialize Perm1_4 with Q P.
431     intros Perm1_4b.
432     specialize Syll2_06 with  $(Q \vee P)$   $(P \vee Q)$   $(P \vee R)$ .
433     intros Syll2_06a.
434     MP Syll2_06a Perm1_4b.
435     Syll Syll2_06a Syll2_05a H.
436     specialize Sum1_6 with P Q R.
437     intros Sum1_6a.
438     Syll Sum1_6a H S.
439     apply S.
440     Qed.
441
442   Theorem n2_4 :  $\forall P Q : \text{Prop}$ ,

```

```

443   (P ∨ (P ∨ Q)) → (P ∨ Q).
444 Proof. intros P Q.
445   specialize n2_31 with P P Q.
446   intros n2_31a.
447   specialize Taut1_2 with P.
448   intros Taut1_2a.
449   specialize n2_38 with Q (P∨P) P.
450   intros n2_38a.
451   MP n2_38a Taut1_2a.
452   Syll n2_31a n2_38a S.
453   apply S.
454 Qed.
455
456 Theorem n2_41 : ∀ P Q : Prop,
457   (Q ∨ (P ∨ Q)) → (P ∨ Q).
458 Proof. intros P Q.
459   specialize Assoc1_5 with Q P Q.
460   intros Assoc1_5a.
461   specialize Taut1_2 with Q.
462   intros Taut1_2a.
463   specialize Sum1_6 with P (Q∨Q) Q.
464   intros Sum1_6a.
465   MP Sum1_6a Taut1_2a.
466   Syll Assoc1_5a Sum1_6a S.
467   apply S.
468 Qed.
469
470 Theorem n2_42 : ∀ P Q : Prop,
471   (¬P ∨ (P → Q)) → (P → Q).
472 Proof. intros P Q.
473   specialize n2_4 with (¬P) Q.
474   intros n2_4a.
475   replace (¬P∨Q) with (P→Q) in n2_4a.
476   apply n2_4a. apply Impl1_01.
477 Qed.
478
479 Theorem n2_43 : ∀ P Q : Prop,
480   (P → (P → Q)) → (P → Q).
481 Proof. intros P Q.
482   specialize n2_42 with P Q.
483   intros n2_42a.
484   replace (¬P ∨ (P→Q)) with (P→(P→Q)) in n2_42a.

```

```

485     apply n2_42a.
486     apply Impl1_01.
487 Qed.
488
489 Theorem n2_45 :  $\forall P Q : \text{Prop},$ 
490    $\neg(P \vee Q) \rightarrow \neg P.$ 
491 Proof. intros P Q.
492   specialize n2_2 with P Q.
493   intros n2_2a.
494   specialize Transp2_16 with P (P $\vee$ Q).
495   intros Transp2_16a.
496   MP n2_2 Transp2_16a.
497   apply Transp2_16a.
498 Qed.
499
500 Theorem n2_46 :  $\forall P Q : \text{Prop},$ 
501    $\neg(P \vee Q) \rightarrow \neg Q.$ 
502 Proof. intros P Q.
503   specialize Add1_3 with P Q.
504   intros Add1_3a.
505   specialize Transp2_16 with Q (P $\vee$ Q).
506   intros Transp2_16a.
507   MP Add1_3a Transp2_16a.
508   apply Transp2_16a.
509 Qed.
510
511 Theorem n2_47 :  $\forall P Q : \text{Prop},$ 
512    $\neg(P \vee Q) \rightarrow (\neg P \vee Q).$ 
513 Proof. intros P Q.
514   specialize n2_45 with P Q.
515   intros n2_45a.
516   specialize n2_2 with ( $\neg P$ ) Q.
517   intros n2_2a.
518   Syll n2_45a n2_2a S.
519   apply S.
520 Qed.
521
522 Theorem n2_48 :  $\forall P Q : \text{Prop},$ 
523    $\neg(P \vee Q) \rightarrow (P \vee \neg Q).$ 
524 Proof. intros P Q.
525   specialize n2_46 with P Q.
526   intros n2_46a.

```

```

527   specialize Add1_3 with P (¬Q).
528   intros Add1_3a.
529   Syll n2_46a Add1_3a S.
530   apply S.
531   Qed.
532
533   Theorem n2_49 : ∀ P Q : Prop,
534     ¬(P ∨ Q) → (¬P ∨ ¬Q).
535   Proof. intros P Q.
536     specialize n2_45 with P Q.
537     intros n2_45a.
538     specialize n2_2 with (¬P) (¬Q).
539     intros n2_2a.
540     Syll n2_45a n2_2a S.
541     apply S.
542     Qed.
543
544   Theorem n2_5 : ∀ P Q : Prop,
545     ¬(P → Q) → (¬P → Q).
546   Proof. intros P Q.
547     specialize n2_47 with (¬P) Q.
548     intros n2_47a.
549     replace (¬P∨Q) with (P→Q) in n2_47a.
550     replace (¬¬P∨Q) with (¬P→Q) in n2_47a.
551     apply n2_47a.
552     apply Impl1_01.
553     apply Impl1_01.
554     Qed.
555
556   Theorem n2_51 : ∀ P Q : Prop,
557     ¬(P → Q) → (P → ¬Q).
558   Proof. intros P Q.
559     specialize n2_48 with (¬P) Q.
560     intros n2_48a.
561     replace (¬P∨Q) with (P→Q) in n2_48a.
562     replace (¬P∨¬Q) with (P→¬Q) in n2_48a.
563     apply n2_48a.
564     apply Impl1_01.
565     apply Impl1_01.
566     Qed.
567
568   Theorem n2_52 : ∀ P Q : Prop,

```

```

569    $\neg(P \rightarrow Q) \rightarrow (\neg P \rightarrow \neg Q).$ 
570 Proof. intros P Q.
571   specialize n2_49 with ( $\neg P$ ) Q.
572   intros n2_49a.
573   replace ( $\neg P \vee Q$ ) with ( $P \rightarrow Q$ ) in n2_49a.
574   replace ( $\neg \neg P \vee \neg Q$ ) with ( $\neg P \rightarrow \neg Q$ ) in n2_49a.
575   apply n2_49a.
576   apply Impl1_01.
577   apply Impl1_01.
578 Qed.
579
580 Theorem n2_521 :  $\forall P Q : \text{Prop},$ 
581    $\neg(P \rightarrow Q) \rightarrow (Q \rightarrow P).$ 
582 Proof. intros P Q.
583   specialize n2_52 with P Q.
584   intros n2_52a.
585   specialize Transp2_17 with Q P.
586   intros Transp2_17a.
587   Syll n2_52a Transp2_17a S.
588   apply S.
589 Qed.
590
591 Theorem n2_53 :  $\forall P Q : \text{Prop},$ 
592    $(P \vee Q) \rightarrow (\neg P \rightarrow Q).$ 
593 Proof. intros P Q.
594   specialize n2_12 with P.
595   intros n2_12a.
596   specialize n2_38 with Q P ( $\neg \neg P$ ).
597   intros n2_38a.
598   MP n2_38a n2_12a.
599   replace ( $\neg \neg P \vee Q$ ) with ( $\neg P \rightarrow Q$ ) in n2_38a.
600   apply n2_38a.
601   apply Impl1_01.
602 Qed.
603
604 Theorem n2_54 :  $\forall P Q : \text{Prop},$ 
605    $(\neg P \rightarrow Q) \rightarrow (P \vee Q).$ 
606 Proof. intros P Q.
607   specialize n2_14 with P.
608   intros n2_14a.
609   specialize n2_38 with Q ( $\neg \neg P$ ) P.
610   intros n2_38a.

```

```

611   MP n2_38a n2_12a.
612   replace ( $\neg\neg P \vee Q$ ) with ( $\neg P \rightarrow Q$ ) in n2_38a.
613   apply n2_38a.
614   apply Impl1_01.
615   Qed.
616
617   Theorem n2_55 :  $\forall P Q : \text{Prop},$ 
618      $\neg P \rightarrow ((P \vee Q) \rightarrow Q).$ 
619   Proof. intros P Q.
620     specialize n2_53 with P Q.
621     intros n2_53a.
622     specialize Comm2_04 with ( $P \vee Q$ ) ( $\neg P$ ) Q.
623     intros Comm2_04a.
624     MP n2_53a Comm2_04a.
625     apply Comm2_04a.
626     Qed.
627
628   Theorem n2_56 :  $\forall P Q : \text{Prop},$ 
629      $\neg Q \rightarrow ((P \vee Q) \rightarrow P).$ 
630   Proof. intros P Q.
631     specialize n2_55 with Q P.
632     intros n2_55a.
633     specialize Perm1_4 with P Q.
634     intros Perm1_4a.
635     specialize Syll12_06 with ( $P \vee Q$ ) ( $Q \vee P$ ) P.
636     intros Syll12_06a.
637     MP Syll12_06a Perm1_4a.
638     Syll n2_55a Syll12_06a Sa.
639     apply Sa.
640     Qed.
641
642   Theorem n2_6 :  $\forall P Q : \text{Prop},$ 
643      $(\neg P \rightarrow Q) \rightarrow ((P \rightarrow Q) \rightarrow Q).$ 
644   Proof. intros P Q.
645     specialize n2_38 with Q ( $\neg P$ ) Q.
646     intros n2_38a.
647     specialize Taut1_2 with Q.
648     intros Taut1_2a.
649     specialize Syll12_05 with ( $\neg P \vee Q$ ) ( $Q \vee Q$ ) Q.
650     intros Syll12_05a.
651     MP Syll12_05a Taut1_2a.
652     Syll n2_38a Syll12_05a S.

```



```

653   replace ( $\neg P \vee Q$ ) with ( $P \rightarrow Q$ ) in S.
654   apply S.
655   apply Impl1_01.
656 Qed.
657
658 Theorem n2_61 :  $\forall P Q : \text{Prop},$ 
659   ( $P \rightarrow Q$ )  $\rightarrow$  ( $(\neg P \rightarrow Q) \rightarrow Q$ ).
660 Proof. intros P Q.
661   specialize n2_6 with P Q.
662   intros n2_6a.
663   specialize Comm2_04 with ( $\neg P \rightarrow Q$ ) ( $P \rightarrow Q$ ) Q.
664   intros Comm2_04a.
665   MP Comm2_04a n2_6a.
666   apply Comm2_04a.
667 Qed.
668
669 Theorem n2_62 :  $\forall P Q : \text{Prop},$ 
670   ( $P \vee Q$ )  $\rightarrow$  ( $(P \rightarrow Q) \rightarrow Q$ ).
671 Proof. intros P Q.
672   specialize n2_53 with P Q.
673   intros n2_53a.
674   specialize n2_6 with P Q.
675   intros n2_6a.
676   Syll n2_53a n2_6a S.
677   apply S.
678 Qed.
679
680 Theorem n2_621 :  $\forall P Q : \text{Prop},$ 
681   ( $P \rightarrow Q$ )  $\rightarrow$  ( $(P \vee Q) \rightarrow Q$ ).
682 Proof. intros P Q.
683   specialize n2_62 with P Q.
684   intros n2_62a.
685   specialize Comm2_04 with ( $P \vee Q$ ) ( $P \rightarrow Q$ ) Q.
686   intros Comm2_04a.
687   MP Comm2_04a n2_62a.
688   apply Comm2_04a.
689 Qed.
690
691 Theorem n2_63 :  $\forall P Q : \text{Prop},$ 
692   ( $P \vee Q$ )  $\rightarrow$  ( $(\neg P \vee Q) \rightarrow Q$ ).
693 Proof. intros P Q.
694   specialize n2_62 with P Q.

```

```

695   intros n2_62a.
696   replace (¬P∨Q) with (P→Q).
697   apply n2_62a.
698   apply Impl1_01.
699   Qed.
700
701   Theorem n2_64 : ∀ P Q : Prop,
702     (P ∨ Q) → ((P ∨ ¬Q) → P).
703   Proof. intros P Q.
704     specialize n2_63 with Q P.
705     intros n2_63a.
706     specialize Perm1_4 with P Q.
707     intros Perm1_4a.
708     Syll n2_63a Perm1_4a Ha.
709     specialize Syll2_06 with (P∨¬Q) (¬Q∨P) P.
710     intros Syll2_06a.
711     specialize Perm1_4 with P (¬Q).
712     intros Perm1_4b.
713     MP Syll2_06a Perm1_4b.
714     Syll Syll2_06a Ha S.
715     apply S.
716   Qed.
717
718   Theorem n2_65 : ∀ P Q : Prop,
719     (P → Q) → ((P → ¬Q) → ¬P).
720   Proof. intros P Q.
721     specialize n2_64 with (¬P) Q.
722     intros n2_64a.
723     replace (¬P∨Q) with (P→Q) in n2_64a.
724     replace (¬P∨¬Q) with (P→¬Q) in n2_64a.
725     apply n2_64a.
726     apply Impl1_01.
727     apply Impl1_01.
728   Qed.
729
730   Theorem n2_67 : ∀ P Q : Prop,
731     ((P ∨ Q) → Q) → (P → Q).
732   Proof. intros P Q.
733     specialize n2_54 with P Q.
734     intros n2_54a.
735     specialize Syll2_06 with (¬P→Q) (P∨Q) Q.
736     intros Syll2_06a.

```

```

737   MP Syll2_06a n2_54a.
738   specialize n2_24 with P Q.
739   intros n2_24.
740   specialize Syll2_06 with P ( $\neg P \rightarrow Q$ ) Q.
741   intros Syll2_06b.
742   MP Syll2_06b n2_24a.
743   Syll Syll2_06b Syll2_06a S.
744   apply S.
745   Qed.
746
747   Theorem n2_68 :  $\forall P Q : \text{Prop}$ ,
748      $((P \rightarrow Q) \rightarrow Q) \rightarrow (P \vee Q)$ .
749   Proof. intros P Q.
750     specialize n2_67 with ( $\neg P$ ) Q.
751     intros n2_67a.
752     replace ( $\neg P \vee Q$ ) with  $(P \rightarrow Q)$  in n2_67a.
753     specialize n2_54 with P Q.
754     intros n2_54a.
755     Syll n2_67a n2_54a S.
756     apply S.
757     apply Impl1_01.
758     Qed.
759
760   Theorem n2_69 :  $\forall P Q : \text{Prop}$ ,
761      $((P \rightarrow Q) \rightarrow Q) \rightarrow ((Q \rightarrow P) \rightarrow P)$ .
762   Proof. intros P Q.
763     specialize n2_68 with P Q.
764     intros n2_68a.
765     specialize Perm1_4 with P Q.
766     intros Perm1_4a.
767     Syll n2_68a Perm1_4a Sa.
768     specialize n2_62 with Q P.
769     intros n2_62a.
770     Syll Sa n2_62a Sb.
771     apply Sb.
772     Qed.
773
774   Theorem n2_73 :  $\forall P Q R : \text{Prop}$ ,
775      $(P \rightarrow Q) \rightarrow ((P \vee Q) \vee R) \rightarrow (Q \vee R)$ .
776   Proof. intros P Q R.
777     specialize n2_621 with P Q.
778     intros n2_621a.

```

```

779   specialize n2_38 with R (P∨Q) Q.
780   intros n2_38a.
781   Syll n2_621a n2_38a S.
782   apply S.
783   Qed.
784
785   Theorem n2_74 : ∀ P Q R : Prop,
786     (Q → P) → ((P ∨ Q) ∨ R) → (P ∨ R).
787   Proof. intros P Q R.
788     specialize n2_73 with Q P R.
789     intros n2_73a.
790     specialize Assoc1_5 with P Q R.
791     intros Assoc1_5a.
792     specialize n2_31 with Q P R.
793     intros n2_31a. (*not cited*)
794     Syll Assoc1_5a n2_31a Sa.
795     specialize n2_32 with P Q R.
796     intros n2_32a. (*not cited*)
797     Syll n2_32a Sa Sb.
798     specialize Syll2_06 with ((P∨Q)∨R) ((Q∨P)∨R) (P∨R).
799     intros Syll2_06a.
800     MP Syll2_06a Sb.
801     Syll n2_73a Syll2_05a H.
802     apply H.
803     Qed.
804
805   Theorem n2_75 : ∀ P Q R : Prop,
806     (P ∨ Q) → ((P ∨ (Q → R)) → (P ∨ R)).
807   Proof. intros P Q R.
808     specialize n2_74 with P (¬Q) R.
809     intros n2_74a.
810     specialize n2_53 with Q P.
811     intros n2_53a.
812     Syll n2_53a n2_74a Sa.
813     specialize n2_31 with P (¬Q) R.
814     intros n2_31a.
815     specialize Syll2_06 with (P∨(¬Q)∨R)((P∨(¬Q))∨R) (P∨R).
816     intros Syll2_06a.
817     MP Syll2_06a n2_31a.
818     Syll Sa Syll2_06a Sb.
819     specialize Perm1_4 with P Q.
820     intros Perm1_4a. (*not cited*)

```

```

821   Syll Perm1_4a Sb Sc.
822   replace ( $\neg Q \vee R$ ) with ( $Q \rightarrow R$ ) in Sc.
823   apply Sc.
824   apply Impl1_01.
825   Qed.
826
827   Theorem n2_76 :  $\forall P Q R : \text{Prop}$ ,
828     ( $P \vee (Q \rightarrow R)$ )  $\rightarrow ((P \vee Q) \rightarrow (P \vee R))$ .
829   Proof. intros P Q R.
830     specialize n2_75 with P Q R.
831     intros n2_75a.
832     specialize Comm2_04 with ( $P \vee Q$ ) ( $P \vee (Q \rightarrow R)$ ) ( $P \vee R$ ).
833     intros Comm2_04a.
834     apply Comm2_04a.
835     apply n2_75a.
836   Qed.
837
838   Theorem n2_77 :  $\forall P Q R : \text{Prop}$ ,
839     ( $P \rightarrow (Q \rightarrow R)$ )  $\rightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R))$ .
840   Proof. intros P Q R.
841     specialize n2_76 with ( $\neg P$ ) Q R.
842     intros n2_76a.
843     replace ( $\neg P \vee (Q \rightarrow R)$ ) with ( $P \rightarrow Q \rightarrow R$ ) in n2_76a.
844     replace ( $\neg P \vee Q$ ) with ( $P \rightarrow Q$ ) in n2_76a.
845     replace ( $\neg P \vee R$ ) with ( $P \rightarrow R$ ) in n2_76a.
846     apply n2_76a.
847     apply Impl1_01.
848     apply Impl1_01.
849     apply Impl1_01.
850   Qed.
851
852   Theorem n2_8 :  $\forall Q R S : \text{Prop}$ ,
853     ( $Q \vee R$ )  $\rightarrow ((\neg R \vee S) \rightarrow (Q \vee S))$ .
854   Proof. intros Q R S.
855     specialize n2_53 with R Q.
856     intros n2_53a.
857     specialize Perm1_4 with Q R.
858     intros Perm1_4a.
859     Syll Perm1_4a n2_53a Ha.
860     specialize n2_38 with S ( $\neg R$ ) Q.
861     intros n2_38a.
862     Syll H n2_38a Hb.

```

```

863     apply Hb.
864 Qed.
865
866 Theorem n2_81 :  $\forall P Q R S : \text{Prop},$ 
867    $(Q \rightarrow (R \rightarrow S)) \rightarrow ((P \vee Q) \rightarrow ((P \vee R) \rightarrow (P \vee S)))$ .
868 Proof. intros P Q R S.
869   specialize Sum1_6 with P Q (R $\rightarrow$ S).
870   intros Sum1_6a.
871   specialize n2_76 with P R S.
872   intros n2_76a.
873   specialize Syll2_05 with (P $\vee$ Q) (P $\vee$ (R $\rightarrow$ S)) ((P $\vee$ R) $\rightarrow$ (P $\vee$ S)).
874   intros Syll2_05a.
875   MP Syll2_05a n2_76a.
876   Syll Sum1_6a Syll2_05a H.
877   apply H.
878 Qed.
879
880 Theorem n2_82 :  $\forall P Q R S : \text{Prop},$ 
881    $(P \vee Q \vee R) \rightarrow ((P \vee \neg R \vee S) \rightarrow (P \vee Q \vee S))$ .
882 Proof. intros P Q R S.
883   specialize n2_8 with Q R S.
884   intros n2_8a.
885   specialize n2_81 with P (Q $\vee$ R) ( $\neg$ R $\vee$ S) (Q $\vee$ S).
886   intros n2_81a.
887   MP n2_81a n2_8a.
888   apply n2_81a.
889 Qed.
890
891 Theorem n2_83 :  $\forall P Q R S : \text{Prop},$ 
892    $(P \rightarrow (Q \rightarrow R)) \rightarrow ((P \rightarrow (R \rightarrow S)) \rightarrow (P \rightarrow (Q \rightarrow S)))$ .
893 Proof. intros P Q R S.
894   specialize n2_82 with ( $\neg$ P) ( $\neg$ Q) R S.
895   intros n2_82a.
896   replace ( $\neg$ Q $\vee$ R) with (Q $\rightarrow$ R) in n2_82a.
897   replace ( $\neg$ P $\vee$ (Q $\rightarrow$ R)) with (P $\rightarrow$ Q $\rightarrow$ R) in n2_82a.
898   replace ( $\neg$ R $\vee$ S) with (R $\rightarrow$ S) in n2_82a.
899   replace ( $\neg$ P $\vee$ (R $\rightarrow$ S)) with (P $\rightarrow$ R $\rightarrow$ S) in n2_82a.
900   replace ( $\neg$ Q $\vee$ S) with (Q $\rightarrow$ S) in n2_82a.
901   replace ( $\neg$ Q $\vee$ S) with (Q $\rightarrow$ S) in n2_82a.
902   replace ( $\neg$ P $\vee$ (Q $\rightarrow$ S)) with (P $\rightarrow$ Q $\rightarrow$ S) in n2_82a.
903   apply n2_82a.
904   apply Impl1_01.

```

```

905   apply Impl1_01.
906   apply Impl1_01.
907   apply Impl1_01.
908   apply Impl1_01.
909   apply Impl1_01.
910   apply Impl1_01.
911 Qed.
912
913 Theorem n2_85 :  $\forall P Q R : \text{Prop},$ 
914    $((P \vee Q) \rightarrow (P \vee R)) \rightarrow (P \vee (Q \rightarrow R)).$ 
915 Proof. intros P Q R.
916   specialize Add1_3 with P Q.
917   intros Add1_3a.
918   specialize Syll2_06 with Q (P $\vee$ Q) R.
919   intros Syll2_06a.
920   MP Syll2_06a Add1_3a.
921   specialize n2_55 with P R.
922   intros n2_55a.
923   specialize Syll2_05 with (P $\vee$ Q) (P $\vee$ R) R.
924   intros Syll2_05a.
925   Syll n2_55a Syll2_05a Ha.
926   specialize n2_83 with ( $\neg$ P) ((P $\vee$ Q) $\rightarrow$ (P $\vee$ R)) ((P $\vee$ Q) $\rightarrow$ R) (Q $\rightarrow$ R).
927   intros n2_83a.
928   MP n2_83a Ha.
929   specialize Comm2_04 with ( $\neg$ P) (P $\vee$ Q $\rightarrow$ P $\vee$ R) (Q $\rightarrow$ R).
930   intros Comm2_04a.
931   Syll Ha Comm2_04a Hb.
932   specialize n2_54 with P (Q $\rightarrow$ R).
933   intros n2_54a.
934   specialize Simp2_02 with ( $\neg$ P) ((P $\vee$ Q $\rightarrow$ R) $\rightarrow$ (Q $\rightarrow$ R)).
935   intros Simp2_02a. (*Not cited*)
936   (*Greg's suggestion per the BRS list on June 25, 2017.*)
937   MP Syll2_06a Simp2_02a.
938   MP Hb Simp2_02a.
939   Syll Hb n2_54a Hc.
940   apply Hc.
941 Qed.
942
943 Theorem n2_86 :  $\forall P Q R : \text{Prop},$ 
944    $((P \rightarrow Q) \rightarrow (P \rightarrow R)) \rightarrow (P \rightarrow (Q \rightarrow R)).$ 
945 Proof. intros P Q R.
946   specialize n2_85 with ( $\neg$ P) Q R.

```

```

947   intros n2_85a.
948   replace (¬P∨Q) with (P→Q) in n2_85a.
949   replace (¬P∨R) with (P→R) in n2_85a.
950   replace (¬P∨(Q→R)) with (P→Q→R) in n2_85a.
951   apply n2_85a.
952   apply Impl1_01.
953   apply Impl1_01.
954   apply Impl1_01.
955   Qed.
956
957   End No2.
958
959   Module No3.
960
961   Import No1.
962   Import No2.
963
964   Axiom Prod3_01 : ∀ P Q : Prop,
965     (P ∧ Q) = ¬(¬P ∨ ¬Q).
966
967   Axiom Abb3_02 : ∀ P Q R : Prop,
968     (P → Q → R) = (P → Q) ∧ (Q → R).
969
970   Theorem Conj3_03 : ∀ P Q : Prop, P → Q → (P∧Q).
971   Proof. intros P Q.
972     specialize n2_11 with (¬P∨¬Q). intros n2_11a.
973     specialize n2_32 with (¬P) (¬Q) (¬(¬P ∨ ¬Q)). intros n2_32a.
974     MP n2_32a n2_11a.
975     replace (¬(¬P∨¬Q)) with (P∧Q) in n2_32a.
976     replace (¬Q ∨ (P∧Q)) with (Q→(P∧Q)) in n2_32a.
977     replace (¬P ∨ (Q → (P∧Q))) with (P→Q→(P∧Q)) in n2_32a.
978     apply n2_32a.
979     apply Impl1_01.
980     apply Impl1_01.
981     apply Prod3_01.
982   Qed.
983   (*3.03 is permits the inference from the theoremhood
984     of P and that of Q to the theoremhood of P and Q. So:*)
985
986   Ltac Conj H1 H2 :=
987     match goal with
988       | [ H1 : ?P, H2 : ?Q |- _ ] =>

```



```

989         assert (P ∧ Q)
990     end.
991
992 Theorem n3_1 : ∀ P Q : Prop,
993     (P ∧ Q) → ¬(¬P ∨ ¬Q).
994 Proof. intros P Q.
995     replace (¬(¬P ∨ ¬Q)) with (P ∧ Q).
996     specialize Id2_08 with (P ∧ Q).
997     intros Id2_08a.
998     apply Id2_08a.
999     apply Prod3_01.
1000 Qed.
1001
1002 Theorem n3_11 : ∀ P Q : Prop,
1003     ¬(¬P ∨ ¬Q) → (P ∧ Q).
1004 Proof. intros P Q.
1005     replace (¬(¬P ∨ ¬Q)) with (P ∧ Q).
1006     specialize Id2_08 with (P ∧ Q).
1007     intros Id2_08a.
1008     apply Id2_08a.
1009     apply Prod3_01.
1010 Qed.
1011
1012 Theorem n3_12 : ∀ P Q : Prop,
1013     (¬P ∨ ¬Q) ∨ (P ∧ Q).
1014 Proof. intros P Q.
1015     specialize n2_11 with (¬P ∨ ¬Q).
1016     intros n2_11a.
1017     replace (¬(¬P ∨ ¬Q)) with (P ∧ Q) in n2_11a.
1018     apply n2_11a.
1019     apply Prod3_01.
1020 Qed.
1021
1022 Theorem n3_13 : ∀ P Q : Prop,
1023     ¬(P ∧ Q) → (¬P ∨ ¬Q).
1024 Proof. intros P Q.
1025     specialize n3_11 with P Q.
1026     intros n3_11a.
1027     specialize Transp2_15 with (¬P ∨ ¬Q) (P ∧ Q).
1028     intros Transp2_15a.
1029     MP Transp2_15a n3_11a.
1030     apply Transp2_15a.

```

```

1031 Qed.
1032
1033 Theorem n3_14 :  $\forall P Q : \text{Prop},$ 
1034    $(\neg P \vee \neg Q) \rightarrow \neg(P \wedge Q).$ 
1035 Proof. intros P Q.
1036   specialize n3_1 with P Q.
1037   intros n3_1a.
1038   specialize Transp2_16 with  $(P \wedge Q) (\neg(\neg P \vee \neg Q)).$ 
1039   intros Transp2_16a.
1040   MP Transp2_16a n3_1a.
1041   specialize n2_12 with  $(\neg P \vee \neg Q).$ 
1042   intros n2_12a.
1043   Syll n2_12a Transp2_16a S.
1044   apply S.
1045 Qed.
1046
1047 Theorem n3_2 :  $\forall P Q : \text{Prop},$ 
1048    $P \rightarrow Q \rightarrow (P \wedge Q).$ 
1049 Proof. intros P Q.
1050   specialize n3_12 with P Q.
1051   intros n3_12a.
1052   specialize n2_32 with  $(\neg P) (\neg Q) (P \wedge Q).$ 
1053   intros n2_32a.
1054   MP n3_32a n3_12a.
1055   replace  $(\neg Q \vee P \wedge Q)$  with  $(Q \rightarrow P \wedge Q)$  in n2_32a.
1056   replace  $(\neg P \vee (Q \rightarrow P \wedge Q))$  with  $(P \rightarrow Q \rightarrow P \wedge Q)$  in n2_32a.
1057   apply n2_32a.
1058   apply Impl1_01.
1059   apply Impl1_01.
1060 Qed.
1061
1062 Theorem n3_21 :  $\forall P Q : \text{Prop},$ 
1063    $Q \rightarrow P \rightarrow (P \wedge Q).$ 
1064 Proof. intros P Q.
1065   specialize n3_2 with P Q.
1066   intros n3_2a.
1067   specialize Comm2_04 with P Q  $(P \wedge Q).$ 
1068   intros Comm2_04a.
1069   MP Comm2_04a n3_2a.
1070   apply Comm2_04a.
1071 Qed.
1072

```

```

1073 Theorem n3_22 :  $\forall$  P Q : Prop,
1074   (P  $\wedge$  Q)  $\rightarrow$  (Q  $\wedge$  P).
1075 Proof. intros P Q.
1076   specialize n3_13 with Q P.
1077   intros n3_13a.
1078   specialize Perm1_4 with ( $\neg$ Q) ( $\neg$ P).
1079   intros Perm1_4a.
1080   Syll n3_13a Perm1_4a Ha.
1081   specialize n3_14 with P Q.
1082   intros n3_14a.
1083   Syll Ha n3_14a Hb.
1084   specialize Transp2_17 with (P $\wedge$ Q) (Q  $\wedge$  P).
1085   intros Transp2_17a.
1086   MP Transp2_17a Hb.
1087   apply Transp2_17a.
1088 Qed.
1089
1090 Theorem n3_24 :  $\forall$  P : Prop,
1091    $\neg$ (P  $\wedge$   $\neg$ P).
1092 Proof. intros P.
1093   specialize n2_11 with ( $\neg$ P).
1094   intros n2_11a.
1095   specialize n3_14 with P ( $\neg$ P).
1096   intros n3_14a.
1097   MP n3_14a n2_11a.
1098   apply n3_14a.
1099 Qed.
1100
1101 Theorem Simp3_26 :  $\forall$  P Q : Prop,
1102   (P  $\wedge$  Q)  $\rightarrow$  P.
1103 Proof. intros P Q.
1104   specialize Simp2_02 with Q P.
1105   intros Simp2_02a.
1106   replace (P $\rightarrow$ (Q $\rightarrow$ P)) with ( $\neg$ P $\vee$ (Q $\rightarrow$ P)) in Simp2_02a.
1107   replace (Q $\rightarrow$ P) with ( $\neg$ Q $\vee$ P) in Simp2_02a.
1108   specialize n2_31 with ( $\neg$ P) ( $\neg$ Q) P.
1109   intros n2_31a.
1110   MP n2_31a Simp2_02a.
1111   specialize n2_53 with ( $\neg$ P $\vee$  $\neg$ Q) P.
1112   intros n2_53a.
1113   MP n2_53a Simp2_02a.
1114   replace ( $\neg$ ( $\neg$ P $\vee$  $\neg$ Q)) with (P $\wedge$ Q) in n2_53a.

```

```

1115   apply n2_53a.
1116   apply Prod3_01.
1117   rewrite <- Impl1_01.
1118   reflexivity.
1119   rewrite <- Impl1_01.
1120   reflexivity.
1121 Qed.
1122
1123 Theorem Simp3_27 :  $\forall$  P Q : Prop,
1124   (P  $\wedge$  Q)  $\rightarrow$  Q.
1125 Proof. intros P Q.
1126   specialize n3_22 with P Q.
1127   intros n3_22a.
1128   specialize Simp3_26 with Q P.
1129   intros Simp3_26a.
1130   Syll n3_22a Simp3_26a S.
1131   apply S.
1132 Qed.
1133
1134 Theorem Exp3_3 :  $\forall$  P Q R : Prop,
1135   ((P  $\wedge$  Q)  $\rightarrow$  R)  $\rightarrow$  (P  $\rightarrow$  (Q  $\rightarrow$  R)).
1136 Proof. intros P Q R.
1137   specialize Id2_08 with ((P $\wedge$ Q) $\rightarrow$ R).
1138   intros Id2_08a. (*This theorem isn't needed.*)
1139   replace (((P  $\wedge$  Q)  $\rightarrow$  R)  $\rightarrow$  ((P  $\wedge$  Q)  $\rightarrow$  R)) with
1140     (((P  $\wedge$  Q)  $\rightarrow$  R)  $\rightarrow$  ( $\neg$ ( $\neg$ P  $\vee$   $\neg$ Q)  $\rightarrow$  R)) in Id2_08a.
1141   specialize Transp2_15 with ( $\neg$ P $\vee$  $\neg$ Q) R.
1142   intros Transp2_15a.
1143   Syll Id2_08a Transp2_15a Sa.
1144   specialize Id2_08 with ( $\neg$ R  $\rightarrow$  ( $\neg$ P  $\vee$   $\neg$ Q)).
1145   intros Id2_08b. (*This theorem isn't needed.*)
1146   Syll Sa Id2_08b Sb.
1147   replace ( $\neg$ P  $\vee$   $\neg$ Q) with (P  $\rightarrow$   $\neg$ Q) in Sb.
1148   specialize Comm2_04 with ( $\neg$ R) P ( $\neg$ Q).
1149   intros Comm2_04a.
1150   Syll Sb Comm2_04a Sc.
1151   specialize Transp2_17 with Q R.
1152   intros Transp2_17a.
1153   specialize Syll2_05 with P ( $\neg$ R  $\rightarrow$   $\neg$ Q) (Q  $\rightarrow$  R).
1154   intros Syll2_05a.
1155   MP Syll2_05a Transp2_17a.
1156   Syll Sa Syll2_05a Sd.

```

```

1157   apply Sd.
1158   rewrite <- Impl1_01.
1159   reflexivity.
1160   rewrite Prod3_01.
1161   reflexivity.
1162 Qed.
1163
1164 Theorem Imp3_31 :  $\forall P Q R : \text{Prop},$ 
1165    $(P \rightarrow (Q \rightarrow R)) \rightarrow (P \wedge Q) \rightarrow R.$ 
1166 Proof. intros P Q R.
1167   specialize Id2_08 with  $(P \rightarrow (Q \rightarrow R)).$ 
1168   intros Id2_08a. (*This use of Id2_08 is redundant.*)
1169   replace  $((P \rightarrow (Q \rightarrow R)) \rightarrow (P \rightarrow (Q \rightarrow R)))$  with
1170      $((P \rightarrow (Q \rightarrow R)) \rightarrow (\neg P \vee (Q \rightarrow R)))$  in Id2_08a.
1171   replace  $(\neg P \vee (Q \rightarrow R))$  with
1172      $(\neg P \vee (\neg Q \vee R))$  in Id2_08a.
1173   specialize n2_31 with  $(\neg P) (\neg Q) R.$ 
1174   intros n2_31a.
1175   Syll Id2_08a n2_31a Sa.
1176   specialize n2_53 with  $(\neg P \vee \neg Q) R.$ 
1177   intros n2_53a.
1178   replace  $(\neg(\neg P \vee \neg Q))$  with  $(P \wedge Q)$  in n2_53a.
1179   Syll n2_31a n2_53a Sb.
1180   apply Sb.
1181   apply Prod3_01.
1182   rewrite Impl1_01.
1183   reflexivity.
1184   rewrite <- Impl1_01.
1185   reflexivity.
1186 Qed.
1187 (*The proof sketch cites Id2_08, but
1188 we did not seem to need it.*)
1189
1190 Theorem Syll3_33 :  $\forall P Q R : \text{Prop},$ 
1191    $((P \rightarrow Q) \wedge (Q \rightarrow R)) \rightarrow (P \rightarrow R).$ 
1192 Proof. intros P Q R.
1193   specialize Syll2_06 with P Q R.
1194   intros Syll2_06a.
1195   specialize Imp3_31 with  $(P \rightarrow Q) (Q \rightarrow R) (P \rightarrow R).$ 
1196   intros Imp3_31a.
1197   MP Imp3_31a Syll2_06a.
1198   apply Imp3_31a.

```

```

1199 Qed.
1200
1201 Theorem Syll3_34 :  $\forall P Q R : \text{Prop}$ ,
1202    $((Q \rightarrow R) \wedge (P \rightarrow Q)) \rightarrow (P \rightarrow R)$ .
1203 Proof. intros P Q R.
1204   specialize Syll2_05 with P Q R.
1205   intros Syll2_05a.
1206   specialize Imp3_31 with  $(Q \rightarrow R)$   $(P \rightarrow Q)$   $(P \rightarrow R)$ .
1207   intros Imp3_31a.
1208   MP Imp3_31a Syll2_05a.
1209   apply Imp3_31a.
1210 Qed.
1211
1212 Theorem Ass3_35 :  $\forall P Q : \text{Prop}$ ,
1213    $(P \wedge (P \rightarrow Q)) \rightarrow Q$ .
1214 Proof. intros P Q.
1215   specialize n2_27 with P Q.
1216   intros n2_27a.
1217   specialize Imp3_31 with P  $(P \rightarrow Q)$  Q.
1218   intros Imp3_31a.
1219   MP Imp3_31a n2_27a.
1220   apply Imp3_31a.
1221 Qed.
1222
1223 Theorem Transp3_37 :  $\forall P Q R : \text{Prop}$ ,
1224    $(P \wedge Q \rightarrow R) \rightarrow (P \wedge \neg R \rightarrow \neg Q)$ .
1225 Proof. intros P Q R.
1226   specialize Transp2_16 with Q R.
1227   intros Transp2_16a.
1228   specialize Syll2_05 with P  $(Q \rightarrow R)$   $(\neg R \rightarrow \neg Q)$ .
1229   intros Syll2_05a.
1230   MP Syll2_05a Transp2_16a.
1231   specialize Exp3_3 with P Q R.
1232   intros Exp3_3a.
1233   Syll Exp3_3a Syll2_05a Sa.
1234   specialize Imp3_31 with P  $(\neg R)$   $(\neg Q)$ .
1235   intros Imp3_31a.
1236   Syll Sa Imp3_31a Sb.
1237   apply Sb.
1238 Qed.
1239
1240 Theorem n3_4 :  $\forall P Q : \text{Prop}$ ,

```

```

1241   (P ∧ Q) → P → Q.
1242 Proof. intros P Q.
1243   specialize n2_51 with P Q.
1244   intros n2_51a.
1245   specialize Transp2_15 with (P→Q) (P→¬Q).
1246   intros Transp2_15a.
1247   MP Transp2_15a n2_51a.
1248   replace (P→¬Q) with (¬P∨¬Q) in Transp2_15a.
1249   replace (¬(¬P∨¬Q)) with (P∧Q) in Transp2_15a.
1250   apply Transp2_15a.
1251   apply Prod3_01.
1252   rewrite <- Impl1_01.
1253   reflexivity.
1254 Qed.
1255
1256 Theorem n3_41 : ∀ P Q R : Prop,
1257   (P → R) → (P ∧ Q → R).
1258 Proof. intros P Q R.
1259   specialize Simp3_26 with P Q.
1260   intros Simp3_26a.
1261   specialize Syll2_06 with (P∧Q) P R.
1262   intros Syll2_06a.
1263   MP Simp3_26a Syll2_06a.
1264   apply Syll2_06a.
1265 Qed.
1266
1267 Theorem n3_42 : ∀ P Q R : Prop,
1268   (Q → R) → (P ∧ Q → R).
1269 Proof. intros P Q R.
1270   specialize Simp3_27 with P Q.
1271   intros Simp3_27a.
1272   specialize Syll2_06 with (P∧Q) Q R.
1273   intros Syll2_06a.
1274   MP Syll2_06a Simp3_27a.
1275   apply Syll2_06a.
1276 Qed.
1277
1278 Theorem Comp3_43 : ∀ P Q R : Prop,
1279   (P → Q) ∧ (P → R) → (P → Q ∧ R).
1280 Proof. intros P Q R.
1281   specialize n3_2 with Q R.
1282   intros n3_2a.

```

```

1283 specialize Syll2_05 with P Q (R→Q∧R).
1284 intros Syll2_05a.
1285 MP Syll2_05a n2_2a.
1286 specialize n2_77 with P R (Q∧R).
1287 intros n2_77a.
1288 Syll Syll2_05a n2_77a Sa.
1289 specialize Imp3_31 with (P→Q) (P→R) (P→Q∧R).
1290 intros Imp3_31a.
1291 MP Sa Imp3_31a.
1292 apply Imp3_31a.
1293 Qed.
1294
1295 Theorem n3_44 : ∀ P Q R : Prop,
1296   (Q → P) ∧ (R → P) → (Q ∨ R → P).
1297 Proof. intros P Q R.
1298   specialize Syll3_33 with (¬Q) R P.
1299   intros Syll3_33a.
1300   specialize n2_6 with Q P.
1301   intros n2_6a.
1302   Syll Syll3_33a n2_6a Sa.
1303   specialize Exp3_3 with (¬Q→R) (R→P) ((Q→P)→P).
1304   intros Exp3_3a.
1305   MP Exp3_3a Sa.
1306   specialize Comm2_04 with (R→P) (Q→P) P.
1307   intros Comm2_04a.
1308   Syll Exp3_3a Comm2_04a Sb.
1309   specialize Imp3_31 with (Q→P) (R→P) P.
1310   intros Imp3_31a.
1311   Syll Sb Imp3_31a Sc.
1312   specialize Comm2_04 with (¬Q→R) ((Q→P)∧(R→P)) P.
1313   intros Comm2_04b.
1314   MP Comm2_04b Sc.
1315   specialize n2_53 with Q R.
1316   intros n2_53a.
1317   specialize Syll2_06 with (Q∨R) (¬Q→R) P.
1318   intros Syll2_06a.
1319   MP Syll2_06a n2_53a.
1320   Syll Comm2_04b Syll2_06a Sd.
1321   apply Sd.
1322 Qed.
1323
1324 Theorem Fact3_45 : ∀ P Q R : Prop,

```



```

1325   (P → Q) → (P ∧ R) → (Q ∧ R).
1326 Proof. intros P Q R.
1327   specialize Syll2_06 with P Q (¬R).
1328   intros Syll2_06a.
1329   specialize Transp2_16 with (Q→¬R) (P→¬R).
1330   intros Transp2_16a.
1331   Syll Syll2_06a Transp2_16a Sa.
1332   specialize Id2_08 with (¬(P→R)→¬(Q→¬R)).
1333   intros Id2_08a.
1334   Syll Sa Id2_08a Sb.
1335   replace (P→¬R) with (¬P∨¬R) in Sb.
1336   replace (Q→¬R) with (¬Q∨¬R) in Sb.
1337   replace (¬(¬P∨¬R)) with (P∧R) in Sb.
1338   replace (¬(¬Q∨¬R)) with (Q∧R) in Sb.
1339   apply Sb.
1340   apply Prod3_01.
1341   apply Prod3_01.
1342   rewrite <- Impl1_01.
1343   reflexivity.
1344   rewrite <- Impl1_01.
1345   reflexivity.
1346 Qed.
1347
1348 Theorem n3_47 : ∀ P Q R S : Prop,
1349   ((P → R) ∧ (Q → S)) → (P ∧ Q) → R ∧ S.
1350 Proof. intros P Q R S.
1351   specialize Simp3_26 with (P→R) (Q→S).
1352   intros Simp3_26a.
1353   specialize Fact3_45 with P R Q.
1354   intros Fact3_45a.
1355   Syll Simp3_26a Fact3_45a Sa.
1356   specialize n3_22 with R Q.
1357   intros n3_22a.
1358   specialize Syll2_05 with (P∧Q) (R∧Q) (Q∧R).
1359   intros Syll2_05a.
1360   MP Syll2_05a n3_22a.
1361   Syll Sa Syll2_05a Sb.
1362   specialize Simp3_27 with (P→R) (Q→S).
1363   intros Simp3_27a.
1364   specialize Fact3_45 with Q S R.
1365   intros Fact3_45b.
1366   Syll Simp3_27a Fact3_45b Sc.

```

```

1367 specialize n3_22 with S R.
1368 intros n3_22b.
1369 specialize Syll2_05 with (Q $\wedge$ R) (S $\wedge$ R) (R $\wedge$ S).
1370 intros Syll2_05b.
1371 MP Syll2_05b n3_22b.
1372 Syll Sc Syll2_05b Sd.
1373 clear Simp3_26a. clear Fact3_45a. clear Sa.
1374   clear n3_22a. clear Fact3_45b.
1375   clear Syll2_05a. clear Simp3_27a.
1376   clear Sc. clear n3_22b. clear Syll2_05b.
1377 Conj Sb Sd.
1378 split.
1379 apply Sb.
1380 apply Sd.
1381 specialize n2_83 with ((P $\rightarrow$ R) $\wedge$ (Q $\rightarrow$ S)) (P $\wedge$ Q) (Q $\wedge$ R) (R $\wedge$ S).
1382 intros n2_83a. (*This with MP works, but it omits Conj3_03.*)
1383 specialize Imp3_31 with (((P $\rightarrow$ R) $\wedge$ (Q $\rightarrow$ S)) $\rightarrow$ ((P $\wedge$ Q) $\rightarrow$ (Q $\wedge$ R)))
1384   (((P $\rightarrow$ R) $\wedge$ (Q $\rightarrow$ S)) $\rightarrow$ ((Q $\wedge$ R) $\rightarrow$ (R $\wedge$ S)))
1385   (((P $\rightarrow$ R) $\wedge$ (Q $\rightarrow$ S)) $\rightarrow$ ((P $\wedge$ Q) $\rightarrow$ (R $\wedge$ S))).
1386 intros Imp3_31a.
1387 MP Imp3_31a n2_83a.
1388 MP Imp3_31a H.
1389 apply Imp3_31a.
1390 Qed.
1391
1392 Theorem n3_48 :  $\forall$  P Q R S : Prop,
1393   ((P  $\rightarrow$  R)  $\wedge$  (Q  $\rightarrow$  S))  $\rightarrow$  (P  $\vee$  Q)  $\rightarrow$  R  $\vee$  S.
1394 Proof. intros P Q R S.
1395   specialize Simp3_26 with (P $\rightarrow$ R) (Q $\rightarrow$ S).
1396   intros Simp3_26a.
1397   specialize Sum1_6 with Q P R.
1398   intros Sum1_6a.
1399   Syll Simp3_26a Sum1_6a Sa.
1400   specialize Perm1_4 with P Q.
1401   intros Perm1_4a.
1402   specialize Syll2_06 with (P $\vee$ Q) (Q $\vee$ P) (Q $\vee$ R).
1403   intros Syll2_06a.
1404   MP Syll2_06a Perm1_4a.
1405   Syll Sa Syll2_06a Sb.
1406   specialize Simp3_27 with (P $\rightarrow$ R) (Q $\rightarrow$ S).
1407   intros Simp3_27a.
1408   specialize Sum1_6 with R Q S.

```

```

1409   intros Sum1_6b.
1410   Syll Simp3_27a Sum1_6b Sc.
1411   specialize Perm1_4 with Q R.
1412   intros Perm1_4b.
1413   specialize Syll2_06 with (Q $\vee$ R) (R $\vee$ Q) (R $\vee$ S).
1414   intros Syll2_06b.
1415   MP Syll2_06b Perm1_4b.
1416   Syll Sc Syll2_06a Sd.
1417   specialize n2_83 with ((P $\rightarrow$ R) $\wedge$ (Q $\rightarrow$ S)) (P $\vee$ Q) (Q $\vee$ R) (R $\vee$ S).
1418   intros n2_83a.
1419   MP n2_83a Sb.
1420   MP n2_83a Sd.
1421   apply n2_83a.
1422   Qed.
1423
1424   End No3.
1425
1426   Module No4.
1427
1428   Import No1.
1429   Import No2.
1430   Import No3.
1431
1432   Axiom Equiv4_01 :  $\forall$  P Q : Prop,
1433     (P  $\leftrightarrow$  Q) = ((P  $\rightarrow$  Q)  $\wedge$  (Q  $\rightarrow$  P)).
1434
1435   Axiom Abb4_02 :  $\forall$  P Q R : Prop,
1436     (P  $\leftrightarrow$  Q  $\leftrightarrow$  R) = ((P  $\leftrightarrow$  Q)  $\wedge$  (Q  $\leftrightarrow$  R)).
1437
1438   Axiom EqBi :  $\forall$  P Q : Prop,
1439     (P = Q)  $\leftrightarrow$  (P  $\leftrightarrow$  Q).
1440
1441   Ltac Equiv H1 :=
1442     match goal with
1443     | [ H1 : (?P $\rightarrow$ ?Q)  $\wedge$  (?Q $\rightarrow$ ?P) |- _ ] =>
1444       replace ((P $\rightarrow$ Q)  $\wedge$  (Q $\rightarrow$ P)) with (P $\leftrightarrow$ Q) in H1
1445   end.
1446
1447   Theorem Transp4_1 :  $\forall$  P Q : Prop,
1448     (P  $\rightarrow$  Q)  $\leftrightarrow$  ( $\neg$ Q  $\rightarrow$   $\neg$ P).
1449   Proof. intros P Q.
1450     specialize Transp2_16 with P Q.

```

```

1451   intros Transp2_16a.
1452   specialize Transp2_17 with P Q.
1453   intros Transp2_17a.
1454   Conj Transp2_16a Transp2_17a.
1455   split.
1456   apply Transp2_16a.
1457   apply Transp2_17a.
1458   Equiv H.
1459   apply H.
1460   apply Equiv4_01.
1461   Qed.
1462
1463   Theorem Transp4_11 :  $\forall$  P Q : Prop,
1464     (P  $\leftrightarrow$  Q)  $\leftrightarrow$  ( $\neg$ P  $\leftrightarrow$   $\neg$ Q).
1465   Proof. intros P Q.
1466     specialize Transp2_16 with P Q.
1467     intros Transp2_16a.
1468     specialize Transp2_16 with Q P.
1469     intros Transp2_16b.
1470     Conj Transp2_16a Transp2_16b.
1471     split.
1472     apply Transp2_16a.
1473     apply Transp2_16b.
1474     specialize n3_47 with (P $\rightarrow$ Q) (Q $\rightarrow$ P) ( $\neg$ Q $\rightarrow$  $\neg$ P) ( $\neg$ P $\rightarrow$  $\neg$ Q).
1475     intros n3_47a.
1476     MP n3_47 H.
1477     specialize n3_22 with ( $\neg$ Q  $\rightarrow$   $\neg$ P) ( $\neg$ P  $\rightarrow$   $\neg$ Q).
1478     intros n3_22a.
1479     Syll n3_47a n3_22a Sa.
1480     replace ((P  $\rightarrow$  Q)  $\wedge$  (Q  $\rightarrow$  P)) with (P $\leftrightarrow$ Q) in Sa.
1481     replace (( $\neg$ P  $\rightarrow$   $\neg$ Q)  $\wedge$  ( $\neg$ Q  $\rightarrow$   $\neg$ P)) with ( $\neg$ P $\leftrightarrow$  $\neg$ Q) in Sa.
1482     clear Transp2_16a. clear H. clear Transp2_16b.
1483     clear n3_22a. clear n3_47a.
1484     specialize Transp2_17 with Q P.
1485     intros Transp2_17a.
1486     specialize Transp2_17 with P Q.
1487     intros Transp2_17b.
1488     Conj Transp2_17a Transp2_17b.
1489     split.
1490     apply Transp2_17a.
1491     apply Transp2_17b.
1492     specialize n3_47 with ( $\neg$ P $\rightarrow$  $\neg$ Q) ( $\neg$ Q $\rightarrow$  $\neg$ P) (Q $\rightarrow$ P) (P $\rightarrow$ Q).

```

```

1493   intros n3_47a.
1494   MP n3_47a H.
1495   specialize n3_22 with (Q→P) (P→Q).
1496   intros n3_22a.
1497   Syll n3_47a n3_22a Sb.
1498   clear Transp2_17a. clear Transp2_17b. clear H.
1499       clear n3_47a. clear n3_22a.
1500   replace ((P → Q) ∧ (Q → P)) with (P↔Q) in Sb.
1501   replace ((¬P → ¬Q) ∧ (¬Q → ¬P)) with (¬P↔¬Q) in Sb.
1502   Conj Sa Sb.
1503   split.
1504   apply Sa.
1505   apply Sb.
1506   Equiv H.
1507   apply H.
1508   apply Equiv4_01.
1509   apply Equiv4_01.
1510   apply Equiv4_01.
1511   apply Equiv4_01.
1512   apply Equiv4_01.
1513   Qed.
1514
1515   Theorem n4_12 : ∀ P Q : Prop,
1516       (P ↔ ¬Q) ↔ (Q ↔ ¬P).
1517   Proof. intros P Q.
1518       specialize Transp2_03 with P Q.
1519       intros Transp2_03a.
1520       specialize Transp2_15 with Q P.
1521       intros Transp2_15a.
1522       Conj Transp2_03a Transp2_15a.
1523       split.
1524       apply Transp2_03a.
1525       apply Transp2_15a.
1526       specialize n3_47 with (P→¬Q) (¬Q→P) (Q→¬P) (¬P→Q).
1527       intros n3_47a.
1528       MP n3_47a H.
1529       specialize Transp2_03 with Q P.
1530       intros Transp2_03b.
1531       specialize Transp2_15 with P Q.
1532       intros Transp2_15b.
1533       Conj Transp2_03b Transp2_15b.
1534       split.

```

```

1535     apply Transp2_03b.
1536     apply Transp2_15b.
1537     specialize n3_47 with (Q → ¬P) (¬P → Q) (P → ¬Q) (¬Q → P).
1538     intros n3_47b.
1539     MP n3_47b H0.
1540     clear Transp2_03a. clear Transp2_15a. clear H.
1541     clear Transp2_03b. clear Transp2_15b. clear H0.
1542     Conj n3_47a n3_47b.
1543     split.
1544     apply n3_47a.
1545     apply n3_47b.
1546     rewrite <- Equiv4_01 in H.
1547     rewrite <- Equiv4_01 in H.
1548     rewrite <- Equiv4_01 in H.
1549     apply H.
1550     Qed.
1551
1552 Theorem n4_13 : ∀ P : Prop,
1553   P ↔ ¬¬P.
1554 Proof. intros P.
1555   specialize n2_12 with P.
1556   intros n2_12a.
1557   specialize n2_14 with P.
1558   intros n2_14a.
1559   Conj n2_12a n2_14a.
1560   split.
1561   apply n2_12a.
1562   apply n2_14a.
1563   Equiv H.
1564   apply H.
1565   apply Equiv4_01.
1566   Qed.
1567
1568 Theorem n4_14 : ∀ P Q R : Prop,
1569   ((P ∧ Q) → R) ↔ ((P ∧ ¬R) → ¬Q).
1570 Proof. intros P Q R.
1571   specialize Transp3_37 with P Q R.
1572   intros Transp3_37a.
1573   specialize Transp3_37 with P (¬R) (¬Q).
1574   intros Transp3_37b.
1575   Conj Transp3_37a Transp3_37b.
1576   split. apply Transp3_37a.

```

```

1577 apply Transp3_37b.
1578 specialize n4_13 with Q.
1579 intros n4_13a.
1580 specialize n4_13 with R.
1581 intros n4_13b.
1582 replace ( $\neg\neg Q$ ) with Q in H.
1583 replace ( $\neg\neg R$ ) with R in H.
1584 Equiv H.
1585 apply H.
1586 apply Equiv4_01.
1587 apply EqBi.
1588 apply n4_13b.
1589 apply EqBi.
1590 apply n4_13a.
1591 Qed.
1592
1593 Theorem n4_15 :  $\forall P Q R : \text{Prop}$ ,
1594   ( $(P \wedge Q) \rightarrow \neg R$ )  $\leftrightarrow$  ( $(Q \wedge R) \rightarrow \neg P$ ).
1595 Proof. intros P Q R.
1596 specialize n4_14 with Q P ( $\neg R$ ).
1597 intros n4_14a.
1598 specialize n3_22 with Q P.
1599 intros n3_22a.
1600 specialize Syll2_06 with  $(Q \wedge P)$   $(P \wedge Q)$  ( $\neg R$ ).
1601 intros Syll2_06a.
1602 MP Syll2_06a n3_22a.
1603 specialize n4_13 with R.
1604 intros n4_13a.
1605 replace ( $\neg\neg R$ ) with R in n4_14a.
1606 rewrite Equiv4_01 in n4_14a.
1607 specialize Simp3_26 with  $((Q \wedge P \rightarrow \neg R) \rightarrow Q \wedge R \rightarrow \neg P)$ 
1608    $((Q \wedge R \rightarrow \neg P) \rightarrow Q \wedge P \rightarrow \neg R)$ .
1609 intros Simp3_26a.
1610 MP Simp3_26a n4_14a.
1611 Syll Syll2_06a Simp3_26a Sa.
1612 specialize Simp3_27 with  $((Q \wedge P \rightarrow \neg R) \rightarrow Q \wedge R \rightarrow \neg P)$ 
1613    $((Q \wedge R \rightarrow \neg P) \rightarrow Q \wedge P \rightarrow \neg R)$ .
1614 intros Simp3_27a.
1615 MP Simp3_27a n4_14a.
1616 specialize n3_22 with P Q.
1617 intros n3_22b.
1618 specialize Syll2_06 with  $(P \wedge Q)$   $(Q \wedge P)$  ( $\neg R$ ).

```

```

1619   intros Syll2_06b.
1620   MP Syll2_06b n3_22b.
1621   Syll Syll2_06b Simp3_27a Sb.
1622   clear n4_14a. clear n3_22a. clear Syll2_06a.
1623       clear n4_13a. clear Simp3_26a. clear n3_22b.
1624       clear Simp3_27a. clear Syll2_06b.
1625   Conj Sa Sb.
1626   split.
1627   apply Sa.
1628   apply Sb.
1629   Equiv H.
1630   apply H.
1631   apply Equiv4_01.
1632   apply EqBi.
1633   apply n4_13a.
1634   Qed.
1635
1636   Theorem n4_2 :  $\forall P : \text{Prop}$ ,
1637        $P \leftrightarrow P$ .
1638   Proof. intros P.
1639       specialize n3_2 with  $(P \rightarrow P) (P \rightarrow P)$ .
1640       intros n3_2a.
1641       specialize Id2_08 with P.
1642       intros Id2_08a.
1643       MP n3_2a Id2_08a.
1644       MP n3_2a Id2_08a.
1645       Equiv n3_2a.
1646       apply n3_2a.
1647       apply Equiv4_01.
1648       Qed.
1649
1650   Theorem n4_21 :  $\forall P Q : \text{Prop}$ ,
1651        $(P \leftrightarrow Q) \leftrightarrow (Q \leftrightarrow P)$ .
1652   Proof. intros P Q.
1653       specialize n3_22 with  $(P \rightarrow Q) (Q \rightarrow P)$ .
1654       intros n3_22a.
1655       replace  $((P \rightarrow Q) \wedge (Q \rightarrow P))$  with  $(P \leftrightarrow Q)$  in n3_22a.
1656       replace  $((Q \rightarrow P) \wedge (P \rightarrow Q))$  with  $(Q \leftrightarrow P)$  in n3_22a.
1657       specialize n3_22 with  $(Q \rightarrow P) (P \rightarrow Q)$ .
1658       intros n3_22b.
1659       replace  $((P \rightarrow Q) \wedge (Q \rightarrow P))$  with  $(P \leftrightarrow Q)$  in n3_22b.
1660       replace  $((Q \rightarrow P) \wedge (P \rightarrow Q))$  with  $(Q \leftrightarrow P)$  in n3_22b.

```



```

1661   Conj n3_22a n3_22b.
1662   split.
1663   apply n3_22a.
1664   apply n3_22b.
1665   Equiv H.
1666   apply H.
1667   apply Equiv4_01.
1668   apply Equiv4_01.
1669   apply Equiv4_01.
1670   apply Equiv4_01.
1671   apply Equiv4_01.
1672   Qed.
1673
1674   Theorem n4_22 :  $\forall P Q R : \text{Prop},$ 
1675      $((P \leftrightarrow Q) \wedge (Q \leftrightarrow R)) \rightarrow (P \leftrightarrow R).$ 
1676   Proof. intros P Q R.
1677     specialize Simp3_26 with  $(P \leftrightarrow Q) (Q \leftrightarrow R).$ 
1678     intros Simp3_26a.
1679     specialize Simp3_26 with  $(P \rightarrow Q) (Q \rightarrow P).$ 
1680     intros Simp3_26b.
1681     replace  $((P \rightarrow Q) \wedge (Q \rightarrow P))$  with  $(P \leftrightarrow Q)$  in Simp3_26b.
1682     Syll Simp3_26a Simp3_26b Sa.
1683     specialize Simp3_27 with  $(P \leftrightarrow Q) (Q \leftrightarrow R).$ 
1684     intros Simp3_27a.
1685     specialize Simp3_26 with  $(Q \rightarrow R) (R \rightarrow Q).$ 
1686     intros Simp3_26c.
1687     replace  $((Q \rightarrow R) \wedge (R \rightarrow Q))$  with  $(Q \leftrightarrow R)$  in Simp3_26c.
1688     Syll Simp3_27a Simp3_26c Sb.
1689     specialize n2_83 with  $((P \leftrightarrow Q) \wedge (Q \leftrightarrow R)) P Q R.$ 
1690     intros n2_83a.
1691     MP n2_83a Sa.
1692     MP n2_83a Sb.
1693     specialize Simp3_27 with  $(P \leftrightarrow Q) (Q \leftrightarrow R).$ 
1694     intros Simp3_27b.
1695     specialize Simp3_27 with  $(Q \rightarrow R) (R \rightarrow Q).$ 
1696     intros Simp3_27c.
1697     replace  $((Q \rightarrow R) \wedge (R \rightarrow Q))$  with  $(Q \leftrightarrow R)$  in Simp3_27c.
1698     Syll Simp3_27b Simp3_27c Sc.
1699     specialize Simp3_26 with  $(P \leftrightarrow Q) (Q \leftrightarrow R).$ 
1700     intros Simp3_26d.
1701     specialize Simp3_27 with  $(P \rightarrow Q) (Q \rightarrow P).$ 
1702     intros Simp3_27d.

```

```

1703   replace ((P→Q) ∧ (Q→P)) with (P↔Q) in Simp3_27d.
1704   Syll Simp3_26d Simp3_27d Sd.
1705   specialize n2_83 with ((P↔Q) ∧ (Q↔R)) R Q P.
1706   intros n2_83b.
1707   MP n2_83b Sc. MP n2_83b Sd.
1708   clear Sd. clear Sb. clear Sc. clear Sa. clear Simp3_26a.
1709       clear Simp3_26b. clear Simp3_26c. clear Simp3_26d.
1710       clear Simp3_27a. clear Simp3_27b. clear Simp3_27c.
1711       clear Simp3_27d.
1712   Conj n2_83a n2_83b.
1713   split.
1714   apply n2_83a.
1715   apply n2_83b.
1716   specialize Comp3_43 with ((P↔Q) ∧ (Q↔R)) (P→R) (R→P).
1717   intros Comp3_43a.
1718   MP Comp3_43a H.
1719   replace ((P→R) ∧ (R→P)) with (P↔R) in Comp3_43a.
1720   apply Comp3_43a.
1721   apply Equiv4_01.
1722   apply Equiv4_01.
1723   apply Equiv4_01.
1724   apply Equiv4_01.
1725   apply Equiv4_01.
1726   Qed.
1727
1728   Theorem n4_24 : ∀ P : Prop,
1729       P ↔ (P ∧ P).
1730   Proof. intros P.
1731   specialize n3_2 with P P.
1732   intros n3_2a.
1733   specialize n2_43 with P (P ∧ P).
1734   intros n2_43a.
1735   MP n3_2a n2_43a.
1736   specialize Simp3_26 with P P.
1737   intros Simp3_26a.
1738   Conj n2_43a Simp3_26a.
1739   split.
1740   apply n2_43a.
1741   apply Simp3_26a.
1742   Equiv H.
1743   apply H.
1744   apply Equiv4_01.

```

```

1745 Qed.
1746
1747 Theorem n4_25 :  $\forall P : \text{Prop},$ 
1748    $P \leftrightarrow (P \vee P).$ 
1749 Proof. intros P.
1750   specialize Add1_3 with P P.
1751   intros Add1_3a.
1752   specialize Taut1_2 with P.
1753   intros Taut1_2a.
1754   Conj Add1_3a Taut1_2a.
1755   split.
1756   apply Add1_3a.
1757   apply Taut1_2a.
1758   Equiv H. apply H.
1759   apply Equiv4_01.
1760 Qed.
1761
1762 Theorem n4_3 :  $\forall P Q : \text{Prop},$ 
1763    $(P \wedge Q) \leftrightarrow (Q \wedge P).$ 
1764 Proof. intros P Q.
1765   specialize n3_22 with P Q.
1766   intros n3_22a.
1767   specialize n3_22 with Q P.
1768   intros n3_22b.
1769   Conj n3_22a n3_22b.
1770   split.
1771   apply n3_22a.
1772   apply n3_22b.
1773   Equiv H. apply H.
1774   apply Equiv4_01.
1775 Qed.
1776
1777 Theorem n4_31 :  $\forall P Q : \text{Prop},$ 
1778    $(P \vee Q) \leftrightarrow (Q \vee P).$ 
1779 Proof. intros P Q.
1780   specialize Perm1_4 with P Q.
1781   intros Perm1_4a.
1782   specialize Perm1_4 with Q P.
1783   intros Perm1_4b.
1784   Conj Perm1_4a Perm1_4b.
1785   split.
1786   apply Perm1_4a.

```

```

1787     apply Perm1_4b.
1788     Equiv H. apply H.
1789     apply Equiv4_01.
1790 Qed.
1791
1792 Theorem n4_32 :  $\forall$  P Q R : Prop,
1793    $((P \wedge Q) \wedge R) \leftrightarrow (P \wedge (Q \wedge R))$ .
1794 Proof. intros P Q R.
1795 specialize n4_15 with P Q R.
1796 intros n4_15a.
1797 specialize Transp4_1 with P  $(\neg(Q \wedge R))$ .
1798 intros Transp4_1a.
1799 replace  $(\neg\neg(Q \wedge R))$  with  $(Q \wedge R)$  in Transp4_1a.
1800 replace  $(Q \wedge R \rightarrow \neg P)$  with  $(P \rightarrow \neg(Q \wedge R))$  in n4_15a.
1801 specialize Transp4_11 with  $(P \wedge Q \rightarrow \neg R)$   $(P \rightarrow \neg(Q \wedge R))$ .
1802 intros Transp4_11a.
1803 replace  $((P \wedge Q \rightarrow \neg R) \leftrightarrow (P \rightarrow \neg(Q \wedge R)))$  with
1804    $(\neg(P \wedge Q \rightarrow \neg R) \leftrightarrow \neg(P \rightarrow \neg(Q \wedge R)))$  in n4_15a.
1805 replace  $(P \wedge Q \rightarrow \neg R)$  with
1806    $(\neg(P \wedge Q) \vee \neg R)$  in n4_15a.
1807 replace  $(P \rightarrow \neg(Q \wedge R))$  with
1808    $(\neg P \vee \neg(Q \wedge R))$  in n4_15a.
1809 replace  $(\neg(\neg(P \wedge Q) \vee \neg R))$  with
1810    $((P \wedge Q) \wedge R)$  in n4_15a.
1811 replace  $(\neg(\neg P \vee \neg(Q \wedge R)))$  with
1812    $(P \wedge (Q \wedge R))$  in n4_15a.
1813 apply n4_15a.
1814 apply Prod3_01.
1815 apply Prod3_01.
1816 rewrite Impl1_01.
1817 reflexivity.
1818 rewrite Impl1_01.
1819 reflexivity.
1820 replace  $(\neg(P \wedge Q \rightarrow \neg R) \leftrightarrow \neg(P \rightarrow \neg(Q \wedge R)))$  with
1821    $((P \wedge Q \rightarrow \neg R) \leftrightarrow (P \rightarrow \neg(Q \wedge R)))$ .
1822 reflexivity.
1823 apply EqBi.
1824 apply Transp4_11a.
1825 apply EqBi.
1826 apply Transp4_1a.
1827 apply EqBi.
1828 specialize n4_13 with  $(Q \wedge R)$ .

```

```

1829     intros n4_13a.
1830     apply n4_13a.
1831     Qed.
1832     (*Note that the actual proof uses n4_12, but
1833       that transposition involves transforming a
1834       biconditional into a conditional. This theorem
1835       may be a misprint. Using Transp4_1 to transpose
1836       a conditional and then applying n4_13 to double
1837       negate does secure the desired formula, though.*/)
1838
1839 Theorem n4_33 :  $\forall$  P Q R : Prop,
1840   (P  $\vee$  (Q  $\vee$  R))  $\leftrightarrow$  ((P  $\vee$  Q)  $\vee$  R).
1841 Proof. intros P Q R.
1842   specialize n2_31 with P Q R.
1843   intros n2_31a.
1844   specialize n2_32 with P Q R.
1845   intros n2_32a.
1846   Conj n2_31a n2_32a.
1847   split.
1848   apply n2_31a.
1849   apply n2_32a.
1850   Equiv H.
1851   apply H.
1852   apply Equiv4_01.
1853   Qed.
1854
1855 Axiom Abb4_34 :  $\forall$  P Q R : Prop,
1856   P  $\wedge$  Q  $\wedge$  R = ((P  $\wedge$  Q)  $\wedge$  R).
1857   (*This axiom ensures left association of brackets.
1858     Coq's default is right association. But Principia
1859     proves associativity of logical product as n4_32.
1860     So in effect, this axiom gives us a derived rule that
1861     allows us to shift between Coq's and Principia's
1862     default rules for brackets of logical products.*/)
1863
1864 Theorem n4_36 :  $\forall$  P Q R : Prop,
1865   (P  $\leftrightarrow$  Q)  $\rightarrow$  ((P  $\wedge$  R)  $\leftrightarrow$  (Q  $\wedge$  R)).
1866 Proof. intros P Q R.
1867   specialize Fact3_45 with P Q R.
1868   intros Fact3_45a.
1869   specialize Fact3_45 with Q P R.
1870   intros Fact3_45b.

```

```

1871 Conj Fact3_45a Fact3_45b.
1872 split.
1873 apply Fact3_45a.
1874 apply Fact3_45b.
1875 specialize n3_47 with (P→Q) (Q→P)
1876       (P ∧ R → Q ∧ R) (Q ∧ R → P ∧ R).
1877 intros n3_47a.
1878 MP n3_47 H.
1879 replace ((P → Q) ∧ (Q → P)) with (P↔Q) in n3_47a.
1880 replace ((P ∧ R → Q ∧ R) ∧ (Q ∧ R → P ∧ R)) with
1881       (P ∧ R ↔ Q ∧ R) in n3_47a.
1882 apply n3_47a.
1883 apply Equiv4_01.
1884 apply Equiv4_01.
1885 Qed.
1886
1887 Theorem n4_37 : ∀ P Q R : Prop,
1888       (P ↔ Q) → ((P ∨ R) ↔ (Q ∨ R)).
1889 Proof. intros P Q R.
1890 specialize Sum1_6 with R P Q.
1891 intros Sum1_6a.
1892 specialize Sum1_6 with R Q P.
1893 intros Sum1_6b.
1894 Conj Sum1_6a Sum1_6b.
1895 split.
1896 apply Sum1_6a.
1897 apply Sum1_6b.
1898 specialize n3_47 with (P → Q) (Q → P)
1899       (R ∨ P → R ∨ Q) (R ∨ Q → R ∨ P).
1900 intros n3_47a.
1901 MP n3_47 H.
1902 replace ((P → Q) ∧ (Q → P)) with (P↔Q) in n3_47a.
1903 replace ((R ∨ P → R ∨ Q) ∧ (R ∨ Q → R ∨ P)) with
1904       (R ∨ P ↔ R ∨ Q) in n3_47a.
1905 replace (R ∨ P) with (P ∨ R) in n3_47a.
1906 replace (R ∨ Q) with (Q ∨ R) in n3_47a.
1907 apply n3_47a.
1908 apply EqBi.
1909 specialize n4_31 with Q R.
1910 intros n4_31a.
1911 apply n4_31a.
1912 apply EqBi.

```

```

1913 specialize n4_31 with P R.
1914 intros n4_31b.
1915 apply n4_31b.
1916 apply Equiv4_01.
1917 apply Equiv4_01.
1918 Qed.
1919
1920 Theorem n4_38 :  $\forall P Q R S : \text{Prop}$ ,
1921    $((P \leftrightarrow R) \wedge (Q \leftrightarrow S)) \rightarrow ((P \wedge Q) \leftrightarrow (R \wedge S))$ .
1922 Proof. intros P Q R S.
1923   specialize n3_47 with P Q R S.
1924   intros n3_47a.
1925   specialize n3_47 with R S P Q.
1926   intros n3_47b.
1927   Conj n3_47a n3_47b.
1928   split.
1929   apply n3_47a.
1930   apply n3_47b.
1931   specialize n3_47 with  $((P \rightarrow R) \wedge (Q \rightarrow S))$ 
1932      $((R \rightarrow P) \wedge (S \rightarrow Q)) (P \wedge Q \rightarrow R \wedge S) (R \wedge S \rightarrow P \wedge Q)$ .
1933   intros n3_47c.
1934   MP n3_47c H.
1935   specialize n4_32 with  $(P \rightarrow R) (Q \rightarrow S) ((R \rightarrow P) \wedge (S \rightarrow Q))$ .
1936   intros n4_32a.
1937   replace  $((P \rightarrow R) \wedge (Q \rightarrow S)) \wedge (R \rightarrow P) \wedge (S \rightarrow Q)$  with
1938      $((P \rightarrow R) \wedge (Q \rightarrow S) \wedge (R \rightarrow P) \wedge (S \rightarrow Q))$  in n3_47c.
1939   specialize n4_32 with  $(Q \rightarrow S) (R \rightarrow P) (S \rightarrow Q)$ .
1940   intros n4_32b.
1941   replace  $((Q \rightarrow S) \wedge (R \rightarrow P) \wedge (S \rightarrow Q))$  with
1942      $((Q \rightarrow S) \wedge (R \rightarrow P)) \wedge (S \rightarrow Q)$  in n3_47c.
1943   specialize n3_22 with  $(Q \rightarrow S) (R \rightarrow P)$ .
1944   intros n3_22a.
1945   specialize n3_22 with  $(R \rightarrow P) (Q \rightarrow S)$ .
1946   intros n3_22b.
1947   Conj n3_22a n3_22b.
1948   split.
1949   apply n3_22a.
1950   apply n3_22b.
1951   Equiv H0.
1952   replace  $((Q \rightarrow S) \wedge (R \rightarrow P))$  with
1953      $((R \rightarrow P) \wedge (Q \rightarrow S))$  in n3_47c.
1954   specialize n4_32 with  $(R \rightarrow P) (Q \rightarrow S) (S \rightarrow Q)$ .

```

```

1955 intros n4_32c.
1956 replace ((R → P) ∧ (Q → S)) ∧ (S → Q) with
1957   ((R → P) ∧ (Q → S) ∧ (S → Q)) in n3_47c.
1958 specialize n4_32 with (P→R) (R → P) ((Q → S) ∧ (S → Q)).
1959 intros n4_32d.
1960 replace ((P → R) ∧ (R → P) ∧ (Q → S) ∧ (S → Q)) with
1961   (((P → R) ∧ (R → P)) ∧ (Q → S) ∧ (S → Q)) in n3_47c.
1962 replace ((P→R) ∧ (R → P)) with (P↔R) in n3_47c.
1963 replace ((Q → S) ∧ (S → Q)) with (Q↔S) in n3_47c.
1964 replace ((P ∧ Q → R ∧ S) ∧ (R ∧ S → P ∧ Q)) with
1965   ((P ∧ Q) ↔ (R ∧ S)) in n3_47c.
1966 apply n3_47c.
1967 apply Equiv4_01.
1968 apply Equiv4_01.
1969 apply Equiv4_01.
1970 apply EqBi.
1971 apply n4_32d.
1972 replace ((R → P) ∧ (Q → S) ∧ (S → Q)) with
1973   ((R → P) ∧ (Q → S)) ∧ (S → Q).
1974 reflexivity.
1975 apply EqBi.
1976 apply n4_32c.
1977 replace ((R → P) ∧ (Q → S)) with ((Q → S) ∧ (R → P)).
1978 reflexivity.
1979 apply EqBi.
1980 apply H0.
1981 apply Equiv4_01.
1982 apply EqBi.
1983 apply n4_32b.
1984 replace ((P → R) ∧ (Q → S) ∧ (R → P) ∧ (S → Q)) with
1985   (((P → R) ∧ (Q → S)) ∧ (R → P) ∧ (S → Q)).
1986 reflexivity.
1987 apply EqBi.
1988 apply n4_32a.
1989 Qed.
1990
1991 Theorem n4_39 : ∀ P Q R S : Prop,
1992   ((P ↔ R) ∧ (Q ↔ S)) → ((P ∨ Q) ↔ (R ∨ S)).
1993 Proof. intros P Q R S.
1994   specialize n3_48 with P Q R S.
1995   intros n3_48a.
1996   specialize n3_48 with R S P Q.

```



```

1997   intros n3_48b.
1998   Conj n3_48a n3_48b.
1999   split.
2000   apply n3_48a.
2001   apply n3_48b.
2002   specialize n3_47 with ((P → R) ∧ (Q → S))
2003     ((R → P) ∧ (S → Q)) (P ∨ Q → R ∨ S) (R ∨ S → P ∨ Q).
2004   intros n3_47a.
2005   MP n3_47a H.
2006   replace ((P ∨ Q → R ∨ S) ∧ (R ∨ S → P ∨ Q)) with
2007     ((P ∨ Q) ↔ (R ∨ S)) in n3_47a.
2008   specialize n4_32 with ((P → R) ∧ (Q → S)) (R → P) (S → Q).
2009   intros n4_32a.
2010   replace (((P → R) ∧ (Q → S)) ∧ (R → P) ∧ (S → Q)) with
2011     (((P → R) ∧ (Q → S)) ∧ (R → P)) ∧ (S → Q) in n3_47a.
2012   specialize n4_32 with (P → R) (Q → S) (R → P).
2013   intros n4_32b.
2014   replace (((P → R) ∧ (Q → S)) ∧ (R → P)) with
2015     ((P → R) ∧ (Q → S) ∧ (R → P)) in n3_47a.
2016   specialize n3_22 with (Q → S) (R → P).
2017   intros n3_22a.
2018   specialize n3_22 with (R → P) (Q → S).
2019   intros n3_22b.
2020   Conj n3_22a n3_22b.
2021   split.
2022   apply n3_22a.
2023   apply n3_22b.
2024   Equiv H0.
2025   replace ((Q → S) ∧ (R → P)) with
2026     ((R → P) ∧ (Q → S)) in n3_47a.
2027   specialize n4_32 with (P → R) (R → P) (Q → S).
2028   intros n4_32c.
2029   replace ((P → R) ∧ (R → P) ∧ (Q → S)) with
2030     (((P → R) ∧ (R → P)) ∧ (Q → S)) in n3_47a.
2031   replace ((P → R) ∧ (R → P)) with (P ↔ R) in n3_47a.
2032   specialize n4_32 with (P ↔ R) (Q → S) (S → Q).
2033   intros n4_32d.
2034   replace (((P ↔ R) ∧ (Q → S)) ∧ (S → Q)) with
2035     ((P ↔ R) ∧ (Q → S) ∧ (S → Q)) in n3_47a.
2036   replace ((Q → S) ∧ (S → Q)) with (Q ↔ S) in n3_47a.
2037   apply n3_47a.
2038   apply Equiv4_01.

```

```

2039   replace ((P ↔ R) ∧ (Q → S) ∧ (S → Q)) with
2040         ((P ↔ R) ∧ (Q → S)) ∧ (S → Q)).
2041   reflexivity.
2042   apply EqBi.
2043   apply n4_32d.
2044   apply Equiv4_01.
2045   apply EqBi.
2046   apply n4_32c.
2047   replace ((R → P) ∧ (Q → S)) with ((Q → S) ∧ (R → P)).
2048   reflexivity.
2049   apply EqBi.
2050   apply H0.
2051   apply Equiv4_01.
2052   replace ((P → R) ∧ (Q → S) ∧ (R → P)) with
2053         ((P → R) ∧ (Q → S)) ∧ (R → P)).
2054   reflexivity.
2055   apply EqBi.
2056   apply n4_32b.
2057   apply EqBi.
2058   apply n4_32a.
2059   apply Equiv4_01.
2060   Qed.
2061
2062   Theorem n4_4 : ∀ P Q R : Prop,
2063     (P ∧ (Q ∨ R)) ↔ ((P ∧ Q) ∨ (P ∧ R)).
2064   Proof. intros P Q R.
2065     specialize n3_2 with P Q.
2066     intros n3_2a.
2067     specialize n3_2 with P R.
2068     intros n3_2b.
2069     Conj n3_2a n3_2b.
2070     split.
2071     apply n3_2a.
2072     apply n3_2b.
2073     specialize Comp3_43 with P (Q → P ∧ Q) (R → P ∧ R).
2074     intros Comp3_43a.
2075     MP Comp3_43a H.
2076     specialize n3_48 with Q R (P ∧ Q) (P ∧ R).
2077     intros n3_48a.
2078     Syll Comp3_43a n3_48a Sa.
2079     specialize Imp3_31 with P (Q ∨ R) ((P ∧ Q) ∨ (P ∧ R)).
2080     intros Imp3_31a.

```

```

2081 MP Imp3_31a Sa.
2082 specialize Simp3_26 with P Q.
2083 intros Simp3_26a.
2084 specialize Simp3_26 with P R.
2085 intros Simp3_26b.
2086 Conj Simp3_26a Simp3_26b.
2087 split.
2088 apply Simp3_26a.
2089 apply Simp3_26b.
2090 specialize n3_44 with P (P $\wedge$ Q) (P $\wedge$ R).
2091 intros n3_44a.
2092 MP n3_44a H0.
2093 specialize Simp3_27 with P Q.
2094 intros Simp3_27a.
2095 specialize Simp3_27 with P R.
2096 intros Simp3_27b.
2097 Conj Simp3_27a Simp3_27b.
2098 split.
2099 apply Simp3_27a.
2100 apply Simp3_27b.
2101 specialize n3_48 with (P $\wedge$ Q) (P $\wedge$ R) Q R.
2102 intros n3_48b.
2103 MP n3_48b H1.
2104 clear H1. clear Simp3_27a. clear Simp3_27b.
2105 Conj n3_44a n3_48b.
2106 split.
2107 apply n3_44a.
2108 apply n3_48b.
2109 specialize Comp3_43 with (P  $\wedge$  Q  $\vee$  P  $\wedge$  R) P (Q $\vee$ R).
2110 intros Comp3_43b.
2111 MP Comp3_43b H1.
2112 clear H1. clear H0. clear n3_44a. clear n3_48b.
2113 clear Simp3_26a. clear Simp3_26b.
2114 Conj Imp3_31a Comp3_43b.
2115 split.
2116 apply Imp3_31a.
2117 apply Comp3_43b.
2118 Equiv H0.
2119 apply H0.
2120 apply Equiv4_01.
2121 Qed.
2122

```

```

2123 Theorem n4_41 :  $\forall$  P Q R : Prop,
2124   (P  $\vee$  (Q  $\wedge$  R))  $\leftrightarrow$  ((P  $\vee$  Q)  $\wedge$  (P  $\vee$  R)).
2125 Proof. intros P Q R.
2126   specialize Simp3_26 with Q R.
2127   intros Simp3_26a.
2128   specialize Sum1_6 with P (Q  $\wedge$  R) Q.
2129   intros Sum1_6a.
2130   MP Simp3_26a Sum1_6a.
2131   specialize Simp3_27 with Q R.
2132   intros Simp3_27a.
2133   specialize Sum1_6 with P (Q  $\wedge$  R) R.
2134   intros Sum1_6b.
2135   MP Simp3_27a Sum1_6b.
2136   clear Simp3_26a. clear Simp3_27a.
2137   Conj Sum1_6a Sum1_6a.
2138   split.
2139   apply Sum1_6a.
2140   apply Sum1_6b.
2141   specialize Comp3_43 with (P  $\vee$  Q  $\wedge$  R) (P  $\vee$  Q) (P  $\vee$  R).
2142   intros Comp3_43a.
2143   MP Comp3_43a H.
2144   specialize n2_53 with P Q.
2145   intros n2_53a.
2146   specialize n2_53 with P R.
2147   intros n2_53b.
2148   Conj n2_53a n2_53b.
2149   split.
2150   apply n2_53a.
2151   apply n2_53b.
2152   specialize n3_47 with (P  $\vee$  Q) (P  $\vee$  R) ( $\neg$ P  $\rightarrow$  Q) ( $\neg$ P  $\rightarrow$  R).
2153   intros n3_47a.
2154   MP n3_47a H0.
2155   specialize Comp3_43 with ( $\neg$ P) Q R.
2156   intros Comp3_43b.
2157   Syll n3_47a Comp3_43b Sa.
2158   specialize n2_54 with P (Q $\wedge$ R).
2159   intros n2_54a.
2160   Syll Sa n2_54a Sb.
2161   clear Sum1_6a. clear Sum1_6b. clear H. clear n2_53a.
2162     clear n2_53b. clear H0. clear n3_47a. clear Sa.
2163     clear Comp3_43b. clear n2_54a.
2164   Conj Comp3_43a Sb.

```

```

2165     split.
2166     apply Comp3_43a.
2167     apply Sb.
2168     Equiv H.
2169     apply H.
2170     apply Equiv4_01.
2171 Qed.
2172
2173 Theorem n4_42 :  $\forall$  P Q : Prop,
2174   P  $\leftrightarrow$  ((P  $\wedge$  Q)  $\vee$  (P  $\wedge$   $\neg$ Q)).
2175 Proof. intros P Q.
2176   specialize n3_21 with P (Q  $\vee$   $\neg$ Q).
2177   intros n3_21a.
2178   specialize n2_11 with Q.
2179   intros n2_11a.
2180   MP n3_21a n2_11a.
2181   specialize Simp3_26 with P (Q  $\vee$   $\neg$ Q).
2182   intros Simp3_26a. clear n2_11a.
2183   Conj n3_21a Simp3_26a.
2184   split.
2185   apply n3_21a.
2186   apply Simp3_26a.
2187   Equiv H.
2188   specialize n4_4 with P Q ( $\neg$ Q).
2189   intros n4_4a.
2190   replace (P  $\wedge$  (Q  $\vee$   $\neg$ Q)) with P in n4_4a.
2191   apply n4_4a.
2192   apply EqBi.
2193   apply H.
2194   apply Equiv4_01.
2195 Qed.
2196
2197 Theorem n4_43 :  $\forall$  P Q : Prop,
2198   P  $\leftrightarrow$  ((P  $\vee$  Q)  $\wedge$  (P  $\vee$   $\neg$ Q)).
2199 Proof. intros P Q.
2200   specialize n2_2 with P Q.
2201   intros n2_2a.
2202   specialize n2_2 with P ( $\neg$ Q).
2203   intros n2_2b.
2204   Conj n2_2a n2_2b.
2205   split.
2206   apply n2_2a.

```

```

2207   apply n2_2b.
2208   specialize Comp3_43 with P (P $\vee$ Q) (P $\vee$  $\neg$ Q).
2209   intros Comp3_43a.
2210   MP Comp3_43a H.
2211   specialize n2_53 with P Q.
2212   intros n2_53a.
2213   specialize n2_53 with P ( $\neg$ Q).
2214   intros n2_53b.
2215   Conj n2_53a n2_53b.
2216   split.
2217   apply n2_53a.
2218   apply n2_53b.
2219   specialize n3_47 with (P $\vee$ Q) (P $\vee$  $\neg$ Q) ( $\neg$ P $\rightarrow$ Q) ( $\neg$ P $\rightarrow$  $\neg$ Q).
2220   intros n3_47a.
2221   MP n3_47a H0.
2222   specialize n2_65 with ( $\neg$ P) Q.
2223   intros n2_65a.
2224   replace ( $\neg$  $\neg$ P) with P in n2_65a.
2225   specialize Imp3_31 with ( $\neg$ P  $\rightarrow$  Q) ( $\neg$ P  $\rightarrow$   $\neg$ Q) (P).
2226   intros Imp3_31a.
2227   MP Imp3_31a n2_65a.
2228   Syll n3_47a Imp3_31a Sa.
2229   clear n2_2a. clear n2_2b. clear H. clear n2_53a.
2230   clear n2_53b. clear H0. clear n2_65a.
2231   clear n3_47a. clear Imp3_31a.
2232   Conj Comp3_43a Sa.
2233   split.
2234   apply Comp3_43a.
2235   apply Sa.
2236   Equiv H.
2237   apply H.
2238   apply Equiv4_01.
2239   apply EqBi.
2240   specialize n4_13 with P.
2241   intros n4_13a.
2242   apply n4_13a.
2243   Qed.
2244
2245   Theorem n4_44 :  $\forall$  P Q : Prop,
2246     P  $\leftrightarrow$  (P  $\vee$  (P  $\wedge$  Q)).
2247   Proof. intros P Q.
2248     specialize n2_2 with P (P $\wedge$ Q).

```

```

2249     intros n2_2a.
2250     specialize Id2_08 with P.
2251     intros Id2_08a.
2252     specialize Simp3_26 with P Q.
2253     intros Simp3_26a.
2254     Conj Id2_08a Simp3_26a.
2255     split.
2256     apply Id2_08a.
2257     apply Simp3_26a.
2258     specialize n3_44 with P P (P ∧ Q).
2259     intros n3_44a.
2260     MP n3_44a H.
2261     clear H. clear Id2_08a. clear Simp3_26a.
2262     Conj n2_2a n3_44a.
2263     split.
2264     apply n2_2a.
2265     apply n3_44a.
2266     Equiv H.
2267     apply H.
2268     apply Equiv4_01.
2269     Qed.
2270
2271 Theorem n4_45 : ∀ P Q : Prop,
2272   P ↔ (P ∧ (P ∨ Q)).
2273 Proof. intros P Q.
2274   specialize n2_2 with (P ∧ P) (P ∧ Q).
2275   intros n2_2a.
2276   replace (P ∧ P ∨ P ∧ Q) with (P ∧ (P ∨ Q)) in n2_2a.
2277   replace (P ∧ P) with P in n2_2a.
2278   specialize Simp3_26 with P (P ∨ Q).
2279   intros Simp3_26a.
2280   Conj n2_2a Simp3_26a.
2281   split.
2282   apply n2_2a.
2283   apply Simp3_26a.
2284   Equiv H.
2285   apply H.
2286   apply Equiv4_01.
2287   specialize n4_24 with P.
2288   intros n4_24a.
2289   apply EqBi.
2290   apply n4_24a.

```

```

2291   specialize n4_4 with P P Q.
2292   intros n4_4a.
2293   apply EqBi.
2294   apply n4_4a.
2295   Qed.
2296
2297   Theorem n4_5 :  $\forall P Q : \text{Prop},$ 
2298      $P \wedge Q \leftrightarrow \neg(\neg P \vee \neg Q).$ 
2299   Proof. intros P Q.
2300     specialize n4_2 with (P  $\wedge$  Q).
2301     intros n4_2a.
2302     rewrite Prod3_01.
2303     replace ( $\neg(\neg P \vee \neg Q)$ ) with (P  $\wedge$  Q).
2304     apply n4_2a.
2305     apply Prod3_01.
2306     Qed.
2307
2308   Theorem n4_51 :  $\forall P Q : \text{Prop},$ 
2309      $\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q).$ 
2310   Proof. intros P Q.
2311     specialize n4_5 with P Q.
2312     intros n4_5a.
2313     specialize n4_12 with (P  $\wedge$  Q) ( $\neg P \vee \neg Q$ ).
2314     intros n4_12a.
2315     replace ((P $\wedge$ Q  $\leftrightarrow \neg(\neg P \vee \neg Q)$ )  $\leftrightarrow (\neg P \vee \neg Q \leftrightarrow \neg(P \wedge Q))$ ) with
2316       ((P $\wedge$ Q  $\leftrightarrow \neg(\neg P \vee \neg Q)$ ) = ( $\neg P \vee \neg Q \leftrightarrow \neg(P \wedge Q)$ )) in n4_12a.
2317     replace (P  $\wedge$  Q  $\leftrightarrow \neg(\neg P \vee \neg Q)$ ) with
2318       ( $\neg P \vee \neg Q \leftrightarrow \neg(P \wedge Q)$ ) in n4_5a.
2319     replace ( $\neg P \vee \neg Q \leftrightarrow \neg(P \wedge Q)$ ) with
2320       ( $\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q)$ ) in n4_5a.
2321     apply n4_5a.
2322     specialize n4_21 with ( $\neg(P \wedge Q)$ ) ( $\neg P \vee \neg Q$ ).
2323     intros n4_21a.
2324     apply EqBi.
2325     specialize n4_21 with ( $\neg(P \wedge Q)$ ) ( $\neg P \vee \neg Q$ ).
2326     intros n4_21b.
2327     apply n4_21b.
2328     apply EqBi.
2329     apply EqBi.
2330     Qed.
2331
2332   Theorem n4_52 :  $\forall P Q : \text{Prop},$ 

```



```

2333   (P ∧ ¬Q) ↔ ¬(¬P ∨ Q).
2334 Proof. intros P Q.
2335   specialize n4_5 with P (¬Q).
2336   intros n4_5a.
2337   replace (¬¬Q) with Q in n4_5a.
2338   apply n4_5a.
2339   specialize n4_13 with Q.
2340   intros n4_13a.
2341   apply EqBi.
2342   apply n4_13a.
2343   Qed.
2344
2345 Theorem n4_53 : ∀ P Q : Prop,
2346   ¬(P ∧ ¬Q) ↔ (¬P ∨ Q).
2347 Proof. intros P Q.
2348   specialize n4_52 with P Q.
2349   intros n4_52a.
2350   specialize n4_12 with (P ∧ ¬Q) ((¬P ∨ Q)).
2351   intros n4_12a.
2352   replace ((P ∧ ¬Q ↔ ¬(¬P ∨ Q)) ↔ (¬P ∨ Q ↔ ¬(P ∧ ¬Q))) with
2353     ((P ∧ ¬Q ↔ ¬(¬P ∨ Q)) = (¬P ∨ Q ↔ ¬(P ∧ ¬Q))) in n4_12a.
2354   replace (P ∧ ¬Q ↔ ¬(¬P ∨ Q)) with
2355     (¬P ∨ Q ↔ ¬(P ∧ ¬Q)) in n4_52a.
2356   replace (¬P ∨ Q ↔ ¬(P ∧ ¬Q)) with
2357     (¬(P ∧ ¬Q) ↔ (¬P ∨ Q)) in n4_52a.
2358   apply n4_52a.
2359   specialize n4_21 with (¬(P ∧ ¬Q)) (¬P ∨ Q).
2360   intros n4_21a.
2361   apply EqBi.
2362   apply n4_21a.
2363   apply EqBi.
2364   apply EqBi.
2365   Qed.
2366
2367 Theorem n4_54 : ∀ P Q : Prop,
2368   (¬P ∧ Q) ↔ ¬(P ∨ ¬Q).
2369 Proof. intros P Q.
2370   specialize n4_5 with (¬P) Q.
2371   intros n4_5a.
2372   specialize n4_13 with P.
2373   intros n4_13a.
2374   replace (¬¬P) with P in n4_5a.

```

```

2375     apply n4_5a.
2376     apply EqBi.
2377     apply n4_13a.
2378 Qed.
2379
2380 Theorem n4_55 :  $\forall P Q : \text{Prop},$ 
2381    $\neg(\neg P \wedge Q) \leftrightarrow (P \vee \neg Q).$ 
2382 Proof. intros P Q.
2383   specialize n4_54 with P Q.
2384   intros n4_54a.
2385   specialize n4_12 with  $(\neg P \wedge Q) (P \vee \neg Q).$ 
2386   intros n4_12a.
2387   replace  $(\neg P \wedge Q \leftrightarrow \neg(P \vee \neg Q))$  with
2388      $(P \vee \neg Q \leftrightarrow \neg(\neg P \wedge Q))$  in n4_54a.
2389   replace  $(P \vee \neg Q \leftrightarrow \neg(\neg P \wedge Q))$  with
2390      $(\neg(\neg P \wedge Q) \leftrightarrow (P \vee \neg Q))$  in n4_54a.
2391   apply n4_54a.
2392   specialize n4_21 with  $(\neg(\neg P \wedge Q)) (P \vee \neg Q).$ 
2393   intros n4_21a. (*Not cited*)
2394   apply EqBi.
2395   apply n4_21a.
2396   apply EqBi.
2397   replace  $((P \vee \neg Q \leftrightarrow \neg(\neg P \wedge Q)) \leftrightarrow (\neg P \wedge Q \leftrightarrow \neg(P \vee \neg Q)))$ 
2398     with  $((\neg P \wedge Q \leftrightarrow \neg(P \vee \neg Q)) \leftrightarrow (P \vee \neg Q \leftrightarrow \neg(\neg P \wedge Q)))$ .
2399   apply n4_12a.
2400   apply EqBi.
2401   specialize n4_21 with  $(P \vee \neg Q \leftrightarrow \neg(\neg P \wedge Q))$ 
2402      $(\neg P \wedge Q \leftrightarrow \neg(P \vee \neg Q)).$ 
2403   intros n4_21b.
2404   apply n4_21.
2405 Qed.
2406
2407 Theorem n4_56 :  $\forall P Q : \text{Prop},$ 
2408    $(\neg P \wedge \neg Q) \leftrightarrow \neg(P \vee Q).$ 
2409 Proof. intros P Q.
2410   specialize n4_54 with P  $(\neg Q).$ 
2411   intros n4_54a.
2412   replace  $(\neg\neg Q)$  with Q in n4_54a.
2413   apply n4_54a.
2414   apply EqBi.
2415   specialize n4_13 with Q.
2416   intros n4_13a.

```

```

2417     apply n4_13a.
2418 Qed.
2419
2420 Theorem n4_57 :  $\forall P Q : \text{Prop},$ 
2421    $\neg(\neg P \wedge \neg Q) \leftrightarrow (P \vee Q).$ 
2422 Proof. intros P Q.
2423   specialize n4_56 with P Q.
2424   intros n4_56a.
2425   specialize n4_12 with  $(\neg P \wedge \neg Q) (P \vee Q).$ 
2426   intros n4_12a.
2427   replace  $(\neg P \wedge \neg Q \leftrightarrow \neg(P \vee Q))$  with
2428      $(P \vee Q \leftrightarrow \neg(\neg P \wedge \neg Q))$  in n4_56a.
2429   replace  $(P \vee Q \leftrightarrow \neg(\neg P \wedge \neg Q))$  with
2430      $(\neg(\neg P \wedge \neg Q) \leftrightarrow P \vee Q)$  in n4_56a.
2431   apply n4_56a.
2432   specialize n4_21 with  $(\neg(\neg P \wedge \neg Q)) (P \vee Q).$ 
2433   intros n4_21a.
2434   apply EqBi.
2435   apply n4_21a.
2436   replace  $((\neg P \wedge \neg Q \leftrightarrow \neg(P \vee Q)) \leftrightarrow (P \vee Q \leftrightarrow \neg(\neg P \wedge \neg Q)))$  with
2437      $((P \vee Q \leftrightarrow \neg(\neg P \wedge \neg Q)) \leftrightarrow (\neg P \wedge \neg Q \leftrightarrow \neg(P \vee Q)))$  in n4_12a.
2438   apply EqBi.
2439   apply n4_12a.
2440   apply EqBi.
2441   specialize n4_21 with
2442      $(P \vee Q \leftrightarrow \neg(\neg P \wedge \neg Q)) (\neg P \wedge \neg Q \leftrightarrow \neg(P \vee Q)).$ 
2443   intros n4_21b.
2444   apply n4_21b.
2445 Qed.
2446
2447 Theorem n4_6 :  $\forall P Q : \text{Prop},$ 
2448    $(P \rightarrow Q) \leftrightarrow (\neg P \vee Q).$ 
2449 Proof. intros P Q.
2450   specialize n4_2 with  $(\neg P \vee Q).$ 
2451   intros n4_2a.
2452   rewrite Impl1_01.
2453   apply n4_2a.
2454 Qed.
2455
2456 Theorem n4_61 :  $\forall P Q : \text{Prop},$ 
2457    $\neg(P \rightarrow Q) \leftrightarrow (P \wedge \neg Q).$ 
2458 Proof. intros P Q.

```

```

2459 specialize n4_6 with P Q.
2460 intros n4_6a.
2461 specialize Transp4_11 with (P → Q) (¬P ∨ Q).
2462 intros Transp4_11a.
2463 specialize n4_52 with P Q.
2464 intros n4_52a.
2465 replace ((P → Q) ↔ ¬P ∨ Q) with
2466   (¬(P → Q) ↔ ¬(¬P ∨ Q)) in n4_6a.
2467 replace (¬(¬P ∨ Q)) with (P ∧ ¬Q) in n4_6a.
2468 apply n4_6a.
2469 apply EqBi.
2470 apply n4_52a.
2471 replace (((P → Q) ↔ ¬P ∨ Q) ↔ (¬(P → Q) ↔ ¬(¬P ∨ Q))) with
2472   ((¬(P → Q) ↔ ¬(¬P ∨ Q)) ↔ ((P → Q) ↔ ¬P ∨ Q)) in Transp4_11a.
2473 apply EqBi.
2474 apply Transp4_11a.
2475 apply EqBi.
2476 specialize n4_21 with (¬(P → Q) ↔ ¬(¬P ∨ Q))
2477   ((P → Q) ↔ (¬P ∨ Q)).
2478 intros n4_21a.
2479 apply n4_21a.
2480 Qed.
2481
2482 Theorem n4_62 : ∀ P Q : Prop,
2483   (P → ¬Q) ↔ (¬P ∨ ¬Q).
2484 Proof. intros P Q.
2485   specialize n4_6 with P (¬Q).
2486   intros n4_6a.
2487   apply n4_6a.
2488   Qed.
2489
2490 Theorem n4_63 : ∀ P Q : Prop,
2491   ¬(P → ¬Q) ↔ (P ∧ Q).
2492 Proof. intros P Q.
2493   specialize n4_62 with P Q.
2494   intros n4_62a.
2495   specialize Transp4_11 with (P → ¬Q) (¬P ∨ ¬Q).
2496   intros Transp4_11a.
2497   specialize n4_5 with P Q.
2498   intros n4_5a.
2499   replace (¬(¬P ∨ ¬Q)) with (P ∧ Q) in Transp4_11a.
2500   replace ((P → ¬Q) ↔ ¬P ∨ ¬Q) with

```

```

2501      ((¬(P → ¬Q) ↔ P ∧ Q)) in n4_62a.
2502  apply n4_62a.
2503  replace ((P→¬Q)↔¬P∨¬Q)↔(¬(P→¬Q)↔P∧Q) with
2504      ((¬(P→¬Q)↔P∧Q)↔((P→¬Q)↔¬P∨¬Q)) in Transp4_11a.
2505  apply EqBi.
2506  apply Transp4_11a.
2507  specialize n4_21 with
2508      (¬(P → ¬Q) ↔ P ∧ Q) ((P → ¬Q) ↔ ¬P ∨ ¬Q).
2509  intros n4_21a.
2510  apply EqBi.
2511  apply n4_21a.
2512  apply EqBi.
2513  apply n4_5a.
2514  Qed.
2515
2516  Theorem n4_64 : ∀ P Q : Prop,
2517      (¬P → Q) ↔ (P ∨ Q).
2518  Proof. intros P Q.
2519      specialize n2_54 with P Q.
2520      intros n2_54a.
2521      specialize n2_53 with P Q.
2522      intros n2_53a.
2523      Conj n2_54a n2_53a.
2524      split.
2525      apply n2_54a.
2526      apply n2_53a.
2527      Equiv H.
2528      apply H.
2529      apply Equiv4_01.
2530  Qed.
2531
2532  Theorem n4_65 : ∀ P Q : Prop,
2533      ¬(¬P → Q) ↔ (¬P ∧ ¬Q).
2534  Proof. intros P Q.
2535      specialize n4_64 with P Q.
2536      intros n4_64a.
2537      specialize Transp4_11 with(¬P → Q) (P ∨ Q).
2538      intros Transp4_11a.
2539      specialize n4_56 with P Q.
2540      intros n4_56a.
2541      replace (((¬P→Q)↔P∨Q)↔(¬(¬P→Q)↔¬(P∨Q))) with
2542          ((¬(¬P→Q)↔¬(P∨Q))↔((¬P→Q)↔P∨Q)) in Transp4_11a.

```

```

2543   replace ((¬P → Q) ↔ P ∨ Q) with
2544       (¬(¬P → Q) ↔ ¬(P ∨ Q)) in n4_64a.
2545   replace (¬(P ∨ Q)) with (¬P ∧ ¬Q) in n4_64a.
2546   apply n4_64a.
2547   apply EqBi.
2548   apply n4_56a.
2549   apply EqBi.
2550   apply Transp4_11a.
2551   apply EqBi.
2552   specialize n4_21 with (¬(¬P → Q) ↔ ¬(P ∨ Q))
2553       ((¬P → Q) ↔ (P ∨ Q)).
2554   intros n4_21a.
2555   apply n4_21a.
2556   Qed.
2557
2558 Theorem n4_66 : ∀ P Q : Prop,
2559   (¬P → ¬Q) ↔ (P ∨ ¬Q).
2560 Proof. intros P Q.
2561   specialize n4_64 with P (¬Q).
2562   intros n4_64a.
2563   apply n4_64a.
2564   Qed.
2565
2566 Theorem n4_67 : ∀ P Q : Prop,
2567   ¬(¬P → ¬Q) ↔ (¬P ∧ Q).
2568 Proof. intros P Q.
2569   specialize n4_66 with P Q.
2570   intros n4_66a.
2571   specialize Transp4_11 with (¬P → ¬Q) (P ∨ ¬Q).
2572   intros Transp4_11a.
2573   replace ((¬P → ¬Q) ↔ P ∨ ¬Q) with
2574       (¬(¬P → ¬Q) ↔ ¬(P ∨ ¬Q)) in n4_66a.
2575   specialize n4_54 with P Q.
2576   intros n4_54a.
2577   replace (¬(P ∨ ¬Q)) with (¬P ∧ Q) in n4_66a.
2578   apply n4_66a.
2579   apply EqBi.
2580   apply n4_54a.
2581   replace (((¬P → ¬Q) ↔ P ∨ ¬Q) ↔ (¬(¬P → ¬Q) ↔ ¬(P ∨ ¬Q))) with
2582       ((¬(¬P → ¬Q) ↔ ¬(P ∨ ¬Q)) ↔ ((¬P → ¬Q) ↔ P ∨ ¬Q)) in Transp4_11a.
2583   apply EqBi.
2584   apply Transp4_11a.

```

```

2585   apply EqBi.
2586   specialize n4_21 with ( $\neg(\neg P \rightarrow \neg Q) \leftrightarrow \neg(P \vee \neg Q)$ )
2587     ( $(\neg P \rightarrow \neg Q) \leftrightarrow (P \vee \neg Q)$ ).
2588   intros n4_21a.
2589   apply n4_21a.
2590   Qed.
2591
2592   (*Return to this proof.*)
2593   (*We did get one half of the  $\leftrightarrow$ .*)
2594 Theorem n4_7 :  $\forall P Q : \text{Prop}$ ,
2595   ( $P \rightarrow Q$ )  $\leftrightarrow$  ( $P \rightarrow (P \wedge Q)$ ).
2596 Proof. intros P Q.
2597   specialize Comp3_43 with P P Q.
2598   intros Comp3_43a.
2599   specialize Exp3_3 with
2600     ( $P \rightarrow P$ ) ( $P \rightarrow Q$ ) ( $P \rightarrow P \wedge Q$ ).
2601   intros Exp3_3a.
2602   MP Exp3_3a Comp3_43a.
2603   specialize Id2_08 with P.
2604   intros Id2_08a.
2605   MP Exp3_3a Id2_08a.
2606   specialize Simp3_27 with P Q.
2607   intros Simp3_27a.
2608   specialize Syll2_05 with P ( $P \wedge Q$ ) Q.
2609   intros Syll2_05a.
2610   MP Syll2_05a Simp3_27a.
2611   clear Id2_08a. clear Comp3_43a. clear Simp3_27a.
2612   Conj Syll2_05a Exp3_3a.
2613   split.
2614   apply Exp3_3a.
2615   apply Syll2_05a.
2616   Equiv H.
2617   apply H.
2618   apply Equiv4_01.
2619   Qed.
2620
2621 Theorem n4_71 :  $\forall P Q : \text{Prop}$ ,
2622   ( $P \rightarrow Q$ )  $\leftrightarrow$  ( $P \leftrightarrow (P \wedge Q)$ ).
2623 Proof. intros P Q.
2624   specialize n4_7 with P Q.
2625   intros n4_7a.
2626   specialize n3_21 with ( $P \rightarrow (P \wedge Q)$ ) ( $(P \wedge Q) \rightarrow P$ ).

```

```

2627 intros n3_21a.
2628 replace ((P → P ∧ Q) ∧ (P ∧ Q → P)) with
2629   (P ↔ (P ∧ Q)) in n3_21a.
2630 specialize Simp3_26 with P Q.
2631 intros Simp3_26a.
2632 MP n3_21a Simp3_26a.
2633 specialize Simp3_26 with (P → (P ∧ Q)) ((P ∧ Q) → P).
2634 intros Simp3_26b.
2635 replace ((P → P ∧ Q) ∧ (P ∧ Q → P)) with
2636   (P ↔ (P ∧ Q)) in Simp3_26b. clear Simp3_26a.
2637 Conj n3_21a Simp3_26b.
2638 split.
2639 apply n3_21a.
2640 apply Simp3_26b.
2641 Equiv H.
2642 clear n3_21a. clear Simp3_26b.
2643 Conj n4_7a H.
2644 split.
2645 apply n4_7a.
2646 apply H.
2647 specialize n4_22 with (P → Q) (P → P ∧ Q) (P ↔ P ∧ Q).
2648 intros n4_22a.
2649 MP n4_22a H0.
2650 apply n4_22a.
2651 apply Equiv4_01.
2652 apply Equiv4_01.
2653 apply Equiv4_01.
2654 Qed.
2655
2656 Theorem n4_72 : ∀ P Q : Prop,
2657   (P → Q) ↔ (Q ↔ (P ∨ Q)).
2658 Proof. intros P Q.
2659 specialize Transp4_1 with P Q.
2660 intros Transp4_1a.
2661 specialize n4_71 with (¬Q) (¬P).
2662 intros n4_71a.
2663 Conj Transp4_1a n4_71a.
2664 split.
2665 apply Transp4_1a.
2666 apply n4_71a.
2667 specialize n4_22 with
2668   (P → Q) (¬Q → ¬P) (¬Q ↔ ¬Q ∧ ¬P).

```



```

2669   intros n4_22a.
2670   MP n4_22a H.
2671   specialize n4_21 with ( $\neg Q$ ) ( $\neg Q \wedge \neg P$ ).
2672   intros n4_21a.
2673   Conj n4_22a n4_21a.
2674   split.
2675   apply n4_22a.
2676   apply n4_21a.
2677   specialize n4_22 with
2678       ( $P \rightarrow Q$ ) ( $\neg Q \leftrightarrow \neg Q \wedge \neg P$ ) ( $\neg Q \wedge \neg P \leftrightarrow \neg Q$ ).
2679   intros n4_22b.
2680   MP n4_22b H0.
2681   specialize n4_12 with ( $\neg Q \wedge \neg P$ ) (Q).
2682   intros n4_12a.
2683   Conj n4_22b n4_12a.
2684   split.
2685   apply n4_22b.
2686   apply n4_12a.
2687   specialize n4_22 with
2688       ( $P \rightarrow Q$ ) ( $(\neg Q \wedge \neg P) \leftrightarrow \neg Q$ ) ( $Q \leftrightarrow \neg(\neg Q \wedge \neg P)$ ).
2689   intros n4_22c.
2690   MP n4_22b H0.
2691   specialize n4_57 with Q P.
2692   intros n4_57a.
2693   replace ( $\neg(\neg Q \wedge \neg P)$ ) with ( $Q \vee P$ ) in n4_22c.
2694   specialize n4_31 with P Q.
2695   intros n4_31a.
2696   replace ( $Q \vee P$ ) with ( $P \vee Q$ ) in n4_22c.
2697   apply n4_22c.
2698   apply EqBi.
2699   apply n4_31a.
2700   apply EqBi.
2701   replace ( $\neg(\neg Q \wedge \neg P) \leftrightarrow Q \vee P$ ) with
2702       ( $Q \vee P \leftrightarrow \neg(\neg Q \wedge \neg P)$ ) in n4_57a.
2703   apply n4_57a.
2704   apply EqBi.
2705   specialize n4_21 with ( $Q \vee P$ ) ( $\neg(\neg Q \wedge \neg P)$ ).
2706   intros n4_21b.
2707   apply n4_21b.
2708   Qed.
2709
2710   Theorem n4_73 :  $\forall P Q : \text{Prop}$ ,

```

```

2711   Q → (P ↔ (P ∧ Q)).
2712   Proof. intros P Q.
2713   specialize Simp2_02 with P Q.
2714   intros Simp2_02a.
2715   specialize n4_71 with P Q.
2716   intros n4_71a.
2717   replace ((P → Q) ↔ (P ↔ P ∧ Q)) with
2718     (((P→Q)→(P↔P∧Q))∧((P↔P∧Q)→(P→Q))) in n4_71a.
2719   specialize Simp3_26 with
2720     ((P → Q) → P ↔ P ∧ Q) (P ↔ P ∧ Q → P → Q).
2721   intros Simp3_26a.
2722   MP Simp3_26a n4_71a.
2723   Syll Simp2_02a Simp3_26a Sa.
2724   apply Sa.
2725   apply Equiv4_01.
2726   Qed.
2727
2728   Theorem n4_74 : ∀ P Q : Prop,
2729     ¬P → (Q ↔ (P ∨ Q)).
2730   Proof. intros P Q.
2731   specialize n2_21 with P Q.
2732   intros n2_21a.
2733   specialize n4_72 with P Q.
2734   intros n4_72a.
2735   replace (P → Q) with (Q ↔ P ∨ Q) in n2_21a.
2736   apply n2_21a.
2737   apply EqBi.
2738   replace ((P → Q) ↔ (Q ↔ P ∨ Q)) with
2739     ((Q ↔ P ∨ Q) ↔ (P → Q)) in n4_72a.
2740   apply n4_72a.
2741   apply EqBi.
2742   specialize n4_21 with (Q↔(P ∨ Q)) (P → Q).
2743   intros n4_21a.
2744   apply n4_21a.
2745   Qed.
2746
2747   Theorem n4_76 : ∀ P Q R : Prop,
2748     ((P → Q) ∧ (P → R)) ↔ (P → (Q ∧ R)).
2749   Proof. intros P Q R.
2750   specialize n4_41 with (¬P) Q R.
2751   intros n4_41a.
2752   replace (¬P ∨ Q) with (P→Q) in n4_41a.

```

```

2753   replace ( $\neg P \vee R$ ) with ( $P \rightarrow R$ ) in n4_41a.
2754   replace ( $\neg P \vee Q \wedge R$ ) with ( $P \rightarrow Q \wedge R$ ) in n4_41a.
2755   replace ( $(P \rightarrow Q \wedge R) \leftrightarrow (P \rightarrow Q) \wedge (P \rightarrow R)$ ) with
2756       ( $(P \rightarrow Q) \wedge (P \rightarrow R) \leftrightarrow (P \rightarrow Q \wedge R)$ ) in n4_41a.
2757   apply n4_41a.
2758   apply EqBi.
2759   specialize n4_21 with ( $(P \rightarrow Q) \wedge (P \rightarrow R) \rightarrow (P \rightarrow Q \wedge R)$ ).
2760   intros n4_21a.
2761   apply n4_21a.
2762   apply Impl1_01.
2763   apply Impl1_01.
2764   apply Impl1_01.
2765   Qed.
2766
2767   Theorem n4_77 :  $\forall P Q R : \text{Prop}$ ,
2768       ( $(Q \rightarrow P) \wedge (R \rightarrow P) \leftrightarrow ((Q \vee R) \rightarrow P)$ ).
2769   Proof. intros P Q R.
2770   specialize n3_44 with P Q R.
2771   intros n3_44a.
2772   specialize n2_2 with Q R.
2773   intros n2_2a.
2774   specialize Add1_3 with Q R.
2775   intros Add1_3a.
2776   specialize Syll2_06 with Q ( $Q \vee R$ ) P.
2777   intros Syll2_06a.
2778   MP Syll2_06a n2_2a.
2779   specialize Syll2_06 with R ( $Q \vee R$ ) P.
2780   intros Syll2_06b.
2781   MP Syll2_06b Add1_3a.
2782   Conj Syll2_06a Syll2_06b.
2783   split.
2784   apply Syll2_06a.
2785   apply Syll2_06b.
2786   specialize Comp3_43 with ( $(Q \vee R) \rightarrow P$ )
2787       ( $Q \rightarrow P$ ) ( $R \rightarrow P$ ).
2788   intros Comp3_43a.
2789   MP Comp3_43a H.
2790   clear n2_2a. clear Add1_3a. clear H.
2791   clear Syll2_06a. clear Syll2_06b.
2792   Conj n3_44a Comp3_43a.
2793   split.
2794   apply n3_44a.

```

```

2795   apply Comp3_43a.
2796   Equiv H.
2797   apply H.
2798   apply Equiv4_01.
2799   Qed.
2800
2801 Theorem n4_78 :  $\forall P Q R : \text{Prop},$ 
2802    $((P \rightarrow Q) \vee (P \rightarrow R)) \leftrightarrow (P \rightarrow (Q \vee R)).$ 
2803 Proof. intros P Q R.
2804 specialize n4_2 with  $((P \rightarrow Q) \vee (P \rightarrow R)).$ 
2805 intros n4_2a.
2806 replace  $((P \rightarrow Q) \vee (P \rightarrow R)) \leftrightarrow ((P \rightarrow Q) \vee (P \rightarrow R))$  with
2807    $((P \rightarrow Q) \vee (P \rightarrow R)) \leftrightarrow ((\neg P \vee Q) \vee \neg P \vee R)$  in n4_2a.
2808 specialize n4_33 with  $(\neg P) Q (\neg P \vee R).$ 
2809 intros n4_33a.
2810 replace  $((\neg P \vee Q) \vee \neg P \vee R)$  with
2811    $(\neg P \vee Q \vee \neg P \vee R)$  in n4_2a.
2812 specialize n4_31 with  $(\neg P) Q.$ 
2813 intros n4_31a.
2814 specialize n4_37 with  $(\neg P \vee Q) (Q \vee \neg P) R.$ 
2815 intros n4_37a.
2816 MP n4_37a n4_31a.
2817 replace  $(Q \vee \neg P \vee R)$  with
2818    $((Q \vee \neg P) \vee R)$  in n4_2a.
2819 replace  $((Q \vee \neg P) \vee R)$  with
2820    $((\neg P \vee Q) \vee R)$  in n4_2a.
2821 specialize n4_33 with  $(\neg P) (\neg P \vee Q) R.$ 
2822 intros n4_33b.
2823 replace  $(\neg P \vee (\neg P \vee Q) \vee R)$  with
2824    $((\neg P \vee (\neg P \vee Q)) \vee R)$  in n4_2a.
2825 specialize n4_25 with  $(\neg P).$ 
2826 intros n4_25a.
2827 specialize n4_37 with
2828    $(\neg P) (\neg P \vee \neg P) (Q \vee R).$ 
2829 intros n4_37b.
2830 MP n4_37b n4_25a.
2831 replace  $(\neg P \vee \neg P \vee Q)$  with
2832    $((\neg P \vee \neg P) \vee Q)$  in n4_2a.
2833 replace  $((\neg P \vee \neg P) \vee Q) \vee R$  with
2834    $((\neg P \vee \neg P) \vee Q \vee R)$  in n4_2a.
2835 replace  $((\neg P \vee \neg P) \vee Q \vee R)$  with
2836    $((\neg P) \vee (Q \vee R))$  in n4_2a.

```

```

2837   replace ( $\neg P \vee Q \vee R$ ) with
2838         ( $P \rightarrow (Q \vee R)$ ) in n4_2a.
2839   apply n4_2a.
2840   apply Impl1_01.
2841   apply EqBi.
2842   apply n4_37b.
2843   apply Abb2_33.
2844   replace ( $(\neg P \vee \neg P) \vee Q$ ) with ( $\neg P \vee \neg P \vee Q$ ).
2845   reflexivity.
2846   apply Abb2_33.
2847   replace ( $(\neg P \vee \neg P \vee Q) \vee R$ ) with
2848         ( $\neg P \vee (\neg P \vee Q) \vee R$ ).
2849   reflexivity.
2850   apply EqBi.
2851   apply n4_33b.
2852   apply EqBi.
2853   apply n4_37a.
2854   replace ( $(Q \vee \neg P) \vee R$ ) with ( $Q \vee \neg P \vee R$ ).
2855   reflexivity.
2856   apply Abb2_33.
2857   apply EqBi.
2858   apply n4_33a.
2859   rewrite <- Impl1_01.
2860   rewrite <- Impl1_01.
2861   reflexivity.
2862   Qed.
2863
2864   Theorem n4_79 :  $\forall P Q R : \text{Prop}$ ,
2865     ( $(Q \rightarrow P) \vee (R \rightarrow P)$ )  $\leftrightarrow$  ( $(Q \wedge R) \rightarrow P$ ).
2866   Proof. intros P Q R.
2867     specialize Transp4_1 with Q P.
2868     intros Transp4_1a.
2869     specialize Transp4_1 with R P.
2870     intros Transp4_1b.
2871     Conj Transp4_1a Transp4_1b.
2872     split.
2873     apply Transp4_1a.
2874     apply Transp4_1b.
2875     specialize n4_39 with
2876       ( $Q \rightarrow P$ ) ( $R \rightarrow P$ ) ( $\neg P \rightarrow \neg Q$ ) ( $\neg P \rightarrow \neg R$ ).
2877     intros n4_39a.
2878     MP n4_39a H.

```

```

2879 specialize n4_78 with (¬P) (¬Q) (¬R).
2880 intros n4_78a.
2881 rewrite Equiv4_01 in n4_78a.
2882 specialize Simp3_26 with
2883   (((¬P→¬Q)∨(¬P→¬R))→(¬P→(¬Q∨¬R)))
2884   ((¬P→(¬Q∨¬R))→((¬P→¬Q)∨(¬P→¬R))).
2885 intros Simp3_26a.
2886 MP Simp3_26a n4_78a.
2887 specialize Transp2_15 with P (¬Q∨¬R).
2888 intros Transp2_15a.
2889 specialize Simp3_27 with
2890   (((¬P→¬Q)∨(¬P→¬R))→(¬P→(¬Q∨¬R)))
2891   ((¬P→(¬Q∨¬R))→((¬P→¬Q)∨(¬P→¬R))).
2892 intros Simp3_27a.
2893 MP Simp3_27a n4_78a.
2894 specialize Transp2_15 with (¬Q∨¬R) P.
2895 intros Transp2_15b.
2896 specialize Syll2_06 with ((¬P→¬Q)∨(¬P→¬R))
2897   (¬P→(¬Q∨¬R)) (¬(¬Q∨¬R)→P).
2898 intros Syll2_06a.
2899 MP Syll2_06a Simp3_26a.
2900 MP Syll2_06a Transp2_15a.
2901 specialize Syll2_06 with (¬(¬Q∨¬R)→P)
2902   (¬P→(¬Q∨¬R)) ((¬P→¬Q)∨(¬P→¬R)).
2903 intros Syll2_06b.
2904 MP Syll2_06b Trans2_15b.
2905 MP Syll2_06b Simp3_27a.
2906 Conj Syll2_06a Syll2_06b.
2907 split.
2908 apply Syll2_06a.
2909 apply Syll2_06b.
2910 Equiv H0.
2911 clear Transp4_1a. clear Transp4_1b. clear H.
2912 clear Simp3_26a. clear Syll2_06b. clear n4_78a.
2913 clear Transp2_15a. clear Simp3_27a.
2914 clear Transp2_15b. clear Syll2_06a.
2915 Conj n4_39a H0.
2916 split.
2917 apply n4_39a.
2918 apply H0.
2919 specialize n4_22 with ((Q→P)∨(R→P))
2920   ((¬P→¬Q)∨(¬P→¬R)) (¬(¬Q∨¬R)→P).

```

```

2921     intros n4_22a.
2922     MP n4_22a H.
2923     specialize n4_2 with ( $\neg(\neg Q \vee \neg R) \rightarrow P$ ).
2924     intros n4_2a.
2925     Conj n4_22a n4_2a.
2926     split.
2927     apply n4_22a.
2928     apply n4_2a.
2929     specialize n4_22 with ( $((Q \rightarrow P) \vee (R \rightarrow P))$ 
2930     ( $\neg(\neg Q \vee \neg R) \rightarrow P$ ) ( $\neg(\neg Q \vee \neg R) \rightarrow P$ )).
2931     intros n4_22b.
2932     MP n4_22b H1.
2933     rewrite <- Prod3_01 in n4_22b.
2934     apply n4_22b.
2935     apply Equiv4_01.
2936     Qed.
2937
2938 Theorem n4_8 :  $\forall P : \text{Prop}$ ,
2939   ( $P \rightarrow \neg P$ )  $\leftrightarrow \neg P$ .
2940 Proof. intros P.
2941   specialize Abs2_01 with P.
2942   intros Abs2_01a.
2943   specialize Simp2_02 with P ( $\neg P$ ).
2944   intros Simp2_02a.
2945   Conj Abs2_01a Simp2_02a.
2946   split.
2947   apply Abs2_01a.
2948   apply Simp2_02a.
2949   Equiv H.
2950   apply H.
2951   apply Equiv4_01.
2952   Qed.
2953
2954 Theorem n4_81 :  $\forall P : \text{Prop}$ ,
2955   ( $\neg P \rightarrow P$ )  $\leftrightarrow P$ .
2956 Proof. intros P.
2957   specialize n2_18 with P.
2958   intros n2_18a.
2959   specialize Simp2_02 with ( $\neg P$ ) P.
2960   intros Simp2_02a.
2961   Conj n2_18a Simp2_02a.
2962   split.

```

```

2963     apply n2_18a.
2964     apply Simp2_02a.
2965     Equiv H.
2966     apply H.
2967     apply Equiv4_01.
2968     Qed.
2969
2970 Theorem n4_82 :  $\forall P Q : \text{Prop}$ ,
2971    $((P \rightarrow Q) \wedge (P \rightarrow \neg Q)) \leftrightarrow \neg P$ .
2972 Proof. intros P Q.
2973   specialize n2_65 with P Q.
2974   intros n2_65a.
2975   specialize Imp3_31 with  $(P \rightarrow Q) (P \rightarrow \neg Q) (\neg P)$ .
2976   intros Imp3_31a.
2977   MP Imp3_31a n2_65a.
2978   specialize n2_21 with P Q.
2979   intros n2_21a.
2980   specialize n2_21 with P  $(\neg Q)$ .
2981   intros n2_21b.
2982   Conj n2_21a n2_21b.
2983   split.
2984   apply n2_21a.
2985   apply n2_21b.
2986   specialize Comp3_43 with  $(\neg P) (P \rightarrow Q) (P \rightarrow \neg Q)$ .
2987   intros Comp3_43a.
2988   MP Comp3_43a H.
2989   clear n2_65a. clear n2_21a.
2990   clear n2_21b. clear H.
2991   Conj Imp3_31a Comp3_43a.
2992   split.
2993   apply Imp3_31a.
2994   apply Comp3_43a.
2995   Equiv H.
2996   apply H.
2997   apply Equiv4_01.
2998   Qed.
2999
3000 Theorem n4_83 :  $\forall P Q : \text{Prop}$ ,
3001    $((P \rightarrow Q) \wedge (\neg P \rightarrow Q)) \leftrightarrow Q$ .
3002 Proof. intros P Q.
3003   specialize n2_61 with P Q.
3004   intros n2_61a.

```



```

3005 specialize Imp3_31 with (P→Q) (¬P→Q) (Q).
3006 intros Imp3_31a.
3007 MP Imp3_31a n2_61a.
3008 specialize Simp2_02 with P Q.
3009 intros Simp2_02a.
3010 specialize Simp2_02 with (¬P) Q.
3011 intros Simp2_02b.
3012 Conj Simp2_02a Simp2_02b.
3013 split.
3014 apply Simp2_02a.
3015 apply Simp2_02b.
3016 specialize Comp3_43 with Q (P→Q) (¬P→Q).
3017 intros Comp3_43a.
3018 MP Comp3_43a H.
3019 clear n2_61a. clear Simp2_02a.
3020 clear Simp2_02b. clear H.
3021 Conj Imp3_31a Comp3_43a.
3022 split.
3023 apply Imp3_31a.
3024 apply Comp3_43a.
3025 Equiv H.
3026 apply H.
3027 apply Equiv4_01.
3028 Qed.
3029
3030 Theorem n4_84 : ∀ P Q R : Prop,
3031   (P ↔ Q) → ((P → R) ↔ (Q → R)).
3032 Proof. intros P Q R.
3033   specialize Syll2_06 with P Q R.
3034   intros Syll2_06a.
3035   specialize Syll2_06 with Q P R.
3036   intros Syll2_06b.
3037   Conj Syll2_06a Syll2_06b.
3038   split.
3039   apply Syll2_06a.
3040   apply Syll2_06b.
3041   specialize n3_47 with
3042     (P→Q) (Q→P) ((Q→R)→P→R) ((P→R)→Q→R).
3043   intros n3_47a.
3044   MP n3_47a H.
3045   replace ((P→Q) ∧ (Q → P)) with
3046     (P↔Q) in n3_47a.

```

```

3047     replace (((Q→R)→P→R)^(P→R)→Q→R)) with
3048         ((Q → R) ↔ (P → R)) in n3_47a.
3049     replace ((Q → R) ↔ (P → R)) with
3050         ((P→R) ↔ (Q → R)) in n3_47a.
3051     apply n3_47a.
3052     apply EqBi.
3053     specialize n4_21 with (P→R) (Q→R).
3054     intros n4_21a.
3055     apply n4_21a.
3056     apply Equiv4_01.
3057     apply Equiv4_01.
3058     Qed.
3059
3060 Theorem n4_85 : ∀ P Q R : Prop,
3061     (P ↔ Q) → ((R → P) ↔ (R → Q)).
3062 Proof. intros P Q R.
3063     specialize Syll2_05 with R P Q.
3064     intros Syll2_05a.
3065     specialize Syll2_05 with R Q P.
3066     intros Syll2_05b.
3067     Conj Syll2_05a Syll2_05b.
3068     split.
3069     apply Syll2_05a.
3070     apply Syll2_05b.
3071     specialize n3_47 with
3072         (P→Q) (Q→P) ((R→P)→R→Q) ((R→Q)→R→P).
3073     intros n3_47a.
3074     MP n3_47a H.
3075     replace ((P→Q) ∧ (Q → P)) with (P↔Q) in n3_47a.
3076     replace (((R→P)→R→Q)^(R→Q)→R→P)) with
3077         ((R → P) ↔ (R → Q)) in n3_47a.
3078     apply n3_47a.
3079     apply Equiv4_01.
3080     apply Equiv4_01.
3081     Qed.
3082
3083 Theorem n4_86 : ∀ P Q R : Prop,
3084     (P ↔ Q) → ((P ↔ R) ↔ (Q ↔ R)).
3085 Proof. intros P Q R.
3086     specialize n4_22 with Q P R.
3087     intros n4_22a.
3088     specialize Exp3_3 with (Q↔P) (P↔R) (Q↔R).

```

```

3089   intros Exp3_3a. (*Not cited*)
3090   MP Exp3_3a n4_22a.
3091   specialize n4_22 with P Q R.
3092   intros n4_22b.
3093   specialize Exp3_3 with (P↔Q) (Q↔R) (P↔R).
3094   intros Exp3_3b.
3095   MP Exp3_3b n4_22b.
3096   clear n4_22a. clear n4_22b.
3097   replace (Q↔P) with (P↔Q) in Exp3_3a.
3098   Conj Exp3_3a Exp3_3b.
3099   split.
3100   apply Exp3_3a.
3101   apply Exp3_3b.
3102   specialize Comp3_43 with (P↔Q)
3103     ((P↔R)→(Q↔R)) ((Q↔R)→(P↔R)).
3104   intros Comp3_43a. (*Not cited*)
3105   MP Comp3_43a H.
3106   replace (((P↔R)→(Q↔R))∧((Q↔R)→(P↔R)))
3107     with ((P↔R)↔(Q↔R)) in Comp3_43a.
3108   apply Comp3_43a.
3109   apply Equiv4_01.
3110   apply EqBi.
3111   specialize n4_21 with P Q.
3112   intros n4_21a.
3113   apply n4_21a.
3114   Qed.
3115
3116   Theorem n4_87 : ∀ P Q R : Prop,
3117     (((P ∧ Q) → R) ↔ (P → Q → R)) ↔
3118     ((Q → (P → R)) ↔ (Q ∧ P → R)).
3119   Proof. intros P Q R.
3120   specialize Exp3_3 with P Q R.
3121   intros Exp3_3a.
3122   specialize Imp3_31 with P Q R.
3123   intros Imp3_31a.
3124   Conj Exp3_3a Imp3_31a.
3125   split.
3126   apply Exp3_3a.
3127   apply Imp3_31a.
3128   Equiv H.
3129   specialize Exp3_3 with Q P R.
3130   intros Exp3_3b.

```

```

3131 specialize Imp3_31 with Q P R.
3132 intros Imp3_31b.
3133 Conj Exp3_3b Imp3_31b.
3134 split.
3135 apply Exp3_3b.
3136 apply Imp3_31b.
3137 Equiv H0.
3138 specialize Comm2_04 with P Q R.
3139 intros Comm2_04a.
3140 specialize Comm2_04 with Q P R.
3141 intros Comm2_04b.
3142 Conj Comm2_04a Comm2_04b.
3143 split.
3144 apply Comm2_04a.
3145 apply Comm2_04b.
3146 Equiv H1.
3147 clear Exp3_3a. clear Imp3_31a. clear Exp3_3b.
3148     clear Imp3_31b. clear Comm2_04a.
3149     clear Comm2_04b.
3150 replace (P $\wedge$ Q $\rightarrow$ R) with (P  $\rightarrow$  Q  $\rightarrow$  R).
3151 replace (Q $\wedge$ P $\rightarrow$ R) with (Q  $\rightarrow$  P  $\rightarrow$  R).
3152 replace (Q $\rightarrow$ P $\rightarrow$ R) with (P  $\rightarrow$  Q  $\rightarrow$  R).
3153 specialize n4_2 with
3154     ((P  $\rightarrow$  Q  $\rightarrow$  R)  $\leftrightarrow$  (P  $\rightarrow$  Q  $\rightarrow$  R)).
3155 intros n4_2a.
3156 apply n4_2a.
3157 apply EqBi.
3158 apply H1.
3159 replace (Q $\rightarrow$ P $\rightarrow$ R) with (Q $\wedge$ P $\rightarrow$ R).
3160 reflexivity.
3161 apply EqBi.
3162 apply H0.
3163 replace (P $\rightarrow$ Q $\rightarrow$ R) with (P $\wedge$ Q $\rightarrow$ R).
3164 reflexivity.
3165 apply EqBi.
3166 apply H.
3167 apply Equiv4_01.
3168 apply Equiv4_01.
3169 apply Equiv4_01.
3170 Qed.
3171
3172 End No4.

```

```

3173
3174 Module No5.
3175
3176 Import No1.
3177 Import No2.
3178 Import No3.
3179 Import No4.
3180
3181 Theorem n5_1 :  $\forall$  P Q : Prop,
3182   (P  $\wedge$  Q)  $\rightarrow$  (P  $\leftrightarrow$  Q).
3183 Proof. intros P Q.
3184   specialize n3_4 with P Q.
3185   intros n3_4a.
3186   specialize n3_4 with Q P.
3187   intros n3_4b.
3188   specialize n3_22 with P Q.
3189   intros n3_22a.
3190   Syll n3_22a n3_4b Sa.
3191   clear n3_22a. clear n3_4b.
3192   Conj n3_4a Sa.
3193   split.
3194   apply n3_4a.
3195   apply Sa.
3196   specialize n4_76 with (P $\wedge$ Q) (P $\rightarrow$ Q) (Q $\rightarrow$ P).
3197   intros n4_76a. (*Not cited*)
3198   replace ((P $\wedge$ Q $\rightarrow$ P $\rightarrow$ Q) $\wedge$ (P $\wedge$ Q $\rightarrow$ Q $\rightarrow$ P)) with
3199     (P  $\wedge$  Q  $\rightarrow$  (P  $\rightarrow$  Q)  $\wedge$  (Q  $\rightarrow$  P)) in H.
3200   replace ((P $\rightarrow$ Q) $\wedge$ (Q $\rightarrow$ P)) with (P $\leftrightarrow$ Q) in H.
3201   apply H.
3202   apply Equiv4_01.
3203   replace (P  $\wedge$  Q  $\rightarrow$  (P  $\rightarrow$  Q)  $\wedge$  (Q  $\rightarrow$  P)) with
3204     ((P  $\wedge$  Q  $\rightarrow$  P  $\rightarrow$  Q)  $\wedge$  (P  $\wedge$  Q  $\rightarrow$  Q  $\rightarrow$  P)).
3205   reflexivity.
3206   apply EqBi.
3207   apply n4_76a.
3208   Qed.
3209
3210 Theorem n5_11 :  $\forall$  P Q : Prop,
3211   (P  $\rightarrow$  Q)  $\vee$  ( $\neg$ P  $\rightarrow$  Q).
3212 Proof. intros P Q.
3213   specialize n2_5 with P Q.
3214   intros n2_5a.

```

```

3215 specialize n2_54 with (P → Q) (¬P → Q).
3216 intros n2_54a.
3217 MP n2_54a n2_5a.
3218 apply n2_54a.
3219 Qed.
3220 (*The proof sketch cites n2_51,
3221 but this may be a misprint.*)
3222
3223 Theorem n5_12 : ∀ P Q : Prop,
3224   (P → Q) ∨ (P → ¬Q).
3225 Proof. intros P Q.
3226 specialize n2_51 with P Q.
3227 intros n2_51a.
3228 specialize n2_54 with ((P → Q)) (P → ¬Q).
3229 intros n2_54a.
3230 MP n2_54a n2_5a.
3231 apply n2_54a.
3232 Qed.
3233 (*The proof sketch cites n2_52,
3234 but this may be a misprint.*)
3235
3236 Theorem n5_13 : ∀ P Q : Prop,
3237   (P → Q) ∨ (Q → P).
3238 Proof. intros P Q.
3239 specialize n2_521 with P Q.
3240 intros n2_521a.
3241 replace (¬(P → Q) → Q → P) with
3242   (¬¬(P → Q) ∨ (Q → P)) in n2_521a.
3243 replace (¬¬(P→Q)) with (P→Q) in n2_521a.
3244 apply n2_521a.
3245 apply EqBi.
3246 specialize n4_13 with (P→Q).
3247 intros n4_13a. (*Not cited*)
3248 apply n4_13a.
3249 rewrite <- Impl1_01.
3250 reflexivity.
3251 Qed.
3252
3253 Theorem n5_14 : ∀ P Q R : Prop,
3254   (P → Q) ∨ (Q → R).
3255 Proof. intros P Q R.
3256 specialize Simp2_02 with P Q.

```

```

3257   intros Simp2_02a.
3258   specialize Transp2_16 with Q (P→Q).
3259   intros Transp2_16a.
3260   MP Transp2_16a Simp2_02a.
3261   specialize n2_21 with Q R.
3262   intros n2_21a.
3263   Syll Transp2_16a n2_21a Sa.
3264   replace (¬(P→Q)→(Q→R)) with
3265     (¬¬(P→Q)∨(Q→R)) in Sa.
3266   replace (¬¬(P→Q)) with (P→Q) in Sa.
3267   apply Sa.
3268   apply EqBi.
3269   specialize n4_13 with (P→Q).
3270   intros n4_13a.
3271   apply n4_13a.
3272   rewrite <- Impl1_01.
3273   reflexivity.
3274   Qed.
3275
3276 Theorem n5_15 : ∀ P Q : Prop,
3277   (P ↔ Q) ∨ (P ↔ ¬Q).
3278 Proof. intros P Q.
3279   specialize n4_61 with P Q.
3280   intros n4_61a.
3281   replace (¬(P → Q) ↔ P ∧ ¬Q) with
3282     ((¬(P→Q)→P∧¬Q)∧((P∧¬Q)→¬(P→Q))) in n4_61a.
3283   specialize Simp3_26 with
3284     (¬(P → Q) → P ∧ ¬Q) ((P ∧ ¬Q) → ¬(P → Q)).
3285   intros Simp3_26a.
3286   MP Simp3_26a n4_61a.
3287   specialize n5_1 with P (¬Q).
3288   intros n5_1a.
3289   Syll Simp3_26a n5_1a Sa.
3290   specialize n2_54 with (P→Q) (P ↔ ¬Q).
3291   intros n2_54a.
3292   MP n2_54a Sa.
3293   specialize n4_61 with Q P.
3294   intros n4_61b.
3295   replace ((¬(Q → P)) ↔ (Q ∧ ¬P)) with
3296     (((¬(Q→P))→(Q∧¬P))∧((Q∧¬P)→¬(Q→P))) in n4_61b.
3297   specialize Simp3_26 with
3298     (¬(Q → P) → (Q ∧ ¬P)) ((Q ∧ ¬P) → ¬(Q → P)).

```

```

3299 intros Simp3_26b.
3300 MP Simp3_26b n4_61b.
3301 specialize n5_1 with Q ( $\neg$ P).
3302 intros n5_1b.
3303 Syll Simp3_26b n5_1b Sb.
3304 specialize n4_12 with P Q.
3305 intros n4_12a.
3306 replace ( $Q \leftrightarrow \neg P$ ) with ( $P \leftrightarrow \neg Q$ ) in Sb.
3307 specialize n2_54 with ( $Q \rightarrow P$ ) ( $P \leftrightarrow \neg Q$ ).
3308 intros n2_54b.
3309 MP n2_54b Sb.
3310 clear n4_61a. clear Simp3_26a. clear n5_1a.
3311 clear n2_54a. clear n4_61b. clear Simp3_26b.
3312 clear n5_1b. clear n4_12a. clear n2_54b.
3313 replace ( $\neg(P \rightarrow Q) \rightarrow P \leftrightarrow \neg Q$ ) with
3314 ( $\neg\neg(P \rightarrow Q) \vee (P \leftrightarrow \neg Q)$ ) in Sa.
3315 replace ( $\neg\neg(P \rightarrow Q)$ ) with ( $P \rightarrow Q$ ) in Sa.
3316 replace ( $\neg(Q \rightarrow P) \rightarrow (P \leftrightarrow \neg Q)$ ) with
3317 ( $\neg\neg(Q \rightarrow P) \vee (P \leftrightarrow \neg Q)$ ) in Sb.
3318 replace ( $\neg\neg(Q \rightarrow P)$ ) with ( $Q \rightarrow P$ ) in Sb.
3319 Conj Sa Sb.
3320 split.
3321 apply Sa.
3322 apply Sb.
3323 specialize n4_41 with ( $P \leftrightarrow \neg Q$ ) ( $P \rightarrow Q$ ) ( $Q \rightarrow P$ ).
3324 intros n4_41a.
3325 replace ( $(P \rightarrow Q) \vee (P \leftrightarrow \neg Q)$ ) with
3326 ( $(P \leftrightarrow \neg Q) \vee (P \rightarrow Q)$ ) in H.
3327 replace ( $(Q \rightarrow P) \vee (P \leftrightarrow \neg Q)$ ) with
3328 ( $(P \leftrightarrow \neg Q) \vee (Q \rightarrow P)$ ) in H.
3329 replace ( $((P \leftrightarrow \neg Q) \vee (P \rightarrow Q)) \wedge ((P \leftrightarrow \neg Q) \vee (Q \rightarrow P))$ ) with
3330 ( $(P \leftrightarrow \neg Q) \vee (P \rightarrow Q) \wedge (Q \rightarrow P)$ ) in H.
3331 replace ( $(P \rightarrow Q) \wedge (Q \rightarrow P)$ ) with ( $P \leftrightarrow Q$ ) in H.
3332 replace ( $(P \leftrightarrow \neg Q) \vee (P \leftrightarrow Q)$ ) with
3333 ( $(P \leftrightarrow Q) \vee (P \leftrightarrow \neg Q)$ ) in H.
3334 apply H.
3335 apply EqBi.
3336 apply n4_31.
3337 apply Equiv4_01.
3338 apply EqBi.
3339 apply n4_41a.
3340 apply EqBi.

```



```

3341   apply n4_31.
3342   apply EqBi.
3343   apply n4_31.
3344   apply EqBi.
3345   specialize n4_13 with (Q→P).
3346   intros n4_13a.
3347   apply n4_13a.
3348   rewrite <- Impl1_01.
3349   reflexivity.
3350   apply EqBi.
3351   specialize n4_13 with (P→Q).
3352   intros n4_13b.
3353   apply n4_13b.
3354   rewrite <- Impl1_01.
3355   reflexivity.
3356   apply EqBi.
3357   apply n4_12a.
3358   apply Equiv4_01.
3359   apply Equiv4_01.
3360   Qed.
3361
3362   Theorem n5_16 : ∀ P Q : Prop,
3363     ¬((P ↔ Q) ∧ (P ↔ ¬Q)).
3364   Proof. intros P Q.
3365     specialize Simp3_26 with ((P→Q) ∧ (P → ¬Q)) (Q→P).
3366     intros Simp3_26a.
3367     specialize Id2_08 with ((P ↔ Q) ∧ (P → ¬Q)).
3368     intros Id2_08a.
3369     replace (((P → Q) ∧ (P → ¬Q)) ∧ (Q → P)) with
3370       ((P→Q) ∧ ((P→¬Q) ∧ (Q→P))) in Simp3_26a.
3371     replace ((P → ¬Q) ∧ (Q → P)) with
3372       ((Q → P) ∧ (P → ¬Q)) in Simp3_26a.
3373     replace ((P→Q) ∧ (Q → P) ∧ (P → ¬Q)) with
3374       (((P→Q) ∧ (Q → P)) ∧ (P → ¬Q)) in Simp3_26a.
3375     replace ((P → Q) ∧ (Q → P)) with
3376       (P↔Q) in Simp3_26a.
3377     Syll Id2_08a Simp3_26a Sa.
3378     specialize n4_82 with P Q.
3379     intros n4_82a.
3380     replace ((P → Q) ∧ (P → ¬Q)) with (¬P) in Sa.
3381     specialize Simp3_27 with
3382       (P→Q) ((Q→P) ∧ (P → ¬Q)).

```

```

3383 intros Simp3_27a.
3384 replace ((P→Q) ∧ (Q → P) ∧ (P → ¬Q)) with
3385   (((P→Q) ∧ (Q → P)) ∧ (P → ¬Q)) in Simp3_27a.
3386 replace ((P → Q) ∧ (Q → P)) with
3387   (P↔Q) in Simp3_27a.
3388 specialize Syll3_33 with Q P (¬Q).
3389 intros Syll3_33a.
3390 Syll Simp3_27a Syll2_06a Sb.
3391 specialize Abs2_01 with Q.
3392 intros Abs2_01a.
3393 Syll Sb Abs2_01a Sc.
3394 clear Sb. clear Simp3_26a. clear Id2_08a.
3395   clear n4_82a. clear Simp3_27a. clear Syll3_33a.
3396   clear Abs2_01a.
3397 Conj Sa Sc.
3398 split.
3399 apply Sa.
3400 apply Sc.
3401 specialize Comp3_43 with
3402   ((P ↔ Q) ∧ (P → ¬Q)) (¬P) (¬Q).
3403 intros Comp3_43a.
3404 MP Comp3_43a H.
3405 specialize n4_65 with Q P.
3406 intros n4_65a.
3407 replace (¬Q ∧ ¬P) with (¬P ∧ ¬Q) in n4_65a.
3408 replace (¬P ∧ ¬Q) with
3409   (¬(¬Q→P)) in Comp3_43a.
3410 specialize Exp3_3 with
3411   (P↔Q) (P→¬Q) (¬(¬Q→P)).
3412 intros Exp3_3a.
3413 MP Exp3_3a Comp3_43a.
3414 replace ((P→¬Q)→¬(¬Q→P)) with
3415   (¬(P→¬Q)∨¬(¬Q→P)) in Exp3_3a.
3416 specialize n4_51 with (P→¬Q) (¬Q→P).
3417 intros n4_51a.
3418 replace (¬(P → ¬Q) ∨ ¬(¬Q → P)) with
3419   (¬((P → ¬Q) ∧ (¬Q → P))) in Exp3_3a.
3420 replace ((P→¬Q) ∧ (¬Q → P)) with
3421   (P↔¬Q) in Exp3_3a.
3422 replace ((P↔Q)→¬(P↔¬Q)) with
3423   (¬(P↔Q)∨¬(P↔¬Q)) in Exp3_3a.
3424 specialize n4_51 with (P↔Q) (P↔¬Q).

```

```

3425 intros n4_51b.
3426 replace ( $\neg(P \leftrightarrow Q) \vee \neg(P \leftrightarrow \neg Q)$ ) with
3427   ( $\neg((P \leftrightarrow Q) \wedge (P \leftrightarrow \neg Q))$ ) in Exp3_3a.
3428 apply Exp3_3a.
3429 apply EqBi.
3430 apply n4_51b.
3431 rewrite <- Impl1_01.
3432 reflexivity.
3433 apply Equiv4_01.
3434 apply EqBi.
3435 apply n4_51a.
3436 rewrite <- Impl1_01.
3437 reflexivity.
3438 apply EqBi.
3439 apply n4_65a.
3440 apply EqBi.
3441 specialize n4_3 with ( $\neg P$ ) ( $\neg Q$ ).
3442 intros n4_3a.
3443 apply n4_3a.
3444 apply Equiv4_01.
3445 apply EqBi.
3446 specialize n4_32 with ( $P \rightarrow Q$ ) ( $Q \rightarrow P$ ) ( $P \rightarrow \neg Q$ ).
3447 intros n4_32a.
3448 apply n4_32a.
3449 replace ( $\neg P$ ) with ( $(P \rightarrow Q) \wedge (P \rightarrow \neg Q)$ ).
3450 reflexivity.
3451 apply EqBi.
3452 apply n4_82a.
3453 apply Equiv4_01.
3454 apply EqBi.
3455 specialize n4_32 with ( $P \rightarrow Q$ ) ( $Q \rightarrow P$ ) ( $P \rightarrow \neg Q$ ).
3456 intros n4_32b.
3457 apply n4_32b.
3458 apply EqBi.
3459 specialize n4_3 with ( $Q \rightarrow P$ ) ( $P \rightarrow \neg Q$ ).
3460 intros n4_3b.
3461 apply n4_3b.
3462 replace ( $(P \rightarrow Q) \wedge (P \rightarrow \neg Q) \wedge (Q \rightarrow P)$ ) with
3463   ( $((P \rightarrow Q) \wedge (P \rightarrow \neg Q)) \wedge (Q \rightarrow P)$ ).
3464 reflexivity.
3465 apply EqBi.
3466 specialize n4_32 with ( $P \rightarrow Q$ ) ( $P \rightarrow \neg Q$ ) ( $Q \rightarrow P$ ).

```

```

3467   intros n4_32a.
3468   apply n4_32a.
3469   Qed.
3470
3471 Theorem n5_17 :  $\forall P Q : \text{Prop}$ ,
3472    $((P \vee Q) \wedge \neg(P \wedge Q)) \leftrightarrow (P \leftrightarrow \neg Q)$ .
3473 Proof. intros P Q.
3474   specialize n4_64 with Q P.
3475   intros n4_64a.
3476   specialize n4_21 with  $(Q \vee P) (\neg Q \rightarrow P)$ .
3477   intros n4_21a.
3478   replace  $((\neg Q \rightarrow P) \leftrightarrow (Q \vee P))$  with
3479      $((Q \vee P) \leftrightarrow (\neg Q \rightarrow P))$  in n4_64a.
3480   replace  $(Q \vee P)$  with  $(P \vee Q)$  in n4_64a.
3481   specialize n4_63 with P Q.
3482   intros n4_63a.
3483   replace  $(\neg(P \rightarrow \neg Q) \leftrightarrow P \wedge Q)$  with
3484      $(P \wedge Q \leftrightarrow \neg(P \rightarrow \neg Q))$  in n4_63a.
3485   specialize Transp4_11 with  $(P \wedge Q) (\neg(P \rightarrow \neg Q))$ .
3486   intros Transp4_11a.
3487   replace  $(\neg\neg(P \rightarrow \neg Q))$  with
3488      $(P \rightarrow \neg Q)$  in Transp4_11a.
3489   replace  $(P \wedge Q \leftrightarrow \neg(P \rightarrow \neg Q))$  with
3490      $(\neg(P \wedge Q) \leftrightarrow (P \rightarrow \neg Q))$  in n4_63a.
3491   clear Transp4_11a. clear n4_21a.
3492   Conj n4_64a n4_63a.
3493   split.
3494   apply n4_64a.
3495   apply n4_63a.
3496   specialize n4_38 with
3497      $(P \vee Q) (\neg(P \wedge Q)) (\neg Q \rightarrow P) (P \rightarrow \neg Q)$ .
3498   intros n4_38a.
3499   MP n4_38a H.
3500   replace  $((\neg Q \rightarrow P) \wedge (P \rightarrow \neg Q))$  with
3501      $(\neg Q \leftrightarrow P)$  in n4_38a.
3502   specialize n4_21 with P  $(\neg Q)$ .
3503   intros n4_21b.
3504   replace  $(\neg Q \leftrightarrow P)$  with  $(P \leftrightarrow \neg Q)$  in n4_38a.
3505   apply n4_38a.
3506   apply EqBi.
3507   apply n4_21b.
3508   apply Equiv4_01.

```

```

3509   replace ( $\neg(P \wedge Q) \leftrightarrow (P \rightarrow \neg Q)$ ) with
3510         ( $P \wedge Q \leftrightarrow \neg(P \rightarrow \neg Q)$ ).
3511   reflexivity.
3512   apply EqBi.
3513   apply Transp4_11a.
3514   apply EqBi.
3515   specialize n4_13 with ( $P \rightarrow \neg Q$ ).
3516   intros n4_13a.
3517   apply n4_13a.
3518   apply EqBi.
3519   specialize n4_21 with ( $P \wedge Q$ ) ( $\neg(P \rightarrow \neg Q)$ ).
3520   intros n4_21b.
3521   apply n4_21b.
3522   apply EqBi.
3523   specialize n4_31 with P Q.
3524   intros n4_31a.
3525   apply n4_31a.
3526   apply EqBi.
3527   apply n4_21a.
3528   Qed.
3529
3530 Theorem n5_18 :  $\forall P Q : \text{Prop}$ ,
3531   ( $P \leftrightarrow Q$ )  $\leftrightarrow \neg(P \leftrightarrow \neg Q)$ .
3532 Proof. intros P Q.
3533   specialize n5_15 with P Q.
3534   intros n5_15a.
3535   specialize n5_16 with P Q.
3536   intros n5_16a.
3537   Conj n5_15a n5_16a.
3538   split.
3539   apply n5_15a.
3540   apply n5_16a.
3541   specialize n5_17 with ( $P \leftrightarrow Q$ ) ( $P \leftrightarrow \neg Q$ ).
3542   intros n5_17a.
3543   replace (( $P \leftrightarrow Q$ )  $\leftrightarrow \neg(P \leftrightarrow \neg Q)$ ) with
3544         (( $(P \leftrightarrow Q) \vee (P \leftrightarrow \neg Q)$ )  $\wedge \neg((P \leftrightarrow Q) \wedge (P \leftrightarrow \neg Q))$ ).
3545   apply H.
3546   apply EqBi.
3547   apply n5_17a.
3548   Qed.
3549
3550 Theorem n5_19 :  $\forall P : \text{Prop}$ ,

```

```

3551    $\neg(P \leftrightarrow \neg P)$ .
3552   Proof. intros P.
3553   specialize n5_18 with P P.
3554   intros n5_18a.
3555   specialize n4_2 with P.
3556   intros n4_2a.
3557   replace ( $\neg(P \leftrightarrow \neg P)$ ) with ( $P \leftrightarrow P$ ).
3558   apply n4_2a.
3559   apply EqBi.
3560   apply n5_18a.
3561   Qed.
3562
3563   Theorem n5_21 :  $\forall P Q : \text{Prop}$ ,
3564     ( $\neg P \wedge \neg Q$ )  $\rightarrow$  ( $P \leftrightarrow Q$ ).
3565   Proof. intros P Q.
3566   specialize n5_1 with ( $\neg P$ ) ( $\neg Q$ ).
3567   intros n5_1a.
3568   specialize Transp4_11 with P Q.
3569   intros Transp4_11a.
3570   replace ( $\neg P \leftrightarrow \neg Q$ ) with ( $P \leftrightarrow Q$ ) in n5_1a.
3571   apply n5_1a.
3572   apply EqBi.
3573   apply Transp4_11a.
3574   Qed.
3575
3576   Theorem n5_22 :  $\forall P Q : \text{Prop}$ ,
3577      $\neg(P \leftrightarrow Q) \leftrightarrow ((P \wedge \neg Q) \vee (Q \wedge \neg P))$ .
3578   Proof. intros P Q.
3579   specialize n4_61 with P Q.
3580   intros n4_61a.
3581   specialize n4_61 with Q P.
3582   intros n4_61b.
3583   Conj n4_61a n4_61b.
3584   split.
3585   apply n4_61a.
3586   apply n4_61b.
3587   specialize n4_39 with
3588     ( $\neg(P \rightarrow Q)$ ) ( $\neg(Q \rightarrow P)$ ) ( $P \wedge \neg Q$ ) ( $Q \wedge \neg P$ ).
3589   intros n4_39a.
3590   MP n4_39a H.
3591   specialize n4_51 with ( $P \rightarrow Q$ ) ( $Q \rightarrow P$ ).
3592   intros n4_51a.

```

```

3593   replace ( $\neg(P \rightarrow Q) \vee \neg(Q \rightarrow P)$ ) with
3594         ( $\neg((P \rightarrow Q) \wedge (Q \rightarrow P))$ ) in n4_39a.
3595   replace (( $P \rightarrow Q$ )  $\wedge$  ( $Q \rightarrow P$ )) with
3596         ( $P \leftrightarrow Q$ ) in n4_39a.
3597   apply n4_39a.
3598   apply Equiv4_01.
3599   apply EqBi.
3600   apply n4_51a.
3601   Qed.
3602
3603   Theorem n5_23 :  $\forall P Q : \text{Prop}$ ,
3604     ( $P \leftrightarrow Q$ )  $\leftrightarrow$  ( $(P \wedge Q) \vee (\neg P \wedge \neg Q)$ ).
3605   Proof. intros P Q.
3606   specialize n5_18 with P Q.
3607   intros n5_18a.
3608   specialize n5_22 with P ( $\neg Q$ ).
3609   intros n5_22a.
3610   Conj n5_18a n5_22a.
3611   split.
3612   apply n5_18a.
3613   apply n5_22a.
3614   specialize n4_22 with ( $P \leftrightarrow Q$ ) ( $\neg(P \leftrightarrow \neg Q)$ )
3615     ( $(P \wedge \neg \neg Q \vee \neg Q \wedge \neg P)$ ).
3616   intros n4_22a.
3617   MP n4_22a H.
3618   replace ( $\neg \neg Q$ ) with Q in n4_22a.
3619   replace ( $\neg Q \wedge \neg P$ ) with ( $\neg P \wedge \neg Q$ ) in n4_22a.
3620   apply n4_22a.
3621   apply EqBi.
3622   specialize n4_3 with ( $\neg P$ ) ( $\neg Q$ ).
3623   intros n4_3a.
3624   apply n4_3a. (*with ( $\neg P$ ) ( $\neg Q$ *)
3625   apply EqBi.
3626   specialize n4_13 with Q.
3627   intros n4_13a.
3628   apply n4_13a.
3629   Qed.
3630   (*The proof sketch in Principia offers n4_36.
3631     This seems to be a misprint. We used n4_3.*)
3632
3633   Theorem n5_24 :  $\forall P Q : \text{Prop}$ ,
3634      $\neg((P \wedge Q) \vee (\neg P \wedge \neg Q)) \leftrightarrow ((P \wedge \neg Q) \vee (Q \wedge \neg P))$ .

```

```

3635 Proof. intros P Q.
3636 specialize n5_22 with P Q.
3637 intros n5_22a.
3638 specialize n5_23 with P Q.
3639 intros n5_23a.
3640 replace ((P↔Q)↔((P ∧ Q) ∨ (¬P ∧ ¬Q))) with
3641   ((¬(P↔Q)↔¬((P ∧ Q) ∨ (¬P ∧ ¬Q)))) in n5_23a.
3642 replace (¬(P↔Q)) with
3643   (¬((P ∧ Q) ∨ (¬P ∧ ¬Q))) in n5_22a.
3644 apply n5_22a.
3645 replace (¬((P ∧ Q) ∨ (¬P ∧ ¬Q))) with (¬(P↔Q)).
3646 reflexivity.
3647 apply EqBi.
3648 apply n5_23a.
3649 replace (¬(P ↔ Q) ↔ ¬(P ∧ Q ∨ ¬P ∧ ¬Q)) with
3650   ((P ↔ Q) ↔ P ∧ Q ∨ ¬P ∧ ¬Q).
3651 reflexivity.
3652 specialize Transp4_11 with
3653   (P↔Q) (P ∧ Q ∨ ¬P ∧ ¬Q).
3654 intros Transp4_11a.
3655 apply EqBi.
3656 apply Transp4_11a. (*Not cited*)
3657 Qed.
3658
3659 Theorem n5_25 : ∀ P Q : Prop,
3660   (P ∨ Q) ↔ ((P → Q) → Q).
3661 Proof. intros P Q.
3662 specialize n2_62 with P Q.
3663 intros n2_62a.
3664 specialize n2_68 with P Q.
3665 intros n2_68a.
3666 Conj n2_62a n2_68a.
3667 split.
3668 apply n2_62a.
3669 apply n2_68a.
3670 Equiv H.
3671 apply H.
3672 apply Equiv4_01.
3673 Qed.
3674
3675 Theorem n5_3 : ∀ P Q R : Prop,
3676   ((P ∧ Q) → R) ↔ ((P ∧ Q) → (P ∧ R)).

```



```

3677 Proof. intros P Q R.
3678 specialize Comp3_43 with (P ∧ Q) P R.
3679 intros Comp3_43a.
3680 specialize Exp3_3 with
3681   (P ∧ Q → P) (P ∧ Q → R) (P ∧ Q → P ∧ R).
3682 intros Exp3_3a. (*Not cited*)
3683 MP Exp3_3a Comp3_43a.
3684 specialize Simp3_26 with P Q.
3685 intros Simp3_26a.
3686 MP Exp3_3a Simp3_26a.
3687 specialize Syll2_05 with (P ∧ Q) (P ∧ R) R.
3688 intros Syll2_05a.
3689 specialize Simp3_27 with P R.
3690 intros Simp3_27a.
3691 MP Syll2_05a Simp3_27a.
3692 clear Comp3_43a. clear Simp3_27a.
3693   clear Simp3_26a.
3694 Conj Exp3_3a Syll2_05a.
3695 split.
3696 apply Exp3_3a.
3697 apply Syll2_05a.
3698 Equiv H.
3699 apply H.
3700 apply Equiv4_01.
3701 Qed.
3702
3703 Theorem n5_31 : ∀ P Q R : Prop,
3704   (R ∧ (P → Q)) → (P → (Q ∧ R)).
3705 Proof. intros P Q R.
3706 specialize Comp3_43 with P Q R.
3707 intros Comp3_43a.
3708 specialize Simp2_02 with P R.
3709 intros Simp2_02a.
3710 specialize Exp3_3 with
3711   (P→R) (P→Q) (P→(Q ∧ R)).
3712 intros Exp3_3a. (*Not cited*)
3713 specialize n3_22 with (P → R) (P → Q). (*Not cited*)
3714 intros n3_22a.
3715 Syll n3_22a Comp3_43a Sa.
3716 MP Exp3_3a Sa.
3717 Syll Simp2_02a Exp3_3a Sb.
3718 specialize Imp3_31 with R (P→Q) (P→(Q ∧ R)).

```

```

3719   intros Imp3_31a. (*Not cited*)
3720   MP Imp3_31a Sb.
3721   apply Imp3_31a.
3722   Qed.
3723
3724   Theorem n5_32 :  $\forall$  P Q R : Prop,
3725     (P  $\rightarrow$  (Q  $\leftrightarrow$  R))  $\leftrightarrow$  ((P  $\wedge$  Q)  $\leftrightarrow$  (P  $\wedge$  R)).
3726   Proof. intros P Q R.
3727     specialize n4_76 with P (Q $\rightarrow$ R) (R $\rightarrow$ Q).
3728     intros n4_76a.
3729     specialize Exp3_3 with P Q R.
3730     intros Exp3_3a.
3731     specialize Imp3_31 with P Q R.
3732     intros Imp3_31a.
3733     Conj Exp3_3a Imp3_31a.
3734     split.
3735     apply Exp3_3a.
3736     apply Imp3_31a.
3737     Equiv H.
3738     specialize Exp3_3 with P R Q.
3739     intros Exp3_3b.
3740     specialize Imp3_31 with P R Q.
3741     intros Imp3_31b.
3742     Conj Exp3_3b Imp3_31b.
3743     split.
3744     apply Exp3_3b.
3745     apply Imp3_31b.
3746     Equiv H0.
3747     specialize n5_3 with P Q R.
3748     intros n5_3a.
3749     specialize n5_3 with P R Q.
3750     intros n5_3b.
3751     replace (P $\rightarrow$ Q $\rightarrow$ R) with (P $\wedge$ Q $\rightarrow$ R) in n4_76a.
3752     replace (P $\wedge$ Q $\rightarrow$ R) with (P $\wedge$ Q $\rightarrow$ P $\wedge$ R) in n4_76a.
3753     replace (P $\rightarrow$ R $\rightarrow$ Q) with (P $\wedge$ R $\rightarrow$ Q) in n4_76a.
3754     replace (P $\wedge$ R $\rightarrow$ Q) with (P $\wedge$ R $\rightarrow$ P $\wedge$ Q) in n4_76a.
3755     replace ((P $\wedge$ Q $\rightarrow$ P $\wedge$ R) $\wedge$ (P $\wedge$ R $\rightarrow$ P $\wedge$ Q)) with
3756       ((P $\wedge$ Q) $\leftrightarrow$ (P $\wedge$ R)) in n4_76a.
3757     replace ((P $\wedge$ Q $\leftrightarrow$ P $\wedge$ R) $\leftrightarrow$ (P $\rightarrow$ (Q $\rightarrow$ R) $\wedge$ (R $\rightarrow$ Q))) with
3758       ((P $\rightarrow$ (Q $\rightarrow$ R) $\wedge$ (R $\rightarrow$ Q)) $\leftrightarrow$ (P $\wedge$ Q  $\leftrightarrow$  P $\wedge$ R)) in n4_76a.
3759     replace ((Q $\rightarrow$ R) $\wedge$ (R $\rightarrow$ Q)) with (Q $\leftrightarrow$ R) in n4_76a.
3760     apply n4_76a.

```

```

3761   apply Equiv4_01.
3762   apply EqBi.
3763   specialize n4_21 with
3764     (P → ((Q → R) ∧ (R → Q))) ((P ∧ Q) ↔ (P ∧ R)).
3765   intros n4_21a.
3766   apply n4_21a. (*to commute the biconditional*)
3767   apply Equiv4_01.
3768   replace (P ∧ R → P ∧ Q) with (P ∧ R → Q).
3769   reflexivity.
3770   apply EqBi.
3771   apply n5_3b.
3772   apply EqBi.
3773   apply H0.
3774   replace (P ∧ Q → P ∧ R) with (P ∧ Q → R).
3775   reflexivity.
3776   apply EqBi.
3777   apply n5_3a.
3778   apply EqBi.
3779   apply H.
3780   apply Equiv4_01.
3781   apply Equiv4_01.
3782   Qed.
3783
3784   Theorem n5_33 : ∀ P Q R : Prop,
3785     (P ∧ (Q → R)) ↔ (P ∧ ((P ∧ Q) → R)).
3786   Proof. intros P Q R.
3787     specialize n5_32 with P (Q → R) ((P ∧ Q) → R).
3788     intros n5_32a.
3789     replace
3790       ((P → (Q → R) ↔ (P ∧ Q → R)) ↔ (P ∧ (Q → R) ↔ P ∧ (P ∧ Q → R)))
3791       with
3792       (((P → (Q → R) ↔ (P ∧ Q → R)) → (P ∧ (Q → R) ↔ P ∧ (P ∧ Q → R)))
3793       ∧
3794       ((P ∧ (Q → R) ↔ P ∧ (P ∧ Q → R) → (P → (Q → R) ↔ (P ∧ Q → R))))))
3795       in n5_32a.
3796     specialize Simp3_26 with
3797       ((P → (Q → R) ↔ (P ∧ Q → R)) → (P ∧ (Q → R) ↔ P ∧ (P ∧ Q → R)))
3798       ((P ∧ (Q → R) ↔ P ∧ (P ∧ Q → R) → (P → (Q → R) ↔ (P ∧ Q → R)))).
3799     intros Simp3_26a. (*Not cited*)
3800     MP Simp3_26a n5_32a.
3801     specialize n4_73 with Q P.
3802     intros n4_73a.

```

```

3803     specialize n4_84 with Q (Q $\wedge$ P) R.
3804     intros n4_84a.
3805     Syll n4_73a n4_84a Sa.
3806     replace (Q $\wedge$ P) with (P $\wedge$ Q) in Sa.
3807     MP Simp3_26a Sa.
3808     apply Simp3_26a.
3809     apply EqBi.
3810     specialize n4_3 with P Q.
3811     intros n4_3a.
3812     apply n4_3a. (*Not cited*)
3813     apply Equiv4_01.
3814     Qed.
3815
3816 Theorem n5_35 :  $\forall$  P Q R : Prop,
3817   ((P  $\rightarrow$  Q)  $\wedge$  (P  $\rightarrow$  R))  $\rightarrow$  (P  $\rightarrow$  (Q  $\leftrightarrow$  R)).
3818 Proof. intros P Q R.
3819     specialize Comp3_43 with P Q R.
3820     intros Comp3_43a.
3821     specialize n5_1 with Q R.
3822     intros n5_1a.
3823     specialize Syll2_05 with P (Q $\wedge$ R) (Q $\leftrightarrow$ R).
3824     intros Syll2_05a.
3825     MP Syll2_05a n5_1a.
3826     Syll Comp3_43a Syll2_05a Sa.
3827     apply Sa.
3828     Qed.
3829
3830 Theorem n5_36 :  $\forall$  P Q : Prop,
3831   (P  $\wedge$  (P  $\leftrightarrow$  Q))  $\leftrightarrow$  (Q  $\wedge$  (P  $\leftrightarrow$  Q)).
3832 Proof. intros P Q.
3833     specialize n5_32 with (P $\leftrightarrow$ Q) P Q.
3834     intros n5_32a.
3835     specialize Id2_08 with (P $\leftrightarrow$ Q).
3836     intros Id2_08a.
3837     replace (P $\leftrightarrow$ Q $\rightarrow$ P $\leftrightarrow$ Q) with
3838       ((P $\leftrightarrow$ Q) $\wedge$ P $\leftrightarrow$ (P $\leftrightarrow$ Q) $\wedge$ Q) in Id2_08a.
3839     replace ((P $\leftrightarrow$ Q) $\wedge$ P) with (P $\wedge$ (P $\leftrightarrow$ Q)) in Id2_08a.
3840     replace ((P $\leftrightarrow$ Q) $\wedge$ Q) with (Q $\wedge$ (P $\leftrightarrow$ Q)) in Id2_08a.
3841     apply Id2_08a.
3842     apply EqBi.
3843     specialize n4_3 with Q (P $\leftrightarrow$ Q).
3844     intros n4_3a.

```

```

3845   apply n4_3a.
3846   apply EqBi.
3847   specialize n4_3 with P (P↔Q).
3848   intros n4_3b.
3849   apply n4_3b.
3850   replace ((P ↔ Q) ∧ P ↔ (P ↔ Q) ∧ Q) with
3851     (P ↔ Q → P ↔ Q).
3852   reflexivity.
3853   apply EqBi.
3854   apply n5_32a.
3855   Qed.
3856   (*The proof sketch cites Ass3_35 and n4_38,
3857     but the sketch was indecipherable.*)
3858
3859   Theorem n5_4 : ∀ P Q : Prop,
3860     (P → (P → Q)) ↔ (P → Q).
3861   Proof. intros P Q.
3862     specialize n2_43 with P Q.
3863     intros n2_43a.
3864     specialize Simp2_02 with (P) (P→Q).
3865     intros Simp2_02a.
3866     Conj n2_43a Simp2_02a.
3867     split.
3868     apply n2_43a.
3869     apply Simp2_02a.
3870     Equiv H.
3871     apply H.
3872     apply Equiv4_01.
3873     Qed.
3874
3875   Theorem n5_41 : ∀ P Q R : Prop,
3876     ((P → Q) → (P → R)) ↔ (P → Q → R).
3877   Proof. intros P Q R.
3878     specialize n2_86 with P Q R.
3879     intros n2_86a.
3880     specialize n2_77 with P Q R.
3881     intros n2_77a.
3882     Conj n2_86a n2_77a.
3883     split.
3884     apply n2_86a.
3885     apply n2_77a.
3886     Equiv H.

```

```

3887   apply H.
3888   apply Equiv4_01.
3889   Qed.
3890
3891 Theorem n5_42 :  $\forall$  P Q R : Prop,
3892    $(P \rightarrow Q \rightarrow R) \leftrightarrow (P \rightarrow Q \rightarrow P \wedge R)$ .
3893 Proof. intros P Q R.
3894   specialize n5_3 with P Q R.
3895   intros n5_3a.
3896   specialize n4_87 with P Q R.
3897   intros n4_87a.
3898   replace  $((P \wedge Q) \rightarrow R)$  with  $(P \rightarrow Q \rightarrow R)$  in n5_3a.
3899   specialize n4_87 with P Q  $(P \wedge R)$ .
3900   intros n4_87b.
3901   replace  $((P \wedge Q) \rightarrow (P \wedge R))$  with
3902      $(P \rightarrow Q \rightarrow (P \wedge R))$  in n5_3a.
3903   apply n5_3a.
3904   specialize Imp3_31 with P Q  $(P \wedge R)$ .
3905   intros Imp3_31b.
3906   specialize Exp3_3 with P Q  $(P \wedge R)$ .
3907   intros Exp3_3b.
3908   Conj Imp3_31b Exp3_3b.
3909   split.
3910   apply Imp3_31b.
3911   apply Exp3_3b.
3912   Equiv H.
3913   apply EqBi.
3914   apply H.
3915   apply Equiv4_01.
3916   specialize Imp3_31 with P Q R.
3917   intros Imp3_31a.
3918   specialize Exp3_3 with P Q R.
3919   intros Exp3_3a.
3920   Conj Imp3_31a Exp3_3.
3921   split.
3922   apply Imp3_31a.
3923   apply Exp3_3a.
3924   Equiv H.
3925   apply EqBi.
3926   apply H.
3927   apply Equiv4_01.
3928   Qed.

```

```

3929
3930 Theorem n5_44 :  $\forall P Q R : \text{Prop},$ 
3931    $(P \rightarrow Q) \rightarrow ((P \rightarrow R) \leftrightarrow (P \rightarrow (Q \wedge R)))$ .
3932 Proof. intros P Q R.
3933 specialize n4_76 with P Q R.
3934 intros n4_76a.
3935 rewrite Equiv4_01 in n4_76a.
3936 specialize Simp3_26 with
3937    $((P \rightarrow Q) \wedge (P \rightarrow R)) \rightarrow (P \rightarrow (Q \wedge R))$ 
3938    $((P \rightarrow (Q \wedge R)) \rightarrow ((P \rightarrow Q) \wedge (P \rightarrow R)))$ .
3939 intros Simp3_26a.
3940 MP Simp3_26a n4_76a.
3941 specialize Simp3_27 with
3942    $((P \rightarrow Q) \wedge (P \rightarrow R)) \rightarrow (P \rightarrow (Q \wedge R))$ 
3943    $((P \rightarrow (Q \wedge R)) \rightarrow ((P \rightarrow Q) \wedge (P \rightarrow R)))$ .
3944 intros Simp3_27a.
3945 MP Simp3_27a n4_76a.
3946 specialize Simp3_27 with  $(P \rightarrow Q) (P \rightarrow Q \wedge R)$ .
3947 intros Simp3_27d.
3948 Syll Simp3_27d Simp3_27a Sa.
3949 specialize n5_3 with  $(P \rightarrow Q) (P \rightarrow R) (P \rightarrow (Q \wedge R))$ .
3950 intros n5_3a.
3951 rewrite Equiv4_01 in n5_3a.
3952 specialize Simp3_26 with
3953    $((((P \rightarrow Q) \wedge (P \rightarrow R)) \rightarrow (P \rightarrow (Q \wedge R))) \rightarrow$ 
3954    $((P \rightarrow Q) \wedge (P \rightarrow R)) \rightarrow ((P \rightarrow Q) \wedge (P \rightarrow (Q \wedge R))))$ 
3955    $((((P \rightarrow Q) \wedge (P \rightarrow R)) \rightarrow ((P \rightarrow Q) \wedge (P \rightarrow (Q \wedge R))))$ 
3956    $\rightarrow ((P \rightarrow Q) \wedge (P \rightarrow R)) \rightarrow (P \rightarrow (Q \wedge R)))$ .
3957 intros Simp3_26b.
3958 MP Simp3_26b n5_3a.
3959 specialize Simp3_27 with
3960    $((((P \rightarrow Q) \wedge (P \rightarrow R)) \rightarrow (P \rightarrow (Q \wedge R))) \rightarrow$ 
3961    $((P \rightarrow Q) \wedge (P \rightarrow R)) \rightarrow ((P \rightarrow Q) \wedge (P \rightarrow (Q \wedge R))))$ 
3962    $((((P \rightarrow Q) \wedge (P \rightarrow R)) \rightarrow ((P \rightarrow Q) \wedge (P \rightarrow (Q \wedge R))))$ 
3963    $\rightarrow ((P \rightarrow Q) \wedge (P \rightarrow R)) \rightarrow (P \rightarrow (Q \wedge R)))$ .
3964 intros Simp3_27b.
3965 MP Simp3_27b n5_3a.
3966 MP Simp3_26a Simp3_26b.
3967 MP Simp3_27a Simp3_27b.
3968 clear n4_76a. clear Simp3_26a. clear Simp3_27a.
3969   clear Simp3_27b. clear Simp3_27d. clear n5_3a.
3970 Conj Simp3_26b Sa.

```

```

3971 split.
3972 apply Sa.
3973 apply Simp3_26b.
3974 Equiv H.
3975 specialize n5_32 with (P→Q) (P→R) (P→(Q∧R)).
3976 intros n5_32a.
3977 rewrite Equiv4_01 in n5_32a.
3978 specialize Simp3_27 with
3979   ((P → Q) → (P → R) ↔ (P → Q ∧ R))
3980   → (P → Q) ∧ (P → R) ↔ (P → Q) ∧ (P → Q ∧ R))
3981   ((P → Q) ∧ (P → R) ↔ (P → Q) ∧ (P → Q ∧ R)
3982   → (P → Q) → (P → R) ↔ (P → Q ∧ R)).
3983 intros Simp3_27c.
3984 MP Simp3_27c n5_32a.
3985 replace (((P→Q)∧(P→(Q∧R)))↔((P→Q)∧(P→R)))
3986   with (((P→Q)∧(P→R))↔((P→Q)∧(P→(Q∧R)))) in H.
3987 MP Simp3_27c H.
3988 apply Simp3_27c.
3989 specialize n4_21 with
3990   ((P→Q)∧(P→R)) ((P→Q)∧(P→(Q∧R))).
3991 intros n4_21a.
3992 apply EqBi.
3993 apply n4_21a.
3994 apply Equiv4_01.
3995 Qed.
3996
3997 Theorem n5_5 : ∀ P Q : Prop,
3998   P → ((P → Q) ↔ Q).
3999 Proof. intros P Q.
4000 specialize Ass3_35 with P Q.
4001 intros Ass3_35a.
4002 specialize Exp3_3 with P (P→Q) Q.
4003 intros Exp3_3a.
4004 MP Exp3_3a Ass3_35a.
4005 specialize Simp2_02 with P Q.
4006 intros Simp2_02a.
4007 specialize Exp3_3 with P Q (P→Q).
4008 intros Exp3_3b.
4009 specialize n3_42 with P Q (P→Q). (*Not cited*)
4010 intros n3_42a.
4011 MP n3_42a Simp2_02a.
4012 MP Exp3_3b n3_42a.

```



```

4013 clear n3_42a. clear Simp2_02a. clear Ass3_35a.
4014 Conj Exp3_3a Exp3_3b.
4015 split.
4016 apply Exp3_3a.
4017 apply Exp3_3b.
4018 specialize n3_47 with P P ((P→Q)→Q) (Q→(P→Q)).
4019 intros n3_47a.
4020 MP n3_47a H.
4021 replace (P∧P) with P in n3_47a.
4022 replace (((P→Q)→Q)∧(Q→(P→Q))) with
4023   ((P→Q)↔Q) in n3_47a.
4024 apply n3_47a.
4025 apply Equiv4_01.
4026 apply EqBi.
4027 specialize n4_24 with P.
4028 intros n4_24a. (*Not cited*)
4029 apply n4_24a.
4030 Qed.
4031
4032 Theorem n5_501 : ∀ P Q : Prop,
4033   P → (Q ↔ (P ↔ Q)).
4034 Proof. intros P Q.
4035 specialize n5_1 with P Q.
4036 intros n5_1a.
4037 specialize Exp3_3 with P Q (P↔Q).
4038 intros Exp3_3a.
4039 MP Exp3_3a n5_1a.
4040 specialize Ass3_35 with P Q.
4041 intros Ass3_35a.
4042 specialize Simp3_26 with (P∧(P→Q)) (Q→P).
4043 intros Simp3_26a. (*Not cited*)
4044 Syll Simp3_26a Ass3_35a Sa.
4045 replace ((P∧(P→Q))∧(Q→P)) with
4046   (P∧((P→Q)∧(Q→P))) in Sa.
4047 replace ((P→Q)∧(Q→P)) with (P↔Q) in Sa.
4048 specialize Exp3_3 with P (P↔Q) Q.
4049 intros Exp3_3b.
4050 MP Exp3_3b Sa.
4051 clear n5_1a. clear Ass3_35a.
4052   clear Simp3_26a. clear Sa.
4053 Conj Exp3_3a Exp3_3b.
4054 split.

```

```

4055 apply Exp3_3a.
4056 apply Exp3_3b.
4057 specialize n4_76 with P (Q → (P ↔ Q)) ((P ↔ Q) → Q).
4058 intros n4_76a. (*Not cited*)
4059 replace ((P → Q → P ↔ Q) ∧ (P → P ↔ Q → Q)) with
4060         ((P → (Q → P ↔ Q) ∧ (P ↔ Q → Q))) in H.
4061 replace ((Q → (P ↔ Q)) ∧ ((P ↔ Q) → Q)) with
4062         (Q ↔ (P ↔ Q)) in H.
4063 apply H.
4064 apply Equiv4_01.
4065 replace (P → (Q → P ↔ Q) ∧ (P ↔ Q → Q)) with
4066         ((P → Q → P ↔ Q) ∧ (P → P ↔ Q → Q)).
4067 reflexivity.
4068 apply EqBi.
4069 apply n4_76a.
4070 apply Equiv4_01.
4071 replace (P ∧ (P → Q) ∧ (Q → P)) with
4072         ((P ∧ (P → Q)) ∧ (Q → P)).
4073 reflexivity.
4074 apply EqBi.
4075 specialize n4_32 with P (P → Q) (Q → P).
4076 intros n4_32a. (*Not cited*)
4077 apply n4_32a.
4078 Qed.
4079
4080 Theorem n5_53 : ∀ P Q R S : Prop,
4081   (((P ∨ Q) ∨ R) → S) ↔ (((P → S) ∧ (Q → S)) ∧ (R → S)).
4082 Proof. intros P Q R S.
4083 specialize n4_77 with S (P ∨ Q) R.
4084 intros n4_77a.
4085 specialize n4_77 with S P Q.
4086 intros n4_77b.
4087 replace (P ∨ Q → S) with
4088         ((P → S) ∧ (Q → S)) in n4_77a.
4089 replace (((P → S) ∧ (Q → S)) ∧ (R → S)) ↔ (((P ∨ Q) ∨ R) → S)
4090 with
4091         (((P ∨ Q) ∨ R) → S) ↔ (((P → S) ∧ (Q → S)) ∧ (R → S))
4092 in n4_77a.
4093 apply n4_77a.
4094 apply EqBi.
4095 specialize n4_21 with ((P ∨ Q) ∨ R → S)
4096         (((P → S) ∧ (Q → S)) ∧ (R → S)).

```

```

4097   intros n4_21a.
4098   apply n4_21a. (*Not cited*)
4099   apply EqBi.
4100   apply n4_77b.
4101   Qed.
4102
4103   Theorem n5_54 :  $\forall P Q : \text{Prop},$ 
4104      $((P \wedge Q) \leftrightarrow P) \vee ((P \wedge Q) \leftrightarrow Q).$ 
4105   Proof. intros P Q.
4106   specialize n4_73 with P Q.
4107   intros n4_73a.
4108   specialize n4_44 with Q P.
4109   intros n4_44a.
4110   specialize Transp2_16 with Q  $(P \leftrightarrow (P \wedge Q)).$ 
4111   intros Transp2_16a.
4112   MP n4_73a Transp2_16a.
4113   specialize Transp4_11 with Q  $(Q \vee (P \wedge Q)).$ 
4114   intros Transp4_11a.
4115   replace  $(Q \wedge P)$  with  $(P \wedge Q)$  in n4_44a.
4116   replace  $(Q \leftrightarrow Q \vee P \wedge Q)$  with
4117      $(\neg Q \leftrightarrow \neg(Q \vee P \wedge Q))$  in n4_44a.
4118   replace  $(\neg Q)$  with  $(\neg(Q \vee P \wedge Q))$  in Transp2_16a.
4119   replace  $(\neg(Q \vee P \wedge Q))$  with
4120      $(\neg Q \wedge \neg(P \wedge Q))$  in Transp2_16a.
4121   specialize n5_1 with  $(\neg Q) (\neg(P \wedge Q)).$ 
4122   intros n5_1a.
4123   Syll Transp2_16a n5_1a Sa.
4124   replace  $(\neg(P \leftrightarrow P \wedge Q) \rightarrow (\neg Q \leftrightarrow \neg(P \wedge Q)))$  with
4125      $(\neg\neg(P \leftrightarrow P \wedge Q) \vee (\neg Q \leftrightarrow \neg(P \wedge Q)))$  in Sa.
4126   replace  $(\neg\neg(P \leftrightarrow P \wedge Q))$  with  $(P \leftrightarrow P \wedge Q)$  in Sa.
4127   specialize Transp4_11 with Q  $(P \wedge Q).$ 
4128   intros Transp4_11b.
4129   replace  $(\neg Q \leftrightarrow \neg(P \wedge Q))$  with  $(Q \leftrightarrow (P \wedge Q))$  in Sa.
4130   replace  $(Q \leftrightarrow (P \wedge Q))$  with  $((P \wedge Q) \leftrightarrow Q)$  in Sa.
4131   replace  $(P \leftrightarrow (P \wedge Q))$  with  $((P \wedge Q) \leftrightarrow P)$  in Sa.
4132   apply Sa.
4133   apply EqBi.
4134   specialize n4_21 with  $(P \wedge Q) P.$ 
4135   intros n4_21a. (*Not cited*)
4136   apply n4_21a.
4137   apply EqBi.
4138   specialize n4_21 with  $(P \wedge Q) Q.$ 

```

```

4139   intros n4_21b. (*Not cited*)
4140   apply n4_21b.
4141   apply EqBi.
4142   apply Transp4_11b.
4143   apply EqBi.
4144   specialize n4_13 with (P  $\leftrightarrow$  (P $\wedge$ Q)).
4145   intros n4_13a. (*Not cited*)
4146   apply n4_13a.
4147   rewrite <- Impl1_01. (*Not cited*)
4148   reflexivity.
4149   apply EqBi.
4150   specialize n4_56 with Q (P $\wedge$ Q).
4151   intros n4_56a. (*Not cited*)
4152   apply n4_56a.
4153   replace ( $\neg$ (Q $\vee$ P $\wedge$ Q)) with ( $\neg$ Q).
4154   reflexivity.
4155   apply EqBi.
4156   apply n4_44a.
4157   replace ( $\neg$ Q $\leftrightarrow$  $\neg$ (Q $\vee$ P $\wedge$ Q)) with (Q $\leftrightarrow$ Q $\vee$ P $\wedge$ Q).
4158   reflexivity.
4159   apply EqBi.
4160   apply Transp4_11a.
4161   apply EqBi.
4162   specialize n4_3 with P Q.
4163   intros n4_3a. (*Not cited*)
4164   apply n4_3a.
4165   Qed.
4166
4167   Theorem n5_55 :  $\forall$  P Q : Prop,
4168     ((P  $\vee$  Q)  $\leftrightarrow$  P)  $\vee$  ((P  $\vee$  Q)  $\leftrightarrow$  Q).
4169   Proof. intros P Q.
4170   specialize Add1_3 with (P $\wedge$ Q) (P).
4171   intros Add1_3a.
4172   replace ((P $\wedge$ Q) $\vee$ P) with ((P $\vee$ P) $\wedge$ (Q $\vee$ P)) in Add1_3a.
4173   replace (P $\vee$ P) with P in Add1_3a.
4174   replace (Q $\vee$ P) with (P $\vee$ Q) in Add1_3a.
4175   specialize n5_1 with P (P $\vee$ Q).
4176   intros n5_1a.
4177   Syll Add1_3a n5_1a Sa.
4178   specialize n4_74 with P Q.
4179   intros n4_74a.
4180   specialize Transp2_15 with P (Q $\leftrightarrow$ P $\vee$ Q).

```

```

4181  intros Transp2_15a. (*Not cited*)
4182  MP Transp2_15a n4_74a.
4183  Syll Transp2_15a Sa Sb.
4184  replace ( $\neg(Q \leftrightarrow (P \vee Q)) \rightarrow (P \leftrightarrow (P \vee Q))$ ) with
4185    ( $\neg(Q \leftrightarrow (P \vee Q)) \vee (P \leftrightarrow (P \vee Q))$ ) in Sb.
4186  replace ( $\neg(Q \leftrightarrow (P \vee Q))$ ) with ( $Q \leftrightarrow (P \vee Q)$ ) in Sb.
4187  replace ( $Q \leftrightarrow (P \vee Q)$ ) with ( $(P \vee Q) \leftrightarrow Q$ ) in Sb.
4188  replace ( $P \leftrightarrow (P \vee Q)$ ) with ( $(P \vee Q) \leftrightarrow P$ ) in Sb.
4189  replace ( $(P \vee Q \leftrightarrow Q) \vee (P \vee Q \leftrightarrow P)$ ) with
4190    ( $(P \vee Q \leftrightarrow P) \vee (P \vee Q \leftrightarrow Q)$ ) in Sb.
4191  apply Sb.
4192  apply EqBi.
4193  specialize n4_31 with ( $P \vee Q \leftrightarrow P$ ) ( $P \vee Q \leftrightarrow Q$ ).
4194  intros n4_31a. (*Not cited*)
4195  apply n4_31a.
4196  apply EqBi.
4197  specialize n4_21 with ( $P \vee Q$ ) P.
4198  intros n4_21a. (*Not cited*)
4199  apply n4_21a.
4200  apply EqBi.
4201  specialize n4_21 with ( $P \vee Q$ ) Q.
4202  intros n4_21b. (*Not cited*)
4203  apply n4_21b.
4204  apply EqBi.
4205  specialize n4_13 with ( $Q \leftrightarrow P \vee Q$ ).
4206  intros n4_13a. (*Not cited*)
4207  apply n4_13a.
4208  rewrite <- Impl1_01.
4209  reflexivity.
4210  apply EqBi.
4211  specialize n4_31 with P Q.
4212  intros n4_31b.
4213  apply n4_31b.
4214  apply EqBi.
4215  specialize n4_25 with P.
4216  intros n4_25a. (*Not cited*)
4217  apply n4_25a.
4218  replace ( $(P \vee P) \wedge (Q \vee P)$ ) with ( $(P \wedge Q) \vee P$ ).
4219  reflexivity.
4220  replace ( $(P \wedge Q) \vee P$ ) with ( $P \vee (P \wedge Q)$ ).
4221  replace ( $Q \vee P$ ) with ( $P \vee Q$ ).
4222  apply EqBi.

```

```

4223 specialize n4_41 with P P Q.
4224 intros n4_41a. (*Not cited*)
4225 apply n4_41a.
4226 apply EqBi.
4227 specialize n4_31 with P Q.
4228 intros n4_31c.
4229 apply n4_31c.
4230 apply EqBi.
4231 specialize n4_31 with P (P ∧ Q).
4232 intros n4_31d. (*Not cited*)
4233 apply n4_31d.
4234 Qed.
4235
4236 Theorem n5_6 : ∀ P Q R : Prop,
4237   ((P ∧ ¬Q) → R) ↔ (P → (Q ∨ R)).
4238 Proof. intros P Q R.
4239 specialize n4_87 with P (¬Q) R.
4240 intros n4_87a.
4241 specialize n4_64 with Q R.
4242 intros n4_64a.
4243 specialize n4_85 with P Q R.
4244 intros n4_85a.
4245 replace (((P ∧ ¬Q → R) ↔ (P → ¬Q → R)) ↔ ((¬Q → P → R) ↔ (¬Q ∧ P → R)))
4246   with
4247     (((P ∧ ¬Q → R) ↔ (P → ¬Q → R)) → ((¬Q → P → R) ↔ (¬Q ∧ P → R)))
4248     ∧
4249     (((¬Q → P → R) ↔ (¬Q ∧ P → R)) → (((P ∧ ¬Q → R) ↔ (P → ¬Q → R))))
4250   in n4_87a.
4251 specialize Simp3_27 with
4252   (((P ∧ ¬Q → R) ↔ (P → ¬Q → R)) → (¬Q → P → R) ↔ (¬Q ∧ P → R))
4253   (((¬Q → P → R) ↔ (¬Q ∧ P → R)) → (P ∧ ¬Q → R) ↔ (P → ¬Q → R)).
4254 intros Simp3_27a.
4255 MP Simp3_27a n4_87a.
4256 specialize Imp3_31 with (¬Q) P R.
4257 intros Imp3_31a.
4258 specialize Exp3_3 with (¬Q) P R.
4259 intros Exp3_3a.
4260 Conj Imp3_31a Exp3_3a.
4261 split.
4262 apply Imp3_31a.
4263 apply Exp3_3a.
4264 Equiv H.

```

```

4265 MP Simp3_27a H.
4266 replace ( $\neg Q \rightarrow R$ ) with ( $Q \vee R$ ) in Simp3_27a.
4267 apply Simp3_27a.
4268 replace ( $Q \vee R$ ) with ( $\neg Q \rightarrow R$ ).
4269 reflexivity.
4270 apply EqBi.
4271 apply n4_64a.
4272 apply Equiv4_01.
4273 apply Equiv4_01.
4274 Qed.
4275
4276 Theorem n5_61 :  $\forall P Q : \text{Prop}$ ,
4277   ( $(P \vee Q) \wedge \neg Q \leftrightarrow (P \wedge \neg Q)$ ).
4278 Proof. intros P Q.
4279 specialize n4_74 with Q P.
4280 intros n4_74a.
4281 specialize n5_32 with ( $\neg Q$ ) P ( $Q \vee P$ ).
4282 intros n5_32a.
4283 replace ( $\neg Q \rightarrow P \leftrightarrow Q \vee P$ ) with
4284   ( $\neg Q \wedge P \leftrightarrow \neg Q \wedge (Q \vee P)$ ) in n4_74a.
4285 replace ( $\neg Q \wedge P$ ) with ( $P \wedge \neg Q$ ) in n4_74a.
4286 replace ( $\neg Q \wedge (Q \vee P)$ ) with ( $(Q \vee P) \wedge \neg Q$ ) in n4_74a.
4287 replace ( $Q \vee P$ ) with ( $P \vee Q$ ) in n4_74a.
4288 replace ( $P \wedge \neg Q \leftrightarrow (P \vee Q) \wedge \neg Q$ ) with
4289   ( $(P \vee Q) \wedge \neg Q \leftrightarrow P \wedge \neg Q$ ) in n4_74a.
4290 apply n4_74a.
4291 apply EqBi.
4292 specialize n4_21 with ( $(P \vee Q) \wedge \neg Q$ ) ( $P \wedge \neg Q$ ).
4293 intros n4_21a. (*Not cited*)
4294 apply n4_21a.
4295 apply EqBi.
4296 specialize n4_31 with P Q.
4297 intros n4_31a. (*Not cited*)
4298 apply n4_31a.
4299 apply EqBi.
4300 specialize n4_3 with ( $Q \vee P$ ) ( $\neg Q$ ).
4301 intros n4_3a. (*Not cited*)
4302 apply n4_3a.
4303 apply EqBi.
4304 specialize n4_3 with P ( $\neg Q$ ).
4305 intros n4_3b. (*Not cited*)
4306 apply n4_3b.

```

```

4307   replace ( $\neg Q \wedge P \leftrightarrow \neg Q \wedge (Q \vee P)$ ) with
4308         ( $\neg Q \rightarrow P \leftrightarrow Q \vee P$ ).
4309   reflexivity.
4310   apply EqBi.
4311   apply n5_32a.
4312   Qed.
4313
4314   Theorem n5_62 :  $\forall P Q : \text{Prop}$ ,
4315     ( $(P \wedge Q) \vee \neg Q$ )  $\leftrightarrow$  ( $P \vee \neg Q$ ).
4316   Proof. intros P Q.
4317     specialize n4_7 with Q P.
4318     intros n4_7a.
4319     replace ( $Q \rightarrow P$ ) with ( $\neg Q \vee P$ ) in n4_7a.
4320     replace ( $Q \rightarrow (Q \wedge P)$ ) with ( $\neg Q \vee (Q \wedge P)$ ) in n4_7a.
4321     replace ( $\neg Q \vee (Q \wedge P)$ ) with ( $(Q \wedge P) \vee \neg Q$ ) in n4_7a.
4322     replace ( $\neg Q \vee P$ ) with ( $P \vee \neg Q$ ) in n4_7a.
4323     replace ( $Q \wedge P$ ) with ( $P \wedge Q$ ) in n4_7a.
4324     replace ( $P \vee \neg Q \leftrightarrow P \wedge Q \vee \neg Q$ ) with
4325           ( $P \wedge Q \vee \neg Q \leftrightarrow P \vee \neg Q$ ) in n4_7a.
4326     apply n4_7a.
4327     apply EqBi.
4328     specialize n4_21 with ( $P \wedge Q \vee \neg Q$ ) ( $P \vee \neg Q$ ).
4329     intros n4_21a. (*Not cited*)
4330     apply n4_21a.
4331     apply EqBi.
4332     specialize n4_3 with P Q.
4333     intros n4_3a. (*Not cited*)
4334     apply n4_3a.
4335     apply EqBi.
4336     specialize n4_31 with P ( $\neg Q$ ).
4337     intros n4_31a. (*Not cited*)
4338     apply n4_31a.
4339     apply EqBi.
4340     specialize n4_31 with ( $Q \wedge P$ ) ( $\neg Q$ ).
4341     intros n4_31b. (*Not cited*)
4342     apply n4_31b.
4343     rewrite <- Impl1_01.
4344     reflexivity.
4345     rewrite <- Impl1_01.
4346     reflexivity.
4347     Qed.
4348

```



```

4349 Theorem n5_63 :  $\forall P Q : \text{Prop},$ 
4350    $(P \vee Q) \leftrightarrow (P \vee (\neg P \wedge Q)).$ 
4351 Proof. intros P Q.
4352 specialize n5_62 with Q ( $\neg P$ ).
4353 intros n5_62a.
4354 replace ( $\neg \neg P$ ) with P in n5_62a.
4355 replace  $(Q \vee P)$  with  $(P \vee Q)$  in n5_62a.
4356 replace  $((Q \wedge \neg P) \vee P)$  with  $(P \vee (Q \wedge \neg P))$  in n5_62a.
4357 replace  $(P \vee Q \wedge \neg P \leftrightarrow P \vee Q)$  with
4358    $(P \vee Q \leftrightarrow P \vee Q \wedge \neg P)$  in n5_62a.
4359 replace  $(Q \wedge \neg P)$  with  $(\neg P \wedge Q)$  in n5_62a.
4360 apply n5_62a.
4361 apply EqBi.
4362 specialize n4_3 with ( $\neg P$ ) Q.
4363 intros n4_3a.
4364 apply n4_3a. (*Not cited*)
4365 apply EqBi.
4366 specialize n4_21 with  $(P \vee Q) (P \vee (Q \wedge \neg P)).$ 
4367 intros n4_21a. (*Not cited*)
4368 apply n4_21a.
4369 apply EqBi.
4370 specialize n4_31 with P  $(Q \wedge \neg P).$ 
4371 intros n4_31a. (*Not cited*)
4372 apply n4_31a.
4373 apply EqBi.
4374 specialize n4_31 with P Q.
4375 intros n4_31b. (*Not cited*)
4376 apply n4_31b.
4377 apply EqBi.
4378 specialize n4_13 with P.
4379 intros n4_13a. (*Not cited*)
4380 apply n4_13a.
4381 Qed.
4382
4383 Theorem n5_7 :  $\forall P Q R : \text{Prop},$ 
4384    $((P \vee R) \leftrightarrow (Q \vee R)) \leftrightarrow (R \vee (P \leftrightarrow Q)).$ 
4385 Proof. intros P Q R.
4386 specialize n4_74 with R P.
4387 intros n4_74a.
4388 specialize n4_74 with R Q.
4389 intros n4_74b. (*Greg's suggestion*)
4390 Conj n4_74a n4_74b.

```

```

4391 split.
4392 apply n4_74a.
4393 apply n4_74b.
4394 specialize Comp3_43 with
4395   ( $\neg R$ ) ( $P \leftrightarrow R \vee P$ ) ( $Q \leftrightarrow R \vee Q$ ).
4396 intros Comp3_43a.
4397 MP Comp3_43a H.
4398 specialize n4_22 with P ( $R \vee P$ ) ( $R \vee Q$ ).
4399 intros n4_22a.
4400 specialize n4_22 with P ( $R \vee Q$ ) Q.
4401 intros n4_22b.
4402 specialize Exp3_3 with ( $P \leftrightarrow (R \vee Q)$ )
4403   ( $(R \vee Q) \leftrightarrow Q$ ) ( $P \leftrightarrow Q$ ).
4404 intros Exp3_3a.
4405 MP Exp3_3a n4_22b.
4406 Syll n4_22a Exp3_3a Sa.
4407 specialize Imp3_31 with ( $(P \leftrightarrow (R \vee P)) \wedge$ 
4408   ( $(R \vee P) \leftrightarrow (R \vee Q)$ )) ( $(R \vee Q) \leftrightarrow Q$ ) ( $P \leftrightarrow Q$ ).
4409 intros Imp3_31a.
4410 MP Imp3_31a Sa.
4411 replace ( $((P \leftrightarrow (R \vee P)) \wedge ((R \vee P) \leftrightarrow$ 
4412   ( $R \vee Q$ )))  $\wedge ((R \vee Q) \leftrightarrow Q)$ ) with
4413   ( $(P \leftrightarrow (R \vee P)) \wedge ((R \vee P) \leftrightarrow$ 
4414   ( $R \vee Q$ ))  $\wedge ((R \vee Q) \leftrightarrow Q)$ ) in Imp3_31a.
4415 replace ( $(R \vee P \leftrightarrow R \vee Q) \wedge (R \vee Q \leftrightarrow Q)$ ) with
4416   ( $(R \vee Q \leftrightarrow Q) \wedge (R \vee P \leftrightarrow R \vee Q)$ ) in Imp3_31a.
4417 replace ( $(P \leftrightarrow (R \vee P)) \wedge$ 
4418   ( $(R \vee Q \leftrightarrow Q) \wedge (R \vee P \leftrightarrow R \vee Q)$ )) with
4419   ( $((P \leftrightarrow (R \vee P)) \wedge (R \vee Q \leftrightarrow Q)) \wedge$ 
4420   ( $R \vee P \leftrightarrow R \vee Q$ )) in Imp3_31a.
4421 specialize Exp3_3 with
4422   ( $(P \leftrightarrow (R \vee P)) \wedge (R \vee Q \leftrightarrow Q)$ )
4423   ( $R \vee P \leftrightarrow R \vee Q$ ) ( $P \leftrightarrow Q$ ).
4424 intros Exp3_3b.
4425 MP Exp3_3b Imp3_31a.
4426 replace ( $Q \leftrightarrow R \vee Q$ ) with ( $R \vee Q \leftrightarrow Q$ ) in Comp3_43a.
4427 Syll Comp3_43a Exp3_3b Sb.
4428 replace ( $R \vee P$ ) with ( $P \vee R$ ) in Sb.
4429 replace ( $R \vee Q$ ) with ( $Q \vee R$ ) in Sb.
4430 specialize Imp3_31 with ( $\neg R$ ) ( $P \vee R \leftrightarrow Q \vee R$ ) ( $P \leftrightarrow Q$ ).
4431 intros Imp3_31b.
4432 MP Imp3_31b Sb.

```

```

4433 replace ( $\neg R \wedge (P \vee R \leftrightarrow Q \vee R)$ ) with
4434 ( $(P \vee R \leftrightarrow Q \vee R) \wedge \neg R$ ) in Imp3_31b.
4435 specialize Exp3_3 with
4436 ( $P \vee R \leftrightarrow Q \vee R$ ) ( $\neg R$ ) ( $P \leftrightarrow Q$ ).
4437 intros Exp3_3c.
4438 MP Exp3_3c Imp3_31b.
4439 replace ( $\neg R \rightarrow (P \leftrightarrow Q)$ ) with
4440 ( $\neg \neg R \vee (P \leftrightarrow Q)$ ) in Exp3_3c.
4441 replace ( $\neg \neg R$ ) with R in Exp3_3c.
4442 specialize Add1_3 with P R.
4443 intros Add1_3a.
4444 specialize Add1_3 with Q R.
4445 intros Add1_3b.
4446 Conj Add1_3a Add1_3b.
4447 split.
4448 apply Add1_3a.
4449 apply Add1_3b.
4450 specialize Comp3_43 with (R) ( $P \vee R$ ) ( $Q \vee R$ ).
4451 intros Comp3_43b.
4452 MP Comp3_43b H0.
4453 specialize n5_1 with ( $P \vee R$ ) ( $Q \vee R$ ).
4454 intros n5_1a.
4455 Syll Comp3_43b n5_1a Sc.
4456 specialize n4_37 with P Q R.
4457 intros n4_37a.
4458 Conj Sc n4_37a.
4459 split.
4460 apply Sc.
4461 apply n4_37a.
4462 specialize n4_77 with ( $P \vee R \leftrightarrow Q \vee R$ )
4463 R ( $P \leftrightarrow Q$ ).
4464 intros n4_77a.
4465 rewrite Equiv4_01 in n4_77a.
4466 specialize Simp3_26 with
4467 ( $(R \rightarrow P \vee R \leftrightarrow Q \vee R) \wedge$ 
4468 ( $P \leftrightarrow Q \rightarrow P \vee R \leftrightarrow Q \vee R$ )
4469  $\rightarrow R \vee (P \leftrightarrow Q) \rightarrow P \vee R \leftrightarrow Q \vee R$ )
4470 ( $(R \vee (P \leftrightarrow Q) \rightarrow P \vee R \leftrightarrow Q \vee R)$ 
4471  $\rightarrow (R \rightarrow P \vee R \leftrightarrow Q \vee R) \wedge$ 
4472 ( $P \leftrightarrow Q \rightarrow P \vee R \leftrightarrow Q \vee R$ )).
4473 intros Simp3_26a.
4474 MP Simp3_26 n4_77a.

```

```

4475 MP Simp3_26a H1.
4476 clear n4_77a. clear H1. clear n4_37a. clear Sa.
4477 clear n5_1a. clear Comp3_43b. clear H0.
4478 clear Add1_3a. clear Add1_3b. clear H. clear Imp3_31b.
4479 clear n4_74a. clear n4_74b. clear Comp3_43a.
4480 clear Imp3_31a. clear n4_22a. clear n4_22b.
4481 clear Exp3_3a. clear Exp3_3b. clear Sb. clear Sc.
4482 Conj Exp3_3c Simp3_26a.
4483 split.
4484 apply Exp3_3c.
4485 apply Simp3_26a.
4486 Equiv H.
4487 apply H.
4488 apply Equiv4_01.
4489 apply EqBi.
4490 apply n4_13. (*With R*)
4491 rewrite <- Impl1_01. (*With (¬R) (P↔Q)*)
4492 reflexivity.
4493 apply EqBi.
4494 apply n4_3. (*With (R ∨ Q ↔ R ∨ P) (¬R)*)
4495 apply EqBi.
4496 apply n4_31. (*With P R*)
4497 apply EqBi.
4498 apply n4_31. (*With Q R*)
4499 apply EqBi.
4500 apply n4_21. (*With (P ∨ R) (Q ∨ R)*)
4501 apply EqBi.
4502 apply n4_32. (*With (P ↔ R ∨ P) (R ∨ Q ↔ Q) (R ∨ P ↔ R ∨ Q)*)
4503 apply EqBi.
4504 apply n4_3. (*With (R ∨ Q ↔ Q) (R ∨ P ↔ R ∨ Q)*)
4505 apply EqBi.
4506 apply n4_21. (*To commute the biconditional.*)
4507 apply n4_32. (*With (P ↔ R ∨ P) (R ∨ P ↔ R ∨ Q) (R ∨ Q ↔ Q)*)
4508 Qed.
4509
4510 Theorem n5_71 : ∀ P Q R : Prop,
4511   (Q → ¬R) → ((P ∨ Q) ∧ R) ↔ (P ∧ R)).
4512 Proof. intros P Q R.
4513 specialize n4_62 with Q R.
4514 intros n4_62a.
4515 specialize n4_51 with Q R.
4516 intros n4_51a.

```

```

4517 specialize n4_21 with ( $\neg(Q \wedge R)$ ) ( $\neg Q \vee \neg R$ ).
4518 intros n4_21a.
4519 rewrite Equiv4_01 in n4_21a.
4520 specialize Simp3_26 with
4521   ( $((\neg(Q \wedge R) \leftrightarrow (\neg Q \vee \neg R)) \rightarrow ((\neg Q \vee \neg R) \leftrightarrow \neg(Q \wedge R)))$ )
4522   ( $((\neg Q \vee \neg R) \leftrightarrow \neg(Q \wedge R)) \rightarrow (\neg(Q \wedge R) \leftrightarrow (\neg Q \vee \neg R)))$ ).
4523 intros Simp3_26a.
4524 MP Simp3_26a n4_21a.
4525 MP Simp3_26a n4_51a.
4526 clear n4_21a. clear n4_51a.
4527 Conj n4_62a Simp3_26a.
4528 split.
4529 apply n4_62a.
4530 apply Simp3_26a.
4531 specialize n4_22 with
4532   ( $(Q \rightarrow \neg R)$  ( $\neg Q \vee \neg R$ ) ( $\neg(Q \wedge R)$ )).
4533 intros n4_22a.
4534 MP n4_22a H.
4535 replace ( $(Q \rightarrow \neg R) \leftrightarrow \neg(Q \wedge R)$ ) with
4536   ( $((Q \rightarrow \neg R) \rightarrow \neg(Q \wedge R))$ 
4537     $\wedge$ 
4538    ( $\neg(Q \wedge R) \rightarrow (Q \rightarrow \neg R)$ )) in n4_22a.
4539 specialize Simp3_26 with
4540   ( $(Q \rightarrow \neg R) \rightarrow \neg(Q \wedge R)$ ) ( $\neg(Q \wedge R) \rightarrow (Q \rightarrow \neg R)$ ).
4541 intros Simp3_26b.
4542 MP Simp3_26b n4_22a.
4543 specialize n4_74 with ( $(Q \wedge R)$ ) ( $(P \wedge R)$ ).
4544 intros n4_74a.
4545 Syll Simp3_26a n4_74a Sa.
4546 replace ( $(P \wedge R) \vee (Q \wedge R)$ ) with
4547   ( $(Q \wedge R) \vee (P \wedge R)$ ) in Sa.
4548 replace ( $(Q \wedge R) \vee (P \wedge R)$ ) with ( $R \wedge (P \vee Q)$ ) in Sa.
4549 replace ( $R \wedge (P \vee Q)$ ) with ( $(P \vee Q) \wedge R$ ) in Sa.
4550 replace ( $(P \wedge R) \leftrightarrow ((P \vee Q) \wedge R)$ ) with
4551   ( $((P \vee Q) \wedge R) \leftrightarrow (P \wedge R)$ ) in Sa.
4552 apply Sa.
4553 apply EqBi.
4554 specialize n4_21 with ( $(P \vee Q) \wedge R$ ) ( $(P \wedge R)$ ).
4555 intros n4_21a. (*Not cited*)
4556 apply n4_21a.
4557 apply EqBi.
4558 specialize n4_3 with ( $P \vee Q$ ) R.

```

```

4559   intros n4_3a.
4560   apply n4_3a. (*Not cited*)
4561   apply EqBi.
4562   specialize n4_4 with R P Q.
4563   intros n4_4a.
4564   replace ((Q $\wedge$ R) $\vee$ (P $\wedge$ R)) with ((P $\wedge$ R) $\vee$ (Q $\wedge$ R)).
4565   replace (Q  $\wedge$  R) with (R  $\wedge$  Q).
4566   replace (P  $\wedge$  R) with (R  $\wedge$  P).
4567   apply n4_4a. (*Not cited*)
4568   apply EqBi.
4569   specialize n4_3 with R P.
4570   intros n4_3a.
4571   apply n4_3a.
4572   apply EqBi.
4573   specialize n4_3 with R Q.
4574   intros n4_3b.
4575   apply n4_3b.
4576   apply EqBi.
4577   specialize n4_31 with (P $\wedge$ R) (Q $\wedge$ R).
4578   intros n4_31a. (*Not cited*)
4579   apply n4_31a.
4580   apply EqBi.
4581   specialize n4_31 with (Q $\wedge$ R) (P $\wedge$ R).
4582   intros n4_31b. (*Not cited*)
4583   apply n4_31b.
4584   apply Equiv4_01.
4585   Qed.
4586
4587   Theorem n5_74 :  $\forall$  P Q R : Prop,
4588     (P  $\rightarrow$  (Q  $\leftrightarrow$  R))  $\leftrightarrow$  ((P  $\rightarrow$  Q)  $\leftrightarrow$  (P  $\rightarrow$  R)).
4589   Proof. intros P Q R.
4590   specialize n5_41 with P Q R.
4591   intros n5_41a.
4592   specialize n5_41 with P R Q.
4593   intros n5_41b.
4594   Conj n5_41a n5_41b.
4595   split.
4596   apply n5_41a.
4597   apply n5_41b.
4598   specialize n4_38 with
4599     ((P $\rightarrow$ Q) $\rightarrow$ (P $\rightarrow$ R)) ((P $\rightarrow$ R) $\rightarrow$ (P $\rightarrow$ Q))
4600     (P $\rightarrow$ Q $\rightarrow$ R) (P $\rightarrow$ R $\rightarrow$ Q).

```

```

4601   intros n4_38a.
4602   MP n4_38a H.
4603   replace (( $(P \rightarrow Q) \rightarrow (P \rightarrow R)$ )  $\wedge$  ( $(P \rightarrow R) \rightarrow (P \rightarrow Q)$ ))
4604     with (( $(P \rightarrow Q) \leftrightarrow (P \rightarrow R)$ ) in n4_38a.
4605   specialize n4_76 with P ( $Q \rightarrow R$ ) ( $R \rightarrow Q$ ).
4606   intros n4_76a.
4607   replace (( $Q \rightarrow R$ )  $\wedge$  ( $R \rightarrow Q$ )) with ( $Q \leftrightarrow R$ ) in n4_76a.
4608   replace (( $P \rightarrow Q \rightarrow R$ )  $\wedge$  ( $P \rightarrow R \rightarrow Q$ )) with
4609     ( $P \rightarrow (Q \leftrightarrow R)$ ) in n4_38a.
4610   replace (( $(P \rightarrow Q) \leftrightarrow (P \rightarrow R)$ )  $\leftrightarrow$  ( $P \rightarrow Q \leftrightarrow R$ )) with
4611     (( $P \rightarrow (Q \leftrightarrow R)$ )  $\leftrightarrow$  ( $(P \rightarrow Q) \leftrightarrow (P \rightarrow R)$ )) in n4_38a.
4612   apply n4_38a.
4613   apply EqBi.
4614   specialize n4_21 with ( $P \rightarrow Q \leftrightarrow R$ )
4615     (( $P \rightarrow Q$ )  $\leftrightarrow$  ( $P \rightarrow R$ )).
4616   intros n4_21a. (*Not cited*)
4617   apply n4_21a.
4618   replace ( $P \rightarrow Q \leftrightarrow R$ ) with (( $P \rightarrow Q \rightarrow R$ )  $\wedge$  ( $P \rightarrow R \rightarrow Q$ )).
4619   reflexivity.
4620   apply EqBi.
4621   apply n4_76a.
4622   apply Equiv4_01.
4623   apply Equiv4_01.
4624   Qed.
4625
4626 Theorem n5_75 :  $\forall P Q R : \text{Prop}$ ,
4627   (( $R \rightarrow \neg Q$ )  $\wedge$  ( $P \leftrightarrow Q \vee R$ ))  $\rightarrow$  (( $P \wedge \neg Q$ )  $\leftrightarrow R$ ).
4628 Proof. intros P Q R.
4629   specialize n5_6 with P Q R.
4630   intros n5_6a.
4631   replace (( $P \wedge \neg Q \rightarrow R$ )  $\leftrightarrow$  ( $P \rightarrow Q \vee R$ )) with
4632     (( $(P \wedge \neg Q \rightarrow R) \rightarrow (P \rightarrow Q \vee R)$ )  $\wedge$ 
4633       ( $(P \rightarrow Q \vee R) \rightarrow (P \wedge \neg Q \rightarrow R)$ )) in n5_6a.
4634   specialize Simp3_27 with
4635     (( $P \wedge \neg Q \rightarrow R$ )  $\rightarrow$  ( $P \rightarrow Q \vee R$ ))
4636     (( $P \rightarrow Q \vee R$ )  $\rightarrow$  ( $P \wedge \neg Q \rightarrow R$ )).
4637   intros Simp3_27a.
4638   MP Simp3_27a n5_6a.
4639   specialize Simp3_26 with
4640     ( $P \rightarrow (Q \vee R)$ ) (( $Q \vee R$ )  $\rightarrow P$ ).
4641   intros Simp3_26a.
4642   replace (( $P \rightarrow (Q \vee R)$ )  $\wedge$  (( $Q \vee R$ )  $\rightarrow P$ )) with

```

```

4643      (P↔(Q∨R)) in Simp3_26a.
4644 Syll Simp3_26a Simp3_27a Sa.
4645 specialize Simp3_27 with
4646   (R→¬Q) (P↔(Q∨R)).
4647 intros Simp3_27b.
4648 Syll Simp3_27b Sa Sb.
4649 specialize Simp3_27 with
4650   (P→(Q∨R)) ((Q∨R)→P).
4651 intros Simp3_27c.
4652 replace ((P→(Q∨R))∧((Q∨R)→P)) with
4653   (P↔(Q∨R)) in Simp3_27c.
4654 Syll Simp3_27b Simp3_27c Sc.
4655 specialize n4_77 with P Q R.
4656 intros n4_77a.
4657 replace (Q∨R→P) with ((Q→P)∧(R→P)) in Sc.
4658 specialize Simp3_27 with (Q→P) (R→P).
4659 intros Simp3_27d.
4660 Syll Sa Simp3_27d Sd.
4661 specialize Simp3_26 with (R→¬Q) (P↔(Q∨R)).
4662 intros Simp3_26b.
4663 Conj Sd Simp3_26b.
4664 split.
4665 apply Sd.
4666 apply Simp3_26b.
4667 specialize Comp3_43 with
4668   ((R→¬Q)∧(P↔(Q∨R))) (R→P) (R→¬Q).
4669 intros Comp3_43a.
4670 MP Comp3_43a H.
4671 specialize Comp3_43 with R P (¬Q).
4672 intros Comp3_43b.
4673 Syll Comp3_43a Comp3_43b Se.
4674 clear n5_6a. clear Simp3_27a.
4675   clear Simp3_27c. clear Simp3_27d.
4676   clear Simp3_26a. clear Comp3_43b.
4677   clear Simp3_26b. clear Comp3_43a.
4678   clear Sa. clear Sc. clear Sd. clear H.
4679   clear n4_77a. clear Simp3_27b.
4680 Conj Sb Se.
4681 split.
4682 apply Sb.
4683 apply Se.
4684 specialize Comp3_43 with

```



```

4685       $((R \rightarrow \neg Q) \wedge (P \leftrightarrow Q \vee R))$ 
4686       $(P \wedge \neg Q \rightarrow R) \quad (R \rightarrow P \wedge \neg Q).$ 
4687      intros Comp3_43c.
4688      MP Comp3_43c H.
4689      replace  $((P \wedge \neg Q \rightarrow R) \wedge (R \rightarrow P \wedge \neg Q))$  with
4690           $(P \wedge \neg Q \leftrightarrow R)$  in Comp3_43c.
4691      apply Comp3_43c.
4692      apply Equiv4_01.
4693      apply EqBi.
4694      apply n4_77a.
4695      apply Equiv4_01.
4696      apply Equiv4_01.
4697      apply Equiv4_01.
4698      Qed.
4699
4700      End No5.

```