

# *Principia Mathematica's* Propositional Logic in *Coq*

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## Abstract

This file contains the *Coq* code for the *Principia* Rewrite project's encoding of the propositional logic given in \*1 – \*5. The Github repository with this *Coq* file is here: <https://github.com/LogicalAtomist/principia>. To receive updates about the project, visit the *Principia Rewrite* project page: <https://www.principiarewrite.com/>. You can also follow the *Principia* Rewrite project on Twitter: <https://twitter.com/thePMrewrite>.

```
1  Require Import Unicode.Utf8.
2  Require Import Classical_Prop.
3  Require Import ClassicalFacts.
4  Require Import PropExtensionality.
5
6  Module No1.
7
8  Import Unicode.Utf8.
9  Import ClassicalFacts.
10 Import Classical_Prop.
11 Import PropExtensionality.
12
13   (*We first give the axioms of Principia
14   for the propositional calculus in *1.*)
15
16 Theorem Impl1_01 :  $\forall$  P Q : Prop,
17   (P  $\rightarrow$  Q) = ( $\neg$ P  $\vee$  Q).
18 Proof. intros P Q.
19   apply propositional_extensionality.
20   split.
21   apply imply_to_or.
22   apply or_to_imply.
```

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23 Qed.
24   (*This is a notational definition in Principia:
25     It is used to switch between " $\vee$ " and " $\rightarrow$ ".*)
26
27 Theorem MP1_1 :  $\forall$  P Q : Prop,
28   (P  $\rightarrow$  Q)  $\rightarrow$  P  $\rightarrow$  Q. (*Modus ponens*)
29 Proof. intros P Q.
30   intros iff_refl.
31   apply iff_refl.
32 Qed.
33   (*1.11 omitted: it is MP for propositions
34     containing variables. Likewise, omitted
35     the well-formedness rules 1.7, 1.71, 1.72*)
36
37 Theorem Taut1_2 :  $\forall$  P : Prop,
38   P  $\vee$  P  $\rightarrow$  P. (*Tautology*)
39 Proof. intros P.
40   apply imply_and_or.
41   apply iff_refl.
42 Qed.
43
44 Theorem Add1_3 :  $\forall$  P Q : Prop,
45   Q  $\rightarrow$  P  $\vee$  Q. (*Addition*)
46 Proof. intros P Q.
47   apply or_intror.
48 Qed.
49
50 Theorem Perm1_4 :  $\forall$  P Q : Prop,
51   P  $\vee$  Q  $\rightarrow$  Q  $\vee$  P. (*Permutation*)
52 Proof. intros P Q.
53   apply or_comm.
54 Qed.
55
56 Theorem Assoc1_5 :  $\forall$  P Q R : Prop,
57   P  $\vee$  (Q  $\vee$  R)  $\rightarrow$  Q  $\vee$  (P  $\vee$  R). (*Association*)
58 Proof. intros P Q R.
59   specialize or_assoc with P Q R.
60   intros or_assoc1.
61   replace (P $\vee$ Q $\vee$ R) with ((P $\vee$ Q) $\vee$ R).
62   specialize or_comm with P Q.
63   intros or_comm1.
64   replace (P $\vee$ Q) with (Q $\vee$ P).

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65   specialize or_assoc with Q P R.
66   intros or_assoc2.
67   replace ((Q∨P)∨R) with (Q∨P∨R).
68   apply iff_refl.
69   apply propositional_extensionality.
70   apply iff_sym.
71   apply or_assoc2.
72   apply propositional_extensionality.
73   apply or_comm.
74   apply propositional_extensionality.
75   apply or_assoc.
76   Qed.
77
78   Theorem Sum1_6 : ∀ P Q R : Prop,
79     (Q → R) → (P ∨ Q → P ∨ R). (*Summation*)
80   Proof. intros P Q R.
81     specialize imply_and_or2 with Q R P.
82     intros imply_and_or2a.
83     replace (P∨Q) with (Q∨P).
84     replace (P∨R) with (R∨P).
85     apply imply_and_or2a.
86     apply propositional_extensionality.
87     apply or_comm.
88     apply propositional_extensionality.
89     apply or_comm.
90   Qed.
91
92   (*These are all the propositional axioms of Principia.*)
93
94   Ltac MP H1 H2 :=
95     match goal with
96     | [ H1 : ?P → ?Q, H2 : ?P |- _ ] =>
97       specialize (H1 H2)
98     end.
99   (*We give this Ltac "MP" to make proofs more human-
100    readable and to more closely mirror Principia's style.*)
101
102   End No1.
103
104   Module No2.
105
106   Import No1.

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107
108  (*We proceed to the deductions of of Principia.*)
109
110  Theorem Abs2_01 :  $\forall P : \text{Prop},$ 
111     $(P \rightarrow \neg P) \rightarrow \neg P.$ 
112  Proof. intros P.
113    specialize Taut1_2 with  $(\neg P).$ 
114    intros Taut1_2.
115    replace  $(\neg P \vee \neg P)$  with  $(P \rightarrow \neg P)$  in Taut1_2
116      by now rewrite Impl1_01.
117    apply Taut1_2.
118  Qed.
119
120  Theorem Simp2_02 :  $\forall P Q : \text{Prop},$ 
121     $Q \rightarrow (P \rightarrow Q).$ 
122  Proof. intros P Q.
123    specialize Add1_3 with  $(\neg P) Q.$ 
124    intros Add1_3.
125    replace  $(\neg P \vee Q)$  with  $(P \rightarrow Q)$  in Add1_3
126      by now rewrite Impl1_01.
127    apply Add1_3.
128  Qed.
129
130  Theorem Transp2_03 :  $\forall P Q : \text{Prop},$ 
131     $(P \rightarrow \neg Q) \rightarrow (Q \rightarrow \neg P).$ 
132  Proof. intros P Q.
133    specialize Perm1_4 with  $(\neg P) (\neg Q).$ 
134    intros Perm1_4.
135    replace  $(\neg P \vee \neg Q)$  with  $(P \rightarrow \neg Q)$  in Perm1_4
136      by now rewrite Impl1_01.
137    replace  $(\neg Q \vee \neg P)$  with  $(Q \rightarrow \neg P)$  in Perm1_4
138      by now rewrite Impl1_01.
139    apply Perm1_4.
140  Qed.
141
142  Theorem Comm2_04 :  $\forall P Q R : \text{Prop},$ 
143     $(P \rightarrow (Q \rightarrow R)) \rightarrow (Q \rightarrow (P \rightarrow R)).$ 
144  Proof. intros P Q R.
145    specialize Assoc1_5 with  $(\neg P) (\neg Q) R.$ 
146    intros Assoc1_5.
147    replace  $(\neg Q \vee R)$  with  $(Q \rightarrow R)$  in Assoc1_5
148      by now rewrite Impl1_01.

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149   replace ( $\neg P \vee (Q \rightarrow R)$ ) with ( $P \rightarrow (Q \rightarrow R)$ ) in Assoc1_5
150   by now rewrite Impl1_01.
151   replace ( $\neg P \vee R$ ) with ( $P \rightarrow R$ ) in Assoc1_5
152   by now rewrite Impl1_01.
153   replace ( $\neg Q \vee (P \rightarrow R)$ ) with ( $Q \rightarrow (P \rightarrow R)$ ) in Assoc1_5
154   by now rewrite Impl1_01.
155   apply Assoc1_5.
156 Qed.
157
158 Theorem Syll2_05 :  $\forall P Q R : \text{Prop}$ ,
159   ( $Q \rightarrow R$ )  $\rightarrow$  (( $P \rightarrow Q$ )  $\rightarrow$  ( $P \rightarrow R$ )).
160 Proof. intros P Q R.
161   specialize Sum1_6 with ( $\neg P$ ) Q R.
162   intros Sum1_6.
163   replace ( $\neg P \vee Q$ ) with ( $P \rightarrow Q$ ) in Sum1_6
164   by now rewrite Impl1_01.
165   replace ( $\neg P \vee R$ ) with ( $P \rightarrow R$ ) in Sum1_6
166   by now rewrite Impl1_01.
167   apply Sum1_6.
168 Qed.
169
170 Theorem Syll2_06 :  $\forall P Q R : \text{Prop}$ ,
171   ( $P \rightarrow Q$ )  $\rightarrow$  (( $Q \rightarrow R$ )  $\rightarrow$  ( $P \rightarrow R$ )).
172 Proof. intros P Q R.
173   specialize Comm2_04 with ( $Q \rightarrow R$ ) ( $P \rightarrow Q$ ) ( $P \rightarrow R$ ).
174   intros Comm2_04.
175   specialize Syll2_05 with P Q R.
176   intros Syll2_05.
177   MP Comm2_04 Syll2_05.
178   apply Comm2_04.
179 Qed.
180
181 Theorem n2_07 :  $\forall P : \text{Prop}$ ,
182    $P \rightarrow (P \vee P)$ .
183 Proof. intros P.
184   specialize Add1_3 with P P.
185   intros Add1_3.
186   apply Add1_3.
187 Qed.
188
189 Theorem Id2_08 :  $\forall P : \text{Prop}$ ,
190    $P \rightarrow P$ .

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191 Proof. intros P.
192   specialize Syll2_05 with P (P ∨ P) P.
193   intros Syll2_05.
194   specialize Taut1_2 with P.
195   intros Taut1_2.
196   MP Syll2_05 Taut1_2.
197   specialize n2_07 with P.
198   intros n2_07.
199   MP Syll2_05 n2_07.
200   apply Syll2_05.
201 Qed.
202
203 Theorem n2_1 : ∀ P : Prop,
204   (¬P) ∨ P.
205 Proof. intros P.
206   specialize Id2_08 with P.
207   intros Id2_08.
208   replace (P → P) with (¬P ∨ P) in Id2_08
209     by now rewrite Impl1_01.
210   apply Id2_08.
211 Qed.
212
213 Theorem n2_11 : ∀ P : Prop,
214   P ∨ ¬P.
215 Proof. intros P.
216   specialize Perm1_4 with (¬P) P.
217   intros Perm1_4.
218   specialize n2_1 with P.
219   intros n2_1.
220   MP Perm1_4 n2_1.
221   apply Perm1_4.
222 Qed.
223
224 Theorem n2_12 : ∀ P : Prop,
225   P → ¬¬P.
226 Proof. intros P.
227   specialize n2_11 with (¬P).
228   intros n2_11.
229   replace (¬P ∨ ¬¬P) with (P → ¬¬P) in n2_11
230     by now rewrite Impl1_01.
231   apply n2_11.
232 Qed.

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233
234 Theorem n2_13 :  $\forall P : \text{Prop},$ 
235    $P \vee \neg\neg\neg P.$ 
236 Proof. intros P.
237   specialize Sum1_6 with P ( $\neg P$ ) ( $\neg\neg\neg P$ ).
238   intros Sum1_6.
239   specialize n2_12 with ( $\neg P$ ).
240   intros n2_12.
241   MP Sum1_6 n2_12.
242   specialize n2_11 with P.
243   intros n2_11.
244   MP Sum1_6 n2_11.
245   apply Sum1_6.
246 Qed.
247
248 Theorem n2_14 :  $\forall P : \text{Prop},$ 
249    $\neg\neg P \rightarrow P.$ 
250 Proof. intros P.
251   specialize Perm1_4 with P ( $\neg\neg\neg P$ ).
252   intros Perm1_4.
253   specialize n2_13 with P.
254   intros n2_13.
255   MP Perm1_4 n2_13.
256   replace ( $\neg\neg\neg P \vee P$ ) with ( $\neg\neg P \rightarrow P$ ) in Perm1_4
257     by now rewrite Impl1_01.
258   apply Perm1_4.
259 Qed.
260
261 Theorem Transp2_15 :  $\forall P Q : \text{Prop},$ 
262    $(\neg P \rightarrow Q) \rightarrow (\neg Q \rightarrow P).$ 
263 Proof. intros P Q.
264   specialize Syll2_05 with ( $\neg P$ ) Q ( $\neg\neg Q$ ).
265   intros Syll2_05a.
266   specialize n2_12 with Q.
267   intros n2_12.
268   MP Syll2_05a n2_12.
269   specialize Transp2_03 with ( $\neg P$ ) ( $\neg Q$ ).
270   intros Transp2_03.
271   specialize Syll2_05 with ( $\neg Q$ ) ( $\neg\neg P$ ) P.
272   intros Syll2_05b.
273   specialize n2_14 with P.
274   intros n2_14.

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275 MP Syll2_05b n2_14.
276 specialize Syll2_05 with ( $\neg P \rightarrow Q$ ) ( $\neg P \rightarrow \neg\neg Q$ ) ( $\neg Q \rightarrow \neg\neg P$ ).
277 intros Syll2_05c.
278 MP Syll2_05c Transp2_03.
279 MP Syll2_05c Syll2_05a.
280 specialize Syll2_05 with ( $\neg P \rightarrow Q$ ) ( $\neg Q \rightarrow \neg\neg P$ ) ( $\neg Q \rightarrow P$ ).
281 intros Syll2_05d.
282 MP Syll2_05d Syll2_05b.
283 MP Syll2_05d Syll2_05c.
284 apply Syll2_05d.
285 Qed.
286
287 Ltac Syll H1 H2 S :=
288   let S := fresh S in match goal with
289     | [ H1 : ?P  $\rightarrow$  ?Q, H2 : ?Q  $\rightarrow$  ?R |- _ ] =>
290       assert (S : P  $\rightarrow$  R) by (intros p; apply (H2 (H1 p)))
291   end.
292
293 Theorem Transp2_16 :  $\forall$  P Q : Prop,
294   (P  $\rightarrow$  Q)  $\rightarrow$  ( $\neg Q \rightarrow \neg P$ ).
295 Proof. intros P Q.
296   specialize n2_12 with Q.
297   intros n2_12a.
298   specialize Syll2_05 with P Q ( $\neg\neg Q$ ).
299   intros Syll2_05a.
300   specialize Transp2_03 with P ( $\neg Q$ ).
301   intros Transp2_03a.
302   MP n2_12a Syll2_05a.
303   Syll Syll2_05a Transp2_03a S.
304   apply S.
305   Qed.
306
307 Theorem Transp2_17 :  $\forall$  P Q : Prop,
308   ( $\neg Q \rightarrow \neg P$ )  $\rightarrow$  (P  $\rightarrow$  Q).
309 Proof. intros P Q.
310   specialize Transp2_03 with ( $\neg Q$ ) P.
311   intros Transp2_03a.
312   specialize n2_14 with Q.
313   intros n2_14a.
314   specialize Syll2_05 with P ( $\neg\neg Q$ ) Q.
315   intros Syll2_05a.
316   MP n2_14a Syll2_05a.

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317   Syll Transp2_03a Syll2_05a S.
318   apply S.
319   Qed.
320
321   Theorem n2_18 :  $\forall$  P : Prop,
322     ( $\neg$ P  $\rightarrow$  P)  $\rightarrow$  P.
323   Proof. intros P.
324     specialize n2_12 with P.
325     intro n2_12a.
326     specialize Syll2_05 with ( $\neg$ P) P ( $\neg\neg$ P).
327     intro Syll2_05a.
328     MP Syll2_05a n2_12.
329     specialize Abs2_01 with ( $\neg$ P).
330     intros Abs2_01a.
331     Syll Syll2_05a Abs2_01a Sa.
332     specialize n2_14 with P.
333     intros n2_14a.
334     Syll H n2_14a Sb.
335     apply Sb.
336     Qed.
337
338   Theorem n2_2 :  $\forall$  P Q : Prop,
339     P  $\rightarrow$  (P  $\vee$  Q).
340   Proof. intros P Q.
341     specialize Add1_3 with Q P.
342     intros Add1_3a.
343     specialize Perm1_4 with Q P.
344     intros Perm1_4a.
345     Syll Add1_3a Perm1_4a S.
346     apply S.
347     Qed.
348
349   Theorem n2_21 :  $\forall$  P Q : Prop,
350      $\neg$ P  $\rightarrow$  (P  $\rightarrow$  Q).
351   Proof. intros P Q.
352     specialize n2_2 with ( $\neg$ P) Q.
353     intros n2_2a.
354     replace ( $\neg$ P $\vee$ Q) with (P $\rightarrow$ Q) in n2_2a
355       by now rewrite Impl1_01.
356     apply n2_2a.
357     Qed.
358

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359 Theorem n2_24 :  $\forall$  P Q : Prop,
360   P  $\rightarrow$  ( $\neg$ P  $\rightarrow$  Q).
361 Proof. intros P Q.
362   specialize n2_21 with P Q.
363   intros n2_21a.
364   specialize Comm2_04 with ( $\neg$ P) P Q.
365   intros Comm2_04a.
366   apply Comm2_04a.
367   apply n2_21a.
368 Qed.
369
370 Theorem n2_25 :  $\forall$  P Q : Prop,
371   P  $\vee$  ((P  $\vee$  Q)  $\rightarrow$  Q).
372 Proof. intros P Q.
373   specialize n2_1 with (P  $\vee$  Q).
374   intros n2_1a.
375   specialize Assoc1_5 with ( $\neg$ (P $\vee$ Q)) P Q.
376   intros Assoc1_5a.
377   MP Assoc1_5a n2_1a.
378   replace ( $\neg$ (P $\vee$ Q) $\vee$ Q) with (P $\vee$ Q $\rightarrow$ Q) in Assoc1_5a
379     by now rewrite Impl1_01.
380   apply Assoc1_5a.
381 Qed.
382
383 Theorem n2_26 :  $\forall$  P Q : Prop,
384    $\neg$ P  $\vee$  ((P  $\rightarrow$  Q)  $\rightarrow$  Q).
385 Proof. intros P Q.
386   specialize n2_25 with ( $\neg$ P) Q.
387   intros n2_25a.
388   replace ( $\neg$ P $\vee$ Q) with (P $\rightarrow$ Q) in n2_25a
389     by now rewrite Impl1_01.
390   apply n2_25a.
391 Qed.
392
393 Theorem n2_27 :  $\forall$  P Q : Prop,
394   P  $\rightarrow$  ((P  $\rightarrow$  Q)  $\rightarrow$  Q).
395 Proof. intros P Q.
396   specialize n2_26 with P Q.
397   intros n2_26a.
398   replace ( $\neg$ P $\vee$ ((P $\rightarrow$ Q) $\rightarrow$ Q)) with (P $\rightarrow$ (P $\rightarrow$ Q) $\rightarrow$ Q)
399     in n2_26a by now rewrite Impl1_01.
400   apply n2_26a.

```

```

401 Qed.
402
403 Theorem n2_3 :  $\forall P Q R : \text{Prop},$ 
404    $(P \vee (Q \vee R)) \rightarrow (P \vee (R \vee Q)).$ 
405 Proof. intros P Q R.
406   specialize Perm1_4 with Q R.
407   intros Perm1_4a.
408   specialize Sum1_6 with P (Q $\vee$ R) (R $\vee$ Q).
409   intros Sum1_6a.
410   MP Sum1_6a Perm1_4a.
411   apply Sum1_6a.
412 Qed.
413
414 Theorem n2_31 :  $\forall P Q R : \text{Prop},$ 
415    $(P \vee (Q \vee R)) \rightarrow ((P \vee Q) \vee R).$ 
416 Proof. intros P Q R.
417   specialize n2_3 with P Q R.
418   intros n2_3a.
419   specialize Assoc1_5 with P R Q.
420   intros Assoc1_5a.
421   specialize Perm1_4 with R (P $\vee$ Q).
422   intros Perm1_4a.
423   Syll Assoc1_5a Perm1_4a Sa.
424   Syll n2_3a Sa Sb.
425   apply Sb.
426 Qed.
427
428 Theorem n2_32 :  $\forall P Q R : \text{Prop},$ 
429    $((P \vee Q) \vee R) \rightarrow (P \vee (Q \vee R)).$ 
430 Proof. intros P Q R.
431   specialize Perm1_4 with (P $\vee$ Q) R.
432   intros Perm1_4a.
433   specialize Assoc1_5 with R P Q.
434   intros Assoc1_5a.
435   specialize n2_3 with P R Q.
436   intros n2_3a.
437   specialize Syll2_06 with ((P $\vee$ Q) $\vee$ R) (R $\vee$ P $\vee$ Q) (P $\vee$ R $\vee$ Q).
438   intros Syll2_06a.
439   MP Syll2_06a Perm1_4a.
440   MP Syll2_06a Assoc1_5a.
441   specialize Syll2_06 with ((P $\vee$ Q) $\vee$ R) (P $\vee$ R $\vee$ Q) (P $\vee$ Q $\vee$ R).
442   intros Syll2_06b.

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443   MP Syll2_06b Syll2_06a.
444   MP Syll2_06b n2_3a.
445   apply Syll2_06b.
446   Qed.
447
448   Theorem Abb2_33 :  $\forall$  P Q R : Prop,
449     (P  $\vee$  Q  $\vee$  R) = ((P  $\vee$  Q)  $\vee$  R).
450   Proof. intros P Q R.
451     apply propositional_extensionality.
452     split.
453     specialize n2_31 with P Q R.
454     intros n2_31.
455     apply n2_31.
456     specialize n2_32 with P Q R.
457     intros n2_32.
458     apply n2_32.
459   Qed.
460   (*This definition makes the default left association.
461     The default in Coq is right association.*)
462
463   Theorem n2_36 :  $\forall$  P Q R : Prop,
464     (Q  $\rightarrow$  R)  $\rightarrow$  ((P  $\vee$  Q)  $\rightarrow$  (R  $\vee$  P)).
465   Proof. intros P Q R.
466     specialize Perm1_4 with P R.
467     intros Perm1_4a.
468     specialize Syll2_05 with (P $\vee$ Q) (P $\vee$ R) (R $\vee$ P).
469     intros Syll2_05a.
470     MP Syll2_05a Perm1_4a.
471     specialize Sum1_6 with P Q R.
472     intros Sum1_6a.
473     Syll Sum1_6a Syll2_05a S.
474     apply S.
475   Qed.
476
477   Theorem n2_37 :  $\forall$  P Q R : Prop,
478     (Q  $\rightarrow$  R)  $\rightarrow$  ((Q  $\vee$  P)  $\rightarrow$  (P  $\vee$  R)).
479   Proof. intros P Q R.
480     specialize Perm1_4 with Q P.
481     intros Perm1_4a.
482     specialize Syll2_06 with (Q $\vee$ P) (P $\vee$ Q) (P $\vee$ R).
483     intros Syll2_06a.
484     MP Syll2_06a Perm1_4a.

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485     specialize Sum1_6 with P Q R.
486     intros Sum1_6a.
487     Syll Sum1_6a Syll12_06a S.
488     apply S.
489     Qed.
490
491     Theorem n2_38 :  $\forall P Q R : \text{Prop},$ 
492        $(Q \rightarrow R) \rightarrow ((Q \vee P) \rightarrow (R \vee P)).$ 
493     Proof. intros P Q R.
494       specialize Perm1_4 with P R.
495       intros Perm1_4a.
496       specialize Syll12_05 with  $(Q \vee P) (P \vee R) (R \vee P).$ 
497       intros Syll12_05a.
498       MP Syll12_05a Perm1_4a.
499       specialize Perm1_4 with Q P.
500       intros Perm1_4b.
501       specialize Syll12_06 with  $(Q \vee P) (P \vee Q) (P \vee R).$ 
502       intros Syll12_06a.
503       MP Syll12_06a Perm1_4b.
504       Syll Syll12_06a Syll12_05a H.
505       specialize Sum1_6 with P Q R.
506       intros Sum1_6a.
507       Syll Sum1_6a H S.
508       apply S.
509       Qed.
510
511     Theorem n2_4 :  $\forall P Q : \text{Prop},$ 
512        $(P \vee (P \vee Q)) \rightarrow (P \vee Q).$ 
513     Proof. intros P Q.
514       specialize n2_31 with P P Q.
515       intros n2_31a.
516       specialize Taut1_2 with P.
517       intros Taut1_2a.
518       specialize n2_38 with Q  $(P \vee P)$  P.
519       intros n2_38a.
520       MP n2_38a Taut1_2a.
521       Syll n2_31a n2_38a S.
522       apply S.
523       Qed.
524
525     Theorem n2_41 :  $\forall P Q : \text{Prop},$ 
526        $(Q \vee (P \vee Q)) \rightarrow (P \vee Q).$ 

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527 Proof. intros P Q.
528   specialize Assoc1_5 with Q P Q.
529   intros Assoc1_5a.
530   specialize Taut1_2 with Q.
531   intros Taut1_2a.
532   specialize Sum1_6 with P (Q∨Q) Q.
533   intros Sum1_6a.
534   MP Sum1_6a Taut1_2a.
535   Syll Assoc1_5a Sum1_6a S.
536   apply S.
537 Qed.
538
539 Theorem n2_42 : ∀ P Q : Prop,
540   (¬P ∨ (P → Q)) → (P → Q).
541 Proof. intros P Q.
542   specialize n2_4 with (¬P) Q.
543   intros n2_4a.
544   replace (¬P∨Q) with (P→Q) in n2_4a
545     by now rewrite Impl1_01.
546   apply n2_4a.
547 Qed.
548
549 Theorem n2_43 : ∀ P Q : Prop,
550   (P → (P → Q)) → (P → Q).
551 Proof. intros P Q.
552   specialize n2_42 with P Q.
553   intros n2_42a.
554   replace (¬P ∨ (P→Q)) with (P→(P→Q))
555     in n2_42a by now rewrite Impl1_01.
556   apply n2_42a.
557 Qed.
558
559 Theorem n2_45 : ∀ P Q : Prop,
560   ¬(P ∨ Q) → ¬P.
561 Proof. intros P Q.
562   specialize n2_2 with P Q.
563   intros n2_2a.
564   specialize Transp2_16 with P (P∨Q).
565   intros Transp2_16a.
566   MP n2_2 Transp2_16a.
567   apply Transp2_16a.
568 Qed.

```

```

569
570 Theorem n2_46 :  $\forall P Q : \text{Prop},$ 
571    $\neg(P \vee Q) \rightarrow \neg Q.$ 
572 Proof. intros P Q.
573   specialize Add1_3 with P Q.
574   intros Add1_3a.
575   specialize Transp2_16 with Q (P $\vee$ Q).
576   intros Transp2_16a.
577   MP Add1_3a Transp2_16a.
578   apply Transp2_16a.
579 Qed.
580
581 Theorem n2_47 :  $\forall P Q : \text{Prop},$ 
582    $\neg(P \vee Q) \rightarrow (\neg P \vee Q).$ 
583 Proof. intros P Q.
584   specialize n2_45 with P Q.
585   intros n2_45a.
586   specialize n2_2 with ( $\neg P$ ) Q.
587   intros n2_2a.
588   Syll n2_45a n2_2a S.
589   apply S.
590 Qed.
591
592 Theorem n2_48 :  $\forall P Q : \text{Prop},$ 
593    $\neg(P \vee Q) \rightarrow (P \vee \neg Q).$ 
594 Proof. intros P Q.
595   specialize n2_46 with P Q.
596   intros n2_46a.
597   specialize Add1_3 with P ( $\neg Q$ ).
598   intros Add1_3a.
599   Syll n2_46a Add1_3a S.
600   apply S.
601 Qed.
602
603 Theorem n2_49 :  $\forall P Q : \text{Prop},$ 
604    $\neg(P \vee Q) \rightarrow (\neg P \vee \neg Q).$ 
605 Proof. intros P Q.
606   specialize n2_45 with P Q.
607   intros n2_45a.
608   specialize n2_2 with ( $\neg P$ ) ( $\neg Q$ ).
609   intros n2_2a.
610   Syll n2_45a n2_2a S.

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611     apply S.
612 Qed.
613
614 Theorem n2_5 :  $\forall P Q : \text{Prop},$ 
615    $\neg(P \rightarrow Q) \rightarrow (\neg P \rightarrow Q).$ 
616 Proof. intros P Q.
617   specialize n2_47 with ( $\neg P$ ) Q.
618   intros n2_47a.
619   replace ( $\neg P \vee Q$ ) with ( $P \rightarrow Q$ ) in n2_47a
620     by now rewrite Impl1_01.
621   replace ( $\neg \neg P \vee Q$ ) with ( $\neg P \rightarrow Q$ ) in n2_47a
622     by now rewrite Impl1_01.
623   apply n2_47a.
624 Qed.
625
626 Theorem n2_51 :  $\forall P Q : \text{Prop},$ 
627    $\neg(P \rightarrow Q) \rightarrow (P \rightarrow \neg Q).$ 
628 Proof. intros P Q.
629   specialize n2_48 with ( $\neg P$ ) Q.
630   intros n2_48a.
631   replace ( $\neg P \vee Q$ ) with ( $P \rightarrow Q$ ) in n2_48a
632     by now rewrite Impl1_01.
633   replace ( $\neg P \vee \neg Q$ ) with ( $P \rightarrow \neg Q$ ) in n2_48a
634     by now rewrite Impl1_01.
635   apply n2_48a.
636 Qed.
637
638 Theorem n2_52 :  $\forall P Q : \text{Prop},$ 
639    $\neg(P \rightarrow Q) \rightarrow (\neg P \rightarrow \neg Q).$ 
640 Proof. intros P Q.
641   specialize n2_49 with ( $\neg P$ ) Q.
642   intros n2_49a.
643   replace ( $\neg P \vee Q$ ) with ( $P \rightarrow Q$ ) in n2_49a
644     by now rewrite Impl1_01.
645   replace ( $\neg \neg P \vee \neg Q$ ) with ( $\neg P \rightarrow \neg Q$ ) in n2_49a
646     by now rewrite Impl1_01.
647   apply n2_49a.
648 Qed.
649
650 Theorem n2_521 :  $\forall P Q : \text{Prop},$ 
651    $\neg(P \rightarrow Q) \rightarrow (Q \rightarrow P).$ 
652 Proof. intros P Q.

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653   specialize n2_52 with P Q.
654   intros n2_52a.
655   specialize Transp2_17 with Q P.
656   intros Transp2_17a.
657   Syll n2_52a Transp2_17a S.
658   apply S.
659   Qed.
660
661   Theorem n2_53 :  $\forall P Q : \text{Prop},$ 
662      $(P \vee Q) \rightarrow (\neg P \rightarrow Q).$ 
663   Proof. intros P Q.
664     specialize n2_12 with P.
665     intros n2_12a.
666     specialize n2_38 with Q P ( $\neg\neg P$ ).
667     intros n2_38a.
668     MP n2_38a n2_12a.
669     replace ( $\neg\neg P \vee Q$ ) with ( $\neg P \rightarrow Q$ ) in n2_38a
670       by now rewrite Impl1_01.
671     apply n2_38a.
672     Qed.
673
674   Theorem n2_54 :  $\forall P Q : \text{Prop},$ 
675      $(\neg P \rightarrow Q) \rightarrow (P \vee Q).$ 
676   Proof. intros P Q.
677     specialize n2_14 with P.
678     intros n2_14a.
679     specialize n2_38 with Q ( $\neg\neg P$ ) P.
680     intros n2_38a.
681     MP n2_38a n2_14a.
682     replace ( $\neg\neg P \vee Q$ ) with ( $\neg P \rightarrow Q$ ) in n2_38a
683       by now rewrite Impl1_01.
684     apply n2_38a.
685     Qed.
686
687   Theorem n2_55 :  $\forall P Q : \text{Prop},$ 
688      $\neg P \rightarrow ((P \vee Q) \rightarrow Q).$ 
689   Proof. intros P Q.
690     specialize n2_53 with P Q.
691     intros n2_53a.
692     specialize Comm2_04 with  $(P \vee Q) (\neg P) Q.$ 
693     intros Comm2_04a.
694     MP n2_53a Comm2_04a.

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695     apply Comm2_04a.
696 Qed.
697
698 Theorem n2_56 :  $\forall P Q : \text{Prop},$ 
699    $\neg Q \rightarrow ((P \vee Q) \rightarrow P).$ 
700 Proof. intros P Q.
701   specialize n2_55 with Q P.
702   intros n2_55a.
703   specialize Perm1_4 with P Q.
704   intros Perm1_4a.
705   specialize Syll2_06 with  $(P \vee Q)$   $(Q \vee P)$  P.
706   intros Syll2_06a.
707   MP Syll2_06a Perm1_4a.
708   Syll n2_55a Syll2_06a Sa.
709   apply Sa.
710 Qed.
711
712 Theorem n2_6 :  $\forall P Q : \text{Prop},$ 
713    $(\neg P \rightarrow Q) \rightarrow ((P \rightarrow Q) \rightarrow Q).$ 
714 Proof. intros P Q.
715   specialize n2_38 with Q  $(\neg P)$  Q.
716   intros n2_38a.
717   specialize Taut1_2 with Q.
718   intros Taut1_2a.
719   specialize Syll2_05 with  $(\neg P \vee Q)$   $(Q \vee Q)$  Q.
720   intros Syll2_05a.
721   MP Syll2_05a Taut1_2a.
722   Syll n2_38a Syll2_05a S.
723   replace  $(\neg P \vee Q)$  with  $(P \rightarrow Q)$  in S
724     by now rewrite Impl1_01.
725   apply S.
726 Qed.
727
728 Theorem n2_61 :  $\forall P Q : \text{Prop},$ 
729    $(P \rightarrow Q) \rightarrow ((\neg P \rightarrow Q) \rightarrow Q).$ 
730 Proof. intros P Q.
731   specialize n2_6 with P Q.
732   intros n2_6a.
733   specialize Comm2_04 with  $(\neg P \rightarrow Q)$   $(P \rightarrow Q)$  Q.
734   intros Comm2_04a.
735   MP Comm2_04a n2_6a.
736   apply Comm2_04a.

```

```

737 Qed.
738
739 Theorem n2_62 :  $\forall P Q : \text{Prop},$ 
740    $(P \vee Q) \rightarrow ((P \rightarrow Q) \rightarrow Q).$ 
741 Proof. intros P Q.
742   specialize n2_53 with P Q.
743   intros n2_53a.
744   specialize n2_6 with P Q.
745   intros n2_6a.
746   Syll n2_53a n2_6a S.
747   apply S.
748 Qed.
749
750 Theorem n2_621 :  $\forall P Q : \text{Prop},$ 
751    $(P \rightarrow Q) \rightarrow ((P \vee Q) \rightarrow Q).$ 
752 Proof. intros P Q.
753   specialize n2_62 with P Q.
754   intros n2_62a.
755   specialize Comm2_04 with  $(P \vee Q) (P \rightarrow Q) Q.$ 
756   intros Comm2_04a.
757   MP Comm2_04a n2_62a.
758   apply Comm2_04a.
759 Qed.
760
761 Theorem n2_63 :  $\forall P Q : \text{Prop},$ 
762    $(P \vee Q) \rightarrow ((\neg P \vee Q) \rightarrow Q).$ 
763 Proof. intros P Q.
764   specialize n2_62 with P Q.
765   intros n2_62a.
766   replace  $(P \rightarrow Q)$  with  $(\neg P \vee Q)$  in n2_62a
767     by now rewrite Impl1_01.
768   apply n2_62a.
769 Qed.
770
771 Theorem n2_64 :  $\forall P Q : \text{Prop},$ 
772    $(P \vee Q) \rightarrow ((P \vee \neg Q) \rightarrow P).$ 
773 Proof. intros P Q.
774   specialize n2_63 with Q P.
775   intros n2_63a.
776   specialize Perm1_4 with P Q.
777   intros Perm1_4a.
778   Syll n2_63a Perm1_4a Ha.

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```

779   specialize Syll2_06 with (P $\vee$  $\neg$ Q) ( $\neg$ Q $\vee$ P) P.
780   intros Syll2_06a.
781   specialize Perm1_4 with P ( $\neg$ Q).
782   intros Perm1_4b.
783   MP Syll2_06a Perm1_4b.
784   Syll Syll2_06a Ha S.
785   apply S.
786   Qed.
787
788   Theorem n2_65 :  $\forall$  P Q : Prop,
789     (P  $\rightarrow$  Q)  $\rightarrow$  ((P  $\rightarrow$   $\neg$ Q)  $\rightarrow$   $\neg$ P).
790   Proof. intros P Q.
791     specialize n2_64 with ( $\neg$ P) Q.
792     intros n2_64a.
793     replace ( $\neg$ P $\vee$ Q) with (P $\rightarrow$ Q) in n2_64a
794       by now rewrite Impl1_01.
795     replace ( $\neg$ P $\vee$  $\neg$ Q) with (P $\rightarrow$  $\neg$ Q) in n2_64a
796       by now rewrite Impl1_01.
797     apply n2_64a.
798     Qed.
799
800   Theorem n2_67 :  $\forall$  P Q : Prop,
801     ((P  $\vee$  Q)  $\rightarrow$  Q)  $\rightarrow$  (P  $\rightarrow$  Q).
802   Proof. intros P Q.
803     specialize n2_54 with P Q.
804     intros n2_54a.
805     specialize Syll2_06 with ( $\neg$ P $\rightarrow$ Q) (P $\vee$ Q) Q.
806     intros Syll2_06a.
807     MP Syll2_06a n2_54a.
808     specialize n2_24 with P Q.
809     intros n2_24.
810     specialize Syll2_06 with P ( $\neg$ P $\rightarrow$ Q) Q.
811     intros Syll2_06b.
812     MP Syll2_06b n2_24a.
813     Syll Syll2_06b Syll2_06a S.
814     apply S.
815     Qed.
816
817   Theorem n2_68 :  $\forall$  P Q : Prop,
818     ((P  $\rightarrow$  Q)  $\rightarrow$  Q)  $\rightarrow$  (P  $\vee$  Q).
819   Proof. intros P Q.
820     specialize n2_67 with ( $\neg$ P) Q.

```

```

821   intros n2_67a.
822   replace ( $\neg P \vee Q$ ) with ( $P \rightarrow Q$ ) in n2_67a
823   by now rewrite Impl1_01.
824   specialize n2_54 with P Q.
825   intros n2_54a.
826   Syll n2_67a n2_54a S.
827   apply S.
828   Qed.
829
830   Theorem n2_69 :  $\forall P Q : \text{Prop}$ ,
831     ( $(P \rightarrow Q) \rightarrow Q$ )  $\rightarrow$  ( $(Q \rightarrow P) \rightarrow P$ ).
832   Proof. intros P Q.
833     specialize n2_68 with P Q.
834     intros n2_68a.
835     specialize Perm1_4 with P Q.
836     intros Perm1_4a.
837     Syll n2_68a Perm1_4a Sa.
838     specialize n2_62 with Q P.
839     intros n2_62a.
840     Syll Sa n2_62a Sb.
841     apply Sb.
842     Qed.
843
844   Theorem n2_73 :  $\forall P Q R : \text{Prop}$ ,
845     ( $P \rightarrow Q$ )  $\rightarrow$  ( $((P \vee Q) \vee R) \rightarrow (Q \vee R)$ ).
846   Proof. intros P Q R.
847     specialize n2_621 with P Q.
848     intros n2_621a.
849     specialize n2_38 with R ( $P \vee Q$ ) Q.
850     intros n2_38a.
851     Syll n2_621a n2_38a S.
852     apply S.
853     Qed.
854
855   Theorem n2_74 :  $\forall P Q R : \text{Prop}$ ,
856     ( $Q \rightarrow P$ )  $\rightarrow$  ( $((P \vee Q) \vee R) \rightarrow (P \vee R)$ ).
857   Proof. intros P Q R.
858     specialize n2_73 with Q P R.
859     intros n2_73a.
860     specialize Assoc1_5 with P Q R.
861     intros Assoc1_5a.
862     specialize n2_31 with Q P R.

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863   intros n2_31a. (*not cited*)
864   Syll Assoc1_5a n2_31a Sa.
865   specialize n2_32 with P Q R.
866   intros n2_32a. (*not cited*)
867   Syll n2_32a Sa Sb.
868   specialize Syll2_06 with ((P∨Q)∨R) ((Q∨P)∨R) (P∨R).
869   intros Syll2_06a.
870   MP Syll2_06a Sb.
871   Syll n2_73a Syll2_05a H.
872   apply H.
873   Qed.
874
875   Theorem n2_75 : ∀ P Q R : Prop,
876     (P ∨ Q) → ((P ∨ (Q → R)) → (P ∨ R)).
877   Proof. intros P Q R.
878     specialize n2_74 with P (¬Q) R.
879     intros n2_74a.
880     specialize n2_53 with Q P.
881     intros n2_53a.
882     Syll n2_53a n2_74a Sa.
883     specialize n2_31 with P (¬Q) R.
884     intros n2_31a.
885     specialize Syll2_06 with (P∨(¬Q)∨R)((P∨(¬Q))∨R) (P∨R).
886     intros Syll2_06a.
887     MP Syll2_06a n2_31a.
888     Syll Sa Syll2_06a Sb.
889     specialize Perm1_4 with P Q.
890     intros Perm1_4a. (*not cited*)
891     Syll Perm1_4a Sb Sc.
892     replace (¬Q∨R) with (Q→R) in Sc
893       by now rewrite Impl1_01.
894     apply Sc.
895     Qed.
896
897   Theorem n2_76 : ∀ P Q R : Prop,
898     (P ∨ (Q → R)) → ((P ∨ Q) → (P ∨ R)).
899   Proof. intros P Q R.
900     specialize n2_75 with P Q R.
901     intros n2_75a.
902     specialize Comm2_04 with (P∨Q) (P∨(Q→R)) (P∨R).
903     intros Comm2_04a.
904     apply Comm2_04a.

```

```

905     apply n2_75a.
906 Qed.
907
908 Theorem n2_77 :  $\forall P Q R : \text{Prop},$ 
909    $(P \rightarrow (Q \rightarrow R)) \rightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R)).$ 
910 Proof. intros P Q R.
911   specialize n2_76 with ( $\neg P$ ) Q R.
912   intros n2_76a.
913   replace ( $\neg P \vee (Q \rightarrow R)$ ) with  $(P \rightarrow Q \rightarrow R)$  in n2_76a
914     by now rewrite Impl1_01.
915   replace ( $\neg P \vee Q$ ) with  $(P \rightarrow Q)$  in n2_76a
916     by now rewrite Impl1_01.
917   replace ( $\neg P \vee R$ ) with  $(P \rightarrow R)$  in n2_76a
918     by now rewrite Impl1_01.
919   apply n2_76a.
920 Qed.
921
922 Theorem n2_8 :  $\forall Q R S : \text{Prop},$ 
923    $(Q \vee R) \rightarrow ((\neg R \vee S) \rightarrow (Q \vee S)).$ 
924 Proof. intros Q R S.
925   specialize n2_53 with R Q.
926   intros n2_53a.
927   specialize Perm1_4 with Q R.
928   intros Perm1_4a.
929   Syll Perm1_4a n2_53a Ha.
930   specialize n2_38 with S ( $\neg R$ ) Q.
931   intros n2_38a.
932   Syll H n2_38a Hb.
933   apply Hb.
934 Qed.
935
936 Theorem n2_81 :  $\forall P Q R S : \text{Prop},$ 
937    $(Q \rightarrow (R \rightarrow S)) \rightarrow ((P \vee Q) \rightarrow ((P \vee R) \rightarrow (P \vee S))).$ 
938 Proof. intros P Q R S.
939   specialize Sum1_6 with P Q  $(R \rightarrow S)$ .
940   intros Sum1_6a.
941   specialize n2_76 with P R S.
942   intros n2_76a.
943   specialize Syll2_05 with  $(P \vee Q)$   $(P \vee (R \rightarrow S))$   $((P \vee R) \rightarrow (P \vee S)).$ 
944   intros Syll2_05a.
945   MP Syll2_05a n2_76a.
946   Syll Sum1_6a Syll2_05a H.

```

```

947   apply H.
948 Qed.
949
950 Theorem n2_82 :  $\forall P Q R S : \text{Prop},$ 
951    $(P \vee Q \vee R) \rightarrow ((P \vee \neg R \vee S) \rightarrow (P \vee Q \vee S)).$ 
952 Proof. intros P Q R S.
953   specialize n2_8 with Q R S.
954   intros n2_8a.
955   specialize n2_81 with P (Q $\vee$ R) ( $\neg$ R $\vee$ S) (Q $\vee$ S).
956   intros n2_81a.
957   MP n2_81a n2_8a.
958   apply n2_81a.
959 Qed.
960
961 Theorem n2_83 :  $\forall P Q R S : \text{Prop},$ 
962    $(P \rightarrow (Q \rightarrow R)) \rightarrow ((P \rightarrow (R \rightarrow S)) \rightarrow (P \rightarrow (Q \rightarrow S))).$ 
963 Proof. intros P Q R S.
964   specialize n2_82 with ( $\neg$ P) ( $\neg$ Q) R S.
965   intros n2_82a.
966   replace ( $\neg$ Q $\vee$ R) with (Q $\rightarrow$ R) in n2_82a
967     by now rewrite Impl1_01.
968   replace ( $\neg$ P $\vee$ (Q $\rightarrow$ R)) with (P $\rightarrow$ Q $\rightarrow$ R) in n2_82a
969     by now rewrite Impl1_01.
970   replace ( $\neg$ R $\vee$ S) with (R $\rightarrow$ S) in n2_82a
971     by now rewrite Impl1_01.
972   replace ( $\neg$ P $\vee$ (R $\rightarrow$ S)) with (P $\rightarrow$ R $\rightarrow$ S) in n2_82a
973     by now rewrite Impl1_01.
974   replace ( $\neg$ Q $\vee$ S) with (Q $\rightarrow$ S) in n2_82a
975     by now rewrite Impl1_01.
976   replace ( $\neg$ Q $\vee$ S) with (Q $\rightarrow$ S) in n2_82a
977     by now rewrite Impl1_01.
978   replace ( $\neg$ P $\vee$ (Q $\rightarrow$ S)) with (P $\rightarrow$ Q $\rightarrow$ S) in n2_82a
979     by now rewrite Impl1_01.
980   apply n2_82a.
981 Qed.
982
983 Theorem n2_85 :  $\forall P Q R : \text{Prop},$ 
984    $((P \vee Q) \rightarrow (P \vee R)) \rightarrow (P \vee (Q \rightarrow R)).$ 
985 Proof. intros P Q R.
986   specialize Add1_3 with P Q.
987   intros Add1_3a.
988   specialize Syll2_06 with Q (P $\vee$ Q) R.

```



```

989   intros Syll2_06a.
990   MP Syll2_06a Add1_3a.
991   specialize n2_55 with P R.
992   intros n2_55a.
993   specialize Syll2_05 with (P∨Q) (P∨R) R.
994   intros Syll2_05a.
995   Syll n2_55a Syll2_05a Ha.
996   specialize n2_83 with (¬P) ((P∨Q)→(P∨R)) ((P∨Q)→R) (Q→R).
997   intros n2_83a.
998   MP n2_83a Ha.
999   specialize Comm2_04 with (¬P) (P∨Q→P∨R) (Q→R).
1000  intros Comm2_04a.
1001  Syll Ha Comm2_04a Hb.
1002  specialize n2_54 with P (Q→R).
1003  intros n2_54a.
1004  specialize Simp2_02 with (¬P) ((P∨Q→R)→(Q→R)).
1005  intros Simp2_02a. (*Not cited*)
1006      (*Greg's suggestion per the BRS list on June 25, 2017.*)
1007  MP Syll2_06a Simp2_02a.
1008  MP Hb Simp2_02a.
1009  Syll Hb n2_54a Hc.
1010  apply Hc.
1011  Qed.
1012
1013  Theorem n2_86 : ∀ P Q R : Prop,
1014    ((P → Q) → (P → R)) → (P → (Q → R)).
1015  Proof. intros P Q R.
1016    specialize n2_85 with (¬P) Q R.
1017    intros n2_85a.
1018    replace (¬P∨Q) with (P→Q) in n2_85a
1019      by now rewrite Impl1_01.
1020    replace (¬P∨R) with (P→R) in n2_85a
1021      by now rewrite Impl1_01.
1022    replace (¬P∨(Q→R)) with (P→Q→R) in n2_85a
1023      by now rewrite Impl1_01.
1024    apply n2_85a.
1025  Qed.
1026
1027  End No2.
1028
1029  Module No3.
1030

```

```

1031 Import No1.
1032 Import No2.
1033
1034
1035 Theorem Prod3_01 :  $\forall$  P Q : Prop,
1036   (P  $\wedge$  Q) = ( $\neg$ ( $\neg$ P  $\vee$   $\neg$ Q)).
1037 Proof. intros P Q.
1038   apply propositional_extensionality.
1039   split.
1040   specialize or_not_and with (P) (Q).
1041   intros or_not_and.
1042   specialize Transp2_03 with ( $\neg$ P  $\vee$   $\neg$ Q) (P  $\wedge$  Q).
1043   intros Transp2_03.
1044   MP Transp2_03 or_not_and.
1045   apply Transp2_03.
1046   specialize not_and_or with (P) (Q).
1047   intros not_and_or.
1048   specialize Transp2_15 with (P  $\wedge$  Q) ( $\neg$ P  $\vee$   $\neg$ Q).
1049   intros Transp2_15.
1050   MP Transp2_15 not_and_or.
1051   apply Transp2_15.
1052 Qed.
1053 (This is a notational definition in Principia;
1054   it is used to switch between " $\wedge$ " and " $\neg\vee\neg$ ".*)
1055
1056 (*Axiom Abb3_02 :  $\forall$  P Q R : Prop,
1057   (P  $\rightarrow$  Q  $\rightarrow$  R) = ((P  $\rightarrow$  Q)  $\wedge$  (Q  $\rightarrow$  R)).*)
1058 (Since Coq forbids such strings as ill-formed, or
1059 else automatically associates to the right,
1060 we leave this notational axiom commented out.*)
1061
1062 Theorem Conj3_03 :  $\forall$  P Q : Prop, P  $\rightarrow$  Q  $\rightarrow$  (P $\wedge$ Q).
1063 Proof. intros P Q.
1064   specialize n2_11 with ( $\neg$ P $\vee$  $\neg$ Q).
1065   intros n2_11a.
1066   specialize n2_32 with ( $\neg$ P) ( $\neg$ Q) ( $\neg$ ( $\neg$ P  $\vee$   $\neg$ Q)).
1067   intros n2_32a.
1068   MP n2_32a n2_11a.
1069   replace ( $\neg$ ( $\neg$ P $\vee$  $\neg$ Q)) with (P $\wedge$ Q) in n2_32a
1070     by now rewrite Prod3_01.
1071   replace ( $\neg$ Q  $\vee$  (P $\wedge$ Q)) with (Q $\rightarrow$ (P $\wedge$ Q)) in n2_32a
1072     by now rewrite Impl1_01.

```

```

1073   replace ( $\neg P \vee (Q \rightarrow (P \wedge Q))$ ) with ( $P \rightarrow Q \rightarrow (P \wedge Q)$ ) in n2_32a
1074   by now rewrite Impl1_01.
1075   apply n2_32a.
1076 Qed.
1077   (*3.03 is permits the inference from the theoremhood
1078     of P and that of Q to the theoremhood of P and Q. So:*)
1079
1080 Ltac Conj H1 H2 :=
1081   match goal with
1082   | [ H1 : ?P, H2 : ?Q |- _ ] =>
1083     assert ( $P \wedge Q$ )
1084 end.
1085
1086 Theorem n3_1 :  $\forall P Q : \text{Prop},$ 
1087   ( $P \wedge Q$ )  $\rightarrow \neg(\neg P \vee \neg Q)$ .
1088 Proof. intros P Q.
1089   specialize Id2_08 with ( $P \wedge Q$ ).
1090   intros Id2_08a.
1091   replace ( $(P \wedge Q) \rightarrow (P \wedge Q)$ ) with ( $(P \wedge Q) \rightarrow \neg(\neg P \vee \neg Q)$ )
1092   in Id2_08a by now rewrite Prod3_01.
1093   apply Id2_08a.
1094 Qed.
1095
1096 Theorem n3_11 :  $\forall P Q : \text{Prop},$ 
1097    $\neg(\neg P \vee \neg Q) \rightarrow (P \wedge Q)$ .
1098 Proof. intros P Q.
1099   specialize Id2_08 with ( $P \wedge Q$ ).
1100   intros Id2_08a.
1101   replace ( $(P \wedge Q) \rightarrow (P \wedge Q)$ ) with ( $\neg(\neg P \vee \neg Q) \rightarrow (P \wedge Q)$ )
1102   in Id2_08a by now rewrite Prod3_01.
1103   apply Id2_08a.
1104 Qed.
1105
1106 Theorem n3_12 :  $\forall P Q : \text{Prop},$ 
1107   ( $\neg P \vee \neg Q$ )  $\vee (P \wedge Q)$ .
1108 Proof. intros P Q.
1109   specialize n2_11 with ( $\neg P \vee \neg Q$ ).
1110   intros n2_11a.
1111   replace ( $\neg(\neg P \vee \neg Q)$ ) with ( $P \wedge Q$ ) in n2_11a
1112   by now rewrite Prod3_01.
1113   apply n2_11a.
1114 Qed.

```

```

1115
1116 Theorem n3_13 :  $\forall P Q : \text{Prop},$ 
1117    $\neg(P \wedge Q) \rightarrow (\neg P \vee \neg Q).$ 
1118 Proof. intros P Q.
1119   specialize n3_11 with P Q.
1120   intros n3_11a.
1121   specialize Transp2_15 with  $(\neg P \vee \neg Q) (P \wedge Q).$ 
1122   intros Transp2_15a.
1123   MP Transp2_15a n3_11a.
1124   apply Transp2_15a.
1125 Qed.
1126
1127 Theorem n3_14 :  $\forall P Q : \text{Prop},$ 
1128    $(\neg P \vee \neg Q) \rightarrow \neg(P \wedge Q).$ 
1129 Proof. intros P Q.
1130   specialize n3_1 with P Q.
1131   intros n3_1a.
1132   specialize Transp2_16 with  $(P \wedge Q) (\neg(\neg P \vee \neg Q)).$ 
1133   intros Transp2_16a.
1134   MP Transp2_16a n3_1a.
1135   specialize n2_12 with  $(\neg P \vee \neg Q).$ 
1136   intros n2_12a.
1137   Syll n2_12a Transp2_16a S.
1138   apply S.
1139 Qed.
1140
1141 Theorem n3_2 :  $\forall P Q : \text{Prop},$ 
1142    $P \rightarrow Q \rightarrow (P \wedge Q).$ 
1143 Proof. intros P Q.
1144   specialize n3_12 with P Q.
1145   intros n3_12a.
1146   specialize n2_32 with  $(\neg P) (\neg Q) (P \wedge Q).$ 
1147   intros n2_32a.
1148   MP n3_32a n3_12a.
1149   replace  $(\neg Q \vee P \wedge Q)$  with  $(Q \rightarrow P \wedge Q)$  in n2_32a
1150     by now rewrite Impl1_01.
1151   replace  $(\neg P \vee (Q \rightarrow P \wedge Q))$  with  $(P \rightarrow Q \rightarrow P \wedge Q)$ 
1152     in n2_32a by now rewrite Impl1_01.
1153   apply n2_32a.
1154 Qed.
1155
1156 Theorem n3_21 :  $\forall P Q : \text{Prop},$ 

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1157    $Q \rightarrow P \rightarrow (P \wedge Q).$ 
1158 Proof. intros P Q.
1159   specialize n3_2 with P Q.
1160   intros n3_2a.
1161   specialize Comm2_04 with P Q (P $\wedge$ Q).
1162   intros Comm2_04a.
1163   MP Comm2_04a n3_2a.
1164   apply Comm2_04a.
1165 Qed.
1166
1167 Theorem n3_22 :  $\forall P Q : \text{Prop},$ 
1168    $(P \wedge Q) \rightarrow (Q \wedge P).$ 
1169 Proof. intros P Q.
1170   specialize n3_13 with Q P.
1171   intros n3_13a.
1172   specialize Perm1_4 with ( $\neg$ Q) ( $\neg$ P).
1173   intros Perm1_4a.
1174   Syll n3_13a Perm1_4a Ha.
1175   specialize n3_14 with P Q.
1176   intros n3_14a.
1177   Syll Ha n3_14a Hb.
1178   specialize Transp2_17 with (P $\wedge$ Q) (Q  $\wedge$  P).
1179   intros Transp2_17a.
1180   MP Transp2_17a Hb.
1181   apply Transp2_17a.
1182 Qed.
1183
1184 Theorem n3_24 :  $\forall P : \text{Prop},$ 
1185    $\neg(P \wedge \neg P).$ 
1186 Proof. intros P.
1187   specialize n2_11 with ( $\neg$ P).
1188   intros n2_11a.
1189   specialize n3_14 with P ( $\neg$ P).
1190   intros n3_14a.
1191   MP n3_14a n2_11a.
1192   apply n3_14a.
1193 Qed.
1194
1195 Theorem Simp3_26 :  $\forall P Q : \text{Prop},$ 
1196    $(P \wedge Q) \rightarrow P.$ 
1197 Proof. intros P Q.
1198   specialize Simp2_02 with Q P.

```

```

1199   intros Simp2_02a.
1200   replace (P→(Q→P)) with (¬P∨(Q→P)) in Simp2_02a
1201   by now rewrite <- Impl1_01.
1202   replace (Q→P) with (¬Q∨P) in Simp2_02a
1203   by now rewrite Impl1_01.
1204   specialize n2_31 with (¬P) (¬Q) P.
1205   intros n2_31a.
1206   MP n2_31a Simp2_02a.
1207   specialize n2_53 with (¬P∨¬Q) P.
1208   intros n2_53a.
1209   MP n2_53a Simp2_02a.
1210   replace (¬(¬P∨¬Q)) with (P∧Q) in n2_53a
1211   by now rewrite Prod3_01.
1212   apply n2_53a.
1213   Qed.
1214
1215   Theorem Simp3_27 : ∀ P Q : Prop,
1216     (P ∧ Q) → Q.
1217   Proof. intros P Q.
1218     specialize n3_22 with P Q.
1219     intros n3_22a.
1220     specialize Simp3_26 with Q P.
1221     intros Simp3_26a.
1222     Syll n3_22a Simp3_26a S.
1223     apply S.
1224     Qed.
1225
1226   Theorem Exp3_3 : ∀ P Q R : Prop,
1227     ((P ∧ Q) → R) → (P → (Q → R)).
1228   Proof. intros P Q R.
1229     specialize Id2_08 with ((P∧Q)→R).
1230     intros Id2_08a. (*This theorem isn't needed.*)
1231     replace (((P ∧ Q) → R) → ((P ∧ Q) → R)) with
1232       (((P ∧ Q) → R) → (¬(¬P ∨ ¬Q) → R)) in Id2_08a
1233     by now rewrite Prod3_01.
1234     specialize Transp2_15 with (¬P∨¬Q) R.
1235     intros Transp2_15a.
1236     Syll Id2_08a Transp2_15a Sa.
1237     specialize Id2_08 with (¬R → (¬P ∨ ¬Q)).
1238     intros Id2_08b. (*This theorem isn't needed.*)
1239     Syll Sa Id2_08b Sb.
1240     replace (¬P ∨ ¬Q) with (P → ¬Q) in Sb

```

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1241     by now rewrite Impl1_01.
1242     specialize Comm2_04 with ( $\neg$ R) P ( $\neg$ Q).
1243     intros Comm2_04a.
1244     Syll Sb Comm2_04a Sc.
1245     specialize Transp2_17 with Q R.
1246     intros Transp2_17a.
1247     specialize Syll2_05 with P ( $\neg$ R  $\rightarrow$   $\neg$ Q) (Q  $\rightarrow$  R).
1248     intros Syll2_05a.
1249     MP Syll2_05a Transp2_17a.
1250     Syll Sa Syll2_05a Sd.
1251     apply Sd.
1252 Qed.
1253
1254 Theorem Imp3_31 :  $\forall$  P Q R : Prop,
1255   (P  $\rightarrow$  (Q  $\rightarrow$  R))  $\rightarrow$  (P  $\wedge$  Q)  $\rightarrow$  R.
1256 Proof. intros P Q R.
1257   specialize Id2_08 with (P  $\rightarrow$  (Q  $\rightarrow$  R)).
1258   intros Id2_08a.
1259   replace ((P  $\rightarrow$  (Q  $\rightarrow$  R)) $\rightarrow$ (P  $\rightarrow$  (Q  $\rightarrow$  R))) with
1260     ((P  $\rightarrow$  (Q  $\rightarrow$  R)) $\rightarrow$ ( $\neg$ P  $\vee$  (Q  $\rightarrow$  R))) in Id2_08a
1261     by now rewrite <- Impl1_01.
1262   replace ( $\neg$ P  $\vee$  (Q  $\rightarrow$  R)) with
1263     ( $\neg$ P  $\vee$  ( $\neg$ Q  $\vee$  R)) in Id2_08a
1264     by now rewrite Impl1_01.
1265   specialize n2_31 with ( $\neg$ P) ( $\neg$ Q) R.
1266   intros n2_31a.
1267   Syll Id2_08a n2_31a Sa.
1268   specialize n2_53 with ( $\neg$ P $\vee$  $\neg$ Q) R.
1269   intros n2_53a.
1270   replace ( $\neg$ ( $\neg$ P $\vee$  $\neg$ Q)) with (P $\wedge$ Q) in n2_53a
1271     by now rewrite Prod3_01.
1272   Syll Sa n2_53a Sb.
1273   apply Sb.
1274 Qed.
1275
1276 Theorem Syll3_33 :  $\forall$  P Q R : Prop,
1277   ((P  $\rightarrow$  Q)  $\wedge$  (Q  $\rightarrow$  R))  $\rightarrow$  (P  $\rightarrow$  R).
1278 Proof. intros P Q R.
1279   specialize Syll2_06 with P Q R.
1280   intros Syll2_06a.
1281   specialize Imp3_31 with (P $\rightarrow$ Q) (Q $\rightarrow$ R) (P $\rightarrow$ R).
1282   intros Imp3_31a.

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1283     MP Imp3_31a Syll2_06a.
1284     apply Imp3_31a.
1285 Qed.
1286
1287 Theorem Syll3_34 :  $\forall P Q R : \text{Prop}$ ,
1288    $((Q \rightarrow R) \wedge (P \rightarrow Q)) \rightarrow (P \rightarrow R)$ .
1289 Proof. intros P Q R.
1290   specialize Syll2_05 with P Q R.
1291   intros Syll2_05a.
1292   specialize Imp3_31 with  $(Q \rightarrow R)$   $(P \rightarrow Q)$   $(P \rightarrow R)$ .
1293   intros Imp3_31a.
1294   MP Imp3_31a Syll2_05a.
1295   apply Imp3_31a.
1296 Qed.
1297
1298 Theorem Ass3_35 :  $\forall P Q : \text{Prop}$ ,
1299    $(P \wedge (P \rightarrow Q)) \rightarrow Q$ .
1300 Proof. intros P Q.
1301   specialize n2_27 with P Q.
1302   intros n2_27a.
1303   specialize Imp3_31 with P  $(P \rightarrow Q)$  Q.
1304   intros Imp3_31a.
1305   MP Imp3_31a n2_27a.
1306   apply Imp3_31a.
1307 Qed.
1308
1309 Theorem Transp3_37 :  $\forall P Q R : \text{Prop}$ ,
1310    $(P \wedge Q \rightarrow R) \rightarrow (P \wedge \neg R \rightarrow \neg Q)$ .
1311 Proof. intros P Q R.
1312   specialize Transp2_16 with Q R.
1313   intros Transp2_16a.
1314   specialize Syll2_05 with P  $(Q \rightarrow R)$   $(\neg R \rightarrow \neg Q)$ .
1315   intros Syll2_05a.
1316   MP Syll2_05a Transp2_16a.
1317   specialize Exp3_3 with P Q R.
1318   intros Exp3_3a.
1319   Syll Exp3_3a Syll2_05a Sa.
1320   specialize Imp3_31 with P  $(\neg R)$   $(\neg Q)$ .
1321   intros Imp3_31a.
1322   Syll Sa Imp3_31a Sb.
1323   apply Sb.
1324 Qed.

```



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1325
1326 Theorem n3_4 :  $\forall P Q : \text{Prop},$ 
1327    $(P \wedge Q) \rightarrow P \rightarrow Q.$ 
1328 Proof. intros P Q.
1329   specialize n2_51 with P Q.
1330   intros n2_51a.
1331   specialize Transp2_15 with  $(P \rightarrow Q) (P \rightarrow \neg Q).$ 
1332   intros Transp2_15a.
1333   MP Transp2_15a n2_51a.
1334   replace  $(P \rightarrow \neg Q)$  with  $(\neg P \vee \neg Q)$  in Transp2_15a
1335     by now rewrite Impl1_01.
1336   replace  $(\neg(\neg P \vee \neg Q))$  with  $(P \wedge Q)$  in Transp2_15a
1337     by now rewrite Prod3_01.
1338   apply Transp2_15a.
1339 Qed.
1340
1341 Theorem n3_41 :  $\forall P Q R : \text{Prop},$ 
1342    $(P \rightarrow R) \rightarrow (P \wedge Q \rightarrow R).$ 
1343 Proof. intros P Q R.
1344   specialize Simp3_26 with P Q.
1345   intros Simp3_26a.
1346   specialize Syll2_06 with  $(P \wedge Q) P R.$ 
1347   intros Syll2_06a.
1348   MP Simp3_26a Syll2_06a.
1349   apply Syll2_06a.
1350 Qed.
1351
1352 Theorem n3_42 :  $\forall P Q R : \text{Prop},$ 
1353    $(Q \rightarrow R) \rightarrow (P \wedge Q \rightarrow R).$ 
1354 Proof. intros P Q R.
1355   specialize Simp3_27 with P Q.
1356   intros Simp3_27a.
1357   specialize Syll2_06 with  $(P \wedge Q) Q R.$ 
1358   intros Syll2_06a.
1359   MP Syll2_06a Simp3_27a.
1360   apply Syll2_06a.
1361 Qed.
1362
1363 Theorem Comp3_43 :  $\forall P Q R : \text{Prop},$ 
1364    $(P \rightarrow Q) \wedge (P \rightarrow R) \rightarrow (P \rightarrow Q \wedge R).$ 
1365 Proof. intros P Q R.
1366   specialize n3_2 with Q R.

```

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1367   intros n3_2a.
1368   specialize Syll2_05 with P Q (R → Q ∧ R).
1369   intros Syll2_05a.
1370   MP Syll2_05a n3_2a.
1371   specialize n2_77 with P R (Q ∧ R).
1372   intros n2_77a.
1373   Syll Syll2_05a n2_77a Sa.
1374   specialize Imp3_31 with (P → Q) (P → R) (P → Q ∧ R).
1375   intros Imp3_31a.
1376   MP Sa Imp3_31a.
1377   apply Imp3_31a.
1378   Qed.
1379
1380   Theorem n3_44 : ∀ P Q R : Prop,
1381     (Q → P) ∧ (R → P) → (Q ∨ R → P).
1382   Proof. intros P Q R.
1383     specialize Syll3_33 with (¬Q) R P.
1384     intros Syll3_33a.
1385     specialize n2_6 with Q P.
1386     intros n2_6a.
1387     Syll Syll3_33a n2_6a Sa.
1388     specialize Exp3_3 with (¬Q → R) (R → P) ((Q → P) → P).
1389     intros Exp3_3a.
1390     MP Exp3_3a Sa.
1391     specialize Comm2_04 with (R → P) (Q → P) P.
1392     intros Comm2_04a.
1393     Syll Exp3_3a Comm2_04a Sb.
1394     specialize Imp3_31 with (Q → P) (R → P) P.
1395     intros Imp3_31a.
1396     Syll Sb Imp3_31a Sc.
1397     specialize Comm2_04 with (¬Q → R) ((Q → P) ∧ (R → P)) P.
1398     intros Comm2_04b.
1399     MP Comm2_04b Sc.
1400     specialize n2_53 with Q R.
1401     intros n2_53a.
1402     specialize Syll2_06 with (Q ∨ R) (¬Q → R) P.
1403     intros Syll2_06a.
1404     MP Syll2_06a n2_53a.
1405     Syll Comm2_04b Syll2_06a Sd.
1406     apply Sd.
1407   Qed.
1408

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```

1409 Theorem Fact3_45 :  $\forall P Q R : \text{Prop},$ 
1410    $(P \rightarrow Q) \rightarrow (P \wedge R) \rightarrow (Q \wedge R).$ 
1411 Proof. intros P Q R.
1412   specialize Syll2_06 with P Q ( $\neg R$ ).
1413   intros Syll2_06a.
1414   specialize Transp2_16 with  $(Q \rightarrow \neg R) (P \rightarrow \neg R).$ 
1415   intros Transp2_16a.
1416   Syll Syll2_06a Transp2_16a Sa.
1417   specialize Id2_08 with  $(\neg(P \rightarrow R) \rightarrow \neg(Q \rightarrow \neg R)).$ 
1418   intros Id2_08a.
1419   Syll Sa Id2_08a Sb.
1420   replace  $(P \rightarrow \neg R)$  with  $(\neg P \vee \neg R)$  in Sb
1421     by now rewrite Impl1_01.
1422   replace  $(Q \rightarrow \neg R)$  with  $(\neg Q \vee \neg R)$  in Sb
1423     by now rewrite Impl1_01.
1424   replace  $(\neg(\neg P \vee \neg R))$  with  $(P \wedge R)$  in Sb
1425     by now rewrite Prod3_01.
1426   replace  $(\neg(\neg Q \vee \neg R))$  with  $(Q \wedge R)$  in Sb
1427     by now rewrite Prod3_01.
1428   apply Sb.
1429 Qed.
1430
1431 Theorem n3_47 :  $\forall P Q R S : \text{Prop},$ 
1432    $((P \rightarrow R) \wedge (Q \rightarrow S)) \rightarrow (P \wedge Q) \rightarrow R \wedge S.$ 
1433 Proof. intros P Q R S.
1434   specialize Simp3_26 with  $(P \rightarrow R) (Q \rightarrow S).$ 
1435   intros Simp3_26a.
1436   specialize Fact3_45 with P R Q.
1437   intros Fact3_45a.
1438   Syll Simp3_26a Fact3_45a Sa.
1439   specialize n3_22 with R Q.
1440   intros n3_22a.
1441   specialize Syll2_05 with  $(P \wedge Q) (R \wedge Q) (Q \wedge R).$ 
1442   intros Syll2_05a.
1443   MP Syll2_05a n3_22a.
1444   Syll Sa Syll2_05a Sb.
1445   specialize Simp3_27 with  $(P \rightarrow R) (Q \rightarrow S).$ 
1446   intros Simp3_27a.
1447   specialize Fact3_45 with Q S R.
1448   intros Fact3_45b.
1449   Syll Simp3_27a Fact3_45b Sc.
1450   specialize n3_22 with S R.

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1451   intros n3_22b.
1452   specialize Syll2_05 with (Q $\wedge$ R) (S $\wedge$ R) (R $\wedge$ S).
1453   intros Syll2_05b.
1454   MP Syll2_05b n3_22b.
1455   Syll Sc Syll2_05b Sd.
1456   clear Simp3_26a. clear Fact3_45a. clear Sa.
1457   clear n3_22a. clear Fact3_45b.
1458   clear Syll2_05a. clear Simp3_27a.
1459   clear Sc. clear n3_22b. clear Syll2_05b.
1460   specialize Conj3_03 with ((P $\rightarrow$ R) $\wedge$ (Q $\rightarrow$ S) $\rightarrow$ P $\wedge$ Q $\rightarrow$ Q $\wedge$ R)
1461     ((P $\rightarrow$ R) $\wedge$ (Q $\rightarrow$ S) $\rightarrow$ Q $\wedge$ R $\rightarrow$ R $\wedge$ S).
1462   intros Conj3_03a.
1463   MP Conj3_03a Sb.
1464   MP Conj3_03a Sd.
1465   specialize n2_83 with ((P $\rightarrow$ R) $\wedge$ (Q $\rightarrow$ S)) (P $\wedge$ Q) (Q $\wedge$ R) (R $\wedge$ S).
1466   intros n2_83a. (*This with MP works, but it omits Conj3_03.*)
1467   specialize Imp3_31 with (((P $\rightarrow$ R) $\wedge$ (Q $\rightarrow$ S)) $\rightarrow$ ((P $\wedge$ Q) $\rightarrow$ (Q $\wedge$ R)))
1468     (((P $\rightarrow$ R) $\wedge$ (Q $\rightarrow$ S)) $\rightarrow$ ((Q $\wedge$ R) $\rightarrow$ (R $\wedge$ S)))
1469     (((P $\rightarrow$ R) $\wedge$ (Q $\rightarrow$ S)) $\rightarrow$ ((P $\wedge$ Q) $\rightarrow$ (R $\wedge$ S))).
1470   intros Imp3_31a.
1471   MP Imp3_31a n2_83a.
1472   MP Imp3_31a Conj3_03a.
1473   apply Imp3_31a.
1474   Qed.
1475
1476   Theorem n3_48 :  $\forall$  P Q R S : Prop,
1477     ((P  $\rightarrow$  R)  $\wedge$  (Q  $\rightarrow$  S))  $\rightarrow$  (P  $\vee$  Q)  $\rightarrow$  R  $\vee$  S.
1478   Proof. intros P Q R S.
1479     specialize Simp3_26 with (P $\rightarrow$ R) (Q $\rightarrow$ S).
1480     intros Simp3_26a.
1481     specialize Sum1_6 with Q P R.
1482     intros Sum1_6a.
1483     Syll Simp3_26a Sum1_6a Sa.
1484     specialize Perm1_4 with P Q.
1485     intros Perm1_4a.
1486     specialize Syll2_06 with (P $\vee$ Q) (Q $\vee$ P) (Q $\vee$ R).
1487     intros Syll2_06a.
1488     MP Syll2_06a Perm1_4a.
1489     Syll Sa Syll2_06a Sb.
1490     specialize Simp3_27 with (P $\rightarrow$ R) (Q $\rightarrow$ S).
1491     intros Simp3_27a.
1492     specialize Sum1_6 with R Q S.

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1493   intros Sum1_6b.
1494   Syll Simp3_27a Sum1_6b Sc.
1495   specialize Perm1_4 with Q R.
1496   intros Perm1_4b.
1497   specialize Syll2_06 with (Q $\vee$ R) (R $\vee$ Q) (R $\vee$ S).
1498   intros Syll2_06b.
1499   MP Syll2_06b Perm1_4b.
1500   Syll Sc Syll2_06a Sd.
1501   specialize n2_83 with ((P $\rightarrow$ R) $\wedge$ (Q $\rightarrow$ S)) (P $\vee$ Q) (Q $\vee$ R) (R $\vee$ S).
1502   intros n2_83a.
1503   MP n2_83a Sb.
1504   MP n2_83a Sd.
1505   apply n2_83a.
1506   Qed.
1507
1508   End No3.
1509
1510   Module No4.
1511
1512   Import No1.
1513   Import No2.
1514   Import No3.
1515
1516   Theorem Equiv4_01 :  $\forall$  P Q : Prop,
1517     (P  $\leftrightarrow$  Q) = ((P  $\rightarrow$  Q)  $\wedge$  (Q  $\rightarrow$  P)).
1518   Proof. intros P Q.
1519   apply propositional_extensionality.
1520   specialize iff_to_and with P Q.
1521   intros iff_to_and.
1522   apply iff_to_and.
1523   Qed.
1524   (*This is a notational definition in Principia;
1525   it is used to switch between " $\leftrightarrow$ " and " $\rightarrow \wedge \leftarrow$ ".*)
1526
1527   (*Axiom Abb4_02 :  $\forall$  P Q R : Prop,
1528     (P  $\leftrightarrow$  Q  $\leftrightarrow$  R) = ((P  $\leftrightarrow$  Q)  $\wedge$  (Q  $\leftrightarrow$  R)).*)
1529   (*Since Coq forbids ill-formed strings, or else
1530   automatically associates to the right, we leave
1531   this notational axiom commented out.*/)
1532
1533   Ltac Equiv H1 :=
1534     match goal with

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1535 | [ H1 : (?P→?Q) ∧ (?Q→?P) |- _ ] =>
1536   replace ((P→Q) ∧ (Q→P)) with (P↔Q) in H1
1537   by now rewrite Equiv4_01
1538 end.
1539
1540 Theorem Transp4_1 : ∀ P Q : Prop,
1541   (P → Q) ↔ (¬Q → ¬P).
1542 Proof. intros P Q.
1543   specialize Transp2_16 with P Q.
1544   intros Transp2_16a.
1545   specialize Transp2_17 with P Q.
1546   intros Transp2_17a.
1547   Conj Transp2_16a Transp2_17a.
1548   split.
1549   apply Transp2_16a.
1550   apply Transp2_17a.
1551   Equiv H.
1552   apply H.
1553 Qed.
1554
1555 Theorem Transp4_11 : ∀ P Q : Prop,
1556   (P ↔ Q) ↔ (¬P ↔ ¬Q).
1557 Proof. intros P Q.
1558   specialize Transp2_16 with P Q.
1559   intros Transp2_16a.
1560   specialize Transp2_16 with Q P.
1561   intros Transp2_16b.
1562   Conj Transp2_16a Transp2_16b.
1563   split.
1564   apply Transp2_16a.
1565   apply Transp2_16b.
1566   specialize n3_47 with (P→Q) (Q→P) (¬Q→¬P) (¬P→¬Q).
1567   intros n3_47a.
1568   MP n3_47 H.
1569   specialize n3_22 with (¬Q → ¬P) (¬P → ¬Q).
1570   intros n3_22a.
1571   Syll n3_47a n3_22a Sa.
1572   replace ((P → Q) ∧ (Q → P)) with (P↔Q) in Sa
1573     by now rewrite Equiv4_01.
1574   replace ((¬P → ¬Q) ∧ (¬Q → ¬P)) with (¬P↔¬Q)
1575     in Sa by now rewrite Equiv4_01.
1576   clear Transp2_16a. clear H. clear Transp2_16b.

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1577   clear n3_22a. clear n3_47a.
1578   specialize Transp2_17 with Q P.
1579   intros Transp2_17a.
1580   specialize Transp2_17 with P Q.
1581   intros Transp2_17b.
1582   Conj Transp2_17a Transp2_17b.
1583   split.
1584   apply Transp2_17a.
1585   apply Transp2_17b.
1586   specialize n3_47 with ( $\neg P \rightarrow \neg Q$ ) ( $\neg Q \rightarrow \neg P$ ) ( $Q \rightarrow P$ ) ( $P \rightarrow Q$ ).
1587   intros n3_47a.
1588   MP n3_47a H.
1589   specialize n3_22 with ( $Q \rightarrow P$ ) ( $P \rightarrow Q$ ).
1590   intros n3_22a.
1591   Syll n3_47a n3_22a Sb.
1592   clear Transp2_17a. clear Transp2_17b. clear H.
1593   clear n3_47a. clear n3_22a.
1594   replace (( $P \rightarrow Q$ )  $\wedge$  ( $Q \rightarrow P$ )) with ( $P \leftrightarrow Q$ ) in Sb
1595   by now rewrite Equiv4_01.
1596   replace (( $\neg P \rightarrow \neg Q$ )  $\wedge$  ( $\neg Q \rightarrow \neg P$ )) with ( $\neg P \leftrightarrow \neg Q$ )
1597   in Sb by now rewrite Equiv4_01.
1598   Conj Sa Sb.
1599   split.
1600   apply Sa.
1601   apply Sb.
1602   Equiv H.
1603   apply H.
1604   Qed.
1605
1606   Theorem n4_12 :  $\forall P Q : \text{Prop}$ ,
1607     ( $P \leftrightarrow \neg Q$ )  $\leftrightarrow$  ( $Q \leftrightarrow \neg P$ ).
1608   Proof. intros P Q.
1609     specialize Transp2_03 with P Q.
1610     intros Transp2_03a.
1611     specialize Transp2_15 with Q P.
1612     intros Transp2_15a.
1613     Conj Transp2_03a Transp2_15a.
1614     split.
1615     apply Transp2_03a.
1616     apply Transp2_15a.
1617     specialize n3_47 with ( $P \rightarrow \neg Q$ ) ( $\neg Q \rightarrow P$ ) ( $Q \rightarrow \neg P$ ) ( $\neg P \rightarrow Q$ ).
1618     intros n3_47a.

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1619     MP n3_47a H.
1620     specialize Transp2_03 with Q P.
1621     intros Transp2_03b.
1622     specialize Transp2_15 with P Q.
1623     intros Transp2_15b.
1624     Conj Transp2_03b Transp2_15b.
1625     split.
1626     apply Transp2_03b.
1627     apply Transp2_15b.
1628     specialize n3_47 with (Q → ¬P) (¬P → Q) (P → ¬Q) (¬Q → P).
1629     intros n3_47b.
1630     MP n3_47b H0.
1631     clear Transp2_03a. clear Transp2_15a. clear H.
1632     clear Transp2_03b. clear Transp2_15b. clear H0.
1633     Conj n3_47a n3_47b.
1634     split.
1635     apply n3_47a.
1636     apply n3_47b.
1637     rewrite <- Equiv4_01 in H.
1638     rewrite <- Equiv4_01 in H.
1639     rewrite <- Equiv4_01 in H.
1640     apply H.
1641 Qed.
1642
1643 Theorem n4_13 : ∀ P : Prop,
1644   P ↔ ¬¬P.
1645 Proof. intros P.
1646   specialize n2_12 with P.
1647   intros n2_12a.
1648   specialize n2_14 with P.
1649   intros n2_14a.
1650   Conj n2_12a n2_14a.
1651   split.
1652   apply n2_12a.
1653   apply n2_14a.
1654   Equiv H.
1655   apply H.
1656 Qed.
1657
1658 Theorem n4_14 : ∀ P Q R : Prop,
1659   ((P ∧ Q) → R) ↔ ((P ∧ ¬R) → ¬Q).
1660 Proof. intros P Q R.

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1661 specialize Transp3_37 with P Q R.
1662 intros Transp3_37a.
1663 specialize Transp3_37 with P ( $\neg$ R) ( $\neg$ Q).
1664 intros Transp3_37b.
1665 Conj Transp3_37a Transp3_37b.
1666 split. apply Transp3_37a.
1667 apply Transp3_37b.
1668 specialize n4_13 with Q.
1669 intros n4_13a.
1670 apply propositional_extensionality in n4_13a.
1671 specialize n4_13 with R.
1672 intros n4_13b.
1673 apply propositional_extensionality in n4_13b.
1674 replace ( $\neg\neg$ Q) with Q in H
1675   by now apply n4_13a.
1676 replace ( $\neg\neg$ R) with R in H
1677   by now apply n4_13b.
1678 Equiv H.
1679 apply H.
1680 Qed.
1681
1682 Theorem n4_15 :  $\forall$  P Q R : Prop,
1683   ((P  $\wedge$  Q)  $\rightarrow$   $\neg$ R)  $\leftrightarrow$  ((Q  $\wedge$  R)  $\rightarrow$   $\neg$ P).
1684 Proof. intros P Q R.
1685 specialize n4_14 with Q P ( $\neg$ R).
1686 intros n4_14a.
1687 specialize n3_22 with Q P.
1688 intros n3_22a.
1689 specialize Syll2_06 with (Q $\wedge$ P) (P $\wedge$ Q) ( $\neg$ R).
1690 intros Syll2_06a.
1691 MP Syll2_06a n3_22a.
1692 specialize n4_13 with R.
1693 intros n4_13a.
1694 apply propositional_extensionality in n4_13a.
1695 replace ( $\neg\neg$ R) with R in n4_14a
1696   by now apply n4_13a.
1697 rewrite Equiv4_01 in n4_14a.
1698 specialize Simp3_26 with ((Q  $\wedge$  P  $\rightarrow$   $\neg$ R)  $\rightarrow$  Q  $\wedge$  R  $\rightarrow$   $\neg$ P)
1699   ((Q  $\wedge$  R  $\rightarrow$   $\neg$ P)  $\rightarrow$  Q  $\wedge$  P  $\rightarrow$   $\neg$ R).
1700 intros Simp3_26a.
1701 MP Simp3_26a n4_14a.
1702 Syll Syll2_06a Simp3_26a Sa.

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1703 specialize Simp3_27 with ((Q ∧ P → ¬R) → Q ∧ R → ¬P)
1704 ((Q ∧ R → ¬P) → Q ∧ P → ¬R).
1705 intros Simp3_27a.
1706 MP Simp3_27a n4_14a.
1707 specialize n3_22 with P Q.
1708 intros n3_22b.
1709 specialize Syll2_06 with (P ∧ Q) (Q ∧ P) (¬R).
1710 intros Syll2_06b.
1711 MP Syll2_06b n3_22b.
1712 Syll Syll2_06b Simp3_27a Sb.
1713 clear n4_14a. clear n3_22a. clear Syll2_06a.
1714 clear n4_13a. clear Simp3_26a. clear n3_22b.
1715 clear Simp3_27a. clear Syll2_06b.
1716 Conj Sa Sb.
1717 split.
1718 apply Sa.
1719 apply Sb.
1720 Equiv H.
1721 apply H.
1722 Qed.
1723
1724 Theorem n4_2 : ∀ P : Prop,
1725   P ↔ P.
1726 Proof. intros P.
1727 specialize n3_2 with (P → P) (P → P).
1728 intros n3_2a.
1729 specialize Id2_08 with P.
1730 intros Id2_08a.
1731 MP n3_2a Id2_08a.
1732 MP n3_2a Id2_08a.
1733 Equiv n3_2a.
1734 apply n3_2a.
1735 Qed.
1736
1737 Theorem n4_21 : ∀ P Q : Prop,
1738   (P ↔ Q) ↔ (Q ↔ P).
1739 Proof. intros P Q.
1740 specialize n3_22 with (P → Q) (Q → P).
1741 intros n3_22a.
1742 replace ((P → Q) ∧ (Q → P)) with (P ↔ Q)
1743   in n3_22a by now rewrite Equiv4_01.
1744 replace ((Q → P) ∧ (P → Q)) with (Q ↔ P)

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1745   in n3_22a by now rewrite Equiv4_01.
1746 specialize n3_22 with (Q→P) (P→Q).
1747 intros n3_22b.
1748 replace ((P → Q) ∧ (Q → P)) with (P↔Q)
1749   in n3_22b by now rewrite Equiv4_01.
1750 replace ((Q → P) ∧ (P → Q)) with (Q↔P)
1751   in n3_22b by now rewrite Equiv4_01.
1752 Conj n3_22a n3_22b.
1753 split.
1754 apply n3_22a.
1755 apply n3_22b.
1756 Equiv H.
1757 apply H.
1758 Qed.
1759
1760 Theorem n4_22 : ∀ P Q R : Prop,
1761   ((P ↔ Q) ∧ (Q ↔ R)) → (P ↔ R).
1762 Proof. intros P Q R.
1763   specialize Simp3_26 with (P↔Q) (Q↔R).
1764   intros Simp3_26a.
1765   specialize Simp3_26 with (P→Q) (Q→P).
1766   intros Simp3_26b.
1767   replace ((P→Q) ∧ (Q→P)) with (P↔Q)
1768     in Simp3_26b by now rewrite Equiv4_01.
1769   Syll Simp3_26a Simp3_26b Sa.
1770   specialize Simp3_27 with (P↔Q) (Q↔R).
1771   intros Simp3_27a.
1772   specialize Simp3_26 with (Q→R) (R→Q).
1773   intros Simp3_26c.
1774   replace ((Q→R) ∧ (R→Q)) with (Q↔R)
1775     in Simp3_26c by now rewrite Equiv4_01.
1776   Syll Simp3_27a Simp3_26c Sb.
1777   specialize n2_83 with ((P↔Q) ∧ (Q↔R)) P Q R.
1778   intros n2_83a.
1779   MP n2_83a Sa.
1780   MP n2_83a Sb.
1781   specialize Simp3_27 with (P↔Q) (Q↔R).
1782   intros Simp3_27b.
1783   specialize Simp3_27 with (Q→R) (R→Q).
1784   intros Simp3_27c.
1785   replace ((Q→R) ∧ (R→Q)) with (Q↔R)
1786     in Simp3_27c by now rewrite Equiv4_01.

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1787 Syll Simp3_27b Simp3_27c Sc.
1788 specialize Simp3_26 with (P↔Q) (Q↔R).
1789 intros Simp3_26d.
1790 specialize Simp3_27 with (P→Q) (Q→P).
1791 intros Simp3_27d.
1792 replace ((P→Q) ∧ (Q→P)) with (P↔Q)
1793   in Simp3_27d by now rewrite Equiv4_01.
1794 Syll Simp3_26d Simp3_27d Sd.
1795 specialize n2_83 with ((P↔Q) ∧ (Q↔R)) R Q P.
1796 intros n2_83b.
1797 MP n2_83b Sc. MP n2_83b Sd.
1798 clear Sd. clear Sb. clear Sc. clear Sa. clear Simp3_26a.
1799   clear Simp3_26b. clear Simp3_26c. clear Simp3_26d.
1800   clear Simp3_27a. clear Simp3_27b. clear Simp3_27c.
1801   clear Simp3_27d.
1802 Conj n2_83a n2_83b.
1803 split.
1804 apply n2_83a.
1805 apply n2_83b.
1806 specialize Comp3_43 with ((P↔Q) ∧ (Q↔R)) (P→R) (R→P).
1807 intros Comp3_43a.
1808 MP Comp3_43a H.
1809 replace ((P→R) ∧ (R→P)) with (P↔R)
1810   in Comp3_43a by now rewrite Equiv4_01.
1811 apply Comp3_43a.
1812 Qed.
1813
1814 Theorem n4_24 : ∀ P : Prop,
1815   P ↔ (P ∧ P).
1816 Proof. intros P.
1817 specialize n3_2 with P P.
1818 intros n3_2a.
1819 specialize n2_43 with P (P ∧ P).
1820 intros n2_43a.
1821 MP n3_2a n2_43a.
1822 specialize Simp3_26 with P P.
1823 intros Simp3_26a.
1824 Conj n2_43a Simp3_26a.
1825 split.
1826 apply n2_43a.
1827 apply Simp3_26a.
1828 Equiv H.

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1829     apply H.
1830 Qed.
1831
1832 Theorem n4_25 :  $\forall P : \text{Prop},$ 
1833    $P \leftrightarrow (P \vee P).$ 
1834 Proof. intros P.
1835   specialize Add1_3 with P P.
1836   intros Add1_3a.
1837   specialize Taut1_2 with P.
1838   intros Taut1_2a.
1839   Conj Add1_3a Taut1_2a.
1840   split.
1841   apply Add1_3a.
1842   apply Taut1_2a.
1843   Equiv H. apply H.
1844 Qed.
1845
1846 Theorem n4_3 :  $\forall P Q : \text{Prop},$ 
1847    $(P \wedge Q) \leftrightarrow (Q \wedge P).$ 
1848 Proof. intros P Q.
1849   specialize n3_22 with P Q.
1850   intros n3_22a.
1851   specialize n3_22 with Q P.
1852   intros n3_22b.
1853   Conj n3_22a n3_22b.
1854   split.
1855   apply n3_22a.
1856   apply n3_22b.
1857   Equiv H. apply H.
1858 Qed.
1859
1860 Theorem n4_31 :  $\forall P Q : \text{Prop},$ 
1861    $(P \vee Q) \leftrightarrow (Q \vee P).$ 
1862 Proof. intros P Q.
1863   specialize Perm1_4 with P Q.
1864   intros Perm1_4a.
1865   specialize Perm1_4 with Q P.
1866   intros Perm1_4b.
1867   Conj Perm1_4a Perm1_4b.
1868   split.
1869   apply Perm1_4a.
1870   apply Perm1_4b.

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1871      Equiv H. apply H.
1872 Qed.
1873
1874 Theorem n4_32 :  $\forall$  P Q R : Prop,
1875   ((P  $\wedge$  Q)  $\wedge$  R)  $\leftrightarrow$  (P  $\wedge$  (Q  $\wedge$  R)).
1876 Proof. intros P Q R.
1877   specialize n4_15 with P Q R.
1878   intros n4_15a.
1879   specialize Transp4_1 with P ( $\neg$ (Q  $\wedge$  R)).
1880   intros Transp4_1a.
1881   apply propositional_extensionality in Transp4_1a.
1882   specialize n4_13 with (Q  $\wedge$  R).
1883   intros n4_13a.
1884   apply propositional_extensionality in n4_13a.
1885   specialize n4_21 with ( $\neg$ (P $\wedge$ Q $\rightarrow$  $\neg$ R) $\leftrightarrow$  $\neg$ (P $\rightarrow$  $\neg$ (Q $\wedge$ R)))
1886     ((P $\wedge$ Q $\rightarrow$  $\neg$ R) $\leftrightarrow$ (P $\rightarrow$  $\neg$ (Q $\wedge$ R))).
1887   intros n4_21a.
1888   apply propositional_extensionality in n4_21a.
1889   replace ( $\neg$  $\neg$ (Q  $\wedge$  R)) with (Q  $\wedge$  R) in Transp4_1a
1890     by now apply n4_13a.
1891   replace (Q  $\wedge$  R $\rightarrow$  $\neg$ P) with (P $\rightarrow$  $\neg$ (Q  $\wedge$  R)) in n4_15a
1892     by now apply Transp4_1a.
1893   specialize Transp4_11 with (P $\wedge$ Q $\rightarrow$  $\neg$ R) (P $\rightarrow$  $\neg$ (Q $\wedge$ R)).
1894   intros Transp4_11a.
1895   apply propositional_extensionality in Transp4_11a.
1896   replace ((P  $\wedge$  Q  $\rightarrow$   $\neg$ R)  $\leftrightarrow$  (P  $\rightarrow$   $\neg$ (Q  $\wedge$  R))) with
1897     ( $\neg$ (P  $\wedge$  Q  $\rightarrow$   $\neg$ R)  $\leftrightarrow$   $\neg$ (P  $\rightarrow$   $\neg$ (Q  $\wedge$  R))) in n4_15a
1898     by now apply Transp4_11a.
1899   replace (P  $\wedge$  Q  $\rightarrow$   $\neg$ R) with
1900     ( $\neg$ (P  $\wedge$  Q)  $\vee$   $\neg$ R) in n4_15a
1901     by now rewrite Impl1_01.
1902   replace (P  $\rightarrow$   $\neg$ (Q  $\wedge$  R)) with
1903     ( $\neg$ P  $\vee$   $\neg$ (Q  $\wedge$  R)) in n4_15a
1904     by now rewrite Impl1_01.
1905   replace ( $\neg$ ( $\neg$ (P  $\wedge$  Q)  $\vee$   $\neg$ R)) with
1906     ((P  $\wedge$  Q)  $\wedge$  R) in n4_15a
1907     by now rewrite Prod3_01.
1908   replace ( $\neg$ ( $\neg$ P  $\vee$   $\neg$ (Q  $\wedge$  R))) with
1909     (P  $\wedge$  (Q  $\wedge$  R)) in n4_15a
1910     by now rewrite Prod3_01.
1911   apply n4_15a.
1912 Qed.

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1913      (*Note that the actual proof uses n4_12, but
1914         that transposition involves transforming a
1915         biconditional into a conditional. This citation
1916         of the lemma may be a misprint. Using
1917         Transp4_1 to transpose a conditional and
1918         then applying n4_13 to double negate does
1919         secure the desired formula.*)
1920
1921 Theorem n4_33 :  $\forall$  P Q R : Prop,
1922   (P  $\vee$  (Q  $\vee$  R))  $\leftrightarrow$  ((P  $\vee$  Q)  $\vee$  R).
1923 Proof. intros P Q R.
1924   specialize n2_31 with P Q R.
1925   intros n2_31a.
1926   specialize n2_32 with P Q R.
1927   intros n2_32a.
1928   Conj n2_31a n2_32a.
1929   split.
1930   apply n2_31a.
1931   apply n2_32a.
1932   Equiv H.
1933   apply H.
1934 Qed.
1935
1936 Theorem Abb4_34 :  $\forall$  P Q R : Prop,
1937   (P  $\wedge$  Q  $\wedge$  R) = ((P  $\wedge$  Q)  $\wedge$  R).
1938 Proof. intros P Q R.
1939   apply propositional_extensionality.
1940   specialize n4_21 with ((P  $\wedge$  Q)  $\wedge$  R) (P  $\wedge$  Q  $\wedge$  R).
1941   intros n4_21.
1942   replace (((P  $\wedge$  Q)  $\wedge$  R  $\leftrightarrow$  P  $\wedge$  Q  $\wedge$  R)  $\leftrightarrow$  (P  $\wedge$  Q  $\wedge$  R  $\leftrightarrow$  (P  $\wedge$  Q)  $\wedge$  R))
1943     with (((P  $\wedge$  Q)  $\wedge$  R  $\leftrightarrow$  P  $\wedge$  Q  $\wedge$  R)  $\rightarrow$  (P  $\wedge$  Q  $\wedge$  R  $\leftrightarrow$  (P  $\wedge$  Q)  $\wedge$  R))
1944      $\wedge$  ((P  $\wedge$  Q  $\wedge$  R  $\leftrightarrow$  (P  $\wedge$  Q)  $\wedge$  R)  $\rightarrow$  ((P  $\wedge$  Q)  $\wedge$  R  $\leftrightarrow$  P  $\wedge$  Q  $\wedge$  R)))
1945     in n4_21 by now rewrite Equiv4_01.
1946   specialize Simp3_26 with
1947     (((P  $\wedge$  Q)  $\wedge$  R  $\leftrightarrow$  P  $\wedge$  Q  $\wedge$  R)  $\rightarrow$  (P  $\wedge$  Q  $\wedge$  R  $\leftrightarrow$  (P  $\wedge$  Q)  $\wedge$  R))
1948     ((P  $\wedge$  Q  $\wedge$  R  $\leftrightarrow$  (P  $\wedge$  Q)  $\wedge$  R)  $\rightarrow$  ((P  $\wedge$  Q)  $\wedge$  R  $\leftrightarrow$  P  $\wedge$  Q  $\wedge$  R)).
1949   intros Simp3_26.
1950   MP Simp3_26 n4_21.
1951   specialize n4_32 with P Q R.
1952   intros n4_32.
1953   MP Simp3_26 n4_32.
1954   apply Simp3_26.

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1955 Qed.
1956
1957 Theorem n4_36 :  $\forall P Q R : \text{Prop},$ 
1958    $(P \leftrightarrow Q) \rightarrow ((P \wedge R) \leftrightarrow (Q \wedge R)).$ 
1959 Proof. intros P Q R.
1960   specialize Fact3_45 with P Q R.
1961   intros Fact3_45a.
1962   specialize Fact3_45 with Q P R.
1963   intros Fact3_45b.
1964   Conj Fact3_45a Fact3_45b.
1965   split.
1966   apply Fact3_45a.
1967   apply Fact3_45b.
1968   specialize n3_47 with  $(P \rightarrow Q) (Q \rightarrow P)$ 
1969      $(P \wedge R \rightarrow Q \wedge R) (Q \wedge R \rightarrow P \wedge R).$ 
1970   intros n3_47a.
1971   MP n3_47 H.
1972   replace  $((P \rightarrow Q) \wedge (Q \rightarrow P))$  with  $(P \leftrightarrow Q)$  in n3_47a
1973     by now rewrite Equiv4_01.
1974   replace  $((P \wedge R \rightarrow Q \wedge R) \wedge (Q \wedge R \rightarrow P \wedge R))$  with  $(P \wedge R \leftrightarrow Q \wedge R)$ 
1975     in n3_47a by now rewrite Equiv4_01.
1976   apply n3_47a.
1977 Qed.
1978
1979 Theorem n4_37 :  $\forall P Q R : \text{Prop},$ 
1980    $(P \leftrightarrow Q) \rightarrow ((P \vee R) \leftrightarrow (Q \vee R)).$ 
1981 Proof. intros P Q R.
1982   specialize Sum1_6 with R P Q.
1983   intros Sum1_6a.
1984   specialize Sum1_6 with R Q P.
1985   intros Sum1_6b.
1986   Conj Sum1_6a Sum1_6b.
1987   split.
1988   apply Sum1_6a.
1989   apply Sum1_6b.
1990   specialize n3_47 with  $(P \rightarrow Q) (Q \rightarrow P)$ 
1991      $(R \vee P \rightarrow R \vee Q) (R \vee Q \rightarrow R \vee P).$ 
1992   intros n3_47a.
1993   MP n3_47 H.
1994   replace  $((P \rightarrow Q) \wedge (Q \rightarrow P))$  with  $(P \leftrightarrow Q)$  in n3_47a
1995     by now rewrite Equiv4_01.
1996   replace  $((R \vee P \rightarrow R \vee Q) \wedge (R \vee Q \rightarrow R \vee P))$  with  $(R \vee P \leftrightarrow R \vee Q)$ 

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1997         in n3_47a by now rewrite Equiv4_01.
1998 specialize n4_31 with Q R.
1999 intros n4_31a.
2000 apply propositional_extensionality in n4_31a.
2001 specialize n4_31 with P R.
2002 intros n4_31b.
2003 apply propositional_extensionality in n4_31b.
2004 replace (R  $\vee$  P) with (P  $\vee$  R) in n3_47a
2005     by now apply n4_31a.
2006 replace (R  $\vee$  Q) with (Q  $\vee$  R) in n3_47a
2007     by now apply n4_31b.
2008 apply n3_47a.
2009 Qed.
2010
2011 Theorem n4_38 :  $\forall$  P Q R S : Prop,
2012   ((P  $\leftrightarrow$  R)  $\wedge$  (Q  $\leftrightarrow$  S))  $\rightarrow$  ((P  $\wedge$  Q)  $\leftrightarrow$  (R  $\wedge$  S)).
2013 Proof. intros P Q R S.
2014   specialize n3_47 with P Q R S.
2015   intros n3_47a.
2016   specialize n3_47 with R S P Q.
2017   intros n3_47b.
2018   Conj n3_47a n3_47b.
2019   split.
2020   apply n3_47a.
2021   apply n3_47b.
2022   specialize n3_47 with ((P $\rightarrow$ R)  $\wedge$  (Q $\rightarrow$ S))
2023     ((R $\rightarrow$ P)  $\wedge$  (S $\rightarrow$ Q)) (P  $\wedge$  Q  $\rightarrow$  R  $\wedge$  S) (R  $\wedge$  S  $\rightarrow$  P  $\wedge$  Q).
2024   intros n3_47c.
2025   MP n3_47c H.
2026   specialize n4_32 with (P $\rightarrow$ R) (Q $\rightarrow$ S) ((R $\rightarrow$ P)  $\wedge$  (S  $\rightarrow$  Q)).
2027   intros n4_32a.
2028   apply propositional_extensionality in n4_32a.
2029   replace ((P  $\rightarrow$  R)  $\wedge$  (Q  $\rightarrow$  S))  $\wedge$  (R  $\rightarrow$  P)  $\wedge$  (S  $\rightarrow$  Q)) with
2030     ((P  $\rightarrow$  R)  $\wedge$  (Q  $\rightarrow$  S)  $\wedge$  (R  $\rightarrow$  P)  $\wedge$  (S  $\rightarrow$  Q)) in n3_47c
2031     by now apply n4_32a.
2032   specialize n4_32 with (Q $\rightarrow$ S) (R $\rightarrow$ P) (S  $\rightarrow$  Q).
2033   intros n4_32b.
2034   apply propositional_extensionality in n4_32b.
2035   replace ((Q  $\rightarrow$  S)  $\wedge$  (R  $\rightarrow$  P)  $\wedge$  (S  $\rightarrow$  Q)) with
2036     (((Q  $\rightarrow$  S)  $\wedge$  (R  $\rightarrow$  P))  $\wedge$  (S  $\rightarrow$  Q)) in n3_47c
2037     by now apply n4_32b.
2038   specialize n3_22 with (Q $\rightarrow$ S) (R $\rightarrow$ P).

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2039   intros n3_22a.
2040   specialize n3_22 with (R→P) (Q→S).
2041   intros n3_22b.
2042   Conj n3_22a n3_22b.
2043   split.
2044   apply n3_22a.
2045   apply n3_22b.
2046   Equiv H0.
2047   specialize n4_3 with (R→P) (Q→S).
2048   intros n4_3a.
2049   apply propositional_extensionality in n4_3a.
2050   replace ((Q → S) ∧ (R → P)) with
2051     ((R → P) ∧ (Q → S)) in n3_47c
2052     by now apply n4_3a.
2053   specialize n4_32 with (R → P) (Q → S) (S → Q).
2054   intros n4_32c.
2055   apply propositional_extensionality in n4_32c.
2056   replace (((R → P) ∧ (Q → S)) ∧ (S → Q)) with
2057     ((R → P) ∧ (Q → S) ∧ (S → Q)) in n3_47c
2058     by now apply n4_32c.
2059   specialize n4_32 with (P→R) (R → P) ((Q → S) ∧ (S → Q)).
2060   intros n4_32d.
2061   apply propositional_extensionality in n4_32d.
2062   replace ((P → R) ∧ (R → P) ∧ (Q → S) ∧ (S → Q)) with
2063     (((P → R) ∧ (R → P)) ∧ (Q → S) ∧ (S → Q)) in n3_47c
2064     by now apply n4_32d.
2065   replace ((P→R) ∧ (R → P)) with (P↔R) in n3_47c
2066     by now rewrite Equiv4_01.
2067   replace ((Q → S) ∧ (S → Q)) with (Q↔S) in n3_47c
2068     by now rewrite Equiv4_01.
2069   replace ((P↔Q→R↔S) ∧ (R↔S→P↔Q)) with ((P↔Q)↔(R↔S))
2070     in n3_47c by now rewrite Equiv4_01.
2071   apply n3_47c.
2072   Qed.
2073
2074   Theorem n4_39 : ∀ P Q R S : Prop,
2075     ((P ↔ R) ∧ (Q ↔ S)) → ((P ∨ Q) ↔ (R ∨ S)).
2076   Proof.  intros P Q R S.
2077     specialize n3_48 with P Q R S.
2078     intros n3_48a.
2079     specialize n3_48 with R S P Q.
2080     intros n3_48b.

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2081 Conj n3_48a n3_48b.
2082 split.
2083 apply n3_48a.
2084 apply n3_48b.
2085 specialize n3_47 with ((P → R) ∧ (Q → S))
2086   ((R → P) ∧ (S → Q)) (P ∨ Q → R ∨ S) (R ∨ S → P ∨ Q).
2087 intros n3_47a.
2088 MP n3_47a H.
2089 replace ((P∨Q→R∨S)∧(R∨S→P∨Q)) with ((P∨Q)↔(R∨S))
2090   in n3_47a by now rewrite Equiv4_01.
2091 specialize n4_32 with ((P → R) ∧ (Q → S)) (R → P) (S → Q).
2092 intros n4_32a.
2093 apply propositional_extensionality in n4_32a.
2094 replace (((P → R) ∧ (Q → S)) ∧ (R → P) ∧ (S → Q)) with
2095   (((P → R) ∧ (Q → S)) ∧ (R → P)) ∧ (S → Q) in n3_47a
2096   by now apply n4_32a.
2097 specialize n4_32 with (P → R) (Q → S) (R → P).
2098 intros n4_32b.
2099 apply propositional_extensionality in n4_32b.
2100 replace (((P → R) ∧ (Q → S)) ∧ (R → P)) with
2101   ((P → R) ∧ (Q → S) ∧ (R → P)) in n3_47a
2102   by now apply n4_32b.
2103 specialize n3_22 with (Q → S) (R → P).
2104 intros n3_22a.
2105 specialize n3_22 with (R → P) (Q → S).
2106 intros n3_22b.
2107 Conj n3_22a n3_22b.
2108 split.
2109 apply n3_22a.
2110 apply n3_22b.
2111 Equiv H0.
2112 apply propositional_extensionality in H0.
2113 replace ((Q → S) ∧ (R → P)) with
2114   ((R → P) ∧ (Q → S)) in n3_47a
2115   by now apply H0.
2116 specialize n4_32 with (P → R) (R → P) (Q → S).
2117 intros n4_32c.
2118 apply propositional_extensionality in n4_32c.
2119 replace ((P → R) ∧ (R → P) ∧ (Q → S)) with
2120   (((P → R) ∧ (R → P)) ∧ (Q → S)) in n3_47a
2121   by now apply n4_32c.
2122 replace ((P → R) ∧ (R → P)) with (P↔R) in n3_47a

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2123     by now rewrite Equiv4_01.
2124 specialize n4_32 with (P↔R) (Q→S) (S→Q).
2125 intros n4_32d.
2126 apply propositional_extensionality in n4_32d.
2127 replace (((P ↔ R) ∧ (Q → S)) ∧ (S → Q)) with
2128     ((P ↔ R) ∧ (Q → S) ∧ (S → Q)) in n3_47a
2129     by now apply n4_32d.
2130 replace ((Q → S) ∧ (S → Q)) with (Q ↔ S) in n3_47a
2131     by now rewrite Equiv4_01.
2132 apply n3_47a.
2133 Qed.
2134
2135 Theorem n4_4 : ∀ P Q R : Prop,
2136     (P ∧ (Q ∨ R)) ↔ ((P ∧ Q) ∨ (P ∧ R)).
2137 Proof. intros P Q R.
2138     specialize n3_2 with P Q.
2139     intros n3_2a.
2140     specialize n3_2 with P R.
2141     intros n3_2b.
2142     Conj n3_2a n3_2b.
2143     split.
2144     apply n3_2a.
2145     apply n3_2b.
2146     specialize Comp3_43 with P (Q→P∧Q) (R→P∧R).
2147     intros Comp3_43a.
2148     MP Comp3_43a H.
2149     specialize n3_48 with Q R (P∧Q) (P∧R).
2150     intros n3_48a.
2151     Syll Comp3_43a n3_48a Sa.
2152     specialize Imp3_31 with P (Q∨R) ((P∧ Q) ∨ (P ∧ R)).
2153     intros Imp3_31a.
2154     MP Imp3_31a Sa.
2155     specialize Simp3_26 with P Q.
2156     intros Simp3_26a.
2157     specialize Simp3_26 with P R.
2158     intros Simp3_26b.
2159     Conj Simp3_26a Simp3_26b.
2160     split.
2161     apply Simp3_26a.
2162     apply Simp3_26b.
2163     specialize n3_44 with P (P∧Q) (P∧R).
2164     intros n3_44a.

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2165 MP n3_44a H0.
2166 specialize Simp3_27 with P Q.
2167 intros Simp3_27a.
2168 specialize Simp3_27 with P R.
2169 intros Simp3_27b.
2170 Conj Simp3_27a Simp3_27b.
2171 split.
2172 apply Simp3_27a.
2173 apply Simp3_27b.
2174 specialize n3_48 with (P ∧ Q) (P ∧ R) Q R.
2175 intros n3_48b.
2176 MP n3_48b H1.
2177 clear H1. clear Simp3_27a. clear Simp3_27b.
2178 Conj n3_44a n3_48b.
2179 split.
2180 apply n3_44a.
2181 apply n3_48b.
2182 specialize Comp3_43 with (P ∧ Q ∨ P ∧ R) P (Q ∨ R).
2183 intros Comp3_43b.
2184 MP Comp3_43b H1.
2185 clear H1. clear H0. clear n3_44a. clear n3_48b.
2186 clear Simp3_26a. clear Simp3_26b.
2187 Conj Imp3_31a Comp3_43b.
2188 split.
2189 apply Imp3_31a.
2190 apply Comp3_43b.
2191 Equiv H0.
2192 apply H0.
2193 Qed.
2194
2195 Theorem n4_41 : ∀ P Q R : Prop,
2196   (P ∨ (Q ∧ R)) ↔ ((P ∨ Q) ∧ (P ∨ R)).
2197 Proof. intros P Q R.
2198   specialize Simp3_26 with Q R.
2199   intros Simp3_26a.
2200   specialize Sum1_6 with P (Q ∧ R) Q.
2201   intros Sum1_6a.
2202   MP Simp3_26a Sum1_6a.
2203   specialize Simp3_27 with Q R.
2204   intros Simp3_27a.
2205   specialize Sum1_6 with P (Q ∧ R) R.
2206   intros Sum1_6b.

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2207 MP Simp3_27a Sum1_6b.
2208 clear Simp3_26a. clear Simp3_27a.
2209 Conj Sum1_6a Sum1_6a.
2210 split.
2211 apply Sum1_6a.
2212 apply Sum1_6b.
2213 specialize Comp3_43 with (P ∨ Q ∧ R) (P ∨ Q) (P ∨ R).
2214 intros Comp3_43a.
2215 MP Comp3_43a H.
2216 specialize n2_53 with P Q.
2217 intros n2_53a.
2218 specialize n2_53 with P R.
2219 intros n2_53b.
2220 Conj n2_53a n2_53b.
2221 split.
2222 apply n2_53a.
2223 apply n2_53b.
2224 specialize n3_47 with (P ∨ Q) (P ∨ R) (¬P → Q) (¬P → R).
2225 intros n3_47a.
2226 MP n3_47a H0.
2227 specialize Comp3_43 with (¬P) Q R.
2228 intros Comp3_43b.
2229 Syll n3_47a Comp3_43b Sa.
2230 specialize n2_54 with P (Q ∧ R).
2231 intros n2_54a.
2232 Syll Sa n2_54a Sb.
2233 clear Sum1_6a. clear Sum1_6b. clear H. clear n2_53a.
2234 clear n2_53b. clear H0. clear n3_47a. clear Sa.
2235 clear Comp3_43b. clear n2_54a.
2236 Conj Comp3_43a Sb.
2237 split.
2238 apply Comp3_43a.
2239 apply Sb.
2240 Equiv H.
2241 apply H.
2242 Qed.
2243
2244 Theorem n4_42 : ∀ P Q : Prop,
2245   P ↔ ((P ∧ Q) ∨ (P ∧ ¬Q)).
2246 Proof. intros P Q.
2247   specialize n3_21 with P (Q ∨ ¬Q).
2248   intros n3_21a.

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2249 specialize n2_11 with Q.
2250 intros n2_11a.
2251 MP n3_21a n2_11a.
2252 specialize Simp3_26 with P (Q  $\vee$   $\neg$ Q).
2253 intros Simp3_26a. clear n2_11a.
2254 Conj n3_21a Simp3_26a.
2255 split.
2256 apply n3_21a.
2257 apply Simp3_26a.
2258 Equiv H.
2259 specialize n4_4 with P Q ( $\neg$ Q).
2260 intros n4_4a.
2261 apply propositional_extensionality in H.
2262 replace (P  $\wedge$  (Q  $\vee$   $\neg$ Q)) with P in n4_4a
2263   by now apply H.
2264 apply n4_4a.
2265 Qed.
2266
2267 Theorem n4_43 :  $\forall$  P Q : Prop,
2268   P  $\leftrightarrow$  ((P  $\vee$  Q)  $\wedge$  (P  $\vee$   $\neg$ Q)).
2269 Proof. intros P Q.
2270 specialize n2_2 with P Q.
2271 intros n2_2a.
2272 specialize n2_2 with P ( $\neg$ Q).
2273 intros n2_2b.
2274 Conj n2_2a n2_2b.
2275 split.
2276 apply n2_2a.
2277 apply n2_2b.
2278 specialize Comp3_43 with P (P $\vee$ Q) (P $\vee$  $\neg$ Q).
2279 intros Comp3_43a.
2280 MP Comp3_43a H.
2281 specialize n2_53 with P Q.
2282 intros n2_53a.
2283 specialize n2_53 with P ( $\neg$ Q).
2284 intros n2_53b.
2285 Conj n2_53a n2_53b.
2286 split.
2287 apply n2_53a.
2288 apply n2_53b.
2289 specialize n3_47 with (P $\vee$ Q) (P $\vee$  $\neg$ Q) ( $\neg$ P $\rightarrow$ Q) ( $\neg$ P $\rightarrow$  $\neg$ Q).
2290 intros n3_47a.

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2291 MP n3_47a H0.
2292 specialize n2_65 with ( $\neg$ P) Q.
2293 intros n2_65a.
2294 specialize n4_13 with P.
2295 intros n4_13a.
2296 apply propositional_extensionality in n4_13a.
2297 replace ( $\neg\neg$ P) with P in n2_65a by now apply n4_13a.
2298 specialize Imp3_31 with ( $\neg$ P  $\rightarrow$  Q) ( $\neg$ P  $\rightarrow$   $\neg$ Q) (P).
2299 intros Imp3_31a.
2300 MP Imp3_31a n2_65a.
2301 Syll n3_47a Imp3_31a Sa.
2302 clear n2_2a. clear n2_2b. clear H. clear n2_53a.
2303   clear n2_53b. clear H0. clear n2_65a.
2304   clear n3_47a. clear Imp3_31a. clear n4_13a.
2305 Conj Comp3_43a Sa.
2306 split.
2307 apply Comp3_43a.
2308 apply Sa.
2309 Equiv H.
2310 apply H.
2311 Qed.
2312
2313 Theorem n4_44 :  $\forall$  P Q : Prop,
2314   P  $\leftrightarrow$  (P  $\vee$  (P  $\wedge$  Q)).
2315 Proof. intros P Q.
2316   specialize n2_2 with P (P $\wedge$ Q).
2317   intros n2_2a.
2318   specialize Id2_08 with P.
2319   intros Id2_08a.
2320   specialize Simp3_26 with P Q.
2321   intros Simp3_26a.
2322   Conj Id2_08a Simp3_26a.
2323   split.
2324   apply Id2_08a.
2325   apply Simp3_26a.
2326   specialize n3_44 with P P (P  $\wedge$  Q).
2327   intros n3_44a.
2328   MP n3_44a H.
2329   clear H. clear Id2_08a. clear Simp3_26a.
2330   Conj n2_2a n3_44a.
2331   split.
2332   apply n2_2a.

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2333     apply n3_44a.
2334     Equiv H.
2335     apply H.
2336 Qed.
2337
2338 Theorem n4_45 :  $\forall$  P Q : Prop,
2339   P  $\leftrightarrow$  (P  $\wedge$  (P  $\vee$  Q)).
2340 Proof. intros P Q.
2341   specialize n2_2 with (P  $\wedge$  P) (P  $\wedge$  Q).
2342   intros n2_2a.
2343   specialize n4_4 with P P Q.
2344   intros n4_4a.
2345   apply propositional_extensionality in n4_4a.
2346   replace (P $\wedge$ P $\vee$ P $\wedge$ Q) with (P $\wedge$ (P $\vee$ Q)) in n2_2a
2347     by now apply n4_4a.
2348   specialize n4_24 with P.
2349   intros n4_24a.
2350   apply propositional_extensionality in n4_24a.
2351   replace (P  $\wedge$  P) with P in n2_2a
2352     by now apply n4_24a.
2353   specialize Simp3_26 with P (P  $\vee$  Q).
2354   intros Simp3_26a.
2355   clear n4_4a. clear n4_24a.
2356   Conj n2_2a Simp3_26a.
2357   split.
2358   apply n2_2a.
2359   apply Simp3_26a.
2360   Equiv H.
2361   apply H.
2362 Qed.
2363
2364 Theorem n4_5 :  $\forall$  P Q : Prop,
2365   P  $\wedge$  Q  $\leftrightarrow$   $\neg$ ( $\neg$ P  $\vee$   $\neg$ Q).
2366 Proof. intros P Q.
2367   specialize n4_2 with (P  $\wedge$  Q).
2368   intros n4_2a.
2369   replace ((P  $\wedge$  Q) $\leftrightarrow$ (P  $\wedge$  Q)) with
2370     ((P  $\wedge$  Q) $\leftrightarrow$  $\neg$ ( $\neg$ P  $\vee$   $\neg$ Q)) in n4_2a
2371     by now rewrite Prod3_01.
2372   apply n4_2a.
2373 Qed.
2374

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2375 Theorem n4_51 :  $\forall$  P Q : Prop,
2376    $\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q)$ .
2377 Proof. intros P Q.
2378   specialize n4_5 with P Q.
2379   intros n4_5a.
2380   specialize n4_12 with (P  $\wedge$  Q) ( $\neg P \vee \neg Q$ ).
2381   intros n4_12a.
2382   specialize Simp3_26 with
2383     ((P  $\wedge$  Q  $\leftrightarrow \neg(\neg P \vee \neg Q)$ )  $\rightarrow$  ( $\neg P \vee \neg Q \leftrightarrow \neg(P \wedge Q)$ ))
2384     (( $\neg P \vee \neg Q \leftrightarrow \neg(P \wedge Q)$ )  $\rightarrow$  ((P  $\wedge$  Q  $\leftrightarrow \neg(\neg P \vee \neg Q)$ ))).
2385   intros Simp3_26a.
2386   replace ((P  $\wedge$  Q  $\leftrightarrow \neg(\neg P \vee \neg Q)$ )  $\leftrightarrow$  ( $\neg P \vee \neg Q \leftrightarrow \neg(P \wedge Q)$ ))
2387     with (((P  $\wedge$  Q  $\leftrightarrow \neg(\neg P \vee \neg Q)$ )  $\rightarrow$  ( $\neg P \vee \neg Q \leftrightarrow \neg(P \wedge Q)$ ))
2388        $\wedge$  (( $\neg P \vee \neg Q \leftrightarrow \neg(P \wedge Q)$ )  $\rightarrow$  ((P  $\wedge$  Q  $\leftrightarrow \neg(\neg P \vee \neg Q)$ ))))
2389     in n4_12a by now rewrite Equiv4_01.
2390   MP Simp3_26a n4_12a.
2391   MP Simp3_26a n4_5a.
2392   specialize n4_21 with ( $\neg(P \wedge Q)$ ) ( $\neg P \vee \neg Q$ ).
2393   intros n4_21a.
2394   specialize Simp3_27 with
2395     ((( $\neg(P \wedge Q) \leftrightarrow \neg P \vee \neg Q$ )  $\rightarrow$  (( $\neg P \vee \neg Q \leftrightarrow \neg(P \wedge Q)$ )))
2396     ((( $\neg P \vee \neg Q \leftrightarrow \neg(P \wedge Q)$ )  $\rightarrow$  (( $\neg(P \wedge Q) \leftrightarrow \neg P \vee \neg Q$ ))).
2397   intros Simp3_27a.
2398   MP Simp3_27a n4_21a.
2399   MP Simp3_27a Simp3_26a.
2400   apply Simp3_27a.
2401 Qed.
2402
2403 Theorem n4_52 :  $\forall$  P Q : Prop,
2404   (P  $\wedge \neg Q$ )  $\leftrightarrow \neg(\neg P \vee Q)$ .
2405 Proof. intros P Q.
2406   specialize n4_5 with P ( $\neg Q$ ).
2407   intros n4_5a.
2408   specialize n4_13 with Q.
2409   intros n4_13a.
2410   apply propositional_extensionality in n4_13a.
2411   replace ( $\neg\neg Q$ ) with Q in n4_5a
2412     by now apply n4_13a.
2413   apply n4_5a.
2414 Qed.
2415
2416 Theorem n4_53 :  $\forall$  P Q : Prop,

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2417  $\neg(P \wedge \neg Q) \leftrightarrow (\neg P \vee Q).$ 
2418 Proof. intros P Q.
2419 specialize n4_52 with P Q.
2420 intros n4_52a.
2421 specialize n4_12 with (P  $\wedge$   $\neg$ Q) ( $\neg$ P  $\vee$  Q).
2422 intros n4_12a.
2423 replace ((P  $\wedge$   $\neg$ Q  $\leftrightarrow$   $\neg$ ( $\neg$ P  $\vee$  Q))  $\leftrightarrow$  ( $\neg$ P  $\vee$  Q  $\leftrightarrow$   $\neg$ (P  $\wedge$   $\neg$ Q)))
2424 with (((P  $\wedge$   $\neg$ Q  $\leftrightarrow$   $\neg$ ( $\neg$ P  $\vee$  Q))  $\rightarrow$  ( $\neg$ P  $\vee$  Q  $\leftrightarrow$   $\neg$ (P  $\wedge$   $\neg$ Q)))
2425  $\wedge$  (( $\neg$ P  $\vee$  Q  $\leftrightarrow$   $\neg$ (P  $\wedge$   $\neg$ Q))  $\rightarrow$  (P  $\wedge$   $\neg$ Q  $\leftrightarrow$   $\neg$ ( $\neg$ P  $\vee$  Q))))
2426 in n4_12a by now rewrite Equiv4_01.
2427 specialize Simp3_26 with
2428 ((P  $\wedge$   $\neg$ Q  $\leftrightarrow$   $\neg$ ( $\neg$ P  $\vee$  Q))  $\rightarrow$  ( $\neg$ P  $\vee$  Q  $\leftrightarrow$   $\neg$ (P  $\wedge$   $\neg$ Q)))
2429 (( $\neg$ P  $\vee$  Q  $\leftrightarrow$   $\neg$ (P  $\wedge$   $\neg$ Q))  $\rightarrow$  (P  $\wedge$   $\neg$ Q  $\leftrightarrow$   $\neg$ ( $\neg$ P  $\vee$  Q))).
2430 intros Simp3_26a.
2431 MP Simp3_26a n4_12a.
2432 MP Simp3_26a n4_52a.
2433 specialize n4_21 with ( $\neg$ (P  $\wedge$   $\neg$ Q)) ( $\neg$ P  $\vee$  Q).
2434 intros n4_21a.
2435 replace (( $\neg$ (P  $\wedge$   $\neg$ Q)  $\leftrightarrow$   $\neg$ P  $\vee$  Q)  $\leftrightarrow$  ( $\neg$ P  $\vee$  Q  $\leftrightarrow$   $\neg$ (P  $\wedge$   $\neg$ Q)))
2436 with ((( $\neg$ (P  $\wedge$   $\neg$ Q)  $\leftrightarrow$   $\neg$ P  $\vee$  Q)  $\rightarrow$  ( $\neg$ P  $\vee$  Q  $\leftrightarrow$   $\neg$ (P  $\wedge$   $\neg$ Q)))
2437  $\wedge$  (( $\neg$ P  $\vee$  Q  $\leftrightarrow$   $\neg$ (P  $\wedge$   $\neg$ Q))  $\rightarrow$  ( $\neg$ (P  $\wedge$   $\neg$ Q)  $\leftrightarrow$   $\neg$ P  $\vee$  Q)))
2438 in n4_21a by now rewrite Equiv4_01.
2439 specialize Simp3_27 with
2440 (( $\neg$ (P  $\wedge$   $\neg$ Q)  $\leftrightarrow$   $\neg$ P  $\vee$  Q)  $\rightarrow$  ( $\neg$ P  $\vee$  Q  $\leftrightarrow$   $\neg$ (P  $\wedge$   $\neg$ Q)))
2441 (( $\neg$ P  $\vee$  Q  $\leftrightarrow$   $\neg$ (P  $\wedge$   $\neg$ Q))  $\rightarrow$  ( $\neg$ (P  $\wedge$   $\neg$ Q)  $\leftrightarrow$   $\neg$ P  $\vee$  Q)).
2442 intros Simp3_27a.
2443 MP Simp3_27a n4_21a.
2444 MP Simp3_27a Simp3_26a.
2445 apply Simp3_27a.
2446 Qed.
2447
2448 Theorem n4_54 :  $\forall$  P Q : Prop,
2449 ( $\neg$ P  $\wedge$  Q)  $\leftrightarrow$   $\neg$ (P  $\vee$   $\neg$ Q).
2450 Proof. intros P Q.
2451 specialize n4_5 with ( $\neg$ P) Q.
2452 intros n4_5a.
2453 specialize n4_13 with P.
2454 intros n4_13a.
2455 apply propositional_extensionality in n4_13a.
2456 replace ( $\neg$  $\neg$ P) with P in n4_5a
2457 by now apply n4_13a.
2458 apply n4_5a.

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```

2459 Qed.
2460
2461 Theorem n4_55 :  $\forall P Q : \text{Prop},$ 
2462    $\neg(\neg P \wedge Q) \leftrightarrow (P \vee \neg Q).$ 
2463 Proof. intros P Q.
2464   specialize n4_54 with P Q.
2465   intros n4_54a.
2466   specialize n4_12 with  $(\neg P \wedge Q) (P \vee \neg Q).$ 
2467   intros n4_12a.
2468   apply propositional_extensionality in n4_12a.
2469   replace  $(\neg P \wedge Q \leftrightarrow \neg(P \vee \neg Q))$  with
2470      $(P \vee \neg Q \leftrightarrow \neg(\neg P \wedge Q))$  in n4_54a
2471     by now apply n4_12a.
2472   specialize n4_21 with  $(\neg(\neg P \wedge Q)) (P \vee \neg Q).$ 
2473   intros n4_21a. (*Not cited*)
2474   apply propositional_extensionality in n4_21a.
2475   replace  $(P \vee \neg Q \leftrightarrow \neg(\neg P \wedge Q))$  with
2476      $(\neg(\neg P \wedge Q) \leftrightarrow (P \vee \neg Q))$  in n4_54a
2477     by now apply n4_21a.
2478   apply n4_54a.
2479 Qed.
2480
2481 Theorem n4_56 :  $\forall P Q : \text{Prop},$ 
2482    $(\neg P \wedge \neg Q) \leftrightarrow \neg(P \vee Q).$ 
2483 Proof. intros P Q.
2484   specialize n4_54 with P  $(\neg Q).$ 
2485   intros n4_54a.
2486   specialize n4_13 with Q.
2487   intros n4_13a.
2488   apply propositional_extensionality in n4_13a.
2489   replace  $(\neg\neg Q)$  with Q in n4_54a
2490     by now apply n4_13a.
2491   apply n4_54a.
2492 Qed.
2493
2494 Theorem n4_57 :  $\forall P Q : \text{Prop},$ 
2495    $\neg(\neg P \wedge \neg Q) \leftrightarrow (P \vee Q).$ 
2496 Proof. intros P Q.
2497   specialize n4_56 with P Q.
2498   intros n4_56a.
2499   specialize n4_12 with  $(\neg P \wedge \neg Q) (P \vee Q).$ 
2500   intros n4_12a.

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2501     apply propositional_extensionality in n4_12a.
2502     replace ( $\neg P \wedge \neg Q \leftrightarrow \neg(P \vee Q)$ ) with
2503         ( $P \vee Q \leftrightarrow \neg(\neg P \wedge \neg Q)$ ) in n4_56a
2504     by now apply n4_12a.
2505     specialize n4_21 with ( $\neg(\neg P \wedge \neg Q)$ ) ( $P \vee Q$ ).
2506     intros n4_21a.
2507     apply propositional_extensionality in n4_21a.
2508     replace ( $P \vee Q \leftrightarrow \neg(\neg P \wedge \neg Q)$ ) with
2509         ( $\neg(\neg P \wedge \neg Q) \leftrightarrow P \vee Q$ ) in n4_56a
2510     by now apply n4_21a.
2511     apply n4_56a.
2512 Qed.
2513
2514 Theorem n4_6 :  $\forall P Q : \text{Prop},$ 
2515     ( $P \rightarrow Q$ )  $\leftrightarrow$  ( $\neg P \vee Q$ ).
2516 Proof. intros P Q.
2517     specialize n4_2 with ( $\neg P \vee Q$ ).
2518     intros n4_2a.
2519     rewrite Impl1_01.
2520     apply n4_2a.
2521 Qed.
2522
2523 Theorem n4_61 :  $\forall P Q : \text{Prop},$ 
2524      $\neg(P \rightarrow Q) \leftrightarrow (P \wedge \neg Q)$ .
2525 Proof. intros P Q.
2526     specialize n4_6 with P Q.
2527     intros n4_6a.
2528     specialize Transp4_11 with ( $P \rightarrow Q$ ) ( $\neg P \vee Q$ ).
2529     intros Transp4_11a.
2530     apply propositional_extensionality in Transp4_11a.
2531     replace (( $P \rightarrow Q$ )  $\leftrightarrow$   $\neg P \vee Q$ ) with
2532         ( $\neg(P \rightarrow Q) \leftrightarrow \neg(\neg P \vee Q)$ ) in n4_6a
2533     by now apply Transp4_11a.
2534     specialize n4_52 with P Q.
2535     intros n4_52a.
2536     apply propositional_extensionality in n4_52a.
2537     replace ( $\neg(\neg P \vee Q)$ ) with ( $P \wedge \neg Q$ ) in n4_6a
2538     by now apply n4_52a.
2539     apply n4_6a.
2540 Qed.
2541
2542 Theorem n4_62 :  $\forall P Q : \text{Prop},$ 

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2543   (P → ¬Q) ↔ (¬P ∨ ¬Q).
2544   Proof. intros P Q.
2545       specialize n4_6 with P (¬Q).
2546       intros n4_6a.
2547       apply n4_6a.
2548   Qed.
2549
2550   Theorem n4_63 : ∀ P Q : Prop,
2551       ¬(P → ¬Q) ↔ (P ∧ Q).
2552   Proof. intros P Q.
2553       specialize n4_62 with P Q.
2554       intros n4_62a.
2555       specialize Transp4_11 with (P → ¬Q) (¬P ∨ ¬Q).
2556       intros Transp4_11a.
2557       specialize n4_5 with P Q.
2558       intros n4_5a.
2559       apply propositional_extensionality in n4_5a.
2560       replace (¬(¬P ∨ ¬Q)) with (P ∧ Q) in Transp4_11a
2561         by now apply n4_5a.
2562       apply propositional_extensionality in Transp4_11a.
2563       replace ((P → ¬Q) ↔ ¬P ∨ ¬Q) with
2564         ((¬(P → ¬Q) ↔ P ∧ Q)) in n4_62a
2565         by now apply Transp4_11a.
2566       apply n4_62a.
2567   Qed.
2568   (*One could use Prod3_01 in lieu of n4_5.*)
2569
2570   Theorem n4_64 : ∀ P Q : Prop,
2571       (¬P → Q) ↔ (P ∨ Q).
2572   Proof. intros P Q.
2573       specialize n2_54 with P Q.
2574       intros n2_54a.
2575       specialize n2_53 with P Q.
2576       intros n2_53a.
2577       Conj n2_54a n2_53a.
2578       split.
2579       apply n2_54a.
2580       apply n2_53a.
2581       Equiv H.
2582       apply H.
2583   Qed.
2584

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2585 Theorem n4_65 :  $\forall$  P Q : Prop,
2586    $\neg(\neg P \rightarrow Q) \leftrightarrow (\neg P \wedge \neg Q)$ .
2587 Proof. intros P Q.
2588 specialize n4_64 with P Q.
2589 intros n4_64a.
2590 specialize Transp4_11 with  $(\neg P \rightarrow Q)$   $(P \vee Q)$ .
2591 intros Transp4_11a.
2592 specialize n4_21 with  $(\neg(\neg P \rightarrow Q) \leftrightarrow \neg(P \vee Q))$ 
2593    $((\neg P \rightarrow Q) \leftrightarrow (P \vee Q))$ .
2594 intros n4_21a.
2595 apply propositional_extensionality in n4_21a.
2596 replace  $((\neg(\neg P \rightarrow Q) \leftrightarrow P \vee Q) \leftrightarrow (\neg(\neg P \rightarrow Q) \leftrightarrow \neg(P \vee Q)))$  with
2597    $((\neg(\neg P \rightarrow Q) \leftrightarrow \neg(P \vee Q)) \leftrightarrow ((\neg P \rightarrow Q) \leftrightarrow P \vee Q))$  in Transp4_11a
2598   by now apply n4_21a.
2599 apply propositional_extensionality in Transp4_11a.
2600 replace  $((\neg P \rightarrow Q) \leftrightarrow P \vee Q)$  with
2601    $(\neg(\neg P \rightarrow Q) \leftrightarrow \neg(P \vee Q))$  in n4_64a
2602   by now apply Transp4_11a.
2603 specialize n4_56 with P Q.
2604 intros n4_56a.
2605 apply propositional_extensionality in n4_56a.
2606 replace  $(\neg(P \vee Q))$  with  $(\neg P \wedge \neg Q)$  in n4_64a
2607   by now apply n4_56a.
2608 apply n4_64a.
2609 Qed.
2610
2611 Theorem n4_66 :  $\forall$  P Q : Prop,
2612    $(\neg P \rightarrow \neg Q) \leftrightarrow (P \vee \neg Q)$ .
2613 Proof. intros P Q.
2614 specialize n4_64 with P  $(\neg Q)$ .
2615 intros n4_64a.
2616 apply n4_64a.
2617 Qed.
2618
2619 Theorem n4_67 :  $\forall$  P Q : Prop,
2620    $\neg(\neg P \rightarrow \neg Q) \leftrightarrow (\neg P \wedge Q)$ .
2621 Proof. intros P Q.
2622 specialize n4_66 with P Q.
2623 intros n4_66a.
2624 specialize Transp4_11 with  $(\neg P \rightarrow \neg Q)$   $(P \vee \neg Q)$ .
2625 intros Transp4_11a.
2626 apply propositional_extensionality in Transp4_11a.

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2627   replace ((¬P → ¬Q) ↔ P ∨ ¬Q) with
2628     (¬(¬P → ¬Q) ↔ ¬(P ∨ ¬Q)) in n4_66a
2629     by now apply Transp4_11a.
2630   specialize n4_54 with P Q.
2631   intros n4_54a.
2632   apply propositional_extensionality in n4_54a.
2633   replace (¬(P ∨ ¬Q)) with (¬P ∧ Q) in n4_66a
2634     by now apply n4_54a.
2635   apply n4_66a.
2636   Qed.
2637
2638   Theorem n4_7 : ∀ P Q : Prop,
2639     (P → Q) ↔ (P → (P ∧ Q)).
2640   Proof. intros P Q.
2641   specialize Comp3_43 with P P Q.
2642   intros Comp3_43a.
2643   specialize Exp3_3 with
2644     (P → P) (P → Q) (P → P ∧ Q).
2645   intros Exp3_3a.
2646   MP Exp3_3a Comp3_43a.
2647   specialize Id2_08 with P.
2648   intros Id2_08a.
2649   MP Exp3_3a Id2_08a.
2650   specialize Simp3_27 with P Q.
2651   intros Simp3_27a.
2652   specialize Syll2_05 with P (P ∧ Q) Q.
2653   intros Syll2_05a.
2654   MP Syll2_05a Simp3_27a.
2655   clear Id2_08a. clear Comp3_43a. clear Simp3_27a.
2656   Conj Syll2_05a Exp3_3a.
2657   split.
2658   apply Exp3_3a.
2659   apply Syll2_05a.
2660   Equiv H.
2661   apply H.
2662   Qed.
2663
2664   Theorem n4_71 : ∀ P Q : Prop,
2665     (P → Q) ↔ (P ↔ (P ∧ Q)).
2666   Proof. intros P Q.
2667   specialize n4_7 with P Q.
2668   intros n4_7a.

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2669 specialize n3_21 with (P → (P ∧ Q)) ((P ∧ Q) → P).
2670 intros n3_21a.
2671 replace ((P → P ∧ Q) ∧ (P ∧ Q → P)) with (P ↔ (P ∧ Q))
2672   in n3_21a by now rewrite Equiv4_01.
2673 specialize Simp3_26 with P Q.
2674 intros Simp3_26a.
2675 MP n3_21a Simp3_26a.
2676 specialize Simp3_26 with (P → (P ∧ Q)) ((P ∧ Q) → P).
2677 intros Simp3_26b.
2678 replace ((P → P ∧ Q) ∧ (P ∧ Q → P)) with (P ↔ (P ∧ Q))
2679   in Simp3_26b by now rewrite Equiv4_01.
2680 clear Simp3_26a.
2681 Conj n3_21a Simp3_26b.
2682 split.
2683 apply n3_21a.
2684 apply Simp3_26b.
2685 Equiv H.
2686 clear n3_21a. clear Simp3_26b.
2687 Conj n4_7a H.
2688 split.
2689 apply n4_7a.
2690 apply H.
2691 specialize n4_22 with (P → Q) (P → P ∧ Q) (P ↔ P ∧ Q).
2692 intros n4_22a.
2693 MP n4_22a H0.
2694 apply n4_22a.
2695 Qed.
2696
2697 Theorem n4_72 : ∀ P Q : Prop,
2698   (P → Q) ↔ (Q ↔ (P ∨ Q)).
2699 Proof. intros P Q.
2700 specialize Transp4_1 with P Q.
2701 intros Transp4_1a.
2702 specialize n4_71 with (¬Q) (¬P).
2703 intros n4_71a.
2704 Conj Transp4_1a n4_71a.
2705 split.
2706 apply Transp4_1a.
2707 apply n4_71a.
2708 specialize n4_22 with
2709   (P → Q) (¬Q → ¬P) (¬Q ↔ ¬Q ∧ ¬P).
2710 intros n4_22a.

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2711 MP n4_22a H.
2712 specialize n4_21 with ( $\neg Q$ ) ( $\neg Q \wedge \neg P$ ).
2713 intros n4_21a.
2714 Conj n4_22a n4_21a.
2715 split.
2716 apply n4_22a.
2717 apply n4_21a.
2718 specialize n4_22 with
2719   ( $P \rightarrow Q$ ) ( $\neg Q \leftrightarrow \neg Q \wedge \neg P$ ) ( $\neg Q \wedge \neg P \leftrightarrow \neg Q$ ).
2720 intros n4_22b.
2721 MP n4_22b H0.
2722 specialize n4_12 with ( $\neg Q \wedge \neg P$ ) (Q).
2723 intros n4_12a.
2724 Conj n4_22b n4_12a.
2725 split.
2726 apply n4_22b.
2727 apply n4_12a.
2728 specialize n4_22 with
2729   ( $P \rightarrow Q$ ) ( $(\neg Q \wedge \neg P) \leftrightarrow \neg Q$ ) ( $Q \leftrightarrow \neg(\neg Q \wedge \neg P)$ ).
2730 intros n4_22c.
2731 MP n4_22b H0.
2732 specialize n4_57 with Q P.
2733 intros n4_57a.
2734 apply propositional_extensionality in n4_57a.
2735 replace ( $\neg(\neg Q \wedge \neg P)$ ) with ( $Q \vee P$ ) in n4_22c
2736   by now apply n4_57a.
2737 specialize n4_31 with P Q.
2738 intros n4_31a.
2739 apply propositional_extensionality in n4_31a.
2740 replace ( $Q \vee P$ ) with ( $P \vee Q$ ) in n4_22c
2741   by now apply n4_22c.
2742 apply n4_22c.
2743 Qed.
2744 (*One could use Prod3_01 in lieu of n4_57.*)
2745
2746 Theorem n4_73 :  $\forall P Q : \text{Prop},$ 
2747    $Q \rightarrow (P \leftrightarrow (P \wedge Q)).$ 
2748 Proof. intros P Q.
2749 specialize Simp2_02 with P Q.
2750 intros Simp2_02a.
2751 specialize n4_71 with P Q.
2752 intros n4_71a.

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2753   replace ((P → Q) ↔ (P ↔ P ∧ Q)) with
2754     (((P→Q)→(P↔P∧Q))∧((P↔P∧Q)→(P→Q)))
2755     in n4_71a by now rewrite Equiv4_01.
2756   specialize Simp3_26 with
2757     ((P → Q) → P ↔ P ∧ Q) (P ↔ P ∧ Q → P → Q).
2758   intros Simp3_26a.
2759   MP Simp3_26a n4_71a.
2760   Syll Simp2_02a Simp3_26a Sa.
2761   apply Sa.
2762 Qed.
2763
2764 Theorem n4_74 : ∀ P Q : Prop,
2765   ¬P → (Q ↔ (P ∨ Q)).
2766 Proof. intros P Q.
2767   specialize n2_21 with P Q.
2768   intros n2_21a.
2769   specialize n4_72 with P Q.
2770   intros n4_72a.
2771   apply propositional_extensionality in n4_72a.
2772   replace (P → Q) with (Q ↔ P ∨ Q) in n2_21a
2773     by now apply n4_72a.
2774   apply n2_21a.
2775 Qed.
2776
2777 Theorem n4_76 : ∀ P Q R : Prop,
2778   ((P → Q) ∧ (P → R)) ↔ (P → (Q ∧ R)).
2779 Proof. intros P Q R.
2780   specialize n4_41 with (¬P) Q R.
2781   intros n4_41a.
2782   replace (¬P ∨ Q) with (P→Q) in n4_41a
2783     by now rewrite Impl1_01.
2784   replace (¬P ∨ R) with (P→R) in n4_41a
2785     by now rewrite Impl1_01.
2786   replace (¬P ∨ Q ∧ R) with (P → Q ∧ R) in n4_41a
2787     by now rewrite Impl1_01.
2788   specialize n4_21 with ((P → Q) ∧ (P → R)) (P → Q ∧ R).
2789   intros n4_21a.
2790   apply propositional_extensionality in n4_21a.
2791   replace ((P → Q ∧ R) ↔ (P → Q) ∧ (P → R)) with
2792     ((P → Q) ∧ (P → R) ↔ (P → Q ∧ R)) in n4_41a
2793     by now apply n4_41a.
2794   apply n4_41a.

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2795 Qed.
2796
2797 Theorem n4_77 :  $\forall P Q R : \text{Prop}$ ,
2798    $((Q \rightarrow P) \wedge (R \rightarrow P)) \leftrightarrow ((Q \vee R) \rightarrow P)$ .
2799 Proof. intros P Q R.
2800   specialize n3_44 with P Q R.
2801   intros n3_44a.
2802   specialize n2_2 with Q R.
2803   intros n2_2a.
2804   specialize Add1_3 with Q R.
2805   intros Add1_3a.
2806   specialize Syll2_06 with Q (Q  $\vee$  R) P.
2807   intros Syll2_06a.
2808   MP Syll2_06a n2_2a.
2809   specialize Syll2_06 with R (Q  $\vee$  R) P.
2810   intros Syll2_06b.
2811   MP Syll2_06b Add1_3a.
2812   Conj Syll2_06a Syll2_06b.
2813   split.
2814   apply Syll2_06a.
2815   apply Syll2_06b.
2816   specialize Comp3_43 with  $((Q \vee R) \rightarrow P)$ 
2817      $(Q \rightarrow P) (R \rightarrow P)$ .
2818   intros Comp3_43a.
2819   MP Comp3_43a H.
2820   clear n2_2a. clear Add1_3a. clear H.
2821   clear Syll2_06a. clear Syll2_06b.
2822   Conj n3_44a Comp3_43a.
2823   split.
2824   apply n3_44a.
2825   apply Comp3_43a.
2826   Equiv H.
2827   apply H.
2828 Qed.
2829
2830 Theorem n4_78 :  $\forall P Q R : \text{Prop}$ ,
2831    $((P \rightarrow Q) \vee (P \rightarrow R)) \leftrightarrow (P \rightarrow (Q \vee R))$ .
2832 Proof. intros P Q R.
2833   specialize n4_2 with  $((P \rightarrow Q) \vee (P \rightarrow R))$ .
2834   intros n4_2a.
2835   replace  $((P \rightarrow Q) \vee (P \rightarrow R)) \leftrightarrow ((P \rightarrow Q) \vee (P \rightarrow R))$  with
2836      $((P \rightarrow Q) \vee (P \rightarrow R)) \leftrightarrow ((P \rightarrow Q) \vee \neg P \vee R)$  in n4_2a

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2837     by now rewrite <- Impl1_01.
2838 replace (((P → Q) ∨ (P → R)) ↔ ((P → Q) ∨ ¬P ∨ R)) with
2839     (((P → Q) ∨ (P → R)) ↔ ((¬P ∨ Q) ∨ ¬P ∨ R)) in n4_2a
2840     by now rewrite <- Impl1_01.
2841 specialize n4_33 with (¬P) Q (¬P ∨ R).
2842 intros n4_33a.
2843 apply propositional_extensionality in n4_33a.
2844 replace ((¬P ∨ Q) ∨ ¬P ∨ R) with
2845     (¬P ∨ Q ∨ ¬P ∨ R) in n4_2a
2846     by now apply n4_33a.
2847 specialize n4_33 with Q (¬P) R.
2848 intros n4_33b.
2849 apply propositional_extensionality in n4_33b.
2850 replace (Q ∨ ¬P ∨ R) with
2851     ((Q ∨ ¬P) ∨ R) in n4_2a
2852     by now apply n4_33b.
2853 specialize n4_31 with (¬P) Q.
2854 intros n4_31a.
2855 specialize n4_37 with (¬P ∨ Q) (Q ∨ ¬P) R.
2856 intros n4_37a.
2857 MP n4_37a n4_31a.
2858 apply propositional_extensionality in n4_37a.
2859 replace ((Q ∨ ¬P) ∨ R) with
2860     ((¬P ∨ Q) ∨ R) in n4_2a
2861     by now apply n4_37a.
2862 specialize n4_33 with (¬P) (¬P ∨ Q) R.
2863 intros n4_33c.
2864 apply propositional_extensionality in n4_33c.
2865 replace (¬P ∨ (¬P ∨ Q) ∨ R) with
2866     ((¬P ∨ (¬P ∨ Q)) ∨ R) in n4_2a
2867     by now apply n4_33c.
2868 specialize n4_33 with (¬P) (¬P) Q.
2869 intros n4_33d.
2870 apply propositional_extensionality in n4_33d.
2871 replace (¬P ∨ ¬P ∨ Q) with
2872     ((¬P ∨ ¬P) ∨ Q) in n4_2a
2873     by now apply n4_33d.
2874 specialize n4_33 with (¬P ∨ ¬P) Q R.
2875 intros n4_33e.
2876 apply propositional_extensionality in n4_33e.
2877 replace (((¬P ∨ ¬P) ∨ Q) ∨ R) with
2878     ((¬P ∨ ¬P) ∨ Q ∨ R) in n4_2a

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2879     by now apply n4_33e.
2880 specialize n4_25 with ( $\neg$ P).
2881 intros n4_25a.
2882 specialize n4_37 with
2883   ( $\neg$ P) ( $\neg$ P  $\vee$   $\neg$ P) (Q  $\vee$  R).
2884 intros n4_37b.
2885 MP n4_37b n4_25a.
2886 apply propositional_extensionality in n4_25a.
2887 replace (( $\neg$ P  $\vee$   $\neg$ P)  $\vee$  Q  $\vee$  R) with
2888   (( $\neg$ P)  $\vee$  (Q  $\vee$  R)) in n4_2a
2889   by now rewrite <- n4_25a.
2890 replace ( $\neg$ P  $\vee$  Q  $\vee$  R) with
2891   (P  $\rightarrow$  (Q  $\vee$  R)) in n4_2a
2892   by now rewrite Impl1_01.
2893 apply n4_2a.
2894 Qed.
2895
2896 Theorem n4_79 :  $\forall$  P Q R : Prop,
2897   ((Q  $\rightarrow$  P)  $\vee$  (R  $\rightarrow$  P))  $\leftrightarrow$  ((Q  $\wedge$  R)  $\rightarrow$  P).
2898 Proof. intros P Q R.
2899   specialize Transp4_1 with Q P.
2900   intros Transp4_1a.
2901   specialize Transp4_1 with R P.
2902   intros Transp4_1b.
2903   Conj Transp4_1a Transp4_1b.
2904   split.
2905   apply Transp4_1a.
2906   apply Transp4_1b.
2907   specialize n4_39 with
2908     (Q $\rightarrow$ P) (R $\rightarrow$ P) ( $\neg$ P $\rightarrow$  $\neg$ Q) ( $\neg$ P $\rightarrow$  $\neg$ R).
2909   intros n4_39a.
2910   MP n4_39a H.
2911   specialize n4_78 with ( $\neg$ P) ( $\neg$ Q) ( $\neg$ R).
2912   intros n4_78a.
2913   rewrite Equiv4_01 in n4_78a.
2914   specialize Simp3_26 with
2915     ((( $\neg$ P $\rightarrow$  $\neg$ Q)  $\vee$  ( $\neg$ P $\rightarrow$  $\neg$ R))  $\rightarrow$  ( $\neg$ P $\rightarrow$ ( $\neg$ Q $\vee$  $\neg$ R)))
2916     (( $\neg$ P $\rightarrow$ ( $\neg$ Q $\vee$  $\neg$ R))  $\rightarrow$  (( $\neg$ P $\rightarrow$  $\neg$ Q)  $\vee$  ( $\neg$ P $\rightarrow$  $\neg$ R))).
2917   intros Simp3_26a.
2918   MP Simp3_26a n4_78a.
2919   specialize Transp2_15 with P ( $\neg$ Q $\vee$  $\neg$ R).
2920   intros Transp2_15a.

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```

2921 specialize Simp3_27 with
2922   (((¬P→¬Q)∨(¬P→¬R))→(¬P→(¬Q∨¬R)))
2923   (((¬P→(¬Q∨¬R))→((¬P→¬Q)∨(¬P→¬R))).
2924 intros Simp3_27a.
2925 MP Simp3_27a n4_78a.
2926 specialize Transp2_15 with (¬Q∨¬R) P.
2927 intros Transp2_15b.
2928 specialize Syll12_06 with ((¬P→¬Q)∨(¬P→¬R))
2929   (¬P→(¬Q∨¬R)) (¬(¬Q∨¬R)→P).
2930 intros Syll12_06a.
2931 MP Syll12_06a Simp3_26a.
2932 MP Syll12_06a Transp2_15a.
2933 specialize Syll12_06 with (¬(¬Q∨¬R)→P)
2934   (¬P→(¬Q∨¬R)) ((¬P→¬Q)∨(¬P→¬R)).
2935 intros Syll12_06b.
2936 MP Syll12_06b Trans2_15b.
2937 MP Syll12_06b Simp3_27a.
2938 Conj Syll12_06a Syll12_06b.
2939 split.
2940 apply Syll12_06a.
2941 apply Syll12_06b.
2942 Equiv H0.
2943 clear Transp4_1a. clear Transp4_1b. clear H.
2944   clear Simp3_26a. clear Syll12_06b. clear n4_78a.
2945   clear Transp2_15a. clear Simp3_27a.
2946   clear Transp2_15b. clear Syll12_06a.
2947 Conj n4_39a H0.
2948 split.
2949 apply n4_39a.
2950 apply H0.
2951 specialize n4_22 with ((Q→P)∨(R→P))
2952   ((¬P→¬Q)∨(¬P→¬R)) (¬(¬Q∨¬R)→P).
2953 intros n4_22a.
2954 MP n4_22a H.
2955 specialize n4_2 with (¬(¬Q∨¬R)→P).
2956 intros n4_2a.
2957 Conj n4_22a n4_2a.
2958 split.
2959 apply n4_22a.
2960 apply n4_2a.
2961 specialize n4_22 with ((Q→P)∨(R→P))
2962   (¬(¬Q∨¬R)→P) (¬(¬Q∨¬R)→P).

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```

2963     intros n4_22b.
2964     MP n4_22b H1.
2965     rewrite <- Prod3_01 in n4_22b.
2966     apply n4_22b.
2967 Qed.
2968
2969 Theorem n4_8 :  $\forall P : \text{Prop},$ 
2970    $(P \rightarrow \neg P) \leftrightarrow \neg P.$ 
2971 Proof. intros P.
2972   specialize Abs2_01 with P.
2973   intros Abs2_01a.
2974   specialize Simp2_02 with P ( $\neg P$ ).
2975   intros Simp2_02a.
2976   Conj Abs2_01a Simp2_02a.
2977   split.
2978   apply Abs2_01a.
2979   apply Simp2_02a.
2980   Equiv H.
2981   apply H.
2982 Qed.
2983
2984 Theorem n4_81 :  $\forall P : \text{Prop},$ 
2985    $(\neg P \rightarrow P) \leftrightarrow P.$ 
2986 Proof. intros P.
2987   specialize n2_18 with P.
2988   intros n2_18a.
2989   specialize Simp2_02 with ( $\neg P$ ) P.
2990   intros Simp2_02a.
2991   Conj n2_18a Simp2_02a.
2992   split.
2993   apply n2_18a.
2994   apply Simp2_02a.
2995   Equiv H.
2996   apply H.
2997 Qed.
2998
2999 Theorem n4_82 :  $\forall P Q : \text{Prop},$ 
3000    $((P \rightarrow Q) \wedge (P \rightarrow \neg Q)) \leftrightarrow \neg P.$ 
3001 Proof. intros P Q.
3002   specialize n2_65 with P Q.
3003   intros n2_65a.
3004   specialize Imp3_31 with  $(P \rightarrow Q)$   $(P \rightarrow \neg Q)$  ( $\neg P$ ).

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3005     intros Imp3_31a.
3006     MP Imp3_31a n2_65a.
3007     specialize n2_21 with P Q.
3008     intros n2_21a.
3009     specialize n2_21 with P ( $\neg$ Q).
3010     intros n2_21b.
3011     Conj n2_21a n2_21b.
3012     split.
3013     apply n2_21a.
3014     apply n2_21b.
3015     specialize Comp3_43 with ( $\neg$ P) (P $\rightarrow$ Q) (P $\rightarrow\neg$ Q).
3016     intros Comp3_43a.
3017     MP Comp3_43a H.
3018     clear n2_65a. clear n2_21a.
3019     clear n2_21b. clear H.
3020     Conj Imp3_31a Comp3_43a.
3021     split.
3022     apply Imp3_31a.
3023     apply Comp3_43a.
3024     Equiv H.
3025     apply H.
3026 Qed.
3027
3028 Theorem n4_83 :  $\forall$  P Q : Prop,
3029   ((P  $\rightarrow$  Q)  $\wedge$  ( $\neg$ P  $\rightarrow$  Q))  $\leftrightarrow$  Q.
3030 Proof. intros P Q.
3031     specialize n2_61 with P Q.
3032     intros n2_61a.
3033     specialize Imp3_31 with (P $\rightarrow$ Q) ( $\neg$ P $\rightarrow$ Q) (Q).
3034     intros Imp3_31a.
3035     MP Imp3_31a n2_61a.
3036     specialize Simp2_02 with P Q.
3037     intros Simp2_02a.
3038     specialize Simp2_02 with ( $\neg$ P) Q.
3039     intros Simp2_02b.
3040     Conj Simp2_02a Simp2_02b.
3041     split.
3042     apply Simp2_02a.
3043     apply Simp2_02b.
3044     specialize Comp3_43 with Q (P $\rightarrow$ Q) ( $\neg$ P $\rightarrow$ Q).
3045     intros Comp3_43a.
3046     MP Comp3_43a H.

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3047 clear n2_61a. clear Simp2_02a.
3048 clear Simp2_02b. clear H.
3049 Conj Imp3_31a Comp3_43a.
3050 split.
3051 apply Imp3_31a.
3052 apply Comp3_43a.
3053 Equiv H.
3054 apply H.
3055 Qed.
3056
3057 Theorem n4_84 :  $\forall$  P Q R : Prop,
3058   (P  $\leftrightarrow$  Q)  $\rightarrow$  ((P  $\rightarrow$  R)  $\leftrightarrow$  (Q  $\rightarrow$  R)).
3059 Proof. intros P Q R.
3060   specialize Syll2_06 with P Q R.
3061   intros Syll2_06a.
3062   specialize Syll2_06 with Q P R.
3063   intros Syll2_06b.
3064   Conj Syll2_06a Syll2_06b.
3065   split.
3066   apply Syll2_06a.
3067   apply Syll2_06b.
3068   specialize n3_47 with
3069     (P $\rightarrow$ Q) (Q $\rightarrow$ P) ((Q $\rightarrow$ R) $\rightarrow$ P $\rightarrow$ R) ((P $\rightarrow$ R) $\rightarrow$ Q $\rightarrow$ R).
3070   intros n3_47a.
3071   MP n3_47a H.
3072   replace ((P $\rightarrow$ Q)  $\wedge$  (Q  $\rightarrow$  P)) with (P $\leftrightarrow$ Q)
3073     in n3_47a by now rewrite Equiv4_01.
3074   replace (((Q $\rightarrow$ R) $\rightarrow$ P $\rightarrow$ R) $\wedge$ ((P $\rightarrow$ R) $\rightarrow$ Q $\rightarrow$ R)) with
3075     ((Q $\rightarrow$ R) $\leftrightarrow$ (P $\rightarrow$ R)) in n3_47a by
3076     now rewrite Equiv4_01.
3077   specialize n4_21 with (P $\rightarrow$ R) (Q $\rightarrow$ R).
3078   intros n4_21a.
3079   apply propositional_extensionality in n4_21a.
3080   replace ((Q  $\rightarrow$  R)  $\leftrightarrow$  (P  $\rightarrow$  R)) with
3081     ((P $\rightarrow$  R)  $\leftrightarrow$  (Q  $\rightarrow$  R)) in n3_47a
3082     by now apply n4_21a.
3083   apply n3_47a.
3084   Qed.
3085
3086 Theorem n4_85 :  $\forall$  P Q R : Prop,
3087   (P  $\leftrightarrow$  Q)  $\rightarrow$  ((R  $\rightarrow$  P)  $\leftrightarrow$  (R  $\rightarrow$  Q)).
3088 Proof. intros P Q R.

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3089 specialize Syll2_05 with R P Q.
3090 intros Syll2_05a.
3091 specialize Syll2_05 with R Q P.
3092 intros Syll2_05b.
3093 Conj Syll2_05a Syll2_05b.
3094 split.
3095 apply Syll2_05a.
3096 apply Syll2_05b.
3097 specialize n3_47 with
3098   (P→Q) (Q→P) ((R→P)→R→Q) ((R→Q)→R→P).
3099 intros n3_47a.
3100 MP n3_47a H.
3101 replace ((P→Q) ∧ (Q → P)) with (P↔Q) in n3_47a
3102 by now rewrite Equiv4_01.
3103 replace (((R→P)→R→Q)∧((R→Q)→R→P)) with
3104   ((R→P)↔(R→Q)) in n3_47a
3105   by now rewrite Equiv4_01.
3106 apply n3_47a.
3107 Qed.
3108
3109 Theorem n4_86 : ∀ P Q R : Prop,
3110   (P ↔ Q) → ((P ↔ R) ↔ (Q ↔ R)).
3111 Proof. intros P Q R.
3112 specialize n4_22 with Q P R.
3113 intros n4_22a.
3114 specialize Exp3_3 with (Q↔P) (P↔R) (Q↔R).
3115 intros Exp3_3a. (*Not cited*)
3116 MP Exp3_3a n4_22a.
3117 specialize n4_22 with P Q R.
3118 intros n4_22b.
3119 specialize Exp3_3 with (P↔Q) (Q↔R) (P↔R).
3120 intros Exp3_3b.
3121 MP Exp3_3b n4_22b.
3122 specialize n4_21 with P Q.
3123 intros n4_21a.
3124 apply propositional_extensionality in n4_21a.
3125 replace (Q↔P) with (P↔Q) in Exp3_3a
3126   by now apply n4_21a.
3127 clear n4_22a. clear n4_22b. clear n4_21a.
3128 Conj Exp3_3a Exp3_3b.
3129 split.
3130 apply Exp3_3a.

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3131 apply Exp3_3b.
3132 specialize Comp3_43 with (P↔Q)
3133   ((P↔R)→(Q↔R)) ((Q↔R)→(P↔R)).
3134 intros Comp3_43a. (*Not cited*)
3135 MP Comp3_43a H.
3136 replace (((P↔R)→(Q↔R))∧((Q↔R)→(P↔R)))
3137   with ((P↔R)↔(Q↔R)) in Comp3_43a
3138   by now rewrite Equiv4_01.
3139 apply Comp3_43a.
3140 Qed.
3141
3142 Theorem n4_87 : ∀ P Q R : Prop,
3143   (((P ∧ Q) → R) ↔ (P → Q → R)) ↔
3144   ((Q → (P → R)) ↔ (Q ∧ P → R)).
3145 Proof. intros P Q R.
3146 specialize Exp3_3 with P Q R.
3147 intros Exp3_3a.
3148 specialize Imp3_31 with P Q R.
3149 intros Imp3_31a.
3150 Conj Exp3_3a Imp3_31a.
3151 split.
3152 apply Exp3_3a.
3153 apply Imp3_31a.
3154 Equiv H.
3155 specialize Exp3_3 with Q P R.
3156 intros Exp3_3b.
3157 specialize Imp3_31 with Q P R.
3158 intros Imp3_31b.
3159 Conj Exp3_3b Imp3_31b.
3160 split.
3161 apply Exp3_3b.
3162 apply Imp3_31b.
3163 Equiv H0.
3164 (*specialize Comm2_04 with P Q R.
3165   intros Comm2_04a.
3166   specialize Comm2_04 with Q P R.
3167   intros Comm2_04b.
3168   Conj Comm2_04a Comm2_04b.
3169   split.
3170   apply Comm2_04a.
3171   apply Comm2_04b.
3172   Equiv H1.*) (*Comm2_04 is cited in proof.
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3173   We have not used it to construct the chain
3174   of biconditionals. *)
3175   specialize n4_21 with (Q → P → R) (Q ∧ P → R).
3176   intros n4_21a.
3177   apply propositional_extensionality in n4_21a.
3178   replace ((Q ∧ P → R) ↔ (Q → P → R)) with
3179     ((Q → P → R) ↔ (Q ∧ P → R)) in H0
3180     by now apply n4_21a.
3181   specialize Simp2_02 with ((P ∧ Q → R) ↔ (P → Q → R))
3182     ((Q → P → R) ↔ (Q ∧ P → R)).
3183   intros Simp2_02a.
3184   MP Simp2_02a H0.
3185   specialize Simp2_02 with ((Q → P → R) ↔ (Q ∧ P → R))
3186     ((P ∧ Q → R) ↔ (P → Q → R)).
3187   intros Simp2_02b.
3188   MP Simp2_02b H.
3189   Conj Simp2_02a Simp2_02b.
3190   split.
3191   apply Simp2_02a.
3192   apply Simp2_02b.
3193   Equiv H1.
3194   apply H1.
3195   Qed.
3196
3197   End No4.
3198
3199   Module No5.
3200
3201   Import No1.
3202   Import No2.
3203   Import No3.
3204   Import No4.
3205
3206   Theorem n5_1 : ∀ P Q : Prop,
3207     (P ∧ Q) → (P ↔ Q).
3208   Proof. intros P Q.
3209     specialize n3_4 with P Q.
3210     intros n3_4a.
3211     specialize n3_4 with Q P.
3212     intros n3_4b.
3213     specialize n3_22 with P Q.
3214     intros n3_22a.

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3215 Syll n3_22a n3_4b Sa.
3216 clear n3_22a. clear n3_4b.
3217 Conj n3_4a Sa.
3218 split.
3219 apply n3_4a.
3220 apply Sa.
3221 specialize n4_76 with (P $\wedge$ Q) (P $\rightarrow$ Q) (Q $\rightarrow$ P).
3222 intros n4_76a. (*Not cited*)
3223 apply propositional_extensionality in n4_76a.
3224 replace ((P $\wedge$ Q $\rightarrow$ P $\rightarrow$ Q) $\wedge$ (P $\wedge$ Q $\rightarrow$ Q $\rightarrow$ P)) with
3225   (P  $\wedge$  Q  $\rightarrow$  (P  $\rightarrow$  Q)  $\wedge$  (Q  $\rightarrow$  P)) in H
3226   by now apply n4_76a.
3227 replace ((P $\rightarrow$ Q) $\wedge$ (Q $\rightarrow$ P)) with (P $\leftrightarrow$ Q) in H
3228   by now rewrite Equiv4_01.
3229 apply H.
3230 Qed.
3231
3232 Theorem n5_11 :  $\forall$  P Q : Prop,
3233   (P  $\rightarrow$  Q)  $\vee$  ( $\neg$ P  $\rightarrow$  Q).
3234 Proof. intros P Q.
3235 specialize n2_5 with P Q.
3236 intros n2_5a.
3237 specialize n2_54 with (P  $\rightarrow$  Q) ( $\neg$ P  $\rightarrow$  Q).
3238 intros n2_54a.
3239 MP n2_54a n2_5a.
3240 apply n2_54a.
3241 Qed.
3242 (*The proof sketch cites n2_51,
3243   but this may be a misprint.*)
3244
3245 Theorem n5_12 :  $\forall$  P Q : Prop,
3246   (P  $\rightarrow$  Q)  $\vee$  (P  $\rightarrow$   $\neg$ Q).
3247 Proof. intros P Q.
3248 specialize n2_51 with P Q.
3249 intros n2_51a.
3250 specialize n2_54 with ((P  $\rightarrow$  Q)) (P  $\rightarrow$   $\neg$ Q).
3251 intros n2_54a.
3252 MP n2_54a n2_5a.
3253 apply n2_54a.
3254 Qed.
3255 (*The proof sketch cites n2_52,
3256   but this may be a misprint.*)

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3257
3258 Theorem n5_13 :  $\forall P Q : \text{Prop},$ 
3259    $(P \rightarrow Q) \vee (Q \rightarrow P).$ 
3260 Proof. intros P Q.
3261 specialize n2_521 with P Q.
3262 intros n2_521a.
3263 replace  $(\neg(P \rightarrow Q) \rightarrow Q \rightarrow P)$  with
3264    $(\neg\neg(P \rightarrow Q) \vee (Q \rightarrow P))$  in n2_521a
3265   by now rewrite <- Impl1_01.
3266 specialize n4_13 with  $(P \rightarrow Q).$ 
3267 intros n4_13a. (*Not cited*)
3268 apply propositional_extensionality in n4_13a.
3269 replace  $(\neg\neg(P \rightarrow Q))$  with  $(P \rightarrow Q)$ 
3270   in n2_521a by now apply n4_13a.
3271 apply n2_521a.
3272 Qed.
3273
3274 Theorem n5_14 :  $\forall P Q R : \text{Prop},$ 
3275    $(P \rightarrow Q) \vee (Q \rightarrow R).$ 
3276 Proof. intros P Q R.
3277 specialize Simp2_02 with P Q.
3278 intros Simp2_02a.
3279 specialize Transp2_16 with Q  $(P \rightarrow Q).$ 
3280 intros Transp2_16a.
3281 MP Transp2_16a Simp2_02a.
3282 specialize n2_21 with Q R.
3283 intros n2_21a.
3284 Syll Transp2_16a n2_21a Sa.
3285 replace  $(\neg(P \rightarrow Q) \rightarrow (Q \rightarrow R))$  with
3286    $(\neg\neg(P \rightarrow Q) \vee (Q \rightarrow R))$  in Sa
3287   by now rewrite <- Impl1_01.
3288 specialize n4_13 with  $(P \rightarrow Q).$ 
3289 intros n4_13a.
3290 apply propositional_extensionality in n4_13a.
3291 replace  $(\neg\neg(P \rightarrow Q))$  with  $(P \rightarrow Q)$  in Sa
3292   by now apply n4_13a.
3293 apply Sa.
3294 Qed.
3295
3296 Theorem n5_15 :  $\forall P Q : \text{Prop},$ 
3297    $(P \leftrightarrow Q) \vee (P \leftrightarrow \neg Q).$ 
3298 Proof. intros P Q.

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3299 specialize n4_61 with P Q.
3300 intros n4_61a.
3301 replace ( $\neg(P \rightarrow Q) \leftrightarrow P \wedge \neg Q$ ) with
3302   ( $(\neg(P \rightarrow Q) \rightarrow P \wedge \neg Q) \wedge ((P \wedge \neg Q) \rightarrow \neg(P \rightarrow Q))$ ) in n4_61a
3303   by now rewrite Equiv4_01.
3304 specialize Simp3_26 with
3305   ( $\neg(P \rightarrow Q) \rightarrow P \wedge \neg Q$ ) ( $(P \wedge \neg Q) \rightarrow \neg(P \rightarrow Q)$ ).
3306 intros Simp3_26a.
3307 MP Simp3_26a n4_61a.
3308 specialize n5_1 with P ( $\neg Q$ ).
3309 intros n5_1a.
3310 Syll Simp3_26a n5_1a Sa.
3311 specialize n2_54 with ( $P \rightarrow Q$ ) ( $P \leftrightarrow \neg Q$ ).
3312 intros n2_54a.
3313 MP n2_54a Sa.
3314 specialize n4_61 with Q P.
3315 intros n4_61b.
3316 replace ( $(\neg(Q \rightarrow P)) \leftrightarrow (Q \wedge \neg P)$ ) with
3317   ( $((\neg(Q \rightarrow P)) \rightarrow (Q \wedge \neg P)) \wedge ((Q \wedge \neg P) \rightarrow (\neg(Q \rightarrow P)))$ )
3318   in n4_61b by now rewrite Equiv4_01.
3319 specialize Simp3_26 with
3320   ( $\neg(Q \rightarrow P) \rightarrow (Q \wedge \neg P)$ ) ( $(Q \wedge \neg P) \rightarrow (\neg(Q \rightarrow P))$ ).
3321 intros Simp3_26b.
3322 MP Simp3_26b n4_61b.
3323 specialize n5_1 with Q ( $\neg P$ ).
3324 intros n5_1b.
3325 Syll Simp3_26b n5_1b Sb.
3326 specialize n4_12 with P Q.
3327 intros n4_12a.
3328 apply propositional_extensionality in n4_12a.
3329 replace ( $Q \leftrightarrow \neg P$ ) with ( $P \leftrightarrow \neg Q$ ) in Sb
3330   by now apply n4_12a.
3331 specialize n2_54 with ( $Q \rightarrow P$ ) ( $P \leftrightarrow \neg Q$ ).
3332 intros n2_54b.
3333 MP n2_54b Sb.
3334 replace ( $\neg(P \rightarrow Q) \rightarrow P \leftrightarrow \neg Q$ ) with
3335   ( $(\neg\neg(P \rightarrow Q) \vee (P \leftrightarrow \neg Q))$ ) in Sa
3336   by now rewrite <- Impl1_01.
3337 specialize n4_13 with ( $P \rightarrow Q$ ).
3338 intros n4_13a.
3339 apply propositional_extensionality in n4_13a.
3340 replace ( $\neg\neg(P \rightarrow Q)$ ) with ( $P \rightarrow Q$ ) in Sa

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3341   by now apply n4_13a.
3342   replace ( $\neg(Q \rightarrow P) \rightarrow (P \leftrightarrow \neg Q)$ ) with
3343     ( $\neg\neg(Q \rightarrow P) \vee (P \leftrightarrow \neg Q)$ ) in Sb
3344   by now rewrite <- Impl1_01.
3345   specialize n4_13 with ( $Q \rightarrow P$ ).
3346   intros n4_13b.
3347   apply propositional_extensionality in n4_13b.
3348   replace ( $\neg\neg(Q \rightarrow P)$ ) with ( $Q \rightarrow P$ ) in Sb
3349   by now apply n4_13b.
3350   clear n4_61a. clear Simp3_26a. clear n5_1a.
3351     clear n2_54a. clear n4_61b. clear Simp3_26b.
3352     clear n5_1b. clear n4_12a. clear n2_54b.
3353     clear n4_13a. clear n4_13b.
3354   Conj Sa Sb.
3355   split.
3356   apply Sa.
3357   apply Sb.
3358   specialize n4_31 with ( $P \rightarrow Q$ ) ( $P \leftrightarrow \neg Q$ ).
3359   intros n4_31a.
3360   apply propositional_extensionality in n4_31a.
3361   replace (( $P \rightarrow Q$ )  $\vee$  ( $P \leftrightarrow \neg Q$ )) with
3362     (( $P \leftrightarrow \neg Q$ )  $\vee$  ( $P \rightarrow Q$ )) in H
3363   by now apply n4_31a.
3364   specialize n4_31 with ( $Q \rightarrow P$ ) ( $P \leftrightarrow \neg Q$ ).
3365   intros n4_31b.
3366   apply propositional_extensionality in n4_31b.
3367   replace (( $Q \rightarrow P$ )  $\vee$  ( $P \leftrightarrow \neg Q$ )) with
3368     (( $P \leftrightarrow \neg Q$ )  $\vee$  ( $Q \rightarrow P$ )) in H
3369   by now apply n4_31b.
3370   specialize n4_41 with ( $P \leftrightarrow \neg Q$ ) ( $P \rightarrow Q$ ) ( $Q \rightarrow P$ ).
3371   intros n4_41a.
3372   apply propositional_extensionality in n4_41a.
3373   replace ((( $P \leftrightarrow \neg Q$ )  $\vee$  ( $P \rightarrow Q$ ))  $\wedge$  (( $P \leftrightarrow \neg Q$ )  $\vee$  ( $Q \rightarrow P$ )))
3374     with (( $P \leftrightarrow \neg Q$ )  $\vee$  ( $P \rightarrow Q$ )  $\wedge$  ( $Q \rightarrow P$ )) in H
3375   by now apply n4_41a.
3376   replace (( $P \rightarrow Q$ )  $\wedge$  ( $Q \rightarrow P$ )) with ( $P \leftrightarrow Q$ ) in H
3377   by now rewrite Equiv4_01.
3378   specialize n4_31 with ( $P \leftrightarrow \neg Q$ ) ( $P \leftrightarrow Q$ ).
3379   intros n4_31c.
3380   apply propositional_extensionality in n4_31c.
3381   replace (( $P \leftrightarrow \neg Q$ )  $\vee$  ( $P \leftrightarrow Q$ )) with
3382     (( $P \leftrightarrow Q$ )  $\vee$  ( $P \leftrightarrow \neg Q$ )) in H

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3383         by now apply n4_31c.
3384     apply H.
3385 Qed.
3386
3387 Theorem n5_16 :  $\forall P Q : \text{Prop},$ 
3388    $\neg((P \leftrightarrow Q) \wedge (P \leftrightarrow \neg Q)).$ 
3389 Proof. intros P Q.
3390 specialize Simp3_26 with  $((P \rightarrow Q) \wedge (P \rightarrow \neg Q)) (Q \rightarrow P).$ 
3391 intros Simp3_26a.
3392 specialize Id2_08 with  $((P \leftrightarrow Q) \wedge (P \rightarrow \neg Q)).$ 
3393 intros Id2_08a.
3394 specialize n4_32 with  $(P \rightarrow Q) (P \rightarrow \neg Q) (Q \rightarrow P).$ 
3395 intros n4_32a.
3396 apply propositional_extensionality in n4_32a.
3397 replace  $((P \rightarrow Q) \wedge (P \rightarrow \neg Q)) \wedge (Q \rightarrow P)$  with
3398    $((P \rightarrow Q) \wedge ((P \rightarrow \neg Q) \wedge (Q \rightarrow P)))$  in Simp3_26a
3399   by now apply n4_32a.
3400 specialize n4_3 with  $(Q \rightarrow P) (P \rightarrow \neg Q).$ 
3401 intros n4_3a.
3402 apply propositional_extensionality in n4_3a.
3403 replace  $((P \rightarrow \neg Q) \wedge (Q \rightarrow P))$  with
3404    $((Q \rightarrow P) \wedge (P \rightarrow \neg Q))$  in Simp3_26a
3405   by now apply n4_3a.
3406 specialize n4_32 with  $(P \rightarrow Q) (Q \rightarrow P) (P \rightarrow \neg Q).$ 
3407 intros n4_32b.
3408 apply propositional_extensionality in n4_32b.
3409 replace  $((P \rightarrow Q) \wedge (Q \rightarrow P) \wedge (P \rightarrow \neg Q))$  with
3410    $((P \rightarrow Q) \wedge (Q \rightarrow P)) \wedge (P \rightarrow \neg Q)$  in Simp3_26a
3411   by now apply n4_32b.
3412 replace  $((P \rightarrow Q) \wedge (Q \rightarrow P))$  with  $(P \leftrightarrow Q)$ 
3413   in Simp3_26a by now rewrite Equiv4_01.
3414 Syll Id2_08a Simp3_26a Sa.
3415 specialize n4_82 with P Q.
3416 intros n4_82a.
3417 apply propositional_extensionality in n4_82a.
3418 replace  $((P \rightarrow Q) \wedge (P \rightarrow \neg Q))$  with  $(\neg P)$  in Sa
3419   by now apply n4_82a.
3420 specialize Simp3_27 with
3421    $(P \rightarrow Q) ((Q \rightarrow P) \wedge (P \rightarrow \neg Q)).$ 
3422 intros Simp3_27a.
3423 replace  $((P \rightarrow Q) \wedge (Q \rightarrow P) \wedge (P \rightarrow \neg Q))$  with
3424    $((P \rightarrow Q) \wedge (Q \rightarrow P)) \wedge (P \rightarrow \neg Q)$  in Simp3_27a

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3425     by now apply n4_32b.
3426 replace ((P → Q) ∧ (Q → P)) with (P↔Q)
3427     in Simp3_27a by now rewrite Equiv4_01.
3428 specialize Syll3_33 with Q P (¬Q).
3429 intros Syll3_33a.
3430 Syll Simp3_27a Syll2_06a Sb.
3431 specialize Abs2_01 with Q.
3432 intros Abs2_01a.
3433 Syll Sb Abs2_01a Sc.
3434 clear Sb. clear Simp3_26a. clear Id2_08a.
3435     clear n4_82a. clear Simp3_27a. clear Syll3_33a.
3436     clear Abs2_01a. clear n4_32a. clear n4_32b. clear n4_3a.
3437 Conj Sa Sc.
3438 split.
3439 apply Sa.
3440 apply Sc.
3441 specialize Comp3_43 with
3442     ((P ↔ Q) ∧ (P → ¬Q)) (¬P) (¬Q).
3443 intros Comp3_43a.
3444 MP Comp3_43a H.
3445 specialize n4_65 with Q P.
3446 intros n4_65a.
3447 specialize n4_3 with (¬P) (¬Q).
3448 intros n4_3a.
3449 apply propositional_extensionality in n4_3a.
3450 replace (¬Q ∧ ¬P) with (¬P ∧ ¬Q) in n4_65a
3451     by now apply n4_3a.
3452 apply propositional_extensionality in n4_65a.
3453 replace (¬P ∧ ¬Q) with (¬(¬Q → P)) in Comp3_43a
3454     by now apply n4_65a.
3455 specialize Exp3_3 with
3456     (P↔Q) (P→¬Q) (¬(¬Q→P)).
3457 intros Exp3_3a.
3458 MP Exp3_3a Comp3_43a.
3459 replace ((P→¬Q)→¬(¬Q→P)) with
3460     (¬(P→¬Q)∨¬(¬Q→P)) in Exp3_3a
3461     by now rewrite <- Impl1_01.
3462 specialize n4_51 with (P→¬Q) (¬Q→P).
3463 intros n4_51a.
3464 apply propositional_extensionality in n4_51a.
3465 replace (¬(P → ¬Q) ∨ ¬(¬Q → P)) with
3466     (¬((P → ¬Q) ∧ (¬Q → P))) in Exp3_3a

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3467     by now apply n4_51a.
3468 replace ((P→¬Q)∧(¬Q→P)) with (P↔¬Q)
3469 in Exp3_3a by now rewrite Equiv4_01.
3470 replace ((P↔Q)→¬(P↔¬Q)) with
3471   (¬(P↔Q)∨¬(P↔¬Q)) in Exp3_3a
3472   by now rewrite Impl1_01.
3473 specialize n4_51 with (P↔Q) (P↔¬Q).
3474 intros n4_51b.
3475 apply propositional_extensionality in n4_51b.
3476 replace (¬(P ↔ Q) ∨ ¬(P ↔ ¬Q)) with
3477   (¬((P ↔ Q) ∧ (P ↔ ¬Q))) in Exp3_3a
3478   by now apply n4_51b.
3479 apply Exp3_3a.
3480 Qed.
3481
3482 Theorem n5_17 : ∀ P Q : Prop,
3483   ((P ∨ Q) ∧ ¬(P ∧ Q)) ↔ (P ↔ ¬Q).
3484 Proof. intros P Q.
3485 specialize n4_64 with Q P.
3486 intros n4_64a.
3487 specialize n4_21 with (Q∨P) (¬Q→P).
3488 intros n4_21a.
3489 apply propositional_extensionality in n4_21a.
3490 replace ((¬Q→P)↔(Q∨P)) with
3491   ((Q∨P)↔(¬Q→P)) in n4_64a
3492   by now apply n4_21a.
3493 specialize n4_31 with P Q.
3494 intros n4_31a.
3495 apply propositional_extensionality in n4_31a.
3496 replace (Q∨P) with (P∨Q) in n4_64a
3497   by now apply n4_31a.
3498 specialize n4_63 with P Q.
3499 intros n4_63a.
3500 specialize n4_21 with (P ∧ Q) (¬(P→¬Q)).
3501 intros n4_21b.
3502 apply propositional_extensionality in n4_21b.
3503 replace (¬(P → ¬Q) ↔ P ∧ Q) with
3504   (P ∧ Q ↔ ¬(P → ¬Q)) in n4_63a
3505   by now apply n4_21b.
3506 specialize Transp4_11 with (P∧Q) (¬(P→¬Q)).
3507 intros Transp4_11a.
3508 specialize n4_13 with (P→¬Q).

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3509   intros n4_13a.
3510   apply propositional_extensionality in n4_13a.
3511   replace ( $\neg\neg(P \rightarrow \neg Q)$ ) with ( $P \rightarrow \neg Q$ )
3512     in Transp4_11a by now apply n4_13a.
3513   apply propositional_extensionality in Transp4_11a.
3514   replace ( $P \wedge Q \leftrightarrow \neg(P \rightarrow \neg Q)$ ) with
3515     ( $\neg(P \wedge Q) \leftrightarrow (P \rightarrow \neg Q)$ ) in n4_63a
3516     by now apply Transp4_11a.
3517   clear Transp4_11a. clear n4_21a.
3518   clear n4_31a. clear n4_21b. clear n4_13a.
3519   Conj n4_64a n4_63a.
3520   split.
3521   apply n4_64a.
3522   apply n4_63a.
3523   specialize n4_38 with
3524     ( $P \vee Q$ ) ( $\neg(P \wedge Q)$ ) ( $\neg Q \rightarrow P$ ) ( $P \rightarrow \neg Q$ ).
3525   intros n4_38a.
3526   MP n4_38a H.
3527   replace ( $(\neg Q \rightarrow P) \wedge (P \rightarrow \neg Q)$ ) with ( $\neg Q \leftrightarrow P$ )
3528     in n4_38a by now rewrite Equiv4_01.
3529   specialize n4_21 with P ( $\neg Q$ ).
3530   intros n4_21c.
3531   apply propositional_extensionality in n4_21c.
3532   replace ( $\neg Q \leftrightarrow P$ ) with ( $P \leftrightarrow \neg Q$ ) in n4_38a
3533     by now apply n4_21c.
3534   apply n4_38a.
3535   Qed.
3536
3537   Theorem n5_18 :  $\forall P Q : \text{Prop}$ ,
3538     ( $P \leftrightarrow Q$ )  $\leftrightarrow \neg(P \leftrightarrow \neg Q)$ .
3539   Proof. intros P Q.
3540   specialize n5_15 with P Q.
3541   intros n5_15a.
3542   specialize n5_16 with P Q.
3543   intros n5_16a.
3544   Conj n5_15a n5_16a.
3545   split.
3546   apply n5_15a.
3547   apply n5_16a.
3548   specialize n5_17 with ( $P \leftrightarrow Q$ ) ( $P \leftrightarrow \neg Q$ ).
3549   intros n5_17a.
3550   rewrite Equiv4_01 in n5_17a.

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3551 specialize Simp3_26 with
3552   (((P↔Q)∨(P↔¬Q))∧¬((P↔Q)∧(P↔¬Q)))
3553   →((P↔Q)↔¬(P↔¬Q))) ((P↔Q)↔¬(P↔¬Q)) →
3554   (((P↔Q)∨(P↔¬Q))∧¬((P↔Q)∧(P↔¬Q))).
3555 intros Simp3_26a. (*not cited*)
3556 MP Simp3_26a n5_17a.
3557 MP Simp3_26a H.
3558 apply Simp3_26a.
3559 Qed.
3560
3561 Theorem n5_19 : ∀ P : Prop,
3562   ¬(P ↔ ¬P).
3563 Proof. intros P.
3564 specialize n5_18 with P P.
3565 intros n5_18a.
3566 specialize n4_2 with P.
3567 intros n4_2a.
3568 rewrite Equiv4_01 in n5_18a.
3569 specialize Simp3_26 with (P↔P→¬(P↔¬P))
3570   (¬(P↔¬P)→P↔P).
3571 intros Simp3_26a. (*not cited*)
3572 MP Simp3_26a n5_18a.
3573 MP Simp3_26a n4_2a.
3574 apply Simp3_26a.
3575 Qed.
3576
3577 Theorem n5_21 : ∀ P Q : Prop,
3578   (¬P ∧ ¬Q) → (P ↔ Q).
3579 Proof. intros P Q.
3580 specialize n5_1 with (¬P) (¬Q).
3581 intros n5_1a.
3582 specialize Transp4_11 with P Q.
3583 intros Transp4_11a.
3584 apply propositional_extensionality in Transp4_11a.
3585 replace (¬P↔¬Q) with (P↔Q) in n5_1a
3586   by now apply Transp4_11a.
3587 apply n5_1a.
3588 Qed.
3589
3590 Theorem n5_22 : ∀ P Q : Prop,
3591   ¬(P ↔ Q) ↔ ((P ∧ ¬Q) ∨ (Q ∧ ¬P)).
3592 Proof. intros P Q.

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3593 specialize n4_61 with P Q.
3594 intros n4_61a.
3595 specialize n4_61 with Q P.
3596 intros n4_61b.
3597 Conj n4_61a n4_61b.
3598 split.
3599 apply n4_61a.
3600 apply n4_61b.
3601 specialize n4_39 with
3602   ( $\neg(P \rightarrow Q)$ ) ( $\neg(Q \rightarrow P)$ ) ( $P \wedge \neg Q$ ) ( $Q \wedge \neg P$ ).
3603 intros n4_39a.
3604 MP n4_39a H.
3605 specialize n4_51 with ( $P \rightarrow Q$ ) ( $Q \rightarrow P$ ).
3606 intros n4_51a.
3607 apply propositional_extensionality in n4_51a.
3608 replace ( $\neg(P \rightarrow Q) \vee \neg(Q \rightarrow P)$ ) with
3609   ( $\neg((P \rightarrow Q) \wedge (Q \rightarrow P))$ ) in n4_39a
3610   by now apply n4_51a.
3611 replace (( $P \rightarrow Q$ )  $\wedge$  ( $Q \rightarrow P$ )) with ( $P \leftrightarrow Q$ )
3612   in n4_39a by now rewrite Equiv4_01.
3613 apply n4_39a.
3614 Qed.
3615
3616 Theorem n5_23 :  $\forall P Q : \text{Prop},$ 
3617   ( $P \leftrightarrow Q$ )  $\leftrightarrow$  ( $(P \wedge Q) \vee (\neg P \wedge \neg Q)$ ).
3618 Proof. intros P Q.
3619 specialize n5_18 with P Q.
3620 intros n5_18a.
3621 specialize n5_22 with P ( $\neg Q$ ).
3622 intros n5_22a.
3623 Conj n5_18a n5_22a.
3624 split.
3625 apply n5_18a.
3626 apply n5_22a.
3627 specialize n4_22 with ( $P \leftrightarrow Q$ ) ( $\neg(P \leftrightarrow \neg Q)$ )
3628   ( $P \wedge \neg\neg Q \vee \neg Q \wedge \neg P$ ).
3629 intros n4_22a.
3630 MP n4_22a H.
3631 specialize n4_13 with Q.
3632 intros n4_13a.
3633 apply propositional_extensionality in n4_13a.
3634 replace ( $\neg\neg Q$ ) with Q in n4_22a by now apply n4_13a.

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3635 specialize n4_3 with (¬P) (¬Q).
3636 intros n4_3a.
3637 apply propositional_extensionality in n4_3a.
3638 replace (¬Q ∧ ¬P) with (¬P ∧ ¬Q) in n4_22a
3639   by now apply n4_3a.
3640 apply n4_22a.
3641 Qed.
3642   (*The proof sketch in Principia offers n4_36.
3643     This seems to be a misprint. We used n4_3.*)
3644
3645 Theorem n5_24 : ∀ P Q : Prop,
3646   ¬((P ∧ Q) ∨ (¬P ∧ ¬Q)) ↔ ((P ∧ ¬Q) ∨ (Q ∧ ¬P)).
3647 Proof. intros P Q.
3648 specialize n5_23 with P Q.
3649 intros n5_23a.
3650 specialize Transp4_11 with
3651   (P↔Q) (P ∧ Q ∨ ¬P ∧ ¬Q).
3652 intros Transp4_11a. (*Not cited*)
3653 rewrite Equiv4_01 in Transp4_11a.
3654 specialize Simp3_26 with (((P↔Q)↔P∧Q∨¬P∧¬Q)
3655   →(¬(P↔Q)↔¬(P∧Q∨¬P∧¬Q)))
3656   ((¬(P↔Q)↔¬(P∧Q∨¬P∧¬Q))
3657   →((P↔Q)↔P∧Q∨¬P∧¬Q)).
3658 intros Simp3_26a.
3659 MP Simp3_26a Transp4_11a.
3660 MP Simp3_26a n5_23a.
3661 specialize n5_22 with P Q.
3662 intros n5_22a.
3663   clear n5_23a. clear Transp4_11a.
3664 Conj Simp3_26a n5_22a.
3665 split.
3666 apply Simp3_26a.
3667 apply n5_22a.
3668 specialize n4_22 with (¬(P∧Q∨¬P∧¬Q))
3669   (¬(P↔Q)) (P∧¬Q∨Q∧¬P).
3670 intros n4_22a.
3671 specialize n4_21 with (¬(P∧Q∨¬P∧¬Q)) (¬(P↔Q)).
3672 intros n4_21a.
3673 apply propositional_extensionality in n4_21a.
3674 replace ((¬(P↔Q))↔(¬((P ∧ Q)∨(¬P ∧ ¬Q))))
3675   with ((¬((P ∧ Q)∨(¬P ∧ ¬Q)))↔(¬(P↔Q))) in H
3676   by now apply n4_21a.

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3677     MP n4_22a H.
3678     apply n4_22a.
3679 Qed.
3680
3681 Theorem n5_25 :  $\forall$  P Q : Prop,
3682   (P  $\vee$  Q)  $\leftrightarrow$  ((P  $\rightarrow$  Q)  $\rightarrow$  Q).
3683 Proof. intros P Q.
3684   specialize n2_62 with P Q.
3685   intros n2_62a.
3686   specialize n2_68 with P Q.
3687   intros n2_68a.
3688   Conj n2_62a n2_68a.
3689   split.
3690   apply n2_62a.
3691   apply n2_68a.
3692   Equiv H.
3693   apply H.
3694 Qed.
3695
3696 Theorem n5_3 :  $\forall$  P Q R : Prop,
3697   ((P  $\wedge$  Q)  $\rightarrow$  R)  $\leftrightarrow$  ((P  $\wedge$  Q)  $\rightarrow$  (P  $\wedge$  R)).
3698 Proof. intros P Q R.
3699   specialize Comp3_43 with (P  $\wedge$  Q) P R.
3700   intros Comp3_43a.
3701   specialize Exp3_3 with
3702     (P  $\wedge$  Q  $\rightarrow$  P) (P  $\wedge$  Q  $\rightarrow$  R) (P  $\wedge$  Q  $\rightarrow$  P  $\wedge$  R).
3703   intros Exp3_3a. (*Not cited*)
3704   MP Exp3_3a Comp3_43a.
3705   specialize Simp3_26 with P Q.
3706   intros Simp3_26a.
3707   MP Exp3_3a Simp3_26a.
3708   specialize Syll2_05 with (P  $\wedge$  Q) (P  $\wedge$  R) R.
3709   intros Syll2_05a.
3710   specialize Simp3_27 with P R.
3711   intros Simp3_27a.
3712   MP Syll2_05a Simp3_27a.
3713   clear Comp3_43a. clear Simp3_27a.
3714   clear Simp3_26a.
3715   Conj Exp3_3a Syll2_05a.
3716   split.
3717   apply Exp3_3a.
3718   apply Syll2_05a.

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3719   Equiv H.
3720   apply H.
3721 Qed.
3722
3723 Theorem n5_31 :  $\forall$  P Q R : Prop,
3724   (R  $\wedge$  (P  $\rightarrow$  Q))  $\rightarrow$  (P  $\rightarrow$  (Q  $\wedge$  R)).
3725 Proof. intros P Q R.
3726   specialize Comp3_43 with P Q R.
3727   intros Comp3_43a.
3728   specialize Simp2_02 with P R.
3729   intros Simp2_02a.
3730   specialize Exp3_3 with
3731     (P $\rightarrow$ R) (P $\rightarrow$ Q) (P $\rightarrow$ (Q  $\wedge$  R)).
3732   intros Exp3_3a. (*Not cited*)
3733   specialize n3_22 with (P  $\rightarrow$  R) (P  $\rightarrow$  Q). (*Not cited*)
3734   intros n3_22a.
3735   Syll n3_22a Comp3_43a Sa.
3736   MP Exp3_3a Sa.
3737   Syll Simp2_02a Exp3_3a Sb.
3738   specialize Imp3_31 with R (P $\rightarrow$ Q) (P $\rightarrow$ (Q  $\wedge$  R)).
3739   intros Imp3_31a. (*Not cited*)
3740   MP Imp3_31a Sb.
3741   apply Imp3_31a.
3742 Qed.
3743
3744 Theorem n5_32 :  $\forall$  P Q R : Prop,
3745   (P  $\rightarrow$  (Q  $\leftrightarrow$  R))  $\leftrightarrow$  ((P  $\wedge$  Q)  $\leftrightarrow$  (P  $\wedge$  R)).
3746 Proof. intros P Q R.
3747   specialize n4_76 with P (Q $\rightarrow$ R) (R $\rightarrow$ Q).
3748   intros n4_76a.
3749   specialize Exp3_3 with P Q R.
3750   intros Exp3_3a.
3751   specialize Imp3_31 with P Q R.
3752   intros Imp3_31a.
3753   Conj Exp3_3a Imp3_31a.
3754   split.
3755   apply Exp3_3a.
3756   apply Imp3_31a.
3757   Equiv H.
3758   specialize Exp3_3 with P R Q.
3759   intros Exp3_3b.
3760   specialize Imp3_31 with P R Q.

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3761   intros Imp3_31b.
3762   Conj Exp3_3b Imp3_31b.
3763   split.
3764   apply Exp3_3b.
3765   apply Imp3_31b.
3766   Equiv H0.
3767   specialize n5_3 with P Q R.
3768   intros n5_3a.
3769   specialize n5_3 with P R Q.
3770   intros n5_3b.
3771   apply propositional_extensionality in H.
3772   replace (P→Q→R) with (P∧Q→R) in n4_76a
3773     by now apply H.
3774   apply propositional_extensionality in H0.
3775   replace (P→R→Q) with (P∧R→Q) in n4_76a
3776     by now apply H0.
3777   apply propositional_extensionality in n5_3a.
3778   replace (P∧Q→R) with (P∧Q→P∧R) in n4_76a
3779     by now apply n5_3a.
3780   apply propositional_extensionality in n5_3b.
3781   replace (P∧R→Q) with (P∧R→P∧Q) in n4_76a
3782     by now apply n5_3b.
3783   replace ((P∧Q→P∧R)∧(P∧R→P∧Q)) with
3784     ((P∧Q)↔(P∧R)) in n4_76a
3785     by now rewrite Equiv4_01.
3786   specialize n4_21 with
3787     (P→((Q→R)∧(R→Q))) ((P∧Q)↔(P∧R)).
3788   intros n4_21a.
3789   apply propositional_extensionality in n4_21a.
3790   replace ((P∧Q↔P∧R)↔(P→(Q→R)∧(R→Q))) with
3791     ((P→(Q→R)∧(R→Q))↔(P∧Q↔P∧R)) in n4_76a
3792     by now apply n4_21a.
3793   replace ((Q→R)∧(R→Q)) with (Q↔R) in n4_76a
3794     by now rewrite Equiv4_01.
3795   apply n4_76a.
3796   Qed.
3797
3798   Theorem n5_33 : ∀ P Q R : Prop,
3799     (P ∧ (Q → R)) ↔ (P ∧ ((P ∧ Q) → R)).
3800   Proof. intros P Q R.
3801     specialize n5_32 with P (Q→R) ((P∧Q)→R).
3802     intros n5_32a.

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3803     replace
3804         ((P → (Q → R) ↔ (P ∧ Q → R)) ↔ (P ∧ (Q → R) ↔ P ∧ (P ∧ Q → R)))
3805     with
3806         (((P → (Q → R) ↔ (P ∧ Q → R)) → (P ∧ (Q → R) ↔ P ∧ (P ∧ Q → R)))
3807         ∧
3808         ((P ∧ (Q → R) ↔ P ∧ (P ∧ Q → R) → (P → (Q → R) ↔ (P ∧ Q → R))))))
3809     in n5_32a by now rewrite Equiv4_01.
3810 specialize Simp3_26 with
3811     ((P → (Q → R) ↔ (P ∧ Q → R)) → (P ∧ (Q → R) ↔ P ∧ (P ∧ Q → R)))
3812     ((P ∧ (Q → R) ↔ P ∧ (P ∧ Q → R) → (P → (Q → R) ↔ (P ∧ Q → R)))).
3813 intros Simp3_26a. (*Not cited*)
3814 MP Simp3_26a n5_32a.
3815 specialize n4_73 with Q P.
3816 intros n4_73a.
3817 specialize n4_84 with Q (Q ∧ P) R.
3818 intros n4_84a.
3819 Syll n4_73a n4_84a Sa.
3820 specialize n4_3 with P Q.
3821 intros n4_3a.
3822 apply propositional_extensionality in n4_3a.
3823 replace (Q ∧ P) with (P ∧ Q) in Sa
3824     by now apply n4_3a. (*Not cited*)
3825 MP Simp3_26a Sa.
3826 apply Simp3_26a.
3827 Qed.
3828
3829 Theorem n5_35 : ∀ P Q R : Prop,
3830     ((P → Q) ∧ (P → R)) → (P → (Q ↔ R)).
3831 Proof. intros P Q R.
3832 specialize Comp3_43 with P Q R.
3833 intros Comp3_43a.
3834 specialize n5_1 with Q R.
3835 intros n5_1a.
3836 specialize Syll2_05 with P (Q ∧ R) (Q ↔ R).
3837 intros Syll2_05a.
3838 MP Syll2_05a n5_1a.
3839 Syll Comp3_43a Syll2_05a Sa.
3840 apply Sa.
3841 Qed.
3842
3843 Theorem n5_36 : ∀ P Q : Prop,
3844     (P ∧ (P ↔ Q)) ↔ (Q ∧ (P ↔ Q)).

```

```

3845 Proof. intros P Q.
3846 specialize Id2_08 with (P↔Q).
3847 intros Id2_08a.
3848 specialize n5_32 with (P↔Q) P Q.
3849 intros n5_32a.
3850 apply propositional_extensionality in n5_32a.
3851 replace (P↔Q→P↔Q) with
3852   ((P↔Q)∧P↔(P↔Q)∧Q) in Id2_08a
3853   by now apply n5_32a.
3854 specialize n4_3 with P (P↔Q).
3855 intros n4_3a.
3856 apply propositional_extensionality in n4_3a.
3857 replace ((P↔Q)∧P) with (P∧(P↔Q)) in Id2_08a
3858   by now apply n4_3a.
3859 specialize n4_3 with Q (P↔Q).
3860 intros n4_3b.
3861 apply propositional_extensionality in n4_3b.
3862 replace ((P↔Q)∧Q) with (Q∧(P↔Q)) in Id2_08a
3863   by now apply n4_3b.
3864 apply Id2_08a.
3865 Qed.
3866 (*The proof sketch cites Ass3_35 and n4_38,
3867   but the sketch was indecipherable.*)
3868
3869 Theorem n5_4 : ∀ P Q : Prop,
3870   (P → (P → Q)) ↔ (P → Q).
3871 Proof. intros P Q.
3872 specialize n2_43 with P Q.
3873 intros n2_43a.
3874 specialize Simp2_02 with (P) (P→Q).
3875 intros Simp2_02a.
3876 Conj n2_43a Simp2_02a.
3877 split.
3878 apply n2_43a.
3879 apply Simp2_02a.
3880 Equiv H.
3881 apply H.
3882 Qed.
3883
3884 Theorem n5_41 : ∀ P Q R : Prop,
3885   ((P → Q) → (P → R)) ↔ (P → Q → R).
3886 Proof. intros P Q R.

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3887 specialize n2_86 with P Q R.
3888 intros n2_86a.
3889 specialize n2_77 with P Q R.
3890 intros n2_77a.
3891 Conj n2_86a n2_77a.
3892 split.
3893 apply n2_86a.
3894 apply n2_77a.
3895 Equiv H.
3896 apply H.
3897 Qed.
3898
3899 Theorem n5_42 :  $\forall$  P Q R : Prop,
3900   (P  $\rightarrow$  Q  $\rightarrow$  R)  $\leftrightarrow$  (P  $\rightarrow$  Q  $\rightarrow$  P  $\wedge$  R).
3901 Proof. intros P Q R.
3902 specialize n5_3 with P Q R.
3903 intros n5_3a.
3904 specialize n4_87 with P Q R.
3905 intros n4_87a.
3906 specialize Imp3_31 with P Q R.
3907 intros Imp3_31a.
3908 specialize Exp3_3 with P Q R.
3909 intros Exp3_3a.
3910 Conj Imp3_31a Exp3_3.
3911 split.
3912 apply Imp3_31a.
3913 apply Exp3_3a.
3914 Equiv H.
3915 apply propositional_extensionality in H.
3916 replace ((P $\wedge$ Q) $\rightarrow$ R) with (P $\rightarrow$ Q $\rightarrow$ R) in n5_3a
3917   by now apply H.
3918 specialize n4_87 with P Q (P $\wedge$ R).
3919 intros n4_87b.
3920 specialize Imp3_31 with P Q (P $\wedge$ R).
3921 intros Imp3_31b.
3922 specialize Exp3_3 with P Q (P $\wedge$ R).
3923 intros Exp3_3b.
3924 Conj Imp3_31b Exp3_3b.
3925 split.
3926 apply Imp3_31b.
3927 apply Exp3_3b.
3928 Equiv H0.

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3929   apply propositional_extensionality in H0.
3930   replace ((P $\wedge$ Q) $\rightarrow$ (P $\wedge$ R)) with
3931         (P $\rightarrow$ Q $\rightarrow$ (P $\wedge$ R)) in n5_3a by now apply H0.
3932   apply n5_3a.
3933   Qed.
3934
3935   Theorem n5_44 :  $\forall$  P Q R : Prop,
3936     (P $\rightarrow$ Q)  $\rightarrow$  ((P  $\rightarrow$  R)  $\leftrightarrow$  (P  $\rightarrow$  (Q  $\wedge$  R))).
3937   Proof. intros P Q R.
3938   specialize n4_76 with P Q R.
3939   intros n4_76a.
3940   rewrite Equiv4_01 in n4_76a.
3941   specialize Simp3_26 with
3942     (((P $\rightarrow$ Q) $\wedge$ (P $\rightarrow$ R)) $\rightarrow$ (P $\rightarrow$ (Q $\wedge$ R)))
3943     ((P $\rightarrow$ (Q $\wedge$ R)) $\rightarrow$ ((P $\rightarrow$ Q) $\wedge$ (P $\rightarrow$ R))).
3944   intros Simp3_26a.
3945   MP Simp3_26a n4_76a.
3946   specialize Simp3_27 with
3947     (((P $\rightarrow$ Q) $\wedge$ (P $\rightarrow$ R)) $\rightarrow$ (P $\rightarrow$ (Q $\wedge$ R)))
3948     ((P $\rightarrow$ (Q $\wedge$ R)) $\rightarrow$ ((P $\rightarrow$ Q) $\wedge$ (P $\rightarrow$ R))).
3949   intros Simp3_27a.
3950   MP Simp3_27a n4_76a.
3951   specialize Simp3_27 with (P $\rightarrow$ Q) (P $\rightarrow$ Q $\wedge$ R).
3952   intros Simp3_27d.
3953   Syll Simp3_27d Simp3_27a Sa.
3954   specialize n5_3 with (P $\rightarrow$ Q) (P $\rightarrow$ R) (P $\rightarrow$ (Q $\wedge$ R)).
3955   intros n5_3a.
3956   rewrite Equiv4_01 in n5_3a.
3957   specialize Simp3_26 with
3958     (((P $\rightarrow$ Q) $\wedge$ (P $\rightarrow$ R)) $\rightarrow$ (P $\rightarrow$ (Q $\wedge$ R))) $\rightarrow$ 
3959     (((P $\rightarrow$ Q) $\wedge$ (P $\rightarrow$ R)) $\rightarrow$ ((P $\rightarrow$ Q) $\wedge$ (P $\rightarrow$ (Q $\wedge$ R))))
3960     (((P $\rightarrow$ Q) $\wedge$ (P $\rightarrow$ R)) $\rightarrow$ ((P $\rightarrow$ Q) $\wedge$ (P $\rightarrow$ (Q $\wedge$ R))))
3961      $\rightarrow$ ((P $\rightarrow$ Q) $\wedge$ (P $\rightarrow$ R)) $\rightarrow$ (P $\rightarrow$ (Q $\wedge$ R))).
3962   intros Simp3_26b.
3963   MP Simp3_26b n5_3a.
3964   specialize Simp3_27 with
3965     (((P $\rightarrow$ Q) $\wedge$ (P $\rightarrow$ R)) $\rightarrow$ (P $\rightarrow$ (Q $\wedge$ R))) $\rightarrow$ 
3966     (((P $\rightarrow$ Q) $\wedge$ (P $\rightarrow$ R)) $\rightarrow$ ((P $\rightarrow$ Q) $\wedge$ (P $\rightarrow$ (Q $\wedge$ R))))
3967     (((P $\rightarrow$ Q) $\wedge$ (P $\rightarrow$ R)) $\rightarrow$ ((P $\rightarrow$ Q) $\wedge$ (P $\rightarrow$ (Q $\wedge$ R))))
3968      $\rightarrow$ ((P $\rightarrow$ Q) $\wedge$ (P $\rightarrow$ R)) $\rightarrow$ (P $\rightarrow$ (Q $\wedge$ R))).
3969   intros Simp3_27b.
3970   MP Simp3_27b n5_3a.

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3971 MP Simp3_26a Simp3_26b.
3972 MP Simp3_27a Simp3_27b.
3973 clear n4_76a. clear Simp3_26a. clear Simp3_27a.
3974 clear Simp3_27b. clear Simp3_27d. clear n5_3a.
3975 Conj Simp3_26b Sa.
3976 split.
3977 apply Sa.
3978 apply Simp3_26b.
3979 Equiv H.
3980 specialize n5_32 with (P→Q) (P→R) (P→(Q∧R)).
3981 intros n5_32a.
3982 rewrite Equiv4_01 in n5_32a.
3983 specialize Simp3_27 with
3984   ((P → Q) → (P → R) ↔ (P → Q ∧ R))
3985   → (P → Q) ∧ (P → R) ↔ (P → Q) ∧ (P → Q ∧ R))
3986   ((P → Q) ∧ (P → R) ↔ (P → Q) ∧ (P → Q ∧ R)
3987   → (P → Q) → (P → R) ↔ (P → Q ∧ R)).
3988 intros Simp3_27c.
3989 MP Simp3_27c n5_32a.
3990 specialize n4_21 with
3991   ((P→Q)∧(P→R)) ((P→Q)∧(P→(Q∧R))).
3992 intros n4_21a.
3993 apply propositional_extensionality in n4_21a.
3994 replace (((P→Q)∧(P→(Q∧R)))↔((P→Q)∧(P→R)))
3995   with (((P→Q)∧(P→R))↔((P→Q)∧(P→(Q∧R))))
3996   in H by now apply n4_21a.
3997 MP Simp3_27c H.
3998 apply Simp3_27c.
3999 Qed.
4000
4001 Theorem n5_5 : ∀ P Q : Prop,
4002   P → ((P → Q) ↔ Q).
4003 Proof. intros P Q.
4004 specialize Ass3_35 with P Q.
4005 intros Ass3_35a.
4006 specialize Exp3_3 with P (P→Q) Q.
4007 intros Exp3_3a.
4008 MP Exp3_3a Ass3_35a.
4009 specialize Simp2_02 with P Q.
4010 intros Simp2_02a.
4011 specialize Exp3_3 with P Q (P→Q).
4012 intros Exp3_3b.

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4013 specialize n3_42 with P Q (P→Q). (*Not cited*)
4014 intros n3_42a.
4015 MP n3_42a Simp2_02a.
4016 MP Exp3_3b n3_42a.
4017 clear n3_42a. clear Simp2_02a. clear Ass3_35a.
4018 Conj Exp3_3a Exp3_3b.
4019 split.
4020 apply Exp3_3a.
4021 apply Exp3_3b.
4022 specialize n3_47 with P P ((P→Q)→Q) (Q→(P→Q)).
4023 intros n3_47a.
4024 MP n3_47a H.
4025 specialize n4_24 with P.
4026 intros n4_24a. (*Not cited*)
4027 apply propositional_extensionality in n4_24a.
4028 replace (P∧P) with P in n3_47a by now apply n4_24a.
4029 replace (((P→Q)→Q)∧(Q→(P→Q))) with ((P→Q)↔Q)
4030   in n3_47a by now rewrite Equiv4_01.
4031 apply n3_47a.
4032 Qed.
4033
4034 Theorem n5_501 : ∀ P Q : Prop,
4035   P → (Q ↔ (P ↔ Q)).
4036 Proof. intros P Q.
4037 specialize n5_1 with P Q.
4038 intros n5_1a.
4039 specialize Exp3_3 with P Q (P↔Q).
4040 intros Exp3_3a.
4041 MP Exp3_3a n5_1a.
4042 specialize Ass3_35 with P Q.
4043 intros Ass3_35a.
4044 specialize Simp3_26 with (P∧(P→Q)) (Q→P).
4045 intros Simp3_26a. (*Not cited*)
4046 Syll Simp3_26a Ass3_35a Sa.
4047 specialize n4_32 with P (P→Q) (Q→P).
4048 intros n4_32a. (*Not cited*)
4049 apply propositional_extensionality in n4_32a.
4050 replace ((P∧(P→Q))∧(Q→P)) with
4051   (P∧((P→Q)∧(Q→P))) in Sa by now apply n4_32a.
4052 replace ((P→Q)∧(Q→P)) with (P↔Q) in Sa
4053   by now rewrite Equiv4_01.
4054 specialize Exp3_3 with P (P↔Q) Q.

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4055   intros Exp3_3b.
4056   MP Exp3_3b Sa.
4057   clear n5_1a. clear Ass3_35a. clear n4_32a.
4058     clear Simp3_26a. clear Sa.
4059   Conj Exp3_3a Exp3_3b.
4060   split.
4061   apply Exp3_3a.
4062   apply Exp3_3b.
4063   specialize n4_76 with P (Q→(P↔Q)) ((P↔Q)→Q).
4064   intros n4_76a. (*Not cited*)
4065   apply propositional_extensionality in n4_76a.
4066   replace ((P→Q→P↔Q)^(P→P↔Q→Q)) with
4067     ((P→(Q→P↔Q)^(P↔Q→Q))) in H
4068     by now apply n4_76a.
4069   replace ((Q→(P↔Q))^(P↔Q)→Q) with
4070     (Q↔(P↔Q)) in H by now rewrite Equiv4_01.
4071   apply H.
4072   Qed.
4073
4074   Theorem n5_53 : ∀ P Q R S : Prop,
4075     (((P ∨ Q) ∨ R) → S) ↔ (((P → S) ∧ (Q → S)) ∧ (R → S)).
4076   Proof. intros P Q R S.
4077   specialize n4_77 with S (P∨Q) R.
4078   intros n4_77a.
4079   specialize n4_77 with S P Q.
4080   intros n4_77b.
4081   apply propositional_extensionality in n4_77b.
4082   replace (P ∨ Q → S) with
4083     ((P→S)^(Q→S)) in n4_77a
4084     by now apply n4_77b. (*Not cited*)
4085   specialize n4_21 with ((P ∨ Q) ∨ R → S)
4086     (((P → S) ∧ (Q → S)) ∧ (R → S)).
4087   intros n4_21a. (*Not cited*)
4088   apply propositional_extensionality in n4_21a.
4089   replace (((P→S)^(Q→S))^(R→S))↔(((P∨Q)∨R)→S))
4090     with
4091     (((P∨Q)∨R)→S)↔(((P→S)^(Q→S))^(R→S))
4092     in n4_77a by now apply n4_21.
4093   apply n4_77a.
4094   Qed.
4095
4096   Theorem n5_54 : ∀ P Q : Prop,

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4097   ((P ∧ Q) ↔ P) ∨ ((P ∧ Q) ↔ Q).
4098   Proof. intros P Q.
4099   specialize n4_73 with P Q.
4100   intros n4_73a.
4101   specialize n4_44 with Q P.
4102   intros n4_44a.
4103   specialize Transp2_16 with Q (P ↔ (P ∧ Q)).
4104   intros Transp2_16a.
4105   MP n4_73a Transp2_16a.
4106   specialize n4_3 with P Q.
4107   intros n4_3a. (*Not cited*)
4108   apply propositional_extensionality in n4_3a.
4109   replace (Q ∧ P) with (P ∧ Q) in n4_44a
4110     by now apply n4_3a.
4111   specialize Transp4_11 with Q (Q ∨ (P ∧ Q)).
4112   intros Transp4_11a.
4113   apply propositional_extensionality in Transp4_11a.
4114   replace (Q ↔ Q ∨ P ∧ Q) with
4115     (¬Q ↔ ¬(Q ∨ P ∧ Q)) in n4_44a by now apply Transp4_11a.
4116   apply propositional_extensionality in n4_44a.
4117   replace (¬Q) with (¬(Q ∨ P ∧ Q)) in Transp2_16a
4118     by now apply n4_44a.
4119   specialize n4_56 with Q (P ∧ Q).
4120   intros n4_56a. (*Not cited*)
4121   apply propositional_extensionality in n4_56a.
4122   replace (¬(Q ∨ P ∧ Q)) with
4123     (¬Q ∧ ¬(P ∧ Q)) in Transp2_16a
4124     by now apply n4_56a.
4125   specialize n5_1 with (¬Q) (¬(P ∧ Q)).
4126   intros n5_1a.
4127   Syll Transp2_16a n5_1a Sa.
4128   replace (¬(P ↔ P ∧ Q) → (¬Q ↔ ¬(P ∧ Q))) with
4129     (¬¬(P ↔ P ∧ Q) ∨ (¬Q ↔ ¬(P ∧ Q))) in Sa
4130     by now rewrite Impl1_01. (*Not cited*)
4131   specialize n4_13 with (P ↔ (P ∧ Q)).
4132   intros n4_13a. (*Not cited*)
4133   apply propositional_extensionality in n4_13a.
4134   replace (¬¬(P ↔ P ∧ Q)) with (P ↔ P ∧ Q) in Sa
4135     by now apply n4_13a.
4136   specialize Transp4_11 with Q (P ∧ Q).
4137   intros Transp4_11b.
4138   apply propositional_extensionality in Transp4_11b.

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4139   replace ( $\neg Q \leftrightarrow \neg (P \wedge Q)$ ) with ( $Q \leftrightarrow (P \wedge Q)$ ) in Sa
4140     by now apply Transp4_11b.
4141   specialize n4_21 with ( $P \wedge Q$ ) Q.
4142   intros n4_21a. (*Not cited*)
4143   apply propositional_extensionality in n4_21a.
4144   replace ( $Q \leftrightarrow (P \wedge Q)$ ) with ( $(P \wedge Q) \leftrightarrow Q$ ) in Sa
4145     by now apply n4_21a.
4146   specialize n4_21 with ( $P \wedge Q$ ) P.
4147   intros n4_21b. (*Not cited*)
4148   apply propositional_extensionality in n4_21b.
4149   replace ( $P \leftrightarrow (P \wedge Q)$ ) with ( $(P \wedge Q) \leftrightarrow P$ ) in Sa
4150     by now apply n4_21b.
4151   apply Sa.
4152 Qed.
4153
4154 Theorem n5_55 :  $\forall P Q : \text{Prop}$ ,
4155   ( $(P \vee Q) \leftrightarrow P$ )  $\vee$  ( $(P \vee Q) \leftrightarrow Q$ ).
4156 Proof. intros P Q.
4157   specialize Add1_3 with ( $P \wedge Q$ ) (P).
4158   intros Add1_3a.
4159   specialize n4_3 with P Q.
4160   intros n4_3a. (*Not cited*)
4161   apply propositional_extensionality in n4_3a.
4162   specialize n4_41 with P Q P.
4163   intros n4_41a. (*Not cited*)
4164   replace ( $Q \wedge P$ ) with ( $P \wedge Q$ ) in n4_41a
4165     by now apply n4_3a.
4166   specialize n4_31 with ( $P \wedge Q$ ) P.
4167   intros n4_31a.
4168   apply propositional_extensionality in n4_31a.
4169   replace ( $P \vee P \wedge Q$ ) with ( $P \wedge Q \vee P$ ) in n4_41a
4170     by now apply n4_31a.
4171   apply propositional_extensionality in n4_41a.
4172   replace ( $(P \wedge Q) \vee P$ ) with ( $(P \vee Q) \wedge (P \vee P)$ ) in Add1_3a
4173     by now apply n4_4a.
4174   specialize n4_25 with P.
4175   intros n4_25a. (*Not cited*)
4176   apply propositional_extensionality in n4_25a.
4177   replace ( $P \vee P$ ) with P in Add1_3a
4178     by now apply n4_25a.
4179   specialize n4_31 with P Q.
4180   intros n4_31b.

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4181 apply propositional_extensionality in n4_31b.
4182 replace (Q∨P) with (P∨Q) in Add1_3a
4183 by now apply n4_31b.
4184 specialize n5_1 with P (P∨Q).
4185 intros n5_1a.
4186 specialize n4_3 with (P ∨ Q) P.
4187 intros n4_3b.
4188 apply propositional_extensionality in n4_3b.
4189 replace ((P ∨ Q) ∧ P) with (P ∧ (P ∨ Q)) in Add1_3a
4190 by now apply n4_3b.
4191 Syll Add1_3a n5_1a Sa.
4192 specialize n4_74 with P Q.
4193 intros n4_74a.
4194 specialize Transp2_15 with P (Q↔P∨Q).
4195 intros Transp2_15a. (*Not cited*)
4196 MP Transp2_15a n4_74a.
4197 Syll Transp2_15a Sa Sb.
4198 replace (¬ (Q ↔ P ∨ Q) → P ↔ P ∨ Q) with
4199 (¬¬(Q ↔ P ∨ Q) ∨ (P ↔ P ∨ Q)) in Sb
4200 by now rewrite Impl1_01.
4201 specialize n4_13 with (Q ↔ P ∨ Q).
4202 intros n4_13a. (*Not cited*)
4203 apply propositional_extensionality in n4_13a.
4204 replace (¬¬(Q↔(P∨Q))) with (Q↔(P∨Q)) in Sb
4205 by now apply n4_13a.
4206 specialize n4_21 with (P ∨ Q) Q.
4207 intros n4_21a. (*Not cited*)
4208 apply propositional_extensionality in n4_21a.
4209 replace (Q↔(P∨Q)) with ((P∨Q)↔Q) in Sb
4210 by now apply n4_21a.
4211 specialize n4_21 with (P ∨ Q) P.
4212 intros n4_21b. (*Not cited*)
4213 apply propositional_extensionality in n4_21b.
4214 replace (P↔(P∨Q)) with ((P∨Q)↔P) in Sb
4215 by now apply n4_21b.
4216 apply n4_31 in Sb.
4217 apply Sb.
4218 Qed.
4219
4220 Theorem n5_6 : ∀ P Q R : Prop,
4221 ((P ∧ ¬Q) → R) ↔ (P → (Q ∨ R)).
4222 Proof. intros P Q R.

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4223 specialize n4_87 with P (¬Q) R.
4224 intros n4_87a.
4225 specialize n4_64 with Q R.
4226 intros n4_64a.
4227 specialize n4_85 with P Q R.
4228 intros n4_85a.
4229 replace (((P ∧ ¬Q → R) ↔ (P → ¬Q → R)) ↔ ((¬Q → P → R) ↔ (¬Q ∧ P → R)))
4230       with
4231       (((P ∧ ¬Q → R) ↔ (P → ¬Q → R)) → ((¬Q → P → R) ↔ (¬Q ∧ P → R)))
4232       ∧
4233       (((¬Q → P → R) ↔ (¬Q ∧ P → R)) → (((P ∧ ¬Q → R) ↔ (P → ¬Q → R))))
4234       in n4_87a by now rewrite Equiv4_01.
4235 specialize Simp3_27 with
4236       (((P ∧ ¬Q → R) ↔ (P → ¬Q → R) → (¬Q → P → R) ↔ (¬Q ∧ P → R)))
4237       (((¬Q → P → R) ↔ (¬Q ∧ P → R) → (P ∧ ¬Q → R) ↔ (P → ¬Q → R))).
4238 intros Simp3_27a.
4239 MP Simp3_27a n4_87a.
4240 specialize Imp3_31 with (¬Q) P R.
4241 intros Imp3_31a.
4242 specialize Exp3_3 with (¬Q) P R.
4243 intros Exp3_3a.
4244 Conj Imp3_31a Exp3_3a.
4245 split.
4246 apply Imp3_31a.
4247 apply Exp3_3a.
4248 Equiv H.
4249 MP Simp3_27a H.
4250 apply propositional_extensionality in n4_64a.
4251 replace (¬Q → R) with (Q ∨ R) in Simp3_27a
4252       by now apply n4_64a.
4253 apply Simp3_27a.
4254 Qed.
4255
4256 Theorem n5_61 : ∀ P Q : Prop,
4257   ((P ∨ Q) ∧ ¬Q) ↔ (P ∧ ¬Q).
4258 Proof. intros P Q.
4259 specialize n4_74 with Q P.
4260 intros n4_74a.
4261 specialize n5_32 with (¬Q) P (Q ∨ P).
4262 intros n5_32a.
4263 apply propositional_extensionality in n5_32a.
4264 replace (¬Q → P ↔ Q ∨ P) with

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4265     ( $\neg Q \wedge P \leftrightarrow \neg Q \wedge (Q \vee P)$ ) in n4_74a
4266     by now apply n5_32a.
4267 specialize n4_3 with P ( $\neg Q$ ).
4268 intros n4_3a. (*Not cited*)
4269 apply propositional_extensionality in n4_3a.
4270 replace ( $\neg Q \wedge P$ ) with ( $P \wedge \neg Q$ ) in n4_74a
4271     by now apply n4_3a.
4272 specialize n4_3 with ( $Q \vee P$ ) ( $\neg Q$ ).
4273 intros n4_3b. (*Not cited*)
4274 apply propositional_extensionality in n4_3b.
4275 replace ( $\neg Q \wedge (Q \vee P)$ ) with ( $((Q \vee P) \wedge \neg Q)$ ) in n4_74a
4276     by now apply n4_3b.
4277 specialize n4_31 with P Q.
4278 intros n4_31a. (*Not cited*)
4279 apply propositional_extensionality in n4_31a.
4280 replace ( $Q \vee P$ ) with ( $P \vee Q$ ) in n4_74a
4281     by now apply n4_31a.
4282 specialize n4_21 with ( $(P \vee Q) \wedge \neg Q$ ) ( $P \wedge \neg Q$ ).
4283 intros n4_21a. (*Not cited*)
4284 apply propositional_extensionality in n4_21a.
4285 replace ( $P \wedge \neg Q \leftrightarrow (P \vee Q) \wedge \neg Q$ ) with
4286     ( $(P \vee Q) \wedge \neg Q \leftrightarrow P \wedge \neg Q$ ) in n4_74a
4287     by now apply n4_21a.
4288 apply n4_74a.
4289 Qed.
4290
4291 Theorem n5_62 :  $\forall P Q : \text{Prop}$ ,
4292     ( $(P \wedge Q) \vee \neg Q$ )  $\leftrightarrow$  ( $P \vee \neg Q$ ).
4293 Proof. intros P Q.
4294 specialize n4_7 with Q P.
4295 intros n4_7a.
4296 replace ( $Q \rightarrow P$ ) with ( $\neg Q \vee P$ ) in n4_7a
4297     by now rewrite Impl1_01.
4298 replace ( $Q \rightarrow (Q \wedge P)$ ) with ( $\neg Q \vee (Q \wedge P)$ ) in n4_7a
4299     by now rewrite Impl1_01.
4300 specialize n4_31 with ( $Q \wedge P$ ) ( $\neg Q$ ).
4301 intros n4_31a. (*Not cited*)
4302 apply propositional_extensionality in n4_31a.
4303 replace ( $\neg Q \vee (Q \wedge P)$ ) with ( $((Q \wedge P) \vee \neg Q)$ ) in n4_7a
4304     by now apply n4_31a.
4305 specialize n4_31 with P ( $\neg Q$ ).
4306 intros n4_31b. (*Not cited*)

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4307   apply propositional_extensionality in n4_31b.
4308   replace ( $\neg Q \vee P$ ) with ( $P \vee \neg Q$ ) in n4_7a
4309   by now apply n4_31b.
4310   specialize n4_3 with P Q.
4311   intros n4_3a. (*Not cited*)
4312   apply propositional_extensionality in n4_3a.
4313   replace ( $Q \wedge P$ ) with ( $P \wedge Q$ ) in n4_7a
4314   by now apply n4_3a.
4315   specialize n4_21 with ( $P \wedge Q \vee \neg Q$ ) ( $P \vee \neg Q$ ).
4316   intros n4_21a. (*Not cited*)
4317   apply propositional_extensionality in n4_21a.
4318   replace ( $P \vee \neg Q \leftrightarrow P \wedge Q \vee \neg Q$ ) with
4319     ( $P \wedge Q \vee \neg Q \leftrightarrow P \vee \neg Q$ ) in n4_7a
4320   by now apply n4_21a.
4321   apply n4_7a.
4322   Qed.
4323
4324   Theorem n5_63 :  $\forall P Q : \text{Prop}$ ,
4325     ( $P \vee Q$ )  $\leftrightarrow$  ( $P \vee (\neg P \wedge Q)$ ).
4326   Proof. intros P Q.
4327   specialize n5_62 with Q ( $\neg P$ ).
4328   intros n5_62a.
4329   specialize n4_13 with P.
4330   intros n4_13a. (*Not cited*)
4331   apply propositional_extensionality in n4_13a.
4332   replace ( $\neg \neg P$ ) with P in n5_62a
4333   by now apply n4_13a.
4334   specialize n4_31 with P Q.
4335   intros n4_31a. (*Not cited*)
4336   apply propositional_extensionality in n4_31a.
4337   replace ( $Q \vee P$ ) with ( $P \vee Q$ ) in n5_62a
4338   by now apply n4_31a.
4339   specialize n4_31 with P ( $Q \wedge \neg P$ ).
4340   intros n4_31b. (*Not cited*)
4341   apply propositional_extensionality in n4_31b.
4342   replace ( $(Q \wedge \neg P) \vee P$ ) with ( $P \vee (Q \wedge \neg P)$ ) in n5_62a
4343   by now apply n4_31b.
4344   specialize n4_21 with ( $P \vee Q$ ) ( $P \vee (Q \wedge \neg P)$ ).
4345   intros n4_21a. (*Not cited*)
4346   apply propositional_extensionality in n4_21a.
4347   replace ( $P \vee Q \wedge \neg P \leftrightarrow P \vee Q$ ) with
4348     ( $P \vee Q \leftrightarrow P \vee Q \wedge \neg P$ ) in n5_62a

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4349     by now apply n4_21a.
4350 specialize n4_3 with ( $\neg$ P) Q.
4351 intros n4_3a. (*Not cited*)
4352 apply propositional_extensionality in n4_3a.
4353 replace (Q $\wedge$  $\neg$ P) with ( $\neg$ P $\wedge$ Q) in n5_62a
4354     by now apply n4_3a.
4355 apply n5_62a.
4356 Qed.
4357
4358 Theorem n5_7 :  $\forall$  P Q R : Prop,
4359   ((P  $\vee$  R)  $\leftrightarrow$  (Q  $\vee$  R))  $\leftrightarrow$  (R  $\vee$  (P  $\leftrightarrow$  Q)).
4360 Proof. intros P Q R.
4361 specialize n4_74 with R P.
4362 intros n4_74a.
4363 specialize n4_74 with R Q.
4364 intros n4_74b. (*Greg's suggestion*)
4365 Conj n4_74a n4_74b.
4366 split.
4367 apply n4_74a.
4368 apply n4_74b.
4369 specialize Comp3_43 with
4370   ( $\neg$ R) (P $\leftrightarrow$ R $\vee$ P) (Q $\leftrightarrow$ R $\vee$ Q).
4371 intros Comp3_43a.
4372 MP Comp3_43a H.
4373 specialize n4_22 with P (R $\vee$ P) (R $\vee$ Q).
4374 intros n4_22a.
4375 specialize n4_22 with P (R $\vee$ Q) Q.
4376 intros n4_22b.
4377 specialize Exp3_3 with (P $\leftrightarrow$ (R $\vee$ Q))
4378   ((R $\vee$ Q) $\leftrightarrow$ Q) (P $\leftrightarrow$ Q).
4379 intros Exp3_3a.
4380 MP Exp3_3a n4_22b.
4381 Syll n4_22a Exp3_3a Sa.
4382 specialize Imp3_31 with ((P $\leftrightarrow$ (R $\vee$ P)) $\wedge$ 
4383   ((R  $\vee$  P)  $\leftrightarrow$  (R  $\vee$  Q))) ((R $\vee$ Q) $\leftrightarrow$ Q) (P $\leftrightarrow$ Q).
4384 intros Imp3_31a.
4385 MP Imp3_31a Sa.
4386 specialize n4_32 with (P  $\leftrightarrow$  R  $\vee$  P) (R  $\vee$  P  $\leftrightarrow$  R  $\vee$  Q) (R  $\vee$  Q  $\leftrightarrow$  Q).
4387 intros n4_32a.
4388 apply propositional_extensionality in n4_32a.
4389 replace (((P $\leftrightarrow$ (R $\vee$ P)) $\wedge$ ((R  $\vee$  P)  $\leftrightarrow$ 
4390   (R  $\vee$  Q)))  $\wedge$  ((R $\vee$ Q) $\leftrightarrow$ Q) with

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4391      ((P↔(R∨P))∧((R ∨ P) ↔
4392      (R ∨ Q)) ∧ ((R∨Q)↔Q))) in Imp3_31a
4393      by now apply n4_32a.
4394      specialize n4_3 with (R ∨ Q ↔ Q) (R ∨ P ↔ R ∨ Q).
4395      intros n4_3a.
4396      apply propositional_extensionality in n4_3a.
4397      replace ((R ∨ P ↔ R ∨ Q) ∧ (R ∨ Q ↔ Q)) with
4398      ((R ∨ Q ↔ Q) ∧ (R ∨ P ↔ R ∨ Q)) in Imp3_31a
4399      by now apply n4_3a.
4400      specialize n4_32 with (P ↔ R ∨ P) (R ∨ Q ↔ Q) (R ∨ P ↔ R ∨ Q).
4401      intros n4_32b.
4402      apply propositional_extensionality in n4_32b.
4403      replace ((P↔(R∨P)) ∧
4404      ((R ∨ Q ↔ Q) ∧ (R ∨ P ↔ R ∨ Q))) with
4405      (((P↔(R∨P)) ∧ (R ∨ Q ↔ Q)) ∧
4406      (R ∨ P ↔ R ∨ Q)) in Imp3_31a
4407      by now apply n4_32b.
4408      specialize Exp3_3 with
4409      ((P↔(R∨P))∧(R∨Q↔Q))
4410      (R ∨ P ↔ R ∨ Q) (P ↔ Q).
4411      intros Exp3_3b.
4412      MP Exp3_3b Imp3_31a.
4413      specialize n4_21 with Q (R∨Q).
4414      intros n4_21a.
4415      apply propositional_extensionality in n4_21a.
4416      replace (Q↔R∨Q) with (R∨Q↔Q) in Comp3_43a
4417      by now apply n4_21a.
4418      Syll Comp3_43a Exp3_3b Sb.
4419      specialize n4_31 with P R.
4420      intros n4_31a.
4421      apply propositional_extensionality in n4_31a.
4422      replace (R∨P) with (P∨R) in Sb by now apply n4_31a.
4423      specialize n4_31 with Q R.
4424      intros n4_31b.
4425      apply propositional_extensionality in n4_31b.
4426      replace (R∨Q) with (Q∨R) in Sb by now apply n4_31b.
4427      specialize Imp3_31 with (¬R) (P∨R↔Q∨R) (P↔Q).
4428      intros Imp3_31b.
4429      MP Imp3_31b Sb.
4430      specialize n4_3 with (P ∨ R ↔ Q ∨ R) (¬R).
4431      intros n4_3b.
4432      apply propositional_extensionality in n4_3b.

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4433 replace ( $\neg R \wedge (P \vee R \leftrightarrow Q \vee R)$ ) with
4434 ( $(P \vee R \leftrightarrow Q \vee R) \wedge \neg R$ ) in Imp3_31b
4435 by now apply n4_3b.
4436 specialize Exp3_3 with
4437 ( $P \vee R \leftrightarrow Q \vee R$ ) ( $\neg R$ ) ( $P \leftrightarrow Q$ ).
4438 intros Exp3_3c.
4439 MP Exp3_3c Imp3_31b.
4440 replace ( $\neg R \rightarrow (P \leftrightarrow Q)$ ) with ( $\neg \neg R \vee (P \leftrightarrow Q)$ )
4441 in Exp3_3c by now rewrite Impl1_01.
4442 specialize n4_13 with R.
4443 intros n4_13a.
4444 apply propositional_extensionality in n4_13a.
4445 replace ( $\neg \neg R$ ) with R in Exp3_3c
4446 by now apply n4_13a.
4447 specialize Add1_3 with P R.
4448 intros Add1_3a.
4449 specialize Add1_3 with Q R.
4450 intros Add1_3b.
4451 Conj Add1_3a Add1_3b.
4452 split.
4453 apply Add1_3a.
4454 apply Add1_3b.
4455 specialize Comp3_43 with (R) ( $P \vee R$ ) ( $Q \vee R$ ).
4456 intros Comp3_43b.
4457 MP Comp3_43b H0.
4458 specialize n5_1 with ( $P \vee R$ ) ( $Q \vee R$ ).
4459 intros n5_1a.
4460 Syll Comp3_43b n5_1a Sc.
4461 specialize n4_37 with P Q R.
4462 intros n4_37a.
4463 Conj Sc n4_37a.
4464 split.
4465 apply Sc.
4466 apply n4_37a.
4467 specialize n4_77 with ( $P \vee R \leftrightarrow Q \vee R$ )
4468 R ( $P \leftrightarrow Q$ ).
4469 intros n4_77a.
4470 rewrite Equiv4_01 in n4_77a.
4471 specialize Simp3_26 with
4472 ( $(R \rightarrow P \vee R \leftrightarrow Q \vee R) \wedge$ 
4473 ( $P \leftrightarrow Q \rightarrow P \vee R \leftrightarrow Q \vee R$ )
4474  $\rightarrow R \vee (P \leftrightarrow Q) \rightarrow P \vee R \leftrightarrow Q \vee R$ )

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4475      ((R ∨ (P ↔ Q) → P ∨ R ↔ Q ∨ R)
4476        → (R → P ∨ R ↔ Q ∨ R) ∧
4477         (P ↔ Q → P ∨ R ↔ Q ∨ R)).
4478  intros Simp3_26a.
4479  MP Simp3_26 n4_77a.
4480  MP Simp3_26a H1.
4481  clear n4_77a. clear H1. clear n4_37a. clear Sa.
4482  clear n5_1a. clear Comp3_43b. clear H0.
4483  clear Add1_3a. clear Add1_3b. clear H. clear Imp3_31b.
4484  clear n4_74a. clear n4_74b. clear Comp3_43a.
4485  clear Imp3_31a. clear n4_22a. clear n4_22b.
4486  clear Exp3_3a. clear Exp3_3b. clear Sb. clear Sc.
4487  clear n4_13a. clear n4_3a. clear n4_3b. clear n4_21a.
4488  clear n4_31a. clear n4_31b. clear n4_32a. clear n4_32b.
4489  Conj Exp3_3c Simp3_26a.
4490  split.
4491  apply Exp3_3c.
4492  apply Simp3_26a.
4493  Equiv H.
4494  apply H.
4495  Qed.
4496
4497  Theorem n5_71 : ∀ P Q R : Prop,
4498    (Q → ¬R) → (((P ∨ Q) ∧ R) ↔ (P ∧ R)).
4499  Proof. intros P Q R.
4500  specialize n4_62 with Q R.
4501  intros n4_62a.
4502  specialize n4_51 with Q R.
4503  intros n4_51a.
4504  specialize n4_21 with (¬(Q ∧ R)) (¬Q ∨ ¬R).
4505  intros n4_21a.
4506  rewrite Equiv4_01 in n4_21a.
4507  specialize Simp3_26 with
4508    ((¬(Q ∧ R) ↔ (¬Q ∨ ¬R)) → ((¬Q ∨ ¬R) ↔ ¬(Q ∧ R)))
4509    (((¬Q ∨ ¬R) ↔ ¬(Q ∧ R)) → (¬(Q ∧ R) ↔ (¬Q ∨ ¬R))).
4510  intros Simp3_26a.
4511  MP Simp3_26a n4_21a.
4512  MP Simp3_26a n4_51a.
4513  clear n4_21a. clear n4_51a.
4514  Conj n4_62a Simp3_26a.
4515  split.
4516  apply n4_62a.

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4517 apply Simp3_26a.
4518 specialize n4_22 with
4519    $(Q \rightarrow \neg R) (\neg Q \vee \neg R) (\neg(Q \wedge R))$ .
4520 intros n4_22a.
4521 MP n4_22a H.
4522 replace  $((Q \rightarrow \neg R) \leftrightarrow \neg(Q \wedge R))$  with
4523    $((Q \rightarrow \neg R) \rightarrow \neg(Q \wedge R))$ 
4524    $\wedge$ 
4525    $(\neg(Q \wedge R) \rightarrow (Q \rightarrow \neg R))$  in n4_22a
4526   by now rewrite Equiv4_01.
4527 specialize Simp3_26 with
4528    $((Q \rightarrow \neg R) \rightarrow \neg(Q \wedge R)) (\neg(Q \wedge R) \rightarrow (Q \rightarrow \neg R))$ .
4529 intros Simp3_26b.
4530 MP Simp3_26b n4_22a.
4531 specialize n4_74 with  $(Q \wedge R) (P \wedge R)$ .
4532 intros n4_74a.
4533 Syll Simp3_26a n4_74a Sa.
4534 specialize n4_31 with  $(Q \wedge R) (P \wedge R)$ .
4535 intros n4_31a. (*Not cited*)
4536 apply propositional_extensionality in n4_31a.
4537 replace  $((P \wedge R) \vee (Q \wedge R))$  with  $((Q \wedge R) \vee (P \wedge R))$ 
4538   in Sa by now rewrite n4_31a.
4539 specialize n4_31 with  $(R \wedge Q) (R \wedge P)$ .
4540 intros n4_31b. (*Not cited*)
4541 apply propositional_extensionality in n4_31b.
4542 specialize n4_21 with  $((P \vee Q) \wedge R) (P \wedge R)$ .
4543 intros n4_21a. (*Not cited*)
4544 apply propositional_extensionality in n4_21a.
4545 specialize n4_4 with R P Q.
4546 intros n4_4a.
4547 replace  $(R \wedge P \vee R \wedge Q)$  with  $(R \wedge Q \vee R \wedge P)$ 
4548   in n4_4a by now apply n4_31b.
4549 specialize n4_3 with P R.
4550 intros n4_3a.
4551 apply propositional_extensionality in n4_3a.
4552 replace  $(R \wedge P)$  with  $(P \wedge R)$  in n4_4a
4553   by now apply n4_3a.
4554 specialize n4_3 with Q R.
4555 intros n4_3b.
4556 apply propositional_extensionality in n4_3b.
4557 replace  $(R \wedge Q)$  with  $(Q \wedge R)$  in n4_4a
4558   by now apply n4_3b.

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4559   apply propositional_extensionality in n4_4a.
4560   replace ((Q $\wedge$ R) $\vee$ (P $\wedge$ R)) with (R $\wedge$ (P $\vee$ Q)) in Sa
4561   by now apply n4_4a.
4562   specialize n4_3 with (P $\vee$ Q) R.
4563   intros n4_3c. (*Not cited*)
4564   apply propositional_extensionality in n4_3c.
4565   replace (R $\wedge$ (P $\vee$ Q)) with ((P $\vee$ Q) $\wedge$ R) in Sa
4566   by now apply n4_3c.
4567   replace ((P $\wedge$ R) $\leftrightarrow$ ((P $\vee$ Q) $\wedge$ R)) with
4568       (((P $\vee$ Q) $\wedge$ R) $\leftrightarrow$ (P $\wedge$ R)) in Sa
4569   by now apply n4_21a.
4570   apply Sa.
4571   Qed.
4572
4573   Theorem n5_74 :  $\forall$  P Q R : Prop,
4574       (P  $\rightarrow$  (Q  $\leftrightarrow$  R))  $\leftrightarrow$  ((P  $\rightarrow$  Q)  $\leftrightarrow$  (P  $\rightarrow$  R)).
4575   Proof. intros P Q R.
4576   specialize n5_41 with P Q R.
4577   intros n5_41a.
4578   specialize n5_41 with P R Q.
4579   intros n5_41b.
4580   Conj n5_41a n5_41b.
4581   split.
4582   apply n5_41a.
4583   apply n5_41b.
4584   specialize n4_38 with
4585       ((P $\rightarrow$ Q) $\rightarrow$ (P $\rightarrow$ R)) ((P $\rightarrow$ R) $\rightarrow$ (P $\rightarrow$ Q))
4586       (P $\rightarrow$ Q $\rightarrow$ R) (P $\rightarrow$ R $\rightarrow$ Q).
4587   intros n4_38a.
4588   MP n4_38a H.
4589   replace (((P $\rightarrow$ Q) $\rightarrow$ (P $\rightarrow$ R)) $\wedge$ ((P $\rightarrow$ R) $\rightarrow$ (P $\rightarrow$ Q)))
4590       with ((P $\rightarrow$ Q) $\leftrightarrow$ (P $\rightarrow$ R)) in n4_38a
4591   by now rewrite Equiv4_01.
4592   specialize n4_76 with P (Q $\rightarrow$ R) (R $\rightarrow$ Q).
4593   intros n4_76a.
4594   replace ((Q $\rightarrow$ R) $\wedge$ (R $\rightarrow$ Q)) with (Q $\leftrightarrow$ R) in n4_76a
4595   by now rewrite Equiv4_01.
4596   apply propositional_extensionality in n4_76a.
4597   replace ((P $\rightarrow$ Q $\rightarrow$ R) $\wedge$ (P $\rightarrow$ R $\rightarrow$ Q)) with
4598       (P $\rightarrow$ (Q $\leftrightarrow$ R)) in n4_38a by now apply n4_76a.
4599   specialize n4_21 with (P $\rightarrow$ Q $\leftrightarrow$ R)
4600       ((P $\rightarrow$ Q) $\leftrightarrow$ (P $\rightarrow$ R)).

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4601   intros n4_21a. (*Not cited*)
4602   apply propositional_extensionality in n4_21a.
4603   replace (((P→Q)↔(P→R))↔(P→Q↔R)) with
4604     ((P→(Q↔R))↔((P→Q)↔(P→R))) in n4_38a
4605     by now apply n4_21a.
4606   apply n4_38a.
4607   Qed.
4608
4609   Theorem n5_75 : ∀ P Q R : Prop,
4610     ((R → ¬Q) ∧ (P ↔ Q ∨ R)) → ((P ∧ ¬Q) ↔ R).
4611   Proof. intros P Q R.
4612   specialize n5_6 with P Q R.
4613   intros n5_6a.
4614   replace ((P ∧ ¬Q → R) ↔ (P → Q ∨ R)) with
4615     (((P ∧ ¬Q → R) → (P → Q ∨ R)) ∧
4616      ((P → Q ∨ R) → (P ∧ ¬Q → R))) in n5_6a
4617     by now rewrite Equiv4_01.
4618   specialize Simp3_27 with
4619     ((P ∧ ¬Q → R) → (P → Q ∨ R))
4620     ((P → Q ∨ R) → (P ∧ ¬Q → R)).
4621   intros Simp3_27a.
4622   MP Simp3_27a n5_6a.
4623   specialize Simp3_26 with
4624     (P → (Q ∨ R)) ((Q ∨ R) → P).
4625   intros Simp3_26a.
4626   replace ((P → (Q ∨ R)) ∧ ((Q ∨ R) → P)) with
4627     (P ↔ (Q ∨ R)) in Simp3_26a
4628     by now rewrite Equiv4_01.
4629   Syll Simp3_26a Simp3_27a Sa.
4630   specialize Simp3_27 with
4631     (R → ¬Q) (P ↔ (Q ∨ R)).
4632   intros Simp3_27b.
4633   Syll Simp3_27b Sa Sb.
4634   specialize Simp3_27 with
4635     (P → (Q ∨ R)) ((Q ∨ R) → P).
4636   intros Simp3_27c.
4637   replace ((P → (Q ∨ R)) ∧ ((Q ∨ R) → P)) with
4638     (P ↔ (Q ∨ R)) in Simp3_27c
4639     by now rewrite Equiv4_01.
4640   Syll Simp3_27b Simp3_27c Sc.
4641   specialize n4_77 with P Q R.
4642   intros n4_77a.

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4643   apply propositional_extensionality in n4_77a.
4644   replace (Q $\vee$ R $\rightarrow$ P) with ((Q $\rightarrow$ P) $\wedge$ (R $\rightarrow$ P)) in Sc
4645   by now apply n4_77a.
4646   specialize Simp3_27 with (Q $\rightarrow$ P) (R $\rightarrow$ P).
4647   intros Simp3_27d.
4648   Syll Sa Simp3_27d Sd.
4649   specialize Simp3_26 with (R $\rightarrow$  $\neg$ Q) (P $\leftrightarrow$ (Q $\vee$ R)).
4650   intros Simp3_26b.
4651   Conj Sd Simp3_26b.
4652   split.
4653   apply Sd.
4654   apply Simp3_26b.
4655   specialize Comp3_43 with
4656     ((R $\rightarrow$  $\neg$ Q) $\wedge$ (P $\leftrightarrow$ (Q $\vee$ R))) (R $\rightarrow$ P) (R $\rightarrow$  $\neg$ Q).
4657   intros Comp3_43a.
4658   MP Comp3_43a H.
4659   specialize Comp3_43 with R P ( $\neg$ Q).
4660   intros Comp3_43b.
4661   Syll Comp3_43a Comp3_43b Se.
4662   clear n5_6a. clear Simp3_27a.
4663     clear Simp3_27c. clear Simp3_27d.
4664     clear Simp3_26a. clear Comp3_43b.
4665     clear Simp3_26b. clear Comp3_43a.
4666     clear Sa. clear Sc. clear Sd. clear H.
4667     clear n4_77a. clear Simp3_27b.
4668   Conj Sb Se.
4669   split.
4670   apply Sb.
4671   apply Se.
4672   specialize Comp3_43 with
4673     ((R $\rightarrow$  $\neg$ Q) $\wedge$ (P $\leftrightarrow$ Q $\vee$ R))
4674     (P $\wedge$  $\neg$ Q $\rightarrow$ R) (R $\rightarrow$ P $\wedge$  $\neg$ Q).
4675   intros Comp3_43c.
4676   MP Comp3_43c H.
4677   replace ((P $\wedge$  $\neg$ Q $\rightarrow$ R) $\wedge$ (R $\rightarrow$ P $\wedge$  $\neg$ Q)) with
4678     (P $\wedge$  $\neg$ Q $\leftrightarrow$ R) in Comp3_43c
4679   by now rewrite Equiv4_01.
4680   apply Comp3_43c.
4681   Qed.
4682
4683   End No5.

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