# *Principia Mathematica*'s Propositional Logic in *Coq*

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**Abstract**

This file contains the *Coq* code for the *Principia* Rewrite project's encoding of the propositional logic given in $*1 - *5$. The Github repository with this *Coq* file is here: https://github.com/LogicalAtomist/principia. To receive updates about the project, visit the *Principia Rewrite* project page: https://www.principiarewrite.com/. You can also follow the *Principia* Rewrite project on Twitter: https://twitter.com/thePMrewrite.

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Require Import Unicode.Utf8.

Module No1.
Import Unicode.Utf8.
  (*We first give the axioms of Principia
for the propositional calculus in *1.*)

Axiom Impl1_01 : ∀ P Q : Prop,
  (P → Q) = (~P ∨ Q).
  (*This is a definition in Principia: there → is a
      defined sign and ∨, ~ are primitive ones. So
      we will use this axiom to switch between
      disjunction and implication.*)

Axiom MP1_1 : ∀  P Q : Prop,
  (P → Q) → P → Q. (*Modus ponens*)

  (*1.11 ommitted: it is MP for propositions
      containing variables. Likewise, ommitted
      the well-formedness rules 1.7, 1.71, 1.72*)

Axiom Taut1_2 : ∀ P : Prop,
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  P ∨ P → P. (*Tautology*)

Axiom Add1_3 : ∀ P Q : Prop,
  Q → P ∨ Q. (*Addition*)

Axiom Perm1_4 : ∀ P Q : Prop,
  P ∨ Q → Q ∨ P. (*Permutation*)

Axiom Assoc1_5 : ∀ P Q R : Prop,
  P ∨ (Q ∨ R) → Q ∨ (P ∨ R). (*Association*)

Axiom Sum1_6: ∀ P Q R : Prop,
  (Q → R) → (P ∨ Q → P ∨ R). (*Summation*)

(*These are all the propositional axioms of Principia.*)

End No1.

Module No2.
Import No1.

(*We proceed to the deductions of of Principia.*)

Theorem Abs2_01 : ∀ P : Prop,
  (P → ~P) → ~P.
Proof. intros P.
  specialize Taut1_2 with (~P).
  replace (~P ∨ ~P) with (P → ~P).
  apply MP1_1.
  apply Impl1_01.
Qed.

Theorem Simp2_02 : ∀ P Q : Prop,
  Q → (P → Q).
Proof. intros P Q.
  specialize Add1_3 with (~P) Q.
  replace (~P ∨ Q) with (P → Q).
  apply (MP1_1 Q (P → Q)).
  apply Impl1_01.
Qed.

Theorem n2_03 : ∀ P Q : Prop,
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  (P → ~Q) → (Q → ~P).
Proof. intros P Q.
  specialize Perm1_4 with (~P) (~Q).
  replace (~P ∨ ~Q) with (P → ~Q).
  replace (~Q ∨ ~P) with (Q → ~P).
  apply (MP1_1 (P → ~Q) (Q → ~P)).
  apply Impl1_01.
  apply Impl1_01.
Qed.

Theorem Comm2_04 : ∀ P Q R : Prop,
  (P → (Q → R)) → (Q → (P → R)).
Proof. intros P Q R.
  specialize Assoc1_5 with (~P) (~Q) R.
  replace (~Q ∨ R) with (Q → R).
  replace (~P ∨ (Q → R)) with (P → (Q → R)).
  replace (~P ∨ R) with (P → R).
  replace (~Q ∨ (P → R)) with (Q → (P → R)).
  apply (MP1_1 (P → Q → R) (Q → P → R)).
  apply Impl1_01.
  apply Impl1_01.
  apply Impl1_01.
  apply Impl1_01.
Qed.

Theorem Syll2_05 : ∀ P Q R : Prop,
  (Q → R) → ((P  → Q) → (P → R)).
Proof. intros P Q R.
  specialize Sum1_6 with (~P) Q R.
  replace (~P ∨ Q) with (P → Q).
  replace (~P ∨ R) with (P → R).
  apply (MP1_1 (Q → R) ((P → Q) → (P → R))).
  apply Impl1_01.
  apply Impl1_01.
Qed.

Theorem Syll2_06 : ∀ P Q R : Prop,
  (P → Q) → ((Q → R) → (P → R)).
Proof. intros P Q R.
  specialize Comm2_04 with (Q → R) (P → Q) (P → R).
  intros Comm2_04.
  specialize Syll2_05 with P Q R.
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  intros Syll2_05.
  specialize MP1_1 with ((Q → R) → (P → Q) → P → R)
      ((P → Q) → ((Q → R) → (P → R))).
  intros MP1_1.
  apply MP1_1.
  apply Comm2_04.
  apply Syll2_05.
Qed.

Theorem n2_07 : ∀ P : Prop,
  P → (P ∨ P).
Proof. intros P.
  specialize Add1_3 with P P.
  apply MP1_1.
Qed.

Theorem n2_08 : ∀ P : Prop,
  P → P.
Proof. intros P.
  specialize Syll2_05 with P (P ∨ P) P.
  intros Syll2_05.
  specialize Taut1_2 with P.
  intros Taut1_2.
  specialize MP1_1 with ((P ∨ P) → P) (P → P).
  intros MP1_1.
  apply Syll2_05.
  apply Taut1_2.
  apply n2_07.
Qed.

Theorem n2_1 : ∀ P : Prop,
  (~P) ∨ P.
Proof. intros P.
  specialize n2_08 with P.
  replace (~P ∨ P) with (P → P).
  apply MP1_1.
  apply Impl1_01.
Qed.

Theorem n2_11 : ∀ P : Prop,
  P ∨ ~P.
Proof. intros P.
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  specialize Perm1_4 with (~P) P.
  intros Perm1_4.
  specialize n2_1 with P.
  intros n2_1.
  apply Perm1_4.
  apply n2_1.
Qed.

Theorem n2_12 : ∀ P : Prop,
  P → ~~P.
Proof. intros P.
  specialize n2_11 with (~P).
  intros n2_11.
  rewrite Impl1_01.
  apply n2_11.
Qed.

Theorem n2_13 : ∀ P : Prop,
  P ∨ ~~~P.
Proof. intros P.
  specialize Sum1_6 with P (~P) (~~~P).
  intros Sum1_6.
  specialize n2_12 with (~P).
  intros n2_12.
  apply Sum1_6.
  apply n2_12.
  specialize n2_11 with P.
  intros n2_11.
  apply n2_11.
Qed.

Theorem n2_14 : ∀ P : Prop,
  ~~P → P.
Proof. intros P.
  specialize Perm1_4 with P (~~~P).
  intros Perm1_4.
  specialize n2_13 with P.
  intros n2_13.
  rewrite Impl1_01.
  apply Perm1_4.
  apply n2_13.
Qed.
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Theorem Trans2_15 : ∀ P Q : Prop,
  (~P → Q) → (~Q → P).
Proof. intros P Q.
  specialize Syll2_05 with (~P) Q (~~Q).
  intros Syll2_05a.
  specialize n2_12 with Q.
  intros n2_12.
  specialize n2_03 with (~P) (~Q).
  intros n2_03.
  specialize Syll2_05 with (~Q) (~~P) P.
  intros Syll2_05b.
  specialize Syll2_05 with (~P → Q) (~P → ~~Q) (~Q → ~~P).
  intros Syll2_05c.
  specialize Syll2_05 with (~P → Q) (~Q → ~~P) (~Q → P).
  intros Syll2_05d.
  apply Syll2_05d.
  apply Syll2_05b.
  specialize n2_14 with P.
  intros n2_14.
  apply n2_14.
  apply Syll2_05c.
  apply n2_03.
  apply Syll2_05a.
  apply n2_12.
Qed.

Ltac Syll H1 H2 S :=
  let S := fresh S in match goal with
    | [ H1 : ?P → ?Q, H2 : ?Q → ?R |- _ ] =>
        assert (S : P → R) by (intros p; apply (H2 (H1 p)))
end.

Ltac MP H1 H2 :=
  match goal with
    | [ H1 : ?P → ?Q, H2 : ?P |- _ ] => specialize (H1 H2)
end.

Theorem Trans2_16 : ∀ P Q : Prop,
  (P → Q) → (~Q → ~P).
Proof. intros P Q.
  specialize n2_12 with Q.
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  intros n2_12a.
  specialize Syll2_05 with P Q (~~Q).
  intros Syll2_05a.
  specialize n2_03 with P (~Q).
  intros n2_03a.
  MP n2_12a Syll2_05a.
  Syll Syll2_05a n2_03a S.
  apply S.
Qed.

Theorem Trans2_17 : ∀ P Q : Prop,
  (~Q → ~P) → (P → Q).
Proof. intros P Q.
  specialize n2_03 with (~Q) P.
  intros n2_03a.
  specialize n2_14 with Q.
  intros n2_14a.
  specialize Syll2_05 with P (~~Q) Q.
  intros Syll2_05a.
  MP n2_14a Syll2_05a.
  Syll n2_03a Syll2_05a S.
  apply S.
Qed.

Theorem n2_18 : ∀ P : Prop,
  (~P → P) → P.
Proof. intros P.
  specialize n2_12 with P.
  intro n2_12a.
  specialize Syll2_05 with (~P) P (~~P).
  intro Syll2_05a.
  MP Syll2_05a n2_12.
  specialize Abs2_01 with (~P).
  intros Abs2_01a.
  Syll Syll2_05a Abs2_01a Sa.
  specialize n2_14 with P.
  intros n2_14a.
  Syll H n2_14a Sb.
  apply Sb.
Qed.

Theorem n2_2 : ∀ P Q : Prop,
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  P → (P ∨ Q).
Proof. intros P Q.
  specialize Add1_3 with Q P.
  intros Add1_3a.
  specialize Perm1_4 with Q P.
  intros Perm1_4a.
  Syll Add1_3a Perm1_4a S.
  apply S.
Qed.

Theorem n2_21 : ∀ P Q : Prop,
  ~P → (P → Q).
Proof. intros P Q.
  specialize n2_2 with (~P) Q.
  intros n2_2a.
  specialize Impl1_01 with P Q.
  intros Impl1_01a.
  replace (~P∨Q) with (P→Q) in n2_2a.
  apply n2_2a.
Qed.

Theorem n2_24 : ∀ P Q : Prop,
  P → (~P → Q).
Proof. intros P Q.
  specialize n2_21 with P Q.
  intros n2_21a.
  specialize Comm2_04 with (~P) P Q.
  intros Comm2_04a.
  apply Comm2_04a.
  apply n2_21a.
Qed.

Theorem n2_25 : ∀ P Q : Prop,
  P ∨ ((P ∨ Q) → Q).
Proof. intros P Q.
  specialize n2_1 with (P ∨ Q).
  intros n2_1a.
  specialize Assoc1_5 with (~(P∨Q)) P Q.
  intros Assoc1_5a.
  MP Assoc1_5a n2_1a.
  replace (~(P∨Q)∨Q) with (P∨Q→Q) in Assoc1_5a.
  apply Assoc1_5a.
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  apply Impl1_01.
Qed.

Theorem n2_26 : ∀ P Q : Prop,
  ~P ∨ ((P → Q) → Q).
Proof. intros P Q.
  specialize n2_25 with (~P) Q.
  intros n2_25a.
  replace (~P∨Q) with (P→Q) in n2_25a.
  apply n2_25a.
  apply Impl1_01.
Qed.

Theorem n2_27 : ∀ P Q : Prop,
  P → ((P → Q) → Q).
Proof. intros P Q.
  specialize n2_26 with P Q.
  intros n2_26a.
  replace (~P∨((P→Q)→Q)) with (P→(P→Q)→Q) in n2_26a.
  apply n2_26a.
  apply Impl1_01.
Qed.

Theorem n2_3 : ∀ P Q R : Prop,
  (P ∨ (Q ∨ R)) → (P ∨ (R ∨ Q)).
Proof. intros P Q R.
  specialize Perm1_4 with Q R.
  intros Perm1_4a.
  specialize Sum1_6 with P (Q∨R) (R∨Q).
  intros Sum1_6a.
  MP Sum1_6a Perm1_4a.
  apply Sum1_6a.
Qed.

Theorem n2_31 : ∀ P Q R : Prop,
  (P ∨ (Q ∨ R)) → ((P ∨ Q) ∨ R).
Proof. intros P Q R.
  specialize n2_3 with P Q R.
  intros n2_3a.
  specialize Assoc1_5 with P R Q.
  intros Assoc1_5a.
  specialize Perm1_4 with R (P∨Q).
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  intros Perm1_4a.
  Syll Assoc1_5a Perm1_4a Sa.
  Syll n2_3a Sa Sb.
  apply Sb.
Qed.

Theorem n2_32 : ∀ P Q R : Prop,
  ((P ∨ Q) ∨ R) → (P ∨ (Q ∨ R)).
Proof. intros P Q R.
  specialize Perm1_4 with (P∨Q) R.
  intros Perm1_4a.
  specialize Assoc1_5 with R P Q.
  intros Assoc1_5a.
  specialize n2_3 with P R Q.
  intros n2_3a.
  specialize Syll2_06 with ((P∨Q)∨R) (R∨P∨Q) (P∨R∨Q).
  intros Syll2_06a.
  MP Syll2_06a Perm1_4a.
  MP Syll2_06a Assoc1_5a.
  specialize Syll2_06 with ((P∨Q)∨R) (P∨R∨Q) (P∨Q∨R).
  intros Syll2_06b.
  MP Syll2_06b Syll2_06a.
  MP Syll2_06b n2_3a.
  apply Syll2_06b.
Qed.

Axiom Abb2_33 : ∀ P Q R : Prop,
  (P ∨ Q ∨ R) = ((P ∨ Q) ∨ R).
  (*This definition makes the default left association.
      The default in Coq is right association.*)

Theorem n2_36 : ∀ P Q R : Prop,
  (Q → R) → ((P ∨ Q) → (R ∨ P)).
Proof. intros P Q R.
  specialize Perm1_4 with P R.
  intros Perm1_4a.
  specialize Syll2_05 with (P∨Q) (P∨R) (R∨P).
  intros Syll2_05a.
  MP Syll2_05a Perm1_4a.
  specialize Sum1_6 with P Q R.
  intros Sum1_6a.
  Syll Sum1_6a Syll2_05a S.
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  apply S.
Qed.

Theorem n2_37 : ∀ P Q R : Prop,
  (Q → R) → ((Q ∨ P) → (P ∨ R)).
Proof. intros P Q R.
  specialize Perm1_4 with Q P.
  intros Perm1_4a.
  specialize Syll2_06 with (Q∨P) (P∨Q) (P∨R).
  intros Syll2_06a.
  MP Syll2_06a Perm1_4a.
  specialize Sum1_6 with P Q R.
  intros Sum1_6a.
  Syll Sum1_6a Syll2_06a S.
  apply S.
Qed.

Theorem n2_38 : ∀ P Q R : Prop,
  (Q → R) → ((Q ∨ P) → (R ∨ P)).
Proof. intros P Q R.
  specialize Perm1_4 with P R.
  intros Perm1_4a.
  specialize Syll2_05 with (Q∨P) (P∨R) (R∨P).
  intros Syll2_05a.
  MP Syll2_05a Perm1_4a.
  specialize Perm1_4 with Q P.
  intros Perm1_4b.
  specialize Syll2_06 with (Q∨P) (P∨Q) (P∨R).
  intros Syll2_06a.
  MP Syll2_06a Perm1_4b.
  Syll Syll2_06a Syll2_05a H.
  specialize Sum1_6 with P Q R.
  intros Sum1_6a.
  Syll Sum1_6a H S.
  apply S.
Qed.

Theorem n2_4 : ∀ P Q : Prop,
  (P ∨ (P ∨ Q)) → (P ∨ Q).
Proof. intros P Q.
  specialize n2_31 with P P Q.
  intros n2_31a.
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  specialize Taut1_2 with P.
  intros Taut1_2a.
  specialize n2_38 with Q (P∨P) P.
  intros n2_38a.
  MP n2_38a Taut1_2a.
  Syll n2_31a n2_38a S.
  apply S.
Qed.

Theorem n2_41 : ∀ P Q : Prop,
  (Q ∨ (P ∨ Q)) → (P ∨ Q).
Proof. intros P Q.
  specialize Assoc1_5 with Q P Q.
  intros Assoc1_5a.
  specialize Taut1_2 with Q.
  intros Taut1_2a.
  specialize Sum1_6 with P (Q∨Q) Q.
  intros Sum1_6a.
  MP Sum1_6a Taut1_2a.
  Syll Assoc1_5a Sum1_6a S.
  apply S.
Qed.

Theorem n2_42 : ∀ P Q : Prop,
  (~P ∨ (P → Q)) → (P → Q).
Proof. intros P Q.
  specialize n2_4 with (~P) Q.
  intros n2_4a.
  replace (~P∨Q) with (P→Q) in n2_4a.
  apply n2_4a. apply Impl1_01.
Qed.

Theorem n2_43 : ∀ P Q : Prop,
  (P → (P → Q)) → (P → Q).
Proof. intros P Q.
  specialize n2_42 with P Q.
  intros n2_42a.
  replace (~P ∨ (P→Q)) with (P→(P→Q)) in n2_42a.
  apply n2_42a.
  apply Impl1_01.
Qed.
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Theorem n2_45 : ∀ P Q : Prop,
  ~(P ∨ Q) → ~P.
Proof. intros P Q.
  specialize n2_2 with P Q.
  intros n2_2a.
  specialize Trans2_16 with P (P∨Q).
  intros Trans2_16a.
  MP n2_2 Trans2_16a.
  apply Trans2_16a.
Qed.

Theorem n2_46 : ∀ P Q : Prop,
  ~(P ∨ Q) → ~Q.
Proof. intros P Q.
  specialize Add1_3 with P Q.
  intros Add1_3a.
  specialize Trans2_16 with Q (P∨Q).
  intros Trans2_16a.
  MP Add1_3a Trans2_16a.
  apply Trans2_16a.
Qed.

Theorem n2_47 : ∀ P Q : Prop,
  ~(P ∨ Q) → (~P ∨ Q).
Proof. intros P Q.
  specialize n2_45 with P Q.
  intros n2_45a.
  specialize n2_2 with (~P) Q.
  intros n2_2a.
  Syll n2_45a n2_2a S.
  apply S.
Qed.

Theorem n2_48 : ∀ P Q : Prop,
  ~(P ∨ Q) → (P ∨ ~Q).
Proof. intros P Q.
  specialize n2_46 with P Q.
  intros n2_46a.
  specialize Add1_3 with P (~Q).
  intros Add1_3a.
  Syll n2_46a Add1_3a S.
  apply S.
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Qed.

Theorem n2_49 : ∀ P Q : Prop,
  ~(P ∨ Q) → (~P ∨ ~Q).
Proof. intros P Q.
  specialize n2_45 with P Q.
  intros n2_45a.
  specialize n2_2 with (~P) (~Q).
  intros n2_2a.
  Syll n2_45a n2_2a S.
  apply S.
Qed.

Theorem n2_5 : ∀ P Q : Prop,
  ~(P → Q) → (~P → Q).
Proof. intros P Q.
  specialize n2_47 with (~P) Q.
  intros n2_47a.
  replace (~P∨Q) with (P→Q) in n2_47a.
  replace (~~P∨Q) with (~P→Q) in n2_47a.
  apply n2_47a.
  apply Impl1_01.
  apply Impl1_01.
Qed.

Theorem n2_51 : ∀ P Q : Prop,
  ~(P → Q) → (P → ~Q).
Proof. intros P Q.
  specialize n2_48 with (~P) Q.
  intros n2_48a.
  replace (~P∨Q) with (P→Q) in n2_48a.
  replace (~P∨~Q) with (P→~Q) in n2_48a.
  apply n2_48a.
  apply Impl1_01.
  apply Impl1_01.
Qed.

Theorem n2_52 : ∀ P Q : Prop,
  ~(P → Q) → (~P → ~Q).
Proof. intros P Q.
  specialize n2_49 with (~P) Q.
  intros n2_49a.
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  replace (~P∨Q) with (P→Q) in n2_49a.
  replace (~~P∨~Q) with (~P→~Q) in n2_49a.
  apply n2_49a.
  apply Impl1_01.
  apply Impl1_01.
Qed.

Theorem n2_521 : ∀ P Q : Prop,
  ~(P→Q)→(Q→P).
Proof. intros P Q.
  specialize n2_52 with P Q.
  intros n2_52a.
  specialize Trans2_17 with Q P.
  intros Trans2_17a.
  Syll n2_52a Trans2_17a S.
  apply S.
Qed.

Theorem n2_53 : ∀ P Q : Prop,
  (P ∨ Q) → (~P → Q).
Proof. intros P Q.
  specialize n2_12 with P.
  intros n2_12a.
  specialize n2_38 with Q P (~~P).
  intros n2_38a.
  MP n2_38a n2_12a.
  replace (~~P∨Q) with (~P→Q) in n2_38a.
  apply n2_38a.
  apply Impl1_01.
Qed.

Theorem n2_54 : ∀ P Q : Prop,
  (~P → Q) → (P ∨ Q).
Proof. intros P Q.
  specialize n2_14 with P.
  intros n2_14a.
  specialize n2_38 with Q (~~P) P.
  intros n2_38a.
  MP n2_38a n2_12a.
  replace (~~P∨Q) with (~P→Q) in n2_38a.
  apply n2_38a.
  apply Impl1_01.
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Qed.

Theorem n2_55 : ∀ P Q : Prop,
  ~P → ((P ∨ Q) → Q).
Proof. intros P Q.
  specialize n2_53 with P Q.
  intros n2_53a.
  specialize Comm2_04 with (P∨Q) (~P) Q.
  intros Comm2_04a.
  MP n2_53a Comm2_04a.
  apply Comm2_04a.
Qed.

Theorem n2_56 : ∀ P Q : Prop,
  ~Q → ((P ∨ Q) → P).
Proof. intros P Q.
  specialize n2_55 with Q P.
  intros n2_55a.
  specialize Perm1_4 with P Q.
  intros Perm1_4a.
  specialize Syll2_06 with (P∨Q) (Q∨P) P.
  intros Syll2_06a.
  MP Syll2_06a Perm1_4a.
  Syll n2_55a Syll2_06a Sa.
  apply Sa.
  Qed.

Theorem n2_6 : ∀ P Q : Prop,
  (~P→Q) → ((P → Q) → Q).
Proof. intros P Q.
  specialize n2_38 with Q (~P) Q.
  intros n2_38a.
  specialize Taut1_2 with Q.
  intros Taut1_2a.
  specialize Syll2_05 with (~P∨Q) (Q∨Q) Q.
  intros Syll2_05a.
  MP Syll2_05a Taut1_2a.
  Syll n2_38a Syll2_05a S.
  replace (~P∨Q) with (P→Q) in S.
  apply S.
  apply Impl1_01.
Qed.
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Theorem n2_61 : ∀ P Q : Prop,
  (P → Q) → ((~P → Q) → Q).
Proof. intros P Q.
  specialize n2_6 with P Q.
  intros n2_6a.
  specialize Comm2_04 with (~P→Q) (P→Q) Q.
  intros Comm2_04a.
  MP Comm2_04a n2_6a.
  apply Comm2_04a.
Qed.

Theorem n2_62 : ∀ P Q : Prop,
  (P ∨ Q) → ((P → Q) → Q).
Proof. intros P Q.
  specialize n2_53 with P Q.
  intros n2_53a.
  specialize n2_6 with P Q.
  intros n2_6a.
  Syll n2_53a n2_6a S.
  apply S.
Qed.

Theorem n2_621 : ∀ P Q : Prop,
  (P → Q) → ((P ∨ Q) → Q).
Proof. intros P Q.
  specialize n2_62 with P Q.
  intros n2_62a.
  specialize Comm2_04 with (P ∨ Q) (P→Q) Q.
  intros Comm2_04a.
  MP Comm2_04a n2_62a.
  apply Comm2_04a.
Qed.

Theorem n2_63 : ∀ P Q : Prop,
  (P ∨ Q) → ((~P ∨ Q) → Q).
Proof. intros P Q.
  specialize n2_62 with P Q.
  intros n2_62a.
  replace (~P∨Q) with (P→Q).
  apply n2_62a.
  apply Impl1_01.
```

```
Qed.

Theorem n2_64 : ∀ P Q : Prop,
  (P ∨ Q) → ((P ∨ ~Q) → P).
Proof. intros P Q.
  specialize n2_63 with Q P.
  intros n2_63a.
  specialize Perm1_4 with P Q.
  intros Perm1_4a.
  Syll n2_63a Perm1_4a Ha.
  specialize Syll2_06 with (P∨~Q) (~Q∨P) P.
  intros Syll2_06a.
  specialize Perm1_4 with P (~Q).
  intros Perm1_4b.
  MP Syll2_06a Perm1_4b.
  Syll Syll2_06a Ha S.
  apply S.
Qed.

Theorem n2_65 : ∀ P Q : Prop,
  (P → Q) → ((P → ~Q) → ~P).
Proof. intros P Q.
  specialize n2_64 with (~P) Q.
  intros n2_64a.
  replace (~P∨Q) with (P→Q) in n2_64a.
  replace (~P∨~Q) with (P→~Q) in n2_64a.
  apply n2_64a.
  apply Impl1_01.
  apply Impl1_01.
Qed.

Theorem n2_67 : ∀ P Q : Prop,
  ((P ∨ Q) → Q) → (P → Q).
Proof. intros P Q.
  specialize n2_54 with P Q.
  intros n2_54a.
  specialize Syll2_06 with (~P→Q) (P∨Q) Q.
  intros Syll2_06a.
  MP Syll2_06a n2_54a.
  specialize n2_24 with  P Q.
  intros n2_24.
  specialize Syll2_06 with P (~P→Q) Q.
```

```
  intros Syll2_06b.
  MP Syll2_06b n2_24a.
  Syll Syll2_06b Syll2_06a S.
  apply S.
Qed.

Theorem n2_68 : ∀ P Q : Prop,
  ((P → Q) → Q) → (P ∨ Q).
Proof. intros P Q.
  specialize n2_67 with (~P) Q.
  intros n2_67a.
  replace (~P∨Q) with (P→Q) in n2_67a.
  specialize n2_54 with P Q.
  intros n2_54a.
  Syll n2_67a n2_54a S.
  apply S.
  apply Impl1_01.
Qed.

Theorem n2_69 : ∀ P Q : Prop,
  ((P → Q) → Q) → ((Q → P) → P).
Proof. intros P Q.
  specialize n2_68 with P Q.
  intros n2_68a.
  specialize Perm1_4 with P Q.
  intros Perm1_4a.
  Syll n2_68a Perm1_4a Sa.
  specialize n2_62 with Q P.
  intros n2_62a.
  Syll Sa n2_62a Sb.
  apply Sb.
Qed.

Theorem n2_73 : ∀ P Q R : Prop,
  (P → Q) → (((P ∨ Q) ∨ R) → (Q ∨ R)).
Proof. intros P Q R.
  specialize n2_621 with P Q.
  intros n2_621a.
  specialize n2_38 with R (P∨Q) Q.
  intros n2_38a.
  Syll n2_621a n2_38a S.
  apply S.
```

```
Qed.

Theorem n2_74 : ∀ P Q R : Prop,
  (Q → P) → ((P ∨ Q) ∨ R) → (P ∨ R).
Proof. intros P Q R.
  specialize n2_73 with Q P R.
  intros n2_73a.
  specialize Assoc1_5 with P Q R.
  intros Assoc1_5a.
  specialize n2_31 with Q P R.
  intros n2_31a. (*not cited*)
  Syll Assoc1_5a n2_31a Sa.
  specialize n2_32 with P Q R.
  intros n2_32a. (*not cited*)
  Syll n2_32a Sa Sb.
  specialize Syll2_06 with ((P∨Q)∨R) ((Q∨P)∨R) (P∨R).
  intros Syll2_06a.
  MP Syll2_06a Sb.
  Syll n2_73a Syll2_05a H.
  apply H.
Qed.

Theorem n2_75 : ∀ P Q R : Prop,
  (P ∨ Q) → ((P ∨ (Q → R)) → (P ∨ R)).
Proof. intros P Q R.
  specialize n2_74 with P (~Q) R.
  intros n2_74a.
  specialize n2_53 with Q P.
  intros n2_53a.
  Syll n2_53a n2_74a Sa.
  specialize n2_31 with P (~Q) R.
  intros n2_31a.
  specialize Syll2_06 with (P∨(~Q)∨R)((P∨(~Q))∨R) (P∨R).
  intros Syll2_06a.
  MP Syll2_06a n2_31a.
  Syll Sa Syll2_06a Sb.
  specialize Perm1_4 with P Q.
  intros Perm1_4a. (*not cited*)
  Syll Perm1_4a Sb Sc.
  replace (~Q∨R) with (Q→R) in Sc.
  apply Sc.
  apply Impl1_01.
```

```
Qed.

Theorem n2_76 : ∀ P Q R : Prop,
  (P ∨ (Q → R)) → ((P ∨ Q) → (P ∨ R)).
Proof. intros P Q R.
  specialize n2_75 with P Q R.
  intros n2_75a.
  specialize Comm2_04 with (P∨Q) (P∨(Q→R)) (P∨R).
  intros Comm2_04a.
  apply Comm2_04a.
  apply n2_75a.
Qed.

Theorem n2_77 : ∀ P Q R : Prop,
  (P → (Q → R)) → ((P → Q) → (P → R)).
Proof. intros P Q R.
  specialize n2_76 with (~P) Q R.
  intros n2_76a.
  replace (~P∨(Q→R)) with (P→Q→R) in n2_76a.
  replace (~P∨Q) with (P→Q) in n2_76a.
  replace (~P∨R) with (P→R) in n2_76a.
  apply n2_76a.
  apply Impl1_01.
  apply Impl1_01.
  apply Impl1_01.
Qed.

Theorem n2_8 : ∀ Q R S : Prop,
  (Q ∨ R) → ((~R ∨ S) → (Q ∨ S)).
Proof. intros Q R S.
  specialize n2_53 with R Q.
  intros n2_53a.
  specialize Perm1_4 with Q R.
  intros Perm1_4a.
  Syll Perm1_4a n2_53a Ha.
  specialize n2_38 with S (~R) Q.
  intros n2_38a.
  Syll H n2_38a Hb.
  apply Hb.
Qed.

Theorem n2_81 : ∀ P Q R S : Prop,
```

```
  (Q → (R → S)) → ((P ∨ Q) → ((P ∨ R) → (P ∨ S))).
Proof. intros P Q R S.
  specialize Sum1_6 with P Q (R→S).
  intros Sum1_6a.
  specialize n2_76 with P R S.
  intros n2_76a.
  specialize Syll2_05 with (P∨Q) (P∨(R→S)) ((P∨R)→(P∨S)).
  intros Syll2_05a.
  MP Syll2_05a n2_76a.
  Syll Sum1_6a Syll2_05a H.
  apply H.
Qed.

Theorem n2_82 : ∀ P Q R S : Prop,
  (P ∨ Q ∨ R)→((P ∨ ~R ∨ S)→(P ∨ Q ∨ S)).
Proof. intros P Q R S.
  specialize n2_8 with Q R S.
  intros n2_8a.
  specialize n2_81 with P (Q∨R) (~R∨S) (Q∨S).
  intros n2_81a.
  MP n2_81a n2_8a.
  apply n2_81a.
Qed.

Theorem n2_83 : ∀ P Q R S : Prop,
  (P→(Q→R))→((P→(R→S))→(P→(Q→S))).
Proof. intros P Q R S.
  specialize n2_82 with (~P) (~Q) R S.
  intros n2_82a.
  replace (~Q∨R) with (Q→R) in n2_82a.
  replace (~P∨(Q→R)) with (P→Q→R) in n2_82a.
  replace (~R∨S) with (R→S) in n2_82a.
  replace (~P∨(R→S)) with (P→R→S) in n2_82a.
  replace (~Q∨S) with (Q→S) in n2_82a.
  replace (~Q∨S) with (Q→S) in n2_82a.
  replace (~P∨(Q→S)) with (P→Q→S) in n2_82a.
  apply n2_82a.
  apply Impl1_01.
  apply Impl1_01.
  apply Impl1_01.
  apply Impl1_01.
  apply Impl1_01.
```

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  apply Impl1_01.
  apply Impl1_01.
Qed.

Theorem n2_85 : ∀ P Q R : Prop,
  ((P ∨ Q) → (P ∨ R)) → (P ∨ (Q → R)).
Proof. intros P Q R.
  specialize Add1_3 with P Q.
  intros Add1_3a.
  specialize Syll2_06 with Q (P∨Q) R.
  intros Syll2_06a.
  MP Syll2_06a Add1_3a.
  specialize n2_55 with P R.
  intros n2_55a.
  specialize Syll2_05 with (P∨Q) (P∨R) R.
  intros Syll2_05a.
  Syll n2_55a Syll2_05a Ha.
  specialize n2_83 with (~P) ((P∨Q)→(P∨R)) ((P∨Q)→R) (Q→R).
  intros n2_83a.
  MP n2_83a Ha.
  specialize Comm2_04 with (~P) (P∨Q→P∨R) (Q→R).
  intros Comm2_04a.
  Syll Ha Comm2_04a Hb.
  specialize n2_54 with P (Q→R).
  intros n2_54a.
  specialize Simp2_02 with (~P) ((P∨Q→R)→(Q→R)).
  intros Simp2_02a. (*Not cited*)
      (*Greg's suggestion per the BRS list on June 25, 2017.*)
  MP Syll2_06a Simp2_02a.
  MP Hb Simp2_02a.
  Syll Hb n2_54a Hc.
  apply Hc.
Qed.

Theorem n2_86 : ∀ P Q R : Prop,
  ((P → Q) → (P → R)) → (P → (Q →  R)).
Proof. intros P Q R.
  specialize n2_85 with (~P) Q R.
  intros n2_85a.
  replace (~P∨Q) with (P→Q) in n2_85a.
  replace (~P∨R) with (P→R) in n2_85a.
  replace (~P∨(Q→R)) with (P→Q→R) in n2_85a.
```

```
  apply n2_85a.
  apply Impl1_01.
  apply Impl1_01.
  apply Impl1_01.
Qed.

End No2.

Module No3.

Import No1.
Import No2.

Axiom Prod3_01 : ∀ P Q : Prop,
  (P ∧ Q) = ~(~P ∨ ~Q).

Axiom Abb3_02 : ∀ P Q R : Prop,
  (P → Q → R) = (P → Q) ∧ (Q → R).

Theorem Conj3_03 : ∀ P Q : Prop, P → Q → (P∧Q).
Proof. intros P Q.
  specialize n2_11 with (~P∨~Q). intros n2_11a.
  specialize n2_32 with (~P) (~Q) (~(~P ∨ ~Q)). intros n2_32a.
  MP n2_32a n2_11a.
  replace (~(~P∨~Q)) with (P∧Q) in n2_32a.
  replace (~Q ∨ (P∧Q)) with (Q→(P∧Q)) in n2_32a.
  replace (~P ∨ (Q → (P∧Q))) with (P→Q→(P∧Q)) in n2_32a.
  apply n2_32a.
  apply Impl1_01.
  apply Impl1_01.
  apply Prod3_01.
Qed.
(*3.03 is permits the inference from the theoremhood
    of P and that of Q to the theoremhood of P and Q. So:*)

Ltac Conj H1 H2 :=
  match goal with
    | [ H1 : ?P, H2 : ?Q |- _ ] =>
      assert (P ∧ Q)
end.

Theorem n3_1 : ∀ P Q : Prop,
```

```
  (P ∧ Q) → ~(~P ∨ ~Q).
Proof. intros P Q.
  replace (~(~P∨~Q)) with (P∧Q).
  specialize n2_08 with (P∧Q).
  intros n2_08a.
  apply n2_08a.
  apply Prod3_01.
Qed.

Theorem n3_11 : ∀ P Q : Prop,
  ~(~P ∨ ~Q) → (P ∧ Q).
Proof. intros P Q.
  replace (~(~P∨~Q)) with (P∧Q).
  specialize n2_08 with (P∧Q).
  intros n2_08a.
  apply n2_08a.
  apply Prod3_01.
Qed.

Theorem n3_12 : ∀ P Q : Prop,
  (~P ∨ ~Q) ∨ (P ∧ Q).
Proof. intros P Q.
  specialize n2_11 with (~P∨~Q).
  intros n2_11a.
  replace (~(~P∨~Q)) with (P∧Q) in n2_11a.
  apply n2_11a.
  apply Prod3_01.
Qed.

Theorem n3_13 : ∀ P Q : Prop,
  ~(P ∧ Q) → (~P ∨ ~Q).
Proof. intros P Q.
  specialize n3_11 with P Q.
  intros n3_11a.
  specialize Trans2_15 with (~P∨~Q) (P∧Q).
  intros Trans2_15a.
  MP Trans2_15a n3_11a.
  apply Trans2_15a.
Qed.

Theorem n3_14 : ∀ P Q : Prop,
  (~P ∨ ~Q) → ~(P ∧ Q).
```

```
Proof. intros P Q.
  specialize n3_1 with P Q.
  intros n3_1a.
  specialize Trans2_16 with (P∧Q) (~(~P∨~Q)).
  intros Trans2_16a.
  MP Trans2_16a n3_1a.
  specialize n2_12 with (~P∨~Q).
  intros n2_12a.
  Syll n2_12a Trans2_16a S.
  apply S.
Qed.

Theorem n3_2 : ∀ P Q : Prop,
  P → Q → (P ∧ Q).
Proof. intros P Q.
  specialize n3_12 with P Q.
  intros n3_12a.
  specialize n2_32 with (~P) (~Q) (P∧Q).
  intros n2_32a.
  MP n3_32a n3_12a.
  replace (~Q ∨ P ∧ Q) with (Q→P∧Q) in n2_32a.
  replace (~P ∨ (Q → P ∧ Q)) with (P→Q→P∧Q) in n2_32a.
  apply n2_32a.
  apply Impl1_01.
  apply Impl1_01.
Qed.

Theorem n3_21 : ∀ P Q : Prop,
  Q → P → (P ∧ Q).
Proof. intros P Q.
  specialize n3_2 with P Q.
  intros n3_2a.
  specialize Comm2_04 with P Q (P∧Q).
  intros Comm2_04a.
  MP Comm2_04a n3_2a.
  apply Comm2_04a.
Qed.

Theorem n3_22 : ∀ P Q : Prop,
  (P ∧ Q) → (Q ∧ P).
Proof. intros P Q.
  specialize n3_13 with Q P.
```

```
  intros n3_13a.
  specialize Perm1_4 with (~Q) (~P).
  intros Perm1_4a.
  Syll n3_13a Perm1_4a Ha.
  specialize n3_14 with P Q.
  intros n3_14a.
  Syll Ha n3_14a Hb.
  specialize Trans2_17 with (P∧Q) (Q ∧ P).
  intros Trans2_17a.
  MP Trans2_17a Hb.
  apply Trans2_17a.
Qed.

Theorem n3_24 : ∀ P : Prop,
  ~(P ∧ ~P).
Proof. intros P.
  specialize n2_11 with (~P).
  intros n2_11a.
  specialize n3_14 with P (~P).
  intros n3_14a.
  MP n3_14a n2_11a.
  apply n3_14a.
Qed.

Theorem Simp3_26 : ∀ P Q : Prop,
  (P ∧ Q) → P.
Proof. intros P Q.
  specialize Simp2_02 with Q P.
  intros Simp2_02a.
  replace (P→(Q→P)) with (~P∨(Q→P)) in Simp2_02a.
  replace (Q→P) with (~Q∨P) in Simp2_02a.
  specialize n2_31 with (~P) (~Q) P.
  intros n2_31a.
  MP n2_31a Simp2_02a.
  specialize n2_53 with (~P∨~Q) P.
  intros n2_53a.
  MP n2_53a Simp2_02a.
  replace (~(~P∨~Q)) with (P∧Q) in n2_53a.
  apply n2_53a.
  apply Prod3_01.
  replace (~Q∨P) with (Q→P).
  reflexivity.
```

```
  apply Impl1_01.
  replace (~P∨(Q→P)) with (P→Q→P).
  reflexivity.
  apply Impl1_01.
Qed.

Theorem Simp3_27 : ∀ P Q : Prop,
  (P ∧ Q) → Q.
Proof. intros P Q.
  specialize n3_22 with P Q.
  intros n3_22a.
  specialize Simp3_26 with Q P.
  intros Simp3_26a.
  Syll n3_22a Simp3_26a S.
  apply S.
Qed.

Theorem Exp3_3 : ∀ P Q R : Prop,
  ((P ∧ Q) → R) → (P → (Q → R)).
Proof. intros P Q R.
  specialize Trans2_15 with (~P∨~Q) R.
  intros Trans2_15a.
  specialize Comm2_04 with (~R) P (~Q).
  intros Comm2_04a.
  replace (P → ~Q) with (~P ∨ ~Q) in Comm2_04a.
  Syll Trans2_15a Comm2_04a Sa.
  specialize Trans2_17 with Q R.
  intros Trans2_17a.
  specialize  Syll2_05 with P (~R→~Q) (Q→R).
  intros Syll2_05a.
  MP Syll2_05a Trans2_17a.
  Syll Sa Syll2_05a Sb.
  replace (~(~P ∨ ~Q)) with (P ∧ Q) in Sb.
  apply Sb.
  apply Prod3_01.
  replace (~P∨~Q) with (P→~Q).
  reflexivity.
  apply Impl1_01.
Qed.
(*The proof sketch cites n2_08, but
   we did not seem to need it.*)
```

```
Theorem Imp3_31 : ∀ P Q R : Prop,
  (P → (Q → R)) → (P ∧ Q) → R.
Proof. intros P Q R.
  specialize n2_31 with (~P) (~Q) R.
  intros n2_31a.
  specialize n2_53 with (~P∨~Q) R.
  intros n2_53a.
  Syll n2_31a n2_53a S.
  replace (~Q∨R) with (Q→R) in S.
  replace (~P∨(Q→R)) with (P→Q→R) in S.
  replace (~(~P∨~Q)) with (P∧Q) in S.
  apply S.
  apply Prod3_01.
  apply Impl1_01.
  apply Impl1_01.
Qed.
(*The proof sketch cites n2_08, but
    we did not seem to need it.*)

Theorem Syll3_33 : ∀ P Q R : Prop,
  ((P → Q) ∧ (Q → R)) → (P → R).
Proof. intros P Q R.
  specialize Syll2_06 with P Q R.
  intros Syll2_06a.
  specialize Imp3_31 with (P→Q) (Q→R) (P→R).
  intros Imp3_31a.
  MP Imp3_31a Syll2_06a.
  apply Imp3_31a.
Qed.

Theorem Syll3_34 : ∀ P Q R : Prop,
  ((Q → R) ∧ (P → Q)) → (P → R).
Proof. intros P Q R.
  specialize Syll2_05 with P Q R.
  intros Syll2_05a.
  specialize Imp3_31 with (Q→R) (P→Q) (P→R).
  intros Imp3_31a.
  MP Imp3_31a Syll2_05a.
  apply Imp3_31a.
Qed.

Theorem Ass3_35 : ∀ P Q : Prop,
```

```
  (P ∧ (P → Q)) → Q.
Proof. intros P Q.
  specialize n2_27 with P Q.
  intros n2_27a.
  specialize Imp3_31 with P (P→Q) Q.
  intros Imp3_31a.
  MP Imp3_31a n2_27a.
  apply Imp3_31a.
Qed.

Theorem n3_37 : ∀ P Q R : Prop,
  (P ∧ Q → R) → (P ∧ ~R → ~Q).
Proof. intros P Q R.
  specialize Trans2_16 with Q R.
  intros Trans2_16a.
  specialize Syll2_05 with P (Q→R) (~R→~Q).
  intros Syll2_05a.
  MP Syll2_05a Trans2_16a.
  specialize Exp3_3 with P Q R.
  intros Exp3_3a.
  Syll Exp3_3a Syll2_05a Sa.
  specialize Imp3_31 with P (~R) (~Q).
  intros Imp3_31a.
  Syll Sa Imp3_31a Sb.
  apply Sb.
Qed.

Theorem n3_4 : ∀ P Q : Prop,
  (P ∧ Q) → P → Q.
Proof. intros P Q.
  specialize n2_51 with P Q.
  intros n2_51a.
  specialize Trans2_15 with (P→Q) (P→~Q).
  intros Trans2_15a.
  MP Trans2_15a n2_51a.
  replace (P→~Q) with (~P∨~Q) in Trans2_15a.
  replace (~(~P∨~Q)) with (P∧Q) in Trans2_15a.
  apply Trans2_15a.
  apply Prod3_01.
  replace (~P∨~Q) with (P→~Q).
  reflexivity.
  apply Impl1_01.
```

```
Qed.

Theorem n3_41 : ∀ P Q R : Prop,
  (P → R) → (P ∧ Q → R).
Proof. intros P Q R.
  specialize Simp3_26 with P Q.
  intros Simp3_26a.
  specialize Syll2_06 with (P∧Q) P R.
  intros Syll2_06a.
  MP Simp3_26a Syll2_06a.
  apply Syll2_06a.
Qed.

Theorem n3_42 : ∀ P Q R : Prop,
  (Q → R) → (P ∧ Q → R).
Proof. intros P Q R.
  specialize Simp3_27 with P Q.
  intros Simp3_27a.
  specialize Syll2_06 with (P∧Q) Q R.
  intros Syll2_06a.
  MP Syll2_06a Simp3_27a.
  apply Syll2_06a.
Qed.

Theorem Comp3_43 : ∀ P Q R : Prop,
  (P → Q) ∧ (P → R) → (P → Q ∧ R).
Proof. intros P Q R.
  specialize n3_2 with Q R.
  intros n3_2a.
  specialize Syll2_05 with P Q (R→Q∧R).
  intros Syll2_05a.
  MP Syll2_05a n3_2a.
  specialize n2_77 with P R (Q∧R).
  intros n2_77a.
  Syll Syll2_05a n2_77a Sa.
  specialize Imp3_31 with (P→Q) (P→R) (P→Q∧R).
  intros Imp3_31a.
  MP Sa Imp3_31a.
  apply Imp3_31a.
Qed.

Theorem n3_44 : ∀ P Q R : Prop,
```

```
  (Q → P) ∧ (R → P) → (Q ∨ R → P).
Proof. intros P Q R.
  specialize Syll3_33 with (~Q) R P.
  intros Syll3_33a.
  specialize n2_6 with Q P.
  intros n2_6a.
  Syll Syll3_33a n2_6a Sa.
  specialize Exp3_3 with (~Q→R) (R→P) ((Q→P)→P).
  intros Exp3_3a.
  MP Exp3_3a Sa.
  specialize Comm2_04 with (R→P) (Q→P) P.
  intros Comm2_04a.
  Syll Exp3_3a Comm2_04a Sb.
  specialize Imp3_31 with (Q→P) (R→P) P.
  intros Imp3_31a.
  Syll Sb Imp3_31a Sc.
  specialize Comm2_04 with (~Q→R) ((Q→P)∧(R→P)) P.
  intros Comm2_04b.
  MP Comm2_04b Sc.
  specialize n2_53 with Q R.
  intros n2_53a.
  specialize Syll2_06 with (Q∨R) (~Q→R) P.
  intros Syll2_06a.
  MP Syll2_06a n2_53a.
  Syll Comm2_04b Syll2_06a Sd.
  apply Sd.
Qed.

Theorem Fact3_45 : ∀ P Q R : Prop,
  (P → Q) → (P ∧ R) → (Q ∧ R).
Proof. intros P Q R.
  specialize Syll2_06 with P Q (~R).
  intros Syll2_06a.
  specialize Trans2_16 with (Q→~R) (P→~R).
  intros Trans2_16a.
  Syll Syll2_06a Trans2_16a S.
  replace (P→~R) with (~P∨~R) in S.
  replace (Q→~R) with (~Q∨~R) in S.
  replace (~(~P∨~R)) with (P∧R) in S.
  replace (~(~Q∨~R)) with (Q∧R) in S.
  apply S.
  apply Prod3_01.
```

```
  apply Prod3_01.
  replace (~Q∨~R) with (Q→~R).
  reflexivity.
  apply Impl1_01.
  replace (~P∨~R) with (P→~R).
  reflexivity.
  apply Impl1_01.
Qed.

Theorem n3_47 : ∀ P Q R S : Prop,
  ((P → R) ∧ (Q → S)) → (P ∧ Q) → R ∧ S.
Proof. intros P Q R S.
  specialize Simp3_26 with (P→R) (Q→S).
  intros Simp3_26a.
  specialize Fact3_45 with P R Q.
  intros Fact3_45a.
  Syll Simp3_26a Fact3_45a Sa.
  specialize n3_22 with R Q.
  intros n3_22a.
  specialize Syll2_05 with (P∧Q) (R∧Q) (Q∧R).
  intros Syll2_05a.
  MP Syll2_05a n3_22a.
  Syll Sa Syll2_05a Sb.
  specialize Simp3_27 with (P→R) (Q→S).
  intros Simp3_27a.
  specialize Fact3_45 with Q S R.
  intros Fact3_45b.
  Syll Simp3_27a Fact3_45b Sc.
  specialize n3_22 with S R.
  intros n3_22b.
  specialize Syll2_05 with (Q∧R) (S∧R) (R∧S).
  intros Syll2_05b.
  MP Syll2_05b n3_22b.
  Syll Sc Syll2_05b Sd.
  specialize n2_83 with ((P→R)∧(Q→S)) (P∧Q) (Q∧R) (R∧S).
  intros n2_83a.
  MP n2_83a Sb.
  MP n2_83 Sd.
  apply n2_83a.
Qed.

Theorem n3_48 : ∀ P Q R S : Prop,
```

```
  ((P → R) ∧ (Q → S)) → (P ∨ Q) → R ∨ S.
Proof. intros P Q R S.
  specialize Simp3_26 with (P→R) (Q→S).
  intros Simp3_26a.
  specialize Sum1_6 with Q P R.
  intros Sum1_6a.
  Syll Simp3_26a Sum1_6a Sa.
  specialize Perm1_4 with P Q.
  intros Perm1_4a.
  specialize Syll2_06 with (P∨Q) (Q∨P) (Q∨R).
  intros Syll2_06a.
  MP Syll2_06a Perm1_4a.
  Syll Sa Syll2_06a Sb.
  specialize Simp3_27 with (P→R) (Q→S).
  intros Simp3_27a.
  specialize Sum1_6 with R Q S.
  intros Sum1_6b.
  Syll Simp3_27a Sum1_6b Sc.
  specialize Perm1_4 with Q R.
  intros Perm1_4b.
  specialize Syll2_06 with (Q∨R) (R∨Q) (R∨S).
  intros Syll2_06b.
  MP Syll2_06b Perm1_4b.
  Syll Sc Syll2_06a Sd.
  specialize n2_83 with ((P→R)∧(Q→S)) (P∨Q) (Q∨R) (R∨S).
  intros n2_83a.
  MP n2_83a Sb.
  MP n2_83a Sd.
  apply n2_83a.
Qed.

End No3.

Module No4.

Import No1.
Import No2.
Import No3.

Axiom Equiv4_01 : ∀ P Q : Prop,
  (P ↔ Q) = ((P → Q) ∧ (Q → P)).

```

```
Axiom Abb4_02 : ∀ P Q R : Prop,
  (P ↔ Q ↔ R) = ((P ↔ Q) ∧ (Q ↔ R)).

Axiom EqBi : ∀ P Q : Prop,
  (P = Q) ↔ (P ↔ Q).

Ltac Equiv H1 :=
  match goal with
    | [ H1 : (?P→?Q) ∧ (?Q→?P) |- _ ] =>
      replace ((P→Q) ∧ (Q→P)) with (P↔Q) in H1
end.

Theorem Trans4_1 : ∀ P Q : Prop,
  (P → Q) ↔ (~Q → ~P).
Proof. intros P Q.
  specialize Trans2_16 with P Q.
  intros Trans2_16a.
  specialize Trans2_17 with P Q.
  intros Trans2_17a.
  Conj Trans2_16a Trans2_17a.
  split.
  apply Trans2_16a.
  apply Trans2_17a.
  Equiv H.
  apply H.
  apply Equiv4_01.
Qed.

Theorem Trans4_11 : ∀ P Q : Prop,
  (P ↔ Q) ↔ (~P ↔ ~Q).
Proof. intros P Q.
  specialize Trans2_16 with P Q.
  intros Trans2_16a.
  specialize Trans2_16 with Q P.
  intros Trans2_16b.
  Conj Trans2_16a Trans2_16b.
  split.
  apply Trans2_16a.
  apply Trans2_16b.
  specialize n3_47 with (P→Q) (Q→P) (~Q→~P) (~P→~Q).
  intros n3_47a.
  MP n3_47 H.
```

```
  specialize n3_22 with (~Q → ~P) (~P → ~Q).
  intros n3_22a.
  Syll n3_47a n3_22a Sa.
  replace ((P → Q) ∧ (Q → P)) with (P↔Q) in Sa.
  replace ((~P → ~Q) ∧ (~Q → ~P)) with (~P↔~Q) in Sa.
  clear Trans2_16a. clear H. clear Trans2_16b.
      clear n3_22a. clear n3_47a.
  specialize Trans2_17 with Q P.
  intros Trans2_17a.
  specialize Trans2_17 with P Q.
  intros Trans2_17b.
  Conj Trans2_17a Trans2_17b.
  split.
  apply Trans2_17a.
  apply Trans2_17b.
  specialize n3_47 with (~P→~Q) (~Q→~P) (Q→P) (P→Q).
  intros n3_47a.
  MP n3_47a H.
  specialize n3_22 with (Q→P) (P→Q).
  intros n3_22a.
  Syll n3_47a n3_22a Sb.
  clear Trans2_17a. clear Trans2_17b. clear H.
      clear n3_47a. clear n3_22a.
  replace ((P → Q) ∧ (Q → P)) with (P↔Q) in Sb.
  replace ((~P → ~Q) ∧ (~Q → ~P)) with (~P↔~Q) in Sb.
  Conj Sa Sb.
  split.
  apply Sa.
  apply Sb.
  Equiv H.
  apply H.
  apply Equiv4_01.
  apply Equiv4_01.
  apply Equiv4_01.
  apply Equiv4_01.
  apply Equiv4_01.
Qed.

Theorem n4_12 : ∀ P Q : Prop,
  (P ↔ ~Q) ↔ (Q ↔ ~P).
  Proof. intros P Q.
    specialize n2_03 with P Q.
```

```
    intros n2_03a.
    specialize Trans2_15 with Q P.
    intros Trans2_15a.
    Conj n2_03a Trans2_15a.
    split.
    apply n2_03a.
    apply Trans2_15a.
    specialize n3_47 with (P→~Q) (~Q→P) (Q→~P) (~P→Q).
    intros n3_47a.
    MP n3_47a H.
    specialize n2_03 with Q P.
    intros n2_03b.
    specialize Trans2_15 with P Q.
    intros Trans2_15b.
    Conj n2_03b Trans2_15b.
    split.
    apply n2_03b.
    apply Trans2_15b.
    specialize n3_47 with (Q→~P) (~P→Q) (P→~Q) (~Q→P).
    intros n3_47b.
    MP n3_47b H0.
    clear n2_03a. clear Trans2_15a. clear H. clear n2_03b.
        clear Trans2_15b. clear H0.
    replace ((P → ~Q) ∧ (~Q → P)) with (P↔~Q) in n3_47a.
    replace ((Q → ~P) ∧ (~P → Q)) with (Q↔~P) in n3_47a.
    replace ((P → ~Q) ∧ (~Q → P)) with (P↔~Q) in n3_47b.
    replace ((Q → ~P) ∧ (~P → Q)) with (Q↔~P) in n3_47b.
    Conj n3_47a n3_47b.
    split.
    apply n3_47a.
    apply n3_47b.
    Equiv H.
    apply H.
    apply Equiv4_01.
    apply Equiv4_01.
    apply Equiv4_01.
    apply Equiv4_01.
    apply Equiv4_01.
  Qed.

Theorem n4_13 : ∀ P : Prop,
  P ↔ ~~P.
```

```
  Proof. intros P.
  specialize n2_12 with P.
  intros n2_12a.
  specialize n2_14 with P.
  intros n2_14a.
  Conj n2_12a n2_14a.
  split.
  apply n2_12a.
  apply n2_14a.
  Equiv H.
  apply H.
  apply Equiv4_01.
  Qed.

Theorem n4_14 : ∀ P Q R : Prop,
  ((P ∧ Q) → R) ↔ ((P ∧ ~R) → ~Q).
Proof. intros P Q R.
specialize n3_37 with P Q R.
intros n3_37a.
specialize n3_37 with P (~R) (~Q).
intros n3_37b.
Conj n3_37a n3_37b.
split. apply n3_37a.
apply n3_37b.
specialize n4_13 with Q.
intros n4_13a.
specialize n4_13 with R.
intros n4_13b.
replace (~~Q) with Q in H.
replace (~~R) with R in H.
Equiv H.
apply H.
apply Equiv4_01.
apply EqBi.
apply n4_13b.
apply EqBi.
apply n4_13a.
Qed.

Theorem n4_15 : ∀ P Q R : Prop,
  ((P ∧ Q) → ~R) ↔ ((Q ∧ R) → ~P).
  Proof. intros P Q R.
```

```
  specialize n4_14 with Q P (~R).
  intros n4_14a.
  specialize n3_22 with Q P.
  intros n3_22a.
  specialize Syll2_06 with (Q∧P) (P∧Q) (~R).
  intros Syll2_06a.
  MP Syll2_06a n3_22a.
  specialize n4_13 with R.
  intros n4_13a.
  replace (~~R) with R in n4_14a.
  rewrite Equiv4_01 in n4_14a.
  specialize Simp3_26 with ((Q ∧ P → ~R) → Q ∧ R → ~P)
      ((Q ∧ R → ~P) → Q ∧ P → ~R).
  intros Simp3_26a.
  MP Simp3_26a n4_14a.
  Syll Syll2_06a Simp3_26a Sa.
  specialize Simp3_27 with ((Q ∧ P → ~R) → Q ∧ R → ~P)
      ((Q ∧ R → ~P) → Q ∧ P → ~R).
  intros Simp3_27a.
  MP Simp3_27a n4_14a.
  specialize n3_22 with P Q.
  intros n3_22b.
  specialize Syll2_06 with (P∧Q) (Q∧P) (~R).
  intros Syll2_06b.
  MP Syll2_06b n3_22b.
  Syll Syll2_06b Simp3_27a Sb.
  clear n4_14a. clear n3_22a. clear Syll2_06a.
      clear n4_13a. clear Simp3_26a. clear n3_22b.
      clear Simp3_27a. clear Syll2_06b.
  Conj Sa Sb.
  split.
  apply Sa.
  apply Sb.
  Equiv H.
  apply H.
  apply Equiv4_01.
  apply EqBi.
  apply n4_13a.
  Qed.

Theorem n4_2 : ∀ P : Prop,
  P ↔ P.
```

```
  Proof. intros P.
  specialize n3_2 with (P→P) (P→P).
  intros n3_2a.
  specialize n2_08 with P.
  intros n2_08a.
  MP n3_2a n2_08a.
  MP n3_2a n2_08a.
  Equiv n3_2a.
  apply n3_2a.
  apply Equiv4_01.
  Qed.

Theorem n4_21 : ∀ P Q : Prop,
  (P ↔ Q) ↔ (Q ↔ P).
  Proof. intros P Q.
  specialize n3_22 with (P→Q) (Q→P).
  intros n3_22a.
  replace ((P → Q) ∧ (Q → P)) with (P↔Q) in n3_22a.
  replace ((Q → P) ∧ (P → Q)) with (Q↔P) in n3_22a.
  specialize n3_22 with (Q→P) (P→Q).
  intros n3_22b.
  replace ((P → Q) ∧ (Q → P)) with (P↔Q) in n3_22b.
  replace ((Q → P) ∧ (P → Q)) with (Q↔P) in n3_22b.
  Conj n3_22a n3_22b.
  split.
  apply n3_22a.
  apply n3_22b.
  Equiv H.
  apply H.
  apply Equiv4_01.
  apply Equiv4_01.
  apply Equiv4_01.
  apply Equiv4_01.
  apply Equiv4_01.
Qed.

Theorem n4_22 : ∀ P Q R : Prop,
  ((P ↔ Q) ∧ (Q ↔ R)) → (P ↔ R).
Proof. intros P Q R.
  specialize Simp3_26 with (P↔Q) (Q↔R).
  intros Simp3_26a.
  specialize Simp3_26 with (P→Q) (Q→P).
```

```
intros Simp3_26b.
replace ((P→Q) ∧ (Q→P)) with (P↔Q) in Simp3_26b.
Syll Simp3_26a Simp3_26b Sa.
specialize Simp3_27 with (P↔Q) (Q↔R).
intros Simp3_27a.
specialize Simp3_26 with (Q→R) (R→Q).
intros Simp3_26c.
replace ((Q→R) ∧ (R→Q)) with (Q↔R) in Simp3_26c.
Syll Simp3_27a Simp3_26c Sb.
specialize n2_83 with ((P↔Q)∧(Q↔R)) P Q R.
intros n2_83a.
MP n2_83a Sa.
MP n2_83a Sb.
specialize Simp3_27 with (P↔Q) (Q↔R).
intros Simp3_27b.
specialize Simp3_27 with (Q→R) (R→Q).
intros Simp3_27c.
replace ((Q→R) ∧ (R→Q)) with (Q↔R) in Simp3_27c.
Syll Simp3_27b Simp3_27c Sc.
specialize Simp3_26 with (P↔Q) (Q↔R).
intros Simp3_26d.
specialize Simp3_27 with (P→Q) (Q→P).
intros Simp3_27d.
replace ((P→Q) ∧ (Q→P)) with (P↔Q) in Simp3_27d.
Syll Simp3_26d Simp3_27d Sd.
specialize n2_83 with ((P↔Q)∧(Q↔R)) R Q P.
intros n2_83b.
MP n2_83b Sc. MP n2_83b Sd.
clear Sd. clear Sb. clear Sc. clear Sa. clear Simp3_26a.
    clear Simp3_26b. clear Simp3_26c. clear Simp3_26d.
    clear Simp3_27a. clear Simp3_27b. clear Simp3_27c.
    clear Simp3_27d.
Conj n2_83a n2_83b.
split.
apply n2_83a.
apply n2_83b.
specialize Comp3_43 with ((P↔Q)∧(Q↔R)) (P→R) (R→P).
intros Comp3_43a.
MP Comp3_43a H.
replace ((P→R) ∧ (R→P)) with (P↔R) in Comp3_43a.
apply Comp3_43a.
apply Equiv4_01.
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  apply Equiv4_01.
  apply Equiv4_01.
  apply Equiv4_01.
  apply Equiv4_01.
Qed.

Theorem n4_24 : ∀ P : Prop,
  P ↔ (P ∧ P).
  Proof. intros P.
  specialize n3_2 with P P.
  intros n3_2a.
  specialize n2_43 with P (P ∧ P).
  intros n2_43a.
  MP n3_2a n2_43a.
  specialize Simp3_26 with P P.
  intros Simp3_26a.
  Conj n2_43a Simp3_26a.
  split.
  apply n2_43a.
  apply Simp3_26a.
  Equiv H.
  apply H.
  apply Equiv4_01.
Qed.

Theorem n4_25 : ∀ P : Prop,
  P ↔ (P ∨ P).
Proof. intros P.
  specialize Add1_3 with P P.
  intros Add1_3a.
  specialize Taut1_2 with P.
  intros Taut1_2a.
  Conj Add1_3a Taut1_2a.
  split.
  apply Add1_3a.
  apply Taut1_2a.
  Equiv H. apply H.
  apply Equiv4_01.
Qed.

Theorem n4_3 : ∀ P Q : Prop,
  (P ∧ Q) ↔ (Q ∧ P).
```

```
Proof. intros P Q.
  specialize n3_22 with P Q.
  intros n3_22a.
  specialize n3_22 with Q P.
  intros n3_22b.
  Conj n3_22a n3_22b.
  split.
  apply n3_22a.
  apply n3_22b.
  Equiv H. apply H.
  apply Equiv4_01.
Qed.

Theorem n4_31 : ∀ P Q : Prop,
  (P ∨ Q) ↔ (Q ∨ P).
  Proof. intros P Q.
    specialize Perm1_4 with P Q.
    intros Perm1_4a.
    specialize Perm1_4 with Q P.
    intros Perm1_4b.
    Conj Perm1_4a Perm1_4b.
    split.
    apply Perm1_4a.
    apply Perm1_4b.
    Equiv H. apply H.
    apply Equiv4_01.
Qed.

  Theorem n4_32 : ∀ P Q R : Prop,
    ((P ∧ Q) ∧ R) ↔ (P ∧ (Q ∧ R)).
    Proof. intros P Q R.
    specialize n4_15 with P Q R.
    intros n4_15a.
    specialize Trans4_1 with P (~(Q ∧ R)).
    intros Trans4_1a.
    replace (~~(Q ∧ R)) with (Q ∧ R) in Trans4_1a.
    replace (Q ∧ R→~P) with (P→~(Q ∧ R)) in n4_15a.
    specialize Trans4_11 with (P ∧ Q → ~R) (P → ~(Q ∧ R)).
    intros Trans4_11a.
    replace ((P ∧ Q → ~R) ↔ (P → ~(Q ∧ R))) with
        (~(P ∧ Q → ~R) ↔ ~(P → ~(Q ∧ R))) in n4_15a.
    replace (P ∧ Q → ~R) with
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        (~(P ∧ Q ) ∨ ~R) in n4_15a.
    replace (P → ~(Q ∧ R)) with
        (~P ∨ ~(Q ∧ R)) in n4_15a.
    replace (~(~(P ∧ Q) ∨ ~R)) with
        ((P ∧ Q) ∧ R) in n4_15a.
    replace (~(~P ∨ ~(Q ∧ R))) with
        (P ∧ (Q ∧ R )) in n4_15a.
    apply n4_15a.
    apply Prod3_01.
    apply Prod3_01.
    rewrite Impl1_01.
    reflexivity.
    rewrite Impl1_01.
    reflexivity.
    replace (~(P ∧ Q → ~R) ↔ ~(P → ~(Q ∧ R))) with
        ((P ∧ Q → ~R) ↔ (P → ~(Q ∧ R))).
    reflexivity.
    apply EqBi.
    apply Trans4_11a.
    apply EqBi.
    apply Trans4_1a.
    apply EqBi.
    specialize n4_13 with (Q ∧ R).
    intros n4_13a.
    apply n4_13a.
    Qed.
    (*Note that the actual proof uses n4_12, but
        that transposition involves transforming a
        biconditional into a conditional. This way
        of doing it - using Trans4_1 to transpose a
        conditional and then applying n4_13 to
        double negate - is easier without a derived
        rule for replacing a biconditional with one
        of its equivalent implications.*)

Theorem n4_33 : ∀ P Q R : Prop,
  (P ∨ (Q ∨ R)) ↔ ((P ∨ Q) ∨ R).
  Proof. intros P Q R.
    specialize n2_31 with P Q R.
    intros n2_31a.
    specialize n2_32 with P Q R.
    intros n2_32a.
```

```
    Conj n2_31a n2_32a.
    split.
    apply n2_31a.
    apply n2_32a.
    Equiv H.
    apply H.
    apply Equiv4_01.
  Qed.

Axiom Abb4_34 : ∀ P Q R : Prop,
  P ∧ Q ∧ R = ((P ∧ Q) ∧ R).
  (*This axiom ensures left association of brackets.
  Coq's default is right association. But Principia
  proves associativity of logical product as n4_32.
  So in effect, this axiom gives us a derived rule that
  allows us to shift between Coq's and Principia's
  default rules for brackets of logical products.*)

Theorem n4_36 : ∀ P Q R : Prop,
  (P ↔ Q) → ((P ∧ R) ↔ (Q ∧ R)).
Proof. intros P Q R.
  specialize Fact3_45 with P Q R.
  intros Fact3_45a.
  specialize Fact3_45 with Q P R.
  intros Fact3_45b.
  Conj Fact3_45a Fact3_45b.
  split.
  apply Fact3_45a.
  apply Fact3_45b.
  specialize n3_47 with (P→Q) (Q→P)
      (P ∧ R → Q ∧ R) (Q ∧ R → P ∧ R).
  intros n3_47a.
  MP n3_47 H.
  replace  ((P → Q) ∧ (Q → P)) with (P↔Q) in n3_47a.
  replace ((P ∧ R → Q ∧ R) ∧ (Q ∧ R → P ∧ R)) with
      (P ∧ R ↔ Q ∧ R) in n3_47a.
  apply n3_47a.
  apply Equiv4_01.
  apply Equiv4_01.
  Qed.

Theorem n4_37 : ∀ P Q R : Prop,
```

```
  (P ↔ Q) → ((P ∨ R) ↔ (Q ∨ R)).
Proof. intros P Q R.
  specialize Sum1_6 with R P Q.
  intros Sum1_6a.
  specialize Sum1_6 with R Q P.
  intros Sum1_6b.
  Conj Sum1_6a Sum1_6b.
  split.
  apply Sum1_6a.
  apply Sum1_6b.
  specialize n3_47 with (P → Q) (Q → P)
      (R ∨ P → R ∨ Q) (R ∨ Q → R ∨ P).
  intros n3_47a.
  MP n3_47 H.
  replace  ((P → Q) ∧ (Q → P)) with (P↔Q) in n3_47a.
  replace ((R ∨ P → R ∨ Q) ∧ (R ∨ Q → R ∨ P)) with
      (R ∨ P ↔ R ∨ Q) in n3_47a.
  replace (R ∨ P) with (P ∨ R) in n3_47a.
  replace (R ∨ Q) with (Q ∨ R) in n3_47a.
  apply n3_47a.
  apply EqBi.
  specialize n4_31 with Q R.
  intros n4_31a.
  apply n4_31a.
  apply EqBi.
  specialize n4_31 with P R.
  intros n4_31b.
  apply n4_31b.
  apply Equiv4_01.
  apply Equiv4_01.
  Qed.

Theorem n4_38 : ∀ P Q R S : Prop,
  ((P ↔ R) ∧ (Q ↔ S)) → ((P ∧ Q) ↔ (R ∧ S)).
Proof. intros P Q R S.
  specialize n3_47 with P Q R S.
  intros n3_47a.
  specialize n3_47 with R S P Q.
  intros n3_47b.
  Conj n3_47a n3_47b.
  split.
  apply n3_47a.
```

```
apply n3_47b.
specialize n3_47 with ((P→R) ∧ (Q→S))
    ((R→P) ∧ (S→Q)) (P ∧ Q → R ∧ S) (R ∧ S → P ∧ Q).
intros n3_47c.
MP n3_47c H.
specialize n4_32 with (P→R) (Q→S) ((R→P) ∧ (S → Q)).
intros n4_32a.
replace (((P → R) ∧ (Q → S)) ∧ (R → P) ∧ (S → Q)) with
    ((P → R) ∧ (Q → S) ∧ (R → P) ∧ (S → Q)) in n3_47c.
specialize n4_32 with (Q→S) (R→P) (S → Q).
intros n4_32b.
replace ((Q → S) ∧ (R → P) ∧ (S → Q)) with
    (((Q → S) ∧ (R → P)) ∧ (S → Q)) in n3_47c.
specialize n3_22 with (Q→S) (R→P).
intros n3_22a.
specialize n3_22 with (R→P) (Q→S).
intros n3_22b.
Conj n3_22a n3_22b.
split.
apply n3_22a.
apply n3_22b.
Equiv H0.
replace ((Q → S) ∧ (R → P)) with
    ((R → P) ∧ (Q → S)) in n3_47c.
specialize n4_32 with (R → P) (Q → S) (S → Q).
intros n4_32c.
replace (((R → P) ∧ (Q → S)) ∧ (S → Q)) with
    ((R → P) ∧ (Q → S) ∧ (S → Q)) in n3_47c.
specialize n4_32 with (P→R) (R → P) ((Q → S)∧(S → Q)).
intros n4_32d.
replace ((P → R) ∧ (R → P) ∧ (Q → S) ∧ (S → Q)) with
    (((P → R) ∧ (R → P)) ∧ (Q → S) ∧ (S → Q)) in n3_47c.
replace ((P→R) ∧ (R → P)) with (P↔R) in n3_47c.
replace ((Q → S) ∧ (S → Q)) with (Q↔S) in n3_47c.
replace ((P ∧ Q → R ∧ S) ∧ (R ∧ S → P ∧ Q)) with
    ((P ∧ Q) ↔ (R ∧ S)) in n3_47c.
apply n3_47c.
apply Equiv4_01.
apply Equiv4_01.
apply Equiv4_01.
apply EqBi.
apply n4_32d.
```

```
  replace ((R → P) ∧ (Q → S) ∧ (S → Q)) with
      (((R → P) ∧ (Q → S)) ∧ (S → Q)).
  reflexivity.
  apply EqBi.
  apply n4_32c.
  replace ((R → P) ∧ (Q → S)) with ((Q → S) ∧ (R → P)).
  reflexivity.
  apply EqBi.
  apply H0.
  apply Equiv4_01.
  apply EqBi.
  apply n4_32b.
  replace ((P → R) ∧ (Q → S) ∧ (R → P) ∧ (S → Q)) with
      (((P → R) ∧ (Q → S)) ∧ (R → P) ∧ (S → Q)).
  reflexivity.
  apply EqBi.
  apply n4_32a.
  Qed.

Theorem n4_39 : ∀ P Q R S : Prop,
  ((P ↔ R) ∧ (Q ↔ S)) → ((P ∨ Q) ↔ (R ∨ S)).
Proof.  intros P Q R S.
  specialize n3_48 with P Q R S.
  intros n3_48a.
  specialize n3_48 with R S P Q.
  intros n3_48b.
  Conj n3_48a n3_48b.
  split.
  apply n3_48a.
  apply n3_48b.
  specialize n3_47 with ((P → R) ∧ (Q → S))
      ((R → P) ∧ (S → Q)) (P ∨ Q → R ∨ S) (R ∨ S → P ∨ Q).
  intros n3_47a.
  MP n3_47a H.
  replace ((P ∨ Q → R ∨ S) ∧ (R ∨ S → P ∨ Q)) with
      ((P ∨ Q) ↔ (R ∨ S)) in n3_47a.
  specialize n4_32 with ((P → R) ∧ (Q → S)) (R → P) (S → Q).
  intros n4_32a.
  replace (((P → R) ∧ (Q → S)) ∧ (R → P) ∧ (S → Q)) with
      ((((P → R) ∧ (Q → S)) ∧ (R → P)) ∧ (S → Q)) in n3_47a.
  specialize n4_32 with (P → R) (Q → S) (R → P).
  intros n4_32b.
```

```
replace (((P → R) ∧ (Q → S)) ∧ (R → P)) with
    ((P → R) ∧ (Q → S) ∧ (R → P)) in n3_47a.
specialize n3_22 with (Q → S) (R → P).
intros n3_22a.
specialize n3_22 with (R → P) (Q → S).
intros n3_22b.
Conj  n3_22a n3_22b.
split.
apply n3_22a.
apply n3_22b.
Equiv H0.
replace ((Q → S) ∧ (R → P)) with
    ((R → P) ∧ (Q → S)) in n3_47a.
specialize n4_32 with (P → R) (R → P) (Q → S).
intros n4_32c.
replace ((P → R) ∧ (R → P) ∧ (Q → S)) with
    (((P → R) ∧ (R → P)) ∧ (Q → S)) in n3_47a.
replace ((P → R) ∧ (R → P)) with (P↔R) in n3_47a.
specialize n4_32 with (P↔R) (Q→S) (S→Q).
intros n4_32d.
replace (((P ↔ R) ∧ (Q → S)) ∧ (S → Q)) with
    ((P ↔ R) ∧ (Q → S) ∧ (S → Q)) in n3_47a.
replace ((Q → S) ∧ (S → Q)) with (Q ↔ S) in n3_47a.
apply n3_47a.
apply Equiv4_01.
replace ((P ↔ R) ∧ (Q → S) ∧ (S → Q)) with
    (((P ↔ R) ∧ (Q → S)) ∧ (S → Q)).
reflexivity.
apply EqBi.
apply n4_32d.
apply Equiv4_01.
apply EqBi.
apply n4_32c.
replace ((R → P) ∧ (Q → S)) with ((Q → S) ∧ (R → P)).
reflexivity.
apply EqBi.
apply H0.
apply Equiv4_01.
replace ((P → R) ∧ (Q → S) ∧ (R → P)) with
    (((P → R) ∧ (Q → S)) ∧ (R → P)).
reflexivity.
apply EqBi.
```

```
  apply n4_32b.
  apply EqBi.
  apply n4_32a.
  apply Equiv4_01.
  Qed.

Theorem n4_4 : ∀ P Q R : Prop,
  (P ∧ (Q ∨ R)) ↔ ((P∧ Q) ∨ (P ∧ R)).
Proof. intros P Q R.
  specialize n3_2 with P Q.
  intros n3_2a.
  specialize n3_2 with P R.
  intros n3_2b.
  Conj n3_2a n3_2b.
  split.
  apply n3_2a.
  apply n3_2b.
  specialize Comp3_43 with P (Q→P∧Q) (R→P∧R).
  intros Comp3_43a.
  MP Comp3_43a H.
  specialize n3_48 with Q R (P∧Q) (P∧R).
  intros n3_48a.
  Syll Comp3_43a n3_48a Sa.
  specialize Imp3_31 with P (Q∨R) ((P∧ Q) ∨ (P ∧ R)).
  intros Imp3_31a.
  MP Imp3_31a Sa.
  specialize Simp3_26 with P Q.
  intros Simp3_26a.
  specialize Simp3_26 with P R.
  intros Simp3_26b.
  Conj Simp3_26a Simp3_26b.
  split.
  apply Simp3_26a.
  apply Simp3_26b.
  specialize n3_44 with P (P∧Q) (P∧R).
  intros n3_44a.
  MP n3_44a H0.
  specialize Simp3_27 with P Q.
  intros Simp3_27a.
  specialize Simp3_27 with P R.
  intros Simp3_27b.
  Conj Simp3_27a Simp3_27b.
```

```
  split.
  apply Simp3_27a.
  apply Simp3_27b.
  specialize n3_48 with (P∧Q) (P∧R) Q R.
  intros n3_48b.
  MP n3_48b H1.
  clear H1. clear Simp3_27a. clear Simp3_27b.
  Conj n3_44a n3_48b.
  split.
  apply n3_44a.
  apply n3_48b.
  specialize Comp3_43 with (P ∧ Q ∨ P ∧ R) P (Q∨R).
  intros Comp3_43b.
  MP Comp3_43b H1.
  clear H1. clear H0. clear n3_44a. clear n3_48b.
      clear Simp3_26a. clear Simp3_26b.
  Conj Imp3_31a Comp3_43b.
  split.
  apply Imp3_31a.
  apply Comp3_43b.
  Equiv H0.
  apply H0.
  apply Equiv4_01.
Qed.

Theorem n4_41 : ∀ P Q R : Prop,
  (P ∨ (Q ∧ R)) ↔ ((P ∨ Q) ∧ (P ∨ R)).
Proof. intros P Q R.
  specialize Simp3_26 with Q R.
  intros Simp3_26a.
  specialize Sum1_6 with P (Q ∧ R) Q.
  intros Sum1_6a.
  MP Simp3_26a Sum1_6a.
  specialize Simp3_27 with Q R.
  intros Simp3_27a.
  specialize Sum1_6 with P (Q ∧ R) R.
  intros Sum1_6b.
  MP Simp3_27a Sum1_6b.
  clear Simp3_26a. clear Simp3_27a.
  Conj Sum1_6a Sum1_6a.
  split.
  apply Sum1_6a.
```

```
  apply Sum1_6b.
  specialize Comp3_43 with (P ∨ Q ∧ R) (P ∨ Q) (P ∨ R).
  intros Comp3_43a.
  MP Comp3_43a H.
  specialize n2_53 with P Q.
  intros n2_53a.
  specialize n2_53 with P R.
  intros n2_53b.
  Conj n2_53a n2_53b.
  split.
  apply n2_53a.
  apply n2_53b.
  specialize n3_47 with (P ∨ Q) (P ∨ R) (~P → Q) (~P → R).
  intros n3_47a.
  MP n3_47a H0.
  specialize Comp3_43 with (~P) Q R.
  intros Comp3_43b.
  Syll n3_47a Comp3_43b Sa.
  specialize n2_54 with P (Q∧R).
  intros n2_54a.
  Syll Sa n2_54a Sb.
  clear Sum1_6a. clear Sum1_6b. clear H. clear n2_53a.
      clear n2_53b. clear H0. clear n3_47a. clear Sa.
      clear Comp3_43b. clear n2_54a.
  Conj Comp3_43a Sb.
  split.
  apply Comp3_43a.
  apply Sb.
  Equiv H.
  apply H.
  apply Equiv4_01.
Qed.

Theorem n4_42 : ∀ P Q : Prop,
  P ↔ ((P ∧ Q) ∨ (P ∧ ~Q)).
Proof. intros P Q.
  specialize n3_21 with P (Q ∨ ~Q).
  intros n3_21a.
  specialize n2_11 with Q.
  intros n2_11a.
  MP n3_21a n2_11a.
  specialize Simp3_26 with P (Q ∨ ~Q).
```

```
  intros Simp3_26a. clear n2_11a.
  Conj n3_21a Simp3_26a.
  split.
  apply n3_21a.
  apply Simp3_26a.
  Equiv H.
  specialize n4_4 with P Q (~Q).
  intros n4_4a.
  replace (P ∧ (Q ∨ ~Q)) with P in n4_4a.
  apply n4_4a.
  apply EqBi.
  apply H.
  apply Equiv4_01.
Qed.

Theorem n4_43 : ∀ P Q : Prop,
  P ↔ ((P ∨ Q) ∧ (P ∨ ~Q)).
Proof. intros P Q.
  specialize n2_2 with P Q.
  intros n2_2a.
  specialize n2_2 with P (~Q).
  intros n2_2b.
  Conj n2_2a n2_2b.
  split.
  apply n2_2a.
  apply n2_2b.
  specialize Comp3_43 with P (P∨Q) (P∨~Q).
  intros Comp3_43a.
  MP Comp3_43a H.
  specialize n2_53 with P Q.
  intros n2_53a.
  specialize n2_53 with P (~Q).
  intros n2_53b.
  Conj n2_53a n2_53b.
  split.
  apply n2_53a.
  apply n2_53b.
  specialize n3_47 with (P∨Q) (P∨~Q) (~P→Q) (~P→~Q).
  intros n3_47a.
  MP n3_47a H0.
  specialize n2_65 with (~P) Q.
  intros n2_65a.
```

```
  replace (~~P) with P in n2_65a.
  specialize Imp3_31 with (~P → Q) (~P → ~Q) (P).
  intros Imp3_31a.
  MP Imp3_31a n2_65a.
  Syll n3_47a Imp3_31a Sa.
  clear n2_2a. clear n2_2b. clear H. clear n2_53a. clear n2_53b.
      clear H0. clear n2_65a. clear n3_47a. clear Imp3_31a.
  Conj Comp3_43a Sa.
  split.
  apply Comp3_43a.
  apply Sa.
  Equiv H.
  apply H.
  apply Equiv4_01.
  apply EqBi.
  specialize n4_13 with P.
  intros n4_13a.
  apply n4_13a.
Qed.

Theorem n4_44 : ∀ P Q : Prop,
  P ↔ (P ∨ (P ∧ Q)).
  Proof. intros P Q.
    specialize n2_2 with P (P∧Q).
    intros n2_2a.
    specialize n2_08 with P.
    intros n2_08a.
    specialize Simp3_26 with P Q.
    intros Simp3_26a.
    Conj n2_08a Simp3_26a.
    split.
    apply n2_08a.
    apply Simp3_26a.
    specialize n3_44 with P P (P ∧ Q).
    intros n3_44a.
    MP n3_44a H.
    clear H. clear n2_08a. clear Simp3_26a.
    Conj n2_2a n3_44a.
    split.
    apply n2_2a.
    apply n3_44a.
    Equiv H.
```

```
    apply H.
    apply Equiv4_01.
  Qed.

Theorem n4_45 : ∀ P Q : Prop,
  P ↔ (P ∧ (P ∨ Q)).
  Proof. intros P Q.
  specialize n2_2 with (P ∧ P) (P ∧ Q).
  intros n2_2a.
  replace (P ∧ P ∨ P ∧ Q) with (P ∧ (P ∨ Q)) in n2_2a.
  replace (P ∧ P) with P in n2_2a.
  specialize Simp3_26 with P (P ∨ Q).
  intros Simp3_26a.
  Conj n2_2a Simp3_26a.
  split.
  apply n2_2a.
  apply Simp3_26a.
  Equiv H.
  apply H.
  apply Equiv4_01.
  specialize n4_24 with P.
  intros n4_24a.
  apply EqBi.
  apply n4_24a.
  specialize n4_4 with P P Q.
  intros n4_4a.
  apply EqBi.
  apply n4_4a.
Qed.

Theorem n4_5 : ∀ P Q : Prop,
  P ∧ Q ↔ ~(~P ∨ ~Q).
  Proof. intros P Q.
    specialize n4_2 with (P ∧ Q).
    intros n4_2a.
    rewrite Prod3_01.
    replace (~(~P ∨ ~Q)) with (P ∧ Q).
    apply n4_2a.
    apply Prod3_01.
  Qed.

Theorem n4_51 : ∀ P Q : Prop,
```

```
  ~(P ∧ Q) ↔ (~P ∨ ~Q).
  Proof. intros P Q.
    specialize n4_5 with P Q.
    intros n4_5a.
    specialize n4_12 with (P ∧ Q) (~P ∨ ~Q).
    intros n4_12a.
    replace ((P∧Q ↔ ~(~P∨~Q))↔(~P∨~Q ↔ ~(P∧Q))) with
        ((P∧Q ↔ ~(~P∨~Q)) = (~P∨~Q ↔ ~(P∧Q))) in n4_12a.
    replace (P ∧ Q ↔ ~(~P ∨ ~Q)) with
        (~P ∨ ~Q ↔ ~(P ∧ Q)) in n4_5a.
    replace (~P ∨ ~Q ↔ ~(P ∧ Q)) with
        (~(P ∧ Q) ↔ (~P ∨ ~Q)) in n4_5a.
    apply n4_5a.
    specialize n4_21 with (~(P ∧ Q)) (~P ∨ ~Q).
    intros n4_21a.
    apply EqBi.
    specialize n4_21 with (~(P∧Q)) (~P ∨ ~Q).
    intros n4_21b.
    apply n4_21b.
    apply EqBi.
    apply EqBi.
  Qed.

Theorem n4_52 : ∀ P Q : Prop,
  (P ∧ ~Q) ↔ ~(~P ∨ Q).
  Proof. intros P Q.
    specialize n4_5 with P (~Q).
    intros n4_5a.
    replace (~~Q) with Q in n4_5a.
    apply n4_5a.
    specialize n4_13 with Q.
    intros n4_13a.
    apply EqBi.
    apply n4_13a.
  Qed.

Theorem n4_53 : ∀ P Q : Prop,
  ~(P ∧ ~Q) ↔ (~P ∨ Q).
  Proof. intros P Q.
    specialize n4_52 with P Q.
    intros n4_52a.
    specialize n4_12 with ( P ∧ ~Q) ((~P ∨ Q)).
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    intros n4_12a.
    replace ((P∧~Q ↔ ~(~P∨Q))↔(~P∨Q ↔ ~(P∧~Q))) with
        ((P∧~Q ↔ ~(~P∨Q)) = (~P∨Q ↔ ~(P∧~Q))) in n4_12a.
    replace (P ∧ ~Q ↔ ~(~P ∨ Q)) with
        (~P ∨ Q ↔ ~(P ∧ ~Q)) in n4_52a.
    replace (~P ∨ Q ↔ ~(P ∧ ~Q)) with
        (~(P ∧ ~Q) ↔ (~P ∨ Q)) in n4_52a.
    apply n4_52a.
    specialize n4_21 with (~(P ∧ ~Q)) (~P ∨ Q).
    intros n4_21a.
    apply EqBi.
    apply n4_21a.
    apply EqBi.
    apply EqBi.
  Qed.

Theorem n4_54 : ∀ P Q : Prop,
  (~P ∧ Q) ↔ ~(P ∨ ~Q).
  Proof. intros P Q.
    specialize n4_5 with (~P) Q.
    intros n4_5a.
    specialize n4_13 with P.
    intros n4_13a.
    replace (~~P) with P in n4_5a.
    apply n4_5a.
    apply EqBi.
    apply n4_13a.
  Qed.

Theorem n4_55 : ∀ P Q : Prop,
  ~(~P ∧ Q) ↔ (P ∨ ~Q).
  Proof. intros P Q.
    specialize n4_54 with P Q.
    intros n4_54a.
    specialize n4_12 with (~P ∧ Q) (P ∨ ~Q).
    intros n4_12a.
    replace (~P ∧ Q ↔ ~(P ∨ ~Q)) with
        (P ∨ ~Q ↔ ~(~P ∧ Q)) in n4_54a.
    replace (P ∨ ~Q ↔ ~(~P ∧ Q)) with
        (~(~P ∧ Q) ↔ (P ∨ ~Q)) in n4_54a.
    apply n4_54a.
    specialize n4_21 with (~(~P ∧ Q)) (P ∨ ~Q).
```

```
    intros n4_21a.
    apply EqBi.
    apply n4_21a.
    replace ((~P ∧ Q ↔ ~(P ∨ ~Q)) ↔ (P ∨ ~Q ↔ ~(~P ∧ Q))) with
        ((~P ∧ Q ↔ ~(P ∨ ~Q)) = (P ∨ ~Q ↔ ~(~P ∧ Q))) in n4_12a.
    rewrite n4_12a.
    reflexivity.
    apply EqBi.
    apply EqBi.
  Qed.

Theorem n4_56 : ∀ P Q : Prop,
  (~P ∧ ~Q) ↔ ~(P ∨ Q).
  Proof. intros P Q.
    specialize n4_54 with P (~Q).
    intros n4_54a.
    replace (~~Q) with Q in n4_54a.
    apply n4_54a.
    apply EqBi.
    specialize n4_13 with Q.
    intros n4_13a.
    apply n4_13a.
  Qed.

Theorem n4_57 : ∀ P Q : Prop,
  ~(~P ∧ ~Q) ↔ (P ∨ Q).
  Proof. intros P Q.
    specialize n4_56 with P Q.
    intros n4_56a.
    specialize n4_12 with (~P ∧ ~Q) (P ∨ Q).
    intros n4_12a.
    replace (~P ∧ ~Q ↔ ~(P ∨ Q)) with
        (P ∨ Q ↔ ~(~P ∧ ~Q)) in n4_56a.
    replace (P ∨ Q ↔ ~(~P ∧ ~Q)) with
        (~(~P ∧ ~Q) ↔ P ∨ Q) in n4_56a.
    apply n4_56a.
    specialize n4_21 with (~(~P ∧ ~Q)) (P ∨ Q).
    intros n4_21a.
    apply EqBi.
    apply n4_21a.
    replace ((~P∧~Q↔~(P∨Q))↔(P∨Q↔~(~P∧~Q))) with
        ((P∨Q↔~(~P∧~Q))↔(~P∧~Q↔~(P∨Q))) in n4_12a.
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    apply EqBi.
    apply n4_12a.
    apply EqBi.
    specialize n4_21 with
        (P ∨ Q ↔ ~(~P ∧ ~Q)) (~P ∧ ~Q ↔ ~(P ∨ Q)).
    intros n4_21b.
    apply n4_21b.
  Qed.

Theorem n4_6 : ∀ P Q : Prop,
  (P → Q) ↔ (~P ∨ Q).
  Proof. intros P Q.
    specialize n4_2 with (~P∨ Q).
    intros n4_2a.
    rewrite Impl1_01.
    apply n4_2a.
  Qed.

Theorem n4_61 : ∀ P Q : Prop,
  ~(P → Q) ↔ (P ∧ ~Q).
  Proof. intros P Q.
  specialize n4_6 with P Q.
  intros n4_6a.
  specialize Trans4_11 with (P→Q) (~P∨Q).
  intros Trans4_11a.
  specialize n4_52 with P Q.
  intros n4_52a.
  replace ((P → Q) ↔ ~P ∨ Q) with
      (~(P → Q) ↔ ~(~P ∨ Q)) in n4_6a.
  replace (~(~P ∨ Q)) with (P ∧ ~Q) in n4_6a.
  apply n4_6a.
  apply EqBi.
  apply n4_52a.
  replace (((P→Q)↔~P∨Q)↔(~(P→Q)↔~(~P∨Q))) with
      ((~(P→Q)↔~(~P∨Q))↔((P→Q)↔~P∨Q)) in Trans4_11a.
  apply EqBi.
  apply Trans4_11a.
  apply EqBi.
  specialize n4_21 with (~(P→Q)↔~(~P∨Q))
      ((P→Q)↔(~P∨Q)).
  intros n4_21a.
  apply n4_21a.
```

```
  Qed.

Theorem n4_62 : ∀ P Q : Prop,
  (P → ~Q) ↔ (~P ∨ ~Q).
  Proof. intros P Q.
    specialize n4_6 with P (~Q).
    intros n4_6a.
    apply n4_6a.
  Qed.

Theorem n4_63 : ∀ P Q : Prop,
  ~(P → ~Q) ↔ (P ∧ Q).
  Proof. intros P Q.
    specialize n4_62 with P Q.
    intros n4_62a.
    specialize Trans4_11 with (P → ~Q) (~P ∨ ~Q).
    intros Trans4_11a.
    specialize n4_5 with P Q.
    intros n4_5a.
    replace (~(~P ∨ ~Q)) with (P ∧ Q) in Trans4_11a.
    replace ((P → ~Q) ↔ ~P ∨ ~Q) with
        ((~(P → ~Q) ↔ P ∧ Q)) in n4_62a.
    apply n4_62a.
    replace (((P→~Q)↔~P∨~Q)↔(~(P→~Q)↔P∧Q)) with
        ((~(P→~Q)↔P∧Q)↔((P→~Q)↔~P∨~Q)) in Trans4_11a.
    apply EqBi.
    apply Trans4_11a.
    specialize n4_21 with
        (~(P → ~Q) ↔ P ∧ Q) ((P → ~Q) ↔ ~P ∨ ~Q).
    intros n4_21a.
    apply EqBi.
    apply n4_21a.
    apply EqBi.
    apply n4_5a.
  Qed.

Theorem n4_64 : ∀ P Q : Prop,
  (~P → Q) ↔ (P ∨ Q).
  Proof. intros P Q.
    specialize n2_54 with P Q.
    intros n2_54a.
    specialize n2_53 with P Q.
```

```
    intros n2_53a.
    Conj n2_54a n2_53a.
    split.
    apply n2_54a.
    apply n2_53a.
    Equiv H.
    apply H.
    apply Equiv4_01.
  Qed.

Theorem n4_65 : ∀ P Q : Prop,
  ~(~P → Q) ↔ (~P ∧ ~Q).
  Proof. intros P Q.
  specialize n4_64 with P Q.
  intros n4_64a.
  specialize Trans4_11 with(~P → Q) (P ∨ Q).
  intros Trans4_11a.
  specialize n4_56 with P Q.
  intros n4_56a.
  replace (((~P→Q)↔P∨Q)↔(~(~P→Q)↔~(P∨Q))) with
      ((~(~P→Q)↔~(P∨Q))↔((~P→Q)↔P∨Q)) in Trans4_11a.
  replace ((~P → Q) ↔ P ∨ Q) with
      (~(~P → Q) ↔ ~(P ∨ Q)) in n4_64a.
  replace (~(P ∨ Q)) with (~P ∧ ~Q) in n4_64a.
  apply n4_64a.
  apply EqBi.
  apply n4_56a.
  apply EqBi.
  apply Trans4_11a.
  apply EqBi.
  specialize n4_21 with (~(~P → Q)↔~(P ∨ Q))
      ((~P → Q)↔(P ∨ Q)).
  intros n4_21a.
  apply n4_21a.
  Qed.

Theorem n4_66 : ∀ P Q : Prop,
  (~P → ~Q) ↔ (P ∨ ~Q).
  Proof. intros P Q.
  specialize n4_64 with P (~Q).
  intros n4_64a.
  apply n4_64a.
```

```
  Qed.

Theorem n4_67 : ∀ P Q : Prop,
  ~(~P → ~Q) ↔ (~P ∧ Q).
  Proof. intros P Q.
  specialize n4_66 with P Q.
  intros n4_66a.
  specialize Trans4_11 with (~P → ~Q) (P ∨ ~Q).
  intros Trans4_11a.
  replace ((~P → ~Q) ↔ P ∨ ~Q) with
      (~(~P → ~Q) ↔ ~(P ∨ ~Q)) in n4_66a.
  specialize n4_54 with P Q.
  intros n4_54a.
  replace (~(P ∨ ~Q)) with (~P ∧ Q) in n4_66a.
  apply n4_66a.
  apply EqBi.
  apply n4_54a.
  replace (((~P→~Q)↔P∨~Q)↔(~(~P→~Q)↔~(P∨~Q))) with
      ((~(~P→~Q)↔~(P∨~Q))↔((~P→~Q)↔P∨~Q)) in Trans4_11a.
  apply EqBi.
  apply Trans4_11a.
  apply EqBi.
  specialize n4_21 with (~(~P → ~Q)↔~(P ∨ ~Q))
     ((~P → ~Q)↔(P ∨ ~Q)).
  intros n4_21a.
  apply n4_21a.
  Qed.

Theorem n4_7 : ∀ P Q : Prop,
  (P → Q) ↔ (P → (P ∧ Q)).
  Proof. intros P Q.
  specialize Comp3_43 with P P Q.
  intros Comp3_43a.
  specialize Exp3_3 with
      (P → P) (P → Q) (P → P ∧ Q).
  intros Exp3_3a.
  MP Exp3_3a Comp3_43a.
  specialize n2_08 with P.
  intros n2_08a.
  MP Exp3_3a n2_08a.
  specialize Simp3_27 with P Q.
  intros Simp3_27a.
```

```
  specialize Syll2_05 with P (P ∧ Q) Q.
  intros Syll2_05a.
  MP Syll2_05a Simp3_27a.
  clear n2_08a. clear Comp3_43a. clear Simp3_27a.
  Conj Syll2_05a Exp3_3a.
  split.
  apply Exp3_3a.
  apply Syll2_05a.
  Equiv H.
  apply H.
  apply Equiv4_01.
  Qed.

Theorem n4_71 : ∀ P Q : Prop,
  (P → Q) ↔ (P ↔ (P ∧ Q)).
  Proof. intros P Q.
  specialize n4_7 with P Q.
  intros n4_7a.
  specialize n3_21 with (P→(P∧Q)) ((P∧Q)→P).
  intros n3_21a.
  replace ((P → P ∧ Q) ∧ (P ∧ Q → P)) with
      (P↔(P ∧ Q)) in n3_21a.
  specialize Simp3_26 with P Q.
  intros Simp3_26a.
  MP n3_21a Simp3_26a.
  specialize Simp3_26 with (P→(P∧Q)) ((P∧Q)→P).
  intros Simp3_26b.
  replace ((P → P ∧ Q) ∧ (P ∧ Q → P)) with
      (P↔(P ∧ Q)) in Simp3_26b. clear Simp3_26a.
  Conj n3_21a Simp3_26b.
  split.
  apply n3_21a.
  apply Simp3_26b.
  Equiv H.
  clear n3_21a. clear Simp3_26b.
  Conj n4_7a H.
  split.
  apply n4_7a.
  apply H.
  specialize n4_22 with (P → Q) (P → P ∧ Q) (P ↔ P ∧ Q).
  intros n4_22a.
  MP n4_22a H0.
```

```
  apply n4_22a.
  apply Equiv4_01.
  apply Equiv4_01.
  apply Equiv4_01.
  Qed.

Theorem n4_72 : ∀ P Q : Prop,
  (P → Q) ↔ (Q ↔ (P ∨ Q)).
  Proof. intros P Q.
  specialize Trans4_1 with P Q.
  intros Trans4_1a.
  specialize n4_71 with (~Q) (~P).
  intros n4_71a.
  Conj Trans4_1a n4_71a.
  split.
  apply Trans4_1a.
  apply n4_71a.
  specialize n4_22 with
      (P→Q) (~Q→~P) (~Q↔~Q ∧ ~P).
  intros n4_22a.
  MP n4_22a H.
  specialize n4_21 with (~Q) (~Q ∧ ~P).
  intros n4_21a.
  Conj n4_22a n4_21a.
  split.
  apply n4_22a.
  apply n4_21a.
  specialize n4_22 with
      (P→Q) (~Q ↔ ~Q ∧ ~P) (~Q ∧ ~P ↔ ~Q).
  intros n4_22b.
  MP n4_22b H0.
  specialize n4_12 with (~Q ∧ ~P) (Q).
  intros n4_12a.
  Conj n4_22b n4_12a.
  split.
  apply n4_22b.
  apply n4_12a.
  specialize n4_22 with
      (P → Q) ((~Q ∧ ~P) ↔ ~Q) (Q ↔ ~(~Q ∧ ~P)).
  intros n4_22c.
  MP n4_22b H0.
  specialize n4_57 with Q P.
```

```
  intros n4_57a.
  replace (~(~Q ∧ ~P)) with (Q ∨ P) in n4_22c.
  specialize n4_31 with P Q.
  intros n4_31a.
  replace (Q ∨ P) with (P ∨ Q) in n4_22c.
  apply n4_22c.
  apply EqBi.
  apply n4_31a.
  apply EqBi.
  replace (~(~Q ∧ ~P) ↔ Q ∨ P) with
      (Q ∨ P ↔~(~Q ∧ ~P)) in n4_57a.
  apply n4_57a.
  apply EqBi.
  specialize n4_21 with (Q ∨ P) (~(~Q ∧ ~P)).
  intros n4_21b.
  apply n4_21b.
  Qed.

Theorem n4_73 : ∀ P Q : Prop,
  Q → (P ↔ (P ∧ Q)).
  Proof. intros P Q.
  specialize Simp2_02 with P Q.
  intros Simp2_02a.
  specialize n4_71 with P Q.
  intros n4_71a.
  replace ((P → Q) ↔ (P ↔ P ∧ Q)) with
      (((P→Q)→(P↔P∧Q))∧((P↔P∧Q)→(P→Q))) in n4_71a.
  specialize Simp3_26 with
      ((P → Q) → P ↔ P ∧ Q) (P ↔ P ∧ Q → P → Q).
  intros Simp3_26a.
  MP Simp3_26a n4_71a.
  Syll Simp2_02a Simp3_26a Sa.
  apply Sa.
  apply Equiv4_01.
  Qed.

Theorem n4_74 : ∀ P Q : Prop,
  ~P → (Q ↔ (P ∨ Q)).
  Proof. intros P Q.
  specialize n2_21 with P Q.
  intros n2_21a.
  specialize n4_72 with P Q.
```

```
  intros n4_72a.
  replace (P → Q) with (Q ↔ P ∨ Q) in n2_21a.
  apply n2_21a.
  apply EqBi.
  replace ((P → Q) ↔ (Q ↔ P ∨ Q)) with
      ((Q ↔ P ∨ Q) ↔ (P → Q)) in n4_72a.
  apply n4_72a.
  apply EqBi.
  specialize n4_21 with (Q↔(P ∨ Q)) (P → Q).
  intros n4_21a.
  apply n4_21a.
  Qed.

Theorem n4_76 : ∀ P Q R : Prop,
  ((P → Q) ∧ (P → R)) ↔ (P → (Q ∧ R)).
  Proof. intros P Q R.
  specialize n4_41 with (~P) Q R.
  intros n4_41a.
  replace (~P ∨ Q) with (P→Q) in n4_41a.
  replace (~P ∨ R) with (P→R) in n4_41a.
  replace (~P ∨ Q ∧ R) with (P → Q ∧ R) in n4_41a.
  replace ((P → Q ∧ R) ↔ (P → Q) ∧ (P → R)) with
      ((P → Q) ∧ (P → R) ↔ (P → Q ∧ R)) in n4_41a.
  apply n4_41a.
  apply EqBi.
  specialize n4_21 with ((P → Q) ∧ (P → R)) (P → Q ∧ R).
  intros n4_21a.
  apply n4_21a.
  apply Impl1_01.
  apply Impl1_01.
  apply Impl1_01.
  Qed.

Theorem n4_77 : ∀ P Q R : Prop,
  ((Q → P) ∧ (R → P)) ↔ ((Q ∨ R) → P).
  Proof. intros P Q R.
  specialize n3_44 with P Q R.
  intros n3_44a.
  specialize n2_08 with (Q ∨ R → P).
  intros n2_08a. (*Not cited*)
  replace ((Q ∨ R → P) → (Q ∨ R → P)) with
      ((Q ∨ R → P) → (~(Q ∨ R) ∨ P)) in n2_08a.
```

```
replace (~(Q ∨ R)) with (~Q ∧ ~R) in n2_08a.
replace (~Q ∧ ~R ∨ P) with
    ((~Q ∨ P) ∧ (~R ∨ P)) in n2_08a.
replace (~Q ∨ P) with (Q → P) in n2_08a.
replace (~R ∨ P) with (R → P) in n2_08a.
Conj n3_44a n2_08a.
split.
apply n3_44a.
apply n2_08a.
Equiv H.
apply H.
apply Equiv4_01.
apply Impl1_01.
apply Impl1_01.
specialize n4_41 with P (~Q) (~R).
intros n4_41a. (*Not cited*)
replace (P ∨ ~Q) with
    (~Q ∨ P) in n4_41a.
replace (P ∨ ~R) with
    (~R ∨ P) in n4_41a.
replace (P ∨ ~Q ∧ ~R) with (~Q ∧ ~R ∨ P) in n4_41a.
replace (~Q ∧ ~R ∨ P ↔ (~Q ∨ P) ∧ (~R ∨ P)) with
    ((~Q ∨ P) ∧ (~R ∨ P) ↔ ~Q ∧ ~R ∨ P) in n4_41a.
apply EqBi.
apply n4_41a.
apply EqBi.
specialize n4_21
    with ((~Q ∨ P) ∧ (~R ∨ P)) (~Q ∧ ~R ∨ P).
intros n4_21a. (*Not cited*)
apply n4_21a.
specialize n4_31 with (~Q ∧ ~R) P.
intros n4_31a. (*Not cited*)
apply EqBi.
apply n4_31a.
specialize n4_31 with (~R) P.
intros n4_31b. (*Not cited*)
apply EqBi.
apply n4_31b.
specialize n4_31 with (~Q) P.
intros n4_31c. (*Not cited*)
apply EqBi.
apply n4_31c. (*Not cited*)
```

```
  apply EqBi.
  specialize n4_56 with Q R.
  intros n4_56a. (*Not cited*)
  apply n4_56a.
  replace (~(Q ∨ R) ∨ P) with (Q∨R→P).
  reflexivity.
  apply Impl1_01. (*Not cited*)
  Qed.
    (*Proof sketch cites Add1_3 + n2_2.*)

Theorem n4_78 : ∀ P Q R : Prop,
  ((P → Q) ∨ (P → R)) ↔ (P → (Q ∨ R)).
  Proof. intros P Q R.
  specialize n4_2 with ((P→Q) ∨ (P → R)).
  intros n4_2a.
  replace (((P→Q)∨(P→R))↔((P→Q)∨(P→R))) with
      (((P → Q) ∨ (P → R))↔((~P ∨ Q) ∨ ~P ∨ R)) in n4_2a.
  specialize n4_33 with (~P) Q (~P ∨ R).
  intros n4_33a.
  replace ((~P ∨ Q) ∨ ~P ∨ R) with
      (~P ∨ Q ∨ ~P ∨ R) in n4_2a.
  specialize n4_31 with (~P) Q.
  intros n4_31a.
  specialize n4_37 with (~P∨Q) (Q ∨ ~P) R.
  intros n4_37a.
  MP n4_37a n4_31a.
  replace (Q ∨ ~P ∨ R) with
      ((Q ∨ ~P) ∨ R) in n4_2a.
  replace ((Q ∨ ~P) ∨ R) with
      ((~P ∨ Q) ∨ R) in n4_2a.
  specialize n4_33 with (~P) (~P∨Q) R.
  intros n4_33b.
  replace (~P ∨ (~P ∨ Q) ∨ R) with
      ((~P ∨ (~P ∨ Q)) ∨ R) in n4_2a.
  specialize n4_25 with (~P).
  intros n4_25a.
  specialize n4_37 with
      (~P) (~P ∨ ~P) (Q ∨ R).
  intros n4_37b.
  MP n4_37b n4_25a.
  replace (~P ∨ ~P ∨ Q) with
      ((~P ∨ ~P) ∨ Q) in n4_2a.
```

```
  replace (((~P ∨ ~P) ∨ Q) ∨ R) with
      ((~P ∨ ~P) ∨ Q ∨ R) in n4_2a.
  replace ((~P ∨ ~P) ∨ Q ∨ R) with
      ((~P) ∨ (Q ∨ R)) in n4_2a.
  replace (~P ∨ Q ∨ R) with
      (P → (Q ∨ R)) in n4_2a.
  apply n4_2a.
  apply Impl1_01.
  apply EqBi.
  apply n4_37b.
  apply Abb2_33.
  replace ((~P ∨ ~P) ∨ Q) with (~P ∨ ~P ∨ Q).
  reflexivity.
  apply Abb2_33.
  replace ((~P ∨ ~P ∨ Q) ∨ R) with
      (~P ∨ (~P ∨ Q) ∨ R).
  reflexivity.
  apply EqBi.
  apply n4_33b.
  apply EqBi.
  apply n4_37a.
  replace ((Q ∨ ~P) ∨ R) with (Q ∨ ~P ∨ R).
  reflexivity.
  apply Abb2_33.
  apply EqBi.
  apply n4_33a.
  replace (~P ∨ Q) with (P→Q).
  replace (~P ∨ R) with (P→R).
  reflexivity.
  apply Impl1_01.
  apply Impl1_01.
  Qed.

Theorem n4_79 : ∀ P Q R : Prop,
  ((Q → P) ∨ (R → P)) ↔ ((Q ∧ R) → P).
  Proof. intros P Q R.
    specialize Trans4_1 with Q P.
    intros Trans4_1a.
    specialize Trans4_1 with R P.
    intros Trans4_1b.
    Conj Trans4_1a Trans4_1b.
    split.
```

```
    apply Trans4_1a.
    apply Trans4_1b.
    specialize n4_39 with
        (Q→P) (R→P) (~P→~Q) (~P→~R).
    intros n4_39a.
    MP n4_39a H.
    specialize n4_78 with (~P) (~Q) (~R).
    intros n4_78a.
    replace ((~P → ~Q) ∨ (~P → ~R)) with
        (~P → ~Q ∨ ~R) in n4_39a.
    specialize Trans4_1 with (~P) (~Q ∨ ~R).
    intros Trans4_1c.
    replace (~P → ~Q ∨ ~R) with
        (~(~Q ∨ ~R) → ~~P) in n4_39a.
    replace (~(~Q ∨ ~R)) with
        (Q ∧ R) in n4_39a.
    replace (~~P) with P in n4_39a.
    apply n4_39a.
    specialize n4_13 with P.
    intros n4_13a.
    apply EqBi.
    apply n4_13a.
    apply Prod3_01.
    replace (~(~Q ∨ ~R) → ~~P) with
        (~P → ~Q ∨ ~R).
    reflexivity.
    apply EqBi.
    apply Trans4_1c.
    replace (~P → ~Q ∨ ~R) with
        ((~P → ~Q) ∨ (~P → ~R)).
    reflexivity.
    apply EqBi.
    apply n4_78a.
  Qed.
  (*The proof sketch cites Trans2_15, but we did
      not need Trans2_15 as a lemma here.*)

Theorem n4_8 : ∀ P : Prop,
  (P → ~P) ↔ ~P.
  Proof. intros P.
    specialize Abs2_01 with P.
    intros Abs2_01a.
```

```
    specialize  Simp2_02 with P (~P).
    intros Simp2_02a.
    Conj Abs2_01a Simp2_02a.
    split.
    apply Abs2_01a.
    apply Simp2_02a.
    Equiv H.
    apply H.
    apply Equiv4_01.
  Qed.

Theorem n4_81 : ∀ P : Prop,
  (~P → P) ↔ P.
  Proof. intros P.
    specialize n2_18 with P.
    intros n2_18a.
    specialize  Simp2_02 with (~P) P.
    intros Simp2_02a.
    Conj n2_18a Simp2_02a.
    split.
    apply n2_18a.
    apply Simp2_02a.
    Equiv H.
    apply H.
    apply Equiv4_01.
  Qed.

Theorem n4_82 : ∀ P Q : Prop,
  ((P → Q) ∧ (P → ~Q)) ↔ ~P.
  Proof. intros P Q.
    specialize n2_65 with P Q.
    intros n2_65a.
    specialize Imp3_31 with (P→Q) (P→~Q) (~P).
    intros Imp3_31a.
    MP Imp3_31a n2_65a.
    specialize n2_21 with P Q.
    intros n2_21a.
    specialize n2_21 with P (~Q).
    intros n2_21b.
    Conj n2_21a n2_21b.
    split.
    apply n2_21a.
```

```
    apply n2_21b.
    specialize Comp3_43 with (~P) (P→Q) (P→~Q).
    intros Comp3_43a.
    MP Comp3_43a H.
    clear n2_65a. clear n2_21a. clear n2_21b.
    clear H.
    Conj Imp3_31a Comp3_43a.
    split.
    apply Imp3_31a.
    apply Comp3_43a.
    Equiv H.
    apply H.
    apply Equiv4_01.
  Qed.

Theorem n4_83 : ∀ P Q : Prop,
  ((P → Q) ∧ (~P → Q)) ↔ Q.
  Proof. intros P Q.
  specialize n2_61 with P Q.
  intros n2_61a.
  specialize Imp3_31 with (P→Q) (~P→Q) (Q).
  intros Imp3_31a.
  MP Imp3_31a n2_61a.
  specialize Simp2_02 with P Q.
  intros Simp2_02a.
  specialize Simp2_02 with (~P) Q.
  intros Simp2_02b.
  Conj Simp2_02a Simp2_02b.
  split.
  apply Simp2_02a.
  apply Simp2_02b.
  specialize Comp3_43 with Q (P→Q) (~P→Q).
  intros Comp3_43a.
  MP Comp3_43a H.
  clear n2_61a. clear Simp2_02a. clear Simp2_02b.
  clear H.
  Conj Imp3_31a Comp3_43a.
  split.
  apply Imp3_31a.
  apply Comp3_43a.
  Equiv H.
  apply H.
```

```
  apply Equiv4_01.
  Qed.

Theorem n4_84 : ∀ P Q R : Prop,
  (P ↔ Q) → ((P → R) ↔ (Q → R)).
  Proof. intros P Q R.
    specialize Syll2_06 with P Q R.
    intros Syll2_06a.
    specialize Syll2_06 with Q P R.
    intros Syll2_06b.
    Conj Syll2_06a Syll2_06b.
    split.
    apply Syll2_06a.
    apply Syll2_06b.
    specialize n3_47 with
        (P→Q) (Q→P) ((Q→R)→P→R) ((P→R)→Q→R).
    intros n3_47a.
    MP n3_47a H.
    replace ((P→Q) ∧ (Q → P)) with
        (P↔Q) in n3_47a.
    replace (((Q→R)→P→R)∧((P→R)→Q→R)) with
        ((Q → R) ↔ (P → R)) in n3_47a.
    replace ((Q → R) ↔ (P → R)) with
        ((P→ R) ↔ (Q → R)) in n3_47a.
    apply n3_47a.
    apply EqBi.
    specialize n4_21 with (P→R) (Q→R).
    intros n4_21a.
    apply n4_21a.
    apply Equiv4_01.
    apply Equiv4_01.
  Qed.

Theorem n4_85 : ∀ P Q R : Prop,
  (P ↔ Q) → ((R → P) ↔ (R → Q)).
  Proof. intros P Q R.
  specialize Syll2_05 with R P Q.
  intros Syll2_05a.
  specialize Syll2_05 with R Q P.
  intros Syll2_05b.
  Conj Syll2_05a Syll2_05b.
  split.
```

```
  apply Syll2_05a.
  apply Syll2_05b.
  specialize n3_47 with
      (P→Q) (Q→P) ((R→P)→R→Q) ((R→Q)→R→P).
  intros n3_47a.
  MP n3_47a H.
  replace ((P→Q) ∧ (Q → P)) with (P↔Q) in n3_47a.
  replace (((R→P)→R→Q)∧((R→Q)→R→P)) with
      ((R → P) ↔ (R → Q)) in n3_47a.
  apply n3_47a.
  apply Equiv4_01.
  apply Equiv4_01.
Qed.

Theorem n4_86 : ∀ P Q R : Prop,
  (P ↔ Q) → ((P ↔ R) ↔ (Q ↔ R)).
  Proof. intros P Q R.
  specialize n4_22 with  Q P R.
  intros n4_22a.
  specialize Exp3_3 with (Q↔P) (P↔R) (Q↔R).
  intros Exp3_3a. (*Not cited*)
  MP Exp3_3a n4_22a.
  specialize n4_22 with  P Q R.
  intros n4_22b.
  specialize Exp3_3 with (P↔Q) (Q↔R) (P↔R).
  intros Exp3_3b.
  MP Exp3_3b n4_22b.
  clear n4_22a.
  clear n4_22b.
  replace (Q↔P) with (P↔Q) in Exp3_3a.
  Conj Exp3_3a Exp3_3b.
  split.
  apply Exp3_3a.
  apply Exp3_3b.
  specialize Comp3_43 with (P↔Q)
      ((P↔R)→(Q↔R)) ((Q↔R)→(P↔R)).
  intros Comp3_43a. (*Not cited*)
  MP Comp3_43a H.
  replace (((P↔R)→(Q↔R))∧((Q↔R)→(P↔R)))
    with ((P↔R)↔(Q↔R)) in Comp3_43a.
  apply Comp3_43a.
  apply Equiv4_01.
```

```
  apply EqBi.
  specialize n4_21 with P Q.
  intros n4_21a.
  apply n4_21a.
  Qed.

Theorem n4_87 : ∀ P Q R : Prop,
  (((P ∧ Q) → R) ↔ (P → Q → R)) ↔
      ((Q → (P → R)) ↔ (Q ∧ P → R)).
  Proof. intros P Q R.
  specialize Exp3_3 with P Q R.
  intros Exp3_3a.
  specialize Imp3_31 with P Q R.
  intros Imp3_31a.
  Conj Exp3_3a Imp3_31a.
  split.
  apply Exp3_3a.
  apply Imp3_31a.
  Equiv H.
  specialize Exp3_3 with Q P R.
  intros Exp3_3b.
  specialize Imp3_31 with Q P R.
  intros Imp3_31b.
  Conj Exp3_3b Imp3_31b.
  split.
  apply Exp3_3b.
  apply Imp3_31b.
  Equiv H0.
  specialize Comm2_04 with P Q R.
  intros Comm2_04a.
  specialize Comm2_04 with Q P R.
  intros Comm2_04b.
  Conj Comm2_04a Comm2_04b.
  split.
  apply Comm2_04a.
  apply Comm2_04b.
  Equiv H1.
  clear Exp3_3a. clear Imp3_31a. clear Exp3_3b.
      clear Imp3_31b. clear Comm2_04a.
      clear Comm2_04b.
  replace (P∧Q→R) with (P → Q → R).
  replace (Q∧P→R) with (Q → P → R).
```

```
  replace (Q→P→R) with (P → Q → R).
  specialize n4_2 with
      ((P → Q → R) ↔ (P → Q → R)).
  intros n4_2a.
  apply n4_2a.
  apply EqBi.
  apply H1.
  replace (Q→P→R) with (Q∧P→R).
  reflexivity.
  apply EqBi.
  apply H0.
  replace (P→Q→R) with (P∧Q→R).
  reflexivity.
  apply EqBi.
  apply H.
  apply Equiv4_01.
  apply Equiv4_01.
  apply Equiv4_01.
  Qed.

End No4.

Module No5.

Import No1.
Import No2.
Import No3.
Import No4.

Theorem n5_1 : ∀ P Q : Prop,
  (P ∧ Q) → (P ↔ Q).
  Proof. intros P Q.
  specialize n3_4 with P Q.
  intros n3_4a.
  specialize n3_4 with Q P.
  intros n3_4b.
  specialize n3_22 with P Q.
  intros n3_22a.
  Syll n3_22a n3_4b Sa.
  clear n3_22a. clear n3_4b.
  Conj n3_4a Sa.
  split.
```

```
  apply n3_4a.
  apply Sa.
  specialize n4_76 with (P∧Q) (P→Q) (Q→P).
  intros n4_76a. (*Not cited*)
  replace ((P∧Q→P→Q)∧(P∧Q→Q→P)) with
      (P ∧ Q → (P → Q) ∧ (Q → P)) in H.
  replace ((P→Q)∧(Q→P)) with (P↔Q) in H.
  apply H.
  apply Equiv4_01.
  replace (P ∧ Q → (P → Q) ∧ (Q → P)) with
      ((P ∧ Q → P → Q) ∧ (P ∧ Q → Q → P)).
  reflexivity.
  apply EqBi.
  apply n4_76a.
  Qed.

Theorem n5_11 : ∀ P Q : Prop,
  (P → Q) ∨ (~P → Q).
  Proof. intros P Q.
  specialize n2_5 with P Q.
  intros n2_5a.
  specialize n2_54 with ((P → Q)) (~P → Q).
  intros n2_54a.
  MP n2_54a n2_5a.
  apply n2_54a.
  Qed.
  (*The proof sketch cites n2_51,
      but this may be a misprint.*)

Theorem n5_12 : ∀ P Q : Prop,
  (P → Q) ∨ (P → ~Q).
  Proof. intros P Q.
  specialize n2_51 with P Q.
  intros n2_51a.
  specialize n2_54 with ((P → Q)) (P → ~Q).
  intros n2_54a.
  MP n2_54a n2_5a.
  apply n2_54a.
  Qed.
  (*The proof sketch cites n2_52,
      but this may be a misprint.*)
```

```
Theorem n5_13 : ∀ P Q : Prop,
  (P → Q) ∨ (Q → P).
  Proof. intros P Q.
  specialize n2_521 with P Q.
  intros n2_521a.
  replace (~(P → Q) → Q → P) with
      (~~(P → Q) ∨ (Q → P)) in n2_521a.
  replace (~~(P→Q)) with (P→Q) in n2_521a.
  apply n2_521a.
  apply EqBi.
  specialize n4_13 with (P→Q).
  intros n4_13a. (*Not cited*)
  apply n4_13a.
  replace (~~(P → Q) ∨ (Q → P)) with
      (~(P → Q) → Q → P).
  reflexivity.
  apply Impl1_01.
  Qed.

Theorem n5_14 : ∀ P Q R : Prop,
  (P → Q) ∨ (Q → R).
  Proof. intros P Q R.
  specialize Simp2_02 with P Q.
  intros Simp2_02a.
  specialize Trans2_16 with Q (P→Q).
  intros Trans2_16a.
  MP Trans2_16a Simp2_02a.
  specialize n2_21 with Q R.
  intros n2_21a.
  Syll Trans2_16a n2_21a Sa.
  replace (~(P→Q)→(Q→R)) with
      (~~(P→Q)∨(Q→R)) in Sa.
  replace (~~(P→Q)) with (P→Q) in Sa.
  apply Sa.
  apply EqBi.
  specialize n4_13 with (P→Q).
  intros n4_13a.
  apply n4_13a.
  replace (~~(P→Q)∨(Q→R)) with
      (~(P→Q)→(Q→R)).
  reflexivity.
  apply Impl1_01.
```

```
  Qed.

Theorem n5_15 : ∀ P Q : Prop,
  (P ↔ Q) ∨ (P ↔ ~Q).
  Proof. intros P Q.
  specialize n4_61 with P Q.
  intros n4_61a.
  replace (~(P → Q) ↔ P ∧ ~Q) with
      ((~(P→Q)→P∧~Q)∧((P∧~Q)→~(P→Q))) in n4_61a.
  specialize Simp3_26 with
      (~(P → Q) → P ∧ ~Q) ((P ∧ ~Q) → ~(P → Q)).
  intros Simp3_26a.
  MP Simp3_26a n4_61a.
  specialize n5_1 with P (~Q).
  intros n5_1a.
  Syll Simp3_26a n5_1a Sa.
  specialize n2_54 with (P→Q) (P ↔ ~Q).
  intros n2_54a.
  MP n2_54a Sa.
  specialize n4_61 with Q P.
  intros n4_61b.
  replace ((~(Q → P)) ↔ (Q ∧ ~P)) with
      (((~(Q→P))→(Q∧~P))∧((Q∧~P)→(~(Q→P)))) in n4_61b.
  specialize Simp3_26 with
      (~(Q → P)→ (Q ∧ ~P)) ((Q ∧ ~P)→ (~(Q → P))).
  intros Simp3_26b.
  MP Simp3_26b n4_61b.
  specialize n5_1 with Q (~P).
  intros n5_1b.
  Syll Simp3_26b n5_1b Sb.
  specialize n4_12 with P Q.
  intros n4_12a.
  replace (Q↔~P) with (P↔~Q) in Sb.
  specialize n2_54 with (Q→P) (P↔~Q).
  intros n2_54b.
  MP n2_54b Sb.
  clear n4_61a. clear Simp3_26a. clear n5_1a.
      clear n2_54a. clear n4_61b. clear Simp3_26b.
      clear n5_1b. clear n4_12a. clear n2_54b.
  replace (~(P → Q) → P ↔ ~Q) with
      (~~(P → Q) ∨ (P ↔ ~Q)) in Sa.
  replace (~~(P→Q)) with (P→Q) in Sa.
```

```
replace (~(Q → P) → (P ↔ ~Q)) with
    (~~(Q → P) ∨ (P ↔ ~Q)) in Sb.
replace (~~(Q→P)) with (Q→P) in Sb.
Conj Sa Sb.
split.
apply Sa.
apply Sb.
specialize n4_41 with (P↔~Q) (P→Q) (Q→P).
intros n4_41a.
replace ((P → Q) ∨ (P ↔ ~Q)) with
    ((P ↔ ~Q) ∨ (P → Q)) in H.
replace ((Q → P) ∨ (P ↔ ~Q)) with
    ((P ↔ ~Q) ∨ (Q → P)) in H.
replace (((P↔~Q)∨(P→Q))∧((P↔~Q)∨(Q→P))) with
    ((P ↔ ~Q) ∨ (P → Q) ∧ (Q → P)) in H.
replace ((P→Q) ∧ (Q → P)) with (P↔Q) in H.
replace ((P ↔ ~Q) ∨ (P ↔ Q)) with
    ((P ↔ Q) ∨ (P ↔ ~Q)) in H.
apply H.
apply EqBi.
apply n4_31.
apply Equiv4_01.
apply EqBi.
apply n4_41a.
apply EqBi.
apply n4_31.
apply EqBi.
apply n4_31.
apply EqBi.
specialize n4_13 with (Q→P).
intros n4_13a.
apply n4_13a.
replace (~~(Q → P) ∨ (P ↔ ~Q)) with
    (~(Q → P) → (P ↔ ~Q)).
reflexivity.
apply Impl1_01.
apply EqBi.
specialize n4_13 with (P→Q).
intros n4_13b.
apply n4_13b.
replace (~~(P → Q) ∨ (P ↔ ~Q)) with
    (~(P → Q) → P ↔ ~Q).
```

```
  reflexivity.
  apply Impl1_01.
  apply EqBi.
  apply n4_12a.
  apply Equiv4_01.
  apply Equiv4_01.
  Qed.

Theorem n5_16 : ∀ P Q : Prop,
  ~((P ↔ Q) ∧ (P ↔ ~Q)).
  Proof. intros P Q.
  specialize Simp3_26 with ((P→Q)∧ (P → ~Q)) (Q→P).
  intros Simp3_26a.
  specialize n2_08 with ((P ↔ Q) ∧ (P → ~Q)).
  intros n2_08a.
  replace (((P → Q) ∧ (P → ~Q)) ∧ (Q → P)) with
      ((P→Q)∧((P→~Q)∧(Q→P))) in Simp3_26a.
  replace ((P → ~Q) ∧ (Q → P)) with
      ((Q → P) ∧ (P → ~Q)) in Simp3_26a.
  replace ((P→Q) ∧ (Q → P)∧ (P → ~Q)) with
      (((P→Q) ∧ (Q → P)) ∧ (P → ~Q)) in Simp3_26a.
  replace ((P → Q) ∧ (Q → P)) with
      (P↔Q) in Simp3_26a.
  Syll n2_08a Simp3_26a Sa.
  specialize n4_82 with P Q.
  intros n4_82a.
  replace ((P → Q) ∧ (P → ~Q)) with (~P) in Sa.
  specialize Simp3_27 with
      (P→Q) ((Q→P)∧ (P → ~Q)).
  intros Simp3_27a.
  replace ((P→Q) ∧ (Q → P)∧ (P → ~Q)) with
      (((P→Q) ∧ (Q → P)) ∧ (P → ~Q)) in Simp3_27a.
  replace ((P → Q) ∧ (Q → P)) with
      (P↔Q) in Simp3_27a.
  specialize Syll3_33 with Q P (~Q).
  intros Syll3_33a.
  Syll Simp3_27a Syll2_06a Sb.
  specialize Abs2_01 with Q.
  intros Abs2_01a.
  Syll Sb Abs2_01a Sc.
  clear Sb. clear Simp3_26a. clear n2_08a.
    clear n4_82a. clear Simp3_27a. clear Syll3_33a.
```

```
    clear Abs2_01a.
Conj Sa Sc.
split.
apply Sa.
apply Sc.
specialize Comp3_43 with
    ((P ↔ Q) ∧ (P → ~Q)) (~P) (~Q).
intros Comp3_43a.
MP Comp3_43a H.
specialize n4_65 with Q P.
intros n4_65a.
replace (~Q ∧ ~P) with (~P ∧ ~Q) in n4_65a.
replace (~P ∧ ~Q) with
    (~(~Q→P)) in Comp3_43a.
specialize Exp3_3 with
    (P↔Q) (P→~Q) (~(~Q→P)).
intros Exp3_3a.
MP Exp3_3a Comp3_43a.
replace ((P→~Q)→~(~Q→P)) with
    (~(P→~Q)∨~(~Q→P)) in Exp3_3a.
specialize n4_51 with (P→~Q) (~Q→P).
intros n4_51a.
replace (~(P → ~Q) ∨ ~(~Q → P)) with
    (~((P → ~Q) ∧ (~Q → P))) in Exp3_3a.
replace ((P→~Q) ∧ (~Q → P)) with
    (P↔~Q) in Exp3_3a.
replace ((P↔Q)→~(P↔~Q)) with
    (~(P↔Q)∨~(P↔~Q)) in Exp3_3a.
specialize n4_51 with (P↔Q) (P↔~Q).
intros n4_51b.
replace (~(P ↔ Q) ∨ ~(P ↔ ~Q)) with
    (~((P ↔ Q) ∧ (P ↔ ~Q))) in Exp3_3a.
apply Exp3_3a.
apply EqBi.
apply n4_51b.
replace (~(P ↔ Q) ∨ ~(P ↔ ~Q)) with
    (P ↔ Q → ~(P ↔ ~Q)).
reflexivity.
apply Impl1_01.
apply Equiv4_01.
apply EqBi.
apply n4_51a.
```

```
  replace (~(P → ~Q) ∨ ~(~Q → P)) with
      ((P → ~Q) → ~(~Q → P)).
  reflexivity.
  apply Impl1_01.
  apply EqBi.
  apply n4_65a.
  apply EqBi.
  specialize n4_3 with (~P) (~Q).
  intros n4_3a.
  apply n4_3a.
  apply Equiv4_01.
  apply EqBi.
  specialize n4_32 with (P→Q) (Q→P) (P→~Q).
  intros n4_32a.
  apply n4_32a.
  replace (~P) with ((P → Q) ∧ (P → ~Q)).
  reflexivity.
  apply EqBi.
  apply n4_82a.
  apply Equiv4_01.
  apply EqBi.
  specialize n4_32 with (P→Q) (Q→P) (P→~Q).
  intros n4_32b.
  apply n4_32b.
  apply EqBi.
  specialize n4_3 with (Q→P) (P→~Q).
  intros n4_3b.
  apply n4_3b.
  replace ((P → Q) ∧ (P → ~Q) ∧ (Q → P)) with
      (((P → Q) ∧ (P → ~Q)) ∧ (Q → P)).
  reflexivity.
  apply EqBi.
  specialize n4_32 with (P→Q) (P→~Q) (Q→P).
  intros n4_32a.
  apply n4_32a.
  Qed.

Theorem n5_17 : ∀ P Q : Prop,
  ((P ∨ Q) ∧ ~(P ∧ Q)) ↔ (P ↔ ~Q).
  Proof. intros P Q.
  specialize n4_64 with Q P.
  intros n4_64a.
```

```
specialize n4_21 with (Q∨P) (~Q→P).
intros n4_21a.
replace ((~Q→P)↔(Q∨P)) with
    ((Q∨P)↔(~Q→P)) in n4_64a.
replace (Q∨P) with (P∨Q) in n4_64a.
specialize n4_63 with P Q.
intros n4_63a.
replace (~(P → ~Q) ↔ P ∧ Q) with
    (P ∧ Q ↔ ~(P → ~Q)) in n4_63a.
specialize Trans4_11 with (P∧Q) (~(P→~Q)).
intros Trans4_11a.
replace (~~(P→~Q)) with
    (P→~Q) in Trans4_11a.
replace (P ∧ Q ↔ ~(P → ~Q)) with
    (~(P ∧ Q) ↔ (P → ~Q)) in n4_63a.
clear Trans4_11a. clear n4_21a.
Conj n4_64a n4_63a.
split.
apply n4_64a.
apply n4_63a.
specialize n4_38 with
    (P ∨ Q) (~(P ∧ Q)) (~Q → P) (P → ~Q).
intros n4_38a.
MP n4_38a H.
replace ((~Q→P) ∧ (P → ~Q)) with
    (~Q↔P) in n4_38a.
specialize n4_21 with P (~Q).
intros n4_21b.
replace (~Q↔P) with (P↔~Q) in n4_38a.
apply n4_38a.
apply EqBi.
apply n4_21b.
apply Equiv4_01.
replace (~(P ∧ Q) ↔ (P → ~Q)) with
    (P ∧ Q ↔ ~(P → ~Q)).
reflexivity.
apply EqBi.
apply Trans4_11a.
apply EqBi.
specialize n4_13 with (P→~Q).
intros n4_13a.
apply n4_13a.
```

```
  apply EqBi.
  specialize n4_21 with (P ∧ Q) (~(P→~Q)).
  intros n4_21b.
  apply n4_21b.
  apply EqBi.
  specialize n4_31 with P Q.
  intros n4_31a.
  apply n4_31a.
  apply EqBi.
  apply n4_21a.
  Qed.

Theorem n5_18 : ∀ P Q : Prop,
  (P ↔ Q) ↔ ~(P ↔ ~Q).
  Proof. intros P Q.
  specialize n5_15 with P Q.
  intros n5_15a.
  specialize n5_16 with P Q.
  intros n5_16a.
  Conj n5_15a n5_16a.
  split.
  apply n5_15a.
  apply n5_16a.
  specialize n5_17 with (P↔Q) (P↔~Q).
  intros n5_17a.
  replace ((P ↔ Q) ↔ ~(P ↔ ~Q)) with
      (((P↔Q)∨(P↔~Q))∧~((P↔Q)∧(P↔~Q))).
  apply H.
  apply EqBi.
  apply n5_17a.
  Qed.

Theorem n5_19 : ∀ P : Prop,
  ~(P ↔ ~P).
  Proof. intros P.
  specialize n5_18 with P P.
  intros n5_18a.
  specialize n4_2 with P.
  intros n4_2a.
  replace (~(P↔~P)) with (P↔P).
  apply n4_2a.
  apply EqBi.
```

```
  apply n5_18a.
  Qed.

Theorem n5_21 : ∀ P Q : Prop,
  (~P ∧ ~Q) → (P ↔ Q).
  Proof. intros P Q.
  specialize n5_1 with (~P) (~Q).
  intros n5_1a.
  specialize Trans4_11 with P Q.
  intros Trans4_11a.
  replace (~P↔~Q) with (P↔Q) in n5_1a.
  apply n5_1a.
  apply EqBi.
  apply Trans4_11a.
  Qed.

Theorem n5_22 : ∀ P Q : Prop,
  ~(P ↔ Q) ↔ ((P ∧ ~Q) ∨ (Q ∧ ~P)).
  Proof. intros P Q.
  specialize n4_61 with P Q.
  intros n4_61a.
  specialize n4_61 with Q P.
  intros n4_61b.
  Conj n4_61a n4_61b.
  split.
  apply n4_61a.
  apply n4_61b.
  specialize n4_39 with
      (~(P → Q)) (~(Q → P)) (P ∧ ~Q) (Q ∧ ~P).
  intros n4_39a.
  MP n4_39a H.
  specialize n4_51 with (P→Q) (Q→P).
  intros n4_51a.
  replace (~(P → Q) ∨ ~(Q → P)) with
      (~((P → Q) ∧ (Q → P))) in n4_39a.
  replace ((P → Q) ∧ (Q → P)) with
      (P↔Q) in n4_39a.
  apply n4_39a.
  apply Equiv4_01.
  apply EqBi.
  apply n4_51a.
  Qed.
```

```
Theorem n5_23 : ∀ P Q : Prop,
  (P ↔ Q) ↔ ((P ∧ Q) ∨ (~P ∧ ~Q)).
  Proof. intros P Q.
  specialize n5_18 with P Q.
  intros n5_18a.
  specialize n5_22 with P (~Q).
  intros n5_22a.
  specialize n4_13 with Q.
  intros n4_13a.
  replace (~(P↔~Q)) with
      ((P ∧ ~~Q) ∨ (~Q ∧ ~P)) in n5_18a.
  replace (~~Q) with Q in n5_18a.
  replace (~Q ∧ ~P) with (~P ∧ ~Q) in n5_18a.
  apply n5_18a.
  apply EqBi.
  specialize n4_3 with (~P) (~Q).
  intros n4_3a.
  apply n4_3a. (*with (~P) (~Q)*)
  apply EqBi.
  apply n4_13a.
  replace (P∧~~Q∨~Q∧~P) with (~(P↔~Q)).
  reflexivity.
  apply EqBi.
  apply n5_22a.
  Qed.
  (*The proof sketch in Principia offers n4_36,
      but we found it far simpler to to use n4_3.*)

Theorem n5_24 : ∀ P Q : Prop,
  ~((P ∧ Q) ∨ (~P ∧ ~Q)) ↔ ((P ∧ ~Q) ∨ (Q ∧ ~P)).
  Proof. intros P Q.
  specialize n5_22 with P Q.
  intros n5_22a.
  specialize n5_23 with P Q.
  intros n5_23a.
  replace ((P↔Q)↔((P∧ Q) ∨(~P ∧ ~Q))) with
      ((~(P↔Q)↔~((P∧ Q)∨(~P ∧ ~Q)))) in n5_23a.
  replace (~(P↔Q)) with
      (~((P ∧ Q) ∨ (~P ∧ ~Q))) in n5_22a.
  apply n5_22a.
  replace (~((P ∧ Q)∨(~P ∧ ~Q))) with (~(P↔Q)).
```

```
  reflexivity.
  apply EqBi.
  apply n5_23a.
  replace (~(P ↔ Q) ↔ ~(P ∧ Q ∨ ~P ∧ ~Q)) with
      ((P ↔ Q) ↔ P ∧ Q ∨ ~P ∧ ~Q).
  reflexivity.
  specialize Trans4_11 with
      (P↔Q) (P ∧ Q ∨ ~P ∧ ~Q).
  intros Trans4_11a.
  apply EqBi.
  apply Trans4_11a. (*Not cited*)
  Qed.

Theorem n5_25 : ∀ P Q : Prop,
  (P ∨ Q) ↔ ((P → Q) → Q).
  Proof. intros P Q.
  specialize n2_62 with P Q.
  intros n2_62a.
  specialize n2_68 with P Q.
  intros n2_68a.
  Conj n2_62a n2_68a.
  split.
  apply n2_62a.
  apply n2_68a.
  Equiv H.
  apply H.
  apply Equiv4_01.
  Qed.

Theorem n5_3 : ∀ P Q R : Prop,
  ((P ∧ Q) → R) ↔ ((P ∧ Q) → (P ∧ R)).
  Proof. intros P Q R.
  specialize Comp3_43 with (P ∧ Q) P R.
  intros Comp3_43a.
  specialize Exp3_3 with
      (P ∧ Q → P) (P ∧ Q →R) (P ∧ Q → P ∧ R).
  intros Exp3_3a. (*Not cited*)
  MP Exp3_3a Comp3_43a.
  specialize Simp3_26 with P Q.
  intros Simp3_26a.
  MP Exp3_3a Simp3_26a.
  specialize Syll2_05 with (P ∧ Q) (P ∧ R) R.
```

```
  intros Syll2_05a.
  specialize Simp3_27 with P R.
  intros Simp3_27a.
  MP Syll2_05a Simp3_27a.
  clear Comp3_43a. clear Simp3_27a.
      clear Simp3_26a.
  Conj Exp3_3a Syll2_05a.
  split.
  apply Exp3_3a.
  apply Syll2_05a.
  Equiv H.
  apply H.
  apply Equiv4_01.
  Qed.

Theorem n5_31 : ∀ P Q R : Prop,
  (R ∧ (P → Q)) → (P → (Q ∧ R)).
  Proof. intros P Q R.
  specialize Comp3_43 with P Q R.
  intros Comp3_43a.
  specialize Simp2_02 with P R.
  intros Simp2_02a.
  specialize Exp3_3 with
      (P→R) (P→Q) (P→(Q ∧ R)).
  intros Exp3_3a. (*Not cited*)
  specialize n3_22 with (P → R) (P → Q). (*Not cited*)
  intros n3_22a.
  Syll n3_22a Comp3_43a Sa.
  MP Exp3_3a Sa.
  Syll Simp2_02a Exp3_3a Sb.
  specialize Imp3_31 with R (P→Q) (P→(Q ∧ R)).
  intros Imp3_31a. (*Not cited*)
  MP Imp3_31a Sb.
  apply Imp3_31a.
  Qed.

Theorem n5_32 : ∀ P Q R : Prop,
  (P → (Q ↔ R)) ↔ ((P ∧ Q) ↔ (P ∧ R)).
  Proof. intros P Q R.
  specialize n4_76 with P (Q→R) (R→Q).
  intros n4_76a.
  specialize Exp3_3 with P Q R.
```

```
intros Exp3_3a.
specialize Imp3_31 with P Q R.
intros Imp3_31a.
Conj Exp3_3a Imp3_31a.
split.
apply Exp3_3a.
apply Imp3_31a.
Equiv H.
specialize Exp3_3 with P R Q.
intros Exp3_3b.
specialize Imp3_31 with P R Q.
intros Imp3_31b.
Conj Exp3_3b Imp3_31b.
split.
apply Exp3_3b.
apply Imp3_31b.
Equiv H0.
specialize n5_3 with P Q R.
intros n5_3a.
specialize n5_3 with P R Q.
intros n5_3b.
replace (P→Q→R) with (P∧Q→R) in n4_76a.
replace (P∧Q→R) with (P∧Q→P∧R) in n4_76a.
replace (P→R→Q) with (P∧R→Q) in n4_76a.
replace (P∧R→Q) with (P∧R→P∧Q) in n4_76a.
replace ((P∧Q→P∧R)∧(P∧R→P∧Q)) with
    ((P∧Q)↔(P∧R)) in n4_76a.
replace ((P∧Q↔P∧R)↔(P→(Q→R)∧(R→Q))) with
    ((P→(Q→R)∧(R→Q))↔(P∧Q ↔ P∧R)) in n4_76a.
replace ((Q→R)∧(R→Q)) with (Q↔R) in n4_76a.
apply n4_76a.
apply Equiv4_01.
apply EqBi.
specialize n4_21 with
    (P→((Q→R)∧(R→Q))) ((P∧Q)↔(P∧R)).
intros n4_21a.
apply n4_21a. (*to commute the biconditional*)
apply Equiv4_01.
replace (P ∧ R → P ∧ Q) with (P ∧ R → Q).
reflexivity.
apply EqBi.
apply n5_3b.
```

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  apply EqBi.
  apply H0.
  replace (P ∧ Q → P ∧ R) with (P ∧ Q → R).
  reflexivity.
  apply EqBi.
  apply n5_3a.
  apply EqBi.
  apply H.
  apply Equiv4_01.
  apply Equiv4_01.
  Qed.

Theorem n5_33 : ∀ P Q R : Prop,
  (P ∧ (Q → R)) ↔ (P ∧ ((P ∧ Q) → R)).
  Proof. intros P Q R.
    specialize n5_32 with P (Q→R) ((P∧Q)→R).
    intros n5_32a.
    replace
        ((P→(Q→R)↔(P∧Q→R))↔(P∧(Q→R)↔P∧(P∧Q→R)))
        with
        (((P→(Q→R)↔(P∧Q→R))→(P∧(Q→R)↔P∧(P∧Q→R)))
        ∧
        ((P∧(Q→R)↔P∧(P∧Q→R)→(P→(Q→R)↔(P∧Q→R)))))
        in n5_32a.
    specialize Simp3_26 with
        ((P→(Q→R)↔(P∧Q→R))→(P∧(Q→R)↔P∧(P∧Q→R)))
        ((P∧(Q→R)↔P∧(P∧Q→R)→(P→(Q→R)↔(P∧Q→R)))).
    intros Simp3_26a. (*Not cited*)
    MP Simp3_26a n5_32a.
    specialize n4_73 with Q P.
    intros n4_73a.
    specialize n4_84 with Q (Q∧P) R.
    intros n4_84a.
    Syll n4_73a n4_84a Sa.
    replace (Q∧P) with (P∧Q) in Sa.
    MP Simp3_26a Sa.
    apply Simp3_26a.
    apply EqBi.
    specialize n4_3 with P Q.
    intros n4_3a.
    apply n4_3a. (*Not cited*)
    apply Equiv4_01.
```

```
  Qed.

Theorem n5_35 : ∀ P Q R : Prop,
  ((P → Q) ∧ (P → R)) → (P → (Q ↔ R)).
  Proof. intros P Q R.
  specialize Comp3_43 with P Q R.
  intros Comp3_43a.
  specialize n5_1 with Q R.
  intros n5_1a.
  specialize Syll2_05 with P (Q∧R) (Q↔R).
  intros Syll2_05a.
  MP Syll2_05a n5_1a.
  Syll Comp3_43a Syll2_05a Sa.
  apply Sa.
  Qed.

Theorem n5_36 : ∀ P Q : Prop,
  (P ∧ (P ↔ Q)) ↔ (Q ∧ (P ↔ Q)).
  Proof. intros P Q.
  specialize n5_32 with (P↔Q) P Q.
  intros n5_32a.
  specialize n2_08 with (P↔Q).
  intros n2_08a.
  replace (P↔Q→P↔Q) with
      ((P↔Q)∧P↔(P↔Q)∧Q) in n2_08a.
  replace ((P↔Q)∧P) with (P∧(P↔Q)) in n2_08a.
  replace ((P↔Q)∧Q) with (Q∧(P↔Q)) in n2_08a.
  apply n2_08a.
  apply EqBi.
  specialize n4_3 with Q (P↔Q).
  intros n4_3a.
  apply n4_3a.
  apply EqBi.
  specialize n4_3 with P (P↔Q).
  intros n4_3b.
  apply n4_3b.
  replace ((P ↔ Q) ∧ P ↔ (P ↔ Q) ∧ Q) with
      (P ↔ Q → P ↔ Q).
  reflexivity.
  apply EqBi.
  apply n5_32a.
  Qed.
```

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  (*The proof sketch cites Ass3_35 and n4_38, but
      the sketch was indecipherable.*)

Theorem n5_4 : ∀ P Q : Prop,
  (P → (P → Q)) ↔ (P → Q).
  Proof. intros P Q.
  specialize n2_43 with P Q.
  intros n2_43a.
  specialize Simp2_02 with (P) (P→Q).
  intros Simp2_02a.
  Conj n2_43a Simp2_02a.
  split.
  apply n2_43a.
  apply Simp2_02a.
  Equiv H.
  apply H.
  apply Equiv4_01.
  Qed.

Theorem n5_41 : ∀ P Q R : Prop,
  ((P → Q) → (P → R)) ↔ (P → Q → R).
  Proof. intros P Q R.
  specialize n2_86 with P Q R.
  intros n2_86a.
  specialize n2_77 with P Q R.
  intros n2_77a.
  Conj n2_86a n2_77a.
  split.
  apply n2_86a.
  apply n2_77a.
  Equiv H.
  apply H.
  apply Equiv4_01.
  Qed.

Theorem n5_42 : ∀ P Q R : Prop,
  (P → Q → R) ↔ (P → Q → P ∧ R).
  Proof. intros P Q R.
  specialize n5_3 with P Q R.
  intros n5_3a.
  specialize n4_87 with P Q R.
  intros n4_87a.
```

```
  replace ((P∧Q)→R) with (P→Q→R) in n5_3a.
  specialize n4_87 with P Q (P∧R).
  intros n4_87b.
  replace ((P∧Q)→(P∧R)) with
      (P→Q→(P∧R)) in n5_3a.
  apply n5_3a.
  specialize Imp3_31 with P Q (P∧R).
  intros Imp3_31b.
  specialize Exp3_3 with P Q (P∧R).
  intros Exp3_3b.
  Conj Imp3_31b Exp3_3b.
  split.
  apply Imp3_31b.
  apply Exp3_3b.
  Equiv H.
  apply EqBi.
  apply H.
  apply Equiv4_01.
  specialize Imp3_31 with P Q R.
  intros Imp3_31a.
  specialize Exp3_3 with P Q R.
  intros Exp3_3a.
  Conj Imp3_31a Exp3_3.
  split.
  apply Imp3_31a.
  apply Exp3_3a.
  Equiv H.
  apply EqBi.
  apply H.
  apply Equiv4_01.
  Qed.

Theorem n5_44 : ∀ P Q R : Prop,
  (P→Q) → ((P → R) ↔ (P → (Q ∧ R))).
  Proof. intros P Q R.
  specialize n4_76 with P Q R.
  intros n4_76a.
  replace ((P→Q)∧(P→R)↔(P→Q∧R)) with
      (((P→Q)∧(P→R)→(P→Q∧R))
      ∧
      ((P→Q∧R)→(P→Q)∧(P→R))) in n4_76a.
  specialize Simp3_26 with
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```
    ((P→Q)∧(P→R)→(P→Q∧R))
    ((P→Q∧R)→(P→Q)∧(P→R)).
intros Simp3_26a. (*Not cited*)
MP Simp3_26a n4_76a.
specialize Exp3_3 with (P→Q) (P→R) (P→Q∧R).
intros Exp3_3a. (*Not cited*)
MP Exp3_3a Simp3_26a.
specialize Simp3_27 with
    ((P→Q)∧(P→R)→(P→Q∧R))
    ((P→Q∧R)→(P→Q)∧(P→R)).
intros Simp3_27a. (*Not cited*)
MP Simp3_27a n4_76a.
specialize Simp3_26 with (P→R) (P→Q).
intros Simp3_26b.
replace ((P→Q)∧(P→R)) with
    ((P→R)∧(P→Q)) in Simp3_27a.
Syll Simp3_27a Simp3_26b Sa.
specialize Simp2_02 with (P→Q) ((P→Q∧R)→P→R).
intros Simp2_02a. (*Not cited*)
MP Simp2_02a Sa.
clear Sa. clear Simp3_26b. clear Simp3_26a.
    clear n4_76a. clear Simp3_27a.
Conj Exp3_3a Simp2_02a.
split.
apply Exp3_3a.
apply Simp2_02a.
specialize n4_76 with (P→Q)
    ((P→R)→(P→(Q∧R))) ((P→(Q∧R))→(P→R)).
intros n4_76b. (*Second use not indicated*)
replace
    (((P→Q)→(P→R)→P→Q∧R)∧((P→Q)→(P→Q∧R)→P→R))
    with
    ((P→Q)→((P→R)→P→Q∧R)∧((P→Q∧R)→P→R)) in H.
replace (((P→R)→P→Q∧R)∧((P→Q∧R)→P→R)) with
    ((P→R)↔(P→Q∧R)) in H.
apply H.
apply Equiv4_01.
replace ((P→Q)→((P→R)→P→Q∧R)∧((P→Q∧R)→P→R)) with
    (((P→Q)→(P→R)→P→Q∧R)∧((P→Q)→(P→Q∧R)→P→R)).
reflexivity.
apply EqBi.
apply n4_76b.
```

```
  apply EqBi.
  specialize n4_3 with (P→R) (P→Q).
  intros n4_3a.
  apply n4_3a. (*Not cited*)
  apply Equiv4_01.
  Qed.

Theorem n5_5 : ∀ P Q : Prop,
  P → ((P → Q) ↔ Q).
  Proof. intros P Q.
  specialize Ass3_35 with P Q.
  intros Ass3_35a.
  specialize Exp3_3 with P (P→Q) Q.
  intros Exp3_3a.
  MP Exp3_3a Ass3_35a.
  specialize Simp2_02 with P Q.
  intros Simp2_02a.
  specialize Exp3_3 with P Q (P→Q).
  intros Exp3_3b.
  specialize n3_42 with P Q (P→Q). (*Not cited*)
  intros n3_42a.
  MP n3_42a Simp2_02a.
  MP Exp3_3b n3_42a.
  clear n3_42a. clear Simp2_02a. clear Ass3_35a.
  Conj Exp3_3a Exp3_3b.
  split.
  apply Exp3_3a.
  apply Exp3_3b.
  specialize n3_47 with P P ((P→Q)→Q) (Q→(P→Q)).
  intros n3_47a.
  MP n3_47a H.
  replace (P∧P) with P in n3_47a.
  replace (((P→Q)→Q)∧(Q→(P→Q))) with
      ((P→Q)↔Q) in n3_47a.
  apply n3_47a.
  apply Equiv4_01.
  apply EqBi.
  specialize n4_24 with P.
  intros n4_24a. (*Not cited*)
  apply n4_24a.
  Qed.
```

```
Theorem n5_501 : ∀ P Q : Prop,
  P → (Q ↔ (P ↔ Q)).
  Proof. intros P Q.
  specialize n5_1 with P Q.
  intros n5_1a.
  specialize Exp3_3 with P Q (P↔Q).
  intros Exp3_3a.
  MP Exp3_3a n5_1a.
  specialize Ass3_35 with P Q.
  intros Ass3_35a.
  specialize Simp3_26 with (P∧(P→Q)) (Q→P).
  intros Simp3_26a. (*Not cited*)
  Syll Simp3_26a Ass3_35a Sa.
  replace ((P∧(P→Q))∧(Q→P)) with
      (P∧((P→Q)∧(Q→P))) in Sa.
  replace ((P→Q)∧(Q→P)) with (P↔Q) in Sa.
  specialize Exp3_3 with P (P↔Q) Q.
  intros Exp3_3b.
  MP Exp3_3b Sa.
  clear n5_1a. clear Ass3_35a.
      clear Simp3_26a. clear Sa.
  Conj Exp3_3a Exp3_3b.
  split.
  apply Exp3_3a.
  apply Exp3_3b.
  specialize n4_76 with P (Q→(P↔Q)) ((P↔Q)→Q).
  intros n4_76a. (*Not cited*)
  replace ((P→Q→P↔Q)∧(P→P↔Q→Q)) with
      ((P→(Q→P↔Q)∧(P↔Q→Q))) in H.
  replace ((Q→(P↔Q))∧((P↔Q)→Q)) with
      (Q↔(P↔Q)) in H.
  apply H.
  apply Equiv4_01.
  replace (P→(Q→P↔Q)∧(P↔Q→Q)) with
      ((P→Q→P↔Q)∧(P→P↔Q→Q)).
  reflexivity.
  apply EqBi.
  apply n4_76a.
  apply Equiv4_01.
  replace (P∧(P→Q)∧(Q→P)) with
      ((P∧(P→Q))∧(Q→P)).
  reflexivity.
```

```
  apply EqBi.
  specialize n4_32 with P (P→Q) (Q→P).
  intros n4_32a. (*Not cited*)
  apply n4_32a.
  Qed.

Theorem n5_53 : ∀ P Q R S : Prop,
  (((P ∨ Q) ∨ R) → S) ↔ (((P → S) ∧ (Q → S)) ∧ (R → S)).
  Proof. intros P Q R S.
  specialize n4_77 with S (P∨Q) R.
  intros n4_77a.
  specialize n4_77 with S P Q.
  intros n4_77b.
  replace (P ∨ Q → S) with
      ((P→S)∧(Q→S)) in n4_77a.
  replace ((((P→S)∧(Q→S))∧(R→S))↔(((P∨Q)∨R)→S))
      with
      ((((P∨Q)∨R)→S)↔(((P→S)∧(Q→S))∧(R→S)))
      in n4_77a.
  apply n4_77a.
  apply EqBi.
  specialize n4_21 with ((P ∨ Q) ∨ R → S)
      (((P → S) ∧ (Q → S)) ∧ (R → S)).
  intros n4_21a.
  apply n4_21a. (*Not cited*)
  apply EqBi.
  apply n4_77b.
  Qed.

Theorem n5_54 : ∀ P Q : Prop,
  ((P ∧ Q) ↔ P) ∨ ((P ∧ Q) ↔ Q).
  Proof. intros P Q.
  specialize n4_73 with P Q.
  intros n4_73a.
  specialize n4_44 with Q P.
  intros n4_44a.
  specialize Trans2_16 with Q (P↔(P∧Q)).
  intros Trans2_16a.
  MP n4_73a Trans2_16a.
  specialize Trans4_11 with Q (Q∨(P∧Q)).
  intros Trans4_11a.
  replace (Q∧P) with (P∧Q) in n4_44a.
```

```
replace (Q↔Q∨P∧Q) with
    (~Q↔~(Q∨P∧Q)) in n4_44a.
replace (~Q) with (~(Q∨P∧Q)) in Trans2_16a.
replace (~(Q∨P∧Q)) with
    (~Q∧~(P∧Q)) in Trans2_16a.
specialize n5_1 with (~Q) (~(P∧Q)).
intros n5_1a.
Syll Trans2_16a n5_1a Sa.
replace (~(P↔P∧Q)→(~Q↔~(P∧Q))) with
    (~~(P↔P∧Q)∨(~Q↔~(P∧Q))) in Sa.
replace (~~(P↔P∧Q)) with (P↔P∧Q) in Sa.
specialize Trans4_11 with Q (P∧Q).
intros Trans4_11b.
replace (~Q↔~(P∧Q)) with (Q↔(P∧Q)) in Sa.
replace (Q↔(P∧Q)) with ((P∧Q)↔Q) in Sa.
replace (P↔(P∧Q)) with ((P∧Q)↔P) in Sa.
apply Sa.
apply EqBi.
specialize n4_21 with (P∧Q) P.
intros n4_21a. (*Not cited*)
apply n4_21a.
apply EqBi.
specialize n4_21 with (P∧Q) Q.
intros n4_21b. (*Not cited*)
apply n4_21b.
apply EqBi.
apply Trans4_11b.
apply EqBi.
specialize n4_13 with (P ↔ (P∧Q)).
intros n4_13a. (*Not cited*)
apply n4_13a.
replace (~~(P↔P∧Q)∨(~Q↔~(P∧Q))) with
    (~(P↔P∧Q)→~Q↔~(P∧Q)).
reflexivity.
apply Impl1_01. (*Not cited*)
apply EqBi.
specialize n4_56 with Q (P∧Q).
intros n4_56a. (*Not cited*)
apply n4_56a.
replace (~(Q∨P∧Q)) with (~Q).
reflexivity.
apply EqBi.
```

```
  apply n4_44a.
  replace (~Q↔~(Q∨P∧Q)) with (Q↔Q∨P∧Q).
  reflexivity.
  apply EqBi.
  apply Trans4_11a.
  apply EqBi.
  specialize n4_3 with P Q.
  intros n4_3a. (*Not cited*)
  apply n4_3a.
  Qed.

Theorem n5_55 : ∀ P Q : Prop,
  ((P ∨ Q) ↔ P) ∨ ((P ∨ Q) ↔ Q).
  Proof. intros P Q.
  specialize Add1_3 with (P∧Q) (P).
  intros Add1_3a.
  replace ((P∧Q)∨P) with ((P∨P)∧(Q∨P)) in Add1_3a.
  replace (P∨P) with P in Add1_3a.
  replace (Q∨P) with (P∨Q) in Add1_3a.
  specialize n5_1 with P (P∨Q).
  intros n5_1a.
  Syll Add1_3a n5_1a Sa.
  specialize n4_74 with P Q.
  intros n4_74a.
  specialize Trans2_15 with P (Q↔P∨Q).
  intros Trans2_15a. (*Not cited*)
  MP Trans2_15a n4_74a.
  Syll Trans2_15a Sa Sb.
  replace (~(Q↔(P∨Q))→(P↔(P∨Q))) with
      (~~(Q↔(P∨Q))∨(P↔(P∨Q))) in Sb.
  replace (~~(Q↔(P∨Q))) with (Q↔(P∨Q)) in Sb.
  replace (Q↔(P∨Q)) with ((P∨Q)↔Q) in Sb.
  replace (P↔(P∨Q)) with ((P∨Q)↔P) in Sb.
  replace ((P∨Q↔Q)∨(P∨Q↔P)) with
      ((P∨Q↔P)∨(P∨Q↔Q)) in Sb.
  apply Sb.
  apply EqBi.
  specialize n4_31 with (P ∨ Q ↔ P) (P ∨ Q ↔ Q).
  intros n4_31a.  (*Not cited*)
  apply n4_31a.
  apply EqBi.
  specialize n4_21 with (P ∨ Q) P.
```

```
  intros n4_21a. (*Not cited*)
  apply n4_21a.
  apply EqBi.
  specialize n4_21 with (P ∨ Q) Q.
  intros n4_21b. (*Not cited*)
  apply n4_21b.
  apply EqBi.
  specialize n4_13 with (Q ↔ P ∨ Q).
  intros n4_13a. (*Not cited*)
  apply n4_13a.
  replace (~~(Q↔P∨Q)∨(P↔P∨Q)) with
      (~(Q↔P∨Q)→P↔P∨Q).
  reflexivity.
  apply Impl1_01.
  apply EqBi.
  specialize n4_31 with P Q.
  intros n4_31b.
  apply n4_31b.
  apply EqBi.
  specialize n4_25 with P.
  intros n4_25a. (*Not cited*)
  apply n4_25a.
  replace ((P∨P)∧(Q∨P)) with ((P∧Q)∨P).
  reflexivity.
  replace ((P∧Q)∨P) with (P∨(P∧Q)).
  replace (Q∨P) with (P∨Q).
  apply EqBi.
  specialize n4_41 with P P Q.
  intros n4_41a. (*Not cited*)
  apply n4_41a.
  apply EqBi.
  specialize n4_31 with P Q.
  intros n4_31c.
  apply n4_31c.
  apply EqBi.
  specialize n4_31 with P (P ∧ Q).
  intros n4_31d. (*Not cited*)
  apply n4_31d.
  Qed.

Theorem n5_6 : ∀ P Q R : Prop,
  ((P ∧ ~Q) → R) ↔ (P → (Q ∨ R)).
```

```
  Proof. intros P Q R.
  specialize n4_87 with P (~Q) R.
  intros n4_87a.
  specialize n4_64 with Q R.
  intros n4_64a.
  specialize n4_85 with P Q R.
  intros n4_85a.
  replace (((P∧~Q→R)↔(P→~Q→R))↔((~Q→P→R)↔(~Q∧P→R)))
       with
       ((((P∧~Q→R)↔(P→~Q→R))→((~Q→P→R)↔(~Q∧P→R)))
       ∧
       ((((~Q→P→R)↔(~Q∧P→R)))→(((P∧~Q→R)↔(P→~Q→R)))))
       in n4_87a.
  specialize Simp3_27 with
      (((P∧~Q→R)↔(P→~Q→R)→(~Q→P→R)↔(~Q∧P→R)))
      (((~Q→P→R)↔(~Q∧P→R)→(P∧~Q→R)↔(P→~Q→R))).
  intros Simp3_27a.
  MP Simp3_27a n4_87a.
  specialize Imp3_31 with (~Q) P R.
  intros Imp3_31a.
  specialize Exp3_3 with (~Q) P R.
  intros Exp3_3a.
  Conj Imp3_31a Exp3_3a.
  split.
  apply Imp3_31a.
  apply Exp3_3a.
  Equiv H.
  MP Simp3_27a H.
  replace (~Q→R) with (Q∨R) in Simp3_27a.
  apply Simp3_27a.
  replace (Q ∨ R) with (~Q → R).
  reflexivity.
  apply EqBi.
  apply n4_64a.
  apply Equiv4_01.
  apply Equiv4_01.
  Qed.

Theorem n5_61 : ∀ P Q : Prop,
  ((P ∨ Q) ∧ ~Q) ↔ (P ∧ ~Q).
  Proof. intros P Q.
  specialize n4_74 with Q P.
```

```
  intros n4_74a.
  specialize n5_32 with (~Q) P (Q∨P).
  intros n5_32a.
  replace (~Q → P ↔ Q ∨ P) with
      (~Q ∧ P ↔ ~Q ∧ (Q ∨ P)) in n4_74a.
  replace (~Q∧P) with (P∧~Q) in n4_74a.
  replace (~Q∧(Q∨P)) with ((Q∨P)∧~Q) in n4_74a.
  replace (Q∨P) with (P∨Q) in n4_74a.
  replace (P ∧ ~Q ↔ (P ∨ Q) ∧ ~Q) with
      ((P ∨ Q) ∧ ~Q ↔ P ∧ ~Q) in n4_74a.
  apply n4_74a.
  apply EqBi.
  specialize n4_21 with ((P∨Q)∧~Q) (P∧~Q).
  intros n4_21a. (*Not cited*)
  apply n4_21a.
  apply EqBi.
  specialize n4_31 with P Q.
  intros n4_31a. (*Not cited*)
  apply n4_31a.
  apply EqBi.
  specialize n4_3 with (Q∨P) (~Q).
  intros n4_3a. (*Not cited*)
  apply n4_3a.
  apply EqBi.
  specialize n4_3 with P (~Q).
  intros n4_3b. (*Not cited*)
  apply n4_3b.
  replace (~Q ∧ P ↔ ~Q ∧ (Q ∨ P)) with
      (~Q → P ↔ Q ∨ P).
  reflexivity.
  apply EqBi.
  apply n5_32a.
  Qed.

Theorem n5_62 : ∀ P Q : Prop,
  ((P ∧ Q) ∨ ~Q) ↔ (P ∨ ~Q).
  Proof. intros P Q.
  specialize n4_7 with Q P.
  intros n4_7a.
  replace (Q→P) with (~Q∨P) in n4_7a.
  replace (Q→(Q∧P)) with (~Q∨(Q∧P)) in n4_7a.
  replace (~Q∨(Q∧P)) with ((Q∧P)∨~Q) in n4_7a.
```

```
  replace (~Q∨P) with (P∨~Q) in n4_7a.
  replace (Q∧P) with (P∧Q) in n4_7a.
  replace (P ∨ ~Q ↔ P ∧ Q ∨ ~Q) with
      (P ∧ Q ∨ ~Q ↔ P ∨ ~Q) in n4_7a.
  apply n4_7a.
  apply EqBi.
  specialize n4_21 with (P ∧ Q ∨ ~Q) (P ∨ ~Q).
  intros n4_21a. (*Not cited*)
  apply n4_21a.
  apply EqBi.
  specialize n4_3 with P Q.
  intros n4_3a. (*Not cited*)
  apply n4_3a.
  apply EqBi.
  specialize n4_31 with P (~Q).
  intros n4_31a. (*Not cited*)
  apply n4_31a.
  apply EqBi.
  specialize n4_31 with (Q ∧ P) (~Q).
  intros n4_31b. (*Not cited*)
  apply n4_31b.
  replace (~Q ∨ Q ∧ P) with (Q → Q ∧ P).
  reflexivity.
  apply EqBi.
  specialize n4_6 with Q (Q∧P).
  intros n4_6a. (*Not cited*)
  apply n4_6a.
  replace (~Q ∨ P) with (Q → P).
  reflexivity.
  apply EqBi.
  specialize n4_6 with Q P.
  intros n4_6b. (*Not cited*)
  apply n4_6b.
  Qed.

Theorem n5_63 : ∀ P Q : Prop,
  (P ∨ Q) ↔ (P ∨ (~P ∧ Q)).
  Proof. intros P Q.
  specialize n5_62 with Q (~P).
  intros n5_62a.
  replace (~~P) with P in n5_62a.
  replace (Q ∨ P) with (P ∨ Q) in n5_62a.
```

```
  replace ((Q∧~P)∨P) with (P∨(Q∧~P)) in n5_62a.
  replace (P ∨ Q ∧ ~P ↔ P ∨ Q) with
      (P ∨ Q ↔ P ∨ Q ∧ ~P) in n5_62a.
  replace (Q∧~P) with (~P∧Q) in n5_62a.
  apply n5_62a.
  apply EqBi.
  specialize n4_3 with (~P) Q.
  intros n4_3a.
  apply n4_3a. (*Not cited*)
  apply EqBi.
  specialize n4_21 with (P∨Q) (P∨(Q∧~P)).
  intros n4_21a. (*Not cited*)
  apply n4_21a.
  apply EqBi.
  specialize n4_31 with P (Q∧~P).
  intros n4_31a. (*Not cited*)
  apply n4_31a.
  apply EqBi.
  specialize n4_31 with P Q.
  intros n4_31b. (*Not cited*)
  apply n4_31b.
  apply EqBi.
  specialize n4_13 with P.
  intros n4_13a. (*Not cited*)
  apply n4_13a.
  Qed.

Theorem n5_7 : ∀ P Q R : Prop,
  ((P ∨ R) ↔ (Q ∨ R)) ↔ (R ∨ (P ↔ Q)).
  Proof. intros P Q R.
  specialize n5_32 with (~R) (~P) (~Q).
  intros n5_32a. (*Not cited*)
  replace (~R∧~P) with (~(R∨P)) in n5_32a.
  replace (~R∧~Q) with (~(R∨Q)) in n5_32a.
  replace ((~(R∨P))↔(~(R∨Q))) with
      ((R∨P)↔(R∨Q)) in n5_32a.
  replace ((~P)↔(~Q)) with (P↔Q) in n5_32a.
  replace (~R→(P↔Q)) with
      (~~R∨(P↔Q)) in n5_32a.
  replace (~~R) with R in n5_32a.
  replace (R∨P) with (P∨R) in n5_32a.
  replace (R∨Q) with (Q∨R) in n5_32a.
```

```
replace ((R∨(P↔Q))↔(P∨R↔Q∨R)) with
    ((P∨R↔Q∨R)↔(R∨(P↔Q))) in n5_32a.
apply n5_32a. (*Not cited*)
apply EqBi.
specialize n4_21 with ((P∨R)↔(Q∨R)) (R∨(P↔Q)).
intros n4_21a.
apply n4_21a. (*Not cited*)
apply EqBi.
specialize n4_31 with Q R.
intros n4_31a. (*Not cited*)
apply n4_31a.
apply EqBi.
specialize n4_31 with P R.
intros n4_31b. (*Not cited*)
apply n4_31b.
apply EqBi.
specialize n4_13 with R.
intros n4_13a. (*Not cited*)
apply n4_13a.
replace (~~R∨(P↔Q)) with (~R→P↔Q).
reflexivity.
apply Impl1_01. (*Not cited*)
apply EqBi.
specialize Trans4_11 with P Q.
intros Trans4_11a. (*Not cited*)
apply Trans4_11a.
apply EqBi.
specialize Trans4_11 with (R ∨ P) (R ∨ Q).
intros Trans4_11a. (*Not cited*)
apply Trans4_11a.
replace (~(R∨Q)) with (~R∧~Q).
reflexivity.
apply EqBi.
specialize n4_56 with R Q.
intros n4_56a. (*Not cited*)
apply n4_56a.
replace (~(R∨P)) with (~R∧~P).
reflexivity.
apply EqBi.
specialize n4_56 with R P.
intros n4_56b. (*Not cited*)
apply n4_56b.
```

```
  Qed.
  (*The proof sketch was indecipherable, but an
      easy proof was available through n5_32.*)

Theorem n5_71 : ∀ P Q R : Prop,
  (Q → ~R) → (((P ∨ Q) ∧ R) ↔ (P ∧ R)).
  Proof. intros P Q R.
  specialize n4_4 with R P Q.
  intros n4_4a.
  specialize n4_62 with Q R.
  intros n4_62a.
  specialize n4_51 with Q R.
  intros n4_51a.
  replace (~Q∨~R) with (~(Q∧R)) in n4_62a.
  replace ((Q→~R)↔~(Q∧R)) with
      (((Q→~R)→~(Q∧R))
      ∧
      (~(Q∧R)→(Q→~R))) in n4_62a.
  specialize Simp3_26 with
      ((Q→~R)→~(Q∧R)) (~(Q∧R)→(Q→~R)).
  intros Simp3_26a.
  MP Simp3_26a n4_62a.
  specialize n4_74 with (Q∧R) (P∧R).
  intros n4_74a.
  Syll Simp3_26a n4_74a Sa.
  replace (R∧P) with (P∧R) in n4_4a.
  replace (R∧Q) with (Q∧R) in n4_4a.
  replace ((P∧R)∨(Q∧R)) with
      ((Q∧R)∨(P∧R)) in n4_4a.
  replace ((Q∧R)∨(P∧R)) with (R∧(P∨Q)) in Sa.
  replace (R∧(P∨Q)) with ((P∨Q)∧R) in Sa.
  replace ((P∧R)↔((P∨Q)∧R)) with
      (((P∨Q)∧R)↔(P∧R)) in Sa.
  apply Sa.
  apply EqBi.
  specialize n4_21 with ((P∨Q)∧R) (P∧R).
  intros n4_21a. (*Not cited*)
  apply n4_21a.
  apply EqBi.
  specialize n4_3 with (P∨Q) R.
  intros n4_3a.
  apply n4_3a. (*Not cited*)
```

```
  apply EqBi.
  apply n4_4a. (*Not cited*)
  apply EqBi.
  specialize n4_31 with (Q∧R) (P∧R).
  intros n4_31a. (*Not cited*)
  apply n4_31a.
  apply EqBi.
  specialize n4_3 with Q R.
  intros n4_3a. (*Not cited*)
  apply n4_3a.
  apply EqBi.
  specialize n4_3 with P R.
  intros n4_3b. (*Not cited*)
  apply n4_3b.
  apply Equiv4_01.
  apply EqBi.
  apply n4_51a.
  Qed.

Theorem n5_74 : ∀ P Q R : Prop,
  (P → (Q ↔ R)) ↔ ((P → Q) ↔ (P → R)).
  Proof. intros P Q R.
  specialize n5_41 with P Q R.
  intros n5_41a.
  specialize n5_41 with P R Q.
  intros n5_41b.
  Conj n5_41a n5_41b.
  split.
  apply n5_41a.
  apply n5_41b.
  specialize n4_38 with
      ((P→Q)→(P→R)) ((P→R)→(P→Q))
      (P→Q→R) (P→R→Q).
  intros n4_38a.
  MP n4_38a H.
  replace (((P→Q)→(P→R))∧((P→R)→(P→Q))) with
      ((P→Q)↔(P→R)) in n4_38a.
  specialize n4_76 with P (Q→R) (R→Q).
  intros n4_76a.
  replace ((Q→R)∧(R→Q)) with (Q↔R) in n4_76a.
  replace ((P→Q→R)∧(P→R→Q)) with
      (P→(Q↔R)) in n4_38a.
```

```
  replace (((P→Q)↔(P→R))↔(P→Q↔R)) with
      ((P→(Q↔R))↔((P→Q)↔(P→R))) in n4_38a.
  apply n4_38a.
  apply EqBi.
  specialize n4_21 with (P→Q↔R) ((P→Q)↔(P→R)).
  intros n4_21a. (*Not cited*)
  apply n4_21a.
  replace (P→Q↔R) with ((P→Q→R)∧(P→R→Q)).
  reflexivity.
  apply EqBi.
  apply n4_76a.
  apply Equiv4_01.
  apply Equiv4_01.
  Qed.

Theorem n5_75 : ∀ P Q R : Prop,
  ((R → ~Q) ∧ (P ↔ Q ∨ R)) → ((P ∧ ~Q) ↔ R).
  Proof. intros P Q R.
  specialize n5_6 with P Q R.
  intros n5_6a.
  replace ((P∧~Q→R)↔(P→Q∨R)) with
      (((P∧~Q→R)→(P→Q∨R))
      ∧
      ((P→Q∨R)→(P∧~Q→R))) in n5_6a.
  specialize Simp3_27 with
      ((P∧~Q→R)→(P→Q∨R)) ((P→Q∨R)→(P∧~Q→R)).
  intros Simp3_27a.
  MP Simp3_27a n5_6a.
  specialize Simp3_26 with (P→(Q∨R)) ((Q∨R)→P).
  intros Simp3_26a.
  replace ((P→(Q∨R))∧((Q∨R)→P)) with
      (P↔(Q∨R)) in Simp3_26a.
  Syll Simp3_26a Simp3_27a Sa.
  specialize Simp3_27 with (R→~Q) (P↔(Q∨R)).
  intros Simp3_27b.
  Syll Simp3_27b Sa Sb.
  specialize Simp3_27 with (P→(Q∨R)) ((Q∨R)→P).
  intros Simp3_27c.
  replace ((P→(Q∨R))∧((Q∨R)→P)) with
      (P↔(Q∨R)) in Simp3_27c.
  Syll Simp3_27b Simp3_27c Sc.
  specialize n4_77 with P Q R.
```

```
  intros n4_77a.
  replace (Q∨R→P) with ((Q→P)∧(R→P)) in Sc.
  specialize Simp3_27 with (Q→P) (R→P).
  intros Simp3_27d.
  Syll Sa Simp3_27d Sd.
  specialize Simp3_26 with (R→~Q) (P↔(Q∨R)).
  intros Simp3_26b.
  Conj Sd Simp3_26b.
  split.
  apply Sd.
  apply Simp3_26b.
  specialize Comp3_43 with
      ((R→~Q)∧(P↔(Q∨R))) (R→P) (R→~Q).
  intros Comp3_43a.
  MP Comp3_43a H.
  specialize Comp3_43 with R P (~Q).
  intros Comp3_43b.
  Syll Comp3_43a Comp3_43b Se.
  clear n5_6a. clear Simp3_27a. clear Simp3_27b.
      clear Simp3_27c. clear Simp3_27d. clear Simp3_26a.
      clear Simp3_26b. clear Comp3_43a. clear Comp3_43b.
      clear Sa. clear Sc. clear Sd. clear H. clear n4_77a.
  Conj Sb Se.
  split.
  apply Sb.
  apply Se.
  specialize Comp3_43 with
      ((R→~Q)∧(P↔Q∨R)) (P∧~Q→R) (R→P∧~Q).
  intros Comp3_43c.
  MP Comp3_43c H.
  replace ((P∧~Q→R)∧(R→P∧~Q)) with
      (P∧~Q↔R) in Comp3_43c.
  apply Comp3_43c.
  apply Equiv4_01.
  apply EqBi.
  apply n4_77a.
  apply Equiv4_01.
  apply Equiv4_01.
  apply Equiv4_01.
  Qed.

End No5.
```