

Principia Mathematica's Propositional Logic in *Coq*

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Abstract

This file contains the *Coq* code for the *Principia* Rewrite project's encoding of the propositional logic given in *1 – *5. The Github repository with this *Coq* file is here: <https://github.com/LogicalAtomist/principia>. To receive updates about the project, visit the *Principia Rewrite* project page: <https://www.principiarewrite.com/>. You can also follow the *Principia* Rewrite project on Twitter: <https://twitter.com/thePMrewrite>.

```
1  Require Import Unicode.Utf8.
2
3  Module No1.
4  Import Unicode.Utf8.
5      (*We first give the axioms of Principia
6      for the propositional calculus in *1.*)
7
8  Axiom Impl1_01 :  $\forall$  P Q : Prop,
9      (P  $\rightarrow$  Q) = ( $\sim$ P  $\vee$  Q).
10      (*This is a definition in Principia: there  $\rightarrow$  is a
11      defined sign and  $\vee$ ,  $\sim$  are primitive ones. So
12      we will use this axiom to switch between
13      disjunction and implication.*)
14
15  Axiom MP1_1 :  $\forall$  P Q : Prop,
16      (P  $\rightarrow$  Q)  $\rightarrow$  P  $\rightarrow$  Q. (*Modus ponens*)
17
18      (*1.11 omitted: it is MP for propositions
19      containing variables. Likewise, omitted
20      the well-formedness rules 1.7, 1.71, 1.72*)
21
22  Axiom Taut1_2 :  $\forall$  P : Prop,
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23   P  $\vee$  P  $\rightarrow$  P. (*Tautology*)
24
25   Axiom Add1_3 :  $\forall$  P Q : Prop,
26     Q  $\rightarrow$  P  $\vee$  Q. (*Addition*)
27
28   Axiom Perm1_4 :  $\forall$  P Q : Prop,
29     P  $\vee$  Q  $\rightarrow$  Q  $\vee$  P. (*Permutation*)
30
31   Axiom Assoc1_5 :  $\forall$  P Q R : Prop,
32     P  $\vee$  (Q  $\vee$  R)  $\rightarrow$  Q  $\vee$  (P  $\vee$  R). (*Association*)
33
34   Axiom Sum1_6 :  $\forall$  P Q R : Prop,
35     (Q  $\rightarrow$  R)  $\rightarrow$  (P  $\vee$  Q  $\rightarrow$  P  $\vee$  R). (*Summation*)
36
37   (*These are all the propositional axioms of Principia.*)
38
39   End No1.
40
41   Module No2.
42   Import No1.
43
44   (*We proceed to the deductions of Principia.*)
45
46   Theorem Abs2_01 :  $\forall$  P : Prop,
47     (P  $\rightarrow$   $\sim$ P)  $\rightarrow$   $\sim$ P.
48   Proof. intros P.
49     specialize Taut1_2 with ( $\sim$ P).
50     replace ( $\sim$ P  $\vee$   $\sim$ P) with (P  $\rightarrow$   $\sim$ P).
51     apply MP1_1.
52     apply Impl1_01.
53   Qed.
54
55   Theorem Simp2_02 :  $\forall$  P Q : Prop,
56     Q  $\rightarrow$  (P  $\rightarrow$  Q).
57   Proof. intros P Q.
58     specialize Add1_3 with ( $\sim$ P) Q.
59     replace ( $\sim$ P  $\vee$  Q) with (P  $\rightarrow$  Q).
60     apply (MP1_1 Q (P  $\rightarrow$  Q)).
61     apply Impl1_01.
62   Qed.
63
64   Theorem n2_03 :  $\forall$  P Q : Prop,

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65   (P → ~Q) → (Q → ~P).
66 Proof. intros P Q.
67   specialize Perm1_4 with (~P) (~Q).
68   replace (~P ∨ ~Q) with (P → ~Q).
69   replace (~Q ∨ ~P) with (Q → ~P).
70   apply (MP1_1 (P → ~Q) (Q → ~P)).
71   apply Impl1_01.
72   apply Impl1_01.
73 Qed.
74
75 Theorem Comm2_04 : ∀ P Q R : Prop,
76   (P → (Q → R)) → (Q → (P → R)).
77 Proof. intros P Q R.
78   specialize Assoc1_5 with (~P) (~Q) R.
79   replace (~Q ∨ R) with (Q → R).
80   replace (~P ∨ (Q → R)) with (P → (Q → R)).
81   replace (~P ∨ R) with (P → R).
82   replace (~Q ∨ (P → R)) with (Q → (P → R)).
83   apply (MP1_1 (P → Q → R) (Q → P → R)).
84   apply Impl1_01.
85   apply Impl1_01.
86   apply Impl1_01.
87   apply Impl1_01.
88 Qed.
89
90 Theorem Syll2_05 : ∀ P Q R : Prop,
91   (Q → R) → ((P → Q) → (P → R)).
92 Proof. intros P Q R.
93   specialize Sum1_6 with (~P) Q R.
94   replace (~P ∨ Q) with (P → Q).
95   replace (~P ∨ R) with (P → R).
96   apply (MP1_1 (Q → R) ((P → Q) → (P → R))).
97   apply Impl1_01.
98   apply Impl1_01.
99 Qed.
100
101 Theorem Syll2_06 : ∀ P Q R : Prop,
102   (P → Q) → ((Q → R) → (P → R)).
103 Proof. intros P Q R.
104   specialize Comm2_04 with (Q → R) (P → Q) (P → R).
105   intros Comm2_04.
106   specialize Syll2_05 with P Q R.

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107   intros Syll2_05.
108   specialize MP1_1 with ((Q → R) → (P → Q) → P → R)
109       ((P → Q) → ((Q → R) → (P → R))).
110   intros MP1_1.
111   apply MP1_1.
112   apply Comm2_04.
113   apply Syll2_05.
114   Qed.
115
116   Theorem n2_07 : ∀ P : Prop,
117       P → (P ∨ P).
118   Proof. intros P.
119       specialize Add1_3 with P P.
120       apply MP1_1.
121   Qed.
122
123   Theorem n2_08 : ∀ P : Prop,
124       P → P.
125   Proof. intros P.
126       specialize Syll2_05 with P (P ∨ P) P.
127       intros Syll2_05.
128       specialize Taut1_2 with P.
129       intros Taut1_2.
130       specialize MP1_1 with ((P ∨ P) → P) (P → P).
131       intros MP1_1.
132       apply Syll2_05.
133       apply Taut1_2.
134       apply n2_07.
135   Qed.
136
137   Theorem n2_1 : ∀ P : Prop,
138       (¬P) ∨ P.
139   Proof. intros P.
140       specialize n2_08 with P.
141       replace (¬P ∨ P) with (P → P).
142       apply MP1_1.
143       apply Impl1_01.
144   Qed.
145
146   Theorem n2_11 : ∀ P : Prop,
147       P ∨ ¬P.
148   Proof. intros P.

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149   specialize Perm1_4 with (~P) P.
150   intros Perm1_4.
151   specialize n2_1 with P.
152   intros n2_1.
153   apply Perm1_4.
154   apply n2_1.
155   Qed.
156
157   Theorem n2_12 :  $\forall$  P : Prop,
158     P  $\rightarrow$   $\sim\sim$ P.
159   Proof. intros P.
160     specialize n2_11 with (~P).
161     intros n2_11.
162     rewrite Impl1_01.
163     apply n2_11.
164     Qed.
165
166   Theorem n2_13 :  $\forall$  P : Prop,
167     P  $\vee$   $\sim\sim\sim$ P.
168   Proof. intros P.
169     specialize Sum1_6 with P (~P) ( $\sim\sim\sim$ P).
170     intros Sum1_6.
171     specialize n2_12 with (~P).
172     intros n2_12.
173     apply Sum1_6.
174     apply n2_12.
175     specialize n2_11 with P.
176     intros n2_11.
177     apply n2_11.
178     Qed.
179
180   Theorem n2_14 :  $\forall$  P : Prop,
181      $\sim\sim$ P  $\rightarrow$  P.
182   Proof. intros P.
183     specialize Perm1_4 with P ( $\sim\sim\sim$ P).
184     intros Perm1_4.
185     specialize n2_13 with P.
186     intros n2_13.
187     rewrite Impl1_01.
188     apply Perm1_4.
189     apply n2_13.
190     Qed.

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191
192 Theorem Trans2_15 :  $\forall P Q : \text{Prop},$ 
193    $(\sim P \rightarrow Q) \rightarrow (\sim Q \rightarrow P).$ 
194 Proof. intros P Q.
195   specialize Syll2_05 with  $(\sim P) Q (\sim\sim Q).$ 
196   intros Syll2_05a.
197   specialize n2_12 with Q.
198   intros n2_12.
199   specialize n2_03 with  $(\sim P) (\sim Q).$ 
200   intros n2_03.
201   specialize Syll2_05 with  $(\sim Q) (\sim\sim P) P.$ 
202   intros Syll2_05b.
203   specialize Syll2_05 with  $(\sim P \rightarrow Q) (\sim P \rightarrow \sim\sim Q) (\sim Q \rightarrow \sim\sim P).$ 
204   intros Syll2_05c.
205   specialize Syll2_05 with  $(\sim P \rightarrow Q) (\sim Q \rightarrow \sim\sim P) (\sim Q \rightarrow P).$ 
206   intros Syll2_05d.
207   apply Syll2_05d.
208   apply Syll2_05b.
209   specialize n2_14 with P.
210   intros n2_14.
211   apply n2_14.
212   apply Syll2_05c.
213   apply n2_03.
214   apply Syll2_05a.
215   apply n2_12.
216 Qed.
217
218 Ltac Syll H1 H2 S :=
219   let S := fresh S in match goal with
220     | [ H1 : ?P  $\rightarrow$  ?Q, H2 : ?Q  $\rightarrow$  ?R |- _ ] =>
221       assert (S : P  $\rightarrow$  R) by (intros p; apply (H2 (H1 p)))
222   end.
223
224 Ltac MP H1 H2 :=
225   match goal with
226     | [ H1 : ?P  $\rightarrow$  ?Q, H2 : ?P |- _ ] => specialize (H1 H2)
227   end.
228
229 Theorem Trans2_16 :  $\forall P Q : \text{Prop},$ 
230    $(P \rightarrow Q) \rightarrow (\sim Q \rightarrow \sim P).$ 
231 Proof. intros P Q.
232   specialize n2_12 with Q.

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233   intros n2_12a.
234   specialize Syll2_05 with P Q (~~Q).
235   intros Syll2_05a.
236   specialize n2_03 with P (~Q).
237   intros n2_03a.
238   MP n2_12a Syll2_05a.
239   Syll Syll2_05a n2_03a S.
240   apply S.
241   Qed.
242
243   Theorem Trans2_17 :  $\forall P Q : \text{Prop},$ 
244      $(\sim Q \rightarrow \sim P) \rightarrow (P \rightarrow Q).$ 
245   Proof. intros P Q.
246     specialize n2_03 with (~Q) P.
247     intros n2_03a.
248     specialize n2_14 with Q.
249     intros n2_14a.
250     specialize Syll2_05 with P (~~Q) Q.
251     intros Syll2_05a.
252     MP n2_14a Syll2_05a.
253     Syll n2_03a Syll2_05a S.
254     apply S.
255     Qed.
256
257   Theorem n2_18 :  $\forall P : \text{Prop},$ 
258      $(\sim P \rightarrow P) \rightarrow P.$ 
259   Proof. intros P.
260     specialize n2_12 with P.
261     intro n2_12a.
262     specialize Syll2_05 with (~P) P (~~P).
263     intro Syll2_05a.
264     MP Syll2_05a n2_12.
265     specialize Abs2_01 with (~P).
266     intros Abs2_01a.
267     Syll Syll2_05a Abs2_01a Sa.
268     specialize n2_14 with P.
269     intros n2_14a.
270     Syll H n2_14a Sb.
271     apply Sb.
272     Qed.
273
274   Theorem n2_2 :  $\forall P Q : \text{Prop},$ 

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275   P → (P ∨ Q).
276 Proof. intros P Q.
277   specialize Add1_3 with Q P.
278   intros Add1_3a.
279   specialize Perm1_4 with Q P.
280   intros Perm1_4a.
281   Syll Add1_3a Perm1_4a S.
282   apply S.
283 Qed.
284
285 Theorem n2_21 : ∀ P Q : Prop,
286   ~P → (P → Q).
287 Proof. intros P Q.
288   specialize n2_2 with (~P) Q.
289   intros n2_2a.
290   specialize Impl1_01 with P Q.
291   intros Impl1_01a.
292   replace (~P∨Q) with (P→Q) in n2_2a.
293   apply n2_2a.
294 Qed.
295
296 Theorem n2_24 : ∀ P Q : Prop,
297   P → (~P → Q).
298 Proof. intros P Q.
299   specialize n2_21 with P Q.
300   intros n2_21a.
301   specialize Comm2_04 with (~P) P Q.
302   intros Comm2_04a.
303   apply Comm2_04a.
304   apply n2_21a.
305 Qed.
306
307 Theorem n2_25 : ∀ P Q : Prop,
308   P ∨ ((P ∨ Q) → Q).
309 Proof. intros P Q.
310   specialize n2_1 with (P ∨ Q).
311   intros n2_1a.
312   specialize Assoc1_5 with (~P∨Q)) P Q.
313   intros Assoc1_5a.
314   MP Assoc1_5a n2_1a.
315   replace (~P∨Q)∨Q with (P∨Q→Q) in Assoc1_5a.
316   apply Assoc1_5a.

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317   apply Impl1_01.
318 Qed.
319
320 Theorem n2_26 :  $\forall P Q : \text{Prop},$ 
321    $\sim P \vee ((P \rightarrow Q) \rightarrow Q).$ 
322 Proof. intros P Q.
323   specialize n2_25 with ( $\sim P$ ) Q.
324   intros n2_25a.
325   replace ( $\sim P \vee Q$ ) with ( $P \rightarrow Q$ ) in n2_25a.
326   apply n2_25a.
327   apply Impl1_01.
328 Qed.
329
330 Theorem n2_27 :  $\forall P Q : \text{Prop},$ 
331    $P \rightarrow ((P \rightarrow Q) \rightarrow Q).$ 
332 Proof. intros P Q.
333   specialize n2_26 with P Q.
334   intros n2_26a.
335   replace ( $\sim P \vee ((P \rightarrow Q) \rightarrow Q)$ ) with ( $P \rightarrow (P \rightarrow Q) \rightarrow Q$ ) in n2_26a.
336   apply n2_26a.
337   apply Impl1_01.
338 Qed.
339
340 Theorem n2_3 :  $\forall P Q R : \text{Prop},$ 
341    $(P \vee (Q \vee R)) \rightarrow (P \vee (R \vee Q)).$ 
342 Proof. intros P Q R.
343   specialize Perm1_4 with Q R.
344   intros Perm1_4a.
345   specialize Sum1_6 with P ( $Q \vee R$ ) ( $R \vee Q$ ).
346   intros Sum1_6a.
347   MP Sum1_6a Perm1_4a.
348   apply Sum1_6a.
349 Qed.
350
351 Theorem n2_31 :  $\forall P Q R : \text{Prop},$ 
352    $(P \vee (Q \vee R)) \rightarrow ((P \vee Q) \vee R).$ 
353 Proof. intros P Q R.
354   specialize n2_3 with P Q R.
355   intros n2_3a.
356   specialize Assoc1_5 with P R Q.
357   intros Assoc1_5a.
358   specialize Perm1_4 with R ( $P \vee Q$ ).

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359   intros Perm1_4a.
360   Syll Assoc1_5a Perm1_4a Sa.
361   Syll n2_3a Sa Sb.
362   apply Sb.
363   Qed.
364
365   Theorem n2_32 :  $\forall$  P Q R : Prop,
366      $((P \vee Q) \vee R) \rightarrow (P \vee (Q \vee R))$ .
367   Proof. intros P Q R.
368     specialize Perm1_4 with  $(P \vee Q)$  R.
369     intros Perm1_4a.
370     specialize Assoc1_5 with R P Q.
371     intros Assoc1_5a.
372     specialize n2_3 with P R Q.
373     intros n2_3a.
374     specialize Syll2_06 with  $((P \vee Q) \vee R)$   $(R \vee P \vee Q)$   $(P \vee R \vee Q)$ .
375     intros Syll2_06a.
376     MP Syll2_06a Perm1_4a.
377     MP Syll2_06a Assoc1_5a.
378     specialize Syll2_06 with  $((P \vee Q) \vee R)$   $(P \vee R \vee Q)$   $(P \vee Q \vee R)$ .
379     intros Syll2_06b.
380     MP Syll2_06b Syll2_06a.
381     MP Syll2_06b n2_3a.
382     apply Syll2_06b.
383     Qed.
384
385   Axiom Abb2_33 :  $\forall$  P Q R : Prop,
386      $(P \vee Q \vee R) = ((P \vee Q) \vee R)$ .
387     (*This definition makes the default left association.
388       The default in Coq is right association.*)
389
390   Theorem n2_36 :  $\forall$  P Q R : Prop,
391      $(Q \rightarrow R) \rightarrow ((P \vee Q) \rightarrow (R \vee P))$ .
392   Proof. intros P Q R.
393     specialize Perm1_4 with P R.
394     intros Perm1_4a.
395     specialize Syll2_05 with  $(P \vee Q)$   $(P \vee R)$   $(R \vee P)$ .
396     intros Syll2_05a.
397     MP Syll2_05a Perm1_4a.
398     specialize Sum1_6 with P Q R.
399     intros Sum1_6a.
400     Syll Sum1_6a Syll2_05a S.

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401     apply S.
402 Qed.
403
404 Theorem n2_37 :  $\forall P Q R : \text{Prop},$ 
405    $(Q \rightarrow R) \rightarrow ((Q \vee P) \rightarrow (P \vee R)).$ 
406 Proof. intros P Q R.
407   specialize Perm1_4 with Q P.
408   intros Perm1_4a.
409   specialize Syll2_06 with  $(Q \vee P)$   $(P \vee Q)$   $(P \vee R)$ .
410   intros Syll2_06a.
411   MP Syll2_06a Perm1_4a.
412   specialize Sum1_6 with P Q R.
413   intros Sum1_6a.
414   Syll Sum1_6a Syll2_06a S.
415   apply S.
416 Qed.
417
418 Theorem n2_38 :  $\forall P Q R : \text{Prop},$ 
419    $(Q \rightarrow R) \rightarrow ((Q \vee P) \rightarrow (R \vee P)).$ 
420 Proof. intros P Q R.
421   specialize Perm1_4 with P R.
422   intros Perm1_4a.
423   specialize Syll2_05 with  $(Q \vee P)$   $(P \vee R)$   $(R \vee P)$ .
424   intros Syll2_05a.
425   MP Syll2_05a Perm1_4a.
426   specialize Perm1_4 with Q P.
427   intros Perm1_4b.
428   specialize Syll2_06 with  $(Q \vee P)$   $(P \vee Q)$   $(P \vee R)$ .
429   intros Syll2_06a.
430   MP Syll2_06a Perm1_4b.
431   Syll Syll2_06a Syll2_05a H.
432   specialize Sum1_6 with P Q R.
433   intros Sum1_6a.
434   Syll Sum1_6a H S.
435   apply S.
436 Qed.
437
438 Theorem n2_4 :  $\forall P Q : \text{Prop},$ 
439    $(P \vee (P \vee Q)) \rightarrow (P \vee Q).$ 
440 Proof. intros P Q.
441   specialize n2_31 with P P Q.
442   intros n2_31a.

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443   specialize Taut1_2 with P.
444   intros Taut1_2a.
445   specialize n2_38 with Q (P $\vee$ P) P.
446   intros n2_38a.
447   MP n2_38a Taut1_2a.
448   Syll n2_31a n2_38a S.
449   apply S.
450 Qed.
451
452 Theorem n2_41 :  $\forall$  P Q : Prop,
453   (Q  $\vee$  (P  $\vee$  Q))  $\rightarrow$  (P  $\vee$  Q).
454 Proof. intros P Q.
455   specialize Assoc1_5 with Q P Q.
456   intros Assoc1_5a.
457   specialize Taut1_2 with Q.
458   intros Taut1_2a.
459   specialize Sum1_6 with P (Q $\vee$ Q) Q.
460   intros Sum1_6a.
461   MP Sum1_6a Taut1_2a.
462   Syll Assoc1_5a Sum1_6a S.
463   apply S.
464 Qed.
465
466 Theorem n2_42 :  $\forall$  P Q : Prop,
467   ( $\sim$ P  $\vee$  (P  $\rightarrow$  Q))  $\rightarrow$  (P  $\rightarrow$  Q).
468 Proof. intros P Q.
469   specialize n2_4 with ( $\sim$ P) Q.
470   intros n2_4a.
471   replace ( $\sim$ P $\vee$ Q) with (P $\rightarrow$ Q) in n2_4a.
472   apply n2_4a. apply Impl1_01.
473 Qed.
474
475 Theorem n2_43 :  $\forall$  P Q : Prop,
476   (P  $\rightarrow$  (P  $\rightarrow$  Q))  $\rightarrow$  (P  $\rightarrow$  Q).
477 Proof. intros P Q.
478   specialize n2_42 with P Q.
479   intros n2_42a.
480   replace ( $\sim$ P  $\vee$  (P $\rightarrow$ Q)) with (P $\rightarrow$ (P $\rightarrow$ Q)) in n2_42a.
481   apply n2_42a.
482   apply Impl1_01.
483 Qed.
484

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485 Theorem n2_45 :  $\forall$  P Q : Prop,
486    $\sim(P \vee Q) \rightarrow \sim P$ .
487 Proof. intros P Q.
488   specialize n2_2 with P Q.
489   intros n2_2a.
490   specialize Trans2_16 with P (P $\vee$ Q).
491   intros Trans2_16a.
492   MP n2_2 Trans2_16a.
493   apply Trans2_16a.
494 Qed.
495
496 Theorem n2_46 :  $\forall$  P Q : Prop,
497    $\sim(P \vee Q) \rightarrow \sim Q$ .
498 Proof. intros P Q.
499   specialize Add1_3 with P Q.
500   intros Add1_3a.
501   specialize Trans2_16 with Q (P $\vee$ Q).
502   intros Trans2_16a.
503   MP Add1_3a Trans2_16a.
504   apply Trans2_16a.
505 Qed.
506
507 Theorem n2_47 :  $\forall$  P Q : Prop,
508    $\sim(P \vee Q) \rightarrow (\sim P \vee Q)$ .
509 Proof. intros P Q.
510   specialize n2_45 with P Q.
511   intros n2_45a.
512   specialize n2_2 with ( $\sim P$ ) Q.
513   intros n2_2a.
514   Syll n2_45a n2_2a S.
515   apply S.
516 Qed.
517
518 Theorem n2_48 :  $\forall$  P Q : Prop,
519    $\sim(P \vee Q) \rightarrow (P \vee \sim Q)$ .
520 Proof. intros P Q.
521   specialize n2_46 with P Q.
522   intros n2_46a.
523   specialize Add1_3 with P ( $\sim Q$ ).
524   intros Add1_3a.
525   Syll n2_46a Add1_3a S.
526   apply S.

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527 Qed.
528
529 Theorem n2_49 :  $\forall P Q : \text{Prop},$ 
530    $\sim(P \vee Q) \rightarrow (\sim P \vee \sim Q).$ 
531 Proof. intros P Q.
532   specialize n2_45 with P Q.
533   intros n2_45a.
534   specialize n2_2 with ( $\sim P$ ) ( $\sim Q$ ).
535   intros n2_2a.
536   Syll n2_45a n2_2a S.
537   apply S.
538 Qed.
539
540 Theorem n2_5 :  $\forall P Q : \text{Prop},$ 
541    $\sim(P \rightarrow Q) \rightarrow (\sim P \rightarrow Q).$ 
542 Proof. intros P Q.
543   specialize n2_47 with ( $\sim P$ ) Q.
544   intros n2_47a.
545   replace ( $\sim P \vee Q$ ) with ( $P \rightarrow Q$ ) in n2_47a.
546   replace ( $\sim \sim P \vee Q$ ) with ( $\sim P \rightarrow Q$ ) in n2_47a.
547   apply n2_47a.
548   apply Impl1_01.
549   apply Impl1_01.
550 Qed.
551
552 Theorem n2_51 :  $\forall P Q : \text{Prop},$ 
553    $\sim(P \rightarrow Q) \rightarrow (P \rightarrow \sim Q).$ 
554 Proof. intros P Q.
555   specialize n2_48 with ( $\sim P$ ) Q.
556   intros n2_48a.
557   replace ( $\sim P \vee Q$ ) with ( $P \rightarrow Q$ ) in n2_48a.
558   replace ( $\sim P \vee \sim Q$ ) with ( $P \rightarrow \sim Q$ ) in n2_48a.
559   apply n2_48a.
560   apply Impl1_01.
561   apply Impl1_01.
562 Qed.
563
564 Theorem n2_52 :  $\forall P Q : \text{Prop},$ 
565    $\sim(P \rightarrow Q) \rightarrow (\sim P \rightarrow \sim Q).$ 
566 Proof. intros P Q.
567   specialize n2_49 with ( $\sim P$ ) Q.
568   intros n2_49a.

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569   replace ( $\sim P \vee Q$ ) with ( $P \rightarrow Q$ ) in n2_49a.
570   replace ( $\sim \sim P \vee \sim Q$ ) with ( $\sim P \rightarrow \sim Q$ ) in n2_49a.
571   apply n2_49a.
572   apply Impl1_01.
573   apply Impl1_01.
574   Qed.
575
576   Theorem n2_521 :  $\forall P Q : \text{Prop},$ 
577      $\sim(P \rightarrow Q) \rightarrow (Q \rightarrow P).$ 
578   Proof. intros P Q.
579     specialize n2_52 with P Q.
580     intros n2_52a.
581     specialize Trans2_17 with Q P.
582     intros Trans2_17a.
583     Syll n2_52a Trans2_17a S.
584     apply S.
585     Qed.
586
587   Theorem n2_53 :  $\forall P Q : \text{Prop},$ 
588      $(P \vee Q) \rightarrow (\sim P \rightarrow Q).$ 
589   Proof. intros P Q.
590     specialize n2_12 with P.
591     intros n2_12a.
592     specialize n2_38 with Q P ( $\sim \sim P$ ).
593     intros n2_38a.
594     MP n2_38a n2_12a.
595     replace ( $\sim \sim P \vee Q$ ) with ( $\sim P \rightarrow Q$ ) in n2_38a.
596     apply n2_38a.
597     apply Impl1_01.
598     Qed.
599
600   Theorem n2_54 :  $\forall P Q : \text{Prop},$ 
601      $(\sim P \rightarrow Q) \rightarrow (P \vee Q).$ 
602   Proof. intros P Q.
603     specialize n2_14 with P.
604     intros n2_14a.
605     specialize n2_38 with Q ( $\sim \sim P$ ) P.
606     intros n2_38a.
607     MP n2_38a n2_14a.
608     replace ( $\sim \sim P \vee Q$ ) with ( $\sim P \rightarrow Q$ ) in n2_38a.
609     apply n2_38a.
610     apply Impl1_01.

```

```

611 Qed.
612
613 Theorem n2_55 :  $\forall P Q : \text{Prop},$ 
614    $\sim P \rightarrow ((P \vee Q) \rightarrow Q).$ 
615 Proof. intros P Q.
616   specialize n2_53 with P Q.
617   intros n2_53a.
618   specialize Comm2_04 with  $(P \vee Q) (\sim P) Q.$ 
619   intros Comm2_04a.
620   MP n2_53a Comm2_04a.
621   apply Comm2_04a.
622 Qed.
623
624 Theorem n2_56 :  $\forall P Q : \text{Prop},$ 
625    $\sim Q \rightarrow ((P \vee Q) \rightarrow P).$ 
626 Proof. intros P Q.
627   specialize n2_55 with Q P.
628   intros n2_55a.
629   specialize Perm1_4 with P Q.
630   intros Perm1_4a.
631   specialize Syll2_06 with  $(P \vee Q) (Q \vee P) P.$ 
632   intros Syll2_06a.
633   MP Syll2_06a Perm1_4a.
634   Syll n2_55a Syll2_06a Sa.
635   apply Sa.
636 Qed.
637
638 Theorem n2_6 :  $\forall P Q : \text{Prop},$ 
639    $(\sim P \rightarrow Q) \rightarrow ((P \rightarrow Q) \rightarrow Q).$ 
640 Proof. intros P Q.
641   specialize n2_38 with Q  $(\sim P) Q.$ 
642   intros n2_38a.
643   specialize Taut1_2 with Q.
644   intros Taut1_2a.
645   specialize Syll2_05 with  $(\sim P \vee Q) (Q \vee Q) Q.$ 
646   intros Syll2_05a.
647   MP Syll2_05a Taut1_2a.
648   Syll n2_38a Syll2_05a S.
649   replace  $(\sim P \vee Q)$  with  $(P \rightarrow Q)$  in S.
650   apply S.
651   apply Impl1_01.
652 Qed.

```



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653
654 Theorem n2_61 :  $\forall P Q : \text{Prop},$ 
655    $(P \rightarrow Q) \rightarrow ((\sim P \rightarrow Q) \rightarrow Q).$ 
656 Proof. intros P Q.
657   specialize n2_6 with P Q.
658   intros n2_6a.
659   specialize Comm2_04 with  $(\sim P \rightarrow Q) (P \rightarrow Q) Q.$ 
660   intros Comm2_04a.
661   MP Comm2_04a n2_6a.
662   apply Comm2_04a.
663 Qed.
664
665 Theorem n2_62 :  $\forall P Q : \text{Prop},$ 
666    $(P \vee Q) \rightarrow ((P \rightarrow Q) \rightarrow Q).$ 
667 Proof. intros P Q.
668   specialize n2_53 with P Q.
669   intros n2_53a.
670   specialize n2_6 with P Q.
671   intros n2_6a.
672   Syll n2_53a n2_6a S.
673   apply S.
674 Qed.
675
676 Theorem n2_621 :  $\forall P Q : \text{Prop},$ 
677    $(P \rightarrow Q) \rightarrow ((P \vee Q) \rightarrow Q).$ 
678 Proof. intros P Q.
679   specialize n2_62 with P Q.
680   intros n2_62a.
681   specialize Comm2_04 with  $(P \vee Q) (P \rightarrow Q) Q.$ 
682   intros Comm2_04a.
683   MP Comm2_04a n2_62a.
684   apply Comm2_04a.
685 Qed.
686
687 Theorem n2_63 :  $\forall P Q : \text{Prop},$ 
688    $(P \vee Q) \rightarrow ((\sim P \vee Q) \rightarrow Q).$ 
689 Proof. intros P Q.
690   specialize n2_62 with P Q.
691   intros n2_62a.
692   replace  $(\sim P \vee Q)$  with  $(P \rightarrow Q).$ 
693   apply n2_62a.
694   apply Impl1_01.

```

```

695 Qed.
696
697 Theorem n2_64 :  $\forall P Q : \text{Prop},$ 
698    $(P \vee Q) \rightarrow ((P \vee \sim Q) \rightarrow P).$ 
699 Proof. intros P Q.
700   specialize n2_63 with Q P.
701   intros n2_63a.
702   specialize Perm1_4 with P Q.
703   intros Perm1_4a.
704   Syll n2_63a Perm1_4a Ha.
705   specialize Syll2_06 with  $(P \vee \sim Q) (\sim Q \vee P) P.$ 
706   intros Syll2_06a.
707   specialize Perm1_4 with P  $(\sim Q).$ 
708   intros Perm1_4b.
709   MP Syll2_06a Perm1_4b.
710   Syll Syll2_06a Ha S.
711   apply S.
712 Qed.
713
714 Theorem n2_65 :  $\forall P Q : \text{Prop},$ 
715    $(P \rightarrow Q) \rightarrow ((P \rightarrow \sim Q) \rightarrow \sim P).$ 
716 Proof. intros P Q.
717   specialize n2_64 with  $(\sim P) Q.$ 
718   intros n2_64a.
719   replace  $(\sim P \vee Q)$  with  $(P \rightarrow Q)$  in n2_64a.
720   replace  $(\sim P \vee \sim Q)$  with  $(P \rightarrow \sim Q)$  in n2_64a.
721   apply n2_64a.
722   apply Impl1_01.
723   apply Impl1_01.
724 Qed.
725
726 Theorem n2_67 :  $\forall P Q : \text{Prop},$ 
727    $((P \vee Q) \rightarrow Q) \rightarrow (P \rightarrow Q).$ 
728 Proof. intros P Q.
729   specialize n2_54 with P Q.
730   intros n2_54a.
731   specialize Syll2_06 with  $(\sim P \rightarrow Q) (P \vee Q) Q.$ 
732   intros Syll2_06a.
733   MP Syll2_06a n2_54a.
734   specialize n2_24 with P Q.
735   intros n2_24.
736   specialize Syll2_06 with P  $(\sim P \rightarrow Q) Q.$ 

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737   intros Syll12_06b.
738   MP Syll12_06b n2_24a.
739   Syll Syll12_06b Syll12_06a S.
740   apply S.
741   Qed.
742
743   Theorem n2_68 :  $\forall P Q : \text{Prop},$ 
744      $((P \rightarrow Q) \rightarrow Q) \rightarrow (P \vee Q).$ 
745   Proof. intros P Q.
746     specialize n2_67 with ( $\sim P$ ) Q.
747     intros n2_67a.
748     replace ( $\sim P \vee Q$ ) with  $(P \rightarrow Q)$  in n2_67a.
749     specialize n2_54 with P Q.
750     intros n2_54a.
751     Syll n2_67a n2_54a S.
752     apply S.
753     apply Impl1_01.
754     Qed.
755
756   Theorem n2_69 :  $\forall P Q : \text{Prop},$ 
757      $((P \rightarrow Q) \rightarrow Q) \rightarrow ((Q \rightarrow P) \rightarrow P).$ 
758   Proof. intros P Q.
759     specialize n2_68 with P Q.
760     intros n2_68a.
761     specialize Perm1_4 with P Q.
762     intros Perm1_4a.
763     Syll n2_68a Perm1_4a Sa.
764     specialize n2_62 with Q P.
765     intros n2_62a.
766     Syll Sa n2_62a Sb.
767     apply Sb.
768     Qed.
769
770   Theorem n2_73 :  $\forall P Q R : \text{Prop},$ 
771      $(P \rightarrow Q) \rightarrow ((P \vee Q) \vee R) \rightarrow (Q \vee R).$ 
772   Proof. intros P Q R.
773     specialize n2_621 with P Q.
774     intros n2_621a.
775     specialize n2_38 with R  $(P \vee Q)$  Q.
776     intros n2_38a.
777     Syll n2_621a n2_38a S.
778     apply S.

```

```

779 Qed.
780
781 Theorem n2_74 :  $\forall P Q R : \text{Prop}$ ,
782    $(Q \rightarrow P) \rightarrow ((P \vee Q) \vee R) \rightarrow (P \vee R)$ .
783 Proof. intros P Q R.
784   specialize n2_73 with Q P R.
785   intros n2_73a.
786   specialize Assoc1_5 with P Q R.
787   intros Assoc1_5a.
788   specialize n2_31 with Q P R.
789   intros n2_31a. (*not cited*)
790   Syll Assoc1_5a n2_31a Sa.
791   specialize n2_32 with P Q R.
792   intros n2_32a. (*not cited*)
793   Syll n2_32a Sa Sb.
794   specialize Syll2_06 with  $((P \vee Q) \vee R) ((Q \vee P) \vee R) (P \vee R)$ .
795   intros Syll2_06a.
796   MP Syll2_06a Sb.
797   Syll n2_73a Syll2_05a H.
798   apply H.
799 Qed.
800
801 Theorem n2_75 :  $\forall P Q R : \text{Prop}$ ,
802    $(P \vee Q) \rightarrow ((P \vee (Q \rightarrow R)) \rightarrow (P \vee R))$ .
803 Proof. intros P Q R.
804   specialize n2_74 with P ( $\sim Q$ ) R.
805   intros n2_74a.
806   specialize n2_53 with Q P.
807   intros n2_53a.
808   Syll n2_53a n2_74a Sa.
809   specialize n2_31 with P ( $\sim Q$ ) R.
810   intros n2_31a.
811   specialize Syll2_06 with  $(P \vee (\sim Q) \vee R) ((P \vee (\sim Q)) \vee R) (P \vee R)$ .
812   intros Syll2_06a.
813   MP Syll2_06a n2_31a.
814   Syll Sa Syll2_06a Sb.
815   specialize Perm1_4 with P Q.
816   intros Perm1_4a. (*not cited*)
817   Syll Perm1_4a Sb Sc.
818   replace  $(\sim Q \vee R)$  with  $(Q \rightarrow R)$  in Sc.
819   apply Sc.
820   apply Impl1_01.

```

```

821 Qed.
822
823 Theorem n2_76 :  $\forall P Q R : \text{Prop}$ ,
824    $(P \vee (Q \rightarrow R)) \rightarrow ((P \vee Q) \rightarrow (P \vee R))$ .
825 Proof. intros P Q R.
826   specialize n2_75 with P Q R.
827   intros n2_75a.
828   specialize Comm2_04 with  $(P \vee Q) (P \vee (Q \rightarrow R)) (P \vee R)$ .
829   intros Comm2_04a.
830   apply Comm2_04a.
831   apply n2_75a.
832 Qed.
833
834 Theorem n2_77 :  $\forall P Q R : \text{Prop}$ ,
835    $(P \rightarrow (Q \rightarrow R)) \rightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R))$ .
836 Proof. intros P Q R.
837   specialize n2_76 with  $(\neg P) Q R$ .
838   intros n2_76a.
839   replace  $(\neg P \vee (Q \rightarrow R))$  with  $(P \rightarrow Q \rightarrow R)$  in n2_76a.
840   replace  $(\neg P \vee Q)$  with  $(P \rightarrow Q)$  in n2_76a.
841   replace  $(\neg P \vee R)$  with  $(P \rightarrow R)$  in n2_76a.
842   apply n2_76a.
843   apply Impl1_01.
844   apply Impl1_01.
845   apply Impl1_01.
846 Qed.
847
848 Theorem n2_8 :  $\forall Q R S : \text{Prop}$ ,
849    $(Q \vee R) \rightarrow ((\neg R \vee S) \rightarrow (Q \vee S))$ .
850 Proof. intros Q R S.
851   specialize n2_53 with R Q.
852   intros n2_53a.
853   specialize Perm1_4 with Q R.
854   intros Perm1_4a.
855   Syll Perm1_4a n2_53a Ha.
856   specialize n2_38 with S  $(\neg R) Q$ .
857   intros n2_38a.
858   Syll H n2_38a Hb.
859   apply Hb.
860 Qed.
861
862 Theorem n2_81 :  $\forall P Q R S : \text{Prop}$ ,

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863    $(Q \rightarrow (R \rightarrow S)) \rightarrow ((P \vee Q) \rightarrow ((P \vee R) \rightarrow (P \vee S)))$ .
864 Proof. intros P Q R S.
865   specialize Sum1_6 with P Q (R→S).
866   intros Sum1_6a.
867   specialize n2_76 with P R S.
868   intros n2_76a.
869   specialize Syll2_05 with (P∨Q) (P∨(R→S)) ((P∨R)→(P∨S)).
870   intros Syll2_05a.
871   MP Syll2_05a n2_76a.
872   Syll Sum1_6a Syll2_05a H.
873   apply H.
874 Qed.
875
876 Theorem n2_82 :  $\forall P Q R S : \text{Prop}$ ,
877    $(P \vee Q \vee R) \rightarrow ((P \vee \sim R \vee S) \rightarrow (P \vee Q \vee S))$ .
878 Proof. intros P Q R S.
879   specialize n2_8 with Q R S.
880   intros n2_8a.
881   specialize n2_81 with P (Q∨R) ( $\sim R \vee S$ ) (Q∨S).
882   intros n2_81a.
883   MP n2_81a n2_8a.
884   apply n2_81a.
885 Qed.
886
887 Theorem n2_83 :  $\forall P Q R S : \text{Prop}$ ,
888    $(P \rightarrow (Q \rightarrow R)) \rightarrow ((P \rightarrow (R \rightarrow S)) \rightarrow (P \rightarrow (Q \rightarrow S)))$ .
889 Proof. intros P Q R S.
890   specialize n2_82 with ( $\sim P$ ) ( $\sim Q$ ) R S.
891   intros n2_82a.
892   replace ( $\sim Q \vee R$ ) with (Q→R) in n2_82a.
893   replace ( $\sim P \vee (Q \rightarrow R)$ ) with (P→Q→R) in n2_82a.
894   replace ( $\sim R \vee S$ ) with (R→S) in n2_82a.
895   replace ( $\sim P \vee (R \rightarrow S)$ ) with (P→R→S) in n2_82a.
896   replace ( $\sim Q \vee S$ ) with (Q→S) in n2_82a.
897   replace ( $\sim Q \vee S$ ) with (Q→S) in n2_82a.
898   replace ( $\sim P \vee (Q \rightarrow S)$ ) with (P→Q→S) in n2_82a.
899   apply n2_82a.
900   apply Impl1_01.
901   apply Impl1_01.
902   apply Impl1_01.
903   apply Impl1_01.
904   apply Impl1_01.

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905   apply Impl1_01.
906   apply Impl1_01.
907 Qed.
908
909 Theorem n2_85 :  $\forall P Q R : \text{Prop},$ 
910    $((P \vee Q) \rightarrow (P \vee R)) \rightarrow (P \vee (Q \rightarrow R)).$ 
911 Proof. intros P Q R.
912   specialize Add1_3 with P Q.
913   intros Add1_3a.
914   specialize Syll2_06 with Q (P $\vee$ Q) R.
915   intros Syll2_06a.
916   MP Syll2_06a Add1_3a.
917   specialize n2_55 with P R.
918   intros n2_55a.
919   specialize Syll2_05 with (P $\vee$ Q) (P $\vee$ R) R.
920   intros Syll2_05a.
921   Syll n2_55a Syll2_05a Ha.
922   specialize n2_83 with ( $\sim$ P) ((P $\vee$ Q) $\rightarrow$ (P $\vee$ R)) ((P $\vee$ Q) $\rightarrow$ R) (Q $\rightarrow$ R).
923   intros n2_83a.
924   MP n2_83a Ha.
925   specialize Comm2_04 with ( $\sim$ P) (P $\vee$ Q $\rightarrow$ P $\vee$ R) (Q $\rightarrow$ R).
926   intros Comm2_04a.
927   Syll Ha Comm2_04a Hb.
928   specialize n2_54 with P (Q $\rightarrow$ R).
929   intros n2_54a.
930   specialize Simp2_02 with ( $\sim$ P) ((P $\vee$ Q $\rightarrow$ R) $\rightarrow$ (Q $\rightarrow$ R)).
931   intros Simp2_02a. (*Not cited*)
932     (*Greg's suggestion per the BRS list on June 25, 2017.*)
933   MP Syll2_06a Simp2_02a.
934   MP Hb Simp2_02a.
935   Syll Hb n2_54a Hc.
936   apply Hc.
937 Qed.
938
939 Theorem n2_86 :  $\forall P Q R : \text{Prop},$ 
940    $((P \rightarrow Q) \rightarrow (P \rightarrow R)) \rightarrow (P \rightarrow (Q \rightarrow R)).$ 
941 Proof. intros P Q R.
942   specialize n2_85 with ( $\sim$ P) Q R.
943   intros n2_85a.
944   replace ( $\sim$ P $\vee$ Q) with (P $\rightarrow$ Q) in n2_85a.
945   replace ( $\sim$ P $\vee$ R) with (P $\rightarrow$ R) in n2_85a.
946   replace ( $\sim$ P $\vee$ (Q $\rightarrow$ R)) with (P $\rightarrow$ Q $\rightarrow$ R) in n2_85a.

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947   apply n2_85a.
948   apply Impl1_01.
949   apply Impl1_01.
950   apply Impl1_01.
951   Qed.
952
953   End No2.
954
955   Module No3.
956
957   Import No1.
958   Import No2.
959
960   Axiom Prod3_01 :  $\forall P Q : \text{Prop},$ 
961      $(P \wedge Q) = \sim(\sim P \vee \sim Q).$ 
962
963   Axiom Abb3_02 :  $\forall P Q R : \text{Prop},$ 
964      $(P \rightarrow Q \rightarrow R) = (P \rightarrow Q) \wedge (Q \rightarrow R).$ 
965
966   Theorem Conj3_03 :  $\forall P Q : \text{Prop}, P \rightarrow Q \rightarrow (P \wedge Q).$ 
967   Proof. intros P Q.
968     specialize n2_11 with  $(\sim P \vee \sim Q).$  intros n2_11a.
969     specialize n2_32 with  $(\sim P) (\sim Q) (\sim(\sim P \vee \sim Q)).$  intros n2_32a.
970     MP n2_32a n2_11a.
971     replace  $(\sim(\sim P \vee \sim Q))$  with  $(P \wedge Q)$  in n2_32a.
972     replace  $(\sim Q \vee (P \wedge Q))$  with  $(Q \rightarrow (P \wedge Q))$  in n2_32a.
973     replace  $(\sim P \vee (Q \rightarrow (P \wedge Q)))$  with  $(P \rightarrow Q \rightarrow (P \wedge Q))$  in n2_32a.
974     apply n2_32a.
975     apply Impl1_01.
976     apply Impl1_01.
977     apply Prod3_01.
978   Qed.
979   (*3.03 is permits the inference from the theoremhood
980     of P and that of Q to the theoremhood of P and Q. So:*)
981
982   Ltac Conj H1 H2 :=
983     match goal with
984       | [ H1 : ?P, H2 : ?Q |- _ ] =>
985         assert  $(P \wedge Q)$ 
986   end.
987
988   Theorem n3_1 :  $\forall P Q : \text{Prop},$ 

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```

989   (P ∧ Q) → ~(~P ∨ ~Q).
990 Proof. intros P Q.
991   replace ~(~P∨~Q) with (P∧Q).
992   specialize n2_08 with (P∧Q).
993   intros n2_08a.
994   apply n2_08a.
995   apply Prod3_01.
996 Qed.
997
998 Theorem n3_11 : ∀ P Q : Prop,
999   ~(~P ∨ ~Q) → (P ∧ Q).
1000 Proof. intros P Q.
1001   replace ~(~P∨~Q) with (P∧Q).
1002   specialize n2_08 with (P∧Q).
1003   intros n2_08a.
1004   apply n2_08a.
1005   apply Prod3_01.
1006 Qed.
1007
1008 Theorem n3_12 : ∀ P Q : Prop,
1009   (~P ∨ ~Q) ∨ (P ∧ Q).
1010 Proof. intros P Q.
1011   specialize n2_11 with (~P∨~Q).
1012   intros n2_11a.
1013   replace ~(~P∨~Q) with (P∧Q) in n2_11a.
1014   apply n2_11a.
1015   apply Prod3_01.
1016 Qed.
1017
1018 Theorem n3_13 : ∀ P Q : Prop,
1019   ~(P ∧ Q) → (~P ∨ ~Q).
1020 Proof. intros P Q.
1021   specialize n3_11 with P Q.
1022   intros n3_11a.
1023   specialize Trans2_15 with (~P∨~Q) (P∧Q).
1024   intros Trans2_15a.
1025   MP Trans2_15a n3_11a.
1026   apply Trans2_15a.
1027 Qed.
1028
1029 Theorem n3_14 : ∀ P Q : Prop,
1030   (~P ∨ ~Q) → ~(P ∧ Q).

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```

1031 Proof. intros P Q.
1032   specialize n3_1 with P Q.
1033   intros n3_1a.
1034   specialize Trans2_16 with (P ∧ Q) (~(~P ∨ ~Q)).
1035   intros Trans2_16a.
1036   MP Trans2_16a n3_1a.
1037   specialize n2_12 with (~P ∨ ~Q).
1038   intros n2_12a.
1039   Syll n2_12a Trans2_16a S.
1040   apply S.
1041 Qed.
1042
1043 Theorem n3_2 : ∀ P Q : Prop,
1044   P → Q → (P ∧ Q).
1045 Proof. intros P Q.
1046   specialize n3_12 with P Q.
1047   intros n3_12a.
1048   specialize n2_32 with (~P) (~Q) (P ∧ Q).
1049   intros n2_32a.
1050   MP n3_32a n3_12a.
1051   replace (~Q ∨ P ∧ Q) with (Q → P ∧ Q) in n2_32a.
1052   replace (~P ∨ (Q → P ∧ Q)) with (P → Q → P ∧ Q) in n2_32a.
1053   apply n2_32a.
1054   apply Impl1_01.
1055   apply Impl1_01.
1056 Qed.
1057
1058 Theorem n3_21 : ∀ P Q : Prop,
1059   Q → P → (P ∧ Q).
1060 Proof. intros P Q.
1061   specialize n3_2 with P Q.
1062   intros n3_2a.
1063   specialize Comm2_04 with P Q (P ∧ Q).
1064   intros Comm2_04a.
1065   MP Comm2_04a n3_2a.
1066   apply Comm2_04a.
1067 Qed.
1068
1069 Theorem n3_22 : ∀ P Q : Prop,
1070   (P ∧ Q) → (Q ∧ P).
1071 Proof. intros P Q.
1072   specialize n3_13 with Q P.

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1073   intros n3_13a.
1074   specialize Perm1_4 with ( $\sim Q$ ) ( $\sim P$ ).
1075   intros Perm1_4a.
1076   Syll n3_13a Perm1_4a Ha.
1077   specialize n3_14 with P Q.
1078   intros n3_14a.
1079   Syll Ha n3_14a Hb.
1080   specialize Trans2_17 with ( $P \wedge Q$ ) ( $Q \wedge P$ ).
1081   intros Trans2_17a.
1082   MP Trans2_17a Hb.
1083   apply Trans2_17a.
1084   Qed.
1085
1086   Theorem n3_24 :  $\forall P : \text{Prop}$ ,
1087      $\sim(P \wedge \sim P)$ .
1088   Proof. intros P.
1089     specialize n2_11 with ( $\sim P$ ).
1090     intros n2_11a.
1091     specialize n3_14 with P ( $\sim P$ ).
1092     intros n3_14a.
1093     MP n3_14a n2_11a.
1094     apply n3_14a.
1095     Qed.
1096
1097   Theorem Simp3_26 :  $\forall P Q : \text{Prop}$ ,
1098      $(P \wedge Q) \rightarrow P$ .
1099   Proof. intros P Q.
1100     specialize Simp2_02 with Q P.
1101     intros Simp2_02a.
1102     replace ( $P \rightarrow (Q \rightarrow P)$ ) with ( $\sim P \vee (Q \rightarrow P)$ ) in Simp2_02a.
1103     replace ( $Q \rightarrow P$ ) with ( $\sim Q \vee P$ ) in Simp2_02a.
1104     specialize n2_31 with ( $\sim P$ ) ( $\sim Q$ ) P.
1105     intros n2_31a.
1106     MP n2_31a Simp2_02a.
1107     specialize n2_53 with ( $\sim P \vee \sim Q$ ) P.
1108     intros n2_53a.
1109     MP n2_53a Simp2_02a.
1110     replace ( $\sim(\sim P \vee \sim Q)$ ) with ( $P \wedge Q$ ) in n2_53a.
1111     apply n2_53a.
1112     apply Prod3_01.
1113     replace ( $\sim Q \vee P$ ) with ( $Q \rightarrow P$ ).
1114     reflexivity.

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1115   apply Impl1_01.
1116   replace ( $\sim P \vee (Q \rightarrow P)$ ) with ( $P \rightarrow Q \rightarrow P$ ).
1117   reflexivity.
1118   apply Impl1_01.
1119   Qed.
1120
1121   Theorem Simp3_27 :  $\forall P Q : \text{Prop}$ ,
1122     ( $P \wedge Q$ )  $\rightarrow Q$ .
1123   Proof. intros P Q.
1124     specialize n3_22 with P Q.
1125     intros n3_22a.
1126     specialize Simp3_26 with Q P.
1127     intros Simp3_26a.
1128     Syll n3_22a Simp3_26a S.
1129     apply S.
1130     Qed.
1131
1132   Theorem Exp3_3 :  $\forall P Q R : \text{Prop}$ ,
1133     (( $P \wedge Q$ )  $\rightarrow R$ )  $\rightarrow (P \rightarrow (Q \rightarrow R))$ .
1134   Proof. intros P Q R.
1135     specialize Trans2_15 with ( $\sim P \vee \sim Q$ ) R.
1136     intros Trans2_15a.
1137     specialize Comm2_04 with ( $\sim R$ ) P ( $\sim Q$ ).
1138     intros Comm2_04a.
1139     replace ( $P \rightarrow \sim Q$ ) with ( $\sim P \vee \sim Q$ ) in Comm2_04a.
1140     Syll Trans2_15a Comm2_04a Sa.
1141     specialize Trans2_17 with Q R.
1142     intros Trans2_17a.
1143     specialize Syll2_05 with P ( $\sim R \rightarrow \sim Q$ ) ( $Q \rightarrow R$ ).
1144     intros Syll2_05a.
1145     MP Syll2_05a Trans2_17a.
1146     Syll Sa Syll2_05a Sb.
1147     replace ( $\sim(\sim P \vee \sim Q)$ ) with ( $P \wedge Q$ ) in Sb.
1148     apply Sb.
1149     apply Prod3_01.
1150     replace ( $\sim P \vee \sim Q$ ) with ( $P \rightarrow \sim Q$ ).
1151     reflexivity.
1152     apply Impl1_01.
1153     Qed.
1154     (*The proof sketch cites n2_08, but
1155       we did not seem to need it.*)
1156

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1157 Theorem Imp3_31 :  $\forall$  P Q R : Prop,
1158   (P  $\rightarrow$  (Q  $\rightarrow$  R))  $\rightarrow$  (P  $\wedge$  Q)  $\rightarrow$  R.
1159 Proof. intros P Q R.
1160   specialize n2_31 with ( $\sim$ P) ( $\sim$ Q) R.
1161   intros n2_31a.
1162   specialize n2_53 with ( $\sim$ P $\vee$  $\sim$ Q) R.
1163   intros n2_53a.
1164   Syll n2_31a n2_53a S.
1165   replace ( $\sim$ Q $\vee$ R) with (Q $\rightarrow$ R) in S.
1166   replace ( $\sim$ P $\vee$ (Q $\rightarrow$ R)) with (P $\rightarrow$ Q $\rightarrow$ R) in S.
1167   replace ( $\sim$ ( $\sim$ P $\vee$  $\sim$ Q)) with (P $\wedge$ Q) in S.
1168   apply S.
1169   apply Prod3_01.
1170   apply Impl1_01.
1171   apply Impl1_01.
1172 Qed.
1173 (*The proof sketch cites n2_08, but
1174   we did not seem to need it.*)
1175
1176 Theorem Syll3_33 :  $\forall$  P Q R : Prop,
1177   ((P  $\rightarrow$  Q)  $\wedge$  (Q  $\rightarrow$  R))  $\rightarrow$  (P  $\rightarrow$  R).
1178 Proof. intros P Q R.
1179   specialize Syll2_06 with P Q R.
1180   intros Syll2_06a.
1181   specialize Imp3_31 with (P $\rightarrow$ Q) (Q $\rightarrow$ R) (P $\rightarrow$ R).
1182   intros Imp3_31a.
1183   MP Imp3_31a Syll2_06a.
1184   apply Imp3_31a.
1185 Qed.
1186
1187 Theorem Syll3_34 :  $\forall$  P Q R : Prop,
1188   ((Q  $\rightarrow$  R)  $\wedge$  (P  $\rightarrow$  Q))  $\rightarrow$  (P  $\rightarrow$  R).
1189 Proof. intros P Q R.
1190   specialize Syll2_05 with P Q R.
1191   intros Syll2_05a.
1192   specialize Imp3_31 with (Q $\rightarrow$ R) (P $\rightarrow$ Q) (P $\rightarrow$ R).
1193   intros Imp3_31a.
1194   MP Imp3_31a Syll2_05a.
1195   apply Imp3_31a.
1196 Qed.
1197
1198 Theorem Ass3_35 :  $\forall$  P Q : Prop,

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1199   (P ∧ (P → Q)) → Q.
1200 Proof. intros P Q.
1201   specialize n2_27 with P Q.
1202   intros n2_27a.
1203   specialize Imp3_31 with P (P→Q) Q.
1204   intros Imp3_31a.
1205   MP Imp3_31a n2_27a.
1206   apply Imp3_31a.
1207 Qed.
1208
1209 Theorem n3_37 : ∀ P Q R : Prop,
1210   (P ∧ Q → R) → (P ∧ ~R → ~Q).
1211 Proof. intros P Q R.
1212   specialize Trans2_16 with Q R.
1213   intros Trans2_16a.
1214   specialize Syll2_05 with P (Q→R) (~R→~Q).
1215   intros Syll2_05a.
1216   MP Syll2_05a Trans2_16a.
1217   specialize Exp3_3 with P Q R.
1218   intros Exp3_3a.
1219   Syll Exp3_3a Syll2_05a Sa.
1220   specialize Imp3_31 with P (~R) (~Q).
1221   intros Imp3_31a.
1222   Syll Sa Imp3_31a Sb.
1223   apply Sb.
1224 Qed.
1225
1226 Theorem n3_4 : ∀ P Q : Prop,
1227   (P ∧ Q) → P → Q.
1228 Proof. intros P Q.
1229   specialize n2_51 with P Q.
1230   intros n2_51a.
1231   specialize Trans2_15 with (P→Q) (P→~Q).
1232   intros Trans2_15a.
1233   MP Trans2_15a n2_51a.
1234   replace (P→~Q) with (~P∨~Q) in Trans2_15a.
1235   replace (~(~P∨~Q)) with (P∧Q) in Trans2_15a.
1236   apply Trans2_15a.
1237   apply Prod3_01.
1238   replace (~P∨~Q) with (P→~Q).
1239   reflexivity.
1240   apply Impl1_01.

```

```

1241 Qed.
1242
1243 Theorem n3_41 :  $\forall$  P Q R : Prop,
1244   (P  $\rightarrow$  R)  $\rightarrow$  (P  $\wedge$  Q  $\rightarrow$  R).
1245 Proof. intros P Q R.
1246   specialize Simp3_26 with P Q.
1247   intros Simp3_26a.
1248   specialize Syll2_06 with (P $\wedge$ Q) P R.
1249   intros Syll2_06a.
1250   MP Simp3_26a Syll2_06a.
1251   apply Syll2_06a.
1252 Qed.
1253
1254 Theorem n3_42 :  $\forall$  P Q R : Prop,
1255   (Q  $\rightarrow$  R)  $\rightarrow$  (P  $\wedge$  Q  $\rightarrow$  R).
1256 Proof. intros P Q R.
1257   specialize Simp3_27 with P Q.
1258   intros Simp3_27a.
1259   specialize Syll2_06 with (P $\wedge$ Q) Q R.
1260   intros Syll2_06a.
1261   MP Syll2_06a Simp3_27a.
1262   apply Syll2_06a.
1263 Qed.
1264
1265 Theorem Comp3_43 :  $\forall$  P Q R : Prop,
1266   (P  $\rightarrow$  Q)  $\wedge$  (P  $\rightarrow$  R)  $\rightarrow$  (P  $\rightarrow$  Q  $\wedge$  R).
1267 Proof. intros P Q R.
1268   specialize n3_2 with Q R.
1269   intros n3_2a.
1270   specialize Syll2_05 with P Q (R $\rightarrow$ Q $\wedge$ R).
1271   intros Syll2_05a.
1272   MP Syll2_05a n3_2a.
1273   specialize n2_77 with P R (Q $\wedge$ R).
1274   intros n2_77a.
1275   Syll Syll2_05a n2_77a Sa.
1276   specialize Imp3_31 with (P $\rightarrow$ Q) (P $\rightarrow$ R) (P $\rightarrow$ Q $\wedge$ R).
1277   intros Imp3_31a.
1278   MP Sa Imp3_31a.
1279   apply Imp3_31a.
1280 Qed.
1281
1282 Theorem n3_44 :  $\forall$  P Q R : Prop,

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1283    $(Q \rightarrow P) \wedge (R \rightarrow P) \rightarrow (Q \vee R \rightarrow P).$ 
1284 Proof. intros P Q R.
1285   specialize Syll3_33 with ( $\sim Q$ ) R P.
1286   intros Syll3_33a.
1287   specialize n2_6 with Q P.
1288   intros n2_6a.
1289   Syll Syll3_33a n2_6a Sa.
1290   specialize Exp3_3 with ( $\sim Q \rightarrow R$ ) ( $R \rightarrow P$ ) ( $(Q \rightarrow P) \rightarrow P$ ).
1291   intros Exp3_3a.
1292   MP Exp3_3a Sa.
1293   specialize Comm2_04 with ( $R \rightarrow P$ ) ( $Q \rightarrow P$ ) P.
1294   intros Comm2_04a.
1295   Syll Exp3_3a Comm2_04a Sb.
1296   specialize Imp3_31 with ( $Q \rightarrow P$ ) ( $R \rightarrow P$ ) P.
1297   intros Imp3_31a.
1298   Syll Sb Imp3_31a Sc.
1299   specialize Comm2_04 with ( $\sim Q \rightarrow R$ ) ( $(Q \rightarrow P) \wedge (R \rightarrow P)$ ) P.
1300   intros Comm2_04b.
1301   MP Comm2_04b Sc.
1302   specialize n2_53 with Q R.
1303   intros n2_53a.
1304   specialize Syll2_06 with ( $Q \vee R$ ) ( $\sim Q \rightarrow R$ ) P.
1305   intros Syll2_06a.
1306   MP Syll2_06a n2_53a.
1307   Syll Comm2_04b Syll2_06a Sd.
1308   apply Sd.
1309 Qed.
1310
1311 Theorem Fact3_45 :  $\forall P Q R : \text{Prop},$ 
1312    $(P \rightarrow Q) \rightarrow (P \wedge R) \rightarrow (Q \wedge R).$ 
1313 Proof. intros P Q R.
1314   specialize Syll2_06 with P Q ( $\sim R$ ).
1315   intros Syll2_06a.
1316   specialize Trans2_16 with ( $Q \rightarrow \sim R$ ) ( $P \rightarrow \sim R$ ).
1317   intros Trans2_16a.
1318   Syll Syll2_06a Trans2_16a S.
1319   replace ( $P \rightarrow \sim R$ ) with ( $\sim P \vee \sim R$ ) in S.
1320   replace ( $Q \rightarrow \sim R$ ) with ( $\sim Q \vee \sim R$ ) in S.
1321   replace ( $\sim(\sim P \vee \sim R)$ ) with ( $P \wedge R$ ) in S.
1322   replace ( $\sim(\sim Q \vee \sim R)$ ) with ( $Q \wedge R$ ) in S.
1323   apply S.
1324   apply Prod3_01.

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1325   apply Prod3_01.
1326   replace (~Q $\vee$ ~R) with (Q $\rightarrow$ ~R).
1327   reflexivity.
1328   apply Impl1_01.
1329   replace (~P $\vee$ ~R) with (P $\rightarrow$ ~R).
1330   reflexivity.
1331   apply Impl1_01.
1332   Qed.
1333
1334   Theorem n3_47 :  $\forall$  P Q R S : Prop,
1335     ((P  $\rightarrow$  R)  $\wedge$  (Q  $\rightarrow$  S))  $\rightarrow$  (P  $\wedge$  Q)  $\rightarrow$  R  $\wedge$  S.
1336   Proof. intros P Q R S.
1337     specialize Simp3_26 with (P $\rightarrow$ R) (Q $\rightarrow$ S).
1338     intros Simp3_26a.
1339     specialize Fact3_45 with P R Q.
1340     intros Fact3_45a.
1341     Syll Simp3_26a Fact3_45a Sa.
1342     specialize n3_22 with R Q.
1343     intros n3_22a.
1344     specialize Syll2_05 with (P $\wedge$ Q) (R $\wedge$ Q) (Q $\wedge$ R).
1345     intros Syll2_05a.
1346     MP Syll2_05a n3_22a.
1347     Syll Sa Syll2_05a Sb.
1348     specialize Simp3_27 with (P $\rightarrow$ R) (Q $\rightarrow$ S).
1349     intros Simp3_27a.
1350     specialize Fact3_45 with Q S R.
1351     intros Fact3_45b.
1352     Syll Simp3_27a Fact3_45b Sc.
1353     specialize n3_22 with S R.
1354     intros n3_22b.
1355     specialize Syll2_05 with (Q $\wedge$ R) (S $\wedge$ R) (R $\wedge$ S).
1356     intros Syll2_05b.
1357     MP Syll2_05b n3_22b.
1358     Syll Sc Syll2_05b Sd.
1359     specialize n2_83 with ((P $\rightarrow$ R) $\wedge$ (Q $\rightarrow$ S)) (P $\wedge$ Q) (Q $\wedge$ R) (R $\wedge$ S).
1360     intros n2_83a.
1361     MP n2_83a Sb.
1362     MP n2_83 Sd.
1363     apply n2_83a.
1364     Qed.
1365
1366   Theorem n3_48 :  $\forall$  P Q R S : Prop,

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1367   ((P → R) ∧ (Q → S)) → (P ∨ Q) → R ∨ S.
1368 Proof. intros P Q R S.
1369   specialize Simp3_26 with (P→R) (Q→S).
1370   intros Simp3_26a.
1371   specialize Sum1_6 with Q P R.
1372   intros Sum1_6a.
1373   Syll Simp3_26a Sum1_6a Sa.
1374   specialize Perm1_4 with P Q.
1375   intros Perm1_4a.
1376   specialize Syll2_06 with (P∨Q) (Q∨P) (Q∨R).
1377   intros Syll2_06a.
1378   MP Syll2_06a Perm1_4a.
1379   Syll Sa Syll2_06a Sb.
1380   specialize Simp3_27 with (P→R) (Q→S).
1381   intros Simp3_27a.
1382   specialize Sum1_6 with R Q S.
1383   intros Sum1_6b.
1384   Syll Simp3_27a Sum1_6b Sc.
1385   specialize Perm1_4 with Q R.
1386   intros Perm1_4b.
1387   specialize Syll2_06 with (Q∨R) (R∨Q) (R∨S).
1388   intros Syll2_06b.
1389   MP Syll2_06b Perm1_4b.
1390   Syll Sc Syll2_06a Sd.
1391   specialize n2_83 with ((P→R) ∧ (Q→S)) (P∨Q) (Q∨R) (R∨S).
1392   intros n2_83a.
1393   MP n2_83a Sb.
1394   MP n2_83a Sd.
1395   apply n2_83a.
1396 Qed.
1397
1398 End No3.
1399
1400 Module No4.
1401
1402 Import No1.
1403 Import No2.
1404 Import No3.
1405
1406 Axiom Equiv4_01 : ∀ P Q : Prop,
1407   (P ↔ Q) = ((P → Q) ∧ (Q → P)).
1408

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1409 Axiom Abb4_02 :  $\forall P Q R : \text{Prop},$ 
1410    $(P \leftrightarrow Q \leftrightarrow R) = ((P \leftrightarrow Q) \wedge (Q \leftrightarrow R)).$ 
1411
1412 Axiom EqBi :  $\forall P Q : \text{Prop},$ 
1413    $(P = Q) \leftrightarrow (P \leftrightarrow Q).$ 
1414
1415 Ltac Equiv H1 :=
1416   match goal with
1417   | [ H1 :  $(?P \rightarrow ?Q) \wedge (?Q \rightarrow ?P) \mid - \_ ] =>$ 
1418     replace  $((P \rightarrow Q) \wedge (Q \rightarrow P))$  with  $(P \leftrightarrow Q)$  in H1
1419 end.
1420
1421 Theorem Trans4_1 :  $\forall P Q : \text{Prop},$ 
1422    $(P \rightarrow Q) \leftrightarrow (\sim Q \rightarrow \sim P).$ 
1423 Proof. intros P Q.
1424   specialize Trans2_16 with P Q.
1425   intros Trans2_16a.
1426   specialize Trans2_17 with P Q.
1427   intros Trans2_17a.
1428   Conj Trans2_16a Trans2_17a.
1429   split.
1430   apply Trans2_16a.
1431   apply Trans2_17a.
1432   Equiv H.
1433   apply H.
1434   apply Equiv4_01.
1435 Qed.
1436
1437 Theorem Trans4_11 :  $\forall P Q : \text{Prop},$ 
1438    $(P \leftrightarrow Q) \leftrightarrow (\sim P \leftrightarrow \sim Q).$ 
1439 Proof. intros P Q.
1440   specialize Trans2_16 with P Q.
1441   intros Trans2_16a.
1442   specialize Trans2_16 with Q P.
1443   intros Trans2_16b.
1444   Conj Trans2_16a Trans2_16b.
1445   split.
1446   apply Trans2_16a.
1447   apply Trans2_16b.
1448   specialize n3_47 with  $(P \rightarrow Q) (Q \rightarrow P) (\sim Q \rightarrow \sim P) (\sim P \rightarrow \sim Q).$ 
1449   intros n3_47a.
1450   MP n3_47 H.

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1451 specialize n3_22 with ( $\sim Q \rightarrow \sim P$ ) ( $\sim P \rightarrow \sim Q$ ).
1452 intros n3_22a.
1453 Syll n3_47a n3_22a Sa.
1454 replace (( $P \rightarrow Q$ )  $\wedge$  ( $Q \rightarrow P$ )) with ( $P \leftrightarrow Q$ ) in Sa.
1455 replace (( $\sim P \rightarrow \sim Q$ )  $\wedge$  ( $\sim Q \rightarrow \sim P$ )) with ( $\sim P \leftrightarrow \sim Q$ ) in Sa.
1456 clear Trans2_16a. clear H. clear Trans2_16b.
1457     clear n3_22a. clear n3_47a.
1458 specialize Trans2_17 with Q P.
1459 intros Trans2_17a.
1460 specialize Trans2_17 with P Q.
1461 intros Trans2_17b.
1462 Conj Trans2_17a Trans2_17b.
1463 split.
1464 apply Trans2_17a.
1465 apply Trans2_17b.
1466 specialize n3_47 with ( $\sim P \rightarrow \sim Q$ ) ( $\sim Q \rightarrow \sim P$ ) ( $Q \rightarrow P$ ) ( $P \rightarrow Q$ ).
1467 intros n3_47a.
1468 MP n3_47a H.
1469 specialize n3_22 with ( $Q \rightarrow P$ ) ( $P \rightarrow Q$ ).
1470 intros n3_22a.
1471 Syll n3_47a n3_22a Sb.
1472 clear Trans2_17a. clear Trans2_17b. clear H.
1473     clear n3_47a. clear n3_22a.
1474 replace (( $P \rightarrow Q$ )  $\wedge$  ( $Q \rightarrow P$ )) with ( $P \leftrightarrow Q$ ) in Sb.
1475 replace (( $\sim P \rightarrow \sim Q$ )  $\wedge$  ( $\sim Q \rightarrow \sim P$ )) with ( $\sim P \leftrightarrow \sim Q$ ) in Sb.
1476 Conj Sa Sb.
1477 split.
1478 apply Sa.
1479 apply Sb.
1480 Equiv H.
1481 apply H.
1482 apply Equiv4_01.
1483 apply Equiv4_01.
1484 apply Equiv4_01.
1485 apply Equiv4_01.
1486 apply Equiv4_01.
1487 Qed.
1488
1489 Theorem n4_12 :  $\forall P Q : \text{Prop}$ ,
1490     ( $P \leftrightarrow \sim Q$ )  $\leftrightarrow$  ( $Q \leftrightarrow \sim P$ ).
1491 Proof. intros P Q.
1492     specialize n2_03 with P Q.

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1493     intros n2_03a.
1494     specialize Trans2_15 with Q P.
1495     intros Trans2_15a.
1496     Conj n2_03a Trans2_15a.
1497     split.
1498     apply n2_03a.
1499     apply Trans2_15a.
1500     specialize n3_47 with (P → ~Q) (~Q → P) (Q → ~P) (~P → Q).
1501     intros n3_47a.
1502     MP n3_47a H.
1503     specialize n2_03 with Q P.
1504     intros n2_03b.
1505     specialize Trans2_15 with P Q.
1506     intros Trans2_15b.
1507     Conj n2_03b Trans2_15b.
1508     split.
1509     apply n2_03b.
1510     apply Trans2_15b.
1511     specialize n3_47 with (Q → ~P) (~P → Q) (P → ~Q) (~Q → P).
1512     intros n3_47b.
1513     MP n3_47b H0.
1514     clear n2_03a. clear Trans2_15a. clear H. clear n2_03b.
1515     clear Trans2_15b. clear H0.
1516     replace ((P → ~Q) ∧ (~Q → P)) with (P ↔ ~Q) in n3_47a.
1517     replace ((Q → ~P) ∧ (~P → Q)) with (Q ↔ ~P) in n3_47a.
1518     replace ((P → ~Q) ∧ (~Q → P)) with (P ↔ ~Q) in n3_47b.
1519     replace ((Q → ~P) ∧ (~P → Q)) with (Q ↔ ~P) in n3_47b.
1520     Conj n3_47a n3_47b.
1521     split.
1522     apply n3_47a.
1523     apply n3_47b.
1524     Equiv H.
1525     apply H.
1526     apply Equiv4_01.
1527     apply Equiv4_01.
1528     apply Equiv4_01.
1529     apply Equiv4_01.
1530     apply Equiv4_01.
1531     Qed.
1532
1533 Theorem n4_13 : ∀ P : Prop,
1534     P ↔ ~~P.

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1535   Proof. intros P.
1536   specialize n2_12 with P.
1537   intros n2_12a.
1538   specialize n2_14 with P.
1539   intros n2_14a.
1540   Conj n2_12a n2_14a.
1541   split.
1542   apply n2_12a.
1543   apply n2_14a.
1544   Equiv H.
1545   apply H.
1546   apply Equiv4_01.
1547   Qed.
1548
1549   Theorem n4_14 :  $\forall P Q R : \text{Prop}$ ,
1550      $((P \wedge Q) \rightarrow R) \leftrightarrow ((P \wedge \sim R) \rightarrow \sim Q)$ .
1551   Proof. intros P Q R.
1552   specialize n3_37 with P Q R.
1553   intros n3_37a.
1554   specialize n3_37 with P ( $\sim R$ ) ( $\sim Q$ ).
1555   intros n3_37b.
1556   Conj n3_37a n3_37b.
1557   split. apply n3_37a.
1558   apply n3_37b.
1559   specialize n4_13 with Q.
1560   intros n4_13a.
1561   specialize n4_13 with R.
1562   intros n4_13b.
1563   replace ( $\sim \sim Q$ ) with Q in H.
1564   replace ( $\sim \sim R$ ) with R in H.
1565   Equiv H.
1566   apply H.
1567   apply Equiv4_01.
1568   apply EqBi.
1569   apply n4_13b.
1570   apply EqBi.
1571   apply n4_13a.
1572   Qed.
1573
1574   Theorem n4_15 :  $\forall P Q R : \text{Prop}$ ,
1575      $((P \wedge Q) \rightarrow \sim R) \leftrightarrow ((Q \wedge R) \rightarrow \sim P)$ .
1576   Proof. intros P Q R.

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1577 specialize n4_14 with Q P (~R).
1578 intros n4_14a.
1579 specialize n3_22 with Q P.
1580 intros n3_22a.
1581 specialize Syll2_06 with (Q^P) (P^Q) (~R).
1582 intros Syll2_06a.
1583 MP Syll2_06a n3_22a.
1584 specialize n4_13 with R.
1585 intros n4_13a.
1586 replace (~~R) with R in n4_14a.
1587 rewrite Equiv4_01 in n4_14a.
1588 specialize Simp3_26 with ((Q ^ P → ~R) → Q ^ R → ~P)
1589      ((Q ^ R → ~P) → Q ^ P → ~R).
1590 intros Simp3_26a.
1591 MP Simp3_26a n4_14a.
1592 Syll Syll2_06a Simp3_26a Sa.
1593 specialize Simp3_27 with ((Q ^ P → ~R) → Q ^ R → ~P)
1594      ((Q ^ R → ~P) → Q ^ P → ~R).
1595 intros Simp3_27a.
1596 MP Simp3_27a n4_14a.
1597 specialize n3_22 with P Q.
1598 intros n3_22b.
1599 specialize Syll2_06 with (P^Q) (Q^P) (~R).
1600 intros Syll2_06b.
1601 MP Syll2_06b n3_22b.
1602 Syll Syll2_06b Simp3_27a Sb.
1603 clear n4_14a. clear n3_22a. clear Syll2_06a.
1604      clear n4_13a. clear Simp3_26a. clear n3_22b.
1605      clear Simp3_27a. clear Syll2_06b.
1606 Conj Sa Sb.
1607 split.
1608 apply Sa.
1609 apply Sb.
1610 Equiv H.
1611 apply H.
1612 apply Equiv4_01.
1613 apply EqBi.
1614 apply n4_13a.
1615 Qed.
1616
1617 Theorem n4_2 : ∀ P : Prop,
1618     P ↔ P.

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1619 Proof. intros P.
1620 specialize n3_2 with (P→P) (P→P).
1621 intros n3_2a.
1622 specialize n2_08 with P.
1623 intros n2_08a.
1624 MP n3_2a n2_08a.
1625 MP n3_2a n2_08a.
1626 Equiv n3_2a.
1627 apply n3_2a.
1628 apply Equiv4_01.
1629 Qed.
1630
1631 Theorem n4_21 : ∀ P Q : Prop,
1632   (P ↔ Q) ↔ (Q ↔ P).
1633 Proof. intros P Q.
1634 specialize n3_22 with (P→Q) (Q→P).
1635 intros n3_22a.
1636 replace ((P → Q) ∧ (Q → P)) with (P↔Q) in n3_22a.
1637 replace ((Q → P) ∧ (P → Q)) with (Q↔P) in n3_22a.
1638 specialize n3_22 with (Q→P) (P→Q).
1639 intros n3_22b.
1640 replace ((P → Q) ∧ (Q → P)) with (P↔Q) in n3_22b.
1641 replace ((Q → P) ∧ (P → Q)) with (Q↔P) in n3_22b.
1642 Conj n3_22a n3_22b.
1643 split.
1644 apply n3_22a.
1645 apply n3_22b.
1646 Equiv H.
1647 apply H.
1648 apply Equiv4_01.
1649 apply Equiv4_01.
1650 apply Equiv4_01.
1651 apply Equiv4_01.
1652 apply Equiv4_01.
1653 Qed.
1654
1655 Theorem n4_22 : ∀ P Q R : Prop,
1656   ((P ↔ Q) ∧ (Q ↔ R)) → (P ↔ R).
1657 Proof. intros P Q R.
1658 specialize Simp3_26 with (P↔Q) (Q↔R).
1659 intros Simp3_26a.
1660 specialize Simp3_26 with (P→Q) (Q→P).

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1661 intros Simp3_26b.
1662 replace ((P→Q) ∧ (Q→P)) with (P↔Q) in Simp3_26b.
1663 Syll Simp3_26a Simp3_26b Sa.
1664 specialize Simp3_27 with (P↔Q) (Q↔R).
1665 intros Simp3_27a.
1666 specialize Simp3_26 with (Q→R) (R→Q).
1667 intros Simp3_26c.
1668 replace ((Q→R) ∧ (R→Q)) with (Q↔R) in Simp3_26c.
1669 Syll Simp3_27a Simp3_26c Sb.
1670 specialize n2_83 with ((P↔Q) ∧ (Q↔R)) P Q R.
1671 intros n2_83a.
1672 MP n2_83a Sa.
1673 MP n2_83a Sb.
1674 specialize Simp3_27 with (P↔Q) (Q↔R).
1675 intros Simp3_27b.
1676 specialize Simp3_27 with (Q→R) (R→Q).
1677 intros Simp3_27c.
1678 replace ((Q→R) ∧ (R→Q)) with (Q↔R) in Simp3_27c.
1679 Syll Simp3_27b Simp3_27c Sc.
1680 specialize Simp3_26 with (P↔Q) (Q↔R).
1681 intros Simp3_26d.
1682 specialize Simp3_27 with (P→Q) (Q→P).
1683 intros Simp3_27d.
1684 replace ((P→Q) ∧ (Q→P)) with (P↔Q) in Simp3_27d.
1685 Syll Simp3_26d Simp3_27d Sd.
1686 specialize n2_83 with ((P↔Q) ∧ (Q↔R)) R Q P.
1687 intros n2_83b.
1688 MP n2_83b Sc. MP n2_83b Sd.
1689 clear Sd. clear Sb. clear Sc. clear Sa. clear Simp3_26a.
1690 clear Simp3_26b. clear Simp3_26c. clear Simp3_26d.
1691 clear Simp3_27a. clear Simp3_27b. clear Simp3_27c.
1692 clear Simp3_27d.
1693 Conj n2_83a n2_83b.
1694 split.
1695 apply n2_83a.
1696 apply n2_83b.
1697 specialize Comp3_43 with ((P↔Q) ∧ (Q↔R)) (P→R) (R→P).
1698 intros Comp3_43a.
1699 MP Comp3_43a H.
1700 replace ((P→R) ∧ (R→P)) with (P↔R) in Comp3_43a.
1701 apply Comp3_43a.
1702 apply Equiv4_01.

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1703     apply Equiv4_01.
1704     apply Equiv4_01.
1705     apply Equiv4_01.
1706     apply Equiv4_01.
1707 Qed.
1708
1709 Theorem n4_24 :  $\forall$  P : Prop,
1710   P  $\leftrightarrow$  (P  $\wedge$  P).
1711 Proof. intros P.
1712   specialize n3_2 with P P.
1713   intros n3_2a.
1714   specialize n2_43 with P (P  $\wedge$  P).
1715   intros n2_43a.
1716   MP n3_2a n2_43a.
1717   specialize Simp3_26 with P P.
1718   intros Simp3_26a.
1719   Conj n2_43a Simp3_26a.
1720   split.
1721   apply n2_43a.
1722   apply Simp3_26a.
1723   Equiv H.
1724   apply H.
1725   apply Equiv4_01.
1726 Qed.
1727
1728 Theorem n4_25 :  $\forall$  P : Prop,
1729   P  $\leftrightarrow$  (P  $\vee$  P).
1730 Proof. intros P.
1731   specialize Add1_3 with P P.
1732   intros Add1_3a.
1733   specialize Taut1_2 with P.
1734   intros Taut1_2a.
1735   Conj Add1_3a Taut1_2a.
1736   split.
1737   apply Add1_3a.
1738   apply Taut1_2a.
1739   Equiv H. apply H.
1740   apply Equiv4_01.
1741 Qed.
1742
1743 Theorem n4_3 :  $\forall$  P Q : Prop,
1744   (P  $\wedge$  Q)  $\leftrightarrow$  (Q  $\wedge$  P).

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1745 Proof. intros P Q.
1746   specialize n3_22 with P Q.
1747   intros n3_22a.
1748   specialize n3_22 with Q P.
1749   intros n3_22b.
1750   Conj n3_22a n3_22b.
1751   split.
1752   apply n3_22a.
1753   apply n3_22b.
1754   Equiv H. apply H.
1755   apply Equiv4_01.
1756 Qed.
1757
1758 Theorem n4_31 :  $\forall$  P Q : Prop,
1759   (P  $\vee$  Q)  $\leftrightarrow$  (Q  $\vee$  P).
1760 Proof. intros P Q.
1761   specialize Perm1_4 with P Q.
1762   intros Perm1_4a.
1763   specialize Perm1_4 with Q P.
1764   intros Perm1_4b.
1765   Conj Perm1_4a Perm1_4b.
1766   split.
1767   apply Perm1_4a.
1768   apply Perm1_4b.
1769   Equiv H. apply H.
1770   apply Equiv4_01.
1771 Qed.
1772
1773 Theorem n4_32 :  $\forall$  P Q R : Prop,
1774   ((P  $\wedge$  Q)  $\wedge$  R)  $\leftrightarrow$  (P  $\wedge$  (Q  $\wedge$  R)).
1775 Proof. intros P Q R.
1776   specialize n4_15 with P Q R.
1777   intros n4_15a.
1778   specialize Trans4_1 with P (~ (Q  $\wedge$  R)).
1779   intros Trans4_1a.
1780   replace (~ (~ (Q  $\wedge$  R))) with (Q  $\wedge$  R) in Trans4_1a.
1781   replace (Q  $\wedge$  R  $\rightarrow$  ~P) with (P  $\rightarrow$  ~ (Q  $\wedge$  R)) in n4_15a.
1782   specialize Trans4_11 with (P  $\wedge$  Q  $\rightarrow$  ~R) (P  $\rightarrow$  ~ (Q  $\wedge$  R)).
1783   intros Trans4_11a.
1784   replace ((P  $\wedge$  Q  $\rightarrow$  ~R)  $\leftrightarrow$  (P  $\rightarrow$  ~ (Q  $\wedge$  R))) with
1785     (~ (P  $\wedge$  Q  $\rightarrow$  ~R)  $\leftrightarrow$  ~ (P  $\rightarrow$  ~ (Q  $\wedge$  R))) in n4_15a.
1786   replace (P  $\wedge$  Q  $\rightarrow$  ~R) with

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1787      ( $\sim(P \wedge Q) \vee \sim R$ ) in n4_15a.
1788  replace ( $P \rightarrow \sim(Q \wedge R)$ ) with
1789      ( $\sim P \vee \sim(Q \wedge R)$ ) in n4_15a.
1790  replace ( $\sim(\sim(P \wedge Q) \vee \sim R)$ ) with
1791      ( $(P \wedge Q) \wedge R$ ) in n4_15a.
1792  replace ( $\sim(\sim P \vee \sim(Q \wedge R))$ ) with
1793      ( $P \wedge (Q \wedge R)$ ) in n4_15a.
1794  apply n4_15a.
1795  apply Prod3_01.
1796  apply Prod3_01.
1797  rewrite Impl1_01.
1798  reflexivity.
1799  rewrite Impl1_01.
1800  reflexivity.
1801  replace ( $\sim(P \wedge Q \rightarrow \sim R) \leftrightarrow \sim(P \rightarrow \sim(Q \wedge R))$ ) with
1802      ( $(P \wedge Q \rightarrow \sim R) \leftrightarrow (P \rightarrow \sim(Q \wedge R))$ ).
1803  reflexivity.
1804  apply EqBi.
1805  apply Trans4_11a.
1806  apply EqBi.
1807  apply Trans4_1a.
1808  apply EqBi.
1809  specialize n4_13 with ( $Q \wedge R$ ).
1810  intros n4_13a.
1811  apply n4_13a.
1812  Qed.
1813  (*Note that the actual proof uses n4_12, but
1814    that transposition involves transforming a
1815    biconditional into a conditional. This way
1816    of doing it - using Trans4_1 to transpose a
1817    conditional and then applying n4_13 to
1818    double negate - is easier without a derived
1819    rule for replacing a biconditional with one
1820    of its equivalent implications.*)
1821
1822  Theorem n4_33 :  $\forall P Q R : \text{Prop}$ ,
1823      ( $P \vee (Q \vee R)$ )  $\leftrightarrow$  ( $(P \vee Q) \vee R$ ).
1824  Proof. intros P Q R.
1825      specialize n2_31 with P Q R.
1826      intros n2_31a.
1827      specialize n2_32 with P Q R.
1828      intros n2_32a.

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1829     Conj n2_31a n2_32a.
1830     split.
1831     apply n2_31a.
1832     apply n2_32a.
1833     Equiv H.
1834     apply H.
1835     apply Equiv4_01.
1836     Qed.
1837
1838 Axiom Abb4_34 :  $\forall$  P Q R : Prop,
1839   P  $\wedge$  Q  $\wedge$  R = ((P  $\wedge$  Q)  $\wedge$  R).
1840   (*This axiom ensures left association of brackets.
1841   Coq's default is right association. But Principia
1842   proves associativity of logical product as n4_32.
1843   So in effect, this axiom gives us a derived rule that
1844   allows us to shift between Coq's and Principia's
1845   default rules for brackets of logical products.*)
1846
1847 Theorem n4_36 :  $\forall$  P Q R : Prop,
1848   (P  $\leftrightarrow$  Q)  $\rightarrow$  ((P  $\wedge$  R)  $\leftrightarrow$  (Q  $\wedge$  R)).
1849 Proof. intros P Q R.
1850   specialize Fact3_45 with P Q R.
1851   intros Fact3_45a.
1852   specialize Fact3_45 with Q P R.
1853   intros Fact3_45b.
1854   Conj Fact3_45a Fact3_45b.
1855   split.
1856   apply Fact3_45a.
1857   apply Fact3_45b.
1858   specialize n3_47 with (P $\rightarrow$ Q) (Q $\rightarrow$ P)
1859     (P  $\wedge$  R  $\rightarrow$  Q  $\wedge$  R) (Q  $\wedge$  R  $\rightarrow$  P  $\wedge$  R).
1860   intros n3_47a.
1861   MP n3_47 H.
1862   replace ((P  $\rightarrow$  Q)  $\wedge$  (Q  $\rightarrow$  P)) with (P $\leftrightarrow$ Q) in n3_47a.
1863   replace ((P  $\wedge$  R  $\rightarrow$  Q  $\wedge$  R)  $\wedge$  (Q  $\wedge$  R  $\rightarrow$  P  $\wedge$  R)) with
1864     (P  $\wedge$  R  $\leftrightarrow$  Q  $\wedge$  R) in n3_47a.
1865   apply n3_47a.
1866   apply Equiv4_01.
1867   apply Equiv4_01.
1868   Qed.
1869
1870 Theorem n4_37 :  $\forall$  P Q R : Prop,

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1871   (P ↔ Q) → ((P ∨ R) ↔ (Q ∨ R)).
1872 Proof. intros P Q R.
1873   specialize Sum1_6 with R P Q.
1874   intros Sum1_6a.
1875   specialize Sum1_6 with R Q P.
1876   intros Sum1_6b.
1877   Conj Sum1_6a Sum1_6b.
1878   split.
1879   apply Sum1_6a.
1880   apply Sum1_6b.
1881   specialize n3_47 with (P → Q) (Q → P)
1882     (R ∨ P → R ∨ Q) (R ∨ Q → R ∨ P).
1883   intros n3_47a.
1884   MP n3_47 H.
1885   replace ((P → Q) ∧ (Q → P)) with (P ↔ Q) in n3_47a.
1886   replace ((R ∨ P → R ∨ Q) ∧ (R ∨ Q → R ∨ P)) with
1887     (R ∨ P ↔ R ∨ Q) in n3_47a.
1888   replace (R ∨ P) with (P ∨ R) in n3_47a.
1889   replace (R ∨ Q) with (Q ∨ R) in n3_47a.
1890   apply n3_47a.
1891   apply EqBi.
1892   specialize n4_31 with Q R.
1893   intros n4_31a.
1894   apply n4_31a.
1895   apply EqBi.
1896   specialize n4_31 with P R.
1897   intros n4_31b.
1898   apply n4_31b.
1899   apply Equiv4_01.
1900   apply Equiv4_01.
1901   Qed.
1902
1903 Theorem n4_38 : ∀ P Q R S : Prop,
1904   ((P ↔ R) ∧ (Q ↔ S)) → ((P ∧ Q) ↔ (R ∧ S)).
1905 Proof. intros P Q R S.
1906   specialize n3_47 with P Q R S.
1907   intros n3_47a.
1908   specialize n3_47 with R S P Q.
1909   intros n3_47b.
1910   Conj n3_47a n3_47b.
1911   split.
1912   apply n3_47a.

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1913 apply n3_47b.
1914 specialize n3_47 with ((P→R) ∧ (Q→S))
1915 ((R→P) ∧ (S→Q)) (P ∧ Q → R ∧ S) (R ∧ S → P ∧ Q).
1916 intros n3_47c.
1917 MP n3_47c H.
1918 specialize n4_32 with (P→R) (Q→S) ((R→P) ∧ (S → Q)).
1919 intros n4_32a.
1920 replace (((P → R) ∧ (Q → S)) ∧ (R → P) ∧ (S → Q)) with
1921 ((P → R) ∧ (Q → S) ∧ (R → P) ∧ (S → Q)) in n3_47c.
1922 specialize n4_32 with (Q→S) (R→P) (S → Q).
1923 intros n4_32b.
1924 replace ((Q → S) ∧ (R → P) ∧ (S → Q)) with
1925 (((Q → S) ∧ (R → P)) ∧ (S → Q)) in n3_47c.
1926 specialize n3_22 with (Q→S) (R→P).
1927 intros n3_22a.
1928 specialize n3_22 with (R→P) (Q→S).
1929 intros n3_22b.
1930 Conj n3_22a n3_22b.
1931 split.
1932 apply n3_22a.
1933 apply n3_22b.
1934 Equiv H0.
1935 replace ((Q → S) ∧ (R → P)) with
1936 ((R → P) ∧ (Q → S)) in n3_47c.
1937 specialize n4_32 with (R → P) (Q → S) (S → Q).
1938 intros n4_32c.
1939 replace (((R → P) ∧ (Q → S)) ∧ (S → Q)) with
1940 ((R → P) ∧ (Q → S) ∧ (S → Q)) in n3_47c.
1941 specialize n4_32 with (P→R) (R → P) ((Q → S) ∧ (S → Q)).
1942 intros n4_32d.
1943 replace ((P → R) ∧ (R → P) ∧ (Q → S) ∧ (S → Q)) with
1944 (((P → R) ∧ (R → P)) ∧ (Q → S) ∧ (S → Q)) in n3_47c.
1945 replace ((P→R) ∧ (R → P)) with (P↔R) in n3_47c.
1946 replace ((Q → S) ∧ (S → Q)) with (Q↔S) in n3_47c.
1947 replace ((P ∧ Q → R ∧ S) ∧ (R ∧ S → P ∧ Q)) with
1948 ((P ∧ Q) ↔ (R ∧ S)) in n3_47c.
1949 apply n3_47c.
1950 apply Equiv4_01.
1951 apply Equiv4_01.
1952 apply Equiv4_01.
1953 apply EqBi.
1954 apply n4_32d.

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1955   replace ((R → P) ∧ (Q → S) ∧ (S → Q)) with
1956         (((R → P) ∧ (Q → S)) ∧ (S → Q)).
1957   reflexivity.
1958   apply EqBi.
1959   apply n4_32c.
1960   replace ((R → P) ∧ (Q → S)) with ((Q → S) ∧ (R → P)).
1961   reflexivity.
1962   apply EqBi.
1963   apply H0.
1964   apply Equiv4_01.
1965   apply EqBi.
1966   apply n4_32b.
1967   replace ((P → R) ∧ (Q → S) ∧ (R → P) ∧ (S → Q)) with
1968         (((P → R) ∧ (Q → S)) ∧ (R → P) ∧ (S → Q)).
1969   reflexivity.
1970   apply EqBi.
1971   apply n4_32a.
1972   Qed.
1973
1974   Theorem n4_39 : ∀ P Q R S : Prop,
1975     ((P ↔ R) ∧ (Q ↔ S)) → ((P ∨ Q) ↔ (R ∨ S)).
1976   Proof.  intros P Q R S.
1977     specialize n3_48 with P Q R S.
1978     intros n3_48a.
1979     specialize n3_48 with R S P Q.
1980     intros n3_48b.
1981     Conj n3_48a n3_48b.
1982     split.
1983     apply n3_48a.
1984     apply n3_48b.
1985     specialize n3_47 with ((P → R) ∧ (Q → S))
1986           ((R → P) ∧ (S → Q)) (P ∨ Q → R ∨ S) (R ∨ S → P ∨ Q).
1987     intros n3_47a.
1988     MP n3_47a H.
1989     replace ((P ∨ Q → R ∨ S) ∧ (R ∨ S → P ∨ Q)) with
1990           ((P ∨ Q) ↔ (R ∨ S)) in n3_47a.
1991     specialize n4_32 with ((P → R) ∧ (Q → S)) (R → P) (S → Q).
1992     intros n4_32a.
1993     replace (((P → R) ∧ (Q → S)) ∧ (R → P) ∧ (S → Q)) with
1994           (((P → R) ∧ (Q → S)) ∧ (R → P)) ∧ (S → Q) in n3_47a.
1995     specialize n4_32 with (P → R) (Q → S) (R → P).
1996     intros n4_32b.

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1997  replace (((P → R) ∧ (Q → S)) ∧ (R → P)) with
1998      ((P → R) ∧ (Q → S) ∧ (R → P)) in n3_47a.
1999  specialize n3_22 with (Q → S) (R → P).
2000  intros n3_22a.
2001  specialize n3_22 with (R → P) (Q → S).
2002  intros n3_22b.
2003  Conj n3_22a n3_22b.
2004  split.
2005  apply n3_22a.
2006  apply n3_22b.
2007  Equiv H0.
2008  replace ((Q → S) ∧ (R → P)) with
2009      ((R → P) ∧ (Q → S)) in n3_47a.
2010  specialize n4_32 with (P → R) (R → P) (Q → S).
2011  intros n4_32c.
2012  replace ((P → R) ∧ (R → P) ∧ (Q → S)) with
2013      (((P → R) ∧ (R → P)) ∧ (Q → S)) in n3_47a.
2014  replace ((P → R) ∧ (R → P)) with (P ↔ R) in n3_47a.
2015  specialize n4_32 with (P ↔ R) (Q → S) (S → Q).
2016  intros n4_32d.
2017  replace (((P ↔ R) ∧ (Q → S)) ∧ (S → Q)) with
2018      ((P ↔ R) ∧ (Q → S) ∧ (S → Q)) in n3_47a.
2019  replace ((Q → S) ∧ (S → Q)) with (Q ↔ S) in n3_47a.
2020  apply n3_47a.
2021  apply Equiv4_01.
2022  replace ((P ↔ R) ∧ (Q → S) ∧ (S → Q)) with
2023      (((P ↔ R) ∧ (Q → S)) ∧ (S → Q)).
2024  reflexivity.
2025  apply EqBi.
2026  apply n4_32d.
2027  apply Equiv4_01.
2028  apply EqBi.
2029  apply n4_32c.
2030  replace ((R → P) ∧ (Q → S)) with ((Q → S) ∧ (R → P)).
2031  reflexivity.
2032  apply EqBi.
2033  apply H0.
2034  apply Equiv4_01.
2035  replace ((P → R) ∧ (Q → S) ∧ (R → P)) with
2036      (((P → R) ∧ (Q → S)) ∧ (R → P)).
2037  reflexivity.
2038  apply EqBi.

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2039   apply n4_32b.
2040   apply EqBi.
2041   apply n4_32a.
2042   apply Equiv4_01.
2043   Qed.
2044
2045   Theorem n4_4 :  $\forall$  P Q R : Prop,
2046     (P  $\wedge$  (Q  $\vee$  R))  $\leftrightarrow$  ((P $\wedge$  Q)  $\vee$  (P  $\wedge$  R)).
2047   Proof. intros P Q R.
2048     specialize n3_2 with P Q.
2049     intros n3_2a.
2050     specialize n3_2 with P R.
2051     intros n3_2b.
2052     Conj n3_2a n3_2b.
2053     split.
2054     apply n3_2a.
2055     apply n3_2b.
2056     specialize Comp3_43 with P (Q $\rightarrow$ P $\wedge$ Q) (R $\rightarrow$ P $\wedge$ R).
2057     intros Comp3_43a.
2058     MP Comp3_43a H.
2059     specialize n3_48 with Q R (P $\wedge$ Q) (P $\wedge$ R).
2060     intros n3_48a.
2061     Syll Comp3_43a n3_48a Sa.
2062     specialize Imp3_31 with P (Q $\vee$ R) ((P $\wedge$  Q)  $\vee$  (P  $\wedge$  R)).
2063     intros Imp3_31a.
2064     MP Imp3_31a Sa.
2065     specialize Simp3_26 with P Q.
2066     intros Simp3_26a.
2067     specialize Simp3_26 with P R.
2068     intros Simp3_26b.
2069     Conj Simp3_26a Simp3_26b.
2070     split.
2071     apply Simp3_26a.
2072     apply Simp3_26b.
2073     specialize n3_44 with P (P $\wedge$ Q) (P $\wedge$ R).
2074     intros n3_44a.
2075     MP n3_44a H0.
2076     specialize Simp3_27 with P Q.
2077     intros Simp3_27a.
2078     specialize Simp3_27 with P R.
2079     intros Simp3_27b.
2080     Conj Simp3_27a Simp3_27b.

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2081 split.
2082 apply Simp3_27a.
2083 apply Simp3_27b.
2084 specialize n3_48 with (P $\wedge$ Q) (P $\wedge$ R) Q R.
2085 intros n3_48b.
2086 MP n3_48b H1.
2087 clear H1. clear Simp3_27a. clear Simp3_27b.
2088 Conj n3_44a n3_48b.
2089 split.
2090 apply n3_44a.
2091 apply n3_48b.
2092 specialize Comp3_43 with (P  $\wedge$  Q  $\vee$  P  $\wedge$  R) P (Q $\vee$ R).
2093 intros Comp3_43b.
2094 MP Comp3_43b H1.
2095 clear H1. clear H0. clear n3_44a. clear n3_48b.
2096 clear Simp3_26a. clear Simp3_26b.
2097 Conj Imp3_31a Comp3_43b.
2098 split.
2099 apply Imp3_31a.
2100 apply Comp3_43b.
2101 Equiv H0.
2102 apply H0.
2103 apply Equiv4_01.
2104 Qed.
2105
2106 Theorem n4_41 :  $\forall$  P Q R : Prop,
2107   (P  $\vee$  (Q  $\wedge$  R))  $\leftrightarrow$  ((P  $\vee$  Q)  $\wedge$  (P  $\vee$  R)).
2108 Proof. intros P Q R.
2109   specialize Simp3_26 with Q R.
2110   intros Simp3_26a.
2111   specialize Sum1_6 with P (Q  $\wedge$  R) Q.
2112   intros Sum1_6a.
2113   MP Simp3_26a Sum1_6a.
2114   specialize Simp3_27 with Q R.
2115   intros Simp3_27a.
2116   specialize Sum1_6 with P (Q  $\wedge$  R) R.
2117   intros Sum1_6b.
2118   MP Simp3_27a Sum1_6b.
2119   clear Simp3_26a. clear Simp3_27a.
2120   Conj Sum1_6a Sum1_6a.
2121   split.
2122   apply Sum1_6a.

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2123   apply Sum1_6b.
2124   specialize Comp3_43 with (P ∨ Q ∧ R) (P ∨ Q) (P ∨ R).
2125   intros Comp3_43a.
2126   MP Comp3_43a H.
2127   specialize n2_53 with P Q.
2128   intros n2_53a.
2129   specialize n2_53 with P R.
2130   intros n2_53b.
2131   Conj n2_53a n2_53b.
2132   split.
2133   apply n2_53a.
2134   apply n2_53b.
2135   specialize n3_47 with (P ∨ Q) (P ∨ R) (~P → Q) (~P → R).
2136   intros n3_47a.
2137   MP n3_47a H0.
2138   specialize Comp3_43 with (~P) Q R.
2139   intros Comp3_43b.
2140   Syll n3_47a Comp3_43b Sa.
2141   specialize n2_54 with P (Q ∧ R).
2142   intros n2_54a.
2143   Syll Sa n2_54a Sb.
2144   clear Sum1_6a. clear Sum1_6b. clear H. clear n2_53a.
2145       clear n2_53b. clear H0. clear n3_47a. clear Sa.
2146       clear Comp3_43b. clear n2_54a.
2147   Conj Comp3_43a Sb.
2148   split.
2149   apply Comp3_43a.
2150   apply Sb.
2151   Equiv H.
2152   apply H.
2153   apply Equiv4_01.
2154   Qed.
2155
2156   Theorem n4_42 : ∀ P Q : Prop,
2157       P ↔ ((P ∧ Q) ∨ (P ∧ ~Q)).
2158   Proof. intros P Q.
2159       specialize n3_21 with P (Q ∨ ~Q).
2160       intros n3_21a.
2161       specialize n2_11 with Q.
2162       intros n2_11a.
2163       MP n3_21a n2_11a.
2164       specialize Simp3_26 with P (Q ∨ ~Q).

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2165   intros Simp3_26a. clear n2_11a.
2166   Conj n3_21a Simp3_26a.
2167   split.
2168   apply n3_21a.
2169   apply Simp3_26a.
2170   Equiv H.
2171   specialize n4_4 with P Q (~Q).
2172   intros n4_4a.
2173   replace (P ∧ (Q ∨ ~Q)) with P in n4_4a.
2174   apply n4_4a.
2175   apply EqBi.
2176   apply H.
2177   apply Equiv4_01.
2178   Qed.
2179
2180   Theorem n4_43 : ∀ P Q : Prop,
2181     P ↔ ((P ∨ Q) ∧ (P ∨ ~Q)).
2182   Proof. intros P Q.
2183     specialize n2_2 with P Q.
2184     intros n2_2a.
2185     specialize n2_2 with P (~Q).
2186     intros n2_2b.
2187     Conj n2_2a n2_2b.
2188     split.
2189     apply n2_2a.
2190     apply n2_2b.
2191     specialize Comp3_43 with P (P∨Q) (P∨~Q).
2192     intros Comp3_43a.
2193     MP Comp3_43a H.
2194     specialize n2_53 with P Q.
2195     intros n2_53a.
2196     specialize n2_53 with P (~Q).
2197     intros n2_53b.
2198     Conj n2_53a n2_53b.
2199     split.
2200     apply n2_53a.
2201     apply n2_53b.
2202     specialize n3_47 with (P∨Q) (P∨~Q) (~P→Q) (~P→~Q).
2203     intros n3_47a.
2204     MP n3_47a H0.
2205     specialize n2_65 with (~P) Q.
2206     intros n2_65a.

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2207   replace (~~P) with P in n2_65a.
2208   specialize Imp3_31 with (~P → Q) (~P → ~Q) (P).
2209   intros Imp3_31a.
2210   MP Imp3_31a n2_65a.
2211   Syll n3_47a Imp3_31a Sa.
2212   clear n2_2a. clear n2_2b. clear H. clear n2_53a. clear n2_53b.
2213       clear H0. clear n2_65a. clear n3_47a. clear Imp3_31a.
2214   Conj Comp3_43a Sa.
2215   split.
2216   apply Comp3_43a.
2217   apply Sa.
2218   Equiv H.
2219   apply H.
2220   apply Equiv4_01.
2221   apply EqBi.
2222   specialize n4_13 with P.
2223   intros n4_13a.
2224   apply n4_13a.
2225   Qed.
2226
2227   Theorem n4_44 : ∀ P Q : Prop,
2228       P ↔ (P ∨ (P ∧ Q)).
2229   Proof. intros P Q.
2230       specialize n2_2 with P (P ∧ Q).
2231       intros n2_2a.
2232       specialize n2_08 with P.
2233       intros n2_08a.
2234       specialize Simp3_26 with P Q.
2235       intros Simp3_26a.
2236       Conj n2_08a Simp3_26a.
2237       split.
2238       apply n2_08a.
2239       apply Simp3_26a.
2240       specialize n3_44 with P P (P ∧ Q).
2241       intros n3_44a.
2242       MP n3_44a H.
2243       clear H. clear n2_08a. clear Simp3_26a.
2244       Conj n2_2a n3_44a.
2245       split.
2246       apply n2_2a.
2247       apply n3_44a.
2248       Equiv H.

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2249     apply H.
2250     apply Equiv4_01.
2251 Qed.
2252
2253 Theorem n4_45 :  $\forall$  P Q : Prop,
2254   P  $\leftrightarrow$  (P  $\wedge$  (P  $\vee$  Q)).
2255 Proof. intros P Q.
2256   specialize n2_2 with (P  $\wedge$  P) (P  $\wedge$  Q).
2257   intros n2_2a.
2258   replace (P  $\wedge$  P  $\vee$  P  $\wedge$  Q) with (P  $\wedge$  (P  $\vee$  Q)) in n2_2a.
2259   replace (P  $\wedge$  P) with P in n2_2a.
2260   specialize Simp3_26 with P (P  $\vee$  Q).
2261   intros Simp3_26a.
2262   Conj n2_2a Simp3_26a.
2263   split.
2264   apply n2_2a.
2265   apply Simp3_26a.
2266   Equiv H.
2267   apply H.
2268   apply Equiv4_01.
2269   specialize n4_24 with P.
2270   intros n4_24a.
2271   apply EqBi.
2272   apply n4_24a.
2273   specialize n4_4 with P P Q.
2274   intros n4_4a.
2275   apply EqBi.
2276   apply n4_4a.
2277 Qed.
2278
2279 Theorem n4_5 :  $\forall$  P Q : Prop,
2280   P  $\wedge$  Q  $\leftrightarrow$   $\sim(\sim P \vee \sim Q)$ .
2281 Proof. intros P Q.
2282   specialize n4_2 with (P  $\wedge$  Q).
2283   intros n4_2a.
2284   rewrite Prod3_01.
2285   replace ( $\sim(\sim P \vee \sim Q)$ ) with (P  $\wedge$  Q).
2286   apply n4_2a.
2287   apply Prod3_01.
2288 Qed.
2289
2290 Theorem n4_51 :  $\forall$  P Q : Prop,

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2291 ~ (P ∧ Q) ↔ (¬P ∨ ¬Q).
2292 Proof. intros P Q.
2293   specialize n4_5 with P Q.
2294   intros n4_5a.
2295   specialize n4_12 with (P ∧ Q) (¬P ∨ ¬Q).
2296   intros n4_12a.
2297   replace ((P ∧ Q ↔ ¬(¬P ∨ ¬Q)) ↔ (¬P ∨ ¬Q ↔ ¬(P ∧ Q))) with
2298     ((P ∧ Q ↔ ¬(¬P ∨ ¬Q)) = (¬P ∨ ¬Q ↔ ¬(P ∧ Q))) in n4_12a.
2299   replace (P ∧ Q ↔ ¬(¬P ∨ ¬Q)) with
2300     (¬P ∨ ¬Q ↔ ¬(P ∧ Q)) in n4_5a.
2301   replace (¬P ∨ ¬Q ↔ ¬(P ∧ Q)) with
2302     (¬(P ∧ Q) ↔ (¬P ∨ ¬Q)) in n4_5a.
2303   apply n4_5a.
2304   specialize n4_21 with (¬(P ∧ Q)) (¬P ∨ ¬Q).
2305   intros n4_21a.
2306   apply EqBi.
2307   specialize n4_21 with (¬(P ∧ Q)) (¬P ∨ ¬Q).
2308   intros n4_21b.
2309   apply n4_21b.
2310   apply EqBi.
2311   apply EqBi.
2312 Qed.
2313
2314 Theorem n4_52 : ∀ P Q : Prop,
2315   (P ∧ ¬Q) ↔ ¬(¬P ∨ Q).
2316 Proof. intros P Q.
2317   specialize n4_5 with P (¬Q).
2318   intros n4_5a.
2319   replace (¬¬Q) with Q in n4_5a.
2320   apply n4_5a.
2321   specialize n4_13 with Q.
2322   intros n4_13a.
2323   apply EqBi.
2324   apply n4_13a.
2325 Qed.
2326
2327 Theorem n4_53 : ∀ P Q : Prop,
2328   ¬(P ∧ ¬Q) ↔ (¬P ∨ Q).
2329 Proof. intros P Q.
2330   specialize n4_52 with P Q.
2331   intros n4_52a.
2332   specialize n4_12 with (P ∧ ¬Q) (¬P ∨ Q).

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```

2333     intros n4_12a.
2334     replace ((P ∧ ~Q ↔ ~(~P ∨ Q)) ↔ (~P ∨ Q ↔ ~(P ∧ ~Q))) with
2335         ((P ∧ ~Q ↔ ~(~P ∨ Q)) = (~P ∨ Q ↔ ~(P ∧ ~Q))) in n4_12a.
2336     replace (P ∧ ~Q ↔ ~(~P ∨ Q)) with
2337         (~P ∨ Q ↔ ~(P ∧ ~Q)) in n4_52a.
2338     replace (~P ∨ Q ↔ ~(P ∧ ~Q)) with
2339         (~P ∧ ~Q ↔ (~P ∨ Q)) in n4_52a.
2340     apply n4_52a.
2341     specialize n4_21 with (~P ∧ ~Q) (~P ∨ Q).
2342     intros n4_21a.
2343     apply EqBi.
2344     apply n4_21a.
2345     apply EqBi.
2346     apply EqBi.
2347     Qed.
2348
2349 Theorem n4_54 : ∀ P Q : Prop,
2350     (~P ∧ Q) ↔ ~(P ∨ ~Q).
2351 Proof. intros P Q.
2352     specialize n4_5 with (~P) Q.
2353     intros n4_5a.
2354     specialize n4_13 with P.
2355     intros n4_13a.
2356     replace (~~P) with P in n4_5a.
2357     apply n4_5a.
2358     apply EqBi.
2359     apply n4_13a.
2360     Qed.
2361
2362 Theorem n4_55 : ∀ P Q : Prop,
2363     ~(~P ∧ Q) ↔ (P ∨ ~Q).
2364 Proof. intros P Q.
2365     specialize n4_54 with P Q.
2366     intros n4_54a.
2367     specialize n4_12 with (~P ∧ Q) (P ∨ ~Q).
2368     intros n4_12a.
2369     replace (~P ∧ Q ↔ ~(P ∨ ~Q)) with
2370         (P ∨ ~Q ↔ ~(~P ∧ Q)) in n4_54a.
2371     replace (P ∨ ~Q ↔ ~(~P ∧ Q)) with
2372         (~(~P ∧ Q) ↔ (P ∨ ~Q)) in n4_54a.
2373     apply n4_54a.
2374     specialize n4_21 with (~(~P ∧ Q)) (P ∨ ~Q).

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2375     intros n4_21a.
2376     apply EqBi.
2377     apply n4_21a.
2378     replace ((~P ∧ Q ↔ ~(P ∨ ~Q)) ↔ (P ∨ ~Q ↔ ~(~P ∧ Q))) with
2379         ((~P ∧ Q ↔ ~(P ∨ ~Q)) = (P ∨ ~Q ↔ ~(~P ∧ Q))) in n4_12a.
2380     rewrite n4_12a.
2381     reflexivity.
2382     apply EqBi.
2383     apply EqBi.
2384     Qed.
2385
2386 Theorem n4_56 : ∀ P Q : Prop,
2387     (~P ∧ ~Q) ↔ ~(P ∨ Q).
2388 Proof. intros P Q.
2389     specialize n4_54 with P (~Q).
2390     intros n4_54a.
2391     replace (~~Q) with Q in n4_54a.
2392     apply n4_54a.
2393     apply EqBi.
2394     specialize n4_13 with Q.
2395     intros n4_13a.
2396     apply n4_13a.
2397     Qed.
2398
2399 Theorem n4_57 : ∀ P Q : Prop,
2400     ~(~P ∧ ~Q) ↔ (P ∨ Q).
2401 Proof. intros P Q.
2402     specialize n4_56 with P Q.
2403     intros n4_56a.
2404     specialize n4_12 with (~P ∧ ~Q) (P ∨ Q).
2405     intros n4_12a.
2406     replace (~P ∧ ~Q ↔ ~(P ∨ Q)) with
2407         (P ∨ Q ↔ ~(~P ∧ ~Q)) in n4_56a.
2408     replace (P ∨ Q ↔ ~(~P ∧ ~Q)) with
2409         (~(~P ∧ ~Q) ↔ P ∨ Q) in n4_56a.
2410     apply n4_56a.
2411     specialize n4_21 with (~(~P ∧ ~Q)) (P ∨ Q).
2412     intros n4_21a.
2413     apply EqBi.
2414     apply n4_21a.
2415     replace ((~P ∧ ~Q ↔ ~(P ∨ Q)) ↔ (P ∨ Q ↔ ~(~P ∧ ~Q))) with
2416         ((P ∨ Q ↔ ~(~P ∧ ~Q)) ↔ (~P ∧ ~Q ↔ ~(P ∨ Q))) in n4_12a.

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2417     apply EqBi.
2418     apply n4_12a.
2419     apply EqBi.
2420     specialize n4_21 with
2421       (P ∨ Q ↔ ~(~P ∧ ~Q)) (~P ∧ ~Q ↔ ~(P ∨ Q)).
2422     intros n4_21b.
2423     apply n4_21b.
2424   Qed.
2425
2426 Theorem n4_6 : ∀ P Q : Prop,
2427   (P → Q) ↔ (~P ∨ Q).
2428 Proof. intros P Q.
2429   specialize n4_2 with (~P ∨ Q).
2430   intros n4_2a.
2431   rewrite Impl1_01.
2432   apply n4_2a.
2433   Qed.
2434
2435 Theorem n4_61 : ∀ P Q : Prop,
2436   ~(P → Q) ↔ (P ∧ ~Q).
2437 Proof. intros P Q.
2438   specialize n4_6 with P Q.
2439   intros n4_6a.
2440   specialize Trans4_11 with (P→Q) (~P∨Q).
2441   intros Trans4_11a.
2442   specialize n4_52 with P Q.
2443   intros n4_52a.
2444   replace ((P → Q) ↔ ~P ∨ Q) with
2445     (~ (P → Q) ↔ ~ (~P ∨ Q)) in n4_6a.
2446   replace (~ (~P ∨ Q)) with (P ∧ ~Q) in n4_6a.
2447   apply n4_6a.
2448   apply EqBi.
2449   apply n4_52a.
2450   replace (((P→Q)↔~P∨Q)↔(~(P→Q)↔~(~P∨Q))) with
2451     ((~(P→Q)↔~(~P∨Q))↔((P→Q)↔~P∨Q)) in Trans4_11a.
2452   apply EqBi.
2453   apply Trans4_11a.
2454   apply EqBi.
2455   specialize n4_21 with (~ (P→Q) ↔ ~ (~P∨Q))
2456     ((P→Q)↔(~P∨Q)).
2457   intros n4_21a.
2458   apply n4_21a.

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2459   Qed.
2460
2461   Theorem n4_62 :  $\forall P Q : \text{Prop},$ 
2462      $(P \rightarrow \sim Q) \leftrightarrow (\sim P \vee \sim Q).$ 
2463   Proof. intros P Q.
2464     specialize n4_6 with P ( $\sim Q$ ).
2465     intros n4_6a.
2466     apply n4_6a.
2467   Qed.
2468
2469   Theorem n4_63 :  $\forall P Q : \text{Prop},$ 
2470      $\sim(P \rightarrow \sim Q) \leftrightarrow (P \wedge Q).$ 
2471   Proof. intros P Q.
2472     specialize n4_62 with P Q.
2473     intros n4_62a.
2474     specialize Trans4_11 with  $(P \rightarrow \sim Q) (\sim P \vee \sim Q).$ 
2475     intros Trans4_11a.
2476     specialize n4_5 with P Q.
2477     intros n4_5a.
2478     replace  $(\sim(\sim P \vee \sim Q))$  with  $(P \wedge Q)$  in Trans4_11a.
2479     replace  $((P \rightarrow \sim Q) \leftrightarrow \sim P \vee \sim Q)$  with
2480        $((\sim(P \rightarrow \sim Q) \leftrightarrow P \wedge Q))$  in n4_62a.
2481     apply n4_62a.
2482     replace  $((\sim(P \rightarrow \sim Q) \leftrightarrow \sim P \vee \sim Q) \leftrightarrow (\sim(P \rightarrow \sim Q) \leftrightarrow P \wedge Q))$  with
2483        $((\sim(P \rightarrow \sim Q) \leftrightarrow P \wedge Q) \leftrightarrow ((P \rightarrow \sim Q) \leftrightarrow \sim P \vee \sim Q))$  in Trans4_11a.
2484     apply EqBi.
2485     apply Trans4_11a.
2486     specialize n4_21 with
2487        $(\sim(P \rightarrow \sim Q) \leftrightarrow P \wedge Q) ((P \rightarrow \sim Q) \leftrightarrow \sim P \vee \sim Q).$ 
2488     intros n4_21a.
2489     apply EqBi.
2490     apply n4_21a.
2491     apply EqBi.
2492     apply n4_5a.
2493   Qed.
2494
2495   Theorem n4_64 :  $\forall P Q : \text{Prop},$ 
2496      $(\sim P \rightarrow Q) \leftrightarrow (P \vee Q).$ 
2497   Proof. intros P Q.
2498     specialize n2_54 with P Q.
2499     intros n2_54a.
2500     specialize n2_53 with P Q.

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2501     intros n2_53a.
2502     Conj n2_54a n2_53a.
2503     split.
2504     apply n2_54a.
2505     apply n2_53a.
2506     Equiv H.
2507     apply H.
2508     apply Equiv4_01.
2509 Qed.
2510
2511 Theorem n4_65 :  $\forall P Q : \text{Prop}$ ,
2512    $\sim(\sim P \rightarrow Q) \leftrightarrow (\sim P \wedge \sim Q)$ .
2513 Proof. intros P Q.
2514 specialize n4_64 with P Q.
2515 intros n4_64a.
2516 specialize Trans4_11 with  $(\sim P \rightarrow Q) (P \vee Q)$ .
2517 intros Trans4_11a.
2518 specialize n4_56 with P Q.
2519 intros n4_56a.
2520 replace  $((\sim P \rightarrow Q) \leftrightarrow P \vee Q) \leftrightarrow (\sim(\sim P \rightarrow Q) \leftrightarrow \sim(P \vee Q))$  with
2521    $((\sim(\sim P \rightarrow Q) \leftrightarrow \sim(P \vee Q)) \leftrightarrow ((\sim P \rightarrow Q) \leftrightarrow P \vee Q))$  in Trans4_11a.
2522 replace  $(\sim P \rightarrow Q) \leftrightarrow P \vee Q$  with
2523    $(\sim(\sim P \rightarrow Q) \leftrightarrow \sim(P \vee Q))$  in n4_64a.
2524 replace  $(\sim(P \vee Q))$  with  $(\sim P \wedge \sim Q)$  in n4_64a.
2525 apply n4_64a.
2526 apply EqBi.
2527 apply n4_56a.
2528 apply EqBi.
2529 apply Trans4_11a.
2530 apply EqBi.
2531 specialize n4_21 with  $(\sim(\sim P \rightarrow Q) \leftrightarrow \sim(P \vee Q))$ 
2532    $((\sim P \rightarrow Q) \leftrightarrow (P \vee Q))$ .
2533 intros n4_21a.
2534 apply n4_21a.
2535 Qed.
2536
2537 Theorem n4_66 :  $\forall P Q : \text{Prop}$ ,
2538    $(\sim P \rightarrow \sim Q) \leftrightarrow (P \vee \sim Q)$ .
2539 Proof. intros P Q.
2540 specialize n4_64 with P  $(\sim Q)$ .
2541 intros n4_64a.
2542 apply n4_64a.

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2543   Qed.
2544
2545   Theorem n4_67 :  $\forall P Q : \text{Prop},$ 
2546      $\sim(\sim P \rightarrow \sim Q) \leftrightarrow (\sim P \wedge Q).$ 
2547   Proof. intros P Q.
2548     specialize n4_66 with P Q.
2549     intros n4_66a.
2550     specialize Trans4_11 with  $(\sim P \rightarrow \sim Q) (P \vee \sim Q).$ 
2551     intros Trans4_11a.
2552     replace  $((\sim P \rightarrow \sim Q) \leftrightarrow P \vee \sim Q)$  with
2553        $(\sim(\sim P \rightarrow \sim Q) \leftrightarrow \sim(P \vee \sim Q))$  in n4_66a.
2554     specialize n4_54 with P Q.
2555     intros n4_54a.
2556     replace  $(\sim(P \vee \sim Q))$  with  $(\sim P \wedge Q)$  in n4_66a.
2557     apply n4_66a.
2558     apply EqBi.
2559     apply n4_54a.
2560     replace  $((\sim P \rightarrow \sim Q) \leftrightarrow P \vee \sim Q) \leftrightarrow (\sim(\sim P \rightarrow \sim Q) \leftrightarrow \sim(P \vee \sim Q))$  with
2561        $((\sim(\sim P \rightarrow \sim Q) \leftrightarrow \sim(P \vee \sim Q)) \leftrightarrow ((\sim P \rightarrow \sim Q) \leftrightarrow P \vee \sim Q))$  in Trans4_11a.
2562     apply EqBi.
2563     apply Trans4_11a.
2564     apply EqBi.
2565     specialize n4_21 with  $(\sim(\sim P \rightarrow \sim Q) \leftrightarrow \sim(P \vee \sim Q))$ 
2566        $((\sim P \rightarrow \sim Q) \leftrightarrow (P \vee \sim Q)).$ 
2567     intros n4_21a.
2568     apply n4_21a.
2569     Qed.
2570
2571   Theorem n4_7 :  $\forall P Q : \text{Prop},$ 
2572      $(P \rightarrow Q) \leftrightarrow (P \rightarrow (P \wedge Q)).$ 
2573   Proof. intros P Q.
2574     specialize Comp3_43 with P P Q.
2575     intros Comp3_43a.
2576     specialize Exp3_3 with
2577        $(P \rightarrow P) (P \rightarrow Q) (P \rightarrow P \wedge Q).$ 
2578     intros Exp3_3a.
2579     MP Exp3_3a Comp3_43a.
2580     specialize n2_08 with P.
2581     intros n2_08a.
2582     MP Exp3_3a n2_08a.
2583     specialize Simp3_27 with P Q.
2584     intros Simp3_27a.

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2585 specialize Syll2_05 with P (P ∧ Q) Q.
2586 intros Syll2_05a.
2587 MP Syll2_05a Simp3_27a.
2588 clear n2_08a. clear Comp3_43a. clear Simp3_27a.
2589 Conj Syll2_05a Exp3_3a.
2590 split.
2591 apply Exp3_3a.
2592 apply Syll2_05a.
2593 Equiv H.
2594 apply H.
2595 apply Equiv4_01.
2596 Qed.
2597
2598 Theorem n4_71 : ∀ P Q : Prop,
2599   (P → Q) ↔ (P ↔ (P ∧ Q)).
2600 Proof. intros P Q.
2601 specialize n4_7 with P Q.
2602 intros n4_7a.
2603 specialize n3_21 with (P→(P∧Q)) ((P∧Q)→P).
2604 intros n3_21a.
2605 replace ((P → P ∧ Q) ∧ (P ∧ Q → P)) with
2606   (P↔(P ∧ Q)) in n3_21a.
2607 specialize Simp3_26 with P Q.
2608 intros Simp3_26a.
2609 MP n3_21a Simp3_26a.
2610 specialize Simp3_26 with (P→(P∧Q)) ((P∧Q)→P).
2611 intros Simp3_26b.
2612 replace ((P → P ∧ Q) ∧ (P ∧ Q → P)) with
2613   (P↔(P ∧ Q)) in Simp3_26b. clear Simp3_26a.
2614 Conj n3_21a Simp3_26b.
2615 split.
2616 apply n3_21a.
2617 apply Simp3_26b.
2618 Equiv H.
2619 clear n3_21a. clear Simp3_26b.
2620 Conj n4_7a H.
2621 split.
2622 apply n4_7a.
2623 apply H.
2624 specialize n4_22 with (P → Q) (P → P ∧ Q) (P ↔ P ∧ Q).
2625 intros n4_22a.
2626 MP n4_22a H0.

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2627   apply n4_22a.
2628   apply Equiv4_01.
2629   apply Equiv4_01.
2630   apply Equiv4_01.
2631   Qed.
2632
2633 Theorem n4_72 :  $\forall P Q : \text{Prop},$ 
2634    $(P \rightarrow Q) \leftrightarrow (Q \leftrightarrow (P \vee Q)).$ 
2635 Proof. intros P Q.
2636   specialize Trans4_1 with P Q.
2637   intros Trans4_1a.
2638   specialize n4_71 with ( $\sim Q$ ) ( $\sim P$ ).
2639   intros n4_71a.
2640   Conj Trans4_1a n4_71a.
2641   split.
2642   apply Trans4_1a.
2643   apply n4_71a.
2644   specialize n4_22 with
2645      $(P \rightarrow Q) (\sim Q \rightarrow \sim P) (\sim Q \leftrightarrow \sim Q \wedge \sim P).$ 
2646   intros n4_22a.
2647   MP n4_22a H.
2648   specialize n4_21 with ( $\sim Q$ ) ( $\sim Q \wedge \sim P$ ).
2649   intros n4_21a.
2650   Conj n4_22a n4_21a.
2651   split.
2652   apply n4_22a.
2653   apply n4_21a.
2654   specialize n4_22 with
2655      $(P \rightarrow Q) (\sim Q \leftrightarrow \sim Q \wedge \sim P) (\sim Q \wedge \sim P \leftrightarrow \sim Q).$ 
2656   intros n4_22b.
2657   MP n4_22b H0.
2658   specialize n4_12 with ( $\sim Q \wedge \sim P$ ) (Q).
2659   intros n4_12a.
2660   Conj n4_22b n4_12a.
2661   split.
2662   apply n4_22b.
2663   apply n4_12a.
2664   specialize n4_22 with
2665      $(P \rightarrow Q) ((\sim Q \wedge \sim P) \leftrightarrow \sim Q) (Q \leftrightarrow \sim(\sim Q \wedge \sim P)).$ 
2666   intros n4_22c.
2667   MP n4_22b H0.
2668   specialize n4_57 with Q P.

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2669   intros n4_57a.
2670   replace (~(~Q ∧ ~P)) with (Q ∨ P) in n4_22c.
2671   specialize n4_31 with P Q.
2672   intros n4_31a.
2673   replace (Q ∨ P) with (P ∨ Q) in n4_22c.
2674   apply n4_22c.
2675   apply EqBi.
2676   apply n4_31a.
2677   apply EqBi.
2678   replace (~(~Q ∧ ~P) ↔ Q ∨ P) with
2679     (Q ∨ P ↔ ~(~Q ∧ ~P)) in n4_57a.
2680   apply n4_57a.
2681   apply EqBi.
2682   specialize n4_21 with (Q ∨ P) (~(~Q ∧ ~P)).
2683   intros n4_21b.
2684   apply n4_21b.
2685   Qed.
2686
2687   Theorem n4_73 : ∀ P Q : Prop,
2688     Q → (P ↔ (P ∧ Q)).
2689   Proof. intros P Q.
2690   specialize Simp2_02 with P Q.
2691   intros Simp2_02a.
2692   specialize n4_71 with P Q.
2693   intros n4_71a.
2694   replace ((P → Q) ↔ (P ↔ P ∧ Q)) with
2695     (((P → Q) → (P ↔ P ∧ Q)) ∧ ((P ↔ P ∧ Q) → (P → Q))) in n4_71a.
2696   specialize Simp3_26 with
2697     ((P → Q) → P ↔ P ∧ Q) (P ↔ P ∧ Q → P → Q).
2698   intros Simp3_26a.
2699   MP Simp3_26a n4_71a.
2700   Syll Simp2_02a Simp3_26a Sa.
2701   apply Sa.
2702   apply Equiv4_01.
2703   Qed.
2704
2705   Theorem n4_74 : ∀ P Q : Prop,
2706     ~P → (Q ↔ (P ∨ Q)).
2707   Proof. intros P Q.
2708   specialize n2_21 with P Q.
2709   intros n2_21a.
2710   specialize n4_72 with P Q.

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2711   intros n4_72a.
2712   replace (P → Q) with (Q ↔ P ∨ Q) in n2_21a.
2713   apply n2_21a.
2714   apply EqBi.
2715   replace ((P → Q) ↔ (Q ↔ P ∨ Q)) with
2716     ((Q ↔ P ∨ Q) ↔ (P → Q)) in n4_72a.
2717   apply n4_72a.
2718   apply EqBi.
2719   specialize n4_21 with (Q ↔ (P ∨ Q)) (P → Q).
2720   intros n4_21a.
2721   apply n4_21a.
2722   Qed.
2723
2724   Theorem n4_76 : ∀ P Q R : Prop,
2725     ((P → Q) ∧ (P → R)) ↔ (P → (Q ∧ R)).
2726   Proof. intros P Q R.
2727     specialize n4_41 with (¬P) Q R.
2728     intros n4_41a.
2729     replace (¬P ∨ Q) with (P → Q) in n4_41a.
2730     replace (¬P ∨ R) with (P → R) in n4_41a.
2731     replace (¬P ∨ Q ∧ R) with (P → Q ∧ R) in n4_41a.
2732     replace ((P → Q ∧ R) ↔ (P → Q) ∧ (P → R)) with
2733       ((P → Q) ∧ (P → R) ↔ (P → Q ∧ R)) in n4_41a.
2734     apply n4_41a.
2735     apply EqBi.
2736     specialize n4_21 with ((P → Q) ∧ (P → R)) (P → Q ∧ R).
2737     intros n4_21a.
2738     apply n4_21a.
2739     apply Impl1_01.
2740     apply Impl1_01.
2741     apply Impl1_01.
2742     Qed.
2743
2744   Theorem n4_77 : ∀ P Q R : Prop,
2745     ((Q → P) ∧ (R → P)) ↔ ((Q ∨ R) → P).
2746   Proof. intros P Q R.
2747     specialize n3_44 with P Q R.
2748     intros n3_44a.
2749     specialize n2_08 with (Q ∨ R → P).
2750     intros n2_08a. (*Not cited*)
2751     replace ((Q ∨ R → P) → (Q ∨ R → P)) with
2752       ((Q ∨ R → P) → (¬(Q ∨ R) ∨ P)) in n2_08a.

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2753   replace ( $\sim(Q \vee R)$ ) with ( $\sim Q \wedge \sim R$ ) in n2_08a.
2754   replace ( $\sim Q \wedge \sim R \vee P$ ) with
2755       ( $(\sim Q \vee P) \wedge (\sim R \vee P)$ ) in n2_08a.
2756   replace ( $\sim Q \vee P$ ) with ( $Q \rightarrow P$ ) in n2_08a.
2757   replace ( $\sim R \vee P$ ) with ( $R \rightarrow P$ ) in n2_08a.
2758   Conj n3_44a n2_08a.
2759   split.
2760   apply n3_44a.
2761   apply n2_08a.
2762   Equiv H.
2763   apply H.
2764   apply Equiv4_01.
2765   apply Impl1_01.
2766   apply Impl1_01.
2767   specialize n4_41 with P ( $\sim Q$ ) ( $\sim R$ ).
2768   intros n4_41a. (*Not cited*)
2769   replace ( $P \vee \sim Q$ ) with
2770       ( $\sim Q \vee P$ ) in n4_41a.
2771   replace ( $P \vee \sim R$ ) with
2772       ( $\sim R \vee P$ ) in n4_41a.
2773   replace ( $P \vee \sim Q \wedge \sim R$ ) with ( $\sim Q \wedge \sim R \vee P$ ) in n4_41a.
2774   replace ( $\sim Q \wedge \sim R \vee P \leftrightarrow (\sim Q \vee P) \wedge (\sim R \vee P)$ ) with
2775       ( $(\sim Q \vee P) \wedge (\sim R \vee P) \leftrightarrow \sim Q \wedge \sim R \vee P$ ) in n4_41a.
2776   apply EqBi.
2777   apply n4_41a.
2778   apply EqBi.
2779   specialize n4_21
2780       with ( $(\sim Q \vee P) \wedge (\sim R \vee P)$ ) ( $\sim Q \wedge \sim R \vee P$ ).
2781   intros n4_21a. (*Not cited*)
2782   apply n4_21a.
2783   specialize n4_31 with ( $\sim Q \wedge \sim R$ ) P.
2784   intros n4_31a. (*Not cited*)
2785   apply EqBi.
2786   apply n4_31a.
2787   specialize n4_31 with ( $\sim R$ ) P.
2788   intros n4_31b. (*Not cited*)
2789   apply EqBi.
2790   apply n4_31b.
2791   specialize n4_31 with ( $\sim Q$ ) P.
2792   intros n4_31c. (*Not cited*)
2793   apply EqBi.
2794   apply n4_31c. (*Not cited*)

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2795   apply EqBi.
2796   specialize n4_56 with Q R.
2797   intros n4_56a. (*Not cited*)
2798   apply n4_56a.
2799   replace (~ (Q ∨ R) ∨ P) with (Q ∨ R → P).
2800   reflexivity.
2801   apply Impl1_01. (*Not cited*)
2802   Qed.
2803   (*Proof sketch cites Add1_3 + n2_2.*)
2804
2805   Theorem n4_78 : ∀ P Q R : Prop,
2806     ((P → Q) ∨ (P → R)) ↔ (P → (Q ∨ R)).
2807   Proof. intros P Q R.
2808     specialize n4_2 with ((P → Q) ∨ (P → R)).
2809     intros n4_2a.
2810     replace (((P → Q) ∨ (P → R)) ↔ ((P → Q) ∨ (P → R))) with
2811       (((P → Q) ∨ (P → R)) ↔ ((~P ∨ Q) ∨ ~P ∨ R)) in n4_2a.
2812     specialize n4_33 with (~P) Q (~P ∨ R).
2813     intros n4_33a.
2814     replace ((~P ∨ Q) ∨ ~P ∨ R) with
2815       (~P ∨ Q ∨ ~P ∨ R) in n4_2a.
2816     specialize n4_31 with (~P) Q.
2817     intros n4_31a.
2818     specialize n4_37 with (~P ∨ Q) (Q ∨ ~P) R.
2819     intros n4_37a.
2820     MP n4_37a n4_31a.
2821     replace (Q ∨ ~P ∨ R) with
2822       ((Q ∨ ~P) ∨ R) in n4_2a.
2823     replace ((Q ∨ ~P) ∨ R) with
2824       ((~P ∨ Q) ∨ R) in n4_2a.
2825     specialize n4_33 with (~P) (~P ∨ Q) R.
2826     intros n4_33b.
2827     replace (~P ∨ (~P ∨ Q) ∨ R) with
2828       ((~P ∨ (~P ∨ Q)) ∨ R) in n4_2a.
2829     specialize n4_25 with (~P).
2830     intros n4_25a.
2831     specialize n4_37 with
2832       (~P) (~P ∨ ~P) (Q ∨ R).
2833     intros n4_37b.
2834     MP n4_37b n4_25a.
2835     replace (~P ∨ ~P ∨ Q) with
2836       ((~P ∨ ~P) ∨ Q) in n4_2a.

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2837   replace (((~P ∨ ~P) ∨ Q) ∨ R) with
2838     ((~P ∨ ~P) ∨ Q ∨ R) in n4_2a.
2839   replace ((~P ∨ ~P) ∨ Q ∨ R) with
2840     ((~P) ∨ (Q ∨ R)) in n4_2a.
2841   replace (~P ∨ Q ∨ R) with
2842     (P → (Q ∨ R)) in n4_2a.
2843   apply n4_2a.
2844   apply Impl1_01.
2845   apply EqBi.
2846   apply n4_37b.
2847   apply Abb2_33.
2848   replace ((~P ∨ ~P) ∨ Q) with (~P ∨ ~P ∨ Q).
2849   reflexivity.
2850   apply Abb2_33.
2851   replace ((~P ∨ ~P ∨ Q) ∨ R) with
2852     (~P ∨ (~P ∨ Q) ∨ R).
2853   reflexivity.
2854   apply EqBi.
2855   apply n4_33b.
2856   apply EqBi.
2857   apply n4_37a.
2858   replace ((Q ∨ ~P) ∨ R) with (Q ∨ ~P ∨ R).
2859   reflexivity.
2860   apply Abb2_33.
2861   apply EqBi.
2862   apply n4_33a.
2863   replace (~P ∨ Q) with (P → Q).
2864   replace (~P ∨ R) with (P → R).
2865   reflexivity.
2866   apply Impl1_01.
2867   apply Impl1_01.
2868   Qed.
2869
2870   Theorem n4_79 : ∀ P Q R : Prop,
2871     ((Q → P) ∨ (R → P)) ↔ ((Q ∧ R) → P).
2872   Proof. intros P Q R.
2873     specialize Trans4_1 with Q P.
2874     intros Trans4_1a.
2875     specialize Trans4_1 with R P.
2876     intros Trans4_1b.
2877     Conj Trans4_1a Trans4_1b.
2878     split.

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2879   apply Trans4_1a.
2880   apply Trans4_1b.
2881   specialize n4_39 with
2882     (Q → P) (R → P) (¬P → ¬Q) (¬P → ¬R).
2883   intros n4_39a.
2884   MP n4_39a H.
2885   specialize n4_78 with (¬P) (¬Q) (¬R).
2886   intros n4_78a.
2887   replace ((¬P → ¬Q) ∨ (¬P → ¬R)) with
2888     (¬P → ¬Q ∨ ¬R) in n4_39a.
2889   specialize Trans4_1 with (¬P) (¬Q ∨ ¬R).
2890   intros Trans4_1c.
2891   replace (¬P → ¬Q ∨ ¬R) with
2892     (¬(¬Q ∨ ¬R) → ¬¬P) in n4_39a.
2893   replace (¬(¬Q ∨ ¬R)) with
2894     (Q ∧ R) in n4_39a.
2895   replace (¬¬P) with P in n4_39a.
2896   apply n4_39a.
2897   specialize n4_13 with P.
2898   intros n4_13a.
2899   apply EqBi.
2900   apply n4_13a.
2901   apply Prod3_01.
2902   replace (¬(¬Q ∨ ¬R) → ¬¬P) with
2903     (¬P → ¬Q ∨ ¬R).
2904   reflexivity.
2905   apply EqBi.
2906   apply Trans4_1c.
2907   replace (¬P → ¬Q ∨ ¬R) with
2908     ((¬P → ¬Q) ∨ (¬P → ¬R)).
2909   reflexivity.
2910   apply EqBi.
2911   apply n4_78a.
2912   Qed.
2913   (*The proof sketch cites Trans2_15, but we did
2914     not need Trans2_15 as a lemma here.*)
2915
2916   Theorem n4_8 : ∀ P : Prop,
2917     (P → ¬P) ↔ ¬P.
2918   Proof. intros P.
2919     specialize Abs2_01 with P.
2920     intros Abs2_01a.

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2921     specialize Simp2_02 with P (~P).
2922     intros Simp2_02a.
2923     Conj Abs2_01a Simp2_02a.
2924     split.
2925     apply Abs2_01a.
2926     apply Simp2_02a.
2927     Equiv H.
2928     apply H.
2929     apply Equiv4_01.
2930 Qed.
2931
2932 Theorem n4_81 :  $\forall P : \text{Prop},$ 
2933    $(\sim P \rightarrow P) \leftrightarrow P.$ 
2934 Proof. intros P.
2935   specialize n2_18 with P.
2936   intros n2_18a.
2937   specialize Simp2_02 with ( $\sim P$ ) P.
2938   intros Simp2_02a.
2939   Conj n2_18a Simp2_02a.
2940   split.
2941   apply n2_18a.
2942   apply Simp2_02a.
2943   Equiv H.
2944   apply H.
2945   apply Equiv4_01.
2946 Qed.
2947
2948 Theorem n4_82 :  $\forall P Q : \text{Prop},$ 
2949    $((P \rightarrow Q) \wedge (P \rightarrow \sim Q)) \leftrightarrow \sim P.$ 
2950 Proof. intros P Q.
2951   specialize n2_65 with P Q.
2952   intros n2_65a.
2953   specialize Imp3_31 with  $(P \rightarrow Q) (P \rightarrow \sim Q) (\sim P).$ 
2954   intros Imp3_31a.
2955   MP Imp3_31a n2_65a.
2956   specialize n2_21 with P Q.
2957   intros n2_21a.
2958   specialize n2_21 with P ( $\sim Q$ ).
2959   intros n2_21b.
2960   Conj n2_21a n2_21b.
2961   split.
2962   apply n2_21a.

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2963     apply n2_21b.
2964     specialize Comp3_43 with ( $\sim P$ ) ( $P \rightarrow Q$ ) ( $P \rightarrow \sim Q$ ).
2965     intros Comp3_43a.
2966     MP Comp3_43a H.
2967     clear n2_65a. clear n2_21a. clear n2_21b.
2968     clear H.
2969     Conj Imp3_31a Comp3_43a.
2970     split.
2971     apply Imp3_31a.
2972     apply Comp3_43a.
2973     Equiv H.
2974     apply H.
2975     apply Equiv4_01.
2976 Qed.
2977
2978 Theorem n4_83 :  $\forall P Q : \text{Prop}$ ,
2979   ( $(P \rightarrow Q) \wedge (\sim P \rightarrow Q)$ )  $\leftrightarrow Q$ .
2980 Proof. intros P Q.
2981   specialize n2_61 with P Q.
2982   intros n2_61a.
2983   specialize Imp3_31 with ( $P \rightarrow Q$ ) ( $\sim P \rightarrow Q$ ) (Q).
2984   intros Imp3_31a.
2985   MP Imp3_31a n2_61a.
2986   specialize Simp2_02 with P Q.
2987   intros Simp2_02a.
2988   specialize Simp2_02 with ( $\sim P$ ) Q.
2989   intros Simp2_02b.
2990   Conj Simp2_02a Simp2_02b.
2991   split.
2992   apply Simp2_02a.
2993   apply Simp2_02b.
2994   specialize Comp3_43 with Q ( $P \rightarrow Q$ ) ( $\sim P \rightarrow Q$ ).
2995   intros Comp3_43a.
2996   MP Comp3_43a H.
2997   clear n2_61a. clear Simp2_02a. clear Simp2_02b.
2998   clear H.
2999   Conj Imp3_31a Comp3_43a.
3000   split.
3001   apply Imp3_31a.
3002   apply Comp3_43a.
3003   Equiv H.
3004   apply H.

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3005     apply Equiv4_01.
3006     Qed.
3007
3008 Theorem n4_84 :  $\forall$  P Q R : Prop,
3009   (P  $\leftrightarrow$  Q)  $\rightarrow$  ((P  $\rightarrow$  R)  $\leftrightarrow$  (Q  $\rightarrow$  R)).
3010 Proof. intros P Q R.
3011     specialize Syll2_06 with P Q R.
3012     intros Syll2_06a.
3013     specialize Syll2_06 with Q P R.
3014     intros Syll2_06b.
3015     Conj Syll2_06a Syll2_06b.
3016     split.
3017     apply Syll2_06a.
3018     apply Syll2_06b.
3019     specialize n3_47 with
3020       (P $\rightarrow$ Q) (Q $\rightarrow$ P) ((Q $\rightarrow$ R) $\rightarrow$ P $\rightarrow$ R) ((P $\rightarrow$ R) $\rightarrow$ Q $\rightarrow$ R).
3021     intros n3_47a.
3022     MP n3_47a H.
3023     replace ((P $\rightarrow$ Q)  $\wedge$  (Q  $\rightarrow$  P)) with
3024       (P $\leftrightarrow$ Q) in n3_47a.
3025     replace (((Q $\rightarrow$ R) $\rightarrow$ P $\rightarrow$ R) $\wedge$ ((P $\rightarrow$ R) $\rightarrow$ Q $\rightarrow$ R)) with
3026       ((Q  $\rightarrow$  R)  $\leftrightarrow$  (P  $\rightarrow$  R)) in n3_47a.
3027     replace ((Q  $\rightarrow$  R)  $\leftrightarrow$  (P  $\rightarrow$  R)) with
3028       ((P $\rightarrow$  R)  $\leftrightarrow$  (Q  $\rightarrow$  R)) in n3_47a.
3029     apply n3_47a.
3030     apply EqBi.
3031     specialize n4_21 with (P $\rightarrow$ R) (Q $\rightarrow$ R).
3032     intros n4_21a.
3033     apply n4_21a.
3034     apply Equiv4_01.
3035     apply Equiv4_01.
3036     Qed.
3037
3038 Theorem n4_85 :  $\forall$  P Q R : Prop,
3039   (P  $\leftrightarrow$  Q)  $\rightarrow$  ((R  $\rightarrow$  P)  $\leftrightarrow$  (R  $\rightarrow$  Q)).
3040 Proof. intros P Q R.
3041     specialize Syll2_05 with R P Q.
3042     intros Syll2_05a.
3043     specialize Syll2_05 with R Q P.
3044     intros Syll2_05b.
3045     Conj Syll2_05a Syll2_05b.
3046     split.

```

```

3047   apply Syll2_05a.
3048   apply Syll2_05b.
3049   specialize n3_47 with
3050     (P→Q) (Q→P) ((R→P)→R→Q) ((R→Q)→R→P).
3051   intros n3_47a.
3052   MP n3_47a H.
3053   replace ((P→Q) ∧ (Q → P)) with (P↔Q) in n3_47a.
3054   replace (((R→P)→R→Q) ∧ ((R→Q)→R→P)) with
3055     ((R → P) ↔ (R → Q)) in n3_47a.
3056   apply n3_47a.
3057   apply Equiv4_01.
3058   apply Equiv4_01.
3059   Qed.
3060
3061   Theorem n4_86 : ∀ P Q R : Prop,
3062     (P ↔ Q) → ((P ↔ R) ↔ (Q ↔ R)).
3063   Proof. intros P Q R.
3064     specialize n4_22 with Q P R.
3065     intros n4_22a.
3066     specialize Exp3_3 with (Q↔P) (P↔R) (Q↔R).
3067     intros Exp3_3a. (*Not cited*)
3068     MP Exp3_3a n4_22a.
3069     specialize n4_22 with P Q R.
3070     intros n4_22b.
3071     specialize Exp3_3 with (P↔Q) (Q↔R) (P↔R).
3072     intros Exp3_3b.
3073     MP Exp3_3b n4_22b.
3074     clear n4_22a.
3075     clear n4_22b.
3076     replace (Q↔P) with (P↔Q) in Exp3_3a.
3077     Conj Exp3_3a Exp3_3b.
3078     split.
3079     apply Exp3_3a.
3080     apply Exp3_3b.
3081     specialize Comp3_43 with (P↔Q)
3082       ((P↔R)→(Q↔R)) ((Q↔R)→(P↔R)).
3083     intros Comp3_43a. (*Not cited*)
3084     MP Comp3_43a H.
3085     replace (((P↔R)→(Q↔R)) ∧ ((Q↔R)→(P↔R)))
3086       with ((P↔R)↔(Q↔R)) in Comp3_43a.
3087     apply Comp3_43a.
3088     apply Equiv4_01.

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3089   apply EqBi.
3090   specialize n4_21 with P Q.
3091   intros n4_21a.
3092   apply n4_21a.
3093   Qed.
3094
3095   Theorem n4_87 :  $\forall$  P Q R : Prop,
3096     (((P  $\wedge$  Q)  $\rightarrow$  R)  $\leftrightarrow$  (P  $\rightarrow$  Q  $\rightarrow$  R))  $\leftrightarrow$ 
3097     ((Q  $\rightarrow$  (P  $\rightarrow$  R))  $\leftrightarrow$  (Q  $\wedge$  P  $\rightarrow$  R)).
3098   Proof. intros P Q R.
3099   specialize Exp3_3 with P Q R.
3100   intros Exp3_3a.
3101   specialize Imp3_31 with P Q R.
3102   intros Imp3_31a.
3103   Conj Exp3_3a Imp3_31a.
3104   split.
3105   apply Exp3_3a.
3106   apply Imp3_31a.
3107   Equiv H.
3108   specialize Exp3_3 with Q P R.
3109   intros Exp3_3b.
3110   specialize Imp3_31 with Q P R.
3111   intros Imp3_31b.
3112   Conj Exp3_3b Imp3_31b.
3113   split.
3114   apply Exp3_3b.
3115   apply Imp3_31b.
3116   Equiv H0.
3117   specialize Comm2_04 with P Q R.
3118   intros Comm2_04a.
3119   specialize Comm2_04 with Q P R.
3120   intros Comm2_04b.
3121   Conj Comm2_04a Comm2_04b.
3122   split.
3123   apply Comm2_04a.
3124   apply Comm2_04b.
3125   Equiv H1.
3126   clear Exp3_3a. clear Imp3_31a. clear Exp3_3b.
3127     clear Imp3_31b. clear Comm2_04a.
3128     clear Comm2_04b.
3129   replace (P $\wedge$ Q $\rightarrow$ R) with (P  $\rightarrow$  Q  $\rightarrow$  R).
3130   replace (Q $\wedge$ P $\rightarrow$ R) with (Q  $\rightarrow$  P  $\rightarrow$  R).

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3131   replace (Q→P→R) with (P → Q → R).
3132   specialize n4_2 with
3133     ((P → Q → R) ↔ (P → Q → R)).
3134   intros n4_2a.
3135   apply n4_2a.
3136   apply EqBi.
3137   apply H1.
3138   replace (Q→P→R) with (Q∧P→R).
3139   reflexivity.
3140   apply EqBi.
3141   apply H0.
3142   replace (P→Q→R) with (P∧Q→R).
3143   reflexivity.
3144   apply EqBi.
3145   apply H.
3146   apply Equiv4_01.
3147   apply Equiv4_01.
3148   apply Equiv4_01.
3149   Qed.
3150
3151 End No4.
3152
3153 Module No5.
3154
3155 Import No1.
3156 Import No2.
3157 Import No3.
3158 Import No4.
3159
3160 Theorem n5_1 : ∀ P Q : Prop,
3161   (P ∧ Q) → (P ↔ Q).
3162 Proof. intros P Q.
3163   specialize n3_4 with P Q.
3164   intros n3_4a.
3165   specialize n3_4 with Q P.
3166   intros n3_4b.
3167   specialize n3_22 with P Q.
3168   intros n3_22a.
3169   Syll n3_22a n3_4b Sa.
3170   clear n3_22a. clear n3_4b.
3171   Conj n3_4a Sa.
3172   split.

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3173   apply n3_4a.
3174   apply Sa.
3175   specialize n4_76 with (P ∧ Q) (P → Q) (Q → P).
3176   intros n4_76a. (*Not cited*)
3177   replace ((P ∧ Q → P → Q) ∧ (P ∧ Q → Q → P)) with
3178     (P ∧ Q → (P → Q) ∧ (Q → P)) in H.
3179   replace ((P → Q) ∧ (Q → P)) with (P ↔ Q) in H.
3180   apply H.
3181   apply Equiv4_01.
3182   replace (P ∧ Q → (P → Q) ∧ (Q → P)) with
3183     ((P ∧ Q → P → Q) ∧ (P ∧ Q → Q → P)).
3184   reflexivity.
3185   apply EqBi.
3186   apply n4_76a.
3187   Qed.
3188
3189   Theorem n5_11 : ∀ P Q : Prop,
3190     (P → Q) ∨ (¬P → Q).
3191   Proof. intros P Q.
3192     specialize n2_5 with P Q.
3193     intros n2_5a.
3194     specialize n2_54 with ((P → Q)) (¬P → Q).
3195     intros n2_54a.
3196     MP n2_54a n2_5a.
3197     apply n2_54a.
3198     Qed.
3199     (*The proof sketch cites n2_51,
3200       but this may be a misprint.*)
3201
3202   Theorem n5_12 : ∀ P Q : Prop,
3203     (P → Q) ∨ (P → ¬Q).
3204   Proof. intros P Q.
3205     specialize n2_51 with P Q.
3206     intros n2_51a.
3207     specialize n2_54 with ((P → Q)) (P → ¬Q).
3208     intros n2_54a.
3209     MP n2_54a n2_5a.
3210     apply n2_54a.
3211     Qed.
3212     (*The proof sketch cites n2_52,
3213       but this may be a misprint.*)
3214

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3215 Theorem n5_13 :  $\forall P Q : \text{Prop},$ 
3216    $(P \rightarrow Q) \vee (Q \rightarrow P).$ 
3217 Proof. intros P Q.
3218 specialize n2_521 with P Q.
3219 intros n2_521a.
3220 replace  $(\sim(P \rightarrow Q) \rightarrow Q \rightarrow P)$  with
3221    $(\sim(P \rightarrow Q) \vee (Q \rightarrow P))$  in n2_521a.
3222 replace  $(\sim(P \rightarrow Q))$  with  $(P \rightarrow Q)$  in n2_521a.
3223 apply n2_521a.
3224 apply EqBi.
3225 specialize n4_13 with  $(P \rightarrow Q).$ 
3226 intros n4_13a. (*Not cited*)
3227 apply n4_13a.
3228 replace  $(\sim(P \rightarrow Q) \vee (Q \rightarrow P))$  with
3229    $(\sim(P \rightarrow Q) \rightarrow Q \rightarrow P).$ 
3230 reflexivity.
3231 apply Impl1_01.
3232 Qed.
3233
3234 Theorem n5_14 :  $\forall P Q R : \text{Prop},$ 
3235    $(P \rightarrow Q) \vee (Q \rightarrow R).$ 
3236 Proof. intros P Q R.
3237 specialize Simp2_02 with P Q.
3238 intros Simp2_02a.
3239 specialize Trans2_16 with Q  $(P \rightarrow Q).$ 
3240 intros Trans2_16a.
3241 MP Trans2_16a Simp2_02a.
3242 specialize n2_21 with Q R.
3243 intros n2_21a.
3244 Syll Trans2_16a n2_21a Sa.
3245 replace  $(\sim(P \rightarrow Q) \rightarrow (Q \rightarrow R))$  with
3246    $(\sim(P \rightarrow Q) \vee (Q \rightarrow R))$  in Sa.
3247 replace  $(\sim(P \rightarrow Q))$  with  $(P \rightarrow Q)$  in Sa.
3248 apply Sa.
3249 apply EqBi.
3250 specialize n4_13 with  $(P \rightarrow Q).$ 
3251 intros n4_13a.
3252 apply n4_13a.
3253 replace  $(\sim(P \rightarrow Q) \vee (Q \rightarrow R))$  with
3254    $(\sim(P \rightarrow Q) \rightarrow (Q \rightarrow R)).$ 
3255 reflexivity.
3256 apply Impl1_01.

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3257   Qed.
3258
3259   Theorem n5_15 :  $\forall P Q : \text{Prop}$ ,
3260      $(P \leftrightarrow Q) \vee (P \leftrightarrow \sim Q)$ .
3261   Proof. intros P Q.
3262     specialize n4_61 with P Q.
3263     intros n4_61a.
3264     replace  $(\sim(P \rightarrow Q) \leftrightarrow P \wedge \sim Q)$  with
3265        $((\sim(P \rightarrow Q) \rightarrow P \wedge \sim Q) \wedge ((P \wedge \sim Q) \rightarrow \sim(P \rightarrow Q)))$  in n4_61a.
3266     specialize Simp3_26 with
3267        $(\sim(P \rightarrow Q) \rightarrow P \wedge \sim Q) ((P \wedge \sim Q) \rightarrow \sim(P \rightarrow Q))$ .
3268     intros Simp3_26a.
3269     MP Simp3_26a n4_61a.
3270     specialize n5_1 with P  $(\sim Q)$ .
3271     intros n5_1a.
3272     Syll Simp3_26a n5_1a Sa.
3273     specialize n2_54 with  $(P \rightarrow Q) (P \leftrightarrow \sim Q)$ .
3274     intros n2_54a.
3275     MP n2_54a Sa.
3276     specialize n4_61 with Q P.
3277     intros n4_61b.
3278     replace  $(\sim(Q \rightarrow P) \leftrightarrow (Q \wedge \sim P))$  with
3279        $((\sim(Q \rightarrow P) \rightarrow (Q \wedge \sim P)) \wedge ((Q \wedge \sim P) \rightarrow \sim(Q \rightarrow P)))$  in n4_61b.
3280     specialize Simp3_26 with
3281        $(\sim(Q \rightarrow P) \rightarrow (Q \wedge \sim P)) ((Q \wedge \sim P) \rightarrow \sim(Q \rightarrow P))$ .
3282     intros Simp3_26b.
3283     MP Simp3_26b n4_61b.
3284     specialize n5_1 with Q  $(\sim P)$ .
3285     intros n5_1b.
3286     Syll Simp3_26b n5_1b Sb.
3287     specialize n4_12 with P Q.
3288     intros n4_12a.
3289     replace  $(Q \leftrightarrow \sim P)$  with  $(P \leftrightarrow \sim Q)$  in Sb.
3290     specialize n2_54 with  $(Q \rightarrow P) (P \leftrightarrow \sim Q)$ .
3291     intros n2_54b.
3292     MP n2_54b Sb.
3293     clear n4_61a. clear Simp3_26a. clear n5_1a.
3294     clear n2_54a. clear n4_61b. clear Simp3_26b.
3295     clear n5_1b. clear n4_12a. clear n2_54b.
3296     replace  $(\sim(P \rightarrow Q) \rightarrow P \leftrightarrow \sim Q)$  with
3297        $(\sim(P \rightarrow Q) \vee (P \leftrightarrow \sim Q))$  in Sa.
3298     replace  $(\sim(P \rightarrow Q))$  with  $(P \rightarrow Q)$  in Sa.

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3299   replace ( $\sim(Q \rightarrow P) \rightarrow (P \leftrightarrow \sim Q)$ ) with
3300         ( $\sim(Q \rightarrow P) \vee (P \leftrightarrow \sim Q)$ ) in Sb.
3301   replace ( $\sim(Q \rightarrow P)$ ) with ( $Q \rightarrow P$ ) in Sb.
3302   Conj Sa Sb.
3303   split.
3304   apply Sa.
3305   apply Sb.
3306   specialize n4_41 with ( $P \leftrightarrow \sim Q$ ) ( $P \rightarrow Q$ ) ( $Q \rightarrow P$ ).
3307   intros n4_41a.
3308   replace ( $(P \rightarrow Q) \vee (P \leftrightarrow \sim Q)$ ) with
3309         ( $(P \leftrightarrow \sim Q) \vee (P \rightarrow Q)$ ) in H.
3310   replace ( $(Q \rightarrow P) \vee (P \leftrightarrow \sim Q)$ ) with
3311         ( $(P \leftrightarrow \sim Q) \vee (Q \rightarrow P)$ ) in H.
3312   replace ( $((P \leftrightarrow \sim Q) \vee (P \rightarrow Q)) \wedge ((P \leftrightarrow \sim Q) \vee (Q \rightarrow P))$ ) with
3313         ( $(P \leftrightarrow \sim Q) \vee (P \rightarrow Q) \wedge (Q \rightarrow P)$ ) in H.
3314   replace ( $(P \rightarrow Q) \wedge (Q \rightarrow P)$ ) with ( $P \leftrightarrow Q$ ) in H.
3315   replace ( $(P \leftrightarrow \sim Q) \vee (P \leftrightarrow Q)$ ) with
3316         ( $(P \leftrightarrow Q) \vee (P \leftrightarrow \sim Q)$ ) in H.
3317   apply H.
3318   apply EqBi.
3319   apply n4_31.
3320   apply Equiv4_01.
3321   apply EqBi.
3322   apply n4_41a.
3323   apply EqBi.
3324   apply n4_31.
3325   apply EqBi.
3326   apply n4_31.
3327   apply EqBi.
3328   specialize n4_13 with ( $Q \rightarrow P$ ).
3329   intros n4_13a.
3330   apply n4_13a.
3331   replace ( $\sim(Q \rightarrow P) \vee (P \leftrightarrow \sim Q)$ ) with
3332         ( $\sim(Q \rightarrow P) \rightarrow (P \leftrightarrow \sim Q)$ ).
3333   reflexivity.
3334   apply Impl1_01.
3335   apply EqBi.
3336   specialize n4_13 with ( $P \rightarrow Q$ ).
3337   intros n4_13b.
3338   apply n4_13b.
3339   replace ( $\sim(P \rightarrow Q) \vee (P \leftrightarrow \sim Q)$ ) with
3340         ( $\sim(P \rightarrow Q) \rightarrow P \leftrightarrow \sim Q$ ).

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3341 reflexivity.
3342 apply Impl1_01.
3343 apply EqBi.
3344 apply n4_12a.
3345 apply Equiv4_01.
3346 apply Equiv4_01.
3347 Qed.
3348
3349 Theorem n5_16 :  $\forall$  P Q : Prop,
3350    $\sim((P \leftrightarrow Q) \wedge (P \leftrightarrow \sim Q))$ .
3351 Proof. intros P Q.
3352 specialize Simp3_26 with  $((P \rightarrow Q) \wedge (P \rightarrow \sim Q)) (Q \rightarrow P)$ .
3353 intros Simp3_26a.
3354 specialize n2_08 with  $((P \leftrightarrow Q) \wedge (P \rightarrow \sim Q))$ .
3355 intros n2_08a.
3356 replace  $((P \rightarrow Q) \wedge (P \rightarrow \sim Q)) \wedge (Q \rightarrow P)$  with
3357    $((P \rightarrow Q) \wedge ((P \rightarrow \sim Q) \wedge (Q \rightarrow P)))$  in Simp3_26a.
3358 replace  $((P \rightarrow \sim Q) \wedge (Q \rightarrow P))$  with
3359    $((Q \rightarrow P) \wedge (P \rightarrow \sim Q))$  in Simp3_26a.
3360 replace  $((P \rightarrow Q) \wedge (Q \rightarrow P) \wedge (P \rightarrow \sim Q))$  with
3361    $((P \rightarrow Q) \wedge (Q \rightarrow P)) \wedge (P \rightarrow \sim Q)$  in Simp3_26a.
3362 replace  $((P \rightarrow Q) \wedge (Q \rightarrow P))$  with
3363    $(P \leftrightarrow Q)$  in Simp3_26a.
3364 Syll n2_08a Simp3_26a Sa.
3365 specialize n4_82 with P Q.
3366 intros n4_82a.
3367 replace  $((P \rightarrow Q) \wedge (P \rightarrow \sim Q))$  with  $(\sim P)$  in Sa.
3368 specialize Simp3_27 with
3369    $(P \rightarrow Q) ((Q \rightarrow P) \wedge (P \rightarrow \sim Q))$ .
3370 intros Simp3_27a.
3371 replace  $((P \rightarrow Q) \wedge (Q \rightarrow P) \wedge (P \rightarrow \sim Q))$  with
3372    $((P \rightarrow Q) \wedge (Q \rightarrow P)) \wedge (P \rightarrow \sim Q)$  in Simp3_27a.
3373 replace  $((P \rightarrow Q) \wedge (Q \rightarrow P))$  with
3374    $(P \leftrightarrow Q)$  in Simp3_27a.
3375 specialize Syll3_33 with Q P  $(\sim Q)$ .
3376 intros Syll3_33a.
3377 Syll Simp3_27a Syll2_06a Sb.
3378 specialize Abs2_01 with Q.
3379 intros Abs2_01a.
3380 Syll Sb Abs2_01a Sc.
3381 clear Sb. clear Simp3_26a. clear n2_08a.
3382   clear n4_82a. clear Simp3_27a. clear Syll3_33a.

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3383     clear Abs2_01a.
3384 Conj Sa Sc.
3385 split.
3386 apply Sa.
3387 apply Sc.
3388 specialize Comp3_43 with
3389    $((P \leftrightarrow Q) \wedge (P \rightarrow \sim Q)) (\sim P) (\sim Q)$ .
3390 intros Comp3_43a.
3391 MP Comp3_43a H.
3392 specialize n4_65 with Q P.
3393 intros n4_65a.
3394 replace  $(\sim Q \wedge \sim P)$  with  $(\sim P \wedge \sim Q)$  in n4_65a.
3395 replace  $(\sim P \wedge \sim Q)$  with
3396    $(\sim(\sim Q \rightarrow P))$  in Comp3_43a.
3397 specialize Exp3_3 with
3398    $(P \leftrightarrow Q) (P \rightarrow \sim Q) (\sim(\sim Q \rightarrow P))$ .
3399 intros Exp3_3a.
3400 MP Exp3_3a Comp3_43a.
3401 replace  $((P \rightarrow \sim Q) \rightarrow \sim(\sim Q \rightarrow P))$  with
3402    $(\sim(P \rightarrow \sim Q) \vee \sim(\sim Q \rightarrow P))$  in Exp3_3a.
3403 specialize n4_51 with  $(P \rightarrow \sim Q) (\sim Q \rightarrow P)$ .
3404 intros n4_51a.
3405 replace  $(\sim(P \rightarrow \sim Q) \vee \sim(\sim Q \rightarrow P))$  with
3406    $(\sim((P \rightarrow \sim Q) \wedge (\sim Q \rightarrow P)))$  in Exp3_3a.
3407 replace  $((P \rightarrow \sim Q) \wedge (\sim Q \rightarrow P))$  with
3408    $(P \leftrightarrow \sim Q)$  in Exp3_3a.
3409 replace  $((P \leftrightarrow Q) \rightarrow \sim(P \leftrightarrow \sim Q))$  with
3410    $(\sim(P \leftrightarrow Q) \vee \sim(P \leftrightarrow \sim Q))$  in Exp3_3a.
3411 specialize n4_51 with  $(P \leftrightarrow Q) (P \leftrightarrow \sim Q)$ .
3412 intros n4_51b.
3413 replace  $(\sim(P \leftrightarrow Q) \vee \sim(P \leftrightarrow \sim Q))$  with
3414    $(\sim((P \leftrightarrow Q) \wedge (P \leftrightarrow \sim Q)))$  in Exp3_3a.
3415 apply Exp3_3a.
3416 apply EqBi.
3417 apply n4_51b.
3418 replace  $(\sim(P \leftrightarrow Q) \vee \sim(P \leftrightarrow \sim Q))$  with
3419    $(P \leftrightarrow Q \rightarrow \sim(P \leftrightarrow \sim Q))$ .
3420 reflexivity.
3421 apply Impl1_01.
3422 apply Equiv4_01.
3423 apply EqBi.
3424 apply n4_51a.

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3425   replace ( $\sim(P \rightarrow \sim Q) \vee \sim(\sim Q \rightarrow P)$ ) with
3426         ( $(P \rightarrow \sim Q) \rightarrow \sim(\sim Q \rightarrow P)$ ).
3427   reflexivity.
3428   apply Impl1_01.
3429   apply EqBi.
3430   apply n4_65a.
3431   apply EqBi.
3432   specialize n4_3 with ( $\sim P$ ) ( $\sim Q$ ).
3433   intros n4_3a.
3434   apply n4_3a.
3435   apply Equiv4_01.
3436   apply EqBi.
3437   specialize n4_32 with ( $P \rightarrow Q$ ) ( $Q \rightarrow P$ ) ( $P \rightarrow \sim Q$ ).
3438   intros n4_32a.
3439   apply n4_32a.
3440   replace ( $\sim P$ ) with ( $(P \rightarrow Q) \wedge (P \rightarrow \sim Q)$ ).
3441   reflexivity.
3442   apply EqBi.
3443   apply n4_82a.
3444   apply Equiv4_01.
3445   apply EqBi.
3446   specialize n4_32 with ( $P \rightarrow Q$ ) ( $Q \rightarrow P$ ) ( $P \rightarrow \sim Q$ ).
3447   intros n4_32b.
3448   apply n4_32b.
3449   apply EqBi.
3450   specialize n4_3 with ( $Q \rightarrow P$ ) ( $P \rightarrow \sim Q$ ).
3451   intros n4_3b.
3452   apply n4_3b.
3453   replace ( $(P \rightarrow Q) \wedge (P \rightarrow \sim Q) \wedge (Q \rightarrow P)$ ) with
3454         ( $((P \rightarrow Q) \wedge (P \rightarrow \sim Q)) \wedge (Q \rightarrow P)$ ).
3455   reflexivity.
3456   apply EqBi.
3457   specialize n4_32 with ( $P \rightarrow Q$ ) ( $P \rightarrow \sim Q$ ) ( $Q \rightarrow P$ ).
3458   intros n4_32a.
3459   apply n4_32a.
3460   Qed.
3461
3462   Theorem n5_17 :  $\forall P Q : \text{Prop},$ 
3463     ( $(P \vee Q) \wedge \sim(P \wedge Q)$ )  $\leftrightarrow (P \leftrightarrow \sim Q)$ .
3464   Proof. intros P Q.
3465     specialize n4_64 with Q P.
3466     intros n4_64a.

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3467 specialize n4_21 with (Q∨P) (¬Q→P).
3468 intros n4_21a.
3469 replace ((¬Q→P)↔(Q∨P)) with
3470   ((Q∨P)↔(¬Q→P)) in n4_64a.
3471 replace (Q∨P) with (P∨Q) in n4_64a.
3472 specialize n4_63 with P Q.
3473 intros n4_63a.
3474 replace (¬(P → ¬Q) ↔ P ∧ Q) with
3475   (P ∧ Q ↔ ¬(P → ¬Q)) in n4_63a.
3476 specialize Trans4_11 with (P∧Q) (¬(P→¬Q)).
3477 intros Trans4_11a.
3478 replace (¬¬(P→¬Q)) with
3479   (P→¬Q) in Trans4_11a.
3480 replace (P ∧ Q ↔ ¬(P → ¬Q)) with
3481   (¬(P ∧ Q) ↔ (P → ¬Q)) in n4_63a.
3482 clear Trans4_11a. clear n4_21a.
3483 Conj n4_64a n4_63a.
3484 split.
3485 apply n4_64a.
3486 apply n4_63a.
3487 specialize n4_38 with
3488   (P ∨ Q) (¬(P ∧ Q)) (¬Q → P) (P → ¬Q).
3489 intros n4_38a.
3490 MP n4_38a H.
3491 replace ((¬Q→P) ∧ (P → ¬Q)) with
3492   (¬Q↔P) in n4_38a.
3493 specialize n4_21 with P (¬Q).
3494 intros n4_21b.
3495 replace (¬Q↔P) with (P↔¬Q) in n4_38a.
3496 apply n4_38a.
3497 apply EqBi.
3498 apply n4_21b.
3499 apply Equiv4_01.
3500 replace (¬(P ∧ Q) ↔ (P → ¬Q)) with
3501   (P ∧ Q ↔ ¬(P → ¬Q)).
3502 reflexivity.
3503 apply EqBi.
3504 apply Trans4_11a.
3505 apply EqBi.
3506 specialize n4_13 with (P→¬Q).
3507 intros n4_13a.
3508 apply n4_13a.

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3509   apply EqBi.
3510   specialize n4_21 with (P ∧ Q) (¬(P→¬Q)).
3511   intros n4_21b.
3512   apply n4_21b.
3513   apply EqBi.
3514   specialize n4_31 with P Q.
3515   intros n4_31a.
3516   apply n4_31a.
3517   apply EqBi.
3518   apply n4_21a.
3519   Qed.
3520
3521 Theorem n5_18 : ∀ P Q : Prop,
3522   (P ↔ Q) ↔ ¬(P ↔ ¬Q).
3523 Proof. intros P Q.
3524   specialize n5_15 with P Q.
3525   intros n5_15a.
3526   specialize n5_16 with P Q.
3527   intros n5_16a.
3528   Conj n5_15a n5_16a.
3529   split.
3530   apply n5_15a.
3531   apply n5_16a.
3532   specialize n5_17 with (P↔Q) (P↔¬Q).
3533   intros n5_17a.
3534   replace ((P ↔ Q) ↔ ¬(P ↔ ¬Q)) with
3535     (((P↔Q)∨(P↔¬Q))∧¬((P↔Q)∧(P↔¬Q))).
3536   apply H.
3537   apply EqBi.
3538   apply n5_17a.
3539   Qed.
3540
3541 Theorem n5_19 : ∀ P : Prop,
3542   ¬(P ↔ ¬P).
3543 Proof. intros P.
3544   specialize n5_18 with P P.
3545   intros n5_18a.
3546   specialize n4_2 with P.
3547   intros n4_2a.
3548   replace (¬(P↔¬P)) with (P↔P).
3549   apply n4_2a.
3550   apply EqBi.

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3551   apply n5_18a.
3552   Qed.
3553
3554   Theorem n5_21 :  $\forall P Q : \text{Prop},$ 
3555      $(\sim P \wedge \sim Q) \rightarrow (P \leftrightarrow Q).$ 
3556   Proof. intros P Q.
3557     specialize n5_1 with ( $\sim P$ ) ( $\sim Q$ ).
3558     intros n5_1a.
3559     specialize Trans4_11 with P Q.
3560     intros Trans4_11a.
3561     replace ( $\sim P \leftrightarrow \sim Q$ ) with  $(P \leftrightarrow Q)$  in n5_1a.
3562     apply n5_1a.
3563     apply EqBi.
3564     apply Trans4_11a.
3565     Qed.
3566
3567   Theorem n5_22 :  $\forall P Q : \text{Prop},$ 
3568      $\sim(P \leftrightarrow Q) \leftrightarrow ((P \wedge \sim Q) \vee (Q \wedge \sim P)).$ 
3569   Proof. intros P Q.
3570     specialize n4_61 with P Q.
3571     intros n4_61a.
3572     specialize n4_61 with Q P.
3573     intros n4_61b.
3574     Conj n4_61a n4_61b.
3575     split.
3576     apply n4_61a.
3577     apply n4_61b.
3578     specialize n4_39 with
3579        $(\sim(P \rightarrow Q)) (\sim(Q \rightarrow P)) (P \wedge \sim Q) (Q \wedge \sim P).$ 
3580     intros n4_39a.
3581     MP n4_39a H.
3582     specialize n4_51 with  $(P \rightarrow Q) (Q \rightarrow P).$ 
3583     intros n4_51a.
3584     replace  $(\sim(P \rightarrow Q) \vee \sim(Q \rightarrow P))$  with
3585        $(\sim((P \rightarrow Q) \wedge (Q \rightarrow P)))$  in n4_39a.
3586     replace  $((P \rightarrow Q) \wedge (Q \rightarrow P))$  with
3587        $(P \leftrightarrow Q)$  in n4_39a.
3588     apply n4_39a.
3589     apply Equiv4_01.
3590     apply EqBi.
3591     apply n4_51a.
3592     Qed.

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3593
3594 Theorem n5_23 :  $\forall P Q : \text{Prop},$ 
3595    $(P \leftrightarrow Q) \leftrightarrow ((P \wedge Q) \vee (\sim P \wedge \sim Q)).$ 
3596 Proof. intros P Q.
3597 specialize n5_18 with P Q.
3598 intros n5_18a.
3599 specialize n5_22 with P ( $\sim Q$ ).
3600 intros n5_22a.
3601 specialize n4_13 with Q.
3602 intros n4_13a.
3603 replace ( $\sim(P \leftrightarrow Q)$ ) with
3604    $((P \wedge \sim\sim Q) \vee (\sim Q \wedge \sim P))$  in n5_18a.
3605 replace ( $\sim\sim Q$ ) with Q in n5_18a.
3606 replace ( $\sim Q \wedge \sim P$ ) with  $(\sim P \wedge \sim Q)$  in n5_18a.
3607 apply n5_18a.
3608 apply EqBi.
3609 specialize n4_3 with ( $\sim P$ ) ( $\sim Q$ ).
3610 intros n4_3a.
3611 apply n4_3a. (*with ( $\sim P$ ) ( $\sim Q$ *)
3612 apply EqBi.
3613 apply n4_13a.
3614 replace  $(P \wedge \sim\sim Q \vee \sim Q \wedge \sim P)$  with  $(\sim(P \leftrightarrow Q))$ .
3615 reflexivity.
3616 apply EqBi.
3617 apply n5_22a.
3618 Qed.
3619 (*The proof sketch in Principia offers n4_36,
3620   but we found it far simpler to use n4_3.*)
3621
3622 Theorem n5_24 :  $\forall P Q : \text{Prop},$ 
3623    $\sim((P \wedge Q) \vee (\sim P \wedge \sim Q)) \leftrightarrow ((P \wedge \sim Q) \vee (Q \wedge \sim P)).$ 
3624 Proof. intros P Q.
3625 specialize n5_22 with P Q.
3626 intros n5_22a.
3627 specialize n5_23 with P Q.
3628 intros n5_23a.
3629 replace  $((P \leftrightarrow Q) \leftrightarrow ((P \wedge Q) \vee (\sim P \wedge \sim Q)))$  with
3630    $((\sim(P \leftrightarrow Q) \leftrightarrow \sim((P \wedge Q) \vee (\sim P \wedge \sim Q))))$  in n5_23a.
3631 replace ( $\sim(P \leftrightarrow Q)$ ) with
3632    $(\sim((P \wedge Q) \vee (\sim P \wedge \sim Q)))$  in n5_22a.
3633 apply n5_22a.
3634 replace  $(\sim((P \wedge Q) \vee (\sim P \wedge \sim Q)))$  with  $(\sim(P \leftrightarrow Q))$ .

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3635 reflexivity.
3636 apply EqBi.
3637 apply n5_23a.
3638 replace (~(P ↔ Q) ↔ ~(P ∧ Q ∨ ~P ∧ ~Q)) with
3639   ((P ↔ Q) ↔ P ∧ Q ∨ ~P ∧ ~Q).
3640 reflexivity.
3641 specialize Trans4_11 with
3642   (P↔Q) (P ∧ Q ∨ ~P ∧ ~Q).
3643 intros Trans4_11a.
3644 apply EqBi.
3645 apply Trans4_11a. (*Not cited*)
3646 Qed.
3647
3648 Theorem n5_25 : ∀ P Q : Prop,
3649   (P ∨ Q) ↔ ((P → Q) → Q).
3650 Proof. intros P Q.
3651 specialize n2_62 with P Q.
3652 intros n2_62a.
3653 specialize n2_68 with P Q.
3654 intros n2_68a.
3655 Conj n2_62a n2_68a.
3656 split.
3657 apply n2_62a.
3658 apply n2_68a.
3659 Equiv H.
3660 apply H.
3661 apply Equiv4_01.
3662 Qed.
3663
3664 Theorem n5_3 : ∀ P Q R : Prop,
3665   ((P ∧ Q) → R) ↔ ((P ∧ Q) → (P ∧ R)).
3666 Proof. intros P Q R.
3667 specialize Comp3_43 with (P ∧ Q) P R.
3668 intros Comp3_43a.
3669 specialize Exp3_3 with
3670   (P ∧ Q → P) (P ∧ Q → R) (P ∧ Q → P ∧ R).
3671 intros Exp3_3a. (*Not cited*)
3672 MP Exp3_3a Comp3_43a.
3673 specialize Simp3_26 with P Q.
3674 intros Simp3_26a.
3675 MP Exp3_3a Simp3_26a.
3676 specialize Syll2_05 with (P ∧ Q) (P ∧ R) R.

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3677   intros Syll2_05a.
3678   specialize Simp3_27 with P R.
3679   intros Simp3_27a.
3680   MP Syll2_05a Simp3_27a.
3681   clear Comp3_43a. clear Simp3_27a.
3682       clear Simp3_26a.
3683   Conj Exp3_3a Syll2_05a.
3684   split.
3685   apply Exp3_3a.
3686   apply Syll2_05a.
3687   Equiv H.
3688   apply H.
3689   apply Equiv4_01.
3690   Qed.
3691
3692   Theorem n5_31 :  $\forall$  P Q R : Prop,
3693     (R  $\wedge$  (P  $\rightarrow$  Q))  $\rightarrow$  (P  $\rightarrow$  (Q  $\wedge$  R)).
3694   Proof. intros P Q R.
3695   specialize Comp3_43 with P Q R.
3696   intros Comp3_43a.
3697   specialize Simp2_02 with P R.
3698   intros Simp2_02a.
3699   specialize Exp3_3 with
3700     (P $\rightarrow$ R) (P $\rightarrow$ Q) (P $\rightarrow$ (Q  $\wedge$  R)).
3701   intros Exp3_3a. (*Not cited*)
3702   specialize n3_22 with (P  $\rightarrow$  R) (P  $\rightarrow$  Q). (*Not cited*)
3703   intros n3_22a.
3704   Syll n3_22a Comp3_43a Sa.
3705   MP Exp3_3a Sa.
3706   Syll Simp2_02a Exp3_3a Sb.
3707   specialize Imp3_31 with R (P $\rightarrow$ Q) (P $\rightarrow$ (Q  $\wedge$  R)).
3708   intros Imp3_31a. (*Not cited*)
3709   MP Imp3_31a Sb.
3710   apply Imp3_31a.
3711   Qed.
3712
3713   Theorem n5_32 :  $\forall$  P Q R : Prop,
3714     (P  $\rightarrow$  (Q  $\leftrightarrow$  R))  $\leftrightarrow$  ((P  $\wedge$  Q)  $\leftrightarrow$  (P  $\wedge$  R)).
3715   Proof. intros P Q R.
3716   specialize n4_76 with P (Q $\rightarrow$ R) (R $\rightarrow$ Q).
3717   intros n4_76a.
3718   specialize Exp3_3 with P Q R.

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3719 intros Exp3_3a.
3720 specialize Imp3_31 with P Q R.
3721 intros Imp3_31a.
3722 Conj Exp3_3a Imp3_31a.
3723 split.
3724 apply Exp3_3a.
3725 apply Imp3_31a.
3726 Equiv H.
3727 specialize Exp3_3 with P R Q.
3728 intros Exp3_3b.
3729 specialize Imp3_31 with P R Q.
3730 intros Imp3_31b.
3731 Conj Exp3_3b Imp3_31b.
3732 split.
3733 apply Exp3_3b.
3734 apply Imp3_31b.
3735 Equiv H0.
3736 specialize n5_3 with P Q R.
3737 intros n5_3a.
3738 specialize n5_3 with P R Q.
3739 intros n5_3b.
3740 replace (P→Q→R) with (P∧Q→R) in n4_76a.
3741 replace (P∧Q→R) with (P∧Q→P∧R) in n4_76a.
3742 replace (P→R→Q) with (P∧R→Q) in n4_76a.
3743 replace (P∧R→Q) with (P∧R→P∧Q) in n4_76a.
3744 replace ((P∧Q→P∧R)∧(P∧R→P∧Q)) with
3745 ((P∧Q)↔(P∧R)) in n4_76a.
3746 replace ((P∧Q↔P∧R)↔(P→(Q→R)∧(R→Q))) with
3747 ((P→(Q→R)∧(R→Q))↔(P∧Q↔P∧R)) in n4_76a.
3748 replace ((Q→R)∧(R→Q)) with (Q↔R) in n4_76a.
3749 apply n4_76a.
3750 apply Equiv4_01.
3751 apply EqBi.
3752 specialize n4_21 with
3753 (P→((Q→R)∧(R→Q))) ((P∧Q)↔(P∧R)).
3754 intros n4_21a.
3755 apply n4_21a. (*to commute the biconditional*)
3756 apply Equiv4_01.
3757 replace (P ∧ R → P ∧ Q) with (P ∧ R → Q).
3758 reflexivity.
3759 apply EqBi.
3760 apply n5_3b.

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3761   apply EqBi.
3762   apply H0.
3763   replace (P ∧ Q → P ∧ R) with (P ∧ Q → R).
3764   reflexivity.
3765   apply EqBi.
3766   apply n5_3a.
3767   apply EqBi.
3768   apply H.
3769   apply Equiv4_01.
3770   apply Equiv4_01.
3771   Qed.
3772
3773   Theorem n5_33 : ∀ P Q R : Prop,
3774     (P ∧ (Q → R)) ↔ (P ∧ ((P ∧ Q) → R)).
3775   Proof. intros P Q R.
3776     specialize n5_32 with P (Q→R) ((P∧Q)→R).
3777     intros n5_32a.
3778     replace
3779       ((P→(Q→R))↔(P∧Q→R))↔(P∧(Q→R)↔P∧(P∧Q→R))
3780     with
3781       (((P→(Q→R))↔(P∧Q→R))→(P∧(Q→R)↔P∧(P∧Q→R)))
3782       ∧
3783       ((P∧(Q→R)↔P∧(P∧Q→R)→(P→(Q→R)↔(P∧Q→R))))
3784     in n5_32a.
3785     specialize Simp3_26 with
3786       ((P→(Q→R))↔(P∧Q→R))→(P∧(Q→R)↔P∧(P∧Q→R))
3787       ((P∧(Q→R)↔P∧(P∧Q→R)→(P→(Q→R)↔(P∧Q→R)))).
3788     intros Simp3_26a. (*Not cited*)
3789     MP Simp3_26a n5_32a.
3790     specialize n4_73 with Q P.
3791     intros n4_73a.
3792     specialize n4_84 with Q (Q∧P) R.
3793     intros n4_84a.
3794     Syll n4_73a n4_84a Sa.
3795     replace (Q∧P) with (P∧Q) in Sa.
3796     MP Simp3_26a Sa.
3797     apply Simp3_26a.
3798     apply EqBi.
3799     specialize n4_3 with P Q.
3800     intros n4_3a.
3801     apply n4_3a. (*Not cited*)
3802     apply Equiv4_01.

```

```

3803   Qed.
3804
3805   Theorem n5_35 :  $\forall$  P Q R : Prop,
3806     ((P  $\rightarrow$  Q)  $\wedge$  (P  $\rightarrow$  R))  $\rightarrow$  (P  $\rightarrow$  (Q  $\leftrightarrow$  R)).
3807   Proof. intros P Q R.
3808     specialize Comp3_43 with P Q R.
3809     intros Comp3_43a.
3810     specialize n5_1 with Q R.
3811     intros n5_1a.
3812     specialize Syll2_05 with P (Q $\wedge$ R) (Q $\leftrightarrow$ R).
3813     intros Syll2_05a.
3814     MP Syll2_05a n5_1a.
3815     Syll Comp3_43a Syll2_05a Sa.
3816     apply Sa.
3817     Qed.
3818
3819   Theorem n5_36 :  $\forall$  P Q : Prop,
3820     (P  $\wedge$  (P  $\leftrightarrow$  Q))  $\leftrightarrow$  (Q  $\wedge$  (P  $\leftrightarrow$  Q)).
3821   Proof. intros P Q.
3822     specialize n5_32 with (P $\leftrightarrow$ Q) P Q.
3823     intros n5_32a.
3824     specialize n2_08 with (P $\leftrightarrow$ Q).
3825     intros n2_08a.
3826     replace (P $\leftrightarrow$ Q $\rightarrow$ P $\leftrightarrow$ Q) with
3827       ((P $\leftrightarrow$ Q) $\wedge$ P $\leftrightarrow$ (P $\leftrightarrow$ Q) $\wedge$ Q) in n2_08a.
3828     replace ((P $\leftrightarrow$ Q) $\wedge$ P) with (P $\wedge$ (P $\leftrightarrow$ Q)) in n2_08a.
3829     replace ((P $\leftrightarrow$ Q) $\wedge$ Q) with (Q $\wedge$ (P $\leftrightarrow$ Q)) in n2_08a.
3830     apply n2_08a.
3831     apply EqBi.
3832     specialize n4_3 with Q (P $\leftrightarrow$ Q).
3833     intros n4_3a.
3834     apply n4_3a.
3835     apply EqBi.
3836     specialize n4_3 with P (P $\leftrightarrow$ Q).
3837     intros n4_3b.
3838     apply n4_3b.
3839     replace ((P  $\leftrightarrow$  Q)  $\wedge$  P  $\leftrightarrow$  (P  $\leftrightarrow$  Q)  $\wedge$  Q) with
3840       (P  $\leftrightarrow$  Q  $\rightarrow$  P  $\leftrightarrow$  Q).
3841     reflexivity.
3842     apply EqBi.
3843     apply n5_32a.
3844     Qed.

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3845   (*The proof sketch cites Ass3_35 and n4_38, but
3846     the sketch was indecipherable.*)
3847
3848 Theorem n5_4 :  $\forall$  P Q : Prop,
3849   (P  $\rightarrow$  (P  $\rightarrow$  Q))  $\leftrightarrow$  (P  $\rightarrow$  Q).
3850 Proof. intros P Q.
3851   specialize n2_43 with P Q.
3852   intros n2_43a.
3853   specialize Simp2_02 with (P) (P $\rightarrow$ Q).
3854   intros Simp2_02a.
3855   Conj n2_43a Simp2_02a.
3856   split.
3857   apply n2_43a.
3858   apply Simp2_02a.
3859   Equiv H.
3860   apply H.
3861   apply Equiv4_01.
3862   Qed.
3863
3864 Theorem n5_41 :  $\forall$  P Q R : Prop,
3865   ((P  $\rightarrow$  Q)  $\rightarrow$  (P  $\rightarrow$  R))  $\leftrightarrow$  (P  $\rightarrow$  Q  $\rightarrow$  R).
3866 Proof. intros P Q R.
3867   specialize n2_86 with P Q R.
3868   intros n2_86a.
3869   specialize n2_77 with P Q R.
3870   intros n2_77a.
3871   Conj n2_86a n2_77a.
3872   split.
3873   apply n2_86a.
3874   apply n2_77a.
3875   Equiv H.
3876   apply H.
3877   apply Equiv4_01.
3878   Qed.
3879
3880 Theorem n5_42 :  $\forall$  P Q R : Prop,
3881   (P  $\rightarrow$  Q  $\rightarrow$  R)  $\leftrightarrow$  (P  $\rightarrow$  Q  $\rightarrow$  P  $\wedge$  R).
3882 Proof. intros P Q R.
3883   specialize n5_3 with P Q R.
3884   intros n5_3a.
3885   specialize n4_87 with P Q R.
3886   intros n4_87a.

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3887   replace ((P ∧ Q) → R) with (P → Q → R) in n5_3a.
3888   specialize n4_87 with P Q (P ∧ R).
3889   intros n4_87b.
3890   replace ((P ∧ Q) → (P ∧ R)) with
3891     (P → Q → (P ∧ R)) in n5_3a.
3892   apply n5_3a.
3893   specialize Imp3_31 with P Q (P ∧ R).
3894   intros Imp3_31b.
3895   specialize Exp3_3 with P Q (P ∧ R).
3896   intros Exp3_3b.
3897   Conj Imp3_31b Exp3_3b.
3898   split.
3899   apply Imp3_31b.
3900   apply Exp3_3b.
3901   Equiv H.
3902   apply EqBi.
3903   apply H.
3904   apply Equiv4_01.
3905   specialize Imp3_31 with P Q R.
3906   intros Imp3_31a.
3907   specialize Exp3_3 with P Q R.
3908   intros Exp3_3a.
3909   Conj Imp3_31a Exp3_3.
3910   split.
3911   apply Imp3_31a.
3912   apply Exp3_3a.
3913   Equiv H.
3914   apply EqBi.
3915   apply H.
3916   apply Equiv4_01.
3917   Qed.
3918
3919   Theorem n5_44 : ∀ P Q R : Prop,
3920     (P → Q) → ((P → R) ↔ (P → (Q ∧ R))).
3921   Proof. intros P Q R.
3922   specialize n4_76 with P Q R.
3923   intros n4_76a.
3924   replace ((P → Q) ∧ (P → R) ↔ (P → Q ∧ R)) with
3925     (((P → Q) ∧ (P → R)) → (P → Q ∧ R))
3926     ∧
3927     ((P → Q ∧ R) → (P → Q) ∧ (P → R)) in n4_76a.
3928   specialize Simp3_26 with

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3929       $((P \rightarrow Q) \wedge (P \rightarrow R) \rightarrow (P \rightarrow Q \wedge R))$ 
3930       $((P \rightarrow Q \wedge R) \rightarrow (P \rightarrow Q) \wedge (P \rightarrow R)).$ 
3931      intros Simp3_26a. (*Not cited*)
3932      MP Simp3_26a n4_76a.
3933      specialize Exp3_3 with  $(P \rightarrow Q)$   $(P \rightarrow R)$   $(P \rightarrow Q \wedge R).$ 
3934      intros Exp3_3a. (*Not cited*)
3935      MP Exp3_3a Simp3_26a.
3936      specialize Simp3_27 with
3937           $((P \rightarrow Q) \wedge (P \rightarrow R) \rightarrow (P \rightarrow Q \wedge R))$ 
3938           $((P \rightarrow Q \wedge R) \rightarrow (P \rightarrow Q) \wedge (P \rightarrow R)).$ 
3939      intros Simp3_27a. (*Not cited*)
3940      MP Simp3_27a n4_76a.
3941      specialize Simp3_26 with  $(P \rightarrow R)$   $(P \rightarrow Q).$ 
3942      intros Simp3_26b.
3943      replace  $((P \rightarrow Q) \wedge (P \rightarrow R))$  with
3944           $((P \rightarrow R) \wedge (P \rightarrow Q))$  in Simp3_27a.
3945      Syll Simp3_27a Simp3_26b Sa.
3946      specialize Simp2_02 with  $(P \rightarrow Q)$   $((P \rightarrow Q \wedge R) \rightarrow P \rightarrow R).$ 
3947      intros Simp2_02a. (*Not cited*)
3948      MP Simp2_02a Sa.
3949      clear Sa. clear Simp3_26b. clear Simp3_26a.
3950      clear n4_76a. clear Simp3_27a.
3951      Conj Exp3_3a Simp2_02a.
3952      split.
3953      apply Exp3_3a.
3954      apply Simp2_02a.
3955      specialize n4_76 with  $(P \rightarrow Q)$ 
3956           $((P \rightarrow R) \rightarrow (P \rightarrow (Q \wedge R)))$   $((P \rightarrow (Q \wedge R)) \rightarrow (P \rightarrow R)).$ 
3957      intros n4_76b. (*Second use not indicated*)
3958      replace
3959           $((P \rightarrow Q) \rightarrow (P \rightarrow R) \rightarrow P \rightarrow Q \wedge R) \wedge ((P \rightarrow Q) \rightarrow (P \rightarrow Q \wedge R) \rightarrow P \rightarrow R))$ 
3960          with
3961           $((P \rightarrow Q) \rightarrow ((P \rightarrow R) \rightarrow P \rightarrow Q \wedge R) \wedge ((P \rightarrow Q \wedge R) \rightarrow P \rightarrow R))$  in H.
3962      replace  $((P \rightarrow R) \rightarrow P \rightarrow Q \wedge R) \wedge ((P \rightarrow Q \wedge R) \rightarrow P \rightarrow R)$  with
3963           $((P \rightarrow R) \leftrightarrow (P \rightarrow Q \wedge R))$  in H.
3964      apply H.
3965      apply Equiv4_01.
3966      replace  $((P \rightarrow Q) \rightarrow ((P \rightarrow R) \rightarrow P \rightarrow Q \wedge R) \wedge ((P \rightarrow Q \wedge R) \rightarrow P \rightarrow R))$  with
3967           $((P \rightarrow Q) \rightarrow (P \rightarrow R) \rightarrow P \rightarrow Q \wedge R) \wedge ((P \rightarrow Q) \rightarrow (P \rightarrow Q \wedge R) \rightarrow P \rightarrow R).$ 
3968      reflexivity.
3969      apply EqBi.
3970      apply n4_76b.

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3971   apply EqBi.
3972   specialize n4_3 with (P→R) (P→Q).
3973   intros n4_3a.
3974   apply n4_3a. (*Not cited*)
3975   apply Equiv4_01.
3976   Qed.
3977
3978 Theorem n5_5 : ∀ P Q : Prop,
3979   P → ((P → Q) ↔ Q).
3980 Proof. intros P Q.
3981   specialize Ass3_35 with P Q.
3982   intros Ass3_35a.
3983   specialize Exp3_3 with P (P→Q) Q.
3984   intros Exp3_3a.
3985   MP Exp3_3a Ass3_35a.
3986   specialize Simp2_02 with P Q.
3987   intros Simp2_02a.
3988   specialize Exp3_3 with P Q (P→Q).
3989   intros Exp3_3b.
3990   specialize n3_42 with P Q (P→Q). (*Not cited*)
3991   intros n3_42a.
3992   MP n3_42a Simp2_02a.
3993   MP Exp3_3b n3_42a.
3994   clear n3_42a. clear Simp2_02a. clear Ass3_35a.
3995   Conj Exp3_3a Exp3_3b.
3996   split.
3997   apply Exp3_3a.
3998   apply Exp3_3b.
3999   specialize n3_47 with P P ((P→Q)→Q) (Q→(P→Q)).
4000   intros n3_47a.
4001   MP n3_47a H.
4002   replace (P∧P) with P in n3_47a.
4003   replace (((P→Q)→Q)∧(Q→(P→Q))) with
4004     ((P→Q)↔Q) in n3_47a.
4005   apply n3_47a.
4006   apply Equiv4_01.
4007   apply EqBi.
4008   specialize n4_24 with P.
4009   intros n4_24a. (*Not cited*)
4010   apply n4_24a.
4011   Qed.
4012

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4013 Theorem n5_501 :  $\forall$  P Q : Prop,
4014   P  $\rightarrow$  (Q  $\leftrightarrow$  (P  $\leftrightarrow$  Q)).
4015 Proof. intros P Q.
4016 specialize n5_1 with P Q.
4017 intros n5_1a.
4018 specialize Exp3_3 with P Q (P $\leftrightarrow$ Q).
4019 intros Exp3_3a.
4020 MP Exp3_3a n5_1a.
4021 specialize Ass3_35 with P Q.
4022 intros Ass3_35a.
4023 specialize Simp3_26 with (P $\wedge$ (P $\rightarrow$ Q)) (Q $\rightarrow$ P).
4024 intros Simp3_26a. (Not cited)
4025 Syll Simp3_26a Ass3_35a Sa.
4026 replace ((P $\wedge$ (P $\rightarrow$ Q)) $\wedge$ (Q $\rightarrow$ P)) with
4027   (P $\wedge$ ((P $\rightarrow$ Q) $\wedge$ (Q $\rightarrow$ P))) in Sa.
4028 replace ((P $\rightarrow$ Q) $\wedge$ (Q $\rightarrow$ P)) with (P $\leftrightarrow$ Q) in Sa.
4029 specialize Exp3_3 with P (P $\leftrightarrow$ Q) Q.
4030 intros Exp3_3b.
4031 MP Exp3_3b Sa.
4032 clear n5_1a. clear Ass3_35a.
4033   clear Simp3_26a. clear Sa.
4034 Conj Exp3_3a Exp3_3b.
4035 split.
4036 apply Exp3_3a.
4037 apply Exp3_3b.
4038 specialize n4_76 with P (Q $\rightarrow$ (P $\leftrightarrow$ Q)) ((P $\leftrightarrow$ Q) $\rightarrow$ Q).
4039 intros n4_76a. (Not cited)
4040 replace ((P $\rightarrow$ Q $\rightarrow$ P $\leftrightarrow$ Q) $\wedge$ (P $\rightarrow$ P $\leftrightarrow$ Q $\rightarrow$ Q)) with
4041   ((P $\rightarrow$ (Q $\rightarrow$ P $\leftrightarrow$ Q) $\wedge$ (P $\leftrightarrow$ Q $\rightarrow$ Q))) in H.
4042 replace ((Q $\rightarrow$ (P $\leftrightarrow$ Q)) $\wedge$ ((P $\leftrightarrow$ Q) $\rightarrow$ Q)) with
4043   (Q $\leftrightarrow$ (P $\leftrightarrow$ Q)) in H.
4044 apply H.
4045 apply Equiv4_01.
4046 replace (P $\rightarrow$ (Q $\rightarrow$ P $\leftrightarrow$ Q) $\wedge$ (P $\leftrightarrow$ Q $\rightarrow$ Q)) with
4047   ((P $\rightarrow$ Q $\rightarrow$ P $\leftrightarrow$ Q) $\wedge$ (P $\rightarrow$ P $\leftrightarrow$ Q $\rightarrow$ Q)).
4048 reflexivity.
4049 apply EqBi.
4050 apply n4_76a.
4051 apply Equiv4_01.
4052 replace (P $\wedge$ (P $\rightarrow$ Q) $\wedge$ (Q $\rightarrow$ P)) with
4053   ((P $\wedge$ (P $\rightarrow$ Q)) $\wedge$ (Q $\rightarrow$ P)).
4054 reflexivity.

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4055   apply EqBi.
4056   specialize n4_32 with P (P→Q) (Q→P).
4057   intros n4_32a. (*Not cited*)
4058   apply n4_32a.
4059   Qed.
4060
4061   Theorem n5_53 : ∀ P Q R S : Prop,
4062     (((P ∨ Q) ∨ R) → S) ↔ (((P → S) ∧ (Q → S)) ∧ (R → S)).
4063   Proof. intros P Q R S.
4064     specialize n4_77 with S (P∨Q) R.
4065     intros n4_77a.
4066     specialize n4_77 with S P Q.
4067     intros n4_77b.
4068     replace (P ∨ Q → S) with
4069       ((P→S)∧(Q→S)) in n4_77a.
4070     replace (((P→S)∧(Q→S))∧(R→S))↔(((P∨Q)∨R)→S))
4071       with
4072       (((P∨Q)∨R)→S)↔(((P→S)∧(Q→S))∧(R→S)))
4073       in n4_77a.
4074     apply n4_77a.
4075     apply EqBi.
4076     specialize n4_21 with ((P ∨ Q) ∨ R → S)
4077       (((P → S) ∧ (Q → S)) ∧ (R → S)).
4078     intros n4_21a.
4079     apply n4_21a. (*Not cited*)
4080     apply EqBi.
4081     apply n4_77b.
4082     Qed.
4083
4084   Theorem n5_54 : ∀ P Q : Prop,
4085     ((P ∧ Q) ↔ P) ∨ ((P ∧ Q) ↔ Q).
4086   Proof. intros P Q.
4087     specialize n4_73 with P Q.
4088     intros n4_73a.
4089     specialize n4_44 with Q P.
4090     intros n4_44a.
4091     specialize Trans2_16 with Q (P↔(P∧Q)).
4092     intros Trans2_16a.
4093     MP n4_73a Trans2_16a.
4094     specialize Trans4_11 with Q (Q∨(P∧Q)).
4095     intros Trans4_11a.
4096     replace (Q∧P) with (P∧Q) in n4_44a.

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4097   replace (Q↔Q∨P∧Q) with
4098     (~Q↔~(Q∨P∧Q)) in n4_44a.
4099   replace (~Q) with (~(Q∨P∧Q)) in Trans2_16a.
4100   replace (~(Q∨P∧Q)) with
4101     (~Q∧~(P∧Q)) in Trans2_16a.
4102   specialize n5_1 with (~Q) (~(P∧Q)).
4103   intros n5_1a.
4104   Syll Trans2_16a n5_1a Sa.
4105   replace (~(P↔P∧Q)→(~Q↔~(P∧Q))) with
4106     (~~(P↔P∧Q)∨(~Q↔~(P∧Q))) in Sa.
4107   replace (~~(P↔P∧Q)) with (P↔P∧Q) in Sa.
4108   specialize Trans4_11 with Q (P∧Q).
4109   intros Trans4_11b.
4110   replace (~Q↔~(P∧Q)) with (Q↔(P∧Q)) in Sa.
4111   replace (Q↔(P∧Q)) with ((P∧Q)↔Q) in Sa.
4112   replace (P↔(P∧Q)) with ((P∧Q)↔P) in Sa.
4113   apply Sa.
4114   apply EqBi.
4115   specialize n4_21 with (P∧Q) P.
4116   intros n4_21a. (*Not cited*)
4117   apply n4_21a.
4118   apply EqBi.
4119   specialize n4_21 with (P∧Q) Q.
4120   intros n4_21b. (*Not cited*)
4121   apply n4_21b.
4122   apply EqBi.
4123   apply Trans4_11b.
4124   apply EqBi.
4125   specialize n4_13 with (P ↔ (P∧Q)).
4126   intros n4_13a. (*Not cited*)
4127   apply n4_13a.
4128   replace (~~(P↔P∧Q)∨(~Q↔~(P∧Q))) with
4129     (~(P↔P∧Q)→~Q↔~(P∧Q)).
4130   reflexivity.
4131   apply Impl1_01. (*Not cited*)
4132   apply EqBi.
4133   specialize n4_56 with Q (P∧Q).
4134   intros n4_56a. (*Not cited*)
4135   apply n4_56a.
4136   replace (~(Q∨P∧Q)) with (~Q).
4137   reflexivity.
4138   apply EqBi.

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4139   apply n4_44a.
4140   replace ( $\sim Q \leftrightarrow \sim (Q \vee P \wedge Q)$ ) with ( $Q \leftrightarrow Q \vee P \wedge Q$ ).
4141   reflexivity.
4142   apply EqBi.
4143   apply Trans4_11a.
4144   apply EqBi.
4145   specialize n4_3 with P Q.
4146   intros n4_3a. (*Not cited*)
4147   apply n4_3a.
4148   Qed.
4149
4150 Theorem n5_55 :  $\forall P Q : \text{Prop}$ ,
4151   ( $(P \vee Q) \leftrightarrow P$ )  $\vee$  ( $(P \vee Q) \leftrightarrow Q$ ).
4152 Proof. intros P Q.
4153   specialize Add1_3 with ( $P \wedge Q$ ) (P).
4154   intros Add1_3a.
4155   replace ( $(P \wedge Q) \vee P$ ) with ( $(P \vee P) \wedge (Q \vee P)$ ) in Add1_3a.
4156   replace ( $P \vee P$ ) with P in Add1_3a.
4157   replace ( $Q \vee P$ ) with ( $P \vee Q$ ) in Add1_3a.
4158   specialize n5_1 with P ( $P \vee Q$ ).
4159   intros n5_1a.
4160   Syll Add1_3a n5_1a Sa.
4161   specialize n4_74 with P Q.
4162   intros n4_74a.
4163   specialize Trans2_15 with P ( $Q \leftrightarrow P \vee Q$ ).
4164   intros Trans2_15a. (*Not cited*)
4165   MP Trans2_15a n4_74a.
4166   Syll Trans2_15a Sa Sb.
4167   replace ( $\sim (Q \leftrightarrow (P \vee Q)) \rightarrow (P \leftrightarrow (P \vee Q))$ ) with
4168     ( $\sim \sim (Q \leftrightarrow (P \vee Q)) \vee (P \leftrightarrow (P \vee Q))$ ) in Sb.
4169   replace ( $\sim \sim (Q \leftrightarrow (P \vee Q))$ ) with ( $Q \leftrightarrow (P \vee Q)$ ) in Sb.
4170   replace ( $Q \leftrightarrow (P \vee Q)$ ) with ( $(P \vee Q) \leftrightarrow Q$ ) in Sb.
4171   replace ( $P \leftrightarrow (P \vee Q)$ ) with ( $(P \vee Q) \leftrightarrow P$ ) in Sb.
4172   replace ( $(P \vee Q \leftrightarrow Q) \vee (P \vee Q \leftrightarrow P)$ ) with
4173     ( $(P \vee Q \leftrightarrow P) \vee (P \vee Q \leftrightarrow Q)$ ) in Sb.
4174   apply Sb.
4175   apply EqBi.
4176   specialize n4_31 with ( $P \vee Q \leftrightarrow P$ ) ( $P \vee Q \leftrightarrow Q$ ).
4177   intros n4_31a. (*Not cited*)
4178   apply n4_31a.
4179   apply EqBi.
4180   specialize n4_21 with ( $P \vee Q$ ) P.

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4181   intros n4_21a. (*Not cited*)
4182   apply n4_21a.
4183   apply EqBi.
4184   specialize n4_21 with (P ∨ Q) Q.
4185   intros n4_21b. (*Not cited*)
4186   apply n4_21b.
4187   apply EqBi.
4188   specialize n4_13 with (Q ↔ P ∨ Q).
4189   intros n4_13a. (*Not cited*)
4190   apply n4_13a.
4191   replace (~~(Q↔P∨Q)∨(P↔P∨Q)) with
4192     (∼(Q↔P∨Q)→P↔P∨Q).
4193   reflexivity.
4194   apply Impl1_01.
4195   apply EqBi.
4196   specialize n4_31 with P Q.
4197   intros n4_31b.
4198   apply n4_31b.
4199   apply EqBi.
4200   specialize n4_25 with P.
4201   intros n4_25a. (*Not cited*)
4202   apply n4_25a.
4203   replace ((P∨P)∧(Q∨P)) with ((P∧Q)∨P).
4204   reflexivity.
4205   replace ((P∧Q)∨P) with (P∨(P∧Q)).
4206   replace (Q∨P) with (P∨Q).
4207   apply EqBi.
4208   specialize n4_41 with P P Q.
4209   intros n4_41a. (*Not cited*)
4210   apply n4_41a.
4211   apply EqBi.
4212   specialize n4_31 with P Q.
4213   intros n4_31c.
4214   apply n4_31c.
4215   apply EqBi.
4216   specialize n4_31 with P (P ∧ Q).
4217   intros n4_31d. (*Not cited*)
4218   apply n4_31d.
4219   Qed.
4220
4221   Theorem n5_6 : ∀ P Q R : Prop,
4222     ((P ∧ ∼Q) → R) ↔ (P → (Q ∨ R)).

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```

4223 Proof. intros P Q R.
4224 specialize n4_87 with P (~Q) R.
4225 intros n4_87a.
4226 specialize n4_64 with Q R.
4227 intros n4_64a.
4228 specialize n4_85 with P Q R.
4229 intros n4_85a.
4230 replace (((P ∧ ~Q → R) ↔ (P → ~Q → R)) ↔ ((~Q → P → R) ↔ (~Q ∧ P → R)))
4231   with
4232     (((P ∧ ~Q → R) ↔ (P → ~Q → R)) → ((~Q → P → R) ↔ (~Q ∧ P → R)))
4233     ∧
4234     (((~Q → P → R) ↔ (~Q ∧ P → R)) → ((P ∧ ~Q → R) ↔ (P → ~Q → R))))
4235   in n4_87a.
4236 specialize Simp3_27 with
4237   (((P ∧ ~Q → R) ↔ (P → ~Q → R) → (~Q → P → R) ↔ (~Q ∧ P → R)))
4238   (((~Q → P → R) ↔ (~Q ∧ P → R) → (P ∧ ~Q → R) ↔ (P → ~Q → R))).
4239 intros Simp3_27a.
4240 MP Simp3_27a n4_87a.
4241 specialize Imp3_31 with (~Q) P R.
4242 intros Imp3_31a.
4243 specialize Exp3_3 with (~Q) P R.
4244 intros Exp3_3a.
4245 Conj Imp3_31a Exp3_3a.
4246 split.
4247 apply Imp3_31a.
4248 apply Exp3_3a.
4249 Equiv H.
4250 MP Simp3_27a H.
4251 replace (~Q → R) with (Q ∨ R) in Simp3_27a.
4252 apply Simp3_27a.
4253 replace (Q ∨ R) with (~Q → R).
4254 reflexivity.
4255 apply EqBi.
4256 apply n4_64a.
4257 apply Equiv4_01.
4258 apply Equiv4_01.
4259 Qed.
4260
4261 Theorem n5_61 : ∀ P Q : Prop,
4262   ((P ∨ Q) ∧ ~Q) ↔ (P ∧ ~Q).
4263 Proof. intros P Q.
4264 specialize n4_74 with Q P.

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4265   intros n4_74a.
4266   specialize n5_32 with (~Q) P (Q ∨ P).
4267   intros n5_32a.
4268   replace (~Q → P ↔ Q ∨ P) with
4269     (~Q ∧ P ↔ ~Q ∧ (Q ∨ P)) in n4_74a.
4270   replace (~Q ∧ P) with (P ∧ ~Q) in n4_74a.
4271   replace (~Q ∧ (Q ∨ P)) with ((Q ∨ P) ∧ ~Q) in n4_74a.
4272   replace (Q ∨ P) with (P ∨ Q) in n4_74a.
4273   replace (P ∧ ~Q ↔ (P ∨ Q) ∧ ~Q) with
4274     ((P ∨ Q) ∧ ~Q ↔ P ∧ ~Q) in n4_74a.
4275   apply n4_74a.
4276   apply EqBi.
4277   specialize n4_21 with ((P ∨ Q) ∧ ~Q) (P ∧ ~Q).
4278   intros n4_21a. (*Not cited*)
4279   apply n4_21a.
4280   apply EqBi.
4281   specialize n4_31 with P Q.
4282   intros n4_31a. (*Not cited*)
4283   apply n4_31a.
4284   apply EqBi.
4285   specialize n4_3 with (Q ∨ P) (~Q).
4286   intros n4_3a. (*Not cited*)
4287   apply n4_3a.
4288   apply EqBi.
4289   specialize n4_3 with P (~Q).
4290   intros n4_3b. (*Not cited*)
4291   apply n4_3b.
4292   replace (~Q ∧ P ↔ ~Q ∧ (Q ∨ P)) with
4293     (~Q → P ↔ Q ∨ P).
4294   reflexivity.
4295   apply EqBi.
4296   apply n5_32a.
4297   Qed.
4298
4299   Theorem n5_62 : ∀ P Q : Prop,
4300     ((P ∧ Q) ∨ ~Q) ↔ (P ∨ ~Q).
4301   Proof. intros P Q.
4302   specialize n4_7 with Q P.
4303   intros n4_7a.
4304   replace (Q → P) with (~Q ∨ P) in n4_7a.
4305   replace (Q → (Q ∧ P)) with (~Q ∨ (Q ∧ P)) in n4_7a.
4306   replace (~Q ∨ (Q ∧ P)) with ((Q ∧ P) ∨ ~Q) in n4_7a.

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4307   replace ( $\sim Q \vee P$ ) with ( $P \vee \sim Q$ ) in n4_7a.
4308   replace ( $Q \wedge P$ ) with ( $P \wedge Q$ ) in n4_7a.
4309   replace ( $P \vee \sim Q \leftrightarrow P \wedge Q \vee \sim Q$ ) with
4310       ( $P \wedge Q \vee \sim Q \leftrightarrow P \vee \sim Q$ ) in n4_7a.
4311   apply n4_7a.
4312   apply EqBi.
4313   specialize n4_21 with ( $P \wedge Q \vee \sim Q$ ) ( $P \vee \sim Q$ ).
4314   intros n4_21a. (*Not cited*)
4315   apply n4_21a.
4316   apply EqBi.
4317   specialize n4_3 with P Q.
4318   intros n4_3a. (*Not cited*)
4319   apply n4_3a.
4320   apply EqBi.
4321   specialize n4_31 with P ( $\sim Q$ ).
4322   intros n4_31a. (*Not cited*)
4323   apply n4_31a.
4324   apply EqBi.
4325   specialize n4_31 with ( $Q \wedge P$ ) ( $\sim Q$ ).
4326   intros n4_31b. (*Not cited*)
4327   apply n4_31b.
4328   replace ( $\sim Q \vee Q \wedge P$ ) with ( $Q \rightarrow Q \wedge P$ ).
4329   reflexivity.
4330   apply EqBi.
4331   specialize n4_6 with Q ( $Q \wedge P$ ).
4332   intros n4_6a. (*Not cited*)
4333   apply n4_6a.
4334   replace ( $\sim Q \vee P$ ) with ( $Q \rightarrow P$ ).
4335   reflexivity.
4336   apply EqBi.
4337   specialize n4_6 with Q P.
4338   intros n4_6b. (*Not cited*)
4339   apply n4_6b.
4340   Qed.
4341
4342   Theorem n5_63 :  $\forall P Q : \text{Prop}$ ,
4343       ( $P \vee Q$ )  $\leftrightarrow$  ( $P \vee (\sim P \wedge Q)$ ).
4344   Proof. intros P Q.
4345   specialize n5_62 with Q ( $\sim P$ ).
4346   intros n5_62a.
4347   replace ( $\sim \sim P$ ) with P in n5_62a.
4348   replace ( $Q \vee P$ ) with ( $P \vee Q$ ) in n5_62a.

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4349  replace ((Q $\wedge$ ~P) $\vee$ P) with (P $\vee$ (Q $\wedge$ ~P)) in n5_62a.
4350  replace (P  $\vee$  Q  $\wedge$  ~P  $\leftrightarrow$  P  $\vee$  Q) with
4351      (P  $\vee$  Q  $\leftrightarrow$  P  $\vee$  Q  $\wedge$  ~P) in n5_62a.
4352  replace (Q $\wedge$ ~P) with (~P $\wedge$ Q) in n5_62a.
4353  apply n5_62a.
4354  apply EqBi.
4355  specialize n4_3 with (~P) Q.
4356  intros n4_3a.
4357  apply n4_3a. (*Not cited*)
4358  apply EqBi.
4359  specialize n4_21 with (P $\vee$ Q) (P $\vee$ (Q $\wedge$ ~P)).
4360  intros n4_21a. (*Not cited*)
4361  apply n4_21a.
4362  apply EqBi.
4363  specialize n4_31 with P (Q $\wedge$ ~P).
4364  intros n4_31a. (*Not cited*)
4365  apply n4_31a.
4366  apply EqBi.
4367  specialize n4_31 with P Q.
4368  intros n4_31b. (*Not cited*)
4369  apply n4_31b.
4370  apply EqBi.
4371  specialize n4_13 with P.
4372  intros n4_13a. (*Not cited*)
4373  apply n4_13a.
4374  Qed.
4375
4376  Theorem n5_7 :  $\forall$  P Q R : Prop,
4377      ((P  $\vee$  R)  $\leftrightarrow$  (Q  $\vee$  R))  $\leftrightarrow$  (R  $\vee$  (P  $\leftrightarrow$  Q)).
4378  Proof. intros P Q R.
4379  specialize n5_32 with (~R) (~P) (~Q).
4380  intros n5_32a. (*Not cited*)
4381  replace (~R $\wedge$ ~P) with (~R $\vee$ P) in n5_32a.
4382  replace (~R $\wedge$ ~Q) with (~R $\vee$ Q) in n5_32a.
4383  replace ((~R $\vee$ P)  $\leftrightarrow$  (~R $\vee$ Q)) with
4384      ((R $\vee$ P)  $\leftrightarrow$  (R $\vee$ Q)) in n5_32a.
4385  replace ((~P)  $\leftrightarrow$  (~Q)) with (P  $\leftrightarrow$  Q) in n5_32a.
4386  replace (~R  $\rightarrow$  (P  $\leftrightarrow$  Q)) with
4387      (~~R  $\vee$  (P  $\leftrightarrow$  Q)) in n5_32a.
4388  replace (~~R) with R in n5_32a.
4389  replace (R $\vee$ P) with (P $\vee$ R) in n5_32a.
4390  replace (R $\vee$ Q) with (Q $\vee$ R) in n5_32a.

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4391  replace ((R∨(P↔Q))↔(P∨R↔Q∨R)) with
4392      ((P∨R↔Q∨R)↔(R∨(P↔Q))) in n5_32a.
4393  apply n5_32a. (*Not cited*)
4394  apply EqBi.
4395  specialize n4_21 with ((P∨R)↔(Q∨R)) (R∨(P↔Q)).
4396  intros n4_21a.
4397  apply n4_21a. (*Not cited*)
4398  apply EqBi.
4399  specialize n4_31 with Q R.
4400  intros n4_31a. (*Not cited*)
4401  apply n4_31a.
4402  apply EqBi.
4403  specialize n4_31 with P R.
4404  intros n4_31b. (*Not cited*)
4405  apply n4_31b.
4406  apply EqBi.
4407  specialize n4_13 with R.
4408  intros n4_13a. (*Not cited*)
4409  apply n4_13a.
4410  replace (~~R∨(P↔Q)) with (~R→P↔Q).
4411  reflexivity.
4412  apply Impl1_01. (*Not cited*)
4413  apply EqBi.
4414  specialize Trans4_11 with P Q.
4415  intros Trans4_11a. (*Not cited*)
4416  apply Trans4_11a.
4417  apply EqBi.
4418  specialize Trans4_11 with (R ∨ P) (R ∨ Q).
4419  intros Trans4_11a. (*Not cited*)
4420  apply Trans4_11a.
4421  replace (~(R∨Q)) with (~R∧~Q).
4422  reflexivity.
4423  apply EqBi.
4424  specialize n4_56 with R Q.
4425  intros n4_56a. (*Not cited*)
4426  apply n4_56a.
4427  replace (~(R∨P)) with (~R∧~P).
4428  reflexivity.
4429  apply EqBi.
4430  specialize n4_56 with R P.
4431  intros n4_56b. (*Not cited*)
4432  apply n4_56b.

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4433 Qed.
4434 (*The proof sketch was indecipherable, but an
4435     easy proof was available through n5_32.*)
4436
4437 Theorem n5_71 :  $\forall$  P Q R : Prop,
4438    $(Q \rightarrow \sim R) \rightarrow ((P \vee Q) \wedge R) \leftrightarrow (P \wedge R)$ .
4439 Proof. intros P Q R.
4440 specialize n4_4 with R P Q.
4441 intros n4_4a.
4442 specialize n4_62 with Q R.
4443 intros n4_62a.
4444 specialize n4_51 with Q R.
4445 intros n4_51a.
4446 replace  $(\sim Q \vee \sim R)$  with  $(\sim(Q \wedge R))$  in n4_62a.
4447 replace  $((Q \rightarrow \sim R) \leftrightarrow \sim(Q \wedge R))$  with
4448    $((Q \rightarrow \sim R) \rightarrow \sim(Q \wedge R))$ 
4449    $\wedge$ 
4450    $(\sim(Q \wedge R) \rightarrow (Q \rightarrow \sim R))$  in n4_62a.
4451 specialize Simp3_26 with
4452    $((Q \rightarrow \sim R) \rightarrow \sim(Q \wedge R)) (\sim(Q \wedge R) \rightarrow (Q \rightarrow \sim R))$ .
4453 intros Simp3_26a.
4454 MP Simp3_26a n4_62a.
4455 specialize n4_74 with  $(Q \wedge R) (P \wedge R)$ .
4456 intros n4_74a.
4457 Syll Simp3_26a n4_74a Sa.
4458 replace  $(R \wedge P)$  with  $(P \wedge R)$  in n4_4a.
4459 replace  $(R \wedge Q)$  with  $(Q \wedge R)$  in n4_4a.
4460 replace  $((P \wedge R) \vee (Q \wedge R))$  with
4461    $((Q \wedge R) \vee (P \wedge R))$  in n4_4a.
4462 replace  $((Q \wedge R) \vee (P \wedge R))$  with  $(R \wedge (P \vee Q))$  in Sa.
4463 replace  $(R \wedge (P \vee Q))$  with  $((P \vee Q) \wedge R)$  in Sa.
4464 replace  $((P \wedge R) \leftrightarrow ((P \vee Q) \wedge R))$  with
4465    $((P \vee Q) \wedge R) \leftrightarrow (P \wedge R)$  in Sa.
4466 apply Sa.
4467 apply EqBi.
4468 specialize n4_21 with  $((P \vee Q) \wedge R) (P \wedge R)$ .
4469 intros n4_21a. (*Not cited*)
4470 apply n4_21a.
4471 apply EqBi.
4472 specialize n4_3 with  $(P \vee Q) R$ .
4473 intros n4_3a.
4474 apply n4_3a. (*Not cited*)

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4475   apply EqBi.
4476   apply n4_4a. (*Not cited*)
4477   apply EqBi.
4478   specialize n4_31 with (Q $\wedge$ R) (P $\wedge$ R).
4479   intros n4_31a. (*Not cited*)
4480   apply n4_31a.
4481   apply EqBi.
4482   specialize n4_3 with Q R.
4483   intros n4_3a. (*Not cited*)
4484   apply n4_3a.
4485   apply EqBi.
4486   specialize n4_3 with P R.
4487   intros n4_3b. (*Not cited*)
4488   apply n4_3b.
4489   apply Equiv4_01.
4490   apply EqBi.
4491   apply n4_51a.
4492   Qed.
4493
4494   Theorem n5_74 :  $\forall$  P Q R : Prop,
4495     (P  $\rightarrow$  (Q  $\leftrightarrow$  R))  $\leftrightarrow$  ((P  $\rightarrow$  Q)  $\leftrightarrow$  (P  $\rightarrow$  R)).
4496   Proof. intros P Q R.
4497   specialize n5_41 with P Q R.
4498   intros n5_41a.
4499   specialize n5_41 with P R Q.
4500   intros n5_41b.
4501   Conj n5_41a n5_41b.
4502   split.
4503   apply n5_41a.
4504   apply n5_41b.
4505   specialize n4_38 with
4506     ((P $\rightarrow$ Q) $\rightarrow$ (P $\rightarrow$ R)) ((P $\rightarrow$ R) $\rightarrow$ (P $\rightarrow$ Q))
4507     (P $\rightarrow$ Q $\rightarrow$ R) (P $\rightarrow$ R $\rightarrow$ Q).
4508   intros n4_38a.
4509   MP n4_38a H.
4510   replace ((P $\rightarrow$ Q) $\rightarrow$ (P $\rightarrow$ R)) $\wedge$ ((P $\rightarrow$ R) $\rightarrow$ (P $\rightarrow$ Q)) with
4511     ((P $\rightarrow$ Q) $\leftrightarrow$ (P $\rightarrow$ R)) in n4_38a.
4512   specialize n4_76 with P (Q $\rightarrow$ R) (R $\rightarrow$ Q).
4513   intros n4_76a.
4514   replace ((Q $\rightarrow$ R) $\wedge$ (R $\rightarrow$ Q)) with (Q $\leftrightarrow$ R) in n4_76a.
4515   replace ((P $\rightarrow$ Q $\rightarrow$ R) $\wedge$ (P $\rightarrow$ R $\rightarrow$ Q)) with
4516     (P $\rightarrow$ (Q $\leftrightarrow$ R)) in n4_38a.

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4517   replace (( $(P \rightarrow Q) \leftrightarrow (P \rightarrow R)$ )  $\leftrightarrow (P \rightarrow Q \leftrightarrow R)$ ) with
4518         ( $(P \rightarrow (Q \leftrightarrow R)) \leftrightarrow ((P \rightarrow Q) \leftrightarrow (P \rightarrow R))$ ) in n4_38a.
4519   apply n4_38a.
4520   apply EqBi.
4521   specialize n4_21 with ( $P \rightarrow Q \leftrightarrow R$ ) ( $(P \rightarrow Q) \leftrightarrow (P \rightarrow R)$ ).
4522   intros n4_21a. (*Not cited*)
4523   apply n4_21a.
4524   replace ( $P \rightarrow Q \leftrightarrow R$ ) with ( $(P \rightarrow Q \rightarrow R) \wedge (P \rightarrow R \rightarrow Q)$ ).
4525   reflexivity.
4526   apply EqBi.
4527   apply n4_76a.
4528   apply Equiv4_01.
4529   apply Equiv4_01.
4530   Qed.
4531
4532   Theorem n5_75 :  $\forall P Q R : \text{Prop}$ ,
4533     ( $(R \rightarrow \sim Q) \wedge (P \leftrightarrow Q \vee R)$ )  $\rightarrow ((P \wedge \sim Q) \leftrightarrow R)$ .
4534   Proof. intros P Q R.
4535   specialize n5_6 with P Q R.
4536   intros n5_6a.
4537   replace (( $P \wedge \sim Q \rightarrow R$ )  $\leftrightarrow (P \rightarrow Q \vee R)$ ) with
4538         (( $(P \wedge \sim Q \rightarrow R) \rightarrow (P \rightarrow Q \vee R)$ )
4539           $\wedge$ 
4540          ( $(P \rightarrow Q \vee R) \rightarrow (P \wedge \sim Q \rightarrow R)$ )) in n5_6a.
4541   specialize Simp3_27 with
4542         (( $P \wedge \sim Q \rightarrow R$ )  $\rightarrow (P \rightarrow Q \vee R)$ ) ( $(P \rightarrow Q \vee R) \rightarrow (P \wedge \sim Q \rightarrow R)$ ).
4543   intros Simp3_27a.
4544   MP Simp3_27a n5_6a.
4545   specialize Simp3_26 with ( $P \rightarrow (Q \vee R)$ ) ( $(Q \vee R) \rightarrow P$ ).
4546   intros Simp3_26a.
4547   replace (( $P \rightarrow (Q \vee R)$ )  $\wedge ((Q \vee R) \rightarrow P)$ ) with
4548         ( $P \leftrightarrow (Q \vee R)$ ) in Simp3_26a.
4549   Syll Simp3_26a Simp3_27a Sa.
4550   specialize Simp3_27 with ( $R \rightarrow \sim Q$ ) ( $P \leftrightarrow (Q \vee R)$ ).
4551   intros Simp3_27b.
4552   Syll Simp3_27b Sa Sb.
4553   specialize Simp3_27 with ( $P \rightarrow (Q \vee R)$ ) ( $(Q \vee R) \rightarrow P$ ).
4554   intros Simp3_27c.
4555   replace (( $P \rightarrow (Q \vee R)$ )  $\wedge ((Q \vee R) \rightarrow P)$ ) with
4556         ( $P \leftrightarrow (Q \vee R)$ ) in Simp3_27c.
4557   Syll Simp3_27b Simp3_27c Sc.
4558   specialize n4_77 with P Q R.

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4559   intros n4_77a.
4560   replace (Q $\vee$ R $\rightarrow$ P) with ((Q $\rightarrow$ P) $\wedge$ (R $\rightarrow$ P)) in Sc.
4561   specialize Simp3_27 with (Q $\rightarrow$ P) (R $\rightarrow$ P).
4562   intros Simp3_27d.
4563   Syll Sa Simp3_27d Sd.
4564   specialize Simp3_26 with (R $\rightarrow$  $\sim$ Q) (P $\leftrightarrow$ (Q $\vee$ R)).
4565   intros Simp3_26b.
4566   Conj Sd Simp3_26b.
4567   split.
4568   apply Sd.
4569   apply Simp3_26b.
4570   specialize Comp3_43 with
4571       ((R $\rightarrow$  $\sim$ Q) $\wedge$ (P $\leftrightarrow$ (Q $\vee$ R))) (R $\rightarrow$ P) (R $\rightarrow$  $\sim$ Q).
4572   intros Comp3_43a.
4573   MP Comp3_43a H.
4574   specialize Comp3_43 with R P ( $\sim$ Q).
4575   intros Comp3_43b.
4576   Syll Comp3_43a Comp3_43b Se.
4577   clear n5_6a. clear Simp3_27a. clear Simp3_27b.
4578       clear Simp3_27c. clear Simp3_27d. clear Simp3_26a.
4579       clear Simp3_26b. clear Comp3_43a. clear Comp3_43b.
4580       clear Sa. clear Sc. clear Sd. clear H. clear n4_77a.
4581   Conj Sb Se.
4582   split.
4583   apply Sb.
4584   apply Se.
4585   specialize Comp3_43 with
4586       ((R $\rightarrow$  $\sim$ Q) $\wedge$ (P $\leftrightarrow$ Q $\vee$ R)) (P $\wedge$  $\sim$ Q $\rightarrow$ R) (R $\rightarrow$ P $\wedge$  $\sim$ Q).
4587   intros Comp3_43c.
4588   MP Comp3_43c H.
4589   replace ((P $\wedge$  $\sim$ Q $\rightarrow$ R) $\wedge$ (R $\rightarrow$ P $\wedge$  $\sim$ Q)) with
4590       (P $\wedge$  $\sim$ Q $\leftrightarrow$ R) in Comp3_43c.
4591   apply Comp3_43c.
4592   apply Equiv4_01.
4593   apply EqBi.
4594   apply n4_77a.
4595   apply Equiv4_01.
4596   apply Equiv4_01.
4597   apply Equiv4_01.
4598   Qed.
4599
4600   End No5.

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