

Principia Mathematica's Propositional Logic in *Coq*

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Abstract

This file contains the *Coq* code for the *Principia* Rewrite project's encoding of the propositional logic given in *1 – *5. The Github repository with this *Coq* file is here: <https://github.com/LogicalAtomist/principia>. To receive updates about the project, visit the *Principia Rewrite* project page: <https://www.principiarewrite.com/>. You can also follow the *Principia* Rewrite project on Twitter: <https://twitter.com/thePMrewrite>.

```
1  Require Import Unicode.Utf8.
2  Require Import Classical_Prop.
3  Require Import ClassicalFacts.
4  Require Import PropExtensionality.
5
6  Module No1.
7
8  Import Unicode.Utf8.
9  Import ClassicalFacts.
10 Import Classical_Prop.
11 Import PropExtensionality.
12
13   (*We first give the axioms of Principia
14   for the propositional calculus in *1.*)
15
16 Theorem Impl1_01 :  $\forall$  P Q : Prop,
17   (P  $\rightarrow$  Q) = ( $\neg$ P  $\vee$  Q).
18 Proof. intros P Q.
19   apply propositional_extensionality.
20   split.
21   apply imply_to_or.
22   apply or_to_imply.
```

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23 Qed.
24   (*This is a notational definition in Principia:
25     It is used to switch between " $\vee$ " and " $\rightarrow$ ".*)
26
27 Theorem MP1_1 :  $\forall$  P Q : Prop,
28   (P  $\rightarrow$  Q)  $\rightarrow$  P  $\rightarrow$  Q. (*Modus ponens*)
29 Proof. intros P Q.
30   intros iff_refl.
31   apply iff_refl.
32 Qed.
33   (*1.11 omitted: it is MP for propositions
34     containing variables. Likewise, omitted
35     the well-formedness rules 1.7, 1.71, 1.72*)
36
37 Theorem Taut1_2 :  $\forall$  P : Prop,
38   P  $\vee$  P  $\rightarrow$  P. (*Tautology*)
39 Proof. intros P.
40   apply imply_and_or.
41   apply iff_refl.
42 Qed.
43
44 Theorem Add1_3 :  $\forall$  P Q : Prop,
45   Q  $\rightarrow$  P  $\vee$  Q. (*Addition*)
46 Proof. intros P Q.
47   apply or_intror.
48 Qed.
49
50 Theorem Perm1_4 :  $\forall$  P Q : Prop,
51   P  $\vee$  Q  $\rightarrow$  Q  $\vee$  P. (*Permutation*)
52 Proof. intros P Q.
53   apply or_comm.
54 Qed.
55
56 Theorem Assoc1_5 :  $\forall$  P Q R : Prop,
57   P  $\vee$  (Q  $\vee$  R)  $\rightarrow$  Q  $\vee$  (P  $\vee$  R). (*Association*)
58 Proof. intros P Q R.
59   specialize or_assoc with P Q R.
60   intros or_assoc1.
61   replace (P $\vee$ Q $\vee$ R) with ((P $\vee$ Q) $\vee$ R).
62   specialize or_comm with P Q.
63   intros or_comm1.
64   replace (P $\vee$ Q) with (Q $\vee$ P).

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65   specialize or_assoc with Q P R.
66   intros or_assoc2.
67   replace ((Q∨P)∨R) with (Q∨P∨R).
68   apply iff_refl.
69   apply propositional_extensionality.
70   apply iff_sym.
71   apply or_assoc2.
72   apply propositional_extensionality.
73   apply or_comm.
74   apply propositional_extensionality.
75   apply or_assoc.
76   Qed.
77
78   Theorem Sum1_6 : ∀ P Q R : Prop,
79     (Q → R) → (P ∨ Q → P ∨ R). (*Summation*)
80   Proof. intros P Q R.
81     specialize imply_and_or2 with Q R P.
82     intros imply_and_or2a.
83     replace (P∨Q) with (Q∨P).
84     replace (P∨R) with (R∨P).
85     apply imply_and_or2a.
86     apply propositional_extensionality.
87     apply or_comm.
88     apply propositional_extensionality.
89     apply or_comm.
90     Qed.
91
92     (*These are all the propositional axioms of Principia.*)
93
94   Ltac MP H1 H2 :=
95     match goal with
96     | [ H1 : ?P → ?Q, H2 : ?P |- _ ] => specialize (H1 H2)
97     end.
98     (*We give this Ltac "MP" to make proofs more human-
99     readable and to more closely mirror Principia's style.*)
100
101   End No1.
102
103   Module No2.
104
105   Import No1.
106

```

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107  (*We proceed to the deductions of of Principia.*)
108
109  Theorem Abs2_01 :  $\forall P : \text{Prop},$ 
110     $(P \rightarrow \neg P) \rightarrow \neg P.$ 
111  Proof. intros P.
112    specialize Taut1_2 with  $(\neg P).$ 
113    intros Taut1_2.
114    replace  $(\neg P \vee \neg P)$  with  $(P \rightarrow \neg P)$  in Taut1_2.
115    apply Taut1_2.
116    apply Impl1_01.
117  Qed.
118
119  Theorem Simp2_02 :  $\forall P Q : \text{Prop},$ 
120     $Q \rightarrow (P \rightarrow Q).$ 
121  Proof. intros P Q.
122    specialize Add1_3 with  $(\neg P) Q.$ 
123    intros Add1_3.
124    replace  $(\neg P \vee Q)$  with  $(P \rightarrow Q)$  in Add1_3.
125    apply Add1_3.
126    apply Impl1_01.
127  Qed.
128
129  Theorem Transp2_03 :  $\forall P Q : \text{Prop},$ 
130     $(P \rightarrow \neg Q) \rightarrow (Q \rightarrow \neg P).$ 
131  Proof. intros P Q.
132    specialize Perm1_4 with  $(\neg P) (\neg Q).$ 
133    intros Perm1_4.
134    replace  $(\neg P \vee \neg Q)$  with  $(P \rightarrow \neg Q)$  in Perm1_4.
135    replace  $(\neg Q \vee \neg P)$  with  $(Q \rightarrow \neg P)$  in Perm1_4.
136    apply Perm1_4.
137    apply Impl1_01.
138    apply Impl1_01.
139  Qed.
140
141  Theorem Comm2_04 :  $\forall P Q R : \text{Prop},$ 
142     $(P \rightarrow (Q \rightarrow R)) \rightarrow (Q \rightarrow (P \rightarrow R)).$ 
143  Proof. intros P Q R.
144    specialize Assoc1_5 with  $(\neg P) (\neg Q) R.$ 
145    intros Assoc1_5.
146    replace  $(\neg Q \vee R)$  with  $(Q \rightarrow R)$  in Assoc1_5.
147    replace  $(\neg P \vee (Q \rightarrow R))$  with  $(P \rightarrow (Q \rightarrow R))$  in Assoc1_5.
148    replace  $(\neg P \vee R)$  with  $(P \rightarrow R)$  in Assoc1_5.

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149   replace ( $\neg Q \vee (P \rightarrow R)$ ) with ( $Q \rightarrow (P \rightarrow R)$ ) in Assoc1_5.
150   apply Assoc1_5.
151   apply Impl1_01.
152   apply Impl1_01.
153   apply Impl1_01.
154   apply Impl1_01.
155   Qed.
156
157   Theorem Syll2_05 :  $\forall P Q R : \text{Prop}$ ,
158     ( $Q \rightarrow R$ )  $\rightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R))$ .
159   Proof. intros P Q R.
160     specialize Sum1_6 with ( $\neg P$ ) Q R.
161     intros Sum1_6.
162     replace ( $\neg P \vee Q$ ) with ( $P \rightarrow Q$ ) in Sum1_6.
163     replace ( $\neg P \vee R$ ) with ( $P \rightarrow R$ ) in Sum1_6.
164     apply Sum1_6.
165     apply Impl1_01.
166     apply Impl1_01.
167     Qed.
168
169   Theorem Syll2_06 :  $\forall P Q R : \text{Prop}$ ,
170     ( $P \rightarrow Q$ )  $\rightarrow ((Q \rightarrow R) \rightarrow (P \rightarrow R))$ .
171   Proof. intros P Q R.
172     specialize Comm2_04 with ( $Q \rightarrow R$ ) ( $P \rightarrow Q$ ) ( $P \rightarrow R$ ).
173     intros Comm2_04.
174     specialize Syll2_05 with P Q R.
175     intros Syll2_05.
176     MP Comm2_04 Syll2_05.
177     apply Comm2_04.
178     Qed.
179
180   Theorem n2_07 :  $\forall P : \text{Prop}$ ,
181      $P \rightarrow (P \vee P)$ .
182   Proof. intros P.
183     specialize Add1_3 with P P.
184     intros Add1_3.
185     apply Add1_3.
186     Qed.
187
188   Theorem Id2_08 :  $\forall P : \text{Prop}$ ,
189      $P \rightarrow P$ .
190   Proof. intros P.

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191   specialize Syll2_05 with P (P ∨ P) P.
192   intros Syll2_05.
193   specialize Taut1_2 with P.
194   intros Taut1_2.
195   MP Syll2_05 Taut1_2.
196   specialize n2_07 with P.
197   intros n2_07.
198   MP Syll2_05 n2_07.
199   apply Syll2_05.
200   Qed.
201
202   Theorem n2_1 : ∀ P : Prop,
203     (¬P) ∨ P.
204   Proof. intros P.
205     specialize Id2_08 with P.
206     intros Id2_08.
207     replace (¬P ∨ P) with (P → P).
208     apply Id2_08.
209     apply Impl1_01.
210   Qed.
211
212   Theorem n2_11 : ∀ P : Prop,
213     P ∨ ¬P.
214   Proof. intros P.
215     specialize Perm1_4 with (¬P) P.
216     intros Perm1_4.
217     specialize n2_1 with P.
218     intros n2_1.
219     MP Perm1_4 n2_1.
220     apply Perm1_4.
221   Qed.
222
223   Theorem n2_12 : ∀ P : Prop,
224     P → ¬¬P.
225   Proof. intros P.
226     specialize n2_11 with (¬P).
227     intros n2_11.
228     replace (¬P ∨ ¬¬P) with (P → ¬¬P) in n2_11.
229     apply n2_11.
230     apply Impl1_01.
231   Qed.
232

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233 Theorem n2_13 :  $\forall$  P : Prop,
234   P  $\vee$   $\neg\neg\neg$ P.
235 Proof. intros P.
236   specialize Sum1_6 with P ( $\neg$ P) ( $\neg\neg\neg$ P).
237   intros Sum1_6.
238   specialize n2_12 with ( $\neg$ P).
239   intros n2_12.
240   MP Sum1_6 n2_12.
241   specialize n2_11 with P.
242   intros n2_11.
243   MP Sum1_6 n2_11.
244   apply Sum1_6.
245 Qed.
246
247 Theorem n2_14 :  $\forall$  P : Prop,
248    $\neg\neg$ P  $\rightarrow$  P.
249 Proof. intros P.
250   specialize Perm1_4 with P ( $\neg\neg\neg$ P).
251   intros Perm1_4.
252   specialize n2_13 with P.
253   intros n2_13.
254   MP Perm1_4 n2_13.
255   replace ( $\neg\neg\neg$ P  $\vee$  P) with ( $\neg\neg$ P  $\rightarrow$  P) in Perm1_4.
256   apply Perm1_4.
257   apply Impl1_01.
258 Qed.
259
260 Theorem Transp2_15 :  $\forall$  P Q : Prop,
261   ( $\neg$ P  $\rightarrow$  Q)  $\rightarrow$  ( $\neg$ Q  $\rightarrow$  P).
262 Proof. intros P Q.
263   specialize Syll2_05 with ( $\neg$ P) Q ( $\neg\neg$ Q).
264   intros Syll2_05a.
265   specialize n2_12 with Q.
266   intros n2_12.
267   MP Syll2_05a n2_12.
268   specialize Transp2_03 with ( $\neg$ P) ( $\neg$ Q).
269   intros Transp2_03.
270   specialize Syll2_05 with ( $\neg$ Q) ( $\neg\neg$ P) P.
271   intros Syll2_05b.
272   specialize n2_14 with P.
273   intros n2_14.
274   MP Syll2_05b n2_14.

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275   specialize Syll2_05 with ( $\neg P \rightarrow Q$ ) ( $\neg P \rightarrow \neg\neg Q$ ) ( $\neg Q \rightarrow \neg\neg P$ ).
276   intros Syll2_05c.
277   MP Syll2_05c Transp2_03.
278   MP Syll2_05c Syll2_05a.
279   specialize Syll2_05 with ( $\neg P \rightarrow Q$ ) ( $\neg Q \rightarrow \neg\neg P$ ) ( $\neg Q \rightarrow P$ ).
280   intros Syll2_05d.
281   MP Syll2_05d Syll2_05b.
282   MP Syll2_05d Syll2_05c.
283   apply Syll2_05d.
284   Qed.
285
286   Ltac Syll H1 H2 S :=
287     let S := fresh S in match goal with
288       | [ H1 : ?P  $\rightarrow$  ?Q, H2 : ?Q  $\rightarrow$  ?R |- _ ] =>
289         assert (S : P  $\rightarrow$  R) by (intros p; apply (H2 (H1 p)))
290     end.
291
292   Theorem Transp2_16 :  $\forall$  P Q : Prop,
293     (P  $\rightarrow$  Q)  $\rightarrow$  ( $\neg Q \rightarrow \neg P$ ).
294   Proof. intros P Q.
295     specialize n2_12 with Q.
296     intros n2_12a.
297     specialize Syll2_05 with P Q ( $\neg\neg Q$ ).
298     intros Syll2_05a.
299     specialize Transp2_03 with P ( $\neg Q$ ).
300     intros Transp2_03a.
301     MP n2_12a Syll2_05a.
302     Syll Syll2_05a Transp2_03a S.
303     apply S.
304     Qed.
305
306   Theorem Transp2_17 :  $\forall$  P Q : Prop,
307     ( $\neg Q \rightarrow \neg P$ )  $\rightarrow$  (P  $\rightarrow$  Q).
308   Proof. intros P Q.
309     specialize Transp2_03 with ( $\neg Q$ ) P.
310     intros Transp2_03a.
311     specialize n2_14 with Q.
312     intros n2_14a.
313     specialize Syll2_05 with P ( $\neg\neg Q$ ) Q.
314     intros Syll2_05a.
315     MP n2_14a Syll2_05a.
316     Syll Transp2_03a Syll2_05a S.

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317     apply S.
318 Qed.
319
320 Theorem n2_18 :  $\forall$  P : Prop,
321   ( $\neg$ P  $\rightarrow$  P)  $\rightarrow$  P.
322 Proof. intros P.
323   specialize n2_12 with P.
324   intro n2_12a.
325   specialize Syll2_05 with ( $\neg$ P) P ( $\neg\neg$ P).
326   intro Syll2_05a.
327   MP Syll2_05a n2_12.
328   specialize Abs2_01 with ( $\neg$ P).
329   intros Abs2_01a.
330   Syll Syll2_05a Abs2_01a Sa.
331   specialize n2_14 with P.
332   intros n2_14a.
333   Syll H n2_14a Sb.
334   apply Sb.
335 Qed.
336
337 Theorem n2_2 :  $\forall$  P Q : Prop,
338   P  $\rightarrow$  (P  $\vee$  Q).
339 Proof. intros P Q.
340   specialize Add1_3 with Q P.
341   intros Add1_3a.
342   specialize Perm1_4 with Q P.
343   intros Perm1_4a.
344   Syll Add1_3a Perm1_4a S.
345   apply S.
346 Qed.
347
348 Theorem n2_21 :  $\forall$  P Q : Prop,
349    $\neg$ P  $\rightarrow$  (P  $\rightarrow$  Q).
350 Proof. intros P Q.
351   specialize n2_2 with ( $\neg$ P) Q.
352   intros n2_2a.
353   specialize Impl1_01 with P Q.
354   intros Impl1_01a.
355   replace ( $\neg$ P $\vee$ Q) with (P $\rightarrow$ Q) in n2_2a.
356   apply n2_2a.
357 Qed.
358

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359 Theorem n2_24 :  $\forall$  P Q : Prop,
360   P  $\rightarrow$  ( $\neg$ P  $\rightarrow$  Q).
361 Proof. intros P Q.
362   specialize n2_21 with P Q.
363   intros n2_21a.
364   specialize Comm2_04 with ( $\neg$ P) P Q.
365   intros Comm2_04a.
366   apply Comm2_04a.
367   apply n2_21a.
368 Qed.
369
370 Theorem n2_25 :  $\forall$  P Q : Prop,
371   P  $\vee$  ((P  $\vee$  Q)  $\rightarrow$  Q).
372 Proof. intros P Q.
373   specialize n2_1 with (P  $\vee$  Q).
374   intros n2_1a.
375   specialize Assoc1_5 with ( $\neg$ (P $\vee$ Q)) P Q.
376   intros Assoc1_5a.
377   MP Assoc1_5a n2_1a.
378   replace ( $\neg$ (P $\vee$ Q) $\vee$ Q) with (P $\vee$ Q $\rightarrow$ Q) in Assoc1_5a.
379   apply Assoc1_5a.
380   apply Impl1_01.
381 Qed.
382
383 Theorem n2_26 :  $\forall$  P Q : Prop,
384    $\neg$ P  $\vee$  ((P  $\rightarrow$  Q)  $\rightarrow$  Q).
385 Proof. intros P Q.
386   specialize n2_25 with ( $\neg$ P) Q.
387   intros n2_25a.
388   replace ( $\neg$ P $\vee$ Q) with (P $\rightarrow$ Q) in n2_25a.
389   apply n2_25a.
390   apply Impl1_01.
391 Qed.
392
393 Theorem n2_27 :  $\forall$  P Q : Prop,
394   P  $\rightarrow$  ((P  $\rightarrow$  Q)  $\rightarrow$  Q).
395 Proof. intros P Q.
396   specialize n2_26 with P Q.
397   intros n2_26a.
398   replace ( $\neg$ P $\vee$ ((P $\rightarrow$ Q) $\rightarrow$ Q)) with (P $\rightarrow$ (P $\rightarrow$ Q) $\rightarrow$ Q) in n2_26a.
399   apply n2_26a.
400   apply Impl1_01.

```

```

401 Qed.
402
403 Theorem n2_3 :  $\forall P Q R : \text{Prop},$ 
404    $(P \vee (Q \vee R)) \rightarrow (P \vee (R \vee Q)).$ 
405 Proof. intros P Q R.
406   specialize Perm1_4 with Q R.
407   intros Perm1_4a.
408   specialize Sum1_6 with P (Q $\vee$ R) (R $\vee$ Q).
409   intros Sum1_6a.
410   MP Sum1_6a Perm1_4a.
411   apply Sum1_6a.
412 Qed.
413
414 Theorem n2_31 :  $\forall P Q R : \text{Prop},$ 
415    $(P \vee (Q \vee R)) \rightarrow ((P \vee Q) \vee R).$ 
416 Proof. intros P Q R.
417   specialize n2_3 with P Q R.
418   intros n2_3a.
419   specialize Assoc1_5 with P R Q.
420   intros Assoc1_5a.
421   specialize Perm1_4 with R (P $\vee$ Q).
422   intros Perm1_4a.
423   Syll Assoc1_5a Perm1_4a Sa.
424   Syll n2_3a Sa Sb.
425   apply Sb.
426 Qed.
427
428 Theorem n2_32 :  $\forall P Q R : \text{Prop},$ 
429    $((P \vee Q) \vee R) \rightarrow (P \vee (Q \vee R)).$ 
430 Proof. intros P Q R.
431   specialize Perm1_4 with (P $\vee$ Q) R.
432   intros Perm1_4a.
433   specialize Assoc1_5 with R P Q.
434   intros Assoc1_5a.
435   specialize n2_3 with P R Q.
436   intros n2_3a.
437   specialize Syll2_06 with ((P $\vee$ Q) $\vee$ R) (R $\vee$ P $\vee$ Q) (P $\vee$ R $\vee$ Q).
438   intros Syll2_06a.
439   MP Syll2_06a Perm1_4a.
440   MP Syll2_06a Assoc1_5a.
441   specialize Syll2_06 with ((P $\vee$ Q) $\vee$ R) (P $\vee$ R $\vee$ Q) (P $\vee$ Q $\vee$ R).
442   intros Syll2_06b.

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443   MP Syll2_06b Syll2_06a.
444   MP Syll2_06b n2_3a.
445   apply Syll2_06b.
446   Qed.
447
448   Theorem Abb2_33 :  $\forall$  P Q R : Prop,
449     (P  $\vee$  Q  $\vee$  R) = ((P  $\vee$  Q)  $\vee$  R).
450   Proof. intros P Q R.
451     apply propositional_extensionality.
452     split.
453     specialize n2_31 with P Q R.
454     intros n2_31.
455     apply n2_31.
456     specialize n2_32 with P Q R.
457     intros n2_32.
458     apply n2_32.
459   Qed.
460   (*This definition makes the default left association.
461     The default in Coq is right association.*)
462
463   Theorem n2_36 :  $\forall$  P Q R : Prop,
464     (Q  $\rightarrow$  R)  $\rightarrow$  ((P  $\vee$  Q)  $\rightarrow$  (R  $\vee$  P)).
465   Proof. intros P Q R.
466     specialize Perm1_4 with P R.
467     intros Perm1_4a.
468     specialize Syll2_05 with (P $\vee$ Q) (P $\vee$ R) (R $\vee$ P).
469     intros Syll2_05a.
470     MP Syll2_05a Perm1_4a.
471     specialize Sum1_6 with P Q R.
472     intros Sum1_6a.
473     Syll Sum1_6a Syll2_05a S.
474     apply S.
475   Qed.
476
477   Theorem n2_37 :  $\forall$  P Q R : Prop,
478     (Q  $\rightarrow$  R)  $\rightarrow$  ((Q  $\vee$  P)  $\rightarrow$  (P  $\vee$  R)).
479   Proof. intros P Q R.
480     specialize Perm1_4 with Q P.
481     intros Perm1_4a.
482     specialize Syll2_06 with (Q $\vee$ P) (P $\vee$ Q) (P $\vee$ R).
483     intros Syll2_06a.
484     MP Syll2_06a Perm1_4a.

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485   specialize Sum1_6 with P Q R.
486   intros Sum1_6a.
487   Syll Sum1_6a Syll12_06a S.
488   apply S.
489   Qed.
490
491   Theorem n2_38 :  $\forall P Q R : \text{Prop},$ 
492      $(Q \rightarrow R) \rightarrow ((Q \vee P) \rightarrow (R \vee P)).$ 
493   Proof. intros P Q R.
494     specialize Perm1_4 with P R.
495     intros Perm1_4a.
496     specialize Syll12_05 with  $(Q \vee P) (P \vee R) (R \vee P).$ 
497     intros Syll12_05a.
498     MP Syll12_05a Perm1_4a.
499     specialize Perm1_4 with Q P.
500     intros Perm1_4b.
501     specialize Syll12_06 with  $(Q \vee P) (P \vee Q) (P \vee R).$ 
502     intros Syll12_06a.
503     MP Syll12_06a Perm1_4b.
504     Syll Syll12_06a Syll12_05a H.
505     specialize Sum1_6 with P Q R.
506     intros Sum1_6a.
507     Syll Sum1_6a H S.
508     apply S.
509     Qed.
510
511   Theorem n2_4 :  $\forall P Q : \text{Prop},$ 
512      $(P \vee (P \vee Q)) \rightarrow (P \vee Q).$ 
513   Proof. intros P Q.
514     specialize n2_31 with P P Q.
515     intros n2_31a.
516     specialize Taut1_2 with P.
517     intros Taut1_2a.
518     specialize n2_38 with Q  $(P \vee P)$  P.
519     intros n2_38a.
520     MP n2_38a Taut1_2a.
521     Syll n2_31a n2_38a S.
522     apply S.
523     Qed.
524
525   Theorem n2_41 :  $\forall P Q : \text{Prop},$ 
526      $(Q \vee (P \vee Q)) \rightarrow (P \vee Q).$ 

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527 Proof. intros P Q.
528   specialize Assoc1_5 with Q P Q.
529   intros Assoc1_5a.
530   specialize Taut1_2 with Q.
531   intros Taut1_2a.
532   specialize Sum1_6 with P (Q∨Q) Q.
533   intros Sum1_6a.
534   MP Sum1_6a Taut1_2a.
535   Syll Assoc1_5a Sum1_6a S.
536   apply S.
537 Qed.
538
539 Theorem n2_42 : ∀ P Q : Prop,
540   (¬P ∨ (P → Q)) → (P → Q).
541 Proof. intros P Q.
542   specialize n2_4 with (¬P) Q.
543   intros n2_4a.
544   replace (¬P∨Q) with (P→Q) in n2_4a.
545   apply n2_4a. apply Impl1_01.
546 Qed.
547
548 Theorem n2_43 : ∀ P Q : Prop,
549   (P → (P → Q)) → (P → Q).
550 Proof. intros P Q.
551   specialize n2_42 with P Q.
552   intros n2_42a.
553   replace (¬P ∨ (P→Q)) with (P→(P→Q)) in n2_42a.
554   apply n2_42a.
555   apply Impl1_01.
556 Qed.
557
558 Theorem n2_45 : ∀ P Q : Prop,
559   ¬(P ∨ Q) → ¬P.
560 Proof. intros P Q.
561   specialize n2_2 with P Q.
562   intros n2_2a.
563   specialize Transp2_16 with P (P∨Q).
564   intros Transp2_16a.
565   MP n2_2 Transp2_16a.
566   apply Transp2_16a.
567 Qed.
568

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```

569 Theorem n2_46 :  $\forall P Q : \text{Prop},$ 
570    $\neg(P \vee Q) \rightarrow \neg Q.$ 
571 Proof. intros P Q.
572   specialize Add1_3 with P Q.
573   intros Add1_3a.
574   specialize Transp2_16 with Q (P $\vee$ Q).
575   intros Transp2_16a.
576   MP Add1_3a Transp2_16a.
577   apply Transp2_16a.
578 Qed.
579
580 Theorem n2_47 :  $\forall P Q : \text{Prop},$ 
581    $\neg(P \vee Q) \rightarrow (\neg P \vee Q).$ 
582 Proof. intros P Q.
583   specialize n2_45 with P Q.
584   intros n2_45a.
585   specialize n2_2 with ( $\neg P$ ) Q.
586   intros n2_2a.
587   Syll n2_45a n2_2a S.
588   apply S.
589 Qed.
590
591 Theorem n2_48 :  $\forall P Q : \text{Prop},$ 
592    $\neg(P \vee Q) \rightarrow (P \vee \neg Q).$ 
593 Proof. intros P Q.
594   specialize n2_46 with P Q.
595   intros n2_46a.
596   specialize Add1_3 with P ( $\neg Q$ ).
597   intros Add1_3a.
598   Syll n2_46a Add1_3a S.
599   apply S.
600 Qed.
601
602 Theorem n2_49 :  $\forall P Q : \text{Prop},$ 
603    $\neg(P \vee Q) \rightarrow (\neg P \vee \neg Q).$ 
604 Proof. intros P Q.
605   specialize n2_45 with P Q.
606   intros n2_45a.
607   specialize n2_2 with ( $\neg P$ ) ( $\neg Q$ ).
608   intros n2_2a.
609   Syll n2_45a n2_2a S.
610   apply S.

```

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611 Qed.
612
613 Theorem n2_5 :  $\forall P Q : \text{Prop},$ 
614    $\neg(P \rightarrow Q) \rightarrow (\neg P \rightarrow Q).$ 
615 Proof. intros P Q.
616   specialize n2_47 with ( $\neg P$ ) Q.
617   intros n2_47a.
618   replace ( $\neg P \vee Q$ ) with ( $P \rightarrow Q$ ) in n2_47a.
619   replace ( $\neg \neg P \vee Q$ ) with ( $\neg P \rightarrow Q$ ) in n2_47a.
620   apply n2_47a.
621   apply Impl1_01.
622   apply Impl1_01.
623 Qed.
624
625 Theorem n2_51 :  $\forall P Q : \text{Prop},$ 
626    $\neg(P \rightarrow Q) \rightarrow (P \rightarrow \neg Q).$ 
627 Proof. intros P Q.
628   specialize n2_48 with ( $\neg P$ ) Q.
629   intros n2_48a.
630   replace ( $\neg P \vee Q$ ) with ( $P \rightarrow Q$ ) in n2_48a.
631   replace ( $\neg P \vee \neg Q$ ) with ( $P \rightarrow \neg Q$ ) in n2_48a.
632   apply n2_48a.
633   apply Impl1_01.
634   apply Impl1_01.
635 Qed.
636
637 Theorem n2_52 :  $\forall P Q : \text{Prop},$ 
638    $\neg(P \rightarrow Q) \rightarrow (\neg P \rightarrow \neg Q).$ 
639 Proof. intros P Q.
640   specialize n2_49 with ( $\neg P$ ) Q.
641   intros n2_49a.
642   replace ( $\neg P \vee Q$ ) with ( $P \rightarrow Q$ ) in n2_49a.
643   replace ( $\neg \neg P \vee \neg Q$ ) with ( $\neg P \rightarrow \neg Q$ ) in n2_49a.
644   apply n2_49a.
645   apply Impl1_01.
646   apply Impl1_01.
647 Qed.
648
649 Theorem n2_521 :  $\forall P Q : \text{Prop},$ 
650    $\neg(P \rightarrow Q) \rightarrow (Q \rightarrow P).$ 
651 Proof. intros P Q.
652   specialize n2_52 with P Q.

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653   intros n2_52a.
654   specialize Transp2_17 with Q P.
655   intros Transp2_17a.
656   Syll n2_52a Transp2_17a S.
657   apply S.
658   Qed.
659
660   Theorem n2_53 :  $\forall P Q : \text{Prop},$ 
661      $(P \vee Q) \rightarrow (\neg P \rightarrow Q).$ 
662   Proof. intros P Q.
663     specialize n2_12 with P.
664     intros n2_12a.
665     specialize n2_38 with Q P ( $\neg\neg P$ ).
666     intros n2_38a.
667     MP n2_38a n2_12a.
668     replace ( $\neg\neg P \vee Q$ ) with ( $\neg P \rightarrow Q$ ) in n2_38a.
669     apply n2_38a.
670     apply Impl1_01.
671     Qed.
672
673   Theorem n2_54 :  $\forall P Q : \text{Prop},$ 
674      $(\neg P \rightarrow Q) \rightarrow (P \vee Q).$ 
675   Proof. intros P Q.
676     specialize n2_14 with P.
677     intros n2_14a.
678     specialize n2_38 with Q ( $\neg\neg P$ ) P.
679     intros n2_38a.
680     MP n2_38a n2_14a.
681     replace ( $\neg\neg P \vee Q$ ) with ( $\neg P \rightarrow Q$ ) in n2_38a.
682     apply n2_38a.
683     apply Impl1_01.
684     Qed.
685
686   Theorem n2_55 :  $\forall P Q : \text{Prop},$ 
687      $\neg P \rightarrow ((P \vee Q) \rightarrow Q).$ 
688   Proof. intros P Q.
689     specialize n2_53 with P Q.
690     intros n2_53a.
691     specialize Comm2_04 with  $(P \vee Q) (\neg P) Q.$ 
692     intros Comm2_04a.
693     MP n2_53a Comm2_04a.
694     apply Comm2_04a.

```

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695 Qed.
696
697 Theorem n2_56 :  $\forall P Q : \text{Prop},$ 
698    $\neg Q \rightarrow ((P \vee Q) \rightarrow P).$ 
699 Proof. intros P Q.
700   specialize n2_55 with Q P.
701   intros n2_55a.
702   specialize Perm1_4 with P Q.
703   intros Perm1_4a.
704   specialize Syll2_06 with  $(P \vee Q) (Q \vee P) P.$ 
705   intros Syll2_06a.
706   MP Syll2_06a Perm1_4a.
707   Syll n2_55a Syll2_06a Sa.
708   apply Sa.
709 Qed.
710
711 Theorem n2_6 :  $\forall P Q : \text{Prop},$ 
712    $(\neg P \rightarrow Q) \rightarrow ((P \rightarrow Q) \rightarrow Q).$ 
713 Proof. intros P Q.
714   specialize n2_38 with Q  $(\neg P) Q.$ 
715   intros n2_38a.
716   specialize Taut1_2 with Q.
717   intros Taut1_2a.
718   specialize Syll2_05 with  $(\neg P \vee Q) (Q \vee Q) Q.$ 
719   intros Syll2_05a.
720   MP Syll2_05a Taut1_2a.
721   Syll n2_38a Syll2_05a S.
722   replace  $(\neg P \vee Q)$  with  $(P \rightarrow Q)$  in S.
723   apply S.
724   apply Impl1_01.
725 Qed.
726
727 Theorem n2_61 :  $\forall P Q : \text{Prop},$ 
728    $(P \rightarrow Q) \rightarrow ((\neg P \rightarrow Q) \rightarrow Q).$ 
729 Proof. intros P Q.
730   specialize n2_6 with P Q.
731   intros n2_6a.
732   specialize Comm2_04 with  $(\neg P \rightarrow Q) (P \rightarrow Q) Q.$ 
733   intros Comm2_04a.
734   MP Comm2_04a n2_6a.
735   apply Comm2_04a.
736 Qed.

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737
738 Theorem n2_62 :  $\forall P Q : \text{Prop},$ 
739    $(P \vee Q) \rightarrow ((P \rightarrow Q) \rightarrow Q).$ 
740 Proof. intros P Q.
741   specialize n2_53 with P Q.
742   intros n2_53a.
743   specialize n2_6 with P Q.
744   intros n2_6a.
745   Syll n2_53a n2_6a S.
746   apply S.
747 Qed.
748
749 Theorem n2_621 :  $\forall P Q : \text{Prop},$ 
750    $(P \rightarrow Q) \rightarrow ((P \vee Q) \rightarrow Q).$ 
751 Proof. intros P Q.
752   specialize n2_62 with P Q.
753   intros n2_62a.
754   specialize Comm2_04 with  $(P \vee Q) (P \rightarrow Q) Q.$ 
755   intros Comm2_04a.
756   MP Comm2_04a n2_62a.
757   apply Comm2_04a.
758 Qed.
759
760 Theorem n2_63 :  $\forall P Q : \text{Prop},$ 
761    $(P \vee Q) \rightarrow ((\neg P \vee Q) \rightarrow Q).$ 
762 Proof. intros P Q.
763   specialize n2_62 with P Q.
764   intros n2_62a.
765   replace  $(\neg P \vee Q)$  with  $(P \rightarrow Q).$ 
766   apply n2_62a.
767   apply Impl1_01.
768 Qed.
769
770 Theorem n2_64 :  $\forall P Q : \text{Prop},$ 
771    $(P \vee Q) \rightarrow ((P \vee \neg Q) \rightarrow P).$ 
772 Proof. intros P Q.
773   specialize n2_63 with Q P.
774   intros n2_63a.
775   specialize Perm1_4 with P Q.
776   intros Perm1_4a.
777   Syll n2_63a Perm1_4a Ha.
778   specialize Syll2_06 with  $(P \vee \neg Q) (\neg Q \vee P) P.$ 

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779   intros Syll2_06a.
780   specialize Perm1_4 with P ( $\neg$ Q).
781   intros Perm1_4b.
782   MP Syll2_06a Perm1_4b.
783   Syll Syll2_06a Ha S.
784   apply S.
785   Qed.
786
787   Theorem n2_65 :  $\forall$  P Q : Prop,
788     (P  $\rightarrow$  Q)  $\rightarrow$  ((P  $\rightarrow$   $\neg$ Q)  $\rightarrow$   $\neg$ P).
789   Proof. intros P Q.
790     specialize n2_64 with ( $\neg$ P) Q.
791     intros n2_64a.
792     replace ( $\neg$ P $\vee$ Q) with (P $\rightarrow$ Q) in n2_64a.
793     replace ( $\neg$ P $\vee$  $\neg$ Q) with (P $\rightarrow$  $\neg$ Q) in n2_64a.
794     apply n2_64a.
795     apply Impl1_01.
796     apply Impl1_01.
797     Qed.
798
799   Theorem n2_67 :  $\forall$  P Q : Prop,
800     ((P  $\vee$  Q)  $\rightarrow$  Q)  $\rightarrow$  (P  $\rightarrow$  Q).
801   Proof. intros P Q.
802     specialize n2_54 with P Q.
803     intros n2_54a.
804     specialize Syll2_06 with ( $\neg$ P $\rightarrow$ Q) (P $\vee$ Q) Q.
805     intros Syll2_06a.
806     MP Syll2_06a n2_54a.
807     specialize n2_24 with P Q.
808     intros n2_24.
809     specialize Syll2_06 with P ( $\neg$ P $\rightarrow$ Q) Q.
810     intros Syll2_06b.
811     MP Syll2_06b n2_24a.
812     Syll Syll2_06b Syll2_06a S.
813     apply S.
814     Qed.
815
816   Theorem n2_68 :  $\forall$  P Q : Prop,
817     ((P  $\rightarrow$  Q)  $\rightarrow$  Q)  $\rightarrow$  (P  $\vee$  Q).
818   Proof. intros P Q.
819     specialize n2_67 with ( $\neg$ P) Q.
820     intros n2_67a.

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821   replace ( $\neg P \vee Q$ ) with ( $P \rightarrow Q$ ) in n2_67a.
822   specialize n2_54 with P Q.
823   intros n2_54a.
824   Syll n2_67a n2_54a S.
825   apply S.
826   apply Impl1_01.
827   Qed.
828
829   Theorem n2_69 :  $\forall P Q : \text{Prop}$ ,
830      $((P \rightarrow Q) \rightarrow Q) \rightarrow ((Q \rightarrow P) \rightarrow P)$ .
831   Proof. intros P Q.
832     specialize n2_68 with P Q.
833     intros n2_68a.
834     specialize Perm1_4 with P Q.
835     intros Perm1_4a.
836     Syll n2_68a Perm1_4a Sa.
837     specialize n2_62 with Q P.
838     intros n2_62a.
839     Syll Sa n2_62a Sb.
840     apply Sb.
841     Qed.
842
843   Theorem n2_73 :  $\forall P Q R : \text{Prop}$ ,
844      $(P \rightarrow Q) \rightarrow ((P \vee Q) \vee R) \rightarrow (Q \vee R)$ .
845   Proof. intros P Q R.
846     specialize n2_621 with P Q.
847     intros n2_621a.
848     specialize n2_38 with R ( $P \vee Q$ ) Q.
849     intros n2_38a.
850     Syll n2_621a n2_38a S.
851     apply S.
852     Qed.
853
854   Theorem n2_74 :  $\forall P Q R : \text{Prop}$ ,
855      $(Q \rightarrow P) \rightarrow ((P \vee Q) \vee R) \rightarrow (P \vee R)$ .
856   Proof. intros P Q R.
857     specialize n2_73 with Q P R.
858     intros n2_73a.
859     specialize Assoc1_5 with P Q R.
860     intros Assoc1_5a.
861     specialize n2_31 with Q P R.
862     intros n2_31a. (*not cited*)

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863   Syll Assoc1_5a n2_31a Sa.
864   specialize n2_32 with P Q R.
865   intros n2_32a. (*not cited*)
866   Syll n2_32a Sa Sb.
867   specialize Syll2_06 with ((P∨Q)∨R) ((Q∨P)∨R) (P∨R).
868   intros Syll2_06a.
869   MP Syll2_06a Sb.
870   Syll n2_73a Syll2_05a H.
871   apply H.
872   Qed.
873
874   Theorem n2_75 : ∀ P Q R : Prop,
875     (P ∨ Q) → ((P ∨ (Q → R)) → (P ∨ R)).
876   Proof. intros P Q R.
877     specialize n2_74 with P (¬Q) R.
878     intros n2_74a.
879     specialize n2_53 with Q P.
880     intros n2_53a.
881     Syll n2_53a n2_74a Sa.
882     specialize n2_31 with P (¬Q) R.
883     intros n2_31a.
884     specialize Syll2_06 with (P∨(¬Q)∨R)((P∨(¬Q))∨R) (P∨R).
885     intros Syll2_06a.
886     MP Syll2_06a n2_31a.
887     Syll Sa Syll2_06a Sb.
888     specialize Perm1_4 with P Q.
889     intros Perm1_4a. (*not cited*)
890     Syll Perm1_4a Sb Sc.
891     replace (¬Q∨R) with (Q→R) in Sc.
892     apply Sc.
893     apply Impl1_01.
894     Qed.
895
896   Theorem n2_76 : ∀ P Q R : Prop,
897     (P ∨ (Q → R)) → ((P ∨ Q) → (P ∨ R)).
898   Proof. intros P Q R.
899     specialize n2_75 with P Q R.
900     intros n2_75a.
901     specialize Comm2_04 with (P∨Q) (P∨(Q→R)) (P∨R).
902     intros Comm2_04a.
903     apply Comm2_04a.
904     apply n2_75a.

```

```

905 Qed.
906
907 Theorem n2_77 :  $\forall P Q R : \text{Prop},$ 
908    $(P \rightarrow (Q \rightarrow R)) \rightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R)).$ 
909 Proof. intros P Q R.
910   specialize n2_76 with ( $\neg P$ ) Q R.
911   intros n2_76a.
912   replace ( $\neg P \vee (Q \rightarrow R)$ ) with  $(P \rightarrow Q \rightarrow R)$  in n2_76a.
913   replace ( $\neg P \vee Q$ ) with  $(P \rightarrow Q)$  in n2_76a.
914   replace ( $\neg P \vee R$ ) with  $(P \rightarrow R)$  in n2_76a.
915   apply n2_76a.
916   apply Impl1_01.
917   apply Impl1_01.
918   apply Impl1_01.
919 Qed.
920
921 Theorem n2_8 :  $\forall Q R S : \text{Prop},$ 
922    $(Q \vee R) \rightarrow ((\neg R \vee S) \rightarrow (Q \vee S)).$ 
923 Proof. intros Q R S.
924   specialize n2_53 with R Q.
925   intros n2_53a.
926   specialize Perm1_4 with Q R.
927   intros Perm1_4a.
928   Syll Perm1_4a n2_53a Ha.
929   specialize n2_38 with S ( $\neg R$ ) Q.
930   intros n2_38a.
931   Syll H n2_38a Hb.
932   apply Hb.
933 Qed.
934
935 Theorem n2_81 :  $\forall P Q R S : \text{Prop},$ 
936    $(Q \rightarrow (R \rightarrow S)) \rightarrow ((P \vee Q) \rightarrow ((P \vee R) \rightarrow (P \vee S))).$ 
937 Proof. intros P Q R S.
938   specialize Sum1_6 with P Q ( $R \rightarrow S$ ).
939   intros Sum1_6a.
940   specialize n2_76 with P R S.
941   intros n2_76a.
942   specialize Syll2_05 with  $(P \vee Q)$   $(P \vee (R \rightarrow S))$   $((P \vee R) \rightarrow (P \vee S)).$ 
943   intros Syll2_05a.
944   MP Syll2_05a n2_76a.
945   Syll Sum1_6a Syll2_05a H.
946   apply H.

```

```

947 Qed.
948
949 Theorem n2_82 :  $\forall P Q R S : \text{Prop},$ 
950    $(P \vee Q \vee R) \rightarrow ((P \vee \neg R \vee S) \rightarrow (P \vee Q \vee S)).$ 
951 Proof. intros P Q R S.
952   specialize n2_8 with Q R S.
953   intros n2_8a.
954   specialize n2_81 with P (Q $\vee$ R) ( $\neg$ R $\vee$ S) (Q $\vee$ S).
955   intros n2_81a.
956   MP n2_81a n2_8a.
957   apply n2_81a.
958 Qed.
959
960 Theorem n2_83 :  $\forall P Q R S : \text{Prop},$ 
961    $(P \rightarrow (Q \rightarrow R)) \rightarrow ((P \rightarrow (R \rightarrow S)) \rightarrow (P \rightarrow (Q \rightarrow S))).$ 
962 Proof. intros P Q R S.
963   specialize n2_82 with ( $\neg$ P) ( $\neg$ Q) R S.
964   intros n2_82a.
965   replace ( $\neg$ Q $\vee$ R) with (Q $\rightarrow$ R) in n2_82a.
966   replace ( $\neg$ P $\vee$ (Q $\rightarrow$ R)) with (P $\rightarrow$ Q $\rightarrow$ R) in n2_82a.
967   replace ( $\neg$ R $\vee$ S) with (R $\rightarrow$ S) in n2_82a.
968   replace ( $\neg$ P $\vee$ (R $\rightarrow$ S)) with (P $\rightarrow$ R $\rightarrow$ S) in n2_82a.
969   replace ( $\neg$ Q $\vee$ S) with (Q $\rightarrow$ S) in n2_82a.
970   replace ( $\neg$ Q $\vee$ S) with (Q $\rightarrow$ S) in n2_82a.
971   replace ( $\neg$ P $\vee$ (Q $\rightarrow$ S)) with (P $\rightarrow$ Q $\rightarrow$ S) in n2_82a.
972   apply n2_82a.
973   apply Impl1_01.
974   apply Impl1_01.
975   apply Impl1_01.
976   apply Impl1_01.
977   apply Impl1_01.
978   apply Impl1_01.
979   apply Impl1_01.
980 Qed.
981
982 Theorem n2_85 :  $\forall P Q R : \text{Prop},$ 
983    $((P \vee Q) \rightarrow (P \vee R)) \rightarrow (P \vee (Q \rightarrow R)).$ 
984 Proof. intros P Q R.
985   specialize Add1_3 with P Q.
986   intros Add1_3a.
987   specialize Syll2_06 with Q (P $\vee$ Q) R.
988   intros Syll2_06a.

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989 MP Syll2_06a Add1_3a.
990 specialize n2_55 with P R.
991 intros n2_55a.
992 specialize Syll2_05 with (P $\vee$ Q) (P $\vee$ R) R.
993 intros Syll2_05a.
994 Syll n2_55a Syll2_05a Ha.
995 specialize n2_83 with ( $\neg$ P) ((P $\vee$ Q) $\rightarrow$ (P $\vee$ R)) ((P $\vee$ Q) $\rightarrow$ R) (Q $\rightarrow$ R).
996 intros n2_83a.
997 MP n2_83a Ha.
998 specialize Comm2_04 with ( $\neg$ P) (P $\vee$ Q $\rightarrow$ P $\vee$ R) (Q $\rightarrow$ R).
999 intros Comm2_04a.
1000 Syll Ha Comm2_04a Hb.
1001 specialize n2_54 with P (Q $\rightarrow$ R).
1002 intros n2_54a.
1003 specialize Simp2_02 with ( $\neg$ P) ((P $\vee$ Q $\rightarrow$ R) $\rightarrow$ (Q $\rightarrow$ R)).
1004 intros Simp2_02a. (*Not cited*)
1005      (*Greg's suggestion per the BRS list on June 25, 2017.*)
1006 MP Syll2_06a Simp2_02a.
1007 MP Hb Simp2_02a.
1008 Syll Hb n2_54a Hc.
1009 apply Hc.
1010 Qed.
1011
1012 Theorem n2_86 :  $\forall$  P Q R : Prop,
1013   ((P  $\rightarrow$  Q)  $\rightarrow$  (P  $\rightarrow$  R))  $\rightarrow$  (P  $\rightarrow$  (Q  $\rightarrow$  R)).
1014 Proof. intros P Q R.
1015   specialize n2_85 with ( $\neg$ P) Q R.
1016   intros n2_85a.
1017   replace ( $\neg$ P $\vee$ Q) with (P $\rightarrow$ Q) in n2_85a.
1018   replace ( $\neg$ P $\vee$ R) with (P $\rightarrow$ R) in n2_85a.
1019   replace ( $\neg$ P $\vee$ (Q $\rightarrow$ R)) with (P $\rightarrow$ Q $\rightarrow$ R) in n2_85a.
1020   apply n2_85a.
1021   apply Impl1_01.
1022   apply Impl1_01.
1023   apply Impl1_01.
1024 Qed.
1025
1026 End No2.
1027
1028 Module No3.
1029
1030 Import No1.

```

```

1031 Import No2.
1032
1033
1034 Theorem Prod3_01 :  $\forall$  P Q : Prop,
1035   (P  $\wedge$  Q) = ( $\neg$ ( $\neg$ P  $\vee$   $\neg$ Q)).
1036 Proof. intros P Q.
1037   apply propositional_extensionality.
1038   split.
1039   specialize or_not_and with (P) (Q).
1040   intros or_not_and.
1041   specialize Transp2_03 with ( $\neg$ P  $\vee$   $\neg$ Q) (P  $\wedge$  Q).
1042   intros Transp2_03.
1043   MP Transp2_03 or_not_and.
1044   apply Transp2_03.
1045   specialize not_and_or with (P) (Q).
1046   intros not_and_or.
1047   specialize Transp2_15 with (P  $\wedge$  Q) ( $\neg$ P  $\vee$   $\neg$ Q).
1048   intros Transp2_15.
1049   MP Transp2_15 not_and_or.
1050   apply Transp2_15.
1051 Qed.
1052 (*This is a notational definition in Principia;
1053   it is used to switch between " $\wedge$ " and " $\neg\vee\neg$ ".*)
1054
1055 (*Axiom Abb3_02 :  $\forall$  P Q R : Prop,
1056   (P  $\rightarrow$  Q  $\rightarrow$  R) = ((P  $\rightarrow$  Q)  $\wedge$  (Q  $\rightarrow$  R)).*)
1057 (*Since Coq forbids such strings as ill-formed, or
1058   else automatically associates to the right,
1059   we leave this notational axiom commented out.*/)
1060
1061 Theorem Conj3_03 :  $\forall$  P Q : Prop, P  $\rightarrow$  Q  $\rightarrow$  (P $\wedge$ Q).
1062 Proof. intros P Q.
1063   specialize n2_11 with ( $\neg$ P $\vee$  $\neg$ Q). intros n2_11a.
1064   specialize n2_32 with ( $\neg$ P) ( $\neg$ Q) ( $\neg$ ( $\neg$ P  $\vee$   $\neg$ Q)). intros n2_32a.
1065   MP n2_32a n2_11a.
1066   replace ( $\neg$ ( $\neg$ P $\vee$  $\neg$ Q)) with (P $\wedge$ Q) in n2_32a.
1067   replace ( $\neg$ Q  $\vee$  (P $\wedge$ Q)) with (Q $\rightarrow$ (P $\wedge$ Q)) in n2_32a.
1068   replace ( $\neg$ P  $\vee$  (Q  $\rightarrow$  (P $\wedge$ Q))) with (P $\rightarrow$ Q $\rightarrow$ (P $\wedge$ Q)) in n2_32a.
1069   apply n2_32a.
1070   apply Impl1_01.
1071   apply Impl1_01.
1072   apply Prod3_01.

```

```

1073 Qed.
1074 (*3.03 is permits the inference from the theoremhood
1075    of P and that of Q to the theoremhood of P and Q. So:*)
1076
1077 Ltac Conj H1 H2 :=
1078   match goal with
1079   | [ H1 : ?P, H2 : ?Q |- _ ] =>
1080     assert (P ∧ Q)
1081 end.
1082
1083 Theorem n3_1 : ∀ P Q : Prop,
1084   (P ∧ Q) → ¬(¬P ∨ ¬Q).
1085 Proof. intros P Q.
1086   replace (¬(¬P ∨ ¬Q)) with (P ∧ Q).
1087   specialize Id2_08 with (P ∧ Q).
1088   intros Id2_08a.
1089   apply Id2_08a.
1090   apply Prod3_01.
1091 Qed.
1092
1093 Theorem n3_11 : ∀ P Q : Prop,
1094   ¬(¬P ∨ ¬Q) → (P ∧ Q).
1095 Proof. intros P Q.
1096   replace (¬(¬P ∨ ¬Q)) with (P ∧ Q).
1097   specialize Id2_08 with (P ∧ Q).
1098   intros Id2_08a.
1099   apply Id2_08a.
1100   apply Prod3_01.
1101 Qed.
1102
1103 Theorem n3_12 : ∀ P Q : Prop,
1104   (¬P ∨ ¬Q) ∨ (P ∧ Q).
1105 Proof. intros P Q.
1106   specialize n2_11 with (¬P ∨ ¬Q).
1107   intros n2_11a.
1108   replace (¬(¬P ∨ ¬Q)) with (P ∧ Q) in n2_11a.
1109   apply n2_11a.
1110   apply Prod3_01.
1111 Qed.
1112
1113 Theorem n3_13 : ∀ P Q : Prop,
1114   ¬(P ∧ Q) → (¬P ∨ ¬Q).

```

```

1115 Proof. intros P Q.
1116   specialize n3_11 with P Q.
1117   intros n3_11a.
1118   specialize Transp2_15 with ( $\neg P \vee \neg Q$ ) ( $P \wedge Q$ ).
1119   intros Transp2_15a.
1120   MP Transp2_15a n3_11a.
1121   apply Transp2_15a.
1122 Qed.
1123
1124 Theorem n3_14 :  $\forall P Q : \text{Prop}$ ,
1125   ( $\neg P \vee \neg Q$ )  $\rightarrow \neg(P \wedge Q)$ .
1126 Proof. intros P Q.
1127   specialize n3_1 with P Q.
1128   intros n3_1a.
1129   specialize Transp2_16 with ( $P \wedge Q$ ) ( $\neg(\neg P \vee \neg Q)$ ).
1130   intros Transp2_16a.
1131   MP Transp2_16a n3_1a.
1132   specialize n2_12 with ( $\neg P \vee \neg Q$ ).
1133   intros n2_12a.
1134   Syll n2_12a Transp2_16a S.
1135   apply S.
1136 Qed.
1137
1138 Theorem n3_2 :  $\forall P Q : \text{Prop}$ ,
1139    $P \rightarrow Q \rightarrow (P \wedge Q)$ .
1140 Proof. intros P Q.
1141   specialize n3_12 with P Q.
1142   intros n3_12a.
1143   specialize n2_32 with ( $\neg P$ ) ( $\neg Q$ ) ( $P \wedge Q$ ).
1144   intros n2_32a.
1145   MP n3_32a n3_12a.
1146   replace ( $\neg Q \vee P \wedge Q$ ) with ( $Q \rightarrow P \wedge Q$ ) in n2_32a.
1147   replace ( $\neg P \vee (Q \rightarrow P \wedge Q)$ ) with ( $P \rightarrow Q \rightarrow P \wedge Q$ ) in n2_32a.
1148   apply n2_32a.
1149   apply Impl1_01.
1150   apply Impl1_01.
1151 Qed.
1152
1153 Theorem n3_21 :  $\forall P Q : \text{Prop}$ ,
1154    $Q \rightarrow P \rightarrow (P \wedge Q)$ .
1155 Proof. intros P Q.
1156   specialize n3_2 with P Q.

```

```

1157   intros n3_2a.
1158   specialize Comm2_04 with P Q (P $\wedge$ Q).
1159   intros Comm2_04a.
1160   MP Comm2_04a n3_2a.
1161   apply Comm2_04a.
1162   Qed.
1163
1164   Theorem n3_22 :  $\forall$  P Q : Prop,
1165     (P  $\wedge$  Q)  $\rightarrow$  (Q  $\wedge$  P).
1166   Proof. intros P Q.
1167     specialize n3_13 with Q P.
1168     intros n3_13a.
1169     specialize Perm1_4 with ( $\neg$ Q) ( $\neg$ P).
1170     intros Perm1_4a.
1171     Syll n3_13a Perm1_4a Ha.
1172     specialize n3_14 with P Q.
1173     intros n3_14a.
1174     Syll Ha n3_14a Hb.
1175     specialize Transp2_17 with (P $\wedge$ Q) (Q  $\wedge$  P).
1176     intros Transp2_17a.
1177     MP Transp2_17a Hb.
1178     apply Transp2_17a.
1179     Qed.
1180
1181   Theorem n3_24 :  $\forall$  P : Prop,
1182      $\neg$ (P  $\wedge$   $\neg$ P).
1183   Proof. intros P.
1184     specialize n2_11 with ( $\neg$ P).
1185     intros n2_11a.
1186     specialize n3_14 with P ( $\neg$ P).
1187     intros n3_14a.
1188     MP n3_14a n2_11a.
1189     apply n3_14a.
1190     Qed.
1191
1192   Theorem Simp3_26 :  $\forall$  P Q : Prop,
1193     (P  $\wedge$  Q)  $\rightarrow$  P.
1194   Proof. intros P Q.
1195     specialize Simp2_02 with Q P.
1196     intros Simp2_02a.
1197     replace (P $\rightarrow$ (Q $\rightarrow$ P)) with ( $\neg$ P $\vee$ (Q $\rightarrow$ P)) in Simp2_02a.
1198     replace (Q $\rightarrow$ P) with ( $\neg$ Q $\vee$ P) in Simp2_02a.

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1199 specialize n2_31 with ( $\neg$ P) ( $\neg$ Q) P.
1200 intros n2_31a.
1201 MP n2_31a Simp2_02a.
1202 specialize n2_53 with ( $\neg$ P $\vee$  $\neg$ Q) P.
1203 intros n2_53a.
1204 MP n2_53a Simp2_02a.
1205 replace ( $\neg$ ( $\neg$ P $\vee$  $\neg$ Q)) with (P $\wedge$ Q) in n2_53a.
1206 apply n2_53a.
1207 apply Prod3_01.
1208 rewrite <- Impl1_01.
1209 reflexivity.
1210 rewrite <- Impl1_01.
1211 reflexivity.
1212 Qed.
1213
1214 Theorem Simp3_27 :  $\forall$  P Q : Prop,
1215   (P  $\wedge$  Q)  $\rightarrow$  Q.
1216 Proof. intros P Q.
1217   specialize n3_22 with P Q.
1218   intros n3_22a.
1219   specialize Simp3_26 with Q P.
1220   intros Simp3_26a.
1221   Syll n3_22a Simp3_26a S.
1222   apply S.
1223   Qed.
1224
1225 Theorem Exp3_3 :  $\forall$  P Q R : Prop,
1226   ((P  $\wedge$  Q)  $\rightarrow$  R)  $\rightarrow$  (P  $\rightarrow$  (Q  $\rightarrow$  R)).
1227 Proof. intros P Q R.
1228   specialize Id2_08 with ((P $\wedge$ Q) $\rightarrow$ R).
1229   intros Id2_08a. (*This theorem isn't needed.*)
1230   replace (((P  $\wedge$  Q)  $\rightarrow$  R)  $\rightarrow$  ((P  $\wedge$  Q)  $\rightarrow$  R)) with
1231     (((P  $\wedge$  Q)  $\rightarrow$  R)  $\rightarrow$  ( $\neg$ ( $\neg$ P  $\vee$   $\neg$ Q)  $\rightarrow$  R)) in Id2_08a.
1232   specialize Transp2_15 with ( $\neg$ P $\vee$  $\neg$ Q) R.
1233   intros Transp2_15a.
1234   Syll Id2_08a Transp2_15a Sa.
1235   specialize Id2_08 with ( $\neg$ R  $\rightarrow$  ( $\neg$ P  $\vee$   $\neg$ Q)).
1236   intros Id2_08b. (*This theorem isn't needed.*)
1237   Syll Sa Id2_08b Sb.
1238   replace ( $\neg$ P  $\vee$   $\neg$ Q) with (P  $\rightarrow$   $\neg$ Q) in Sb.
1239   specialize Comm2_04 with ( $\neg$ R) P ( $\neg$ Q).
1240   intros Comm2_04a.

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1241 Syll Sb Comm2_04a Sc.
1242 specialize Transp2_17 with Q R.
1243 intros Transp2_17a.
1244 specialize Syll2_05 with P ( $\neg R \rightarrow \neg Q$ ) ( $Q \rightarrow R$ ).
1245 intros Syll2_05a.
1246 MP Syll2_05a Transp2_17a.
1247 Syll Sa Syll2_05a Sd.
1248 apply Sd.
1249 rewrite <- Impl1_01.
1250 reflexivity.
1251 rewrite Prod3_01.
1252 reflexivity.
1253 Qed.
1254
1255 Theorem Imp3_31 :  $\forall P Q R : \text{Prop}$ ,
1256   ( $P \rightarrow (Q \rightarrow R)$ )  $\rightarrow (P \wedge Q) \rightarrow R$ .
1257 Proof. intros P Q R.
1258   specialize Id2_08 with ( $P \rightarrow (Q \rightarrow R)$ ).
1259   intros Id2_08a. (*This use of Id2_08 is redundant.*)
1260   replace (( $P \rightarrow (Q \rightarrow R)$ ) $\rightarrow(P \rightarrow (Q \rightarrow R))$ ) with
1261     (( $P \rightarrow (Q \rightarrow R)$ ) $\rightarrow(\neg P \vee (Q \rightarrow R))$ ) in Id2_08a.
1262   replace ( $\neg P \vee (Q \rightarrow R)$ ) with
1263     ( $\neg P \vee (\neg Q \vee R)$ ) in Id2_08a.
1264   specialize n2_31 with ( $\neg P$ ) ( $\neg Q$ ) R.
1265   intros n2_31a.
1266   Syll Id2_08a n2_31a Sa.
1267   specialize n2_53 with ( $\neg P \vee \neg Q$ ) R.
1268   intros n2_53a.
1269   replace ( $\neg(\neg P \vee \neg Q)$ ) with ( $P \wedge Q$ ) in n2_53a.
1270   Syll n2_31a n2_53a Sb.
1271   apply Sb.
1272   apply Prod3_01.
1273   rewrite Impl1_01.
1274   reflexivity.
1275   rewrite <- Impl1_01.
1276   reflexivity.
1277 Qed.
1278 (*The proof sketch cites Id2_08, but
1279   we did not seem to need it.*/)
1280
1281 Theorem Syll3_33 :  $\forall P Q R : \text{Prop}$ ,
1282   (( $P \rightarrow Q$ )  $\wedge (Q \rightarrow R)$ )  $\rightarrow (P \rightarrow R)$ .

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1283 Proof. intros P Q R.
1284   specialize Syll2_06 with P Q R.
1285   intros Syll2_06a.
1286   specialize Imp3_31 with (P→Q) (Q→R) (P→R).
1287   intros Imp3_31a.
1288   MP Imp3_31a Syll2_06a.
1289   apply Imp3_31a.
1290 Qed.
1291
1292 Theorem Syll3_34 : ∀ P Q R : Prop,
1293   ((Q → R) ∧ (P → Q)) → (P → R).
1294 Proof. intros P Q R.
1295   specialize Syll2_05 with P Q R.
1296   intros Syll2_05a.
1297   specialize Imp3_31 with (Q→R) (P→Q) (P→R).
1298   intros Imp3_31a.
1299   MP Imp3_31a Syll2_05a.
1300   apply Imp3_31a.
1301 Qed.
1302
1303 Theorem Ass3_35 : ∀ P Q : Prop,
1304   (P ∧ (P → Q)) → Q.
1305 Proof. intros P Q.
1306   specialize n2_27 with P Q.
1307   intros n2_27a.
1308   specialize Imp3_31 with P (P→Q) Q.
1309   intros Imp3_31a.
1310   MP Imp3_31a n2_27a.
1311   apply Imp3_31a.
1312 Qed.
1313
1314 Theorem Transp3_37 : ∀ P Q R : Prop,
1315   (P ∧ Q → R) → (P ∧ ¬R → ¬Q).
1316 Proof. intros P Q R.
1317   specialize Transp2_16 with Q R.
1318   intros Transp2_16a.
1319   specialize Syll2_05 with P (Q→R) (¬R→¬Q).
1320   intros Syll2_05a.
1321   MP Syll2_05a Transp2_16a.
1322   specialize Exp3_3 with P Q R.
1323   intros Exp3_3a.
1324   Syll Exp3_3a Syll2_05a Sa.

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1325   specialize Imp3_31 with P ( $\neg$ R) ( $\neg$ Q).
1326   intros Imp3_31a.
1327   Syll Sa Imp3_31a Sb.
1328   apply Sb.
1329   Qed.
1330
1331   Theorem n3_4 :  $\forall$  P Q : Prop,
1332     (P  $\wedge$  Q)  $\rightarrow$  P  $\rightarrow$  Q.
1333   Proof. intros P Q.
1334     specialize n2_51 with P Q.
1335     intros n2_51a.
1336     specialize Transp2_15 with (P $\rightarrow$ Q) (P $\rightarrow\neg$ Q).
1337     intros Transp2_15a.
1338     MP Transp2_15a n2_51a.
1339     replace (P $\rightarrow\neg$ Q) with ( $\neg$ P $\vee\neg$ Q) in Transp2_15a.
1340     replace ( $\neg(\neg$ P $\vee\neg$ Q)) with (P $\wedge$ Q) in Transp2_15a.
1341     apply Transp2_15a.
1342     apply Prod3_01.
1343     rewrite <- Impl1_01.
1344     reflexivity.
1345   Qed.
1346
1347   Theorem n3_41 :  $\forall$  P Q R : Prop,
1348     (P  $\rightarrow$  R)  $\rightarrow$  (P  $\wedge$  Q  $\rightarrow$  R).
1349   Proof. intros P Q R.
1350     specialize Simp3_26 with P Q.
1351     intros Simp3_26a.
1352     specialize Syll2_06 with (P $\wedge$ Q) P R.
1353     intros Syll2_06a.
1354     MP Simp3_26a Syll2_06a.
1355     apply Syll2_06a.
1356   Qed.
1357
1358   Theorem n3_42 :  $\forall$  P Q R : Prop,
1359     (Q  $\rightarrow$  R)  $\rightarrow$  (P  $\wedge$  Q  $\rightarrow$  R).
1360   Proof. intros P Q R.
1361     specialize Simp3_27 with P Q.
1362     intros Simp3_27a.
1363     specialize Syll2_06 with (P $\wedge$ Q) Q R.
1364     intros Syll2_06a.
1365     MP Syll2_06a Simp3_27a.
1366     apply Syll2_06a.

```

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1367 Qed.
1368
1369 Theorem Comp3_43 :  $\forall P Q R : \text{Prop}$ ,
1370    $(P \rightarrow Q) \wedge (P \rightarrow R) \rightarrow (P \rightarrow Q \wedge R)$ .
1371 Proof. intros P Q R.
1372   specialize n3_2 with Q R.
1373   intros n3_2a.
1374   specialize Syll2_05 with P Q  $(R \rightarrow Q \wedge R)$ .
1375   intros Syll2_05a.
1376   MP Syll2_05a n3_2a.
1377   specialize n2_77 with P R  $(Q \wedge R)$ .
1378   intros n2_77a.
1379   Syll Syll2_05a n2_77a Sa.
1380   specialize Imp3_31 with  $(P \rightarrow Q) (P \rightarrow R) (P \rightarrow Q \wedge R)$ .
1381   intros Imp3_31a.
1382   MP Sa Imp3_31a.
1383   apply Imp3_31a.
1384 Qed.
1385
1386 Theorem n3_44 :  $\forall P Q R : \text{Prop}$ ,
1387    $(Q \rightarrow P) \wedge (R \rightarrow P) \rightarrow (Q \vee R \rightarrow P)$ .
1388 Proof. intros P Q R.
1389   specialize Syll3_33 with  $(\neg Q) R P$ .
1390   intros Syll3_33a.
1391   specialize n2_6 with Q P.
1392   intros n2_6a.
1393   Syll Syll3_33a n2_6a Sa.
1394   specialize Exp3_3 with  $(\neg Q \rightarrow R) (R \rightarrow P) ((Q \rightarrow P) \rightarrow P)$ .
1395   intros Exp3_3a.
1396   MP Exp3_3a Sa.
1397   specialize Comm2_04 with  $(R \rightarrow P) (Q \rightarrow P) P$ .
1398   intros Comm2_04a.
1399   Syll Exp3_3a Comm2_04a Sb.
1400   specialize Imp3_31 with  $(Q \rightarrow P) (R \rightarrow P) P$ .
1401   intros Imp3_31a.
1402   Syll Sb Imp3_31a Sc.
1403   specialize Comm2_04 with  $(\neg Q \rightarrow R) ((Q \rightarrow P) \wedge (R \rightarrow P)) P$ .
1404   intros Comm2_04b.
1405   MP Comm2_04b Sc.
1406   specialize n2_53 with Q R.
1407   intros n2_53a.
1408   specialize Syll2_06 with  $(Q \vee R) (\neg Q \rightarrow R) P$ .

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1409   intros Syll2_06a.
1410   MP Syll2_06a n2_53a.
1411   Syll Comm2_04b Syll2_06a Sd.
1412   apply Sd.
1413   Qed.
1414
1415   Theorem Fact3_45 :  $\forall$  P Q R : Prop,
1416     (P  $\rightarrow$  Q)  $\rightarrow$  (P  $\wedge$  R)  $\rightarrow$  (Q  $\wedge$  R).
1417   Proof. intros P Q R.
1418     specialize Syll2_06 with P Q ( $\neg$ R).
1419     intros Syll2_06a.
1420     specialize Transp2_16 with (Q $\rightarrow$  $\neg$ R) (P $\rightarrow$  $\neg$ R).
1421     intros Transp2_16a.
1422     Syll Syll2_06a Transp2_16a Sa.
1423     specialize Id2_08 with ( $\neg$ (P $\rightarrow$ R) $\rightarrow$  $\neg$ (Q $\rightarrow$  $\neg$ R)).
1424     intros Id2_08a.
1425     Syll Sa Id2_08a Sb.
1426     replace (P $\rightarrow$  $\neg$ R) with ( $\neg$ P $\vee$  $\neg$ R) in Sb.
1427     replace (Q $\rightarrow$  $\neg$ R) with ( $\neg$ Q $\vee$  $\neg$ R) in Sb.
1428     replace ( $\neg$ ( $\neg$ P $\vee$  $\neg$ R)) with (P $\wedge$ R) in Sb.
1429     replace ( $\neg$ ( $\neg$ Q $\vee$  $\neg$ R)) with (Q $\wedge$ R) in Sb.
1430     apply Sb.
1431     apply Prod3_01.
1432     apply Prod3_01.
1433     rewrite <- Impl1_01.
1434     reflexivity.
1435     rewrite <- Impl1_01.
1436     reflexivity.
1437   Qed.
1438
1439   Theorem n3_47 :  $\forall$  P Q R S : Prop,
1440     ((P  $\rightarrow$  R)  $\wedge$  (Q  $\rightarrow$  S))  $\rightarrow$  (P  $\wedge$  Q)  $\rightarrow$  R  $\wedge$  S.
1441   Proof. intros P Q R S.
1442     specialize Simp3_26 with (P $\rightarrow$ R) (Q $\rightarrow$ S).
1443     intros Simp3_26a.
1444     specialize Fact3_45 with P R Q.
1445     intros Fact3_45a.
1446     Syll Simp3_26a Fact3_45a Sa.
1447     specialize n3_22 with R Q.
1448     intros n3_22a.
1449     specialize Syll2_05 with (P $\wedge$ Q) (R $\wedge$ Q) (Q $\wedge$ R).
1450     intros Syll2_05a.

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1451 MP Syll2_05a n3_22a.
1452 Syll Sa Syll2_05a Sb.
1453 specialize Simp3_27 with (P→R) (Q→S).
1454 intros Simp3_27a.
1455 specialize Fact3_45 with Q S R.
1456 intros Fact3_45b.
1457 Syll Simp3_27a Fact3_45b Sc.
1458 specialize n3_22 with S R.
1459 intros n3_22b.
1460 specialize Syll2_05 with (Q∧R) (S∧R) (R∧S).
1461 intros Syll2_05b.
1462 MP Syll2_05b n3_22b.
1463 Syll Sc Syll2_05b Sd.
1464 clear Simp3_26a. clear Fact3_45a. clear Sa.
1465   clear n3_22a. clear Fact3_45b.
1466   clear Syll2_05a. clear Simp3_27a.
1467   clear Sc. clear n3_22b. clear Syll2_05b.
1468 Conj Sb Sd.
1469 split.
1470 apply Sb.
1471 apply Sd.
1472 specialize n2_83 with ((P→R)∧(Q→S)) (P∧Q) (Q∧R) (R∧S).
1473 intros n2_83a. (*This with MP works, but it omits Conj3_03.*)
1474 specialize Imp3_31 with (((P→R)∧(Q→S))→((P∧Q)→(Q∧R)))
1475   (((P→R)∧(Q→S))→((Q∧R)→(R∧S)))
1476   (((P→R)∧(Q→S))→((P∧Q)→(R∧S))).
1477 intros Imp3_31a.
1478 MP Imp3_31a n2_83a.
1479 MP Imp3_31a H.
1480 apply Imp3_31a.
1481 Qed.
1482
1483 Theorem n3_48 : ∀ P Q R S : Prop,
1484   ((P → R) ∧ (Q → S)) → (P ∨ Q) → R ∨ S.
1485 Proof. intros P Q R S.
1486   specialize Simp3_26 with (P→R) (Q→S).
1487   intros Simp3_26a.
1488   specialize Sum1_6 with Q P R.
1489   intros Sum1_6a.
1490   Syll Simp3_26a Sum1_6a Sa.
1491   specialize Perm1_4 with P Q.
1492   intros Perm1_4a.

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1493 specialize Syll2_06 with (P $\vee$ Q) (Q $\vee$ P) (Q $\vee$ R).
1494 intros Syll2_06a.
1495 MP Syll2_06a Perm1_4a.
1496 Syll Sa Syll2_06a Sb.
1497 specialize Simp3_27 with (P $\rightarrow$ R) (Q $\rightarrow$ S).
1498 intros Simp3_27a.
1499 specialize Sum1_6 with R Q S.
1500 intros Sum1_6b.
1501 Syll Simp3_27a Sum1_6b Sc.
1502 specialize Perm1_4 with Q R.
1503 intros Perm1_4b.
1504 specialize Syll2_06 with (Q $\vee$ R) (R $\vee$ Q) (R $\vee$ S).
1505 intros Syll2_06b.
1506 MP Syll2_06b Perm1_4b.
1507 Syll Sc Syll2_06a Sd.
1508 specialize n2_83 with ((P $\rightarrow$ R) $\wedge$ (Q $\rightarrow$ S)) (P $\vee$ Q) (Q $\vee$ R) (R $\vee$ S).
1509 intros n2_83a.
1510 MP n2_83a Sb.
1511 MP n2_83a Sd.
1512 apply n2_83a.
1513 Qed.
1514
1515 End No3.
1516
1517 Module No4.
1518
1519 Import No1.
1520 Import No2.
1521 Import No3.
1522
1523 Theorem Equiv4_01 :  $\forall$  P Q : Prop,
1524   (P  $\leftrightarrow$  Q) = ((P  $\rightarrow$  Q)  $\wedge$  (Q  $\rightarrow$  P)).
1525 Proof. intros P Q.
1526 apply propositional_extensionality.
1527 specialize iff_to_and with P Q.
1528 intros iff_to_and.
1529 apply iff_to_and.
1530 Qed.
1531 (*This is a notational definition in Principia;
1532 it is used to switch between " $\leftrightarrow$ " and " $\rightarrow \wedge \leftarrow$ ".*)
1533
1534 (*Axiom Abb4_02 :  $\forall$  P Q R : Prop,

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1535   (P ↔ Q ↔ R) = ((P ↔ Q) ∧ (Q ↔ R)).*)
1536   (*Since Coq forbids ill-formed strings, or else
1537   automatically associates to the right, we leave
1538   this notational axiom commented out.*/)
1539
1540 Ltac Equiv H1 :=
1541   match goal with
1542   | [ H1 : (?P→?Q) ∧ (?Q→?P) | - _ ] =>
1543     replace ((P→Q) ∧ (Q→P)) with (P↔Q) in H1
1544 end.
1545
1546 Theorem Transp4_1 : ∀ P Q : Prop,
1547   (P → Q) ↔ (¬Q → ¬P).
1548 Proof. intros P Q.
1549   specialize Transp2_16 with P Q.
1550   intros Transp2_16a.
1551   specialize Transp2_17 with P Q.
1552   intros Transp2_17a.
1553   Conj Transp2_16a Transp2_17a.
1554   split.
1555   apply Transp2_16a.
1556   apply Transp2_17a.
1557   Equiv H.
1558   apply H.
1559   apply Equiv4_01.
1560 Qed.
1561
1562 Theorem Transp4_11 : ∀ P Q : Prop,
1563   (P ↔ Q) ↔ (¬P ↔ ¬Q).
1564 Proof. intros P Q.
1565   specialize Transp2_16 with P Q.
1566   intros Transp2_16a.
1567   specialize Transp2_16 with Q P.
1568   intros Transp2_16b.
1569   Conj Transp2_16a Transp2_16b.
1570   split.
1571   apply Transp2_16a.
1572   apply Transp2_16b.
1573   specialize n3_47 with (P→Q) (Q→P) (¬Q→¬P) (¬P→¬Q).
1574   intros n3_47a.
1575   MP n3_47 H.
1576   specialize n3_22 with (¬Q → ¬P) (¬P → ¬Q).

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1577   intros n3_22a.
1578   Syll n3_47a n3_22a Sa.
1579   replace ((P → Q) ∧ (Q → P)) with (P ↔ Q) in Sa.
1580   replace ((¬P → ¬Q) ∧ (¬Q → ¬P)) with (¬P ↔ ¬Q) in Sa.
1581   clear Transp2_16a. clear H. clear Transp2_16b.
1582   clear n3_22a. clear n3_47a.
1583   specialize Transp2_17 with Q P.
1584   intros Transp2_17a.
1585   specialize Transp2_17 with P Q.
1586   intros Transp2_17b.
1587   Conj Transp2_17a Transp2_17b.
1588   split.
1589   apply Transp2_17a.
1590   apply Transp2_17b.
1591   specialize n3_47 with (¬P → ¬Q) (¬Q → ¬P) (Q → P) (P → Q).
1592   intros n3_47a.
1593   MP n3_47a H.
1594   specialize n3_22 with (Q → P) (P → Q).
1595   intros n3_22a.
1596   Syll n3_47a n3_22a Sb.
1597   clear Transp2_17a. clear Transp2_17b. clear H.
1598   clear n3_47a. clear n3_22a.
1599   replace ((P → Q) ∧ (Q → P)) with (P ↔ Q) in Sb.
1600   replace ((¬P → ¬Q) ∧ (¬Q → ¬P)) with (¬P ↔ ¬Q) in Sb.
1601   Conj Sa Sb.
1602   split.
1603   apply Sa.
1604   apply Sb.
1605   Equiv H.
1606   apply H.
1607   apply Equiv4_01.
1608   apply Equiv4_01.
1609   apply Equiv4_01.
1610   apply Equiv4_01.
1611   apply Equiv4_01.
1612   Qed.
1613
1614   Theorem n4_12 : ∀ P Q : Prop,
1615     (P ↔ ¬Q) ↔ (Q ↔ ¬P).
1616   Proof. intros P Q.
1617     specialize Transp2_03 with P Q.
1618     intros Transp2_03a.

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1619 specialize Transp2_15 with Q P.
1620 intros Transp2_15a.
1621 Conj Transp2_03a Transp2_15a.
1622 split.
1623 apply Transp2_03a.
1624 apply Transp2_15a.
1625 specialize n3_47 with (P → ¬Q) (¬Q → P) (Q → ¬P) (¬P → Q).
1626 intros n3_47a.
1627 MP n3_47a H.
1628 specialize Transp2_03 with Q P.
1629 intros Transp2_03b.
1630 specialize Transp2_15 with P Q.
1631 intros Transp2_15b.
1632 Conj Transp2_03b Transp2_15b.
1633 split.
1634 apply Transp2_03b.
1635 apply Transp2_15b.
1636 specialize n3_47 with (Q → ¬P) (¬P → Q) (P → ¬Q) (¬Q → P).
1637 intros n3_47b.
1638 MP n3_47b H0.
1639 clear Transp2_03a. clear Transp2_15a. clear H.
1640 clear Transp2_03b. clear Transp2_15b. clear H0.
1641 Conj n3_47a n3_47b.
1642 split.
1643 apply n3_47a.
1644 apply n3_47b.
1645 rewrite <- Equiv4_01 in H.
1646 rewrite <- Equiv4_01 in H.
1647 rewrite <- Equiv4_01 in H.
1648 apply H.
1649 Qed.
1650
1651 Theorem n4_13 : ∀ P : Prop,
1652   P ↔ ¬¬P.
1653 Proof. intros P.
1654 specialize n2_12 with P.
1655 intros n2_12a.
1656 specialize n2_14 with P.
1657 intros n2_14a.
1658 Conj n2_12a n2_14a.
1659 split.
1660 apply n2_12a.

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```

1661   apply n2_14a.
1662   Equiv H.
1663   apply H.
1664   apply Equiv4_01.
1665   Qed.
1666
1667   Theorem n4_14 :  $\forall P Q R : \text{Prop},$ 
1668      $((P \wedge Q) \rightarrow R) \leftrightarrow ((P \wedge \neg R) \rightarrow \neg Q).$ 
1669   Proof. intros P Q R.
1670   specialize Transp3_37 with P Q R.
1671   intros Transp3_37a.
1672   specialize Transp3_37 with P ( $\neg R$ ) ( $\neg Q$ ).
1673   intros Transp3_37b.
1674   Conj Transp3_37a Transp3_37b.
1675   split. apply Transp3_37a.
1676   apply Transp3_37b.
1677   specialize n4_13 with Q.
1678   intros n4_13a.
1679   specialize n4_13 with R.
1680   intros n4_13b.
1681   replace ( $\neg\neg Q$ ) with Q in H.
1682   replace ( $\neg\neg R$ ) with R in H.
1683   Equiv H.
1684   apply H.
1685   apply Equiv4_01.
1686   apply propositional_extensionality.
1687   apply n4_13b.
1688   apply propositional_extensionality.
1689   apply n4_13a.
1690   Qed.
1691
1692   Theorem n4_15 :  $\forall P Q R : \text{Prop},$ 
1693      $((P \wedge Q) \rightarrow \neg R) \leftrightarrow ((Q \wedge R) \rightarrow \neg P).$ 
1694   Proof. intros P Q R.
1695   specialize n4_14 with Q P ( $\neg R$ ).
1696   intros n4_14a.
1697   specialize n3_22 with Q P.
1698   intros n3_22a.
1699   specialize Syll2_06 with  $(Q \wedge P)$   $(P \wedge Q)$  ( $\neg R$ ).
1700   intros Syll2_06a.
1701   MP Syll2_06a n3_22a.
1702   specialize n4_13 with R.

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1703   intros n4_13a.
1704   replace ( $\neg\neg R$ ) with R in n4_14a.
1705   rewrite Equiv4_01 in n4_14a.
1706   specialize Simp3_26 with ( $((Q \wedge P \rightarrow \neg R) \rightarrow Q \wedge R \rightarrow \neg P)$ 
1707      $((Q \wedge R \rightarrow \neg P) \rightarrow Q \wedge P \rightarrow \neg R)$ ).
1708   intros Simp3_26a.
1709   MP Simp3_26a n4_14a.
1710   Syll Syll2_06a Simp3_26a Sa.
1711   specialize Simp3_27 with ( $((Q \wedge P \rightarrow \neg R) \rightarrow Q \wedge R \rightarrow \neg P)$ 
1712      $((Q \wedge R \rightarrow \neg P) \rightarrow Q \wedge P \rightarrow \neg R)$ ).
1713   intros Simp3_27a.
1714   MP Simp3_27a n4_14a.
1715   specialize n3_22 with P Q.
1716   intros n3_22b.
1717   specialize Syll2_06 with  $(P \wedge Q) (Q \wedge P) (\neg R)$ .
1718   intros Syll2_06b.
1719   MP Syll2_06b n3_22b.
1720   Syll Syll2_06b Simp3_27a Sb.
1721   clear n4_14a. clear n3_22a. clear Syll2_06a.
1722     clear n4_13a. clear Simp3_26a. clear n3_22b.
1723     clear Simp3_27a. clear Syll2_06b.
1724   Conj Sa Sb.
1725   split.
1726   apply Sa.
1727   apply Sb.
1728   Equiv H.
1729   apply H.
1730   apply Equiv4_01.
1731   apply propositional_extensionality.
1732   apply n4_13a.
1733   Qed.
1734
1735   Theorem n4_2 :  $\forall P : \text{Prop}$ ,
1736      $P \leftrightarrow P$ .
1737   Proof. intros P.
1738     specialize n3_2 with  $(P \rightarrow P) (P \rightarrow P)$ .
1739     intros n3_2a.
1740     specialize Id2_08 with P.
1741     intros Id2_08a.
1742     MP n3_2a Id2_08a.
1743     MP n3_2a Id2_08a.
1744     Equiv n3_2a.

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1745   apply n3_2a.
1746   apply Equiv4_01.
1747 Qed.
1748
1749 Theorem n4_21 :  $\forall$  P Q : Prop,
1750   (P  $\leftrightarrow$  Q)  $\leftrightarrow$  (Q  $\leftrightarrow$  P).
1751 Proof. intros P Q.
1752   specialize n3_22 with (P $\rightarrow$ Q) (Q $\rightarrow$ P).
1753   intros n3_22a.
1754   replace ((P  $\rightarrow$  Q)  $\wedge$  (Q  $\rightarrow$  P)) with (P $\leftrightarrow$ Q) in n3_22a.
1755   replace ((Q  $\rightarrow$  P)  $\wedge$  (P  $\rightarrow$  Q)) with (Q $\leftrightarrow$ P) in n3_22a.
1756   specialize n3_22 with (Q $\rightarrow$ P) (P $\rightarrow$ Q).
1757   intros n3_22b.
1758   replace ((P  $\rightarrow$  Q)  $\wedge$  (Q  $\rightarrow$  P)) with (P $\leftrightarrow$ Q) in n3_22b.
1759   replace ((Q  $\rightarrow$  P)  $\wedge$  (P  $\rightarrow$  Q)) with (Q $\leftrightarrow$ P) in n3_22b.
1760   Conj n3_22a n3_22b.
1761   split.
1762   apply n3_22a.
1763   apply n3_22b.
1764   Equiv H.
1765   apply H.
1766   apply Equiv4_01.
1767   apply Equiv4_01.
1768   apply Equiv4_01.
1769   apply Equiv4_01.
1770   apply Equiv4_01.
1771 Qed.
1772
1773 Theorem n4_22 :  $\forall$  P Q R : Prop,
1774   ((P  $\leftrightarrow$  Q)  $\wedge$  (Q  $\leftrightarrow$  R))  $\rightarrow$  (P  $\leftrightarrow$  R).
1775 Proof. intros P Q R.
1776   specialize Simp3_26 with (P $\leftrightarrow$ Q) (Q $\leftrightarrow$ R).
1777   intros Simp3_26a.
1778   specialize Simp3_26 with (P $\rightarrow$ Q) (Q $\rightarrow$ P).
1779   intros Simp3_26b.
1780   replace ((P $\rightarrow$ Q)  $\wedge$  (Q $\rightarrow$ P)) with (P $\leftrightarrow$ Q) in Simp3_26b.
1781   Syll Simp3_26a Simp3_26b Sa.
1782   specialize Simp3_27 with (P $\leftrightarrow$ Q) (Q $\leftrightarrow$ R).
1783   intros Simp3_27a.
1784   specialize Simp3_26 with (Q $\rightarrow$ R) (R $\rightarrow$ Q).
1785   intros Simp3_26c.
1786   replace ((Q $\rightarrow$ R)  $\wedge$  (R $\rightarrow$ Q)) with (Q $\leftrightarrow$ R) in Simp3_26c.

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1787 Syll Simp3_27a Simp3_26c Sb.
1788 specialize n2_83 with ((P↔Q)^(Q↔R)) P Q R.
1789 intros n2_83a.
1790 MP n2_83a Sa.
1791 MP n2_83a Sb.
1792 specialize Simp3_27 with (P↔Q) (Q↔R).
1793 intros Simp3_27b.
1794 specialize Simp3_27 with (Q→R) (R→Q).
1795 intros Simp3_27c.
1796 replace ((Q→R) ∧ (R→Q)) with (Q↔R) in Simp3_27c.
1797 Syll Simp3_27b Simp3_27c Sc.
1798 specialize Simp3_26 with (P↔Q) (Q↔R).
1799 intros Simp3_26d.
1800 specialize Simp3_27 with (P→Q) (Q→P).
1801 intros Simp3_27d.
1802 replace ((P→Q) ∧ (Q→P)) with (P↔Q) in Simp3_27d.
1803 Syll Simp3_26d Simp3_27d Sd.
1804 specialize n2_83 with ((P↔Q)^(Q↔R)) R Q P.
1805 intros n2_83b.
1806 MP n2_83b Sc. MP n2_83b Sd.
1807 clear Sd. clear Sb. clear Sc. clear Sa. clear Simp3_26a.
1808     clear Simp3_26b. clear Simp3_26c. clear Simp3_26d.
1809     clear Simp3_27a. clear Simp3_27b. clear Simp3_27c.
1810     clear Simp3_27d.
1811 Conj n2_83a n2_83b.
1812 split.
1813 apply n2_83a.
1814 apply n2_83b.
1815 specialize Comp3_43 with ((P↔Q)^(Q↔R)) (P→R) (R→P).
1816 intros Comp3_43a.
1817 MP Comp3_43a H.
1818 replace ((P→R) ∧ (R→P)) with (P↔R) in Comp3_43a.
1819 apply Comp3_43a.
1820 apply Equiv4_01.
1821 apply Equiv4_01.
1822 apply Equiv4_01.
1823 apply Equiv4_01.
1824 apply Equiv4_01.
1825 Qed.
1826
1827 Theorem n4_24 : ∀ P : Prop,
1828     P ↔ (P ∧ P).

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```

1829 Proof. intros P.
1830 specialize n3_2 with P P.
1831 intros n3_2a.
1832 specialize n2_43 with P (P ∧ P).
1833 intros n2_43a.
1834 MP n3_2a n2_43a.
1835 specialize Simp3_26 with P P.
1836 intros Simp3_26a.
1837 Conj n2_43a Simp3_26a.
1838 split.
1839 apply n2_43a.
1840 apply Simp3_26a.
1841 Equiv H.
1842 apply H.
1843 apply Equiv4_01.
1844 Qed.
1845
1846 Theorem n4_25 : ∀ P : Prop,
1847   P ↔ (P ∨ P).
1848 Proof. intros P.
1849   specialize Add1_3 with P P.
1850   intros Add1_3a.
1851   specialize Taut1_2 with P.
1852   intros Taut1_2a.
1853   Conj Add1_3a Taut1_2a.
1854   split.
1855   apply Add1_3a.
1856   apply Taut1_2a.
1857   Equiv H. apply H.
1858   apply Equiv4_01.
1859   Qed.
1860
1861 Theorem n4_3 : ∀ P Q : Prop,
1862   (P ∧ Q) ↔ (Q ∧ P).
1863 Proof. intros P Q.
1864   specialize n3_22 with P Q.
1865   intros n3_22a.
1866   specialize n3_22 with Q P.
1867   intros n3_22b.
1868   Conj n3_22a n3_22b.
1869   split.
1870   apply n3_22a.

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1871   apply n3_22b.
1872   Equiv H. apply H.
1873   apply Equiv4_01.
1874 Qed.
1875
1876 Theorem n4_31 :  $\forall$  P Q : Prop,
1877   (P  $\vee$  Q)  $\leftrightarrow$  (Q  $\vee$  P).
1878 Proof. intros P Q.
1879   specialize Perm1_4 with P Q.
1880   intros Perm1_4a.
1881   specialize Perm1_4 with Q P.
1882   intros Perm1_4b.
1883   Conj Perm1_4a Perm1_4b.
1884   split.
1885   apply Perm1_4a.
1886   apply Perm1_4b.
1887   Equiv H. apply H.
1888   apply Equiv4_01.
1889 Qed.
1890
1891 Theorem n4_32 :  $\forall$  P Q R : Prop,
1892   ((P  $\wedge$  Q)  $\wedge$  R)  $\leftrightarrow$  (P  $\wedge$  (Q  $\wedge$  R)).
1893 Proof. intros P Q R.
1894   specialize n4_15 with P Q R.
1895   intros n4_15a.
1896   specialize Transp4_1 with P ( $\neg$ (Q  $\wedge$  R)).
1897   intros Transp4_1a.
1898   replace ( $\neg$ (Q  $\wedge$  R)) with (Q  $\wedge$  R) in Transp4_1a.
1899   replace (Q  $\wedge$  R  $\rightarrow$   $\neg$ P) with (P  $\rightarrow$   $\neg$ (Q  $\wedge$  R)) in n4_15a.
1900   specialize Transp4_11 with (P  $\wedge$  Q  $\rightarrow$   $\neg$ R) (P  $\rightarrow$   $\neg$ (Q  $\wedge$  R)).
1901   intros Transp4_11a.
1902   replace ((P  $\wedge$  Q  $\rightarrow$   $\neg$ R)  $\leftrightarrow$  (P  $\rightarrow$   $\neg$ (Q  $\wedge$  R))) with
1903     ( $\neg$ (P  $\wedge$  Q  $\rightarrow$   $\neg$ R)  $\leftrightarrow$   $\neg$ (P  $\rightarrow$   $\neg$ (Q  $\wedge$  R))) in n4_15a.
1904   replace (P  $\wedge$  Q  $\rightarrow$   $\neg$ R) with
1905     ( $\neg$ (P  $\wedge$  Q)  $\vee$   $\neg$ R) in n4_15a.
1906   replace (P  $\rightarrow$   $\neg$ (Q  $\wedge$  R)) with
1907     ( $\neg$ P  $\vee$   $\neg$ (Q  $\wedge$  R)) in n4_15a.
1908   replace ( $\neg$ ( $\neg$ (P  $\wedge$  Q)  $\vee$   $\neg$ R)) with
1909     ((P  $\wedge$  Q)  $\wedge$  R) in n4_15a.
1910   replace ( $\neg$ ( $\neg$ P  $\vee$   $\neg$ (Q  $\wedge$  R))) with
1911     (P  $\wedge$  (Q  $\wedge$  R)) in n4_15a.
1912   apply n4_15a.

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1913   apply Prod3_01.
1914   apply Prod3_01.
1915   rewrite Impl1_01.
1916   reflexivity.
1917   rewrite Impl1_01.
1918   reflexivity.
1919   replace (¬(P ∧ Q → ¬R) ↔ ¬(P → ¬(Q ∧ R))) with
1920         ((P ∧ Q → ¬R) ↔ (P → ¬(Q ∧ R))).
1921   reflexivity.
1922   apply propositional_extensionality.
1923   apply Transp4_11a.
1924   apply propositional_extensionality.
1925   apply Transp4_1a.
1926   apply propositional_extensionality.
1927   specialize n4_13 with (Q ∧ R).
1928   intros n4_13a.
1929   apply n4_13a.
1930 Qed.
1931   (*Note that the actual proof uses n4_12, but
1932     that transposition involves transforming a
1933     biconditional into a conditional. This citation
1934     of the lemma may be a misprint. Using
1935     Transp4_1 to transpose a conditional and
1936     then applying n4_13 to double negate does
1937     secure the desired formula.*)
1938
1939 Theorem n4_33 : ∀ P Q R : Prop,
1940   (P ∨ (Q ∨ R)) ↔ ((P ∨ Q) ∨ R).
1941 Proof. intros P Q R.
1942   specialize n2_31 with P Q R.
1943   intros n2_31a.
1944   specialize n2_32 with P Q R.
1945   intros n2_32a.
1946   Conj n2_31a n2_32a.
1947   split.
1948   apply n2_31a.
1949   apply n2_32a.
1950   Equiv H.
1951   apply H.
1952   apply Equiv4_01.
1953 Qed.
1954

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1955 Theorem Abb4_34 :  $\forall$  P Q R : Prop,
1956   (P  $\wedge$  Q  $\wedge$  R) = ((P  $\wedge$  Q)  $\wedge$  R).
1957 Proof. intros P Q R.
1958 apply propositional_extensionality.
1959 specialize n4_21 with ((P  $\wedge$  Q)  $\wedge$  R) (P  $\wedge$  Q  $\wedge$  R).
1960 intros n4_21.
1961 replace (((P  $\wedge$  Q)  $\wedge$  R  $\leftrightarrow$  P  $\wedge$  Q  $\wedge$  R)  $\leftrightarrow$  (P  $\wedge$  Q  $\wedge$  R  $\leftrightarrow$  (P  $\wedge$  Q)  $\wedge$  R))
1962   with (((P  $\wedge$  Q)  $\wedge$  R  $\leftrightarrow$  P  $\wedge$  Q  $\wedge$  R)  $\rightarrow$  (P  $\wedge$  Q  $\wedge$  R  $\leftrightarrow$  (P  $\wedge$  Q)  $\wedge$  R))
1963    $\wedge$  ((P  $\wedge$  Q  $\wedge$  R  $\leftrightarrow$  (P  $\wedge$  Q)  $\wedge$  R)  $\rightarrow$  ((P  $\wedge$  Q)  $\wedge$  R  $\leftrightarrow$  P  $\wedge$  Q  $\wedge$  R)))
1964   in n4_21.
1965 specialize Simp3_26 with
1966   (((P  $\wedge$  Q)  $\wedge$  R  $\leftrightarrow$  P  $\wedge$  Q  $\wedge$  R)  $\rightarrow$  (P  $\wedge$  Q  $\wedge$  R  $\leftrightarrow$  (P  $\wedge$  Q)  $\wedge$  R))
1967   ((P  $\wedge$  Q  $\wedge$  R  $\leftrightarrow$  (P  $\wedge$  Q)  $\wedge$  R)  $\rightarrow$  ((P  $\wedge$  Q)  $\wedge$  R  $\leftrightarrow$  P  $\wedge$  Q  $\wedge$  R)).
1968 intros Simp3_26.
1969 MP Simp3_26 n4_21.
1970 specialize n4_32 with P Q R.
1971 intros n4_32.
1972 MP Simp3_26 n4_32.
1973 apply Simp3_26.
1974 apply Equiv4_01.
1975 Qed.
1976
1977 Theorem n4_36 :  $\forall$  P Q R : Prop,
1978   (P  $\leftrightarrow$  Q)  $\rightarrow$  ((P  $\wedge$  R)  $\leftrightarrow$  (Q  $\wedge$  R)).
1979 Proof. intros P Q R.
1980 specialize Fact3_45 with P Q R.
1981 intros Fact3_45a.
1982 specialize Fact3_45 with Q P R.
1983 intros Fact3_45b.
1984 Conj Fact3_45a Fact3_45b.
1985 split.
1986 apply Fact3_45a.
1987 apply Fact3_45b.
1988 specialize n3_47 with (P $\rightarrow$ Q) (Q $\rightarrow$ P)
1989   (P  $\wedge$  R  $\rightarrow$  Q  $\wedge$  R) (Q  $\wedge$  R  $\rightarrow$  P  $\wedge$  R).
1990 intros n3_47a.
1991 MP n3_47 H.
1992 replace ((P  $\rightarrow$  Q)  $\wedge$  (Q  $\rightarrow$  P)) with (P $\leftrightarrow$ Q) in n3_47a.
1993 replace ((P  $\wedge$  R  $\rightarrow$  Q  $\wedge$  R)  $\wedge$  (Q  $\wedge$  R  $\rightarrow$  P  $\wedge$  R)) with
1994   (P  $\wedge$  R  $\leftrightarrow$  Q  $\wedge$  R) in n3_47a.
1995 apply n3_47a.
1996 apply Equiv4_01.

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1997     apply Equiv4_01.
1998 Qed.
1999
2000 Theorem n4_37 :  $\forall$  P Q R : Prop,
2001   (P  $\leftrightarrow$  Q)  $\rightarrow$  ((P  $\vee$  R)  $\leftrightarrow$  (Q  $\vee$  R)).
2002 Proof. intros P Q R.
2003   specialize Sum1_6 with R P Q.
2004   intros Sum1_6a.
2005   specialize Sum1_6 with R Q P.
2006   intros Sum1_6b.
2007   Conj Sum1_6a Sum1_6b.
2008   split.
2009   apply Sum1_6a.
2010   apply Sum1_6b.
2011   specialize n3_47 with (P  $\rightarrow$  Q) (Q  $\rightarrow$  P)
2012     (R  $\vee$  P  $\rightarrow$  R  $\vee$  Q) (R  $\vee$  Q  $\rightarrow$  R  $\vee$  P).
2013   intros n3_47a.
2014   MP n3_47 H.
2015   replace ((P  $\rightarrow$  Q)  $\wedge$  (Q  $\rightarrow$  P)) with (P $\leftrightarrow$ Q) in n3_47a.
2016   replace ((R  $\vee$  P  $\rightarrow$  R  $\vee$  Q)  $\wedge$  (R  $\vee$  Q  $\rightarrow$  R  $\vee$  P)) with
2017     (R  $\vee$  P  $\leftrightarrow$  R  $\vee$  Q) in n3_47a.
2018   replace (R  $\vee$  P) with (P  $\vee$  R) in n3_47a.
2019   replace (R  $\vee$  Q) with (Q  $\vee$  R) in n3_47a.
2020   apply n3_47a.
2021   apply propositional_extensionality.
2022   specialize n4_31 with Q R.
2023   intros n4_31a.
2024   apply n4_31a.
2025   apply propositional_extensionality.
2026   specialize n4_31 with P R.
2027   intros n4_31b.
2028   apply n4_31b.
2029   apply Equiv4_01.
2030   apply Equiv4_01.
2031 Qed.
2032
2033 Theorem n4_38 :  $\forall$  P Q R S : Prop,
2034   ((P  $\leftrightarrow$  R)  $\wedge$  (Q  $\leftrightarrow$  S))  $\rightarrow$  ((P  $\wedge$  Q)  $\leftrightarrow$  (R  $\wedge$  S)).
2035 Proof. intros P Q R S.
2036   specialize n3_47 with P Q R S.
2037   intros n3_47a.
2038   specialize n3_47 with R S P Q.

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2039   intros n3_47b.
2040   Conj n3_47a n3_47b.
2041   split.
2042   apply n3_47a.
2043   apply n3_47b.
2044   specialize n3_47 with ((P→R) ∧ (Q→S))
2045     ((R→P) ∧ (S→Q)) (P ∧ Q → R ∧ S) (R ∧ S → P ∧ Q).
2046   intros n3_47c.
2047   MP n3_47c H.
2048   specialize n4_32 with (P→R) (Q→S) ((R→P) ∧ (S → Q)).
2049   intros n4_32a.
2050   replace (((P → R) ∧ (Q → S)) ∧ (R → P) ∧ (S → Q)) with
2051     ((P → R) ∧ (Q → S) ∧ (R → P) ∧ (S → Q)) in n3_47c.
2052   specialize n4_32 with (Q→S) (R→P) (S → Q).
2053   intros n4_32b.
2054   replace ((Q → S) ∧ (R → P) ∧ (S → Q)) with
2055     (((Q → S) ∧ (R → P)) ∧ (S → Q)) in n3_47c.
2056   specialize n3_22 with (Q→S) (R→P).
2057   intros n3_22a.
2058   specialize n3_22 with (R→P) (Q→S).
2059   intros n3_22b.
2060   Conj n3_22a n3_22b.
2061   split.
2062   apply n3_22a.
2063   apply n3_22b.
2064   Equiv H0.
2065   replace ((Q → S) ∧ (R → P)) with
2066     ((R → P) ∧ (Q → S)) in n3_47c.
2067   specialize n4_32 with (R → P) (Q → S) (S → Q).
2068   intros n4_32c.
2069   replace (((R → P) ∧ (Q → S)) ∧ (S → Q)) with
2070     ((R → P) ∧ (Q → S) ∧ (S → Q)) in n3_47c.
2071   specialize n4_32 with (P→R) (R → P) ((Q → S) ∧ (S → Q)).
2072   intros n4_32d.
2073   replace ((P → R) ∧ (R → P) ∧ (Q → S) ∧ (S → Q)) with
2074     (((P → R) ∧ (R → P)) ∧ (Q → S) ∧ (S → Q)) in n3_47c.
2075   replace ((P→R) ∧ (R → P)) with (P↔R) in n3_47c.
2076   replace ((Q → S) ∧ (S → Q)) with (Q↔S) in n3_47c.
2077   replace ((P ∧ Q → R ∧ S) ∧ (R ∧ S → P ∧ Q)) with
2078     ((P ∧ Q) ↔ (R ∧ S)) in n3_47c.
2079   apply n3_47c.
2080   apply Equiv4_01.

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2081   apply Equiv4_01.
2082   apply Equiv4_01.
2083   apply propositional_extensionality.
2084   apply n4_32d.
2085   replace ((R → P) ∧ (Q → S) ∧ (S → Q)) with
2086     (((R → P) ∧ (Q → S)) ∧ (S → Q)).
2087   reflexivity.
2088   apply propositional_extensionality.
2089   apply n4_32c.
2090   replace ((R → P) ∧ (Q → S)) with ((Q → S) ∧ (R → P)).
2091   reflexivity.
2092   apply propositional_extensionality.
2093   apply H0.
2094   apply Equiv4_01.
2095   apply propositional_extensionality.
2096   apply n4_32b.
2097   replace ((P → R) ∧ (Q → S) ∧ (R → P) ∧ (S → Q)) with
2098     (((P → R) ∧ (Q → S)) ∧ (R → P) ∧ (S → Q)).
2099   reflexivity.
2100   apply propositional_extensionality.
2101   apply n4_32a.
2102   Qed.
2103
2104   Theorem n4_39 : ∀ P Q R S : Prop,
2105     ((P ↔ R) ∧ (Q ↔ S)) → ((P ∨ Q) ↔ (R ∨ S)).
2106   Proof.  intros P Q R S.
2107     specialize n3_48 with P Q R S.
2108     intros n3_48a.
2109     specialize n3_48 with R S P Q.
2110     intros n3_48b.
2111     Conj n3_48a n3_48b.
2112     split.
2113     apply n3_48a.
2114     apply n3_48b.
2115     specialize n3_47 with ((P → R) ∧ (Q → S))
2116       ((R → P) ∧ (S → Q)) (P ∨ Q → R ∨ S) (R ∨ S → P ∨ Q).
2117     intros n3_47a.
2118     MP n3_47a H.
2119     replace ((P ∨ Q → R ∨ S) ∧ (R ∨ S → P ∨ Q)) with
2120       ((P ∨ Q) ↔ (R ∨ S)) in n3_47a.
2121     specialize n4_32 with ((P → R) ∧ (Q → S)) (R → P) (S → Q).
2122     intros n4_32a.

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2123 replace (((P → R) ∧ (Q → S)) ∧ (R → P) ∧ (S → Q)) with
2124       (((P → R) ∧ (Q → S)) ∧ (R → P)) ∧ (S → Q) in n3_47a.
2125 specialize n4_32 with (P → R) (Q → S) (R → P).
2126 intros n4_32b.
2127 replace (((P → R) ∧ (Q → S)) ∧ (R → P)) with
2128       ((P → R) ∧ (Q → S) ∧ (R → P)) in n3_47a.
2129 specialize n3_22 with (Q → S) (R → P).
2130 intros n3_22a.
2131 specialize n3_22 with (R → P) (Q → S).
2132 intros n3_22b.
2133 Conj n3_22a n3_22b.
2134 split.
2135 apply n3_22a.
2136 apply n3_22b.
2137 Equiv H0.
2138 replace ((Q → S) ∧ (R → P)) with
2139       ((R → P) ∧ (Q → S)) in n3_47a.
2140 specialize n4_32 with (P → R) (R → P) (Q → S).
2141 intros n4_32c.
2142 replace ((P → R) ∧ (R → P) ∧ (Q → S)) with
2143       (((P → R) ∧ (R → P)) ∧ (Q → S)) in n3_47a.
2144 replace ((P → R) ∧ (R → P)) with (P ↔ R) in n3_47a.
2145 specialize n4_32 with (P ↔ R) (Q → S) (S → Q).
2146 intros n4_32d.
2147 replace (((P ↔ R) ∧ (Q → S)) ∧ (S → Q)) with
2148       ((P ↔ R) ∧ (Q → S) ∧ (S → Q)) in n3_47a.
2149 replace ((Q → S) ∧ (S → Q)) with (Q ↔ S) in n3_47a.
2150 apply n3_47a.
2151 apply Equiv4_01.
2152 replace ((P ↔ R) ∧ (Q → S) ∧ (S → Q)) with
2153       (((P ↔ R) ∧ (Q → S)) ∧ (S → Q)).
2154 reflexivity.
2155 apply propositional_extensionality.
2156 apply n4_32d.
2157 apply Equiv4_01.
2158 apply propositional_extensionality.
2159 apply n4_32c.
2160 replace ((R → P) ∧ (Q → S)) with ((Q → S) ∧ (R → P)).
2161 reflexivity.
2162 apply propositional_extensionality.
2163 apply H0.
2164 apply Equiv4_01.

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2165   replace ((P → R) ∧ (Q → S) ∧ (R → P)) with
2166         (((P → R) ∧ (Q → S)) ∧ (R → P)).
2167   reflexivity.
2168   apply propositional_extensionality.
2169   apply n4_32b.
2170   apply propositional_extensionality.
2171   apply n4_32a.
2172   apply Equiv4_01.
2173   Qed.
2174
2175   Theorem n4_4 : ∀ P Q R : Prop,
2176     (P ∧ (Q ∨ R)) ↔ ((P ∧ Q) ∨ (P ∧ R)).
2177   Proof. intros P Q R.
2178     specialize n3_2 with P Q.
2179     intros n3_2a.
2180     specialize n3_2 with P R.
2181     intros n3_2b.
2182     Conj n3_2a n3_2b.
2183     split.
2184     apply n3_2a.
2185     apply n3_2b.
2186     specialize Comp3_43 with P (Q → P ∧ Q) (R → P ∧ R).
2187     intros Comp3_43a.
2188     MP Comp3_43a H.
2189     specialize n3_48 with Q R (P ∧ Q) (P ∧ R).
2190     intros n3_48a.
2191     Syll Comp3_43a n3_48a Sa.
2192     specialize Imp3_31 with P (Q ∨ R) ((P ∧ Q) ∨ (P ∧ R)).
2193     intros Imp3_31a.
2194     MP Imp3_31a Sa.
2195     specialize Simp3_26 with P Q.
2196     intros Simp3_26a.
2197     specialize Simp3_26 with P R.
2198     intros Simp3_26b.
2199     Conj Simp3_26a Simp3_26b.
2200     split.
2201     apply Simp3_26a.
2202     apply Simp3_26b.
2203     specialize n3_44 with P (P ∧ Q) (P ∧ R).
2204     intros n3_44a.
2205     MP n3_44a H0.
2206     specialize Simp3_27 with P Q.

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2207   intros Simp3_27a.
2208   specialize Simp3_27 with P R.
2209   intros Simp3_27b.
2210   Conj Simp3_27a Simp3_27b.
2211   split.
2212   apply Simp3_27a.
2213   apply Simp3_27b.
2214   specialize n3_48 with (P ∧ Q) (P ∧ R) Q R.
2215   intros n3_48b.
2216   MP n3_48b H1.
2217   clear H1. clear Simp3_27a. clear Simp3_27b.
2218   Conj n3_44a n3_48b.
2219   split.
2220   apply n3_44a.
2221   apply n3_48b.
2222   specialize Comp3_43 with (P ∧ Q ∨ P ∧ R) P (Q ∨ R).
2223   intros Comp3_43b.
2224   MP Comp3_43b H1.
2225   clear H1. clear H0. clear n3_44a. clear n3_48b.
2226   clear Simp3_26a. clear Simp3_26b.
2227   Conj Imp3_31a Comp3_43b.
2228   split.
2229   apply Imp3_31a.
2230   apply Comp3_43b.
2231   Equiv H0.
2232   apply H0.
2233   apply Equiv4_01.
2234   Qed.
2235
2236   Theorem n4_41 : ∀ P Q R : Prop,
2237     (P ∨ (Q ∧ R)) ↔ ((P ∨ Q) ∧ (P ∨ R)).
2238   Proof. intros P Q R.
2239     specialize Simp3_26 with Q R.
2240     intros Simp3_26a.
2241     specialize Sum1_6 with P (Q ∧ R) Q.
2242     intros Sum1_6a.
2243     MP Simp3_26a Sum1_6a.
2244     specialize Simp3_27 with Q R.
2245     intros Simp3_27a.
2246     specialize Sum1_6 with P (Q ∧ R) R.
2247     intros Sum1_6b.
2248     MP Simp3_27a Sum1_6b.

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2249 clear Simp3_26a. clear Simp3_27a.
2250 Conj Sum1_6a Sum1_6a.
2251 split.
2252 apply Sum1_6a.
2253 apply Sum1_6b.
2254 specialize Comp3_43 with (P  $\vee$  Q  $\wedge$  R) (P  $\vee$  Q) (P  $\vee$  R).
2255 intros Comp3_43a.
2256 MP Comp3_43a H.
2257 specialize n2_53 with P Q.
2258 intros n2_53a.
2259 specialize n2_53 with P R.
2260 intros n2_53b.
2261 Conj n2_53a n2_53b.
2262 split.
2263 apply n2_53a.
2264 apply n2_53b.
2265 specialize n3_47 with (P  $\vee$  Q) (P  $\vee$  R) ( $\neg$ P  $\rightarrow$  Q) ( $\neg$ P  $\rightarrow$  R).
2266 intros n3_47a.
2267 MP n3_47a H0.
2268 specialize Comp3_43 with ( $\neg$ P) Q R.
2269 intros Comp3_43b.
2270 Syll n3_47a Comp3_43b Sa.
2271 specialize n2_54 with P (Q $\wedge$ R).
2272 intros n2_54a.
2273 Syll Sa n2_54a Sb.
2274 clear Sum1_6a. clear Sum1_6b. clear H. clear n2_53a.
2275 clear n2_53b. clear H0. clear n3_47a. clear Sa.
2276 clear Comp3_43b. clear n2_54a.
2277 Conj Comp3_43a Sb.
2278 split.
2279 apply Comp3_43a.
2280 apply Sb.
2281 Equiv H.
2282 apply H.
2283 apply Equiv4_01.
2284 Qed.
2285
2286 Theorem n4_42 :  $\forall$  P Q : Prop,
2287   P  $\leftrightarrow$  ((P  $\wedge$  Q)  $\vee$  (P  $\wedge$   $\neg$ Q)).
2288 Proof. intros P Q.
2289   specialize n3_21 with P (Q  $\vee$   $\neg$ Q).
2290   intros n3_21a.

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2291 specialize n2_11 with Q.
2292 intros n2_11a.
2293 MP n3_21a n2_11a.
2294 specialize Simp3_26 with P (Q  $\vee$   $\neg$ Q).
2295 intros Simp3_26a. clear n2_11a.
2296 Conj n3_21a Simp3_26a.
2297 split.
2298 apply n3_21a.
2299 apply Simp3_26a.
2300 Equiv H.
2301 specialize n4_4 with P Q ( $\neg$ Q).
2302 intros n4_4a.
2303 replace (P  $\wedge$  (Q  $\vee$   $\neg$ Q)) with P in n4_4a.
2304 apply n4_4a.
2305 apply propositional_extensionality.
2306 apply H.
2307 apply Equiv4_01.
2308 Qed.
2309
2310 Theorem n4_43 :  $\forall$  P Q : Prop,
2311   P  $\leftrightarrow$  ((P  $\vee$  Q)  $\wedge$  (P  $\vee$   $\neg$ Q)).
2312 Proof. intros P Q.
2313 specialize n2_2 with P Q.
2314 intros n2_2a.
2315 specialize n2_2 with P ( $\neg$ Q).
2316 intros n2_2b.
2317 Conj n2_2a n2_2b.
2318 split.
2319 apply n2_2a.
2320 apply n2_2b.
2321 specialize Comp3_43 with P (P $\vee$ Q) (P $\vee$  $\neg$ Q).
2322 intros Comp3_43a.
2323 MP Comp3_43a H.
2324 specialize n2_53 with P Q.
2325 intros n2_53a.
2326 specialize n2_53 with P ( $\neg$ Q).
2327 intros n2_53b.
2328 Conj n2_53a n2_53b.
2329 split.
2330 apply n2_53a.
2331 apply n2_53b.
2332 specialize n3_47 with (P $\vee$ Q) (P $\vee$  $\neg$ Q) ( $\neg$ P $\rightarrow$ Q) ( $\neg$ P $\rightarrow$  $\neg$ Q).

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2333   intros n3_47a.
2334   MP n3_47a H0.
2335   specialize n2_65 with ( $\neg$ P) Q.
2336   intros n2_65a.
2337   replace ( $\neg\neg$ P) with P in n2_65a.
2338   specialize Imp3_31 with ( $\neg$ P  $\rightarrow$  Q) ( $\neg$ P  $\rightarrow$   $\neg$ Q) (P).
2339   intros Imp3_31a.
2340   MP Imp3_31a n2_65a.
2341   Syll n3_47a Imp3_31a Sa.
2342   clear n2_2a. clear n2_2b. clear H. clear n2_53a.
2343     clear n2_53b. clear H0. clear n2_65a.
2344     clear n3_47a. clear Imp3_31a.
2345   Conj Comp3_43a Sa.
2346   split.
2347   apply Comp3_43a.
2348   apply Sa.
2349   Equiv H.
2350   apply H.
2351   apply Equiv4_01.
2352   apply propositional_extensionality.
2353   specialize n4_13 with P.
2354   intros n4_13a.
2355   apply n4_13a.
2356   Qed.
2357
2358   Theorem n4_44 :  $\forall$  P Q : Prop,
2359     P  $\leftrightarrow$  (P  $\vee$  (P  $\wedge$  Q)).
2360   Proof. intros P Q.
2361     specialize n2_2 with P (P $\wedge$ Q).
2362     intros n2_2a.
2363     specialize Id2_08 with P.
2364     intros Id2_08a.
2365     specialize Simp3_26 with P Q.
2366     intros Simp3_26a.
2367     Conj Id2_08a Simp3_26a.
2368     split.
2369     apply Id2_08a.
2370     apply Simp3_26a.
2371     specialize n3_44 with P P (P  $\wedge$  Q).
2372     intros n3_44a.
2373     MP n3_44a H.
2374     clear H. clear Id2_08a. clear Simp3_26a.

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2375     Conj n2_2a n3_44a.
2376     split.
2377     apply n2_2a.
2378     apply n3_44a.
2379     Equiv H.
2380     apply H.
2381     apply Equiv4_01.
2382 Qed.
2383
2384 Theorem n4_45 :  $\forall$  P Q : Prop,
2385   P  $\leftrightarrow$  (P  $\wedge$  (P  $\vee$  Q)).
2386 Proof. intros P Q.
2387   specialize n2_2 with (P  $\wedge$  P) (P  $\wedge$  Q).
2388   intros n2_2a.
2389   replace (P  $\wedge$  P  $\vee$  P  $\wedge$  Q) with (P  $\wedge$  (P  $\vee$  Q)) in n2_2a.
2390   replace (P  $\wedge$  P) with P in n2_2a.
2391   specialize Simp3_26 with P (P  $\vee$  Q).
2392   intros Simp3_26a.
2393   Conj n2_2a Simp3_26a.
2394   split.
2395   apply n2_2a.
2396   apply Simp3_26a.
2397   Equiv H.
2398   apply H.
2399   apply Equiv4_01.
2400   specialize n4_24 with P.
2401   intros n4_24a.
2402   apply propositional_extensionality.
2403   apply n4_24a.
2404   specialize n4_4 with P P Q.
2405   intros n4_4a.
2406   apply propositional_extensionality.
2407   apply n4_4a.
2408 Qed.
2409
2410 Theorem n4_5 :  $\forall$  P Q : Prop,
2411   P  $\wedge$  Q  $\leftrightarrow$   $\neg(\neg P \vee \neg Q)$ .
2412 Proof. intros P Q.
2413   specialize n4_2 with (P  $\wedge$  Q).
2414   intros n4_2a.
2415   rewrite Prod3_01.
2416   replace ( $\neg(\neg P \vee \neg Q)$ ) with (P  $\wedge$  Q).

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2417     apply n4_2a.
2418     apply Prod3_01.
2419 Qed.
2420
2421 Theorem n4_51 :  $\forall P Q : \text{Prop},$ 
2422    $\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q).$ 
2423 Proof. intros P Q.
2424     specialize n4_5 with P Q.
2425     intros n4_5a.
2426     specialize n4_12 with (P  $\wedge$  Q) ( $\neg P \vee \neg Q$ ).
2427     intros n4_12a.
2428     specialize Simp3_26 with
2429       ((P  $\wedge$  Q  $\leftrightarrow \neg(\neg P \vee \neg Q)$ )  $\rightarrow (\neg P \vee \neg Q \leftrightarrow \neg(P \wedge Q))$ )
2430       (( $\neg P \vee \neg Q \leftrightarrow \neg(P \wedge Q)$ )  $\rightarrow ((P \wedge Q \leftrightarrow \neg(\neg P \vee \neg Q))$ )).
2431     intros Simp3_26a.
2432     replace ((P  $\wedge$  Q  $\leftrightarrow \neg(\neg P \vee \neg Q)$ )  $\leftrightarrow (\neg P \vee \neg Q \leftrightarrow \neg(P \wedge Q))$ )
2433       with (((P  $\wedge$  Q  $\leftrightarrow \neg(\neg P \vee \neg Q)$ )  $\rightarrow (\neg P \vee \neg Q \leftrightarrow \neg(P \wedge Q))$ )
2434          $\wedge ((\neg P \vee \neg Q \leftrightarrow \neg(P \wedge Q)) \rightarrow ((P \wedge Q \leftrightarrow \neg(\neg P \vee \neg Q))$ ))
2435       in n4_12a.
2436     MP Simp3_26a n4_12a.
2437     MP Simp3_26a n4_5a.
2438     specialize n4_21 with ( $\neg(P \wedge Q)$ ) ( $\neg P \vee \neg Q$ ).
2439     intros n4_21a.
2440     specialize Simp3_27 with
2441       ((( $\neg(P \wedge Q) \leftrightarrow \neg P \vee \neg Q$ )  $\rightarrow ((\neg P \vee \neg Q \leftrightarrow \neg(P \wedge Q))$ ))
2442       ((( $\neg P \vee \neg Q \leftrightarrow \neg(P \wedge Q)$ )  $\rightarrow ((\neg(P \wedge Q) \leftrightarrow \neg P \vee \neg Q)$ )).
2443     intros Simp3_27a.
2444     MP Simp3_27a n4_21a.
2445     MP Simp3_27a Simp3_26a.
2446     apply Simp3_27a.
2447     apply Equiv4_01.
2448 Qed.
2449
2450 Theorem n4_52 :  $\forall P Q : \text{Prop},$ 
2451   (P  $\wedge \neg Q$ )  $\leftrightarrow \neg(\neg P \vee Q).$ 
2452 Proof. intros P Q.
2453     specialize n4_5 with P ( $\neg Q$ ).
2454     intros n4_5a.
2455     replace ( $\neg\neg Q$ ) with Q in n4_5a.
2456     apply n4_5a.
2457     specialize n4_13 with Q.
2458     intros n4_13a.

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2459     apply propositional_extensionality.
2460     apply n4_13a.
2461 Qed.
2462
2463 Theorem n4_53 :  $\forall P Q : \text{Prop},$ 
2464    $\neg(P \wedge \neg Q) \leftrightarrow (\neg P \vee Q).$ 
2465 Proof. intros P Q.
2466   specialize n4_52 with P Q.
2467   intros n4_52a.
2468   specialize n4_12 with (P  $\wedge$   $\neg Q$ ) ( $\neg P \vee Q$ ).
2469   intros n4_12a.
2470   replace ((P  $\wedge$   $\neg Q \leftrightarrow \neg(\neg P \vee Q) \leftrightarrow (\neg P \vee Q \leftrightarrow \neg(P \wedge \neg Q))$ )
2471     with (((P  $\wedge$   $\neg Q \leftrightarrow \neg(\neg P \vee Q) \rightarrow (\neg P \vee Q \leftrightarrow \neg(P \wedge \neg Q))$ )
2472        $\wedge ((\neg P \vee Q \leftrightarrow \neg(P \wedge \neg Q)) \rightarrow (P \wedge \neg Q \leftrightarrow \neg(\neg P \vee Q)))$ )
2473     in n4_12a.
2474   specialize Simp3_26 with
2475     ((P  $\wedge$   $\neg Q \leftrightarrow \neg(\neg P \vee Q) \rightarrow (\neg P \vee Q \leftrightarrow \neg(P \wedge \neg Q))$ )
2476     (( $\neg P \vee Q \leftrightarrow \neg(P \wedge \neg Q) \rightarrow (P \wedge \neg Q \leftrightarrow \neg(\neg P \vee Q))$ )).
2477   intros Simp3_26a.
2478   MP Simp3_26a n4_12a.
2479   MP Simp3_26a n4_52a.
2480   specialize n4_21 with ( $\neg(P \wedge \neg Q)$ ) ( $\neg P \vee Q$ ).
2481   intros n4_21a.
2482   replace (( $\neg(P \wedge \neg Q) \leftrightarrow \neg P \vee Q \leftrightarrow (\neg P \vee Q \leftrightarrow \neg(P \wedge \neg Q))$ )
2483     with ((( $\neg(P \wedge \neg Q) \leftrightarrow \neg P \vee Q \rightarrow (\neg P \vee Q \leftrightarrow \neg(P \wedge \neg Q))$ )
2484        $\wedge ((\neg P \vee Q \leftrightarrow \neg(P \wedge \neg Q)) \rightarrow (\neg(P \wedge \neg Q) \leftrightarrow \neg P \vee Q))$ )
2485     in n4_21a.
2486   specialize Simp3_27 with
2487     (( $\neg(P \wedge \neg Q) \leftrightarrow \neg P \vee Q \rightarrow (\neg P \vee Q \leftrightarrow \neg(P \wedge \neg Q))$ )
2488     (( $\neg P \vee Q \leftrightarrow \neg(P \wedge \neg Q) \rightarrow (\neg(P \wedge \neg Q) \leftrightarrow \neg P \vee Q)$ )).
2489   intros Simp3_27a.
2490   MP Simp3_27a n4_21a.
2491   MP Simp3_27a Simp3_26a.
2492   apply Simp3_27a.
2493   apply Equiv4_01.
2494   apply Equiv4_01.
2495 Qed.
2496
2497 Theorem n4_54 :  $\forall P Q : \text{Prop},$ 
2498    $(\neg P \wedge Q) \leftrightarrow \neg(P \vee \neg Q).$ 
2499 Proof. intros P Q.
2500   specialize n4_5 with ( $\neg P$ ) Q.

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2501     intros n4_5a.
2502     specialize n4_13 with P.
2503     intros n4_13a.
2504     replace (¬¬P) with P in n4_5a.
2505     apply n4_5a.
2506     apply propositional_extensionality.
2507     apply n4_13a.
2508 Qed.
2509
2510 Theorem n4_55 : ∀ P Q : Prop,
2511   ¬(¬P ∧ Q) ↔ (P ∨ ¬Q).
2512 Proof. intros P Q.
2513     specialize n4_54 with P Q.
2514     intros n4_54a.
2515     specialize n4_12 with (¬P ∧ Q) (P ∨ ¬Q).
2516     intros n4_12a.
2517     replace (¬P ∧ Q ↔ ¬(P ∨ ¬Q)) with
2518       (P ∨ ¬Q ↔ ¬(¬P ∧ Q)) in n4_54a.
2519     replace (P ∨ ¬Q ↔ ¬(¬P ∧ Q)) with
2520       (¬(¬P ∧ Q) ↔ (P ∨ ¬Q)) in n4_54a.
2521     apply n4_54a.
2522     specialize n4_21 with (¬(¬P ∧ Q)) (P ∨ ¬Q).
2523     intros n4_21a. (*Not cited*)
2524     apply propositional_extensionality.
2525     apply n4_21a.
2526     apply propositional_extensionality.
2527     replace ((P ∨ ¬Q ↔ ¬(¬P ∧ Q)) ↔ (¬P ∧ Q ↔ ¬(P ∨ ¬Q)))
2528       with ((¬P ∧ Q ↔ ¬(P ∨ ¬Q)) ↔ (P ∨ ¬Q ↔ ¬(¬P ∧ Q))).
2529     apply n4_12a.
2530     apply propositional_extensionality.
2531     specialize n4_21 with (P ∨ ¬Q ↔ ¬(¬P ∧ Q))
2532       (¬P ∧ Q ↔ ¬(P ∨ ¬Q)).
2533     intros n4_21b.
2534     apply n4_21.
2535 Qed.
2536
2537 Theorem n4_56 : ∀ P Q : Prop,
2538   (¬P ∧ ¬Q) ↔ ¬(P ∨ Q).
2539 Proof. intros P Q.
2540     specialize n4_54 with P (¬Q).
2541     intros n4_54a.
2542     replace (¬¬Q) with Q in n4_54a.

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2543     apply n4_54a.
2544     apply propositional_extensionality.
2545     specialize n4_13 with Q.
2546     intros n4_13a.
2547     apply n4_13a.
2548 Qed.
2549
2550 Theorem n4_57 :  $\forall P Q : \text{Prop},$ 
2551    $\neg(\neg P \wedge \neg Q) \leftrightarrow (P \vee Q).$ 
2552 Proof. intros P Q.
2553   specialize n4_56 with P Q.
2554   intros n4_56a.
2555   specialize n4_12 with  $(\neg P \wedge \neg Q) (P \vee Q).$ 
2556   intros n4_12a.
2557   replace  $(\neg P \wedge \neg Q \leftrightarrow \neg(P \vee Q))$  with
2558      $(P \vee Q \leftrightarrow \neg(\neg P \wedge \neg Q))$  in n4_56a.
2559   replace  $(P \vee Q \leftrightarrow \neg(\neg P \wedge \neg Q))$  with
2560      $(\neg(\neg P \wedge \neg Q) \leftrightarrow P \vee Q)$  in n4_56a.
2561   apply n4_56a.
2562   specialize n4_21 with  $(\neg(\neg P \wedge \neg Q)) (P \vee Q).$ 
2563   intros n4_21a.
2564   apply propositional_extensionality.
2565   apply n4_21a.
2566   replace  $((\neg P \wedge \neg Q \leftrightarrow \neg(P \vee Q)) \leftrightarrow (P \vee Q \leftrightarrow \neg(\neg P \wedge \neg Q)))$  with
2567      $((P \vee Q \leftrightarrow \neg(\neg P \wedge \neg Q)) \leftrightarrow (\neg P \wedge \neg Q \leftrightarrow \neg(P \vee Q)))$  in n4_12a.
2568   apply propositional_extensionality.
2569   apply n4_12a.
2570   apply propositional_extensionality.
2571   specialize n4_21 with
2572      $(P \vee Q \leftrightarrow \neg(\neg P \wedge \neg Q)) (\neg P \wedge \neg Q \leftrightarrow \neg(P \vee Q)).$ 
2573   intros n4_21b.
2574   apply n4_21b.
2575 Qed.
2576
2577 Theorem n4_6 :  $\forall P Q : \text{Prop},$ 
2578    $(P \rightarrow Q) \leftrightarrow (\neg P \vee Q).$ 
2579 Proof. intros P Q.
2580   specialize n4_2 with  $(\neg P \vee Q).$ 
2581   intros n4_2a.
2582   rewrite Impl1_01.
2583   apply n4_2a.
2584 Qed.

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2585
2586 Theorem n4_61 :  $\forall P Q : \text{Prop},$ 
2587    $\neg(P \rightarrow Q) \leftrightarrow (P \wedge \neg Q).$ 
2588 Proof. intros P Q.
2589   specialize n4_6 with P Q.
2590   intros n4_6a.
2591   specialize Transp4_11 with  $(P \rightarrow Q) (\neg P \vee Q).$ 
2592   intros Transp4_11a.
2593   specialize n4_52 with P Q.
2594   intros n4_52a.
2595   replace  $((P \rightarrow Q) \leftrightarrow \neg P \vee Q)$  with
2596      $(\neg(P \rightarrow Q) \leftrightarrow \neg(\neg P \vee Q))$  in n4_6a.
2597   replace  $(\neg(\neg P \vee Q))$  with  $(P \wedge \neg Q)$  in n4_6a.
2598   apply n4_6a.
2599   apply propositional_extensionality.
2600   apply n4_52a.
2601   replace  $((\neg(P \rightarrow Q) \leftrightarrow \neg P \vee Q) \leftrightarrow (\neg(P \rightarrow Q) \leftrightarrow \neg(\neg P \vee Q)))$  with
2602      $((\neg(P \rightarrow Q) \leftrightarrow \neg(\neg P \vee Q)) \leftrightarrow (\neg(P \rightarrow Q) \leftrightarrow P \vee Q))$  in Transp4_11a.
2603   apply propositional_extensionality.
2604   apply Transp4_11a.
2605   apply propositional_extensionality.
2606   specialize n4_21 with  $(\neg(P \rightarrow Q) \leftrightarrow \neg(\neg P \vee Q))$ 
2607      $((P \rightarrow Q) \leftrightarrow (\neg P \vee Q)).$ 
2608   intros n4_21a.
2609   apply n4_21a.
2610 Qed.
2611
2612 Theorem n4_62 :  $\forall P Q : \text{Prop},$ 
2613    $(P \rightarrow \neg Q) \leftrightarrow (\neg P \vee \neg Q).$ 
2614 Proof. intros P Q.
2615   specialize n4_6 with P  $(\neg Q).$ 
2616   intros n4_6a.
2617   apply n4_6a.
2618 Qed.
2619
2620 Theorem n4_63 :  $\forall P Q : \text{Prop},$ 
2621    $\neg(P \rightarrow \neg Q) \leftrightarrow (P \wedge Q).$ 
2622 Proof. intros P Q.
2623   specialize n4_62 with P Q.
2624   intros n4_62a.
2625   specialize Transp4_11 with  $(P \rightarrow \neg Q) (\neg P \vee \neg Q).$ 
2626   intros Transp4_11a.

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2627 specialize n4_5 with P Q.
2628 intros n4_5a.
2629 replace ( $\neg(\neg P \vee \neg Q)$ ) with  $(P \wedge Q)$  in Transp4_11a.
2630 replace  $((P \rightarrow \neg Q) \leftrightarrow \neg P \vee \neg Q)$  with
2631  $((\neg(P \rightarrow \neg Q) \leftrightarrow P \wedge Q))$  in n4_62a.
2632 apply n4_62a.
2633 replace  $((\neg(P \rightarrow \neg Q) \leftrightarrow \neg P \vee \neg Q) \leftrightarrow (\neg(P \rightarrow \neg Q) \leftrightarrow P \wedge Q))$  with
2634  $((\neg(P \rightarrow \neg Q) \leftrightarrow P \wedge Q) \leftrightarrow ((P \rightarrow \neg Q) \leftrightarrow \neg P \vee \neg Q))$  in Transp4_11a.
2635 apply propositional_extensionality.
2636 apply Transp4_11a.
2637 specialize n4_21 with
2638  $(\neg(P \rightarrow \neg Q) \leftrightarrow P \wedge Q) ((P \rightarrow \neg Q) \leftrightarrow \neg P \vee \neg Q)$ .
2639 intros n4_21a.
2640 apply propositional_extensionality.
2641 apply n4_21a.
2642 apply propositional_extensionality.
2643 apply n4_5a.
2644 Qed.
2645
2646 Theorem n4_64 :  $\forall P Q : \text{Prop}$ ,
2647  $(\neg P \rightarrow Q) \leftrightarrow (P \vee Q)$ .
2648 Proof. intros P Q.
2649 specialize n2_54 with P Q.
2650 intros n2_54a.
2651 specialize n2_53 with P Q.
2652 intros n2_53a.
2653 Conj n2_54a n2_53a.
2654 split.
2655 apply n2_54a.
2656 apply n2_53a.
2657 Equiv H.
2658 apply H.
2659 apply Equiv4_01.
2660 Qed.
2661
2662 Theorem n4_65 :  $\forall P Q : \text{Prop}$ ,
2663  $\neg(\neg P \rightarrow Q) \leftrightarrow (\neg P \wedge \neg Q)$ .
2664 Proof. intros P Q.
2665 specialize n4_64 with P Q.
2666 intros n4_64a.
2667 specialize Transp4_11 with  $(\neg P \rightarrow Q) (P \vee Q)$ .
2668 intros Transp4_11a.

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2669 specialize n4_56 with P Q.
2670 intros n4_56a.
2671 replace (((¬P→Q)↔P∨Q)↔(¬(¬P→Q)↔¬(P∨Q))) with
2672   ((¬(¬P→Q)↔¬(P∨Q))↔((¬P→Q)↔P∨Q)) in Transp4_11a.
2673 replace ((¬P → Q) ↔ P ∨ Q) with
2674   (¬(¬P → Q) ↔ ¬(P ∨ Q)) in n4_64a.
2675 replace (¬(P ∨ Q)) with (¬P ∧ ¬Q) in n4_64a.
2676 apply n4_64a.
2677 apply propositional_extensionality.
2678 apply n4_56a.
2679 apply propositional_extensionality.
2680 apply Transp4_11a.
2681 apply propositional_extensionality.
2682 specialize n4_21 with (¬(¬P → Q)↔¬(P ∨ Q))
2683   ((¬P → Q)↔(P ∨ Q)).
2684 intros n4_21a.
2685 apply n4_21a.
2686 Qed.
2687
2688 Theorem n4_66 : ∀ P Q : Prop,
2689   (¬P → ¬Q) ↔ (P ∨ ¬Q).
2690 Proof. intros P Q.
2691 specialize n4_64 with P (¬Q).
2692 intros n4_64a.
2693 apply n4_64a.
2694 Qed.
2695
2696 Theorem n4_67 : ∀ P Q : Prop,
2697   ¬(¬P → ¬Q) ↔ (¬P ∧ Q).
2698 Proof. intros P Q.
2699 specialize n4_66 with P Q.
2700 intros n4_66a.
2701 specialize Transp4_11 with (¬P → ¬Q) (P ∨ ¬Q).
2702 intros Transp4_11a.
2703 replace ((¬P → ¬Q) ↔ P ∨ ¬Q) with
2704   (¬(¬P → ¬Q) ↔ ¬(P ∨ ¬Q)) in n4_66a.
2705 specialize n4_54 with P Q.
2706 intros n4_54a.
2707 replace (¬(P ∨ ¬Q)) with (¬P ∧ Q) in n4_66a.
2708 apply n4_66a.
2709 apply propositional_extensionality.
2710 apply n4_54a.

```

```

2711   replace (((¬P→¬Q)↔P∨¬Q)↔(¬(¬P→¬Q)↔¬(P∨¬Q))) with
2712         ((¬(¬P→¬Q)↔¬(P∨¬Q))↔((¬P→¬Q)↔P∨¬Q)) in Transp4_11a.
2713   apply propositional_extensionality.
2714   apply Transp4_11a.
2715   apply propositional_extensionality.
2716   specialize n4_21 with (¬(¬P → ¬Q)↔¬(P ∨ ¬Q))
2717         ((¬P → ¬Q)↔(P ∨ ¬Q)).
2718   intros n4_21a.
2719   apply n4_21a.
2720   Qed.
2721
2722   (*Return to this proof.*)
2723   (*We did get one half of the ↔.*)
2724   Theorem n4_7 : ∀ P Q : Prop,
2725     (P → Q) ↔ (P → (P ∧ Q)).
2726   Proof. intros P Q.
2727     specialize Comp3_43 with P P Q.
2728     intros Comp3_43a.
2729     specialize Exp3_3 with
2730       (P → P) (P → Q) (P → P ∧ Q).
2731     intros Exp3_3a.
2732     MP Exp3_3a Comp3_43a.
2733     specialize Id2_08 with P.
2734     intros Id2_08a.
2735     MP Exp3_3a Id2_08a.
2736     specialize Simp3_27 with P Q.
2737     intros Simp3_27a.
2738     specialize Syll2_05 with P (P ∧ Q) Q.
2739     intros Syll2_05a.
2740     MP Syll2_05a Simp3_27a.
2741     clear Id2_08a. clear Comp3_43a. clear Simp3_27a.
2742     Conj Syll2_05a Exp3_3a.
2743     split.
2744     apply Exp3_3a.
2745     apply Syll2_05a.
2746     Equiv H.
2747     apply H.
2748     apply Equiv4_01.
2749     Qed.
2750
2751   Theorem n4_71 : ∀ P Q : Prop,
2752     (P → Q) ↔ (P ↔ (P ∧ Q)).

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2753 Proof. intros P Q.
2754 specialize n4_7 with P Q.
2755 intros n4_7a.
2756 specialize n3_21 with (P → (P ∧ Q)) ((P ∧ Q) → P).
2757 intros n3_21a.
2758 replace ((P → P ∧ Q) ∧ (P ∧ Q → P)) with
2759   (P ↔ (P ∧ Q)) in n3_21a.
2760 specialize Simp3_26 with P Q.
2761 intros Simp3_26a.
2762 MP n3_21a Simp3_26a.
2763 specialize Simp3_26 with (P → (P ∧ Q)) ((P ∧ Q) → P).
2764 intros Simp3_26b.
2765 replace ((P → P ∧ Q) ∧ (P ∧ Q → P)) with
2766   (P ↔ (P ∧ Q)) in Simp3_26b. clear Simp3_26a.
2767 Conj n3_21a Simp3_26b.
2768 split.
2769 apply n3_21a.
2770 apply Simp3_26b.
2771 Equiv H.
2772 clear n3_21a. clear Simp3_26b.
2773 Conj n4_7a H.
2774 split.
2775 apply n4_7a.
2776 apply H.
2777 specialize n4_22 with (P → Q) (P → P ∧ Q) (P ↔ P ∧ Q).
2778 intros n4_22a.
2779 MP n4_22a H0.
2780 apply n4_22a.
2781 apply Equiv4_01.
2782 apply Equiv4_01.
2783 apply Equiv4_01.
2784 Qed.
2785
2786 Theorem n4_72 : ∀ P Q : Prop,
2787   (P → Q) ↔ (Q ↔ (P ∨ Q)).
2788 Proof. intros P Q.
2789 specialize Transp4_1 with P Q.
2790 intros Transp4_1a.
2791 specialize n4_71 with (¬Q) (¬P).
2792 intros n4_71a.
2793 Conj Transp4_1a n4_71a.
2794 split.

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2795   apply Transp4_1a.
2796   apply n4_71a.
2797   specialize n4_22 with
2798     (P → Q) (¬Q → ¬P) (¬Q ↔ ¬Q ∧ ¬P).
2799   intros n4_22a.
2800   MP n4_22a H.
2801   specialize n4_21 with (¬Q) (¬Q ∧ ¬P).
2802   intros n4_21a.
2803   Conj n4_22a n4_21a.
2804   split.
2805   apply n4_22a.
2806   apply n4_21a.
2807   specialize n4_22 with
2808     (P → Q) (¬Q ↔ ¬Q ∧ ¬P) (¬Q ∧ ¬P ↔ ¬Q).
2809   intros n4_22b.
2810   MP n4_22b H0.
2811   specialize n4_12 with (¬Q ∧ ¬P) (Q).
2812   intros n4_12a.
2813   Conj n4_22b n4_12a.
2814   split.
2815   apply n4_22b.
2816   apply n4_12a.
2817   specialize n4_22 with
2818     (P → Q) ((¬Q ∧ ¬P) ↔ ¬Q) (Q ↔ ¬(¬Q ∧ ¬P)).
2819   intros n4_22c.
2820   MP n4_22b H0.
2821   specialize n4_57 with Q P.
2822   intros n4_57a.
2823   replace (¬(¬Q ∧ ¬P)) with (Q ∨ P) in n4_22c.
2824   specialize n4_31 with P Q.
2825   intros n4_31a.
2826   replace (Q ∨ P) with (P ∨ Q) in n4_22c.
2827   apply n4_22c.
2828   apply propositional_extensionality.
2829   apply n4_31a.
2830   apply propositional_extensionality.
2831   replace (¬(¬Q ∧ ¬P) ↔ Q ∨ P) with
2832     (Q ∨ P ↔ ¬(¬Q ∧ ¬P)) in n4_57a.
2833   apply n4_57a.
2834   apply propositional_extensionality.
2835   specialize n4_21 with (Q ∨ P) (¬(¬Q ∧ ¬P)).
2836   intros n4_21b.

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2837     apply n4_21b.
2838 Qed.
2839
2840 Theorem n4_73 :  $\forall P Q : \text{Prop},$ 
2841    $Q \rightarrow (P \leftrightarrow (P \wedge Q)).$ 
2842 Proof. intros P Q.
2843   specialize Simp2_02 with P Q.
2844   intros Simp2_02a.
2845   specialize n4_71 with P Q.
2846   intros n4_71a.
2847   replace  $((P \rightarrow Q) \leftrightarrow (P \leftrightarrow P \wedge Q))$  with
2848      $((P \rightarrow Q) \rightarrow (P \leftrightarrow P \wedge Q)) \wedge ((P \leftrightarrow P \wedge Q) \rightarrow (P \rightarrow Q))$  in n4_71a.
2849   specialize Simp3_26 with
2850      $((P \rightarrow Q) \rightarrow P \leftrightarrow P \wedge Q) (P \leftrightarrow P \wedge Q \rightarrow P \rightarrow Q).$ 
2851   intros Simp3_26a.
2852   MP Simp3_26a n4_71a.
2853   Syll Simp2_02a Simp3_26a Sa.
2854   apply Sa.
2855   apply Equiv4_01.
2856 Qed.
2857
2858 Theorem n4_74 :  $\forall P Q : \text{Prop},$ 
2859    $\neg P \rightarrow (Q \leftrightarrow (P \vee Q)).$ 
2860 Proof. intros P Q.
2861   specialize n2_21 with P Q.
2862   intros n2_21a.
2863   specialize n4_72 with P Q.
2864   intros n4_72a.
2865   replace  $(P \rightarrow Q)$  with  $(Q \leftrightarrow P \vee Q)$  in n2_21a.
2866   apply n2_21a.
2867   apply propositional_extensionality.
2868   replace  $((P \rightarrow Q) \leftrightarrow (Q \leftrightarrow P \vee Q))$  with
2869      $((Q \leftrightarrow P \vee Q) \leftrightarrow (P \rightarrow Q))$  in n4_72a.
2870   apply n4_72a.
2871   apply propositional_extensionality.
2872   specialize n4_21 with  $(Q \leftrightarrow (P \vee Q)) (P \rightarrow Q).$ 
2873   intros n4_21a.
2874   apply n4_21a.
2875 Qed.
2876
2877 Theorem n4_76 :  $\forall P Q R : \text{Prop},$ 
2878    $((P \rightarrow Q) \wedge (P \rightarrow R)) \leftrightarrow (P \rightarrow (Q \wedge R)).$ 

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2879 Proof. intros P Q R.
2880 specialize n4_41 with ( $\neg$ P) Q R.
2881 intros n4_41a.
2882 replace ( $\neg$ P  $\vee$  Q) with (P $\rightarrow$ Q) in n4_41a.
2883 replace ( $\neg$ P  $\vee$  R) with (P $\rightarrow$ R) in n4_41a.
2884 replace ( $\neg$ P  $\vee$  Q  $\wedge$  R) with (P  $\rightarrow$  Q  $\wedge$  R) in n4_41a.
2885 replace ((P  $\rightarrow$  Q  $\wedge$  R)  $\leftrightarrow$  (P  $\rightarrow$  Q)  $\wedge$  (P  $\rightarrow$  R)) with
2886       ((P  $\rightarrow$  Q)  $\wedge$  (P  $\rightarrow$  R)  $\leftrightarrow$  (P  $\rightarrow$  Q  $\wedge$  R)) in n4_41a.
2887 apply n4_41a.
2888 apply propositional_extensionality.
2889 specialize n4_21 with ((P  $\rightarrow$  Q)  $\wedge$  (P  $\rightarrow$  R)) (P  $\rightarrow$  Q  $\wedge$  R).
2890 intros n4_21a.
2891 apply n4_21a.
2892 apply Impl1_01.
2893 apply Impl1_01.
2894 apply Impl1_01.
2895 Qed.
2896
2897 Theorem n4_77 :  $\forall$  P Q R : Prop,
2898   ((Q  $\rightarrow$  P)  $\wedge$  (R  $\rightarrow$  P))  $\leftrightarrow$  ((Q  $\vee$  R)  $\rightarrow$  P).
2899 Proof. intros P Q R.
2900 specialize n3_44 with P Q R.
2901 intros n3_44a.
2902 specialize n2_2 with Q R.
2903 intros n2_2a.
2904 specialize Add1_3 with Q R.
2905 intros Add1_3a.
2906 specialize Syll2_06 with Q (Q  $\vee$  R) P.
2907 intros Syll2_06a.
2908 MP Syll2_06a n2_2a.
2909 specialize Syll2_06 with R (Q  $\vee$  R) P.
2910 intros Syll2_06b.
2911 MP Syll2_06b Add1_3a.
2912 Conj Syll2_06a Syll2_06b.
2913 split.
2914 apply Syll2_06a.
2915 apply Syll2_06b.
2916 specialize Comp3_43 with ((Q  $\vee$  R) $\rightarrow$ P)
2917       (Q $\rightarrow$ P) (R $\rightarrow$ P).
2918 intros Comp3_43a.
2919 MP Comp3_43a H.
2920 clear n2_2a. clear Add1_3a. clear H.

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2921     clear Syll2_06a. clear Syll2_06b.
2922 Conj n3_44a Comp3_43a.
2923 split.
2924 apply n3_44a.
2925 apply Comp3_43a.
2926 Equiv H.
2927 apply H.
2928 apply Equiv4_01.
2929 Qed.
2930
2931 Theorem n4_78 :  $\forall P Q R : \text{Prop}$ ,
2932    $((P \rightarrow Q) \vee (P \rightarrow R)) \leftrightarrow (P \rightarrow (Q \vee R))$ .
2933 Proof. intros P Q R.
2934 specialize n4_2 with  $((P \rightarrow Q) \vee (P \rightarrow R))$ .
2935 intros n4_2a.
2936 replace  $((P \rightarrow Q) \vee (P \rightarrow R)) \leftrightarrow ((P \rightarrow Q) \vee (P \rightarrow R))$  with
2937    $((P \rightarrow Q) \vee (P \rightarrow R)) \leftrightarrow ((\neg P \vee Q) \vee \neg P \vee R)$  in n4_2a.
2938 specialize n4_33 with  $(\neg P) Q (\neg P \vee R)$ .
2939 intros n4_33a.
2940 replace  $((\neg P \vee Q) \vee \neg P \vee R)$  with
2941    $(\neg P \vee Q \vee \neg P \vee R)$  in n4_2a.
2942 specialize n4_31 with  $(\neg P) Q$ .
2943 intros n4_31a.
2944 specialize n4_37 with  $(\neg P \vee Q) (Q \vee \neg P) R$ .
2945 intros n4_37a.
2946 MP n4_37a n4_31a.
2947 replace  $(Q \vee \neg P \vee R)$  with
2948    $((Q \vee \neg P) \vee R)$  in n4_2a.
2949 replace  $((Q \vee \neg P) \vee R)$  with
2950    $((\neg P \vee Q) \vee R)$  in n4_2a.
2951 specialize n4_33 with  $(\neg P) (\neg P \vee Q) R$ .
2952 intros n4_33b.
2953 replace  $(\neg P \vee (\neg P \vee Q) \vee R)$  with
2954    $((\neg P \vee (\neg P \vee Q)) \vee R)$  in n4_2a.
2955 specialize n4_25 with  $(\neg P)$ .
2956 intros n4_25a.
2957 specialize n4_37 with
2958    $(\neg P) (\neg P \vee \neg P) (Q \vee R)$ .
2959 intros n4_37b.
2960 MP n4_37b n4_25a.
2961 replace  $(\neg P \vee \neg P \vee Q)$  with
2962    $((\neg P \vee \neg P) \vee Q)$  in n4_2a.

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2963   replace (((¬P ∨ ¬P) ∨ Q) ∨ R) with
2964     ((¬P ∨ ¬P) ∨ Q ∨ R) in n4_2a.
2965   replace ((¬P ∨ ¬P) ∨ Q ∨ R) with
2966     ((¬P) ∨ (Q ∨ R)) in n4_2a.
2967   replace (¬P ∨ Q ∨ R) with
2968     (P → (Q ∨ R)) in n4_2a.
2969   apply n4_2a.
2970   apply Impl1_01.
2971   apply propositional_extensionality.
2972   apply n4_37b.
2973   apply Abb2_33.
2974   replace ((¬P ∨ ¬P) ∨ Q) with (¬P ∨ ¬P ∨ Q).
2975   reflexivity.
2976   apply Abb2_33.
2977   replace ((¬P ∨ ¬P ∨ Q) ∨ R) with
2978     (¬P ∨ (¬P ∨ Q) ∨ R).
2979   reflexivity.
2980   apply propositional_extensionality.
2981   apply n4_33b.
2982   apply propositional_extensionality.
2983   apply n4_37a.
2984   replace ((Q ∨ ¬P) ∨ R) with (Q ∨ ¬P ∨ R).
2985   reflexivity.
2986   apply Abb2_33.
2987   apply propositional_extensionality.
2988   apply n4_33a.
2989   rewrite <- Impl1_01.
2990   rewrite <- Impl1_01.
2991   reflexivity.
2992   Qed.
2993
2994   Theorem n4_79 : ∀ P Q R : Prop,
2995     ((Q → P) ∨ (R → P)) ↔ ((Q ∧ R) → P).
2996   Proof. intros P Q R.
2997     specialize Transp4_1 with Q P.
2998     intros Transp4_1a.
2999     specialize Transp4_1 with R P.
3000     intros Transp4_1b.
3001     Conj Transp4_1a Transp4_1b.
3002     split.
3003     apply Transp4_1a.
3004     apply Transp4_1b.

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3005 specialize n4_39 with
3006   (Q→P) (R→P) (¬P→¬Q) (¬P→¬R).
3007 intros n4_39a.
3008 MP n4_39a H.
3009 specialize n4_78 with (¬P) (¬Q) (¬R).
3010 intros n4_78a.
3011 rewrite Equiv4_01 in n4_78a.
3012 specialize Simp3_26 with
3013   (((¬P→¬Q)∨(¬P→¬R))→(¬P→(¬Q∨¬R)))
3014   ((¬P→(¬Q∨¬R))→((¬P→¬Q)∨(¬P→¬R))).
3015 intros Simp3_26a.
3016 MP Simp3_26a n4_78a.
3017 specialize Transp2_15 with P (¬Q∨¬R).
3018 intros Transp2_15a.
3019 specialize Simp3_27 with
3020   (((¬P→¬Q)∨(¬P→¬R))→(¬P→(¬Q∨¬R)))
3021   ((¬P→(¬Q∨¬R))→((¬P→¬Q)∨(¬P→¬R))).
3022 intros Simp3_27a.
3023 MP Simp3_27a n4_78a.
3024 specialize Transp2_15 with (¬Q∨¬R) P.
3025 intros Transp2_15b.
3026 specialize Syll2_06 with ((¬P→¬Q)∨(¬P→¬R))
3027   (¬P→(¬Q∨¬R)) (¬(¬Q∨¬R)→P).
3028 intros Syll2_06a.
3029 MP Syll2_06a Simp3_26a.
3030 MP Syll2_06a Transp2_15a.
3031 specialize Syll2_06 with (¬(¬Q∨¬R)→P)
3032   (¬P→(¬Q∨¬R)) ((¬P→¬Q)∨(¬P→¬R)).
3033 intros Syll2_06b.
3034 MP Syll2_06b Trans2_15b.
3035 MP Syll2_06b Simp3_27a.
3036 Conj Syll2_06a Syll2_06b.
3037 split.
3038 apply Syll2_06a.
3039 apply Syll2_06b.
3040 Equiv H0.
3041 clear Transp4_1a. clear Transp4_1b. clear H.
3042 clear Simp3_26a. clear Syll2_06b. clear n4_78a.
3043 clear Transp2_15a. clear Simp3_27a.
3044 clear Transp2_15b. clear Syll2_06a.
3045 Conj n4_39a H0.
3046 split.

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3047     apply n4_39a.
3048     apply H0.
3049     specialize n4_22 with ((Q→P)∨(R→P))
3050       ((¬P→¬Q)∨(¬P→¬R)) (¬(¬Q∨¬R)→P).
3051     intros n4_22a.
3052     MP n4_22a H.
3053     specialize n4_2 with (¬(¬Q∨¬R)→P).
3054     intros n4_2a.
3055     Conj n4_22a n4_2a.
3056     split.
3057     apply n4_22a.
3058     apply n4_2a.
3059     specialize n4_22 with ((Q→P)∨(R→P))
3060       (¬(¬Q∨¬R)→P) (¬(¬Q∨¬R)→P).
3061     intros n4_22b.
3062     MP n4_22b H1.
3063     rewrite <- Prod3_01 in n4_22b.
3064     apply n4_22b.
3065     apply Equiv4_01.
3066   Qed.
3067
3068   Theorem n4_8 : ∀ P : Prop,
3069     (P → ¬P) ↔ ¬P.
3070   Proof. intros P.
3071     specialize Abs2_01 with P.
3072     intros Abs2_01a.
3073     specialize Simp2_02 with P (¬P).
3074     intros Simp2_02a.
3075     Conj Abs2_01a Simp2_02a.
3076     split.
3077     apply Abs2_01a.
3078     apply Simp2_02a.
3079     Equiv H.
3080     apply H.
3081     apply Equiv4_01.
3082   Qed.
3083
3084   Theorem n4_81 : ∀ P : Prop,
3085     (¬P → P) ↔ P.
3086   Proof. intros P.
3087     specialize n2_18 with P.
3088     intros n2_18a.

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3089     specialize Simp2_02 with (¬P) P.
3090     intros Simp2_02a.
3091     Conj n2_18a Simp2_02a.
3092     split.
3093     apply n2_18a.
3094     apply Simp2_02a.
3095     Equiv H.
3096     apply H.
3097     apply Equiv4_01.
3098 Qed.
3099
3100 Theorem n4_82 : ∀ P Q : Prop,
3101   ((P → Q) ∧ (P → ¬Q)) ↔ ¬P.
3102 Proof. intros P Q.
3103   specialize n2_65 with P Q.
3104   intros n2_65a.
3105   specialize Imp3_31 with (P→Q) (P→¬Q) (¬P).
3106   intros Imp3_31a.
3107   MP Imp3_31a n2_65a.
3108   specialize n2_21 with P Q.
3109   intros n2_21a.
3110   specialize n2_21 with P (¬Q).
3111   intros n2_21b.
3112   Conj n2_21a n2_21b.
3113   split.
3114   apply n2_21a.
3115   apply n2_21b.
3116   specialize Comp3_43 with (¬P) (P→Q) (P→¬Q).
3117   intros Comp3_43a.
3118   MP Comp3_43a H.
3119   clear n2_65a. clear n2_21a.
3120   clear n2_21b. clear H.
3121   Conj Imp3_31a Comp3_43a.
3122   split.
3123   apply Imp3_31a.
3124   apply Comp3_43a.
3125   Equiv H.
3126   apply H.
3127   apply Equiv4_01.
3128 Qed.
3129
3130 Theorem n4_83 : ∀ P Q : Prop,

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3131   ((P → Q) ∧ (¬P → Q)) ↔ Q.
3132 Proof. intros P Q.
3133 specialize n2_61 with P Q.
3134 intros n2_61a.
3135 specialize Imp3_31 with (P→Q) (¬P→Q) (Q).
3136 intros Imp3_31a.
3137 MP Imp3_31a n2_61a.
3138 specialize Simp2_02 with P Q.
3139 intros Simp2_02a.
3140 specialize Simp2_02 with (¬P) Q.
3141 intros Simp2_02b.
3142 Conj Simp2_02a Simp2_02b.
3143 split.
3144 apply Simp2_02a.
3145 apply Simp2_02b.
3146 specialize Comp3_43 with Q (P→Q) (¬P→Q).
3147 intros Comp3_43a.
3148 MP Comp3_43a H.
3149 clear n2_61a. clear Simp2_02a.
3150   clear Simp2_02b. clear H.
3151 Conj Imp3_31a Comp3_43a.
3152 split.
3153 apply Imp3_31a.
3154 apply Comp3_43a.
3155 Equiv H.
3156 apply H.
3157 apply Equiv4_01.
3158 Qed.
3159
3160 Theorem n4_84 : ∀ P Q R : Prop,
3161   (P ↔ Q) → ((P → R) ↔ (Q → R)).
3162 Proof. intros P Q R.
3163   specialize Syll2_06 with P Q R.
3164   intros Syll2_06a.
3165   specialize Syll2_06 with Q P R.
3166   intros Syll2_06b.
3167   Conj Syll2_06a Syll2_06b.
3168   split.
3169   apply Syll2_06a.
3170   apply Syll2_06b.
3171   specialize n3_47 with
3172     (P→Q) (Q→P) ((Q→R)→P→R) ((P→R)→Q→R).

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3173     intros n3_47a.
3174     MP n3_47a H.
3175     replace ((P→Q) ∧ (Q → P)) with
3176         (P↔Q) in n3_47a.
3177     replace (((Q→R)→P→R)∧((P→R)→Q→R)) with
3178         ((Q → R) ↔ (P → R)) in n3_47a.
3179     replace ((Q → R) ↔ (P → R)) with
3180         ((P→R) ↔ (Q → R)) in n3_47a.
3181     apply n3_47a.
3182     apply propositional_extensionality.
3183     specialize n4_21 with (P→R) (Q→R).
3184     intros n4_21a.
3185     apply n4_21a.
3186     apply Equiv4_01.
3187     apply Equiv4_01.
3188     Qed.
3189
3190     Theorem n4_85 : ∀ P Q R : Prop,
3191         (P ↔ Q) → ((R → P) ↔ (R → Q)).
3192     Proof. intros P Q R.
3193     specialize Syll2_05 with R P Q.
3194     intros Syll2_05a.
3195     specialize Syll2_05 with R Q P.
3196     intros Syll2_05b.
3197     Conj Syll2_05a Syll2_05b.
3198     split.
3199     apply Syll2_05a.
3200     apply Syll2_05b.
3201     specialize n3_47 with
3202         (P→Q) (Q→P) ((R→P)→R→Q) ((R→Q)→R→P).
3203     intros n3_47a.
3204     MP n3_47a H.
3205     replace ((P→Q) ∧ (Q → P)) with (P↔Q) in n3_47a.
3206     replace (((R→P)→R→Q)∧((R→Q)→R→P)) with
3207         ((R → P) ↔ (R → Q)) in n3_47a.
3208     apply n3_47a.
3209     apply Equiv4_01.
3210     apply Equiv4_01.
3211     Qed.
3212
3213     Theorem n4_86 : ∀ P Q R : Prop,
3214         (P ↔ Q) → ((P ↔ R) ↔ (Q ↔ R)).

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3215 Proof. intros P Q R.
3216 specialize n4_22 with Q P R.
3217 intros n4_22a.
3218 specialize Exp3_3 with (Q↔P) (P↔R) (Q↔R).
3219 intros Exp3_3a. (*Not cited*)
3220 MP Exp3_3a n4_22a.
3221 specialize n4_22 with P Q R.
3222 intros n4_22b.
3223 specialize Exp3_3 with (P↔Q) (Q↔R) (P↔R).
3224 intros Exp3_3b.
3225 MP Exp3_3b n4_22b.
3226 clear n4_22a. clear n4_22b.
3227 replace (Q↔P) with (P↔Q) in Exp3_3a.
3228 Conj Exp3_3a Exp3_3b.
3229 split.
3230 apply Exp3_3a.
3231 apply Exp3_3b.
3232 specialize Comp3_43 with (P↔Q)
3233   ((P↔R)→(Q↔R)) ((Q↔R)→(P↔R)).
3234 intros Comp3_43a. (*Not cited*)
3235 MP Comp3_43a H.
3236 replace (((P↔R)→(Q↔R))∧((Q↔R)→(P↔R)))
3237   with ((P↔R)↔(Q↔R)) in Comp3_43a.
3238 apply Comp3_43a.
3239 apply Equiv4_01.
3240 apply propositional_extensionality.
3241 specialize n4_21 with P Q.
3242 intros n4_21a.
3243 apply n4_21a.
3244 Qed.
3245
3246 Theorem n4_87 : ∀ P Q R : Prop,
3247   (((P ∧ Q) → R) ↔ (P → Q → R)) ↔
3248   ((Q → (P → R)) ↔ (Q ∧ P → R)).
3249 Proof. intros P Q R.
3250 specialize Exp3_3 with P Q R.
3251 intros Exp3_3a.
3252 specialize Imp3_31 with P Q R.
3253 intros Imp3_31a.
3254 Conj Exp3_3a Imp3_31a.
3255 split.
3256 apply Exp3_3a.

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3257   apply Imp3_31a.
3258   Equiv H.
3259   specialize Exp3_3 with Q P R.
3260   intros Exp3_3b.
3261   specialize Imp3_31 with Q P R.
3262   intros Imp3_31b.
3263   Conj Exp3_3b Imp3_31b.
3264   split.
3265   apply Exp3_3b.
3266   apply Imp3_31b.
3267   Equiv H0.
3268   specialize Comm2_04 with P Q R.
3269   intros Comm2_04a.
3270   specialize Comm2_04 with Q P R.
3271   intros Comm2_04b.
3272   Conj Comm2_04a Comm2_04b.
3273   split.
3274   apply Comm2_04a.
3275   apply Comm2_04b.
3276   Equiv H1.
3277   clear Exp3_3a. clear Imp3_31a. clear Exp3_3b.
3278       clear Imp3_31b. clear Comm2_04a.
3279       clear Comm2_04b.
3280   replace (P $\wedge$ Q $\rightarrow$ R) with (P  $\rightarrow$  Q  $\rightarrow$  R).
3281   replace (Q $\wedge$ P $\rightarrow$ R) with (Q  $\rightarrow$  P  $\rightarrow$  R).
3282   replace (Q $\rightarrow$ P $\rightarrow$ R) with (P  $\rightarrow$  Q  $\rightarrow$  R).
3283   specialize n4_2 with
3284       ((P  $\rightarrow$  Q  $\rightarrow$  R)  $\leftrightarrow$  (P  $\rightarrow$  Q  $\rightarrow$  R)).
3285   intros n4_2a.
3286   apply n4_2a.
3287   apply propositional_extensionality.
3288   apply H1.
3289   replace (Q $\rightarrow$ P $\rightarrow$ R) with (Q $\wedge$ P $\rightarrow$ R).
3290   reflexivity.
3291   apply propositional_extensionality.
3292   apply H0.
3293   replace (P $\rightarrow$ Q $\rightarrow$ R) with (P $\wedge$ Q $\rightarrow$ R).
3294   reflexivity.
3295   apply propositional_extensionality.
3296   apply H.
3297   apply Equiv4_01.
3298   apply Equiv4_01.

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3299     apply Equiv4_01.
3300 Qed.
3301
3302 End No4.
3303
3304 Module No5.
3305
3306 Import No1.
3307 Import No2.
3308 Import No3.
3309 Import No4.
3310
3311 Theorem n5_1 :  $\forall P Q : \text{Prop}$ ,
3312    $(P \wedge Q) \rightarrow (P \leftrightarrow Q)$ .
3313 Proof. intros P Q.
3314   specialize n3_4 with P Q.
3315   intros n3_4a.
3316   specialize n3_4 with Q P.
3317   intros n3_4b.
3318   specialize n3_22 with P Q.
3319   intros n3_22a.
3320   Syll n3_22a n3_4b Sa.
3321   clear n3_22a. clear n3_4b.
3322   Conj n3_4a Sa.
3323   split.
3324   apply n3_4a.
3325   apply Sa.
3326   specialize n4_76 with  $(P \wedge Q) (P \rightarrow Q) (Q \rightarrow P)$ .
3327   intros n4_76a. (*Not cited*)
3328   replace  $((P \wedge Q \rightarrow P \rightarrow Q) \wedge (P \wedge Q \rightarrow Q \rightarrow P))$  with
3329      $(P \wedge Q \rightarrow (P \rightarrow Q) \wedge (Q \rightarrow P))$  in H.
3330   replace  $((P \rightarrow Q) \wedge (Q \rightarrow P))$  with  $(P \leftrightarrow Q)$  in H.
3331   apply H.
3332   apply Equiv4_01.
3333   replace  $(P \wedge Q \rightarrow (P \rightarrow Q) \wedge (Q \rightarrow P))$  with
3334      $((P \wedge Q \rightarrow P \rightarrow Q) \wedge (P \wedge Q \rightarrow Q \rightarrow P))$ .
3335   reflexivity.
3336   apply propositional_extensionality.
3337   apply n4_76a.
3338 Qed.
3339
3340 Theorem n5_11 :  $\forall P Q : \text{Prop}$ ,

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3341   (P → Q) ∨ (¬P → Q).
3342   Proof. intros P Q.
3343   specialize n2_5 with P Q.
3344   intros n2_5a.
3345   specialize n2_54 with (P → Q) (¬P → Q).
3346   intros n2_54a.
3347   MP n2_54a n2_5a.
3348   apply n2_54a.
3349   Qed.
3350   (*The proof sketch cites n2_51,
3351     but this may be a misprint.*/)
3352
3353   Theorem n5_12 : ∀ P Q : Prop,
3354     (P → Q) ∨ (P → ¬Q).
3355   Proof. intros P Q.
3356   specialize n2_51 with P Q.
3357   intros n2_51a.
3358   specialize n2_54 with ((P → Q)) (P → ¬Q).
3359   intros n2_54a.
3360   MP n2_54a n2_5a.
3361   apply n2_54a.
3362   Qed.
3363   (*The proof sketch cites n2_52,
3364     but this may be a misprint.*/)
3365
3366   Theorem n5_13 : ∀ P Q : Prop,
3367     (P → Q) ∨ (Q → P).
3368   Proof. intros P Q.
3369   specialize n2_521 with P Q.
3370   intros n2_521a.
3371   replace (¬(P → Q) → Q → P) with
3372     (¬¬(P → Q) ∨ (Q → P)) in n2_521a.
3373   replace (¬¬(P → Q)) with (P → Q) in n2_521a.
3374   apply n2_521a.
3375   apply propositional_extensionality.
3376   specialize n4_13 with (P → Q).
3377   intros n4_13a. (*Not cited*)
3378   apply n4_13a.
3379   rewrite <- Impl1_01.
3380   reflexivity.
3381   Qed.
3382

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```

3383 Theorem n5_14 :  $\forall P Q R : \text{Prop},$ 
3384    $(P \rightarrow Q) \vee (Q \rightarrow R).$ 
3385 Proof. intros P Q R.
3386 specialize Simp2_02 with P Q.
3387 intros Simp2_02a.
3388 specialize Transp2_16 with Q  $(P \rightarrow Q).$ 
3389 intros Transp2_16a.
3390 MP Transp2_16a Simp2_02a.
3391 specialize n2_21 with Q R.
3392 intros n2_21a.
3393 Syll Transp2_16a n2_21a Sa.
3394 replace  $(\neg(P \rightarrow Q) \rightarrow (Q \rightarrow R))$  with
3395    $(\neg\neg(P \rightarrow Q) \vee (Q \rightarrow R))$  in Sa.
3396 replace  $(\neg\neg(P \rightarrow Q))$  with  $(P \rightarrow Q)$  in Sa.
3397 apply Sa.
3398 apply propositional_extensionality.
3399 specialize n4_13 with  $(P \rightarrow Q).$ 
3400 intros n4_13a.
3401 apply n4_13a.
3402 rewrite <- Impl1_01.
3403 reflexivity.
3404 Qed.
3405
3406 Theorem n5_15 :  $\forall P Q : \text{Prop},$ 
3407    $(P \leftrightarrow Q) \vee (P \leftrightarrow \neg Q).$ 
3408 Proof. intros P Q.
3409 specialize n4_61 with P Q.
3410 intros n4_61a.
3411 replace  $(\neg(P \rightarrow Q) \leftrightarrow P \wedge \neg Q)$  with
3412    $((\neg(P \rightarrow Q) \rightarrow P \wedge \neg Q) \wedge ((P \wedge \neg Q) \rightarrow \neg(P \rightarrow Q)))$  in n4_61a.
3413 specialize Simp3_26 with
3414    $(\neg(P \rightarrow Q) \rightarrow P \wedge \neg Q) ((P \wedge \neg Q) \rightarrow \neg(P \rightarrow Q)).$ 
3415 intros Simp3_26a.
3416 MP Simp3_26a n4_61a.
3417 specialize n5_1 with P  $(\neg Q).$ 
3418 intros n5_1a.
3419 Syll Simp3_26a n5_1a Sa.
3420 specialize n2_54 with  $(P \rightarrow Q) (P \leftrightarrow \neg Q).$ 
3421 intros n2_54a.
3422 MP n2_54a Sa.
3423 specialize n4_61 with Q P.
3424 intros n4_61b.

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3425 replace ((¬(Q → P)) ↔ (Q ∧ ¬P)) with
3426 (((¬(Q→P))→(Q∧¬P))∧((Q∧¬P)→(¬(Q→P)))) in n4_61b.
3427 specialize Simp3_26 with
3428   (¬(Q → P)→ (Q ∧ ¬P)) ((Q ∧ ¬P)→ (¬(Q → P))).
3429 intros Simp3_26b.
3430 MP Simp3_26b n4_61b.
3431 specialize n5_1 with Q (¬P).
3432 intros n5_1b.
3433 Syll Simp3_26b n5_1b Sb.
3434 specialize n4_12 with P Q.
3435 intros n4_12a.
3436 replace (Q↔¬P) with (P↔¬Q) in Sb.
3437 specialize n2_54 with (Q→P) (P↔¬Q).
3438 intros n2_54b.
3439 MP n2_54b Sb.
3440 clear n4_61a. clear Simp3_26a. clear n5_1a.
3441   clear n2_54a. clear n4_61b. clear Simp3_26b.
3442   clear n5_1b. clear n4_12a. clear n2_54b.
3443 replace (¬(P → Q) → P ↔ ¬Q) with
3444   (¬¬(P → Q) ∨ (P ↔ ¬Q)) in Sa.
3445 replace (¬¬(P→Q)) with (P→Q) in Sa.
3446 replace (¬(Q → P) → (P ↔ ¬Q)) with
3447   (¬¬(Q → P) ∨ (P ↔ ¬Q)) in Sb.
3448 replace (¬¬(Q→P)) with (Q→P) in Sb.
3449 Conj Sa Sb.
3450 split.
3451 apply Sa.
3452 apply Sb.
3453 specialize n4_41 with (P↔¬Q) (P→Q) (Q→P).
3454 intros n4_41a.
3455 replace ((P → Q) ∨ (P ↔ ¬Q)) with
3456   ((P ↔ ¬Q) ∨ (P → Q)) in H.
3457 replace ((Q → P) ∨ (P ↔ ¬Q)) with
3458   ((P ↔ ¬Q) ∨ (Q → P)) in H.
3459 replace (((P↔¬Q)∨(P→Q))∧((P↔¬Q)∨(Q→P))) with
3460   ((P ↔ ¬Q) ∨ (P → Q) ∧ (Q → P)) in H.
3461 replace ((P→Q) ∧ (Q → P)) with (P↔Q) in H.
3462 replace ((P ↔ ¬Q) ∨ (P ↔ Q)) with
3463   ((P ↔ Q) ∨ (P ↔ ¬Q)) in H.
3464 apply H.
3465 apply propositional_extensionality.
3466 apply n4_31.

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3467   apply Equiv4_01.
3468   apply propositional_extensionality.
3469   apply n4_41a.
3470   apply propositional_extensionality.
3471   apply n4_31.
3472   apply propositional_extensionality.
3473   apply n4_31.
3474   apply propositional_extensionality.
3475   specialize n4_13 with (Q→P).
3476   intros n4_13a.
3477   apply n4_13a.
3478   rewrite <- Impl1_01.
3479   reflexivity.
3480   apply propositional_extensionality.
3481   specialize n4_13 with (P→Q).
3482   intros n4_13b.
3483   apply n4_13b.
3484   rewrite <- Impl1_01.
3485   reflexivity.
3486   apply propositional_extensionality.
3487   apply n4_12a.
3488   apply Equiv4_01.
3489   apply Equiv4_01.
3490   Qed.
3491
3492   Theorem n5_16 : ∀ P Q : Prop,
3493     ¬((P ↔ Q) ∧ (P ↔ ¬Q)).
3494   Proof. intros P Q.
3495     specialize Simp3_26 with ((P→Q) ∧ (P → ¬Q)) (Q→P).
3496     intros Simp3_26a.
3497     specialize Id2_08 with ((P ↔ Q) ∧ (P → ¬Q)).
3498     intros Id2_08a.
3499     replace (((P → Q) ∧ (P → ¬Q)) ∧ (Q → P)) with
3500       ((P→Q) ∧ ((P→¬Q) ∧ (Q→P))) in Simp3_26a.
3501     replace ((P → ¬Q) ∧ (Q → P)) with
3502       ((Q → P) ∧ (P → ¬Q)) in Simp3_26a.
3503     replace ((P→Q) ∧ (Q → P) ∧ (P → ¬Q)) with
3504       (((P→Q) ∧ (Q → P)) ∧ (P → ¬Q)) in Simp3_26a.
3505     replace ((P → Q) ∧ (Q → P)) with
3506       (P↔Q) in Simp3_26a.
3507     Syll Id2_08a Simp3_26a Sa.
3508     specialize n4_82 with P Q.

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3509 intros n4_82a.
3510 replace ((P → Q) ∧ (P → ¬Q)) with (¬P) in Sa.
3511 specialize Simp3_27 with
3512   (P→Q) ((Q→P) ∧ (P → ¬Q)).
3513 intros Simp3_27a.
3514 replace ((P→Q) ∧ (Q → P) ∧ (P → ¬Q)) with
3515   (((P→Q) ∧ (Q → P)) ∧ (P → ¬Q)) in Simp3_27a.
3516 replace ((P → Q) ∧ (Q → P)) with
3517   (P↔Q) in Simp3_27a.
3518 specialize Syll3_33 with Q P (¬Q).
3519 intros Syll3_33a.
3520 Syll Simp3_27a Syll2_06a Sb.
3521 specialize Abs2_01 with Q.
3522 intros Abs2_01a.
3523 Syll Sb Abs2_01a Sc.
3524 clear Sb. clear Simp3_26a. clear Id2_08a.
3525   clear n4_82a. clear Simp3_27a. clear Syll3_33a.
3526   clear Abs2_01a.
3527 Conj Sa Sc.
3528 split.
3529 apply Sa.
3530 apply Sc.
3531 specialize Comp3_43 with
3532   ((P ↔ Q) ∧ (P → ¬Q)) (¬P) (¬Q).
3533 intros Comp3_43a.
3534 MP Comp3_43a H.
3535 specialize n4_65 with Q P.
3536 intros n4_65a.
3537 replace (¬Q ∧ ¬P) with (¬P ∧ ¬Q) in n4_65a.
3538 replace (¬P ∧ ¬Q) with
3539   (¬(¬Q→P)) in Comp3_43a.
3540 specialize Exp3_3 with
3541   (P↔Q) (P→¬Q) (¬(¬Q→P)).
3542 intros Exp3_3a.
3543 MP Exp3_3a Comp3_43a.
3544 replace ((P→¬Q)→¬(¬Q→P)) with
3545   (¬(P→¬Q) ∨ ¬(¬Q→P)) in Exp3_3a.
3546 specialize n4_51 with (P→¬Q) (¬Q→P).
3547 intros n4_51a.
3548 replace (¬(P → ¬Q) ∨ ¬(¬Q → P)) with
3549   (¬((P → ¬Q) ∧ (¬Q → P))) in Exp3_3a.
3550 replace ((P→¬Q) ∧ (¬Q → P)) with

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3551      (P↔¬Q) in Exp3_3a.
3552 replace ((P↔Q)→¬(P↔¬Q)) with
3553      (¬(P↔Q)∨¬(P↔¬Q)) in Exp3_3a.
3554 specialize n4_51 with (P↔Q) (P↔¬Q).
3555 intros n4_51b.
3556 replace (¬(P ↔ Q) ∨ ¬(P ↔ ¬Q)) with
3557      (¬((P ↔ Q) ∧ (P ↔ ¬Q))) in Exp3_3a.
3558 apply Exp3_3a.
3559 apply propositional_extensionality.
3560 apply n4_51b.
3561 rewrite <- Impl1_01.
3562 reflexivity.
3563 apply Equiv4_01.
3564 apply propositional_extensionality.
3565 apply n4_51a.
3566 rewrite <- Impl1_01.
3567 reflexivity.
3568 apply propositional_extensionality.
3569 apply n4_65a.
3570 apply propositional_extensionality.
3571 specialize n4_3 with (¬P) (¬Q).
3572 intros n4_3a.
3573 apply n4_3a.
3574 apply Equiv4_01.
3575 apply propositional_extensionality.
3576 specialize n4_32 with (P→Q) (Q→P) (P→¬Q).
3577 intros n4_32a.
3578 apply n4_32a.
3579 replace (¬P) with ((P → Q) ∧ (P → ¬Q)).
3580 reflexivity.
3581 apply propositional_extensionality.
3582 apply n4_82a.
3583 apply Equiv4_01.
3584 apply propositional_extensionality.
3585 specialize n4_32 with (P→Q) (Q→P) (P→¬Q).
3586 intros n4_32b.
3587 apply n4_32b.
3588 apply propositional_extensionality.
3589 specialize n4_3 with (Q→P) (P→¬Q).
3590 intros n4_3b.
3591 apply n4_3b.
3592 replace ((P → Q) ∧ (P → ¬Q) ∧ (Q → P)) with

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3593      (((P → Q) ∧ (P → ¬Q)) ∧ (Q → P)).
3594 reflexivity.
3595 apply propositional_extensionality.
3596 specialize n4_32 with (P→Q) (P→¬Q) (Q→P).
3597 intros n4_32a.
3598 apply n4_32a.
3599 Qed.
3600
3601 Theorem n5_17 : ∀ P Q : Prop,
3602   ((P ∨ Q) ∧ ¬(P ∧ Q)) ↔ (P ↔ ¬Q).
3603 Proof. intros P Q.
3604 specialize n4_64 with Q P.
3605 intros n4_64a.
3606 specialize n4_21 with (Q∨P) (¬Q→P).
3607 intros n4_21a.
3608 replace ((¬Q→P)↔(Q∨P)) with
3609   ((Q∨P)↔(¬Q→P)) in n4_64a.
3610 replace (Q∨P) with (P∨Q) in n4_64a.
3611 specialize n4_63 with P Q.
3612 intros n4_63a.
3613 replace (¬(P → ¬Q) ↔ P ∧ Q) with
3614   (P ∧ Q ↔ ¬(P → ¬Q)) in n4_63a.
3615 specialize Transp4_11 with (P∧Q) (¬(P→¬Q)).
3616 intros Transp4_11a.
3617 replace (¬¬(P→¬Q)) with
3618   (P→¬Q) in Transp4_11a.
3619 replace (P ∧ Q ↔ ¬(P → ¬Q)) with
3620   (¬(P ∧ Q) ↔ (P → ¬Q)) in n4_63a.
3621 clear Transp4_11a. clear n4_21a.
3622 Conj n4_64a n4_63a.
3623 split.
3624 apply n4_64a.
3625 apply n4_63a.
3626 specialize n4_38 with
3627   (P ∨ Q) (¬(P ∧ Q)) (¬Q → P) (P → ¬Q).
3628 intros n4_38a.
3629 MP n4_38a H.
3630 replace ((¬Q→P) ∧ (P → ¬Q)) with
3631   (¬Q↔P) in n4_38a.
3632 specialize n4_21 with P (¬Q).
3633 intros n4_21b.
3634 replace (¬Q↔P) with (P↔¬Q) in n4_38a.

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3635 apply n4_38a.
3636 apply propositional_extensionality.
3637 apply n4_21b.
3638 apply Equiv4_01.
3639 replace ( $\neg(P \wedge Q) \leftrightarrow (P \rightarrow \neg Q)$ ) with
3640 ( $P \wedge Q \leftrightarrow \neg(P \rightarrow \neg Q)$ ).
3641 reflexivity.
3642 apply propositional_extensionality.
3643 apply Transp4_11a.
3644 apply propositional_extensionality.
3645 specialize n4_13 with ( $P \rightarrow \neg Q$ ).
3646 intros n4_13a.
3647 apply n4_13a.
3648 apply propositional_extensionality.
3649 specialize n4_21 with ( $P \wedge Q$ ) ( $\neg(P \rightarrow \neg Q)$ ).
3650 intros n4_21b.
3651 apply n4_21b.
3652 apply propositional_extensionality.
3653 specialize n4_31 with P Q.
3654 intros n4_31a.
3655 apply n4_31a.
3656 apply propositional_extensionality.
3657 apply n4_21a.
3658 Qed.
3659
3660 Theorem n5_18 :  $\forall P Q : \text{Prop},$ 
3661 ( $P \leftrightarrow Q$ )  $\leftrightarrow \neg(P \leftrightarrow \neg Q)$ .
3662 Proof. intros P Q.
3663 specialize n5_15 with P Q.
3664 intros n5_15a.
3665 specialize n5_16 with P Q.
3666 intros n5_16a.
3667 Conj n5_15a n5_16a.
3668 split.
3669 apply n5_15a.
3670 apply n5_16a.
3671 specialize n5_17 with ( $P \leftrightarrow Q$ ) ( $P \leftrightarrow \neg Q$ ).
3672 intros n5_17a.
3673 replace (( $P \leftrightarrow Q$ )  $\leftrightarrow \neg(P \leftrightarrow \neg Q)$ ) with
3674 (( $(P \leftrightarrow Q) \vee (P \leftrightarrow \neg Q)$ )  $\wedge \neg((P \leftrightarrow Q) \wedge (P \leftrightarrow \neg Q))$ ).
3675 apply H.
3676 apply propositional_extensionality.

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3677     apply n5_17a.
3678 Qed.
3679
3680 Theorem n5_19 :  $\forall P : \text{Prop},$ 
3681    $\neg(P \leftrightarrow \neg P).$ 
3682 Proof. intros P.
3683   specialize n5_18 with P P.
3684   intros n5_18a.
3685   specialize n4_2 with P.
3686   intros n4_2a.
3687   replace ( $\neg(P \leftrightarrow \neg P)$ ) with ( $P \leftrightarrow P$ ).
3688   apply n4_2a.
3689   apply propositional_extensionality.
3690   apply n5_18a.
3691 Qed.
3692
3693 Theorem n5_21 :  $\forall P Q : \text{Prop},$ 
3694    $(\neg P \wedge \neg Q) \rightarrow (P \leftrightarrow Q).$ 
3695 Proof. intros P Q.
3696   specialize n5_1 with ( $\neg P$ ) ( $\neg Q$ ).
3697   intros n5_1a.
3698   specialize Transp4_11 with P Q.
3699   intros Transp4_11a.
3700   replace ( $\neg P \leftrightarrow \neg Q$ ) with ( $P \leftrightarrow Q$ ) in n5_1a.
3701   apply n5_1a.
3702   apply propositional_extensionality.
3703   apply Transp4_11a.
3704 Qed.
3705
3706 Theorem n5_22 :  $\forall P Q : \text{Prop},$ 
3707    $\neg(P \leftrightarrow Q) \leftrightarrow ((P \wedge \neg Q) \vee (Q \wedge \neg P)).$ 
3708 Proof. intros P Q.
3709   specialize n4_61 with P Q.
3710   intros n4_61a.
3711   specialize n4_61 with Q P.
3712   intros n4_61b.
3713   Conj n4_61a n4_61b.
3714   split.
3715   apply n4_61a.
3716   apply n4_61b.
3717   specialize n4_39 with
3718     ( $\neg(P \rightarrow Q)$ ) ( $\neg(Q \rightarrow P)$ ) ( $P \wedge \neg Q$ ) ( $Q \wedge \neg P$ ).

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3719   intros n4_39a.
3720   MP n4_39a H.
3721   specialize n4_51 with (P→Q) (Q→P).
3722   intros n4_51a.
3723   replace (¬(P → Q) ∨ ¬(Q → P)) with
3724     (¬((P → Q) ∧ (Q → P))) in n4_39a.
3725   replace ((P → Q) ∧ (Q → P)) with
3726     (P↔Q) in n4_39a.
3727   apply n4_39a.
3728   apply Equiv4_01.
3729   apply propositional_extensionality.
3730   apply n4_51a.
3731   Qed.
3732
3733   Theorem n5_23 : ∀ P Q : Prop,
3734     (P ↔ Q) ↔ ((P ∧ Q) ∨ (¬P ∧ ¬Q)).
3735   Proof. intros P Q.
3736   specialize n5_18 with P Q.
3737   intros n5_18a.
3738   specialize n5_22 with P (¬Q).
3739   intros n5_22a.
3740   Conj n5_18a n5_22a.
3741   split.
3742   apply n5_18a.
3743   apply n5_22a.
3744   specialize n4_22 with (P↔Q) (¬(P↔¬Q))
3745     (P ∧ ¬¬Q ∨ ¬Q ∧ ¬P).
3746   intros n4_22a.
3747   MP n4_22a H.
3748   replace (¬¬Q) with Q in n4_22a.
3749   replace (¬Q ∧ ¬P) with (¬P ∧ ¬Q) in n4_22a.
3750   apply n4_22a.
3751   apply propositional_extensionality.
3752   specialize n4_3 with (¬P) (¬Q).
3753   intros n4_3a.
3754   apply n4_3a. (*with (¬P) (¬Q)*)
3755   apply propositional_extensionality.
3756   specialize n4_13 with Q.
3757   intros n4_13a.
3758   apply n4_13a.
3759   Qed.
3760   (*The proof sketch in Principia offers n4_36.

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3761      This seems to be a misprint. We used n4_3.*)
3762
3763 Theorem n5_24 :  $\forall P Q : \text{Prop},$ 
3764    $\neg((P \wedge Q) \vee (\neg P \wedge \neg Q)) \leftrightarrow ((P \wedge \neg Q) \vee (Q \wedge \neg P)).$ 
3765 Proof. intros P Q.
3766   specialize n5_22 with P Q.
3767   intros n5_22a.
3768   specialize n5_23 with P Q.
3769   intros n5_23a.
3770   replace ((P $\leftrightarrow$ Q) $\leftrightarrow$ ((P $\wedge$  Q)  $\vee$ ( $\neg$ P  $\wedge$   $\neg$ Q))) with
3771     (( $\neg$ (P $\leftrightarrow$ Q) $\leftrightarrow$  $\neg$ ((P $\wedge$  Q) $\vee$ ( $\neg$ P  $\wedge$   $\neg$ Q)))) in n5_23a.
3772   replace ( $\neg$ (P $\leftrightarrow$ Q)) with
3773     ( $\neg((P \wedge Q) \vee (\neg P \wedge \neg Q))$ ) in n5_22a.
3774   apply n5_22a.
3775   replace ( $\neg((P \wedge Q) \vee (\neg P \wedge \neg Q))$ ) with ( $\neg$ (P $\leftrightarrow$ Q)).
3776   reflexivity.
3777   apply propositional_extensionality.
3778   apply n5_23a.
3779   replace ( $\neg$ (P  $\leftrightarrow$  Q)  $\leftrightarrow$   $\neg$ (P  $\wedge$  Q  $\vee$   $\neg$ P  $\wedge$   $\neg$ Q)) with
3780     ((P  $\leftrightarrow$  Q)  $\leftrightarrow$  P  $\wedge$  Q  $\vee$   $\neg$ P  $\wedge$   $\neg$ Q).
3781   reflexivity.
3782   specialize Transp4_11 with
3783     (P $\leftrightarrow$ Q) (P  $\wedge$  Q  $\vee$   $\neg$ P  $\wedge$   $\neg$ Q).
3784   intros Transp4_11a.
3785   apply propositional_extensionality.
3786   apply Transp4_11a. (*Not cited*)
3787 Qed.
3788
3789 Theorem n5_25 :  $\forall P Q : \text{Prop},$ 
3790   (P  $\vee$  Q)  $\leftrightarrow$  ((P  $\rightarrow$  Q)  $\rightarrow$  Q).
3791 Proof. intros P Q.
3792   specialize n2_62 with P Q.
3793   intros n2_62a.
3794   specialize n2_68 with P Q.
3795   intros n2_68a.
3796   Conj n2_62a n2_68a.
3797   split.
3798   apply n2_62a.
3799   apply n2_68a.
3800   Equiv H.
3801   apply H.
3802   apply Equiv4_01.

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3803 Qed.
3804
3805 Theorem n5_3 :  $\forall$  P Q R : Prop,
3806    $((P \wedge Q) \rightarrow R) \leftrightarrow ((P \wedge Q) \rightarrow (P \wedge R))$ .
3807 Proof. intros P Q R.
3808   specialize Comp3_43 with (P  $\wedge$  Q) P R.
3809   intros Comp3_43a.
3810   specialize Exp3_3 with
3811     (P  $\wedge$  Q  $\rightarrow$  P) (P  $\wedge$  Q  $\rightarrow$  R) (P  $\wedge$  Q  $\rightarrow$  P  $\wedge$  R).
3812   intros Exp3_3a. (*Not cited*)
3813   MP Exp3_3a Comp3_43a.
3814   specialize Simp3_26 with P Q.
3815   intros Simp3_26a.
3816   MP Exp3_3a Simp3_26a.
3817   specialize Syll2_05 with (P  $\wedge$  Q) (P  $\wedge$  R) R.
3818   intros Syll2_05a.
3819   specialize Simp3_27 with P R.
3820   intros Simp3_27a.
3821   MP Syll2_05a Simp3_27a.
3822   clear Comp3_43a. clear Simp3_27a.
3823   clear Simp3_26a.
3824   Conj Exp3_3a Syll2_05a.
3825   split.
3826   apply Exp3_3a.
3827   apply Syll2_05a.
3828   Equiv H.
3829   apply H.
3830   apply Equiv4_01.
3831 Qed.
3832
3833 Theorem n5_31 :  $\forall$  P Q R : Prop,
3834    $(R \wedge (P \rightarrow Q)) \rightarrow (P \rightarrow (Q \wedge R))$ .
3835 Proof. intros P Q R.
3836   specialize Comp3_43 with P Q R.
3837   intros Comp3_43a.
3838   specialize Simp2_02 with P R.
3839   intros Simp2_02a.
3840   specialize Exp3_3 with
3841     (P  $\rightarrow$  R) (P  $\rightarrow$  Q) (P  $\rightarrow$  (Q  $\wedge$  R)).
3842   intros Exp3_3a. (*Not cited*)
3843   specialize n3_22 with (P  $\rightarrow$  R) (P  $\rightarrow$  Q). (*Not cited*)
3844   intros n3_22a.

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3845 Syll n3_22a Comp3_43a Sa.
3846 MP Exp3_3a Sa.
3847 Syll Simp2_02a Exp3_3a Sb.
3848 specialize Imp3_31 with R (P→Q) (P→(Q ∧ R)).
3849 intros Imp3_31a. (*Not cited*)
3850 MP Imp3_31a Sb.
3851 apply Imp3_31a.
3852 Qed.
3853
3854 Theorem n5_32 : ∀ P Q R : Prop,
3855   (P → (Q ↔ R)) ↔ ((P ∧ Q) ↔ (P ∧ R)).
3856 Proof. intros P Q R.
3857 specialize n4_76 with P (Q→R) (R→Q).
3858 intros n4_76a.
3859 specialize Exp3_3 with P Q R.
3860 intros Exp3_3a.
3861 specialize Imp3_31 with P Q R.
3862 intros Imp3_31a.
3863 Conj Exp3_3a Imp3_31a.
3864 split.
3865 apply Exp3_3a.
3866 apply Imp3_31a.
3867 Equiv H.
3868 specialize Exp3_3 with P R Q.
3869 intros Exp3_3b.
3870 specialize Imp3_31 with P R Q.
3871 intros Imp3_31b.
3872 Conj Exp3_3b Imp3_31b.
3873 split.
3874 apply Exp3_3b.
3875 apply Imp3_31b.
3876 Equiv H0.
3877 specialize n5_3 with P Q R.
3878 intros n5_3a.
3879 specialize n5_3 with P R Q.
3880 intros n5_3b.
3881 replace (P→Q→R) with (P∧Q→R) in n4_76a.
3882 replace (P∧Q→R) with (P∧Q→P∧R) in n4_76a.
3883 replace (P→R→Q) with (P∧R→Q) in n4_76a.
3884 replace (P∧R→Q) with (P∧R→P∧Q) in n4_76a.
3885 replace ((P∧Q→P∧R)∧(P∧R→P∧Q)) with
3886   ((P∧Q)↔(P∧R)) in n4_76a.

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3887   replace ((P ∧ Q ↔ P ∧ R) ↔ (P → (Q → R) ∧ (R → Q))) with
3888         ((P → (Q → R) ∧ (R → Q)) ↔ (P ∧ Q ↔ P ∧ R)) in n4_76a.
3889   replace ((Q → R) ∧ (R → Q)) with (Q ↔ R) in n4_76a.
3890   apply n4_76a.
3891   apply Equiv4_01.
3892   apply propositional_extensionality.
3893   specialize n4_21 with
3894         (P → ((Q → R) ∧ (R → Q))) ((P ∧ Q) ↔ (P ∧ R)).
3895   intros n4_21a.
3896   apply n4_21a. (*to commute the biconditional*)
3897   apply Equiv4_01.
3898   replace (P ∧ R → P ∧ Q) with (P ∧ R → Q).
3899   reflexivity.
3900   apply propositional_extensionality.
3901   apply n5_3b.
3902   apply propositional_extensionality.
3903   apply H0.
3904   replace (P ∧ Q → P ∧ R) with (P ∧ Q → R).
3905   reflexivity.
3906   apply propositional_extensionality.
3907   apply n5_3a.
3908   apply propositional_extensionality.
3909   apply H.
3910   apply Equiv4_01.
3911   apply Equiv4_01.
3912   Qed.
3913
3914   Theorem n5_33 : ∀ P Q R : Prop,
3915     (P ∧ (Q → R)) ↔ (P ∧ ((P ∧ Q) → R)).
3916   Proof. intros P Q R.
3917     specialize n5_32 with P (Q → R) ((P ∧ Q) → R).
3918     intros n5_32a.
3919     replace
3920       ((P → (Q → R) ↔ (P ∧ Q → R)) ↔ (P ∧ (Q → R) ↔ P ∧ (P ∧ Q → R)))
3921       with
3922       (((P → (Q → R) ↔ (P ∧ Q → R)) → (P ∧ (Q → R) ↔ P ∧ (P ∧ Q → R)))
3923        ∧
3924        ((P ∧ (Q → R) ↔ P ∧ (P ∧ Q → R) → (P → (Q → R) ↔ (P ∧ Q → R))))))
3925       in n5_32a.
3926     specialize Simp3_26 with
3927       ((P → (Q → R) ↔ (P ∧ Q → R)) → (P ∧ (Q → R) ↔ P ∧ (P ∧ Q → R)))
3928       ((P ∧ (Q → R) ↔ P ∧ (P ∧ Q → R) → (P → (Q → R) ↔ (P ∧ Q → R)))).

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3929     intros Simp3_26a. (*Not cited*)
3930     MP Simp3_26a n5_32a.
3931     specialize n4_73 with Q P.
3932     intros n4_73a.
3933     specialize n4_84 with Q (Q $\wedge$ P) R.
3934     intros n4_84a.
3935     Syll n4_73a n4_84a Sa.
3936     replace (Q $\wedge$ P) with (P $\wedge$ Q) in Sa.
3937     MP Simp3_26a Sa.
3938     apply Simp3_26a.
3939     apply propositional_extensionality.
3940     specialize n4_3 with P Q.
3941     intros n4_3a.
3942     apply n4_3a. (*Not cited*)
3943     apply Equiv4_01.
3944 Qed.
3945
3946 Theorem n5_35 :  $\forall$  P Q R : Prop,
3947   ((P  $\rightarrow$  Q)  $\wedge$  (P  $\rightarrow$  R))  $\rightarrow$  (P  $\rightarrow$  (Q  $\leftrightarrow$  R)).
3948 Proof. intros P Q R.
3949     specialize Comp3_43 with P Q R.
3950     intros Comp3_43a.
3951     specialize n5_1 with Q R.
3952     intros n5_1a.
3953     specialize Syll2_05 with P (Q $\wedge$ R) (Q $\leftrightarrow$ R).
3954     intros Syll2_05a.
3955     MP Syll2_05a n5_1a.
3956     Syll Comp3_43a Syll2_05a Sa.
3957     apply Sa.
3958 Qed.
3959
3960 Theorem n5_36 :  $\forall$  P Q : Prop,
3961   (P  $\wedge$  (P  $\leftrightarrow$  Q))  $\leftrightarrow$  (Q  $\wedge$  (P  $\leftrightarrow$  Q)).
3962 Proof. intros P Q.
3963     specialize n5_32 with (P $\leftrightarrow$ Q) P Q.
3964     intros n5_32a.
3965     specialize Id2_08 with (P $\leftrightarrow$ Q).
3966     intros Id2_08a.
3967     replace (P $\leftrightarrow$ Q $\rightarrow$ P $\leftrightarrow$ Q) with
3968       ((P $\leftrightarrow$ Q) $\wedge$ P $\leftrightarrow$ (P $\leftrightarrow$ Q) $\wedge$ Q) in Id2_08a.
3969     replace ((P $\leftrightarrow$ Q) $\wedge$ P) with (P $\wedge$ (P $\leftrightarrow$ Q)) in Id2_08a.
3970     replace ((P $\leftrightarrow$ Q) $\wedge$ Q) with (Q $\wedge$ (P $\leftrightarrow$ Q)) in Id2_08a.

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3971   apply Id2_08a.
3972   apply propositional_extensionality.
3973   specialize n4_3 with Q (P↔Q).
3974   intros n4_3a.
3975   apply n4_3a.
3976   apply propositional_extensionality.
3977   specialize n4_3 with P (P↔Q).
3978   intros n4_3b.
3979   apply n4_3b.
3980   replace ((P ↔ Q) ∧ P ↔ (P ↔ Q) ∧ Q) with
3981     (P ↔ Q → P ↔ Q).
3982   reflexivity.
3983   apply propositional_extensionality.
3984   apply n5_32a.
3985   Qed.
3986   (*The proof sketch cites Ass3_35 and n4_38,
3987     but the sketch was indecipherable.*)
3988
3989   Theorem n5_4 : ∀ P Q : Prop,
3990     (P → (P → Q)) ↔ (P → Q).
3991   Proof. intros P Q.
3992     specialize n2_43 with P Q.
3993     intros n2_43a.
3994     specialize Simp2_02 with (P) (P→Q).
3995     intros Simp2_02a.
3996     Conj n2_43a Simp2_02a.
3997     split.
3998     apply n2_43a.
3999     apply Simp2_02a.
4000     Equiv H.
4001     apply H.
4002     apply Equiv4_01.
4003   Qed.
4004
4005   Theorem n5_41 : ∀ P Q R : Prop,
4006     ((P → Q) → (P → R)) ↔ (P → Q → R).
4007   Proof. intros P Q R.
4008     specialize n2_86 with P Q R.
4009     intros n2_86a.
4010     specialize n2_77 with P Q R.
4011     intros n2_77a.
4012     Conj n2_86a n2_77a.

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4013   split.
4014   apply n2_86a.
4015   apply n2_77a.
4016   Equiv H.
4017   apply H.
4018   apply Equiv4_01.
4019   Qed.
4020
4021   Theorem n5_42 :  $\forall$  P Q R : Prop,
4022     (P  $\rightarrow$  Q  $\rightarrow$  R)  $\leftrightarrow$  (P  $\rightarrow$  Q  $\rightarrow$  P  $\wedge$  R).
4023   Proof. intros P Q R.
4024     specialize n5_3 with P Q R.
4025     intros n5_3a.
4026     specialize n4_87 with P Q R.
4027     intros n4_87a.
4028     replace ((P $\wedge$ Q) $\rightarrow$ R) with (P $\rightarrow$ Q $\rightarrow$ R) in n5_3a.
4029     specialize n4_87 with P Q (P $\wedge$ R).
4030     intros n4_87b.
4031     replace ((P $\wedge$ Q) $\rightarrow$ (P $\wedge$ R)) with
4032       (P $\rightarrow$ Q $\rightarrow$ (P $\wedge$ R)) in n5_3a.
4033     apply n5_3a.
4034     specialize Imp3_31 with P Q (P $\wedge$ R).
4035     intros Imp3_31b.
4036     specialize Exp3_3 with P Q (P $\wedge$ R).
4037     intros Exp3_3b.
4038     Conj Imp3_31b Exp3_3b.
4039     split.
4040     apply Imp3_31b.
4041     apply Exp3_3b.
4042     Equiv H.
4043     apply propositional_extensionality.
4044     apply H.
4045     apply Equiv4_01.
4046     specialize Imp3_31 with P Q R.
4047     intros Imp3_31a.
4048     specialize Exp3_3 with P Q R.
4049     intros Exp3_3a.
4050     Conj Imp3_31a Exp3_3.
4051     split.
4052     apply Imp3_31a.
4053     apply Exp3_3a.
4054     Equiv H.

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4055   apply propositional_extensionality.
4056   apply H.
4057   apply Equiv4_01.
4058 Qed.
4059
4060 Theorem n5_44 :  $\forall$  P Q R : Prop,
4061    $(P \rightarrow Q) \rightarrow ((P \rightarrow R) \leftrightarrow (P \rightarrow (Q \wedge R)))$ .
4062 Proof. intros P Q R.
4063 specialize n4_76 with P Q R.
4064 intros n4_76a.
4065 rewrite Equiv4_01 in n4_76a.
4066 specialize Simp3_26 with
4067    $((P \rightarrow Q) \wedge (P \rightarrow R)) \rightarrow (P \rightarrow (Q \wedge R))$ 
4068    $((P \rightarrow (Q \wedge R)) \rightarrow ((P \rightarrow Q) \wedge (P \rightarrow R)))$ .
4069 intros Simp3_26a.
4070 MP Simp3_26a n4_76a.
4071 specialize Simp3_27 with
4072    $((P \rightarrow Q) \wedge (P \rightarrow R)) \rightarrow (P \rightarrow (Q \wedge R))$ 
4073    $((P \rightarrow (Q \wedge R)) \rightarrow ((P \rightarrow Q) \wedge (P \rightarrow R)))$ .
4074 intros Simp3_27a.
4075 MP Simp3_27a n4_76a.
4076 specialize Simp3_27 with  $(P \rightarrow Q)$   $(P \rightarrow Q \wedge R)$ .
4077 intros Simp3_27d.
4078 Syll Simp3_27d Simp3_27a Sa.
4079 specialize n5_3 with  $(P \rightarrow Q)$   $(P \rightarrow R)$   $(P \rightarrow (Q \wedge R))$ .
4080 intros n5_3a.
4081 rewrite Equiv4_01 in n5_3a.
4082 specialize Simp3_26 with
4083    $((((P \rightarrow Q) \wedge (P \rightarrow R)) \rightarrow (P \rightarrow (Q \wedge R))) \rightarrow$ 
4084    $((P \rightarrow Q) \wedge (P \rightarrow R)) \rightarrow ((P \rightarrow Q) \wedge (P \rightarrow (Q \wedge R))))$ 
4085    $((((P \rightarrow Q) \wedge (P \rightarrow R)) \rightarrow ((P \rightarrow Q) \wedge (P \rightarrow (Q \wedge R))))$ 
4086    $\rightarrow ((P \rightarrow Q) \wedge (P \rightarrow R)) \rightarrow (P \rightarrow (Q \wedge R)))$ .
4087 intros Simp3_26b.
4088 MP Simp3_26b n5_3a.
4089 specialize Simp3_27 with
4090    $((((P \rightarrow Q) \wedge (P \rightarrow R)) \rightarrow (P \rightarrow (Q \wedge R))) \rightarrow$ 
4091    $((P \rightarrow Q) \wedge (P \rightarrow R)) \rightarrow ((P \rightarrow Q) \wedge (P \rightarrow (Q \wedge R))))$ 
4092    $((((P \rightarrow Q) \wedge (P \rightarrow R)) \rightarrow ((P \rightarrow Q) \wedge (P \rightarrow (Q \wedge R))))$ 
4093    $\rightarrow ((P \rightarrow Q) \wedge (P \rightarrow R)) \rightarrow (P \rightarrow (Q \wedge R)))$ .
4094 intros Simp3_27b.
4095 MP Simp3_27b n5_3a.
4096 MP Simp3_26a Simp3_26b.

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4097 MP Simp3_27a Simp3_27b.
4098 clear n4_76a. clear Simp3_26a. clear Simp3_27a.
4099 clear Simp3_27b. clear Simp3_27d. clear n5_3a.
4100 Conj Simp3_26b Sa.
4101 split.
4102 apply Sa.
4103 apply Simp3_26b.
4104 Equiv H.
4105 specialize n5_32 with (P→Q) (P→R) (P→(Q∧R)).
4106 intros n5_32a.
4107 rewrite Equiv4_01 in n5_32a.
4108 specialize Simp3_27 with
4109   (((P → Q) → (P → R) ↔ (P → Q ∧ R))
4110    → (P → Q) ∧ (P → R) ↔ (P → Q) ∧ (P → Q ∧ R))
4111   ((P → Q) ∧ (P → R) ↔ (P → Q) ∧ (P → Q ∧ R)
4112    → (P → Q) → (P → R) ↔ (P → Q ∧ R)).
4113 intros Simp3_27c.
4114 MP Simp3_27c n5_32a.
4115 replace (((P→Q)∧(P→(Q∧R)))↔((P→Q)∧(P→R)))
4116   with (((P→Q)∧(P→R))↔((P→Q)∧(P→(Q∧R)))) in H.
4117 MP Simp3_27c H.
4118 apply Simp3_27c.
4119 specialize n4_21 with
4120   ((P→Q)∧(P→R)) ((P→Q)∧(P→(Q∧R))).
4121 intros n4_21a.
4122 apply propositional_extensionality.
4123 apply n4_21a.
4124 apply Equiv4_01.
4125 Qed.
4126
4127 Theorem n5_5 : ∀ P Q : Prop,
4128   P → ((P → Q) ↔ Q).
4129 Proof. intros P Q.
4130 specialize Ass3_35 with P Q.
4131 intros Ass3_35a.
4132 specialize Exp3_3 with P (P→Q) Q.
4133 intros Exp3_3a.
4134 MP Exp3_3a Ass3_35a.
4135 specialize Simp2_02 with P Q.
4136 intros Simp2_02a.
4137 specialize Exp3_3 with P Q (P→Q).
4138 intros Exp3_3b.

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4139 specialize n3_42 with P Q (P→Q). (*Not cited*)
4140 intros n3_42a.
4141 MP n3_42a Simp2_02a.
4142 MP Exp3_3b n3_42a.
4143 clear n3_42a. clear Simp2_02a. clear Ass3_35a.
4144 Conj Exp3_3a Exp3_3b.
4145 split.
4146 apply Exp3_3a.
4147 apply Exp3_3b.
4148 specialize n3_47 with P P ((P→Q)→Q) (Q→(P→Q)).
4149 intros n3_47a.
4150 MP n3_47a H.
4151 replace (P∧P) with P in n3_47a.
4152 replace (((P→Q)→Q)∧(Q→(P→Q))) with
4153   ((P→Q)↔Q) in n3_47a.
4154 apply n3_47a.
4155 apply Equiv4_01.
4156 apply propositional_extensionality.
4157 specialize n4_24 with P.
4158 intros n4_24a. (*Not cited*)
4159 apply n4_24a.
4160 Qed.
4161
4162 Theorem n5_501 : ∀ P Q : Prop,
4163   P → (Q ↔ (P ↔ Q)).
4164 Proof. intros P Q.
4165 specialize n5_1 with P Q.
4166 intros n5_1a.
4167 specialize Exp3_3 with P Q (P↔Q).
4168 intros Exp3_3a.
4169 MP Exp3_3a n5_1a.
4170 specialize Ass3_35 with P Q.
4171 intros Ass3_35a.
4172 specialize Simp3_26 with (P∧(P→Q)) (Q→P).
4173 intros Simp3_26a. (*Not cited*)
4174 Syll Simp3_26a Ass3_35a Sa.
4175 replace ((P∧(P→Q))∧(Q→P)) with
4176   (P∧((P→Q)∧(Q→P))) in Sa.
4177 replace ((P→Q)∧(Q→P)) with (P↔Q) in Sa.
4178 specialize Exp3_3 with P (P↔Q) Q.
4179 intros Exp3_3b.
4180 MP Exp3_3b Sa.

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4181 clear n5_1a. clear Ass3_35a.
4182 clear Simp3_26a. clear Sa.
4183 Conj Exp3_3a Exp3_3b.
4184 split.
4185 apply Exp3_3a.
4186 apply Exp3_3b.
4187 specialize n4_76 with P (Q → (P ↔ Q)) ((P ↔ Q) → Q).
4188 intros n4_76a. (*Not cited*)
4189 replace ((P → Q → P ↔ Q) ∧ (P → P ↔ Q → Q)) with
4190 ((P → (Q → P ↔ Q) ∧ (P ↔ Q → Q))) in H.
4191 replace ((Q → (P ↔ Q)) ∧ ((P ↔ Q) → Q)) with
4192 (Q ↔ (P ↔ Q)) in H.
4193 apply H.
4194 apply Equiv4_01.
4195 replace (P → (Q → P ↔ Q) ∧ (P ↔ Q → Q)) with
4196 ((P → Q → P ↔ Q) ∧ (P → P ↔ Q → Q)).
4197 reflexivity.
4198 apply propositional_extensionality.
4199 apply n4_76a.
4200 apply Equiv4_01.
4201 replace (P ∧ (P → Q) ∧ (Q → P)) with
4202 ((P ∧ (P → Q)) ∧ (Q → P)).
4203 reflexivity.
4204 apply propositional_extensionality.
4205 specialize n4_32 with P (P → Q) (Q → P).
4206 intros n4_32a. (*Not cited*)
4207 apply n4_32a.
4208 Qed.
4209
4210 Theorem n5_53 : ∀ P Q R S : Prop,
4211 ((P ∨ Q) ∨ R) → S ↔ ((P → S) ∧ (Q → S)) ∧ (R → S)).
4212 Proof. intros P Q R S.
4213 specialize n4_77 with S (P ∨ Q) R.
4214 intros n4_77a.
4215 specialize n4_77 with S P Q.
4216 intros n4_77b.
4217 replace (P ∨ Q → S) with
4218 ((P → S) ∧ (Q → S)) in n4_77a.
4219 replace (((P → S) ∧ (Q → S)) ∧ (R → S)) ↔ (((P ∨ Q) ∨ R) → S))
4220 with
4221 (((P ∨ Q) ∨ R) → S) ↔ ((P → S) ∧ (Q → S)) ∧ (R → S))
4222 in n4_77a.

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4223   apply n4_77a.
4224   apply propositional_extensionality.
4225   specialize n4_21 with ((P ∨ Q) ∨ R → S)
4226       ((P → S) ∧ (Q → S)) ∧ (R → S)).
4227   intros n4_21a.
4228   apply n4_21a. (*Not cited*)
4229   apply propositional_extensionality.
4230   apply n4_77b.
4231   Qed.
4232
4233   Theorem n5_54 : ∀ P Q : Prop,
4234       ((P ∧ Q) ↔ P) ∨ ((P ∧ Q) ↔ Q).
4235   Proof. intros P Q.
4236   specialize n4_73 with P Q.
4237   intros n4_73a.
4238   specialize n4_44 with Q P.
4239   intros n4_44a.
4240   specialize Transp2_16 with Q (P ↔ (P ∧ Q)).
4241   intros Transp2_16a.
4242   MP n4_73a Transp2_16a.
4243   specialize Transp4_11 with Q (Q ∨ (P ∧ Q)).
4244   intros Transp4_11a.
4245   replace (Q ∧ P) with (P ∧ Q) in n4_44a.
4246   replace (Q ↔ Q ∨ P ∧ Q) with
4247       (¬Q ↔ ¬(Q ∨ P ∧ Q)) in n4_44a.
4248   replace (¬Q) with (¬(Q ∨ P ∧ Q)) in Transp2_16a.
4249   replace (¬(Q ∨ P ∧ Q)) with
4250       (¬Q ∧ ¬(P ∧ Q)) in Transp2_16a.
4251   specialize n5_1 with (¬Q) (¬(P ∧ Q)).
4252   intros n5_1a.
4253   Syll Transp2_16a n5_1a Sa.
4254   replace (¬(P ↔ P ∧ Q) → (¬Q ↔ ¬(P ∧ Q))) with
4255       (¬¬(P ↔ P ∧ Q) ∨ (¬Q ↔ ¬(P ∧ Q))) in Sa.
4256   replace (¬¬(P ↔ P ∧ Q)) with (P ↔ P ∧ Q) in Sa.
4257   specialize Transp4_11 with Q (P ∧ Q).
4258   intros Transp4_11b.
4259   replace (¬Q ↔ ¬(P ∧ Q)) with (Q ↔ (P ∧ Q)) in Sa.
4260   replace (Q ↔ (P ∧ Q)) with ((P ∧ Q) ↔ Q) in Sa.
4261   replace (P ↔ (P ∧ Q)) with ((P ∧ Q) ↔ P) in Sa.
4262   apply Sa.
4263   apply propositional_extensionality.
4264   specialize n4_21 with (P ∧ Q) P.

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4265   intros n4_21a. (*Not cited*)
4266   apply n4_21a.
4267   apply propositional_extensionality.
4268   specialize n4_21 with (P $\wedge$ Q) Q.
4269   intros n4_21b. (*Not cited*)
4270   apply n4_21b.
4271   apply propositional_extensionality.
4272   apply Transp4_11b.
4273   apply propositional_extensionality.
4274   specialize n4_13 with (P  $\leftrightarrow$  (P $\wedge$ Q)).
4275   intros n4_13a. (*Not cited*)
4276   apply n4_13a.
4277   rewrite <- Impl1_01. (*Not cited*)
4278   reflexivity.
4279   apply propositional_extensionality.
4280   specialize n4_56 with Q (P $\wedge$ Q).
4281   intros n4_56a. (*Not cited*)
4282   apply n4_56a.
4283   replace ( $\neg$ (Q $\vee$ P $\wedge$ Q)) with ( $\neg$ Q).
4284   reflexivity.
4285   apply propositional_extensionality.
4286   apply n4_44a.
4287   replace ( $\neg$ Q $\leftrightarrow$  $\neg$ (Q $\vee$ P $\wedge$ Q)) with (Q $\leftrightarrow$ Q $\vee$ P $\wedge$ Q).
4288   reflexivity.
4289   apply propositional_extensionality.
4290   apply Transp4_11a.
4291   apply propositional_extensionality.
4292   specialize n4_3 with P Q.
4293   intros n4_3a. (*Not cited*)
4294   apply n4_3a.
4295   Qed.
4296
4297   Theorem n5_55 :  $\forall$  P Q : Prop,
4298     ((P  $\vee$  Q)  $\leftrightarrow$  P)  $\vee$  ((P  $\vee$  Q)  $\leftrightarrow$  Q).
4299   Proof. intros P Q.
4300   specialize Add1_3 with (P $\wedge$ Q) (P).
4301   intros Add1_3a.
4302   replace ((P $\wedge$ Q) $\vee$ P) with ((P $\vee$ P) $\wedge$ (Q $\vee$ P)) in Add1_3a.
4303   replace (P $\vee$ P) with P in Add1_3a.
4304   replace (Q $\vee$ P) with (P $\vee$ Q) in Add1_3a.
4305   specialize n5_1 with P (P $\vee$ Q).
4306   intros n5_1a.

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4307 Syll Add1_3a n5_1a Sa.
4308 specialize n4_74 with P Q.
4309 intros n4_74a.
4310 specialize Transp2_15 with P (Q $\leftrightarrow$ P $\vee$ Q).
4311 intros Transp2_15a. (*Not cited*)
4312 MP Transp2_15a n4_74a.
4313 Syll Transp2_15a Sa Sb.
4314 replace ( $\neg$ (Q $\leftrightarrow$ (P $\vee$ Q)) $\rightarrow$ (P $\leftrightarrow$ (P $\vee$ Q))) with
4315   ( $\neg$ (Q $\leftrightarrow$ (P $\vee$ Q)) $\vee$ (P $\leftrightarrow$ (P $\vee$ Q))) in Sb.
4316 replace ( $\neg$ (Q $\leftrightarrow$ (P $\vee$ Q))) with (Q $\leftrightarrow$ (P $\vee$ Q)) in Sb.
4317 replace (Q $\leftrightarrow$ (P $\vee$ Q)) with ((P $\vee$ Q) $\leftrightarrow$ Q) in Sb.
4318 replace (P $\leftrightarrow$ (P $\vee$ Q)) with ((P $\vee$ Q) $\leftrightarrow$ P) in Sb.
4319 replace ((P $\vee$ Q $\leftrightarrow$ Q) $\vee$ (P $\vee$ Q $\leftrightarrow$ P)) with
4320   ((P $\vee$ Q $\leftrightarrow$ P) $\vee$ (P $\vee$ Q $\leftrightarrow$ Q)) in Sb.
4321 apply Sb.
4322 apply propositional_extensionality.
4323 specialize n4_31 with (P  $\vee$  Q  $\leftrightarrow$  P) (P  $\vee$  Q  $\leftrightarrow$  Q).
4324 intros n4_31a. (*Not cited*)
4325 apply n4_31a.
4326 apply propositional_extensionality.
4327 specialize n4_21 with (P  $\vee$  Q) P.
4328 intros n4_21a. (*Not cited*)
4329 apply n4_21a.
4330 apply propositional_extensionality.
4331 specialize n4_21 with (P  $\vee$  Q) Q.
4332 intros n4_21b. (*Not cited*)
4333 apply n4_21b.
4334 apply propositional_extensionality.
4335 specialize n4_13 with (Q  $\leftrightarrow$  P  $\vee$  Q).
4336 intros n4_13a. (*Not cited*)
4337 apply n4_13a.
4338 rewrite <- Impl1_01.
4339 reflexivity.
4340 apply propositional_extensionality.
4341 specialize n4_31 with P Q.
4342 intros n4_31b.
4343 apply n4_31b.
4344 apply propositional_extensionality.
4345 specialize n4_25 with P.
4346 intros n4_25a. (*Not cited*)
4347 apply n4_25a.
4348 replace ((P $\vee$ P) $\wedge$ (Q $\vee$ P)) with ((P $\wedge$ Q) $\vee$ P).

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4349 reflexivity.
4350 replace ((P $\wedge$ Q) $\vee$ P) with (P $\vee$ (P $\wedge$ Q)).
4351 replace (Q $\vee$ P) with (P $\vee$ Q).
4352 apply propositional_extensionality.
4353 specialize n4_41 with P P Q.
4354 intros n4_41a. (*Not cited*)
4355 apply n4_41a.
4356 apply propositional_extensionality.
4357 specialize n4_31 with P Q.
4358 intros n4_31c.
4359 apply n4_31c.
4360 apply propositional_extensionality.
4361 specialize n4_31 with P (P  $\wedge$  Q).
4362 intros n4_31d. (*Not cited*)
4363 apply n4_31d.
4364 Qed.
4365
4366 Theorem n5_6 :  $\forall$  P Q R : Prop,
4367   ((P  $\wedge$   $\neg$ Q)  $\rightarrow$  R)  $\leftrightarrow$  (P  $\rightarrow$  (Q  $\vee$  R)).
4368 Proof. intros P Q R.
4369 specialize n4_87 with P ( $\neg$ Q) R.
4370 intros n4_87a.
4371 specialize n4_64 with Q R.
4372 intros n4_64a.
4373 specialize n4_85 with P Q R.
4374 intros n4_85a.
4375 replace (((P $\wedge$  $\neg$ Q $\rightarrow$ R) $\leftrightarrow$ (P $\rightarrow$  $\neg$ Q $\rightarrow$ R)) $\leftrightarrow$ (( $\neg$ Q $\rightarrow$ P $\rightarrow$ R) $\leftrightarrow$ ( $\neg$ Q $\wedge$ P $\rightarrow$ R)))
4376   with
4377     (((P $\wedge$  $\neg$ Q $\rightarrow$ R) $\leftrightarrow$ (P $\rightarrow$  $\neg$ Q $\rightarrow$ R)) $\rightarrow$ (( $\neg$ Q $\rightarrow$ P $\rightarrow$ R) $\leftrightarrow$ ( $\neg$ Q $\wedge$ P $\rightarrow$ R)))
4378      $\wedge$ 
4379     (((( $\neg$ Q $\rightarrow$ P $\rightarrow$ R) $\leftrightarrow$ ( $\neg$ Q $\wedge$ P $\rightarrow$ R))) $\rightarrow$ ((P $\wedge$  $\neg$ Q $\rightarrow$ R) $\leftrightarrow$ (P $\rightarrow$  $\neg$ Q $\rightarrow$ R))))
4380   in n4_87a.
4381 specialize Simp3_27 with
4382   (((P $\wedge$  $\neg$ Q $\rightarrow$ R) $\leftrightarrow$ (P $\rightarrow$  $\neg$ Q $\rightarrow$ R) $\rightarrow$ ( $\neg$ Q $\rightarrow$ P $\rightarrow$ R) $\leftrightarrow$ ( $\neg$ Q $\wedge$ P $\rightarrow$ R)))
4383   ((( $\neg$ Q $\rightarrow$ P $\rightarrow$ R) $\leftrightarrow$ ( $\neg$ Q $\wedge$ P $\rightarrow$ R) $\rightarrow$ (P $\wedge$  $\neg$ Q $\rightarrow$ R) $\leftrightarrow$ (P $\rightarrow$  $\neg$ Q $\rightarrow$ R))).
4384 intros Simp3_27a.
4385 MP Simp3_27a n4_87a.
4386 specialize Imp3_31 with ( $\neg$ Q) P R.
4387 intros Imp3_31a.
4388 specialize Exp3_3 with ( $\neg$ Q) P R.
4389 intros Exp3_3a.
4390 Conj Imp3_31a Exp3_3a.

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4391 split.
4392 apply Imp3_31a.
4393 apply Exp3_3a.
4394 Equiv H.
4395 MP Simp3_27a H.
4396 replace ( $\neg Q \rightarrow R$ ) with ( $Q \vee R$ ) in Simp3_27a.
4397 apply Simp3_27a.
4398 replace ( $Q \vee R$ ) with ( $\neg Q \rightarrow R$ ).
4399 reflexivity.
4400 apply propositional_extensionality.
4401 apply n4_64a.
4402 apply Equiv4_01.
4403 apply Equiv4_01.
4404 Qed.
4405
4406 Theorem n5_61 :  $\forall P Q : \text{Prop}$ ,
4407   ( $(P \vee Q) \wedge \neg Q \leftrightarrow (P \wedge \neg Q)$ ).
4408 Proof. intros P Q.
4409 specialize n4_74 with Q P.
4410 intros n4_74a.
4411 specialize n5_32 with ( $\neg Q$ ) P ( $Q \vee P$ ).
4412 intros n5_32a.
4413 replace ( $\neg Q \rightarrow P \leftrightarrow Q \vee P$ ) with
4414   ( $\neg Q \wedge P \leftrightarrow \neg Q \wedge (Q \vee P)$ ) in n4_74a.
4415 replace ( $\neg Q \wedge P$ ) with ( $P \wedge \neg Q$ ) in n4_74a.
4416 replace ( $\neg Q \wedge (Q \vee P)$ ) with ( $(Q \vee P) \wedge \neg Q$ ) in n4_74a.
4417 replace ( $Q \vee P$ ) with ( $P \vee Q$ ) in n4_74a.
4418 replace ( $P \wedge \neg Q \leftrightarrow (P \vee Q) \wedge \neg Q$ ) with
4419   ( $(P \vee Q) \wedge \neg Q \leftrightarrow P \wedge \neg Q$ ) in n4_74a.
4420 apply n4_74a.
4421 apply propositional_extensionality.
4422 specialize n4_21 with ( $(P \vee Q) \wedge \neg Q$ ) ( $P \wedge \neg Q$ ).
4423 intros n4_21a. (*Not cited*)
4424 apply n4_21a.
4425 apply propositional_extensionality.
4426 specialize n4_31 with P Q.
4427 intros n4_31a. (*Not cited*)
4428 apply n4_31a.
4429 apply propositional_extensionality.
4430 specialize n4_3 with ( $Q \vee P$ ) ( $\neg Q$ ).
4431 intros n4_3a. (*Not cited*)
4432 apply n4_3a.

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4433   apply propositional_extensionality.
4434   specialize n4_3 with P (¬Q).
4435   intros n4_3b. (*Not cited*)
4436   apply n4_3b.
4437   replace (¬Q ∧ P ↔ ¬Q ∧ (Q ∨ P)) with
4438     (¬Q → P ↔ Q ∨ P).
4439   reflexivity.
4440   apply propositional_extensionality.
4441   apply n5_32a.
4442   Qed.
4443
4444   Theorem n5_62 : ∀ P Q : Prop,
4445     ((P ∧ Q) ∨ ¬Q) ↔ (P ∨ ¬Q).
4446   Proof. intros P Q.
4447     specialize n4_7 with Q P.
4448     intros n4_7a.
4449     replace (Q → P) with (¬Q ∨ P) in n4_7a.
4450     replace (Q → (Q ∧ P)) with (¬Q ∨ (Q ∧ P)) in n4_7a.
4451     replace (¬Q ∨ (Q ∧ P)) with ((Q ∧ P) ∨ ¬Q) in n4_7a.
4452     replace (¬Q ∨ P) with (P ∨ ¬Q) in n4_7a.
4453     replace (Q ∧ P) with (P ∧ Q) in n4_7a.
4454     replace (P ∨ ¬Q ↔ P ∧ Q ∨ ¬Q) with
4455       (P ∧ Q ∨ ¬Q ↔ P ∨ ¬Q) in n4_7a.
4456     apply n4_7a.
4457     apply propositional_extensionality.
4458     specialize n4_21 with (P ∧ Q ∨ ¬Q) (P ∨ ¬Q).
4459     intros n4_21a. (*Not cited*)
4460     apply n4_21a.
4461     apply propositional_extensionality.
4462     specialize n4_3 with P Q.
4463     intros n4_3a. (*Not cited*)
4464     apply n4_3a.
4465     apply propositional_extensionality.
4466     specialize n4_31 with P (¬Q).
4467     intros n4_31a. (*Not cited*)
4468     apply n4_31a.
4469     apply propositional_extensionality.
4470     specialize n4_31 with (Q ∧ P) (¬Q).
4471     intros n4_31b. (*Not cited*)
4472     apply n4_31b.
4473     rewrite <- Impl1_01.
4474     reflexivity.

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4475     rewrite <- Impl1_01.
4476     reflexivity.
4477 Qed.
4478
4479 Theorem n5_63 :  $\forall P Q : \text{Prop}$ ,
4480    $(P \vee Q) \leftrightarrow (P \vee (\neg P \wedge Q))$ .
4481 Proof. intros P Q.
4482   specialize n5_62 with Q ( $\neg P$ ).
4483   intros n5_62a.
4484   replace ( $\neg\neg P$ ) with P in n5_62a.
4485   replace  $(Q \vee P)$  with  $(P \vee Q)$  in n5_62a.
4486   replace  $((Q \wedge \neg P) \vee P)$  with  $(P \vee (Q \wedge \neg P))$  in n5_62a.
4487   replace  $(P \vee Q \wedge \neg P \leftrightarrow P \vee Q)$  with
4488      $(P \vee Q \leftrightarrow P \vee Q \wedge \neg P)$  in n5_62a.
4489   replace  $(Q \wedge \neg P)$  with  $(\neg P \wedge Q)$  in n5_62a.
4490   apply n5_62a.
4491   apply propositional_extensionality.
4492   specialize n4_3 with ( $\neg P$ ) Q.
4493   intros n4_3a.
4494   apply n4_3a. (*Not cited*)
4495   apply propositional_extensionality.
4496   specialize n4_21 with  $(P \vee Q) (P \vee (Q \wedge \neg P))$ .
4497   intros n4_21a. (*Not cited*)
4498   apply n4_21a.
4499   apply propositional_extensionality.
4500   specialize n4_31 with P  $(Q \wedge \neg P)$ .
4501   intros n4_31a. (*Not cited*)
4502   apply n4_31a.
4503   apply propositional_extensionality.
4504   specialize n4_31 with P Q.
4505   intros n4_31b. (*Not cited*)
4506   apply n4_31b.
4507   apply propositional_extensionality.
4508   specialize n4_13 with P.
4509   intros n4_13a. (*Not cited*)
4510   apply n4_13a.
4511 Qed.
4512
4513 Theorem n5_7 :  $\forall P Q R : \text{Prop}$ ,
4514    $((P \vee R) \leftrightarrow (Q \vee R)) \leftrightarrow (R \vee (P \leftrightarrow Q))$ .
4515 Proof. intros P Q R.
4516   specialize n4_74 with R P.

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4517  intros n4_74a.
4518  specialize n4_74 with R Q.
4519  intros n4_74b. (*Greg's suggestion*)
4520  Conj n4_74a n4_74b.
4521  split.
4522  apply n4_74a.
4523  apply n4_74b.
4524  specialize Comp3_43 with
4525    ( $\neg R$ ) ( $P \leftrightarrow R \vee P$ ) ( $Q \leftrightarrow R \vee Q$ ).
4526  intros Comp3_43a.
4527  MP Comp3_43a H.
4528  specialize n4_22 with P ( $R \vee P$ ) ( $R \vee Q$ ).
4529  intros n4_22a.
4530  specialize n4_22 with P ( $R \vee Q$ ) Q.
4531  intros n4_22b.
4532  specialize Exp3_3 with ( $P \leftrightarrow (R \vee Q)$ )
4533    ( $(R \vee Q) \leftrightarrow Q$ ) ( $P \leftrightarrow Q$ ).
4534  intros Exp3_3a.
4535  MP Exp3_3a n4_22b.
4536  Syll n4_22a Exp3_3a Sa.
4537  specialize Imp3_31 with (( $P \leftrightarrow (R \vee P)$ )  $\wedge$ 
4538    ( $(R \vee P) \leftrightarrow (R \vee Q)$ )) ( $(R \vee Q) \leftrightarrow Q$ ) ( $P \leftrightarrow Q$ ).
4539  intros Imp3_31a.
4540  MP Imp3_31a Sa.
4541  replace ((( $P \leftrightarrow (R \vee P)$ )  $\wedge$  ( $(R \vee P) \leftrightarrow$ 
4542    ( $R \vee Q$ )))  $\wedge$  ( $(R \vee Q) \leftrightarrow Q$ )) with
4543    (( $P \leftrightarrow (R \vee P)$ )  $\wedge$  ( $(R \vee P) \leftrightarrow$ 
4544    ( $R \vee Q$ ))  $\wedge$  ( $(R \vee Q) \leftrightarrow Q$ )) in Imp3_31a.
4545  replace (( $(R \vee P \leftrightarrow R \vee Q) \wedge (R \vee Q \leftrightarrow Q)$ ) with
4546    ( $(R \vee Q \leftrightarrow Q) \wedge (R \vee P \leftrightarrow R \vee Q)$ ) in Imp3_31a.
4547  replace (( $P \leftrightarrow (R \vee P)$ )  $\wedge$ 
4548    ( $(R \vee Q \leftrightarrow Q) \wedge (R \vee P \leftrightarrow R \vee Q)$ )) with
4549    ((( $P \leftrightarrow (R \vee P)$ )  $\wedge$  ( $R \vee Q \leftrightarrow Q$ ))  $\wedge$ 
4550    ( $R \vee P \leftrightarrow R \vee Q$ )) in Imp3_31a.
4551  specialize Exp3_3 with
4552    ( $(P \leftrightarrow (R \vee P)) \wedge (R \vee Q \leftrightarrow Q)$ )
4553    ( $(R \vee P \leftrightarrow R \vee Q) (P \leftrightarrow Q)$ ).
4554  intros Exp3_3b.
4555  MP Exp3_3b Imp3_31a.
4556  replace ( $Q \leftrightarrow R \vee Q$ ) with ( $R \vee Q \leftrightarrow Q$ ) in Comp3_43a.
4557  Syll Comp3_43a Exp3_3b Sb.
4558  replace ( $R \vee P$ ) with ( $P \vee R$ ) in Sb.

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4559 replace (R∨Q) with (Q∨R) in Sb.
4560 specialize Imp3_31 with (¬R) (P∨R↔Q∨R) (P↔Q).
4561 intros Imp3_31b.
4562 MP Imp3_31b Sb.
4563 replace (¬R ∧ (P ∨ R ↔ Q ∨ R)) with
4564   ((P ∨ R ↔ Q ∨ R) ∧ ¬R) in Imp3_31b.
4565 specialize Exp3_3 with
4566   (P ∨ R ↔ Q ∨ R) (¬R) (P ↔ Q).
4567 intros Exp3_3c.
4568 MP Exp3_3c Imp3_31b.
4569 replace (¬R→(P↔Q)) with
4570   (¬¬R∨(P↔Q)) in Exp3_3c.
4571 replace (¬¬R) with R in Exp3_3c.
4572 specialize Add1_3 with P R.
4573 intros Add1_3a.
4574 specialize Add1_3 with Q R.
4575 intros Add1_3b.
4576 Conj Add1_3a Add1_3b.
4577 split.
4578 apply Add1_3a.
4579 apply Add1_3b.
4580 specialize Comp3_43 with (R) (P∨R) (Q∨R).
4581 intros Comp3_43b.
4582 MP Comp3_43b H0.
4583 specialize n5_1 with (P ∨ R) (Q ∨ R).
4584 intros n5_1a.
4585 Syll Comp3_43b n5_1a Sc.
4586 specialize n4_37 with P Q R.
4587 intros n4_37a.
4588 Conj Sc n4_37a.
4589 split.
4590 apply Sc.
4591 apply n4_37a.
4592 specialize n4_77 with (P ∨ R ↔ Q ∨ R)
4593   R (P↔Q).
4594 intros n4_77a.
4595 rewrite Equiv4_01 in n4_77a.
4596 specialize Simp3_26 with
4597   ((R → P ∨ R ↔ Q ∨ R) ∧
4598     (P ↔ Q → P ∨ R ↔ Q ∨ R)
4599     → R ∨ (P ↔ Q) → P ∨ R ↔ Q ∨ R)
4600   ((R ∨ (P ↔ Q) → P ∨ R ↔ Q ∨ R)

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4601      → (R → P ∨ R ↔ Q ∨ R) ∧
4602      (P ↔ Q → P ∨ R ↔ Q ∨ R)).
4603 intros Simp3_26a.
4604 MP Simp3_26 n4_77a.
4605 MP Simp3_26a H1.
4606 clear n4_77a. clear H1. clear n4_37a. clear Sa.
4607   clear n5_1a. clear Comp3_43b. clear H0.
4608   clear Add1_3a. clear Add1_3b. clear H. clear Imp3_31b.
4609   clear n4_74a. clear n4_74b. clear Comp3_43a.
4610   clear Imp3_31a. clear n4_22a. clear n4_22b.
4611   clear Exp3_3a. clear Exp3_3b. clear Sb. clear Sc.
4612 Conj Exp3_3c Simp3_26a.
4613 split.
4614 apply Exp3_3c.
4615 apply Simp3_26a.
4616 Equiv H.
4617 apply H.
4618 apply Equiv4_01.
4619 apply propositional_extensionality.
4620 apply n4_13. (*With R*)
4621 rewrite <- Impl1_01. (*With (¬R) (P↔Q)*)
4622 reflexivity.
4623 apply propositional_extensionality.
4624 apply n4_3. (*With (R ∨ Q ↔ R ∨ P) (¬R)*)
4625 apply propositional_extensionality.
4626 apply n4_31. (*With P R*)
4627 apply propositional_extensionality.
4628 apply n4_31. (*With Q R*)
4629 apply propositional_extensionality.
4630 apply n4_21. (*With (P ∨ R) (Q ∨ R)*)
4631 apply propositional_extensionality.
4632 apply n4_32. (*With (P ↔ R ∨ P) (R ∨ Q ↔ Q) (R ∨ P ↔ R ∨ Q)*)
4633 apply propositional_extensionality.
4634 apply n4_3. (*With (R ∨ Q ↔ Q) (R ∨ P ↔ R ∨ Q)*)
4635 apply propositional_extensionality.
4636 apply n4_21. (*To commute the biconditional.*)
4637 apply n4_32. (*With (P ↔ R ∨ P) (R ∨ P ↔ R ∨ Q) (R ∨ Q ↔ Q)*)
4638 Qed.
4639
4640 Theorem n5_71 : ∀ P Q R : Prop,
4641   (Q → ¬R) → (((P ∨ Q) ∧ R) ↔ (P ∧ R)).
4642 Proof. intros P Q R.

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4643 specialize n4_62 with Q R.
4644 intros n4_62a.
4645 specialize n4_51 with Q R.
4646 intros n4_51a.
4647 specialize n4_21 with ( $\neg(Q \wedge R)$ ) ( $\neg Q \vee \neg R$ ).
4648 intros n4_21a.
4649 rewrite Equiv4_01 in n4_21a.
4650 specialize Simp3_26 with
4651   ( $(\neg(Q \wedge R) \leftrightarrow (\neg Q \vee \neg R)) \rightarrow ((\neg Q \vee \neg R) \leftrightarrow \neg(Q \wedge R))$ )
4652   ( $((\neg Q \vee \neg R) \leftrightarrow \neg(Q \wedge R)) \rightarrow (\neg(Q \wedge R) \leftrightarrow (\neg Q \vee \neg R)))$ ).
4653 intros Simp3_26a.
4654 MP Simp3_26a n4_21a.
4655 MP Simp3_26a n4_51a.
4656 clear n4_21a. clear n4_51a.
4657 Conj n4_62a Simp3_26a.
4658 split.
4659 apply n4_62a.
4660 apply Simp3_26a.
4661 specialize n4_22 with
4662   ( $(Q \rightarrow \neg R)$ ) ( $\neg Q \vee \neg R$ ) ( $\neg(Q \wedge R)$ ).
4663 intros n4_22a.
4664 MP n4_22a H.
4665 replace (( $(Q \rightarrow \neg R) \leftrightarrow \neg(Q \wedge R)$ ) with
4666   ( $((Q \rightarrow \neg R) \rightarrow \neg(Q \wedge R))$ 
4667    $\wedge$ 
4668   ( $\neg(Q \wedge R) \rightarrow (Q \rightarrow \neg R)$ )) in n4_22a.
4669 specialize Simp3_26 with
4670   ( $(Q \rightarrow \neg R) \rightarrow \neg(Q \wedge R)$ ) ( $\neg(Q \wedge R) \rightarrow (Q \rightarrow \neg R)$ ).
4671 intros Simp3_26b.
4672 MP Simp3_26b n4_22a.
4673 specialize n4_74 with ( $(Q \wedge R)$ ) ( $(P \wedge R)$ ).
4674 intros n4_74a.
4675 Syll Simp3_26a n4_74a Sa.
4676 replace (( $(P \wedge R) \vee (Q \wedge R)$ ) with
4677   ( $((Q \wedge R) \vee (P \wedge R))$  in Sa.
4678 replace (( $(Q \wedge R) \vee (P \wedge R)$ ) with ( $(R \wedge (P \vee Q))$  in Sa.
4679 replace ( $(R \wedge (P \vee Q))$  with ( $(P \vee Q) \wedge R$ ) in Sa.
4680 replace (( $(P \wedge R) \leftrightarrow ((P \vee Q) \wedge R)$ ) with
4681   ( $((P \vee Q) \wedge R) \leftrightarrow (P \wedge R)$ ) in Sa.
4682 apply Sa.
4683 apply propositional_extensionality.
4684 specialize n4_21 with (( $(P \vee Q) \wedge R$ ) ( $(P \wedge R)$ ).

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4685   intros n4_21a. (*Not cited*)
4686   apply n4_21a.
4687   apply propositional_extensionality.
4688   specialize n4_3 with (P $\vee$ Q) R.
4689   intros n4_3a.
4690   apply n4_3a. (*Not cited*)
4691   apply propositional_extensionality.
4692   specialize n4_4 with R P Q.
4693   intros n4_4a.
4694   replace ((Q $\wedge$ R) $\vee$ (P $\wedge$ R)) with ((P $\wedge$ R) $\vee$ (Q $\wedge$ R)).
4695   replace (Q  $\wedge$  R) with (R  $\wedge$  Q).
4696   replace (P  $\wedge$  R) with (R  $\wedge$  P).
4697   apply n4_4a. (*Not cited*)
4698   apply propositional_extensionality.
4699   specialize n4_3 with R P.
4700   intros n4_3a.
4701   apply n4_3a.
4702   apply propositional_extensionality.
4703   specialize n4_3 with R Q.
4704   intros n4_3b.
4705   apply n4_3b.
4706   apply propositional_extensionality.
4707   specialize n4_31 with (P $\wedge$ R) (Q $\wedge$ R).
4708   intros n4_31a. (*Not cited*)
4709   apply n4_31a.
4710   apply propositional_extensionality.
4711   specialize n4_31 with (Q $\wedge$ R) (P $\wedge$ R).
4712   intros n4_31b. (*Not cited*)
4713   apply n4_31b.
4714   apply Equiv4_01.
4715   Qed.
4716
4717   Theorem n5_74 :  $\forall$  P Q R : Prop,
4718     (P  $\rightarrow$  (Q  $\leftrightarrow$  R))  $\leftrightarrow$  ((P  $\rightarrow$  Q)  $\leftrightarrow$  (P  $\rightarrow$  R)).
4719   Proof. intros P Q R.
4720   specialize n5_41 with P Q R.
4721   intros n5_41a.
4722   specialize n5_41 with P R Q.
4723   intros n5_41b.
4724   Conj n5_41a n5_41b.
4725   split.
4726   apply n5_41a.

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4727 apply n5_41b.
4728 specialize n4_38 with
4729   ((P→Q)→(P→R)) ((P→R)→(P→Q))
4730   (P→Q→R) (P→R→Q).
4731 intros n4_38a.
4732 MP n4_38a H.
4733 replace (((P→Q)→(P→R))∧((P→R)→(P→Q)))
4734   with ((P→Q)↔(P→R)) in n4_38a.
4735 specialize n4_76 with P (Q→R) (R→Q).
4736 intros n4_76a.
4737 replace ((Q→R)∧(R→Q)) with (Q↔R) in n4_76a.
4738 replace ((P→Q→R)∧(P→R→Q)) with
4739   (P→(Q↔R)) in n4_38a.
4740 replace (((P→Q)↔(P→R))↔(P→Q↔R)) with
4741   ((P→(Q↔R))↔((P→Q)↔(P→R))) in n4_38a.
4742 apply n4_38a.
4743 apply propositional_extensionality.
4744 specialize n4_21 with (P→Q↔R)
4745   ((P→Q)↔(P→R)).
4746 intros n4_21a. (*Not cited*)
4747 apply n4_21a.
4748 replace (P→Q↔R) with ((P→Q→R)∧(P→R→Q)).
4749 reflexivity.
4750 apply propositional_extensionality.
4751 apply n4_76a.
4752 apply Equiv4_01.
4753 apply Equiv4_01.
4754 Qed.
4755
4756 Theorem n5_75 : ∀ P Q R : Prop,
4757   ((R → ¬Q) ∧ (P ↔ Q ∨ R)) → ((P ∧ ¬Q) ↔ R).
4758 Proof. intros P Q R.
4759 specialize n5_6 with P Q R.
4760 intros n5_6a.
4761 replace ((P∧¬Q→R)↔(P→Q∨R)) with
4762   (((P∧¬Q→R)→(P→Q∨R)) ∧
4763     ((P→Q∨R)→(P∧¬Q→R))) in n5_6a.
4764 specialize Simp3_27 with
4765   ((P∧¬Q→R)→(P→Q∨R))
4766   ((P→Q∨R)→(P∧¬Q→R)).
4767 intros Simp3_27a.
4768 MP Simp3_27a n5_6a.

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4769 specialize Simp3_26 with
4770    $(P \rightarrow (Q \vee R)) \ ((Q \vee R) \rightarrow P)$ .
4771 intros Simp3_26a.
4772 replace  $((P \rightarrow (Q \vee R)) \wedge ((Q \vee R) \rightarrow P))$  with
4773    $(P \leftrightarrow (Q \vee R))$  in Simp3_26a.
4774 Syll Simp3_26a Simp3_27a Sa.
4775 specialize Simp3_27 with
4776    $(R \rightarrow \neg Q) \ (P \leftrightarrow (Q \vee R))$ .
4777 intros Simp3_27b.
4778 Syll Simp3_27b Sa Sb.
4779 specialize Simp3_27 with
4780    $(P \rightarrow (Q \vee R)) \ ((Q \vee R) \rightarrow P)$ .
4781 intros Simp3_27c.
4782 replace  $((P \rightarrow (Q \vee R)) \wedge ((Q \vee R) \rightarrow P))$  with
4783    $(P \leftrightarrow (Q \vee R))$  in Simp3_27c.
4784 Syll Simp3_27b Simp3_27c Sc.
4785 specialize n4_77 with P Q R.
4786 intros n4_77a.
4787 replace  $(Q \vee R \rightarrow P)$  with  $((Q \rightarrow P) \wedge (R \rightarrow P))$  in Sc.
4788 specialize Simp3_27 with  $(Q \rightarrow P) \ (R \rightarrow P)$ .
4789 intros Simp3_27d.
4790 Syll Sa Simp3_27d Sd.
4791 specialize Simp3_26 with  $(R \rightarrow \neg Q) \ (P \leftrightarrow (Q \vee R))$ .
4792 intros Simp3_26b.
4793 Conj Sd Simp3_26b.
4794 split.
4795 apply Sd.
4796 apply Simp3_26b.
4797 specialize Comp3_43 with
4798    $((R \rightarrow \neg Q) \wedge (P \leftrightarrow (Q \vee R))) \ (R \rightarrow P) \ (R \rightarrow \neg Q)$ .
4799 intros Comp3_43a.
4800 MP Comp3_43a H.
4801 specialize Comp3_43 with R P  $(\neg Q)$ .
4802 intros Comp3_43b.
4803 Syll Comp3_43a Comp3_43b Se.
4804 clear n5_6a. clear Simp3_27a.
4805   clear Simp3_27c. clear Simp3_27d.
4806   clear Simp3_26a. clear Comp3_43b.
4807   clear Simp3_26b. clear Comp3_43a.
4808   clear Sa. clear Sc. clear Sd. clear H.
4809   clear n4_77a. clear Simp3_27b.
4810 Conj Sb Se.

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4811 split.
4812 apply Sb.
4813 apply Se.
4814 specialize Comp3_43 with
4815   (( $R \rightarrow \neg Q$ )  $\wedge$  ( $P \leftrightarrow Q \vee R$ ))
4816   ( $P \wedge \neg Q \rightarrow R$ ) ( $R \rightarrow P \wedge \neg Q$ ).
4817 intros Comp3_43c.
4818 MP Comp3_43c H.
4819 replace (( $P \wedge \neg Q \rightarrow R$ )  $\wedge$  ( $R \rightarrow P \wedge \neg Q$ )) with
4820   ( $P \wedge \neg Q \leftrightarrow R$ ) in Comp3_43c.
4821 apply Comp3_43c.
4822 apply Equiv4_01.
4823 apply propositional_extensionality.
4824 apply n4_77a.
4825 apply Equiv4_01.
4826 apply Equiv4_01.
4827 apply Equiv4_01.
4828 Qed.
4829
4830 End No5.

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