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Module No3.

Import No1.
Import No2.

Axiom Prod3_01 : ∀ P Q : Prop, (P ∧ Q) = ~(~P ∨ ~Q).

Axiom Abb3_02 : ∀ P Q R : Prop, (P→Q→R)=(P→Q)∧(Q→R).

Theorem Conj3_03 : ∀ P Q : Prop, P → Q → (P∧Q).
(*3.03 is a meta-
theorem allowing one to move from the theoremhood of P and theoremh
ood of Q to the theoremhood of P and Q.*)
Proof. intros P Q.
  specialize n2_11 with (~P∨~Q). intros n2_11a.
  specialize n2_32 with (~P) (~Q) (~(~P ∨ ~Q)). intros n2_32a.
  MP n2_32a n2_11a.
  replace (~(~P∨~Q)) with (P∧Q) in n2_32a.
  replace (~Q ∨ (P∧Q)) with (Q→(P∧Q)) in n2_32a.
  replace (~P ∨ (Q → (P∧Q))) with (P→Q→(P∧Q)) in n2_32a.
  apply n2_32a.
  apply Impl1_01.
  apply Impl1_01.
  apply Prod3_01.
Qed.

Ltac Prod H1 H2 :=
  match goal with
    | [ H1 : ?P, H2 : ?Q |- _ ] =>
      assert (P ∧ Q) by (specialize Conj3_03 with P Q;
        intros Conj3_03; MP Conj3_03 P; MP Conj3_03 Q)
end.
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Theorem n3_1 : ∀ P Q : Prop,
  (P ∧ Q) → ~(~P ∨ ~Q).
Proof. intros P Q.
  replace (~(~P∨~Q)) with (P∧Q).
  specialize n2_08 with (P∧Q). intros n2_08a.
  apply n2_08a.
  apply Prod3_01.
Qed.

Theorem n3_11 : ∀ P Q : Prop,
  ~(~P ∨ ~Q) → (P ∧ Q).
Proof. intros P Q.
  replace (~(~P∨~Q)) with (P∧Q).
  specialize n2_08 with (P∧Q). intros n2_08a.
  apply n2_08a.
  apply Prod3_01.
Qed.

Theorem n3_12 : ∀ P Q : Prop,
  (~P ∨ ~Q) ∨ (P ∧ Q).
Proof. intros P Q.
  specialize n2_11 with (~P∨~Q). intros n2_11a.
  replace (~(~P∨~Q)) with (P∧Q) in n2_11a.
  apply n2_11a.
  apply Prod3_01.
Qed.

Theorem n3_13 : ∀ P Q : Prop,
  ~(P ∧ Q) → (~P ∨ ~Q).
Proof. intros P Q.
  specialize n3_11 with P Q. intros n3_11a.
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  specialize Trans2_15 with (~P∨~Q) (P∧Q). intros Trans2_15a.
  MP Trans2_16a n3_11a.
  apply Trans2_15a.
Qed.

Theorem n3_14 : ∀ P Q : Prop,
  (~P ∨ ~Q) → ~(P ∧ Q).
Proof. intros P Q.
  specialize n3_1 with P Q. intros n3_1a.
  specialize Trans2_16 with (P∧Q) (~(~P∨~Q)). intros Trans2_16a.
  MP Trans2_16a n3_1a.
  specialize n2_12 with (~P∨~Q). intros n2_12a.
  Syll n2_12a Trans2_16a S.
  apply S.
Qed.

Theorem n3_2 : ∀ P Q : Prop,
  P → Q → (P ∧ Q).
Proof. intros P Q.
  specialize n3_12 with P Q. intros n3_12a.
  specialize n2_32 with (~P) (~Q) (P∧Q). intros n2_32a.
  MP n3_32a n3_12a.
  replace (~Q ∨ P ∧ Q) with (Q→P∧Q) in n2_32a.
  replace (~P ∨ (Q → P ∧ Q)) with (P→Q→P∧Q) in n2_32a.
  apply n2_32a.
  apply Impl1_01. apply Impl1_01.
Qed.

Theorem n3_21 : ∀ P Q : Prop,
  Q → P → (P ∧ Q).
Proof. intros P Q.
  specialize n3_2 with P Q. intros n3_2a.
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  specialize Comm2_04 with P Q (P∧Q). intros Comm2_04a.
  MP Comm2_04a n3_2a.
  apply Comm2_04a.
Qed.

Theorem n3_22 : ∀ P Q : Prop,
  (P ∧ Q) → (Q ∧ P).
Proof. intros P Q.
  specialize n3_13 with Q P. intros n3_13a.
  specialize Perm1_4 with (~Q) (~P). intros Perm1_4a.
  Syll n3_13a Perm1_4a Ha.
  specialize n3_14 with P Q. intros n3_14a.
  Syll Ha n3_14a Hb.
  specialize Trans2_17 with (P∧Q) (Q ∧ P). intros Trans2_17a.
  MP Trans2_17a Hb.
  apply Trans2_17a.
Qed.

Theorem n3_24 : ∀ P : Prop,
  ~(P ∧ ~P).
Proof. intros P.
  specialize n2_11 with (~P). intros n2_11a.
  specialize n3_14 with P (~P). intros n3_14a.
  MP n3_14a n2_11a.
  apply n3_14a.
Qed.

Theorem Simp3_26 : ∀ P Q : Prop,
  (P ∧ Q) → P.
Proof. intros P Q.
  specialize n2_02 with Q P. intros n2_02a.
  replace (P→(Q→P)) with (~P∨(Q→P)) in n2_02a.
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  replace (Q→P) with (~Q∨P) in n2_02a.
  specialize n2_31 with (~P) (~Q) P. intros n2_31a.
  MP n2_31a n2_02a.
  specialize n2_53 with (~P∨~Q) P. intros n2_53a.
  MP n2_53a n2_02a.
  replace (~(~P∨~Q)) with (P∧Q) in n2_53a.
  apply n2_53a.
  apply Prod3_01.
  replace (~Q∨P) with (Q→P).
  reflexivity.
  apply Impl1_01.
  replace (~P∨(Q→P)) with (P→Q→P).
  reflexivity.
  apply Impl1_01.
Qed.

Theorem Simp3_27 : ∀ P Q : Prop,
  (P ∧ Q) → Q.
Proof. intros P Q.
  specialize n3_22 with P Q. intros n3_22a.
  specialize Simp3_26 with Q P. intros Simp3_26a.
  Syll n3_22a Simp3_26a S.
  apply S.
Qed.

Theorem Exp3_3 : ∀ P Q R : Prop,
  ((P ∧ Q) → R) → (P → (Q → R)).
Proof. intros P Q R.
  specialize Trans2_15 with (~P∨~Q) R. intros Trans2_15a.
  replace (~R→(~P∨~Q)) with (~R→(P→~Q)) in Trans2_15a.
  specialize Comm2_04 with (~R) P (~Q). intros Comm2_04a.
  Syll Trans2_15a Comm2_04a Sa.
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  specialize Trans2_17 with Q R. intros Trans2_17a.
  specialize  Syll2_05 with P (~R→~Q) (Q→R). intros Syll2_05a.
  MP Syll2_05a Trans2_17a.
  Syll Sa Syll2_05a Sb.
  replace (~(~P∨~Q)) with (P∧Q) in Sb.
  apply Sb.
  apply Prod3_01.
  replace (~P∨~Q) with (P→~Q).
  reflexivity.
  apply Impl1_01.
Qed.

Theorem Imp3_31 : ∀ P Q R : Prop,
  (P → (Q → R)) → (P ∧ Q) → R.
Proof. intros P Q R.
  specialize n2_31 with (~P) (~Q) R. intros n2_31a.
  specialize n2_53 with (~P∨~Q) R. intros n2_53a.
  Syll n2_31a n2_53a S.
  replace (~Q∨R) with (Q→R) in S.
  replace (~P∨(Q→R)) with (P→Q→R) in S.
  replace (~(~P∨~Q)) with (P∧Q) in S.
  apply S.
  apply Prod3_01.
  apply Impl1_01.
  apply Impl1_01.
Qed.

Theorem Syll3_33 : ∀ P Q R : Prop,
  ((P → Q) ∧ (Q → R)) → (P → R).
Proof. intros P Q R.
  specialize Syll2_06 with P Q R. intros Syll2_06a.
  specialize Imp3_31 with (P→Q) (Q→R) (P→R). intros Imp3_31a.
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  MP Imp3_31a Syll2_06a.
  apply Imp3_31a.
Qed.

Theorem Syll3_34 : ∀ P Q R : Prop,
  ((Q → R) ∧ (P → Q)) → (P → R).
Proof. intros P Q R.
  specialize Syll2_05 with P Q R. intros Syll2_05a.
  specialize Imp3_31 with (Q→R) (P→Q) (P→R). intros Imp3_31a.
  MP Imp3_31a Syll2_05a.
  apply Imp3_31a.
Qed.

Theorem Ass3_35 : ∀ P Q : Prop,
  (P ∧ (P → Q)) → Q.
Proof. intros P Q.
  specialize n2_27 with P Q. intros n2_27a.
  specialize Imp3_31 with P (P→Q) Q. intros Imp3_31a.
  MP Imp3_31a n2_27a.
  apply Imp3_31a.
Qed.

Theorem n3_37 : ∀ P Q R : Prop,
  (P ∧ Q → R) → (P ∧ ~R → ~Q).
Proof. intros P Q R.
  specialize Trans2_16 with Q R. intros Trans2_16a.
  specialize Syll2_05 with P (Q→R) (~R→~Q). intros Syll2_05a.
  MP Syll2_05a Trans2_16a.
  specialize Exp3_3 with P Q R. intros Exp3_3a.
  Syll Exp3_3a Syll2_05a Sa.
  specialize Imp3_31 with P (~R) (~Q). intros Imp3_31a.
  Syll Sa Imp3_31a Sb.
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  apply Sb.
Qed.

Theorem n3_4 : ∀ P Q : Prop,
  (P ∧ Q) → P → Q.
Proof. intros P Q.
  specialize n2_51 with P Q. intros n2_51a.
  specialize Trans2_15 with (P→Q) (P→~Q). intros Trans2_15a.
  MP Trans2_15a n2_51a.
  replace (P→~Q) with (~P∨~Q) in Trans2_15a.
  replace (~(~P∨~Q)) with (P∧Q) in Trans2_15a.
  apply Trans2_15a.
  apply Prod3_01.
  replace (~P∨~Q) with (P→~Q).
  reflexivity.
  apply Impl1_01.
Qed.

Theorem n3_41 : ∀ P Q R : Prop,
  (P → R) → (P ∧ Q → R).
Proof. intros P Q R.
  specialize Simp3_26 with P Q. intros Simp3_26a.
  specialize Syll2_06 with (P∧Q) P R. intros Syll2_06a.
  MP Simp3_26a Syll2_06a.
  apply Syll2_06a.
Qed.

Theorem n3_42 : ∀ P Q R : Prop,
  (Q → R) → (P ∧ Q → R).
Proof. intros P Q R.
  specialize Simp3_27 with P Q. intros Simp3_27a.
  specialize Syll2_06 with (P∧Q) Q R. intros Syll2_06a.
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  MP Syll2_05a Simp3_27a.
  apply Syll2_06a.
Qed.

Theorem Comp3_43 : ∀ P Q R : Prop,
  (P → Q) ∧ (P → R) → (P → Q ∧ R).
Proof. intros P Q R.
  specialize n3_2 with Q R. intros n3_2a.
  specialize Syll2_05 with P Q (R→Q∧R). intros Syll2_05a.
  MP Syll2_05a n3_2a.
  specialize n2_77 with P R (Q∧R). intros n2_77a.
  Syll Syll2_05a n2_77a Sa.
  specialize Imp3_31 with (P→Q) (P→R) (P→Q∧R). intros Imp3_31a.
  MP Sa Imp3_31a.
  apply Imp3_31a.
Qed.

Theorem n3_44 : ∀ P Q R : Prop,
  (Q → P) ∧ (R → P) → (Q ∨ R → P).
Proof. intros P Q R.
  specialize Syll3_33 with (~Q) R P. intros Syll3_33a.
  specialize n2_6 with Q P. intros n2_6a.
  Syll Syll3_33a n2_6a Sa.
  specialize Exp3_3 with (~Q→R) (R→P) ((Q→P)→P). intros Exp3_3a.
  MP Exp3_3a Sa.
  specialize Comm2_04 with (R→P) (Q→P) P. intros Comm2_04a.
  Syll Exp3_3a Comm2_04a Sb.
  specialize Imp3_31 with (Q→P) (R→P) P. intros Imp3_31a.
  Syll Sb Imp3_31a Sc.
  specialize Comm2_04 with (~Q→R) ((Q→P)∧(R→P)) P. intros Comm2_0
4b.
  MP Comm2_04b Sc.
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  specialize n2_53 with Q R. intros n2_53a.
  specialize Syll2_06 with (Q∨R) (~Q→R) P. intros Syll2_06a.
  MP Syll2_06a n2_53a.
  Syll Comm2_04b Syll2_06a Sd.
  apply Sd.
Qed.

Theorem Fact3_45 : ∀ P Q R : Prop,
  (P → Q) → (P ∧ R) → (Q ∧ R).
Proof. intros P Q R.
  specialize Syll2_06 with P Q (~R). intros Syll2_06a.
  specialize Trans2_16 with (Q→~R) (P→~R). intros Trans2_16a.
  Syll Syll2_06a Trans2_16a S.
  replace (P→~R) with (~P∨~R) in S.
  replace (Q→~R) with (~Q∨~R) in S.
  replace (~(~P∨~R)) with (P∧R) in S.
  replace (~(~Q∨~R)) with (Q∧R) in S.
  apply S.
  apply Prod3_01.
  apply Prod3_01.
  replace (~Q∨~R) with (Q→~R).
  reflexivity.
  apply Impl1_01.
  replace (~P∨~R) with (P→~R).
  reflexivity.
  apply Impl1_01.
Qed.

Theorem n3_47 : ∀ P Q R S : Prop,
  ((P → R) ∧ (Q → S)) → (P ∧ Q) → R ∧ S.
Proof. intros P Q R S.
  specialize Simp3_26 with (P→R) (Q→S). intros Simp3_26a.
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  specialize Fact3_45 with P R Q. intros Fact3_45a.
  Syll Simp3_26a Fact3_45a Sa.
  specialize n3_22 with R Q. intros n3_22a.
  specialize Syll2_05 with (P∧Q) (R∧Q) (Q∧R). intros Syll2_05a.
  MP Syll2_05a n3_22a.
  Syll Sa Syll2_05a Sb.
  specialize Simp3_27 with (P→R) (Q→S). intros Simp3_27a.
  specialize Fact3_45 with Q S R. intros Fact3_45b.
  Syll Simp3_27a Fact3_45b Sc.
  specialize n3_22 with S R. intros n3_22b.
  specialize Syll2_05 with (Q∧R) (S∧R) (R∧S). intros Syll2_05b.
  MP Syll2_05b n3_22b.
  Syll Sc Syll2_05b Sd.
  specialize n2_83 with ((P→R)∧(Q→S)) (P∧Q) (Q∧R) (R∧S). intros n2_83
a.
  MP n2_83a Sb.
  MP n2_83 Sd.
  apply n2_83a.
Qed.

Theorem n3_48 : ∀ P Q R S : Prop,
  ((P → R) ∧ (Q → S)) → (P ∨ Q) → R ∨ S.
Proof. intros P Q R S.
  specialize Simp3_26 with (P→R) (Q→S). intros Simp3_26a.
  specialize Sum1_6 with Q P R. intros Sum1_6a.
  Syll Simp3_26a Sum1_6a Sa.
  specialize Perm1_4 with P Q. intros Perm1_4a.
  specialize Syll2_06 with (P∨Q) (Q∨P) (Q∨R). intros Syll2_06a.
  MP Syll2_06a Perm1_4a.
  Syll Sa Syll2_06a Sb.
  specialize Simp3_27 with (P→R) (Q→S). intros Simp3_27a.
  specialize Sum1_6 with R Q S. intros Sum1_6b.
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  Syll Simp3_27a Sum1_6b Sc.
  specialize Perm1_4 with Q R. intros Perm1_4b.
  specialize Syll2_06 with (Q∨R) (R∨Q) (R∨S). intros Syll2_06b.
  MP Syll2_06b Perm1_4b.
  Syll Sc Syll2_06a Sd.
  specialize n2_83 with ((P→R)∧(Q→S)) (P∨Q) (Q∨R) (R∨S). intros n2_83
a.
  MP n2_83a Sb.
  MP n2_83a Sd.
  apply n2_83a.
Qed.

End No3.
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